Thickly Resolvable Designs

by

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B.Sc., University of Victoria, 2007
M.Sc., University of Victoria, 2009

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ABSTRACT

In this dissertation, we consider a generalization of the historically significant problem posed in 1850 by Reverend Thomas Kirkman which asked whether it was possible for 15 schoolgirls to walk in lines of three to school for seven days so that no two of them appear in the same line on multiple days. This puzzle spawned the study of what we now call resolvable pairwise balanced designs, which balance pair coverage of points within blocks while also demanding that the blocks can be grouped in such a way that each group partitions the point-set. Our generalization aims to relax this condition slightly, so that each group of blocks balances point-wise coverage but each point occurs in each group σ times (instead of just once). We call these objects *thickly-resolvable designs*. Here we show that the necessary divisibility conditions for the existence of thickly-resolvable designs are also sufficient when the size of the point set is large enough. A few variations of this problem are considered as well.

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Chapter 1

Introduction

In 1844 the following question was posed by W. S. B. Woolhouse in the Lady's and Gentlemen's Diary:

> How many triads can be made out of n symbols, so that no pair of symbols shall be comprised more than once amongst them? [49]

A solution to this problem was published by Reverend Thomas Kirkman in 1847 in [28] which showed how to construct an example of such a system whenever $n \equiv 1, 3 \pmod 6$. He also noticed in some cases that the 3-sets of symbols could be grouped so that each grouping partitioned the n symbols; and so, in 1850, he decided to submit the following puzzle, a version of this problem for the case when $n = 15$:

> Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily, so that no two shall walk twice abreast. [29]

This puzzle, known as the Kirkman Schoolgirl Problem, generated much interest, especially within the mathematical community, and is thought to be the origin of the study of what we now call resolvable designs. We will now define these objects in more generality.

A *pairwise balanced design* with parameters v, K, and λ, or $\text{PBD}_\lambda(v, K)$, is a pair $(V, \mathcal{B})$, where V is a v-set of *points*, K is a set of positive integers, $\mathcal{B}$ is a collection of subsets of V whose cardinalities belong to K (the subsets are called *blocks*), and such that every pair of distinct points occur together in exactly λ blocks of $\mathcal{B}$. Notice that we refer to $\mathcal{B}$ as a collection of blocks rather than a set because blocks may be repeated. The parameters v, k, λ are often called the *order*, *block size*, and *index*, respectively. When $K = \{k\}$, we write $\text{PBD}_\lambda(v, k)$ or $\text{BIBD}(v, k, \lambda)$ since these are also known as *balanced incomplete block designs* when the block size is constant. Also, we will often omit the index λ when it equals 1. Throughout this dissertation, we will mainly be interested in designs with constant block size.

Independent of Kirkman's earlier work, in 1853 in [40] Jacob Steiner also introduced and studied designs with constant block size 3 and index 1 and, as his work was better known at the time, these objects were named in his honour. Thus, a $\text{PBD}(v, 3)$ is more commonly referred to as a *Steiner triple system* of order v and denoted $\text{STS}(v)$.

Example 1.1. Here we present a $\text{PBD}(7, 3)$, also known as an $\text{STS}(7)$, which (up to isomorphism) is unique and is the smallest non-trivial PBD:

$$\begin{aligned} V &= \{0, 1, 2, 3, 4, 5, 6\} \quad \text{and} \\ \mathcal{B} &= \{\{0, 2, 6\}, \{1, 2, 4\}, \{1, 3, 6\}, \{2, 3, 5\}, \{0, 1, 5\}, \{0, 3, 4\}, \{4, 5, 6\}\}. \end{aligned}$$

There is a useful representation of this PBD called the Fano Plane or the projective plane of order 2; see Figure 1.1. The blocks are the lines (including the circle) of the diagram.

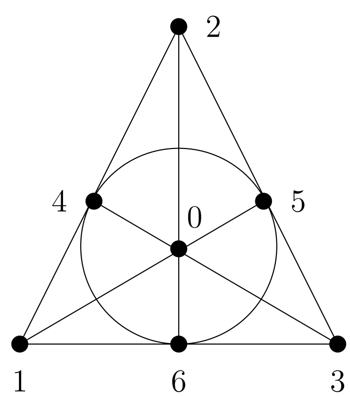

Figure 1.1: The Fano Plane: A $\text{PBD}(7, 3)$

Since Kirkman's first work in this area, people have wondered for what values of v does a $\text{PBD}_\lambda(v, k)$ exist. As Kirkman (and Steiner) proved, this question is completely settled when $k = 3$ and $\lambda = 1$.

Theorem 1.2. *An STS*(v) *exists if and only if* $v \equiv 1$ *or* $3 \pmod 6$, $v \geq 1$.

We now turn our attention to more general block sizes.

Example 1.3. Below is an example of a PBD$(11, \{3, 5\})$ with two different block sizes, 3 and 5. In fact, since this PBD has exactly one block of size 5, it is often denoted as a PBD$(11, \{3, 5^*\})$.

A PBD$(11, \{3, 5\})$		
{1,6,7}	{1,8,11}	{1,9,10}
{2,6,8}	{2,7,9}	{2,10,11}
{3,6,9}	{3,7,11}	{3,8,10}
{4,6,10}	{4,7,8}	{4,9,11}
{5,6,11}	{5,7,10}	{5,8,9}
	{1,2,3,4,5}	

Table 1.1: A PBD$(11, \{3, 5^*\})$

The existence question for a set of block sizes K (or even just a general block size k) is much more challenging. There has been quite a lot of work done here, but with the exception of various small or special (sets of) block sizes, in general there are no necessary and sufficient conditions that guarantee PBDs exist.

There are necessary conditions on v however that must be satisfied in order for a PBD$_\lambda(v, K)$ to exist.

First, each block of size k covers $\binom{k}{2}$ pairs of points and since we need to cover each of the $\binom{v}{2}$ pairs exactly λ times each, we must demand that

$$\lambda v(v-1) \equiv 0 \pmod{\beta(K)}, \tag{1.1}$$

where $\beta(K) = \gcd\{k(k-1) : k \in K\}$. This condition is often referred to as the *global condition* since it guarantees the overall average number of blocks covering each pair of points is an integer.

We also need to ensure that it is possible for a point to appear within some block with all $v-1$ other points exactly λ times each. Since each time a point is included within a block of size k, it appears with $k-1$ other points, we need

$$\lambda(v-1) \equiv 0 \pmod{\alpha(K)}, \tag{1.2}$$

where $\alpha(K) = \gcd\{k-1 : k \in K\}$. This is usually referred to as the *local condition* since it focusses on pair coverage at a particular point.

Notice when $K = \{k\}$, these conditions simplify to $\lambda v(v-1) \equiv 0 \pmod{k(k-1)}$ and $\lambda(v-1) \equiv 0 \pmod{k-1}$, respectively.

As done in Theorem 1.2 for $k = 3$, Hanani proved in [23] that the congruences given in (1.1) and (1.2) are not only necessary, but also sufficient for $k = 4$. He was also able to prove the congruences are sufficient for $k = 5$ for all admissible values except $(v, k, \lambda) = (15, 5, 2)$ in [24]. The non-existence of a $\text{PBD}_2(15, 5)$ was a fact discovered in 1945 by Nandi in [34].

Although these two conditions are not in general sufficient for the existence of a $\text{PBD}_\lambda(v, K)$, they are known to be 'asymptotically' sufficient. In other words, when v is large enough (bigger than some constant $v_0(K, \lambda)$) these designs exist for all values of v satisfying the necessary arithmetic conditions. Richard M. Wilson was the first to carefully consider the question of asymptotic existence of designs and has since settled asymptotic existence for many types of designs. Wilson's Theorem 1 in [45] was a monumental result of this type.

Theorem 1.4. [45] *Given a set K of positive integers and a positive integer λ, there exists a PBD(v, K, λ) for all sufficiently large integers v satisfying* (1.1) *and* (1.2)*.*

Another important necessary condition for the existence of a $\text{PBD}_\lambda(v, k)$ is known as Fisher's Inequality and is stated below in Theorem 1.5.

Theorem 1.5. *In any PBD$_\lambda(v, k)$ we must have $b \geq v$, where $b = \frac{\lambda v(v-1)}{k(k-1)}$ is the number of blocks in the design.*

We would like to note that it is possible for parameters v, k, and λ to satisfy the global and local conditions, while failing the one given in Theorem 1.5. For example see Example 1.6.

Example 1.6. Consider the parameters $(v, k, \lambda) = (16, 6, 1)$. It seems possible from (1.1) and (1.2) for a PBD$(16, 6)$ to exist, since

$$\lambda(v-1) = 15 \equiv 0 \pmod{5} \quad \text{and} \quad \lambda v(v-1) = 240 \equiv 0 \pmod{30};$$

however,

$$b = \frac{\lambda v(v-1)}{k(k-1)} = \frac{240}{30} = 8.$$

Thus, Theorem 1.5 is not satisfied so there does not exist a PBD$(16, 6)$.

In general, the necessary conditions given in (1.1), (1.2), and Theorem 1.5 are not also sufficient (although Fisher's inequality is always satisfied when v is sufficiently large).

We would like to mention that there are numerous ways to extend the idea of pairwise balanced designs, some of which will be discussed in detail in Chapter 2. One particular extension, beyond what is needed in this dissertation, is that of a t-design. Here we ask that every t-set of elements occur exactly λ times (so pairwise balanced designs are equivalent to 2-designs). Not much was known about this extension until 2014 when Keevash proved a generalization of the existence result given in Theorem 1.4 for constant block size k and higher values of t using randomized algorithms and probabilistic algebraic constructions. In this dissertation, we will only be concerned with the $t = 2$ case.

Many designs have additional structure. Even the problem that Kirkman posed in 1844 asked for more than just balanced pair coverage; the solution had to be able to partition the girls into groups of 3 each day. The designs given in Example 1.1 and Example 1.3 do not boast the extra structure that Kirkman asked for in his schoolgirl problem; namely, the blocks cannot be grouped in such a way that each group partitions the point set. This can be seen easily in Example 1.1 by noting that each block intersects every other block in exactly one point and in Example 1.3 that the block of size 5 intersects all other blocks. A design satisfying Kirkman's schoolgirl property is called *resolvable* and a grouping of blocks partitioning the point set is called a *parallel class* or *resolution class*. This concept is also studied in finite geometries, especially in 3-dimensional projective space. In this context, resolvability is studied under the term *parallelism* and resolution classes of lines are known as *spreads*.

Kirkman's schoolgirl problem is an example of a resolvable $\mathrm{PBD}(15, 3)$, or equivalently a resolvable STS(15), and in [28] he exhibited a solution for the 15 girls. However, the problem remained open for v girls until 1971 when Ray-Chaudhuri and Wilson showed that resolvable triple systems exist whenever $v \equiv 3 \pmod 6$ (stated below). Resolvable Steiner triple systems are now often referred to as *Kirkman triple systems*.

Theorem 1.7. [35] *A resolvable STS*(v) *exists if and only if* $v \equiv 3 \pmod 6$, $v \geq 3$.

Example 1.8. Here we present an example of a resolvable STS(9) (or Kirkman Triple System of order 9), which we know exists by Theoreom 1.7.

$$\begin{aligned} V &= \{1,2,3,4,5,6,7,8,9\} \quad \text{and} \\ \mathcal{B} &= \left\{ \begin{array}{cccccc} \{1,2,3\}, & \{1,4,7\}, & \{1,5,9\}, & \{1,6,8\}, & \{2,4,9\}, & \{2,5,8\}, \\ \{2,6,7\}, & \{3,4,8\}, & \{3,5,7\}, & \{3,6,9\}, & \{4,5,6\}, & \{7,8,9\} \end{array} \right\}. \end{aligned}$$

As in Example 1.1, this PBD can also be represented diagrammatically; see Figure 1.2. The blocks of each parallel class are drawn in the same colour. Despite having

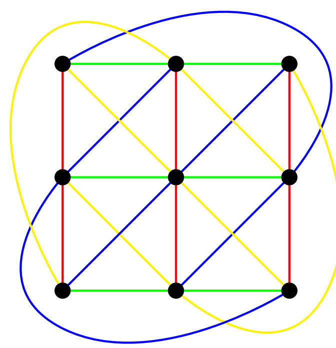

Figure 1.2: The Affine Plane of order 3: A PBD$(9,3)$

a complete existence result for the $k=3$ and $\lambda=1$ case, the existence question for larger values of k and λ remained open until 1973, when D.K. Ray-Chaudhuri and R.M. Wilson made some progress by proving the necessary conditions were asymptotically sufficient in v for arbitrary values of k with $\lambda=1$; their result is stated below as Theorem 1.9. Since resolvable designs are simply designs containing extra structure, the parameters must still satisfy (1.1) and (1.2); however, to ensure we are able to partition the v points into blocks of size k we must also require that

$$v \equiv 0 \pmod{k}. \tag{1.3}$$

This congruence is referred to as the *resolvabilty condition*. Notice, in the $K=\{k\}$ case, the local condition, $\lambda(v-1) \equiv 0 \pmod{k-1}$, and the resolvability condition, $v \equiv 0 \pmod{k}$, together imply the global condition; and so, the global condition is often omitted. Also, when $\lambda=1$, the local and resolvability conditions can be more succinctly written as $v \equiv k \pmod{k(k-1)}$.

Theorem 1.9. [36] *Let $k \geq 2$ be an integer. For sufficiently large v, there exists a resolvable PBD(v,k) if and only if $v \equiv k \pmod{k(k-1)}$.*

In 1984, J.X. Lu extended this result in [32] to allow for arbitrary values of the index λ.

Another necessary condition, which is similar to Fisher's Inequality (Theorem 1.5), was found in 1942 by Bose.

Theorem 1.10. *If there exists a resolvable PBD$_\lambda(v,k)$, then $b \geq v + r - 1$.*

We again note that these necessary conditions for the existence of a resolvable PBD are not in general sufficient. However, Theorem 1.12 gives a necessary and sufficient condition for the existence of a resolvable PBD$(v,2)$.

Remark 1.11. A PBD$(v,2)$ could be constructed trivially by taking each of the $\binom{v}{2}$ pairs as its own block. However, the extra resolvability condition results in a more difficult and interesting concept, namely that of a 1-factorization.

Theorem 1.12. *A resolvable PBD$(v,2)$ exists if and only if v is an even integer and $v \geq 2$.*

The resolvable PBDs in Theorem 1.12 can be constructed for all even values of $v = 2n$ by using the *patterned starter* $\{0,\infty\},\{1,-1\},\{2,-2\},\ldots,\{n-1,n\}$ on the points of $\mathbb{Z}_{2n-1} \bigcup \{\infty\}$ and developing modulo $2n-1$. This can be seen in Figure 1.3 for $v = 10$.

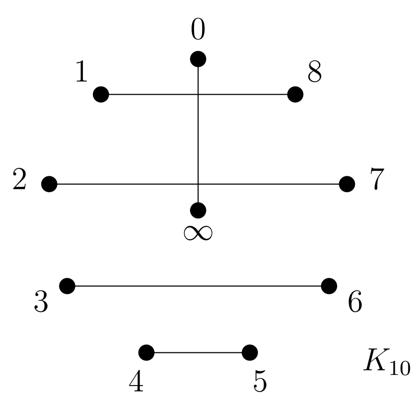

Figure 1.3: A Resolvable PBD$(10,2)$

The $k = 3$ case was settled by Ray-Chaudhuri and Wilson, stated above as Theorem 1.7. Later, in [25], Hanani *et al.* settled the $k = 4$ case and showed that resolvable PBD$(v, 4)$s exist whenever $v \equiv 4 \pmod{12}$.

In this dissertation, we will be interested in a parameterized weakening of resolvability. Here we will focus on designs with the additional property that blocks can be resolved so that each class covers every point of V exactly σ times. More precisely, we say that a *σ-parallel class* in a PBD $(V, \mathcal{B})$ is a sub-collection of blocks $\mathcal{A}$ such that every point of V belongs to exactly σ blocks in $\mathcal{A}$ and we refer to σ as the *thickness* of the class. Then, a PBD$_\lambda(v, k)$ $(V, \mathcal{B})$ is *σ-resolvable* if $\mathcal{B}$ admits a partition into σ-parallel classes. We will also refer to σ-resolvable designs more generally as *thickly resolvable designs*.

In the spirit of Kirkman's schoolgirl problem, a scenario where thickly resolvable designs would be useful is a meeting where v countries each send σ delegates. In each time slot, each representative must attend a caucus meeting containing k delegates and each country must have a representative attend a caucus meeting with a representative from each other country exactly λ times. A σ-resolvable design on v points, block size k, and index λ would give such a schedule.

Before we discuss the necessary conditions and results for σ-resolvable designs, we would like to note that most authors on this topic use the parameter α in place of σ to denote the thickness of each class. We have chosen to use σ instead to avoid conflict with the standard parameter $\alpha(K) = \gcd\{k - 1 : k \in K\}$, first introduced in [43] within the necessary conditions of PBDs with multiple block sizes from a set K. Since we chose to make use of this meaning for α, we have elected to use σ for the class thickness.

The necessary divisibility conditions for the existence of a σ-resolvable design are given in (1.4) and (1.5), and are easy extensions of the necessary conditions for resolvable designs (where $\sigma = 1$) given above. For the resolvability condition here we need to ensure that we can cover, counting multiplicity, σv points (each point in V σ times) using blocks of size k, yielding the condition

$$\sigma v \equiv 0 \pmod{k}. \tag{1.4}$$

This can also be thought of as counting the number of blocks in each σ-parallel class, which obviously must be an integer.

For the local condition, we want to make sure that it is (numerically) possible for a point v to occur with each other point in V within exactly λ blocks. Each point $v \in V$ must occur with exactly $\lambda(v-1)$ other points (including multiplicity) and in each σ-parallel class v will appear within a block with $\sigma(k-1)$ other vertices ($k-1$ vertices in each of the σ blocks containing v); thus, $\frac{\lambda(v-1)}{\sigma(k-1)}$ counts the σ-parallel classes and we must demand that

$$\lambda(v-1) \equiv 0 \pmod{\sigma(k-1)}, \tag{1.5}$$

so that the number of σ-parallel classes is an integer. Or equivalently, we must have that σ divides the replication number $r = \frac{\lambda(v-1)}{k-1}$, the number of blocks containing a specific point.

As in the $\sigma = 1$ case, these two conditions together imply the third (global) condition, $\lambda v(v-1) \equiv 0 \pmod{k(k-1)}$, which results from the fact that we need to cover each of the $\binom{v}{2}$ pairs of points using only blocks that each cover $\binom{k}{2}$ pairs of points.

Example 1.13. One Thick-Parallel Class.
Our first example is not particularly enlightening, but we would like to note that all designs are r-resolvable; that is, if we put all the blocks into one class, each point will occur r times (since r is the replication number of the design).

Example 1.14. Resolvable PBDs.
One way to construct examples of thickly-resolvable designs for $\sigma > 1$ (perhaps uninterestingly) is to use resolvable designs in which the number of parallel classes is a multiple of our desired thickness. Here we can merge the parallel classes so that each aggregate has σ of the classes. Each original parallel class will contribute one occurrence of each point; and hence, this will result in a σ-resolvable design

For instance, we can group the 4 parallel classes of the resolvable PBD$(9,3)$ given in Example 1.8 so that each group contains the blocks of 2 of the classes to obtain a 2-resolvable PBD$(9,3)$. In Figure 1.4, we show the 2-resolvable PBD$(9,3)$ that results by converting the four parallel classes of the resolvable PBD$(9,3)$ in Example 1.8 into two 2-parallel classes (so that each point appears in exactly 2 blocks of each 2-class).

We would like to note that despite the construction method used in Example 1.14, σ-resolvable designs cannot always be constructed using a resolvable design with the same parameters. In fact, thickly-resolvable designs might be most useful for parameter choices in which a resolvable design is not arithmetically possible.

Thick-resolvability enables us to generalize resolvability and obtain designs that are 'close' to resolvable.

Remark 1.15. Another possible weakening of resolvability is to consider the chromatic index of designs. A design is said to be *s-block-colourable* if it is possible to colour the blocks using s colours so that no two intersecting blocks receive the same colour. Then, all blocks of a common colour would form a partial resolution class (usually referred to as a chromatic class). The minimum number of colours required to colour all the blocks in this way is referred to as the *chromatic index* and the design would be resolvable if and only if the chromatic index is equal to the replication number of the design (the number of parallel classes). For Steiner triple systems, partial colour classes of nontrivial thickness have been considered in the literature as 'block colourings of type π', where π is an integer partition of the replication number; see [10] for details.

Example 1.16. Resolvable Cycle Decompositions.
A 2-resolvable $\text{PBD}_\lambda(v, 2)$ is equivalent to a resolvable decomposition of the complete graph on v vertices with λ edges between every pair of vertices, K_v^λ, into cycles. We use each edge of a cycle in the decomposition as a block of the design and we group all the edges resulting from cycles in each parallel class of the decomposition together to make our 2-parallel classes. Figure 1.5 shows a decomposition of K_7 into Hamiltonian cycles (cycles passing through every vertex exactly once). The edges of each coloured cycle in Figure 1.5 make up a 2-parallel class in a 2-resolvable PBD$(7, 2)$.

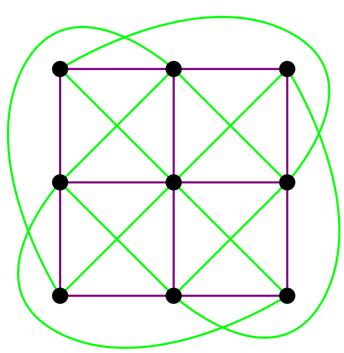

Figure 1.4: 2-Resolvable PBD$(9, 3)$ constructed using the method outlined in Example 1.14

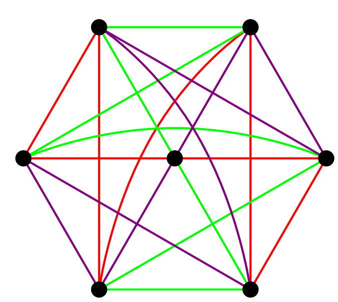

Figure 1.5: 2-Resolvable PBD$(7, 2)$ constructed using the method outlined in Example 1.16

Resolvable cycle decompositions are also related to the famous *Oberwolfach Problem*, which asks whether it is possible to seat an odd number m of mathematicians at n round tables in $\frac{m-1}{2}$ meals so that each mathematician sits next to everyone else exactly once if the n round tables are of sizes $k_1, k_2, \ldots, k_n$ (with $k_1+k_2+\cdots+k_n = m$). A resolvable cycle decomposition would be equivalent to such a seating arrangement where we decompose K_m into cycles of length $k_1, k_2, \ldots, k_n$ in which one cycle of each length is used in each parallel class. The cycle decomposition given in Figure 1.5 gives a solution to the Oberwolfach Problem when there are seven mathematicians and one round table for them all to sit at.

Example 1.17. Cyclic PBDs.
A k-resolvable PBD(v, k) can be obtained from a cyclic PBD(v, k) (that is, one possessing a transitive cyclic automorphism) in which no base block has a short orbit. In other words, there are k unique translate blocks for each base block. We develop each of the base blocks additively modulo v with each resulting in a k-parallel class since each point is a translate of each of the k elements of the base block and so will appear in exactly k of the developed blocks.

For example, we could start with the cyclic PBD$(13, 3)$ with two generator blocks: $\{1, 3, 9\}$ and $\{2, 5, 6\}$. When we additively develop these base blocks modulo 13, each one results in a 3-parallel class. So this will yield a 3-resolvable PBD$(13, 3)$ that has two 3-parallel classes. To see this more explicitly, in Table 1.2 we have written out the blocks grouped in their 3-classes.

Blocks of a cyclic PBD(13, 3)			
{1,3,9}	{2,4,10}	{2,5,6}	{3,6,7}
{3,5,11}	{4,6,12}	{4,7,8}	{5,8,9}
{5,7,0}	{6,8,1}	{6,9,10}	{7,10,11}
{7,9,2}	{8,10,3}	{8,11,12}	{9,12,0}
{9,11,4}	{10,12,5}	{10,0,1}	{11,1,2}
{11,0,6}	{12,1,7}	{12,2,3}	{0,3,4}
{0,2,8}		{1,4,5}	

Table 1.2: A 3-Resolvable PBD$(13, 3)$

Example 1.18. Computer Generated.

Table 1.3 gives the blocks of a $\text{PBD}_2(10,3)$ found by Royle (and presented in [26]), where the groups of two columns partition the blocks into 3-parallel classes, making this design 3-resolvable. In fact, Royle showed that each of the 960 two-fold triple systems on 10 points (i.e. $\text{PBD}_2(10,3)$s) are 3-resolvable!

Blocks of a $\text{PBD}_2(10,3)$					
{0,1,2}	{2,6,7}	{0,2,3}	{3,4,8}	{0,3,1}	{1,5,9}
{0,4,5}	{3,5,9}	{0,5,6}	{1,6,7}	{0,6,4}	{2,4,8}
{0,7,8}	{3,6,8}	{0,8,9}	{1,4,9}	{0,9,7}	{2,5,7}
{1,2,4}	{3,7,9}	{2,3,5}	{1,8,7}	{3,1,6}	{2,9,8}
{1,5,8}	{4,6,9}	{2,6,9}	{5,4,7}	{3,4,7}	{6,5,8}

Table 1.3: A 3-Resolvable $\text{PBD}_2(10,3)$

There has not been a lot of progress made on the existence question for thickly-resolvable designs with $\sigma > 1$. Analogous to the $\sigma = 1$ case, the first results in this area were for designs with block sizes three and four.

In 1991, Jungnickel, Mullin, and Vanstone showed in [26] that there exists a σ-resolvable $\text{PBD}_\lambda(v,3)$ for all choices of v, σ, and λ that satisfy (1.4) and (1.5) except when $v = 6$, $\sigma = 1$ and $\lambda \equiv 2 \pmod 4$.

A similar result was proved by Vasiga, Furino, and Ling in [42] for thickly-resolvable designs with blocksize 4, except this time with only one exception: $v = 10$ where $\lambda = \sigma = 2$. Both the results in [26] and [42] were proved using frames and some small constructed examples. We will discuss frames in Chapter 2 and use them throughout the dissertation.

The main goal of this dissertation is to prove that σ-resolvable designs exist asymptotically in v whenever it is admissible for a given choice of k, λ, and σ. Here is the statement of our main result. This work has appeared as published work with Dukes and Ling in [15].

Theorem 1.19. [15] *Let $k \geq 2$, $\sigma \geq 1$, and $\lambda \geq 0$ be integers. There exists a σ-resolvable $PBD_\lambda(v,k)$ for all sufficiently large v satisfying* (1.4) *and* (1.5)*; that is, $\sigma v \equiv 0 \pmod k$ and $\lambda(v-1) \equiv 0 \pmod{\sigma(k-1)}$.*

We say that the integers v satisfying (1.4) and (1.5) are *admissible* for the particular choice of σ, k, and λ. Also, for convenience, we will use the variables

$a = \frac{\sigma(k-1)}{\gcd(\sigma(k-1),\lambda)}$ and $b = \frac{k}{\gcd(k,\sigma)}$ so that our necessary conditions can be written more succinctly as

$$\begin{aligned} v &\equiv 1 \pmod{a} \quad \text{and} \\ v &\equiv 0 \pmod{b}. \end{aligned}$$

Taking advantage of this rephrasing, we can assume that a and b are relatively prime integers, since otherwise there are no admissible v values for that choice of parameters. Thus, the admissible orders v will be periodic with least period $\pi = ab$.

Our proof of Theorem 1.19 uses asymptotic existence theories for combinatorial configurations, resolvable graph decompositions, and frames. The background for these structures will be discussed in Chapter 2. Then in Chapter 3 we focus on one very powerful result by Lamken and Wilson in [30] on coloured graph decompositions, which we apply in order to prove the asymptotic existence of σ-frames (a very crucial component within the proof of Theorem 1.19). In Chapter 4 we discuss non-uniform σ-frames and adapt a couple of classical constructions within design theory to include the parameter σ. Chapter 5 is where the proof of Theorem 1.19 can be found. We begin by constructing our first examples of thickly-resolvable designs using resolvable graph decompositions with a graph cleverly constructed making use of combinatorial configurations. In Section 5.2 we use σ-frames to obtain examples in each admissible congruence classes modulo a large period. These are extended in Section 5.3 to close out each congruence class, making use of non-uniform σ-frames, and obtain eventual periodicity modulo ab. In Section 5.4, we put all these pieces together to prove Theorem 1.19. Finally, in Chapter 6 we discuss a few applications and extensions of Theorem 1.19, including an application to incomplete designs as well as extending to group divisible designs and graph designs.

Chapter 2

Background

In this chapter we will consider some generalizations of pairwise balanced designs; specifically, we will focus on the ones that we will use within the proof of Theorem 1.19.

2.1 Graph Decompositions

In this section, we present our first generalization of pairwise balanced designs, where we allow blocks to cover only some of the pairs of points they contain. We make use of graphs to show which pairs will be covered (and which will not) for each block.

Definition 2.1. A *G-decomposition* (or *G-design*) of *order* v and *index* λ is a pair $(V, \mathcal{B})$, where V is a set of points, $\mathcal{B}$ is a collection of graphs on vertices in V, where each is isomorphic to G, and such that every unordered pair of points in V is an edge of exactly λ graphs in $\mathcal{B}$. The members of $\mathcal{B}$ are called *G-blocks*.

In other words, a G-decomposition is an edge-decomposition of K_v^λ (the λ-fold complete graph on v vertices) into graphs each isomorphic to G. In a G-block, only pairs of vertices with an edge between them will be 'covered'. Consequently, when $G = K_k$, a G-decomposition is a equivalent to a $\text{PBD}_\lambda(v, k)$ and because of this connection we will abbreviate a G-design of order v and index λ to a $\text{PBD}_\lambda(v, G)$. In this context, we can also interpret a $\text{PBD}_\lambda(v, K)$ as a partition of the edge set of K_v^λ into k-cliques where $k \in K$.

Notice that Example 1.1 is an example of a K_3-decomposition of K_7 and this can be seen pictorially as in Figure 1.1 except with triangles instead of the lines.

Example 2.2. When v is odd, K_v admits an Eulerian trail. If additionally, $\binom{v}{2}$ is even, then such a trail can be cut up into copies of P_3, the path on two edges, resulting in a PBD(v, P_3).

Example 2.3. If we think of a PBD$(v, 4)$ as an edge-decomposition of K_v into copies of K_4, then we can break up the K_4-blocks into 2 copies of the path P_4 on three edges, as can be seen in Figure 2.1.

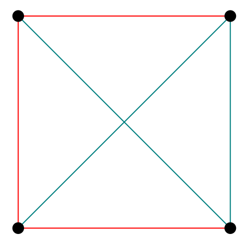

Figure 2.1: K_4 edge-decomposed into two paths P_4

Example 2.4. Walecki Hamiltonian Decompositions.
In a classical result of graph theory, Walecki showed when v is odd that K_v can be decomposed into Hamiltonian cycles. See [1] for details into the construction Walecki used. Figure 1.5 is an example of such a decomposition. In Example 1.16, Figure 1.5 is an example of a 2-resolvable PBD$(7, 2)$ where the blocks are the edges. Here, we use each Hamilton cycle as a C_7-block, so this is also a PBD$(7, C_7)$.

The arithmetic necessary conditions for a PBD$_\lambda(v, k)$ can be derived using this framework. First, for the global condition (given in (1.1) but with $K = \{k\}$), we note that there are $\lambda\binom{v}{2}$ edges in K_v^λ which must be partitioned into blocks each containing $\binom{k}{2}$ edges. To obtain the local condition in (1.2), we note that we must be able to write the degree at a vertex in K_v^λ, $\lambda(v-1)$, using blocks each using $k-1$ edges at a vertex.

For general graphs, these necessary conditions are a bit more complicated. Suppose we want a G-decomposition for a simple graph G with k vertices, e edges, and vertex degrees $d_1, d_2, \ldots, d_k$. First, because we need to partition the number of edges of K_v^λ into graphs isomorphic to G (so each having e edges), we must demand that

$$\lambda v(v-1) \equiv 0 \pmod{2e}; \tag{2.1}$$

thus, obtaining our global condition.

Second, for the local condition, we need to make sure that the degree of a vertex in K_v^λ can be comprised of degrees in G; and so, we need to be able to write $\lambda(v-1)$ as a positive integral linear combination of degrees in G. In other words, we need

$$\lambda(v-1) \equiv 0 \pmod{g} \tag{2.2}$$

where g is the greatest common divisor of the degrees in G. We would like to point out that when G is d-regular, (2.2) simplifies to $\lambda(v-1) \equiv 0 \pmod{d}$; thus, when $G = K_k$ this is equivalent to (1.2) for constant block size k.

Example 2.5. In contrast to Example 2.4, for v even (and $\lambda = 1$) there cannot be a Hamiltonian decomposition of K_v since (2.2) demands that $v \equiv 1 \pmod{2}$.

Example 2.6. Consider $G = K_4 \setminus \{e\}$, the complete graph on 4 vertices minus an edge. For this G we have $k = 4$, $e = 5$ and degrees 2 and 3. Thus, the necessary conditions for this graph become

$$\lambda v(v-1) \equiv 0 \pmod{10} \quad \text{and} \quad \lambda(v-1) \equiv 0 \pmod{1}.$$

Notice that the local condition here is satisfied trivially for all v (whereas, in the clique case, we would need to ensure $3 \mid \lambda(v-1)$).

There are some known results for particular graphs G, especially when G is a cycle, path or tree.

Theorem 2.7. [2, 38] *For $\lambda = 1, 2$ and $m \geq 3$, there exists a PBD$_\lambda(v, C_m)$ if and only if v satisfies* (2.1) *and* (2.2)*.*

Theorem 2.7 was proved in two papers. In order to decompose into cycles, the necessary conditions demand that v be odd (since cycles are 2-regular graphs). It was shown by Alspach and Gavlas that such decompositions exist whenever the cycle length is odd in [2] and the even length cycle case was completed by Šajna in [38].

Wilson considered this connection between designs and graph decompositions and wondered if it was possible to get an asymptotic existence theory of graph decompositions for any graph G. It did not take him long to prove Theorem 2.8, and settle the asymptotic existence of G-decompositions with $\lambda = 1$. This is among Wilson's most famous results.

Theorem 2.8. [48] *Given a graph G with k vertices, e edges, and degrees $d_1, d_2, \dots, d_k$, there exists a G-decomposition of K_v for all sufficiently large values of v satisfying* (2.1) *and* (2.2) *with $\lambda = 1$.*

We would like to point out when G has no edges ($e = 0$) that (2.1) forces either $v = 1$ or $\lambda = 0$. Also, Theorem 2.8 for the case of a general index λ was settled by very similar methods, although not explicitly contained in [48].

We can also define resolvability in this context: a G-decomposition of K_v^λ is said to be *resolvable* when the G-blocks can be grouped so that each set of blocks partitions the vertices of K_v^λ. Analogously, each set of vertex-disjoint G-blocks that span the vertices of K_v^λ is called a *parallel class*.

As in the PBD case, this extra structure adds an extra necessary condition. The resolvability condition here is that the number of vertices of G must divide the number of vertices in K_v^λ so that a single parallel class is possible; hence, we must have that

$$v \equiv 0 \pmod{k}. \tag{2.3}$$

However, in addition to this extra divisibility constraint, there is a new phenomenon that occurs with the local condition. We must ensure that the design is *equireplicate*; that is, each vertex must be incident with the same number of G-blocks in the design. We will refer to this number as the replication number. The number of parallel classes must equal the replication number, since a vertex will appear in exactly one G-block within each parallel class.

There are $\lambda\binom{v}{2}$ edges in K_v^λ and each G-block accounts for e of them; thus, a G-decomposition has $\frac{\lambda}{e}\binom{v}{2}$ G-blocks in total. Now, the number of G-blocks per parallel class is $\frac{v}{k}$, so, by dividing, there must be $\frac{\lambda(v-1)k}{2e}$ parallel classes. Since this necessarily equals the replication number of the design, we need this many vertex degrees in G to make up $\lambda(v-1)$, the degree in K_v^λ. This means we need

$$\begin{aligned} d_1x_1 + d_2x_2 + \cdots + d_kx_k &= \lambda(v-1) \\ \text{and} \qquad x_1 + x_2 + \cdots + x_k &= \frac{\lambda(v-1)k}{2e} \end{aligned}$$

to have simultaneous integer solutions x_i. Or equivalently, we must demand that

$$\lambda(v-1) \equiv 0 \pmod{\alpha^*}, \tag{2.4}$$

where α^* is the least positive integer A such that

$$A \begin{bmatrix} 1 \\ \frac{k}{2e} \end{bmatrix} = \begin{bmatrix} d_1 \\ 1 \end{bmatrix} \mathbb{Z} + \begin{bmatrix} d_2 \\ 1 \end{bmatrix} \mathbb{Z} + \cdots + \begin{bmatrix} d_k \\ 1 \end{bmatrix} \mathbb{Z}. \tag{2.5}$$

Observe that, for d-regular graphs, we will have $\alpha^* = d$ since the sum of the degrees is $kd = 2e$, so taking $A = d$ is the smallest integer in which $\frac{kA}{2e}$ is an integer and (2.5) is satisfied. Thus, when $G = K_k$, (2.4) simplifies to the congruence given in (1.2) for PBDs.

We would also like to note that $A = 2e$ satisfies (2.5) since the sum of the degrees in a graph is equal to twice the number of edges; thus, we have that $\alpha^* \mid 2e$. Now, since $2e$ is a solution, we know that the ideal of solutions generated by α^* is nonempty. In addition, since $2e$ can be written as an integral linear combination of the degrees in G, we must have that the $\gcd\{d_i : i = 1, 2, \ldots, k\} \mid 2e$. Also, since it is an integral span in (2.5), the equation resulting from the first coordinate implies that α^* must be an integral linear combination of the degrees in G; and hence, α^* must be divisible by the greatest common divisor of the degrees. The equation resulting from the second coordinate in (2.5) guarantees that $2e \mid \alpha^* k$. This divisibility clearly implies that (2.1) be satisfied since $\alpha^* \mid \lambda(v-1)$ and $k \mid v$.

Example 2.9. A Resolvable PBD$(9, P_3)$.
There is a resolvable decomposition of K_9 into paths P_3 on two edges. In Figure 2.2, we give two parallel classes each consisting of three copies of P_3. We can develop each of them horizontally twice to obtain two other parallel classes. Together these six parallel classes account for all edges of K_9.

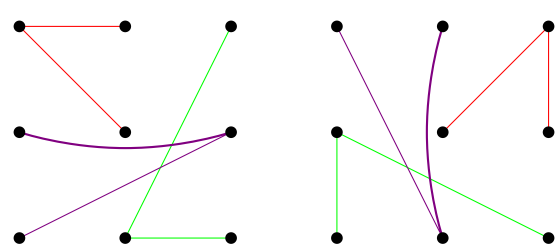

Figure 2.2: A Resolvable K_9 edge-decomposed into paths P_3

Example 2.10. No Resolvable PBD(v, C_4).
Notice C_4 has $k = 4$ vertices, $e = 4$ edges, and is regular of degree 2. Thus the necessary conditions are

$$v \equiv 0 \pmod 4 \quad \text{and} \quad v - 1 \equiv 0 \pmod 2.$$

The first condition implies that v must be even, while the second gives v odd; thus, there are no values of v that satisfy both congruences. Note that this is not the case for other values of λ, and in fact, there is a resolvable $\text{PBD}_2(4, C_4)$ and a resolvable $\text{PBD}_2(8, C_4)$. In Figure 2.3 we exhibit an example of a resolvable $\text{PBD}_2(4, C_4)$ with each of the three resolution classes depicted in their own colour. Figure 2.4 shows the C_4 blocks of one parallel class of a resolvable $\text{PBD}_2(8, C_4)$. The other parallel classes are simply translates modulo 7 of the one shown.

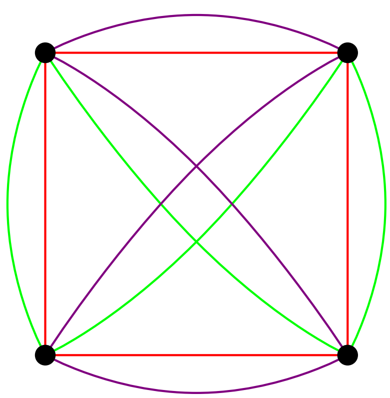

Figure 2.3: A Resolvable $\text{PBD}_2(4, C_4)$

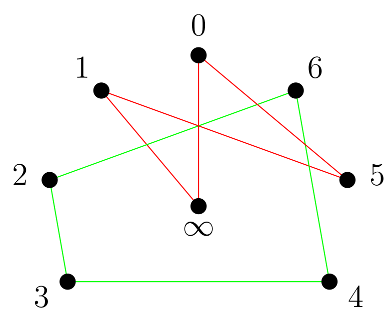

Figure 2.4: A Resolvable $\text{PBD}_2(8, C_4)$

Example 2.11. We refer to Example 2.6 and see how resolvability affects the necessary conditions when $G = K_4 \setminus \{e\}$. Here we need $v \equiv 0 \pmod 4$ and $\lambda(v-1) \equiv 0 \pmod{\alpha^*}$ where α^* is the least positive integer A so that

$$A \begin{bmatrix} 1 \\ \frac{2}{5} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \mathbb{Z} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} \mathbb{Z}.$$

Since this is an integral span, we must have $A \geq 5$. And since

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ \frac{2}{5} \end{bmatrix},$$

$A = 5$ is the smallest such solution; thus, $\alpha^* = 5$. So, in order for a resolvable $\mathrm{PBD}_\lambda(v, G)$ to possibly exist we must have $\lambda v(v-1) \equiv 0 \pmod{20}$.

Asymptotic existence of resolvable graph designs remained open until 2007 when Dukes and Ling were able to show that (2.3) and (2.4) were asymptoticly sufficient conditions.

Theorem 2.12. [14] *Let $\lambda \in \mathbb{Z}, \lambda \geq 0$. Suppose G is a graph with k vertices, e edges with no multiple edges and degree sequence $d_1, d_2, \ldots, d_k$. Then there exists a resolvable G-decomposition for all sufficiently large values of v satisfying* (2.3) *and* (2.4).

The proof of Theorem 2.12 used the existence theory of *equireplicate* G-designs proved in [16] and also adapted a clever construction idea of Rolf Rees from [37].

Another important result in the area of graph decompositions is that of Lamken and Wilson in [30]. We will wait to discuss this result in more detail in Chapter 3, where we will also use it to prove the asymptotic existence of σ-frames (defined in Section 2.3) which will be crucial in the proof of Theorem 1.19.

2.2 Combinatorial Configurations

In this section, we allow for non-complete pair-coverage. In other words, every pair of points is covered by at most one block (but may not be covered at all). These are called *combinatorial configurations*, which we now define formally.

Definition 2.13. An (n_r, m_k)-*configuration* is a triple $(U, \mathcal{A}, \iota)$, where U is an n-set of *points*, $\mathcal{A}$ is an m-set of *lines*, and $\iota \subseteq U \times \mathcal{A}$ is a relation called *incidence* such that

- every line is incident with exactly k points,
- every point is incident with exactly r lines, and
- every pair of distinct points are together incident with at most one line.

Figure 2.5 shows a $(10_3, 10_3)$-configuration. Notice that each (straight) line is on precisely three points, each point is on precisely three lines, but not all pairs appear together on a line, as in PBDs.

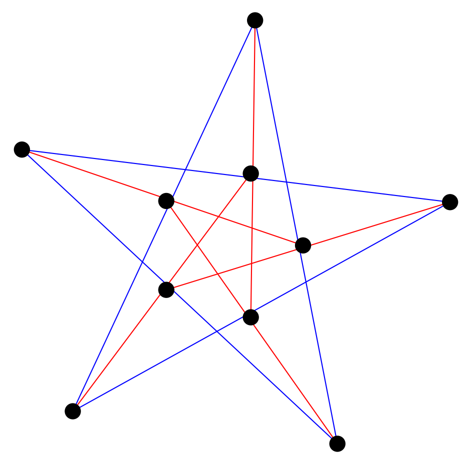

Figure 2.5: A $(10_3, 10_3)$-Configuation - A Non-Desargues Configuration

One of the most famous $(10_3, 10_3)$-configuations is known as the Desargues Configuration, named after Girard Desargues who is considered to be a founder of projective geometry. The example given in Figure 2.5 is called Non-Desargues since it is not isomorphic to the Desargues Configuration. In the example given here, each point has three other points that are not collinear with it and form a triangle of three straight lines; whereas, in the Desargues Configuration, the three non-collinear points are always collinear with each other.

We would like to note that both the Desargues configuration and the one given in Figure 2.5 have geometric realizations with straight lines, however we do not require this for our purposes. For more information on geometric configurations please refer to [18].

Within the proof of Theorem 1.19, we would like to make use of the following asymptotic existence result for combinatorial configurations. We will use this result to construct a graph we will use as a graph block within a graph decomposition in order to obtain our first examples of thickly-resolvable designs.

Theorem 2.14. [4] *Given integers $k \geq 2$ and $r \geq 1$, there exists a combinatorial (n_r, m_k)-configuration for all sufficiently large integers $n \equiv 0 \pmod{\frac{k}{\gcd(k,r)}}$.*

We would like to comment briefly that the n in Theorem 2.14 need not be extremely large as in Wilson's asymptotic results (for instance the main result in [48] stated here as Theorem 2.8). The n here is at worst a mild polynomial in k and r.

2.3 Group Divisible Designs and Frames

In this section, we consider a generalization of PBDs in which some pairs of points are never covered by a block. Here there are groups of points that never occur together while all others are still covered exactly λ times. We now present the formal definition of a group divisible design.

Definition 2.15. Let v and λ be positive integers and let K be a set of positive integers. A *group divisible design* of order v and index λ, denoted $\mathrm{GDD}_\lambda(v, K)$, is a triple $(X, \mathcal{G}, \mathcal{B})$, where

- X is a set of v elements,
- $\mathcal{G} = \{G_1, G_2, \ldots, G_u\}$ is a set of subsets of X which partition X called *groups*,
- $\mathcal{B}$ is a family of subsets of X each of cardinality from K called *blocks*, and
- every pair of elements from X is in exactly λ blocks if they are from different groups or 0 blocks if they are from the same group.

We say the GDD has *type* $g_1^{u_1} g_2^{u_2} \cdots g_l^{u_l}$ when there are u_1 groups of size g_1, u_2 groups of size g_2, $\ldots$, u_l groups of size g_l, where $u_1 + u_2 + \cdots + u_l = u$. This is the *exponential notation* for the group type. And when $|G_i| = g$ for some $g \in \mathbb{N}$ and all $1 \le i \le u$, we call the GDD *uniform* of type g^u.

Example 2.16. A $\mathrm{GDD}(10, \{3, 4\})$ of type $3^3 1^1$.
Here there are three groups of size 3, so none of those 9 pairs of points appear within a block together, while all other pairs appear together in exactly one block. The groups and blocks are shown in Table 2.1.

A GDD(10, {3,4}) of type $3^3 1^1$				
Groups		Blocks		
{1, 2, 3}	{4, 5, 6}	{1, 4, 7, 10}	{2, 5, 8, 10}	{3, 6, 9, 10}
{7, 8, 9}	{10}	{1, 5, 9}	{2, 6, 7}	{3, 4, 8}
		{1, 6, 8}	{2, 4, 9}	{3, 5, 7}

Table 2.1: A $\mathrm{GDD}(10, \{3, 4\})$ of type $3^3 1^1$

Notice a uniform $\text{GDD}_\lambda(v, K)$ of type 1^v is equivalent to a $\text{PBD}_\lambda(v, K)$ since in this case, all pairs will occur together in exactly λ blocks. GDDs can also be viewed from a graph decomposition perspective as well. A $\text{GDD}_\lambda(v, K)$ is equivalent to an edge-decomposition of the complete multipartite graph in which the partite set sizes are determined by the group sizes and we decompose into cliques K_k for $k \in K$.

In this dissertation, we will mainly be concerned with uniform GDDs of type g^u with constant block size $K = \{k\}$, so we will restrict our attention to this case. We now take some time to discuss how this generalization affects the necessary conditions.

First the global condition, there are $\lambda\binom{u}{2}g^2$ pairs (or edges) to cover with blocks (or cliques) of size k. Since each block covers $\binom{k}{2}$ pairs we have

$$\lambda u(u-1)g^2 \equiv 0 \pmod{k(k-1)}. \tag{2.6}$$

For the local condition, we notice that the degree of each vertex in the complete multipartite graph to be decomposed is $\lambda g(u-1)$ since there are g vertices in each of the other $u-1$ groups that need to be covered with a given vertex λ times each. Each K_k-block that includes the given vertex accounts for $k-1$ edges at that vertex; hence,

$$\lambda(u-1)g \equiv 0 \pmod{k-1}. \tag{2.7}$$

We also must have $u \geq k$ since each block (of size k) can take at most one point from each of the u groups. When $u = k$, we get a special GDD that we call a *transversal design* and use the notation $\text{TD}_\lambda(k, n)$, which is equivalent to a $\text{GDD}_\lambda(nk, \{k\})$ of type n^k (so there are k groups of size n with block size k). In this case, (2.6) and (2.7) are trivially satisfied, so transversal designs are possible on any number of points. In fact, Chowla, Erdős, and Straus, proved in 1960 that transversal designs exist asymptotically for all v values.

Theorem 2.17. [8] *For any positive integer k, there exists a TD(k, n) whenever n is sufficiently large.*

For $u > k$, in his thesis in 1976, Chang showed that GDDs exist whenever u is large enough and satisfies (2.6) and (2.7). Chang's thesis was never published and his result follows as an application of the main result in [30].

Theorem 2.18. [7, 30] *Let integers g, k, λ be given with $g, k \geq 2$ and $\lambda \geq 1$. A GDD$_\lambda(gu, k)$ of type g^u exists for all sufficiently large values u satisfying* (2.6) *and* (2.7).

Theorem 2.18 was extended by Liu in [31] to include multiple block sizes from a set K.

We now give a well-known recursive construction by Wilson that we will adapt in Chapter 4 to include the parameter σ.

Theorem 2.19. [43] (Wilson's Fundamental Construction) *Let* $(V, \mathcal{G}, \mathcal{B})$ *be a GDD with index* λ_1 *and groups* $G_1, G_2, \ldots, G_t$*. Suppose there exists a function* $w : V \to \mathbb{Z}^+ \bigcup \{0\}$ *so that for each block* $B = \{x_1, x_2, \ldots, x_k\} \in \mathcal{B}$ *there exists a GDD*$_{\lambda_2}$ *with block sizes from* K *of type* $[w(x_1), w(x_2), \ldots, w(x_k)]$*. Then there exists a GDD*$_{\lambda_1\lambda_2}$ *with block sizes from* K *of type*

$$\left[\sum_{x\in G_1} w(x), \sum_{x\in G_2} w(x), \ldots, \sum_{x\in G_t} w(x)\right].$$

Analogous to pairwise balanced designs, when the blocks of a group divisible design of type g^u can be partitioned into parallel classes so that each point occurs exactly once in each class, we call the GDD *resolvable* and denote it as RGDD$_\lambda(v, K)$ of type g^u. We would like to note that non-uniform GDDs have no hope of being resolvable, since the replication numbers of points in different-sized groups are different (this is an issue since each parallel class will cover all points exactly once). As with PBDs, this adds the necessary condition that

$$gu \equiv 0 \pmod{k} \tag{2.8}$$

since in order for a single parallel class to exist we must be able to partition the gu vertices into K_k-blocks. Notice that (2.7) and (2.8) clearly imply that (2.6) is also satisfied.

Example 2.20. RGDD(v, k) of type $k^{\frac{v}{k}}$ from a RPBD(v, k)
We can construct a resolvable GDD on v points with blocksize k from a resolvable PBD with the same parameters by taking the blocks from one parallel class to be the groups of the GDD. Since $\lambda = 1$ in the PBD, the pairs of points in these groups will no longer be covered by a block (since we are using the only block that covered them as a group instead). Below we give the groups and blocks of a RGDD$(9, 3)$ constructed in this way from the RPBD$(9, 3)$ given in Example 1.8 and seen in Figure 1.2. The parallel classes are listed as rows in Table 2.2.

A uniform RGDD$(9,3)$ of type 3^3			
Groups	Blocks		
{1, 2, 3}	{1, 4, 7}	{2, 5, 8}	{3, 6, 9}
{4, 5, 6}	{1, 5, 9}	{2, 6, 7}	{3, 4, 8}
{7, 8, 9}	{1, 6, 8}	{2, 4, 9}	{3, 5, 7}

Table 2.2: A RGDD$(9,3)$ of type 3^3

It was first shown in [31] by Liu, that (2.7) and (2.8) were asymptotically sufficient when the number of groups u was large enough **and** the group size g was also large enough, but he conjectured that the conditions were asymptotically sufficient for a fixed group size where only u needs to be large.

Theorem 2.21. [31] *Given integers $k \geq 2$ and $\lambda \geq 1$, there exists an RGDD$_\lambda(gu,k)$ of type g^u for all sufficiently large integers g and u satisfying* (2.7) *and* (2.8).

Liu's conjecture was later proved by Chan, Dukes, Lamken, and Ling in [6].

Theorem 2.22. [6] *Given integers $k \geq 2$ and $g, \lambda \geq 1$, there exists an RGDD$_\lambda(gu,k)$ of type g^u for all sufficiently large integers u satisfying* (2.7) *and* (2.8).

If we relax this condition slightly so the blocks can be partitioned into classes with each class missing precisely one group each, we have a (K,λ)-*frame*, defined formally below in Definition 2.23.

Definition 2.23. Let X be a set of v elements and $\mathcal{G} = \{G_1, G_2, \ldots, G_u\}$ be a partition of X. Let $\lambda \geq 1$ and K be a set of positive integers. A (K,λ)-*frame* is a group divisible design $(X, \mathcal{G}, \mathcal{B})$ whose blocks are subsets of X each of cardinality k for some $k \in K$ where pairs of points from different groups are covered precisely λ times each (points from the same group are not covered at all) and blocks can be partitioned into partial parallel classes so that each partial parallel class partitions $X - G_i$, for some $G_i \in \mathcal{G}$.

Since frames are simply GDDs with a special partial resolution condition, we are able to refer to their *type* and whether or not they are *uniform* in the same way as we did for GDDs. We would also like to note that a $(\{k\}, k-1)$-frame of type 1^v is also know as a *near-resolvable design*, where exactly one point is missed in each parallel class.

Example 2.24. A $(\{3\},1)$-Frame of type 2^4.
In Table 2.3 is an example of a frame on 8 points with groups of size 2 and blocks of size 3. As in Table 2.2, the frame partial classes are displayed in the rows of the table (with the group they miss in the same row).

A $(\{3\},1)$-Frame of type 2^4		
Groups	Blocks	
{1, 5}	{2, 6, 7}	{3, 4, 8}
{2, 4}	{1, 6, 8}	{3, 4, 7}
{3, 6}	{1, 4, 7}	{2, 5, 8}
{7, 8}	{1, 2, 3}	{4, 5, 6}

Table 2.3: A $(\{3\},1)$-Frame of type 2^4

As with GDDs, we will mainly be concerned with uniform frames with constant block size here unless otherwise stated (in Chapter 5 we will use non-uniform frames and in Chapter 6 we will discuss frames with graph blocks). The necessary conditions for frames are very similar to those for GDDs. Here we need $u \geq k+1$ instead of $u \geq k$ since we need to able to miss a group in each partial parallel class. Now in order to have a partial parallel class that misses a single group we must be able to partition the $g(u-1)$ points into blocks of size k; thus, we need

$$g(u-1) \equiv 0 \pmod{k}. \tag{2.9}$$

Notice that each group must be missed by the same number of partial parallel classes. In what follows, we will refer to this number as m. Each time a group is not missed, there are an equal number of pairs (one point in the group) covered since we have a uniform frame and so all groups have the same number of points. Now since each point in the design needs to occur with the same number of other points, we must have that each group is missed m times each. We would now like to calculate m, because we will need an arithmetic condition to ensure that m is integral.

First, we note that each group is touched by precisely $m(u-1)$ frame classes. This is because there are $u-1$ other groups missed (uniquely) by each of m classes. Thus, we must have the replication number $r = m(u-1)$, since a point will appear within exactly one block in each partial class that does not miss its group. We need gu points to occur r times each within the $\frac{\lambda u(u-1)g^2}{k(k-1)}$ blocks each containing k points;

thus,

$$rgu = \frac{\lambda u(u-1)g^2}{k-1}.$$

Plugging in $r = m(u-1)$ and solving for m gives

$$m = \frac{\lambda g}{k-1};$$

and hence, to ensure that m is integral we need

$$\lambda g \equiv 0 \pmod{k-1}. \tag{2.10}$$

We would like to point out that (2.9) and (2.10) guarantee that (2.6) and (2.7) will also be satisfied.

The asymptotic existence of uniform frames of index λ is simply an application of the main result by Lamken and Wilson in [30] on edge-coloured graph decompositions, which we will discuss in more detail in the next chapter. Details on how to apply the result in [30] to uniform frames can be found in [5] and a similar method will be used in Chapter 3 when we apply the results of [30] to prove the asymptotic existence of σ-frames.

Theorem 2.25. [5] *Let k, g, λ be positive integers with $k \geq 2$ satisfying* (2.10). *There exists a $(\{k\}, \lambda)$-frame of type g^u for all sufficiently large integers u satisfying* (2.9).

Before we discuss σ-frames, we give a well-known construction of resolvable PBDs that makes use of frames. In Chapter 4 we will extend this construction to obtain σ-resolvable PBDs using σ-frames, defined below.

Theorem 2.26. (Filling in groups) *Suppose there is a frame with group sizes g_i for $i = 1, 2, \ldots, u$. Suppose also that there exists a resolvable $PBD_\lambda(g_i + h, k)$ with a resolvable subdesign of order h for $i = 1, 2, \ldots, u-1$, and a resolvable $PBD_\lambda(g_u + h, k)$ with no condition needed on subdesigns. Then there exists a resolvable $PBD_\lambda(g_1 + g_2 + \cdots + g_u + h, k)$. Furthermore, each ingredient PBD occurs as a subdesign in the resultant design.*

We would like to relax the resolvability condition even further so that classes cover points exactly σ times in all but one group of the GDD and call this a *σ-resolvable (K, λ)-frame*, or simply a *σ-frame*.

Definition 2.27. Let X be a set of v elements and $\mathcal{G} = \{G_1, G_2, \ldots, G_u\}$ be a partition of X. Let $\lambda \geq 1$ and K be a set of positive integers. A σ-resolvable (K, λ)-frame is a group divisible design $(X, \mathcal{G}, \mathcal{B})$ whose blocks are subsets of X each of cardinality k for some $k \in K$ where pairs of points from different groups are covered precisely λ times each (points from the same group are not covered at all) and blocks can be partitioned into classes so that each class contains the points in $X - G_i$, for some $G_i \in \mathcal{G}$ exactly σ times each (and none of the points of G_i).

Clearly, a 1-frame is simply a frame, since each of the classes will cover the points in every group except one exactly once.

We will discuss σ-frames in more detail in the next chapters; in particular, the necessary conditions required for the existence of σ-frames will be given in Section 3.2 and a proof of their asymptotic existence will be given in Section 3.3. The proof will apply the result by Lamken and Wilson on graph decompositions in [30].

Chapter 3

Coloured Graph Decompositions and σ-Frames

3.1 The Lamken-Wilson Theorem

In 2000, a very important extension to Wilson's earlier work was published by Lamken and Wilson in [30]. They were able to extend the main result in [48] to allow arcs of different colours as well as the ability to choose G-blocks from a family of directed graphs $\mathcal{G}$. In other words, they wanted to decompose the *edge-r-coloured complete digraph*, $K_v^{(r)}$, (the directed complete graph on v vertices that has all possible arcs for each of r distinct colours) into subgraphs isomorphic to graphs in $\mathcal{G}$, a family of edge-r-coloured subdigraphs. All the previous results focussed on decomposing into a single graph. These variations require more complicated necessary conditions than Wilson needed in Theorem 2.8 that we will now discuss in detail.

We begin with some necessary notation. Akin to what Wilson did with one colour in [48], we need the number of arcs of each colour in $E(G)$ (for all $G \in \mathcal{G}$) to divide the total number of arcs of each colour in $K_v^{(r)}$, namely $v(v-1)$. So, define $\kappa(G) = (m_1, m_2, \ldots, m_r)$ for each $G \in \mathcal{G}$, where m_i is the number of arcs of colour i in G, and let $\beta(\mathcal{G})$ be the greatest common divisor of the integers m such that $(m, m, \ldots, m)$ is in the integer span of the vectors $\kappa(G)$ for $G \in \mathcal{G}$. Since we need $(v(v-1), v(v-1), \ldots, v(v-1))$ to be in this integer span, we must demand that

$$v(v-1) \equiv 0 \pmod{\beta(\mathcal{G})}.$$

Next, we need to ensure that the analogous local condition is satisfied. Since Lamken and Wilson were considering directed graphs, the greatest common divisor of the degrees in G (or the family of subgraphs $\mathcal{G}$) must be calculated using the in and out degrees at each vertex for each colour, so we let $\deg_i^-(x)$ be the in-degree and $\deg_i^+(x)$ the out-degree of colour i at vertex x. We will also need

$$\tau(x) = (\deg_1^+(x), \deg_1^-(x), \deg_2^+(x), \deg_2^-(x), \ldots, \deg_r^+(x), \deg_r^-(x))$$

to represent the degree vector for vertex x. Then we define $\alpha(\mathcal{G})$ to be the greatest common divisor of the integers t such that the $2r$-vector $(t, t, \ldots, t)$ is in the integer span of the degree vectors $\tau(x)$ for all $x \in V(K_v^{(r)})$. This gives the second congruence in Theorem 3.2 and ensures $v-1$ can be written as a linear combination of the degrees of the vertices in $\mathcal{G}$ for each of the r colours.

Now we have established the necessary conditions for the extra edge-colour constraint, we also need to make sure that the family $\mathcal{G}$ of subgraphs is 'admissible'. Lamken and Wilson call $\mathcal{G}$ an *admissible* family if there exists a positive rational linear relation

$$(1, 1, \ldots, 1) = \sum_{G \in \mathcal{G}} c_G \kappa(G) \qquad \text{with all} \quad c_G > 0.$$

This condition guarantees that there are no graphs in $\mathcal{G}$ that cannot be used in the decomposition.

Example 3.1. Non-Admissible Family of Graphs.
Consider a family of graphs each coloured with two colours (say, red and blue) occurring in an equal number of edges. Then, if in addition, we also include a single extra graph with more red edges than blue edges, this extra graph will be useless in the family. It will never be able to be used within the decomposition, since there will be no way to 'catch-up' the number of blue edges used. Figure 3.1 shows such a family of graphs with the right-most graph being useless within the family.

Note, though, that having both a graph with more red edges than blue and a graph with more blue edges than red is possibly admissible.

We are now able to state the theorem that gives asymptotic existence of $\mathcal{G}$-decompositions of $K_v^{(r)}$.

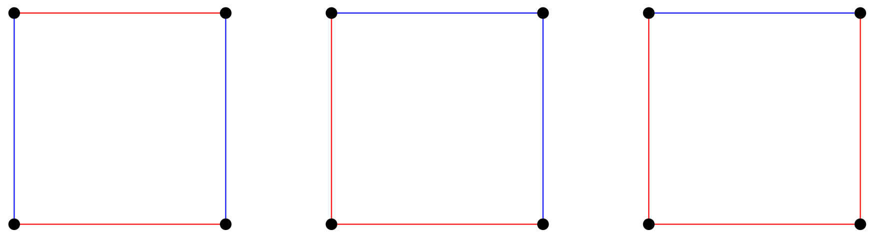

Figure 3.1: A Non-Admissible Family of Graphs

Theorem 3.2. [30] *Let $\mathcal{G}$ be an admissible family of edge-r-coloured digraphs with no multiple arcs in any of the r colours. Then there exists a $\mathcal{G}$-decomposition of $K_v^{(r)}$ for all sufficiently large v satisfying the congruences*

$$\begin{aligned} v(v-1) &\equiv 0 \pmod{\beta(\mathcal{G})} \\ v-1 &\equiv 0 \pmod{\alpha(\mathcal{G})}. \end{aligned}$$

Remark 3.3. Theorem 3.2 is stated here for $\lambda = 1$, but was proved for a vector of possible λ values (allowing a different λ for each colour) in [30] using this result and an extension involving only elementary techniques.

We will apply this very important theorem in Section 3.3 to prove the asymptotic existence of uniform thick-frames (or σ-frames); however, before this is possible, we must determine the necessary arithmetic conditions.

3.2 Necessary Conditions for σ-Frames

Consider a uniform σ-frame F of type g^u with points from X, groups $G_1, G_2, \ldots, G_u$ and blocks of size k.

Since F is a σ-frame, the blocks can be partitioned into *partial σ-parallel classes*; that is, each class covers all points exactly σ times in all but one particular group (for which it does not cover the points at all). Thus, since F is uniform, each class must cover $g(u-1)$ points σ times each using only blocks of size k and hence, we must have

$$\sigma g(u-1) \equiv 0 \pmod{k}. \tag{3.1}$$

As we did in the $\sigma = 1$ case, we let m be the number of partial parallel classes that miss a G_i and as usual, r is the number of times a point is covered in the design.

Note, as before, when the frame is uniform m is the same for each group. In Chapter 4 we will calculate $m(G)$ in the non-uniform case, where groups of different sizes will be missed a different number of times.

Now, consider $x \in X$, and in particular let $x \in G_i$. In each class that does not miss G_i, x is covered exactly σ times. The number of classes in which G_i is not missed must be $m(u-1)$, since G_i is not missed whenever another G_j is missed and each of the other $u-1$ groups is missed in exactly m classes; and hence,

$$r = \sigma m(u-1). \tag{3.2}$$

Notice that F must have

$$b = \frac{\lambda\binom{u}{2}g^2}{\binom{k}{2}}$$

blocks in order to cover all the pairs exactly λ times. Since b must be an integer, we obtain our *global* necessary condition:

$$\lambda g^2 u(u-1) \equiv 0 \pmod{k(k-1)}. \tag{3.3}$$

Also, we would like to ensure we can cover each of the gu points exactly r times overall, with each block covering k points, yielding

$$rgu = \frac{\lambda u(u-1)g^2}{k(k-1)}k,$$

and thus, after solving for r, we have that

$$r = \frac{\lambda g(u-1)}{k-1}. \tag{3.4}$$

Putting (3.2) and (3.4) together and solving for m we see that

$$m = \frac{\lambda g}{\sigma(k-1)}; \tag{3.5}$$

and so, in order for a uniform σ-frame to exist we have the following condition on our parameters:

$$\lambda g \equiv 0 \pmod{\sigma(k-1)}. \tag{3.6}$$

Notice that this congruence does not depend on u and so only restricts our choice of initial parameters.

In order for the frame to exist, we must be able to cover the pairs containing a particular point $x \in X$ using blocks of size k, which cover $k-1$ pairs containing x whenever x appears within a block. Notice that for each $x \in X$, we would like to pair x with precisely $g(u-1)$ other points (every point in a different group from x) λ times each using blocks that cover $k-1$ of the pairs each time x occurs within a block; hence

$$\lambda g(u-1) \equiv 0 \pmod{k-1}.$$

However, we need to also be able to group the blocks containing x into σ-partial classes where x will appear in σ blocks. Thus, we obtain our *local* necessary condition:

$$\lambda g(u-1) \equiv 0 \pmod{\sigma(k-1)}. \tag{3.7}$$

We would like to point out that knowing (3.1) and (3.6) are satisfied implies that (3.3) and (3.7) are also satisfied. Clearly, (3.7) follows directly from (3.6), since $\lambda g(u-1)$ is simply a multiple of λg. Now, (3.3) follows from the fact that (3.1) and (3.6) imply $\lambda\sigma g^2(u-1) \equiv 0 \pmod{\sigma k(k-1)}$, or equivalently $\lambda g^2(u-1) \equiv 0 \pmod{k(k-1)}$. In the following section, we will prove that if (3.1) and (3.6) are satisfied and u is large enough, then a σ-frame of type g^u exists whenever possible. This result is written more formally below.

Theorem 3.4. *Let λ, g, k be positive integers where $\lambda g \equiv 0 \pmod{\sigma(k-1)}$. Then there exists a σ-frame of type g^u, block size k, and index λ for all sufficiently large u satisfying*

$$\sigma g(u-1) \equiv 0 \pmod{k}.$$

We will prove Theorem 3.4 in Section 3.3 by showing it is implied by the existence of a particular graph decomposition and using the main result by Lamken and Wilson in [30].

3.3 Graph Decompositions and σ-Frames

We begin this section, and the proof of Theorem 3.4, by defining a family of directed coloured graphs involved in the decomposition that we claim is equivalent to a σ-frame. Once we have described the decomposition and demonstrated that the two are equivalent, we will then show that the decomposition exists by verifying that it satisfies the necessary conditions given in Theorem 3.2.

Suppose we have positive integers λ, σ, g and k with $\lambda g \equiv 0 \pmod{\sigma(k-1)}$. Define $m = \frac{\lambda g}{\sigma(k-1)}$, $S = \{1, 2, \ldots, g\}$ and $M = \{1^*, 2^*, \ldots, m^*\}$ (we use * notation only to differentiate the elements of S from the elements of M). Let $f : V(K_k) \to S$ be a vertex-labelling of K_k using labels from S and let $l^* \in M$. Now for each labelling f and element $l^* \in M$, define an edge-coloured directed graph, G_{fl^*}, isomorphic to K_k with an extra vertex, denoted ∞, where for each $x, y \in V(K_k)$ the directed edge $(x, y) \in E(G_{fl^*})$ and is coloured $(f(x), f(y)) \in S \times S$ and for each $x \in V(K_k)$ the directed edge $(x, \infty) \in E(G_{fl^*})$ and is coloured $(f(x), l^*) \in S \times M$.

Let $\mathcal{G}$ be the collection of all the G_{fl^*} over all possible labellings f and elements l^* of M. Now, for u large enough and satisfying the necessary conditions (3.1), we wish to decompose $K_u^{\boldsymbol{\lambda}}$, where $\boldsymbol{\lambda} = (\lambda \mathbf{j}_{g^2}, \sigma \mathbf{j}_{gm})$, into graphs from $\mathcal{G}$. Here $\boldsymbol{\lambda}$ gives the number of arcs of each colour at each vertex in the

Fact 3.5. *The existence of a decomposition of $K_u^{\boldsymbol{\lambda}}$, where $\boldsymbol{\lambda} = (\lambda \mathbf{j}_{g^2}, \sigma \mathbf{j}_{gm})$, into graphs from $\mathcal{G}$ is equivalent to the existence of a σ-frame of type g^u, block size k and index λ.*

To see the equivalence we let

- the vertices of $K_u^{\boldsymbol{\lambda}}$ represent the groups of the frame,
- the arcs of $K_u^{\boldsymbol{\lambda}}$ with colours in $S \times S$ represent edges from one group to another in the frame, with the colours indicating exactly which vertices within each group the edge connects,
- the arcs of $K_u^{\boldsymbol{\lambda}}$ with colours in $S \times M$ are used to distinguish which group each partial parallel class misses.

In other words, an edge in $K_u^{\boldsymbol{\lambda}}$ coloured $(f(x), f(y))$ represents an edge from level $f(x)$ of group x to level $f(y)$ in group y.

Now we would like to show that there exists a decomposition of $K_u^{\boldsymbol{\lambda}}$ into graphs from $\mathcal{G}$. To do this we will apply Lamken and Wilson's main result from [30], so we need to ensure that u satisfies the necessary conditions for this choice of $\boldsymbol{\lambda}$ and $\mathcal{G}$. Thus, we need to show

1. that the chosen family of graphs $\mathcal{G}$ is admissible (i.e. $\boldsymbol{\lambda}$ can be written as a positive rational linear combination of $\mu(G)$ for $G \in \mathcal{G}$),

2. that u satisifies the global condition (i.e. $\mu(K_u^{\boldsymbol{\lambda}}) = u(u-1)\boldsymbol{\lambda}$ can be written as an integral linear combination of the vectors $\mu(G)$ for $G \in \mathcal{G}$),

3. that u satisfies the local condition (i.e. $\tau(K_u^{\lambda}) = (u-1)(\boldsymbol{\lambda}, \boldsymbol{\lambda})$ can be written as an integral linear combination of the vectors $\tau(G, x)$ for $G \in \mathcal{G}$ and $x \in V(G)$).

3.3.1 Admissibility

In order to show that $\mu(K_u^{\boldsymbol{\lambda}})$ can be written as a positive rational linear combination of the $\mu(G_{fl^*})$'s, we will begin by computing $\sum_{f,l^*} \mu(G_{fl^*})$. We first count how many of the G_{kl^*} have a fixed edge (x, y) coloured $(i, j) \in S \times S$. If we know that (x, y) is coloured (i, j), then we know that $f(x) = i$ and $f(y) = j$; there are g^{k-2} functions f with x and y labelled this way (since we can label the other $k-2$ vertices with any of the g possible labels). We can also label the ∞ vertex using one of the m options. Doing this over all $k(k-1)$ (directed) edges, we get that there are $k(k-1)mg^{k-2}$ edges with colour $(i, j) \in S \times S$ within all the G_{fl^*}'s.

Now we focus our attention on the colours in $S \times M$ that are used to colour the (x, ∞) edges. We use a similar count as before, so let's fix colour (i, l^*) on edge (x, ∞). Thus, we know that $f(x) = i$ and that the ∞ vertex is labelled l^*. There are g^{k-1} ways to label the other $k-1$ vertices; and so, when we sum over all k edges (x, ∞), we find that the colour $(i, l^*) \in S \times M$ is used kg^{k-1} on edges over all the G_{fl^*}'s.

Given the two counts above, we know that

$$\begin{aligned} \sum_{f,l^*} \mu(G_{fl^*}) &= (k(k-1)mg^{k-2}\mathbf{j}_{g^2},\ kg^{k-1}\mathbf{j}_{gm}) \\ &= kg^{k-1}\left(\frac{(k-1)m}{g}\mathbf{j}_{g^2},\ \mathbf{j}_{gm}\right). \end{aligned}$$

And after plugging in $m = \frac{\lambda g}{\sigma(k-1)}$ we have

$$\begin{aligned} \sum_{f,l^*} \mu(G_{fl^*}) &= kg^{k-1}\left(\frac{\lambda}{\sigma}\mathbf{j}_{g^2},\ \mathbf{j}_{gm}\right) \\ &= \frac{kg^{k-1}}{\sigma}(\lambda\mathbf{j}_{g^2},\ \sigma\boldsymbol{j}_{gm}) \\ &= \frac{kg^{k-1}}{\sigma}\boldsymbol{\lambda}. \end{aligned}$$

This proves that $\boldsymbol{\lambda}$ can be written as a positive rational linear combination of the $\mu(G_{fl^*})$'s (since $k, g, \sigma \in \mathbb{N}$).

3.3.2 The Global Condition

Here our goal is to show that $\mu(K_u^{\boldsymbol{\lambda}}) = u(u-1)\boldsymbol{\lambda}$ can be written as an integral linear combination of the vectors $\mu(G)$ for $G \in \mathcal{G}$, which we will do by taking advantage of the following lemma, see [39] for a proof.

Lemma 3.6. *Given an $m \times n$ rational matrix M and a rational vector $\boldsymbol{c}$ of length m, the equation $M\boldsymbol{x} = \boldsymbol{c}$ has an integral solution $\boldsymbol{x}$ if and only if for all rational vectors $\boldsymbol{y}$ such that $\boldsymbol{y}^\intercal M$ is integral whenever $\boldsymbol{y}^\intercal \cdot \boldsymbol{c}$ is integral.*

We apply the lemma by letting M be a matrix that contains the vectors $\mu(G_{fl^*})$ as columns (so M has size $(g^2 + gm) \times (g^k m)$) and $\boldsymbol{c} = u(u-1)\boldsymbol{\lambda}$. Now suppose that there is a $\boldsymbol{y} \in \mathbb{Q}^{g^2+gm}$ with $\boldsymbol{y}^\intercal M \in \mathbb{Z}^{g^k m}$. For notational convenience we will denote $\boldsymbol{y}$ as the concatenation of two vectors: $\boldsymbol{w}$ of length g^2 with entries w_{ij} indexed by the colours in $S \times S$ and $\boldsymbol{z}$ of length gm with entries z_{il^*} indexed by the colours in $S \times M$, so $\boldsymbol{y} = (\boldsymbol{w}, \boldsymbol{z})$. Notice that $\boldsymbol{y}^\intercal M$ integral means that $\boldsymbol{y}^\intercal \cdot \mu(G_{fl^*}) \in \mathbb{Z}$ for all $G_{fl^*} \in \mathcal{G}$. So in other words, for each particular G_{fl^*} we have that

$$\sum_{(i,j)\in S\times S} |\{(b,c) : f(b) = i, f(c) = j\}| w_{ij} + \sum_{i\in S} |\{b : f(b) = i\}|\, z_{il^*} \equiv 0 \tag{3.8}$$

where $b, c \in V(K_k)$, $(b,c) \in E(G_{fl^*})$ and the congruence is taken modulo 1 and should be considered as such unless otherwise stated.

Notice that for a particular G_{fl^*}, f and l^* are determined, so the second sum above does not need to sum over all $(i, j^*) \in S \times M$ since there are no edges coloured (i, j^*) in G_{fl^*} for $j^* \neq l^*$ so those entries in $\mu(G_{fl^*})$ will be zero.

Now we would like to show that $\boldsymbol{y}^\intercal \cdot \boldsymbol{c} \in \mathbb{Z}$. In other words, we need to show that

$$\boldsymbol{y}^\intercal \cdot \boldsymbol{c} = \lambda u(u-1) \sum_{(i,j)\in S\times S} w_{ij} + \sigma u(u-1) \sum_{(i,l^*)\in S\times M} z_{il^*} \equiv 0.$$

We begin by choosing particular labellings, that use only two labels i and j, and let $l^* \in M$ vary. First, we consider the labelling f_1 where $f_1(x) = i$ for all $x \in V(K_k)$.

Applying the equivalence in (3.8) to graphs in $\mathcal{G}$ that use this labelling we obtain

$$k(k-1)w_{ii} + kz_{il^*} \equiv 0 \tag{3.9}$$

since all $k(k-1)$ directed edges between vertices in $V(K_k)$ will be coloured (i,i) and arcs from each of the k vertices in K_k to the ∞ vertex will all be coloured (i,l^*).

Before we move on to our next labelling, we would like to note that by varying our choice of $l^* \in M$ in (3.9) and subtracting, we obtain that

$$kz_{il^*} \equiv kz_{ir^*}. \tag{3.10}$$

Now, the next labelling we consider has only one vertex labelled j. Let $x_0 \in V(K_k)$ and consider the labelling f_2 that labels all vertices in K_k with i except x_0 which is labelled j. Then, we apply (3.8) to $G_{f_2 l^*}$ noticing that there will be $k-1$ (the indegree/outdegree at each vertex in K_k) edges with colour (i,j) and (j,i), with the remaining $k(k-1) - 2(k-1) = (k-1)(k-2)$ edges in K_k getting the colour (i,i); thus, yielding the following equivalence

$$(k-1)(w_{ij} + w_{ji}) + (k-1)(k-2)w_{ii} + (k-1)z_{il^*} + z_{jl*} \equiv 0. \tag{3.11}$$

When exactly two vertices of K_k are labelled j (with the rest labelled i), there will be $2k-4$ edges coloured (i,j) and (j,i), two edges coloured (j,j), and $(k-2)(k-3)$ edges (between the remaining $k-2$ vertices) coloured (i,i). Thus,

$$(2k-4)(w_{ij} + w_{ji}) + 2w_{jj} + (k-2)(k-3)w_{ii} + (k-2)z_{il^*} + 2z_{jl^*} \equiv 0. \tag{3.12}$$

Notice that if we add (3.9) to (3.12) and subtract twice (3.11) we obtain

$$2(w_{ii} + w_{jj}) \equiv 2(w_{ij} + w_{ji}). \tag{3.13}$$

Also, (3.11) minus (3.9) yields

$$(k-1)(w_{ij} + w_{ji}) + z_{jl^*} \equiv 2(k-1)w_{ii} + z_{il^*}. \tag{3.14}$$

Note that $\sigma mu(u-1) = \frac{\lambda gu(u-1)}{k-1} \in \mathbb{Z}$; and hence, we know that $k-1$ divides $\lambda gu(u-1)$. Also, since $\frac{\lambda gu(u-1)}{k-1}$ is a multiple of m and we can choose whatever label $l^* \in M$ we desire, we can evenly distribute our selection so that each label l^* occurs equally often ($\sigma u(u-1)$ times each). In other words, we can add (3.14) to itself $\frac{\lambda gu(u-1)}{k-1}$ times varying $l^* \in M$ so that each one occurs $\sigma u(u-1)$ times each; thus,

$$\lambda gu(u-1)(w_{ij}+w_{ji}) + \sigma u(u-1)\sum_{l^*\in M} z_{jl^*} \equiv 2\lambda gu(u-1)w_{ii} + \sigma u(u-1)\sum_{l^*\in M} z_{il^*}.$$

Now, since $u(u-1)$ is necessarily even, we can apply (3.13) to the above equivalence and obtain

$$\lambda gu(u-1)(w_{ii}+w_{jj}) + \sigma u(u-1)\sum_{l^*\in M} z_{jl^*} \equiv 2\lambda gu(u-1)w_{ii} + \sigma u(u-1)\sum_{l^*\in M} z_{il^*},$$

which implies that

$$\lambda gu(u-1)w_{jj} + \sigma u(u-1)\sum_{l^*\in M} z_{jl^*} \equiv \lambda gu(u-1)w_{ii} + \sigma u(u-1)\sum_{l^*\in M} z_{il^*}. \tag{3.15}$$

Now we turn our focus back to $\boldsymbol{y}^\intercal \cdot \boldsymbol{c}$.

$$\begin{aligned}\boldsymbol{y}^\intercal \cdot \boldsymbol{c} &= \lambda u(u-1)\sum_{(i,j)\in S\times S} w_{ij} + \sigma u(u-1)\sum_{(i,l^*)\in S\times M} z_{il^*}\\ &= \frac{\lambda u(u-1)}{2}\sum_{(i,j)\in S\times S} 2w_{ij} + \sigma u(u-1)\sum_{(i,l^*)\in S\times M} z_{il^*}.\end{aligned}$$

We pair up the terms $2w_{ij}$ and $2w_{ji}$ in the first sum above and apply (3.13). Notice that each $i \in S$ will appear in exactly $g-1$ such pairs with another label in S, so after applying (3.13), w_{ii} will occur in the subsequent sum g times each. Therefore, we have

$$\begin{aligned}\boldsymbol{y}^\intercal \cdot \boldsymbol{c} &\equiv \frac{\lambda u(u-1)}{2}\sum_{i\in S} 2gw_{ii} + \sigma u(u-1)\sum_{(i,l^*)\in S\times M} z_{il^*}\\ &= \lambda gu(u-1)\sum_{i\in S} w_{ii} + \sigma u(u-1)\sum_{(i,l^*)\in S\times M} z_{il^*}\\ &= \sum_{i\in S}\left(\lambda gu(u-1)w_{ii} + \sigma u(u-1)\sum_{l^*\in M} z_{il^*}\right) \qquad (3.16)\end{aligned}$$

Now, given (3.15), we know that (3.16) is equivalent (modulo 1) for each $i \in S$

and so can be simplified using a specific label in S, say 1. Thus,

$$\boldsymbol{y}^{\intercal} \cdot \boldsymbol{c} \equiv \lambda g^2 u(u-1) w_{11} + \sigma g u(u-1) \sum_{l^* \in M} z_{1l^*}. \tag{3.17}$$

Using a similar argument we can further simplify (3.17) by combining the fact that k divides $\sigma g(u-1)$ (due to the necessary condition (3.1)) with (3.10) we see that the integrality of kz_{1l^*} does not depend on the choice of l^*, so again we may use any label from M, say 1^* and obtain

$$\begin{aligned} \boldsymbol{y}^{\intercal} \cdot \boldsymbol{c} &= \lambda g^2 u(u-1) w_{11} + \frac{\sigma g u(u-1)}{k} \sum_{l^* \in M} k z_{1l^*} \\ &\equiv \lambda g^2 u(u-1) w_{11} + \sigma g m u(u-1) z_{11^*}. \end{aligned} \tag{3.18}$$

Finally, if we take $i = 1$ and $l^* = 1^*$ in (3.9) we have $k(k-1)w_{11} + kz_{11^*} \equiv 0$. Now combining this with the fact that k divides $\sigma g u(u-1)$ (required by the necessary condition (3.1)), we know that

$$\frac{\sigma m g u(u-1)}{k} \left(k(k-1)w_{11} + kz_{11^*}\right) \equiv 0;$$

and hence,

$$\lambda g^2 u(u-1) w_{11} + \sigma m g u(u-1) z_{11^*} \equiv 0.$$

Therefore, given the expression for $\boldsymbol{y}^{\intercal} \cdot \boldsymbol{c}$ in (3.18), we can finally conclude that $\boldsymbol{y}^{\intercal} \cdot \boldsymbol{c} \equiv 0 \pmod 1$ and hence $\boldsymbol{y}^{\intercal} \cdot \boldsymbol{c} \in \mathbb{Z}$. And thus, given Lemma 3.6, there exists an integer solution $\boldsymbol{x}$ in $M\boldsymbol{x} = \boldsymbol{c}$, so $u(u-1)\boldsymbol{\lambda} \in \operatorname{span}_{\mathbb{Z}} \{\mu(G) \mid G \in \mathcal{G}\}$.

3.3.3 The Local Condition

In this section we will again take of advantage of Lemma 3.6 in order to show that the vector $(u-1)(\boldsymbol{\lambda}, \boldsymbol{\lambda})$ can be written as an integer linear combination of the vectors $\tau(G_{fl^*}, x)$, where $G_{fl^*} \in \mathcal{G}$, $x \in V(G_{fl^*})$, and $(\boldsymbol{\lambda}, \boldsymbol{\lambda})$ is the concatenation of $\boldsymbol{\lambda}$ with itself.

Here we let M be the $2(g^2 + gm) \times m(k+1)g^k$ matrix that contains all the $\tau(G, x)$ vectors as its columns, $\boldsymbol{c} = (u-1)(\boldsymbol{\lambda}, \boldsymbol{\lambda})$, and let $\boldsymbol{y} \in \mathbb{Q}^{2(g^2+gm)}$ such that $\boldsymbol{y}^{\intercal} \cdot M \in \mathbb{Z}^{m(k+1)g^k}$. As we did when proving the global condition, for notational convenience we will denote $\boldsymbol{y}$ as the concatenation of the vectors $\boldsymbol{w}^-$ and $\boldsymbol{w}^+$, each of length g^2 with entries w_{ij}^- and w_{ij}^+ respectively, indexed by the colours in $S \times S$, and $\boldsymbol{z}^-$ and $\boldsymbol{z}^+$, each of length gm with entries $z_{il^*}^-$ and $z_{il^*}^+$ respectively, indexed by the colours in $S \times M$, so $\boldsymbol{y} = (\boldsymbol{w}^-,\ \boldsymbol{z}^-,\ \boldsymbol{w}^+,\ \boldsymbol{z}^+)$.

Now $\boldsymbol{y}^\intercal \cdot M$ being integral implies that $\boldsymbol{y}^\intercal \cdot \tau(G_{fl^*}, x)$ is an integer for all $G_{fl^*} \in \mathcal{G}$ and all $x \in V(G_{fl^*})$. So in other words, when x is not the infinity vertex of G_{fl^*} and is labelled with $j \in S$, we must have

$$\sum_{i \in S} |\{b : f(b) = i\}|(w_{ij}^- + w_{ij}^+) + z_{jl^*}^+ \equiv 0. \tag{3.19}$$

Notice that in $\tau(G_{fl^*}, x)$ the only colour in $S \times M$ present at x is (j, l^*) and hence all the terms indexed by other colours in $S \times M$ will be 0.

Also, when x is the infinity vertex of G_{fl^*}, we have

$$\sum_{i \in S} |\{b : f(b) = i\}| z_{il^*}^- \equiv 0. \tag{3.20}$$

Now, as we did for the global condition, we will consider particular labellings and therefore particular graphs in $\mathcal{G}$. First we take the labelling f_1 that gives all vertices of K_k label i, so (3.19) and (3.20) with $G_{f_1 l^*}$ tells us that

$$(k-1)(w_{ii}^- + w_{ii}^+) + z_{il^*}^+ \equiv 0 \text{ and} \tag{3.21}$$

$$k z_{il^*}^- \equiv 0. \tag{3.22}$$

If we instead use the labelling f_2 where exactly one vertex x is labelled j and all others are labelled i we obtain

$$(k-1)(w_{ij}^- + w_{ij}^+) + z_{jl^*}^+ \equiv 0 \text{ and} \tag{3.23}$$

$$(k-1) z_{il^*}^- + z_{jl^*}^- \equiv 0. \tag{3.24}$$

And lastly, let f_3 label exactly two vertices with j and all others with i; then

$$(k-2)(w_{ij}^- + w_{ij}^+) + (w_{jj}^- + w_{jj}^+) + z_{jl^*}^+ \equiv 0. \tag{3.25}$$

Before we begin to consider $\boldsymbol{y}^\intercal \cdot \boldsymbol{c}$ we will combine these integral conditions containing the entries of $\boldsymbol{y}$ in various ways to be used later. First, we subtract (3.23) from (3.25) which yields

$$w_{ij}^- + w_{ij}^+ \equiv w_{jj}^- + w_{jj}^+ \tag{3.26}$$

and subtract (3.22) from (3.24) to obtain

$$z_{jl^*}^- \equiv z_{il^*}^-. \tag{3.27}$$

Next we subtract two versions of (3.21) for different choices $l^*, r^* \in M$ to obtain

$$z_{il^*}^+ \equiv z_{ir^*}^+. \tag{3.28}$$

Now if we turn our attention to $\boldsymbol{y}^\intercal \cdot \boldsymbol{c}$, we obtain

$$\begin{aligned}
\boldsymbol{y}^\intercal \cdot \boldsymbol{c} &= \lambda(u-1) \sum_{(i,j)\in S\times S} (w_{ij}^- + w_{ij}^+) + \sigma(u-1) \sum_{(i,l^*)\in S\times M} z_{il^*}^+ + \sigma(u-1) \sum_{(i,l^*)\in S\times M} z_{il^*}^- \\
&\equiv \lambda g(u-1) \sum_{i\in S} (w_{ii}^- + w_{ii}^+) + \sigma(u-1) \sum_{(i,l^*)\in S\times M} z_{il^*}^+ + \sigma(u-1) \sum_{(i,l^*)\in S\times M} z_{il^*}^-,
\end{aligned}$$

where the equivalence follows from (3.26) and the fact that each label i will occur in the initial sum g times in total (once with itself and once with each of the $g-1$ other labels). Next, we apply (3.28) and (3.27) which gives

$$\begin{aligned}
\boldsymbol{y}^\intercal \cdot \boldsymbol{c} &\equiv \lambda g(u-1) \sum_{i\in S} (w_{ii}^- + w_{ii}^+) + \sigma m(u-1) \sum_{i\in S} z_{i1^*}^+ + \sigma g(u-1) \sum_{l^*\in M} z_{1l^*}^-, \\
&= \sum_{i\in S} \left(\lambda g(u-1)(w_{ii}^- + w_{ii}^+) + \sigma m(u-1) z_{i1^*}^+\right) + \sigma g(u-1) \sum_{l^*\in M} z_{1l^*}^-.
\end{aligned}$$

Now, since we have that $k \mid \sigma g(u-1)$, we can write

$$\begin{aligned}
\boldsymbol{y}^\intercal \cdot \boldsymbol{c} &= \sum_{i\in S} \left(\lambda g(u-1)(w_{ii}^- + w_{ii}^+) + \sigma m(u-1) z_{i1^*}^+\right) + \frac{\sigma g(u-1)}{k} \sum_{l^*\in M} k z_{1l^*}^- \\
&\equiv \sum_{i\in S} \left(\lambda g(u-1)(w_{ii}^- + w_{ii}^+) + \sigma m(u-1) z_{i1^*}^+\right), \qquad (3.29)
\end{aligned}$$

where the congruence follows from (3.22).

If we take (3.25) minus (3.21) and use the fact given in (3.26) we obtain

$$(k-1)(w_{jj}^- + w_{jj}^+) + z_{jl^*} \equiv (k-1)(w_{ii}^- + w_{ii}^+) + z_{il^*}. \tag{3.30}$$

Applying the congruence given in (3.30) to (3.29) we are able to simplify the sum by selecting any label for i, say 1; thus,

$$\boldsymbol{y}^{\intercal} \cdot \boldsymbol{c} \equiv g\left(\lambda g(u-1)(w_{11}^{-}+w_{11}^{+})+\sigma m(u-1)z_{11^*}^{+}\right). \tag{3.31}$$

Finally, if we take $i=1$ and $l^*=1^*$ in (3.21), we have $(k-1)(w_{11}^{-}+w_{11}^{+})+z_{11^*}^{+} \equiv 0$. Multiplying this by the integer $\sigma m(u-1)$ gives

$$\begin{aligned} \sigma m(u-1)(k-1)(w_{11}^{-}+w_{11}^{+})+\sigma m(u-1)z_{11^*}^{+} &\equiv 0 \\ \sigma\left(\frac{\lambda g}{\sigma(k-1)}\right)(u-1)(k-1)(w_{11}^{-}+w_{11}^{+})+\sigma m(u-1)z_{11^*}^{+} &\equiv 0 \\ \lambda g(u-1)(w_{11}^{-}+w_{11}^{+})+\sigma m(u-1)z_{11^*}^{+} &\equiv 0; \end{aligned}$$

and hence, we must have that $\boldsymbol{y}^{\intercal} \cdot \boldsymbol{c} \equiv 0 \pmod 1$ or in other words, $\boldsymbol{y}^{\intercal} \cdot \boldsymbol{c} \in \mathbb{Z}$. And thus, given Lemma 3.6, we can conclude that the local condition is satisfied.

Chapter 4

Thickly Resolvable Constructions

This chapter is devoted to proving facts about thickly-resolvable designs as well as adapting some famous constructions to include thick-resolvability.

In Section 3.2 we saw in a uniform σ-frame that each group is missed by partial σ-classes an equal number of times, in particular, each group will be missed exactly $m = \frac{\lambda g}{\sigma(k-1)}$ times. We begin this chapter by generalizing this result to include the case of non-uniform frames. In order to do this, we define $m(G)$ to be the number of partial σ-classes that miss the group G of a σ-frame.

Lemma 4.1. *Consider a σ-resolvable $(\{k\}, \lambda)$-frame $(X, \Pi, \mathcal{B})$, where $|X| = v$. For each $G \in \Pi$,*

$$m(G) = \frac{\lambda |G|}{\sigma(k-1)}.$$

Proof. We begin our proof by counting the number of blocks in the frame in two ways. First, there are $\frac{\sigma(v-|G|)}{k}$ blocks in each partial σ-class that misses G since each of the $v - |G|$ points is covered σ times each by blocks of size k; thus,

$$b = \sum_{G \in \Pi} m(G) \frac{\sigma(v - |G|)}{k}. \tag{4.1}$$

We can also count the number of blocks by considering how many times each point occurs. Let r_x be the number of blocks containing $x \in X$. Now, in any GDD, we know that each point not in G must occur in exactly λ blocks with x and in each block that x occurs, it occurs with $k - 1$ other points, giving us

$$r_x(k-1) = \lambda(v - |G|). \tag{4.2}$$

Thus,

$$\begin{aligned} bk &= \sum_{G\in\Pi}\sum_{x\in G} r_x. \\ &= \sum_{G\in\Pi} |G|\frac{\lambda(v-|G|)}{k-1}; \end{aligned}$$

and so, we also have that

$$b = \sum_{G\in\Pi} \frac{|G|\lambda(v-|G|)}{k(k-1)}. \tag{4.3}$$

Putting (4.1) and (4.3) together and moving everything to one side, we obtain

$$\sum_{G\in\Pi} \frac{v-|G|}{k}\left(\sigma\cdot m(G) - \frac{\lambda|G|}{k-1}\right) = 0. \tag{4.4}$$

Also, since each $x \in G$ occurs σ times in each partial σ-class except those that miss G, we must have that $r_x = \left(\sum_{H\in\Pi} \sigma\cdot m(H)\right) - \sigma\cdot m(G)$. Now, using (4.2), we have

$$\frac{\lambda(v-|G|)}{k-1} = \left(\sum_{H\in\Pi}\sigma\cdot m(H)\right) - \sigma\cdot m(G),$$

which implies

$$\left(\sum_{H\in\Pi}\sigma\cdot m(H)\right) - \frac{\lambda v}{k-1} = \sigma\cdot m(G) - \frac{\lambda|G|}{k-1}. \tag{4.5}$$

Plugging (4.5) into (4.4), we get

$$\sum_{G\in\Pi}\frac{v-|G|}{k}\left(\sum_{H\in\Pi}\sigma\cdot m(H) - \frac{\lambda v}{k-1}\right) = 0.$$

Now, since $\frac{v-|G|}{k}$ is positive for each $G \in \Pi$ and the bracketed portion is a constant, we must have that

$$\sum_{H\in\Pi}\sigma\cdot m(H) - \frac{\lambda v}{k-1} = 0,$$

which implies that

$$\sum_{H\in\Pi} m(H) = \frac{\lambda v}{\sigma(k-1)}.$$

Thus, returning to (4.5) and solving for $m(G)$ we have

$$\begin{aligned} m(G) &= \frac{1}{\sigma}\sum_{H\in\Pi}\sigma\cdot m(H) - \frac{\lambda v}{\sigma(k-1)} + \frac{\lambda|G|}{\sigma(k-1)} \\ &= \frac{\lambda v}{\sigma(k-1)} - \frac{\lambda v}{\sigma(k-1)} + \frac{\lambda|G|}{\sigma(k-1)} \\ &= \frac{\lambda|G|}{\sigma(k-1)}. \end{aligned}$$

□

The next result is an extension of a famous and extremely useful construction of Richard Wilson in [46], stated as Theorem 2.19 in Chapter 2. We will make use of this extension in Section 5.3 to prove the existence of certain non-uniform σ-frames that are crucial within the proof of Theorem 1.19.

Lemma 4.2. (Wilson's Fundamental Construction) *Let $(X,\Pi,\mathcal{B})$ be a GDD with block sizes from K and index $\lambda=1$, and let $w: X\to\mathbb{N}\cup\{0\}$ be a weight function on X. Suppose, for each block $B\in\mathcal{B}$, that there exists a σ-resolvable $(\{k\},\lambda)$-frame with group sizes $w(x)$ for $x\in B$. Then there exists a σ-resolvable $(\{k\},\lambda)$-frame with group sizes $\sum_{x\in G} w(x)$ for each group $G\in\Pi$.*

Proof. We will construct the new σ-frame $(X',\Pi',\mathcal{B}')$. We begin by defining the new point set: for each $x\in X$, let $S(x)=\{x_1,x_2,\dots,x_{w(x)}\}$ and let

$$X' = \bigcup_{x\in X} S(x),$$

so $|X'|=\sum_{x\in X} w(x)$. Now, we define the new group set to be

$$\Pi' = \left\{ \bigcup_{x\in G} S(x) \;:\; G\in\Pi \right\}.$$

It is clear that these groups have the desired sizes and so the frame will have the desired type. Finally, the more difficult part, we must form the new block set and check that it is pairwise-balanced and partitions into partial σ-classes. For each block $B\in\mathcal{B}$ of the initial GDD, we consider the σ-frame to be on the point set $\cup_{x\in B}S(x)$ with groups $S(x)$ for each $x\in B$ and block set $\mathcal{B}(B)$. Then we set

$$\mathcal{B}' = \bigcup_{B\in\mathcal{B}} \mathcal{B}(B).$$

In order to show that $\mathcal{B}'$ is pairwise-balanced, we need to show that pairs of points from different groups of Π' occur in exactly λ blocks and pairs of points from the same group do not occur together in any block. We would like to point out that it is well-known that this type of construction will result in a pairwise-balanced GDD, but we include a brief discussion for completeness.

So, let's consider any pair $\{x_i, y_j\}$, where $x_i \in S(x)$ and $y_j \in S(y)$ for $x, y \in X$.

- *Suppose x_i and y_j are from different groups of* Π': In this case, x and y must have been from different groups of Π. Thus, there is exactly one block $B \in \mathcal{B}$ containing $\{x, y\}$ and within the σ-frame on $\cup_{x \in B} S(x)$ the pair $\{x_i, y_j\}$ will be in exactly λ blocks of $\mathcal{B}(B)$ and hence in exactly λ blocks of $\mathcal{B}'$.

- *Suppose x_i and y_j are from the same group of* Π': In this case, either x and y were from the same group of Π (and hence there is no block in $\mathcal{B}$ containing $\{x, y\}$) or $y_j = x_l$ for some $l \neq i \in \{1, 2, \ldots, w(x)\}$. In the latter case, we note that in all blocks of $\mathcal{B}$ containing x, $x_i, x_l \in S(x)$ will occur in the same group in all the frames associated with each block on x and hence will never occur together within a block.

Now, it only remains to show that the blocks of $\mathcal{B}'$ resolve into partial σ-parallel classes; in other words, we wish to show that the blocks can be partitioned so that each point in $X' \setminus G$, for some $G \in \Pi'$, occurs exactly σ times.

We first note that, according to Lemma 4.1 and the fact that $|S(x)| = w(x)$, there are exactly $m(x) = \frac{\lambda \cdot w(x)}{\sigma(k-1)}$ partial σ-classes in each of the ingredient σ-frames that miss the group $S(x)$. We label each of these partial resolution classes arbitrarily from 1 to $m(x)$ and define $R^B_{x,i}$ to be the i-th partial σ-resolution class missing the group $S(x)$ in the ingredient σ-frame associated with block B.

Now, for each $x \in G_i$, we consider $\bigcup_{B \ni x} R^B_{x,p}$ for $1 \leq p \leq m(x)$. Notice that x will appear with each other point $y \in G_j$ (for $i \neq j$) in exactly one block B of $\mathcal{B}$. Therefore, each $y_l \in S(y)$ will occur in $R^B_{x,p}$ exactly σ times for each $p \in \{1, 2, \ldots, m(x)\}$ and will not appear in any other $R^{B'}_{x,p'}$ (since B is the only block in the original GDD containing both x and y). Thus, for each $x \in G_i \in \Pi$ and each $1 \leq p \leq m(x)$, $\bigcup_{B \ni x} R^B_{x,p}$ forms a partial σ-parallel class that misses the group $G'_i = \{\bigcup_{x \in G_i} S(x)\} \in \Pi'$. So, for each $x \in G_i$, there are $m(x)$ partial σ-resolution classes that miss $G'_i \in \Pi'$. Finally, for each $G' \in \Pi'$,

$$\begin{aligned} m(G'_i) &= \sum_{x \in G_i} m(x) \\ &= \frac{\lambda \sum_{x \in G_i} w(x)}{\sigma(k-1)} \\ &= \frac{\lambda |G'_i|}{\sigma(k-1)}, \end{aligned}$$

as desired.

Clearly the blocks have been partitioned so that each block belongs to exactly one of the frame classes. Thus, $(X', \Pi', \mathcal{B}')$ is a σ-(k, λ)-frame. □

We conclude this chapter with a construction that will be used frequently throughout Chapters 5 and 6, mainly with subdesigns of order 1. For a PBD $(V, \mathcal{B})$ we say that the PBD $(V_0, \mathcal{B}_0)$ (with index λ) is a *subdesign* if $V_0 \subseteq V$, $\mathcal{B}_0 \subseteq \mathcal{B}$, and some partition of $\mathcal{B}$ into σ-parallel classes induces a partition of $\mathcal{B}_0$ into σ-parallel classes. This construction is an adaptation of a widely-used construction within the field of asymptotic design theory that now handles the σ parameter (for any $\sigma \geq 1$).

Lemma 4.3. (Filling in groups) *Suppose there is a σ-frame with group sizes g_i for $i = 1, 2, \ldots, u$. Suppose also that there exist a σ-resolvable PBD$_\lambda(g_i + h, k)$ with a σ-resolvable subdesign of order h for $i = 1, 2, \ldots, u-1$, and a σ-resolvable PBD$_\lambda(g_u + h, k)$ with no condition needed on subdesigns. Then there exists a σ-resolvable PBD$_\lambda(g_1 + g_2 + \cdots + g_u + h, k)$. Furthermore, each ingredient PBD occurs as a subdesign in the resultant design.*

Proof. We begin by establishing some notation. We let $(X, \Pi, \mathcal{B})$ be the ingredient σ-frame, where $\Pi = \{G_i : |G_i| = g_i \text{ and } 1 \leq i \leq u\}$ and $X = \bigcup_{i=1}^{u} G_i$. We also define a new set of points H so that $|H| = h$ and $H \cap G_i = \emptyset$ for all i.

Now to form the σ-resolvable PBD of order $g_1 + \cdots + g_u + h$, for each $1 \leq i \leq u$ we place the σ-resolvable PBD$(g_i + h, k, \lambda)$ onto $G_i \cup H$ and identify each of the $u-1$ ingredient σ-resolvable subdesigns of order h to H. Notice that in the PBDs with subdesigns on H, blocks covering two points of H must cover only points of H, so we will denote the set of blocks for the subdesign as $\mathcal{B}(H)$ and the set of remaining blocks of the PBD (which will each contain at most one point of H) as $\mathcal{B}_i$. We will now show that the set of points $X' = X \cup H$ together with the block set $\mathcal{B}' = \mathcal{B} \cup \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_u$ forms the desired σ-resolvable PBD.

Clearly X' and blocks in $\mathcal{B}'$ have the desired sizes, so we just need to show that the pairs are each covered λ times and that the blocks can be resolved. It is not difficult to show that pairs of points are covered λ times each, so we will only briefly discuss it. Pairs from different groups will only be covered by the frame blocks in $\mathcal{B}$, pairs from the same group G_i will only be covered by blocks in $\mathcal{B}_i$, pairs with one point in G_i and one in H will also only be covered by blocks in $\mathcal{B}_i$, and pairs from H will only be covered by blocks in $\mathcal{B}_u$ (since blocks containing more than one point of H were removed from $\mathcal{B}_i$ for $1 \leq i \leq u-1$). Thus, so far, we know that $(X', \mathcal{B}')$ is a PBD, but it remains to show that it is also σ-resolvable.

We would first like to note that the subdesigns in each of the $u-1$ ingredient PBDs are themselves also σ-resolvable; and so, we know that each subdesign has $\frac{\lambda(h-1)}{\sigma(k-1)}$ σ-parallel classes.

Thus, in the PBD on $G_i \cup H$ for $1 \leq i \leq u-1$, since the blocks in $\mathcal{B}(H)$ were removed, there are precisely $\frac{\lambda(h-1)}{\sigma(k-1}$ σ-parallel classes that are missing blocks and containing only points of G_i. Thus, we can conclude there are $\frac{\lambda((g_i+h)-1)}{\sigma(k-1)} - \frac{\lambda(h-1)}{\sigma(k-1)} = \frac{\lambda g_i}{\sigma(k-1)}$ complete σ-parallel classes (that contain points of $G_i \cup H$ exactly σ times each). Now to begin constructing σ-resolvable parallel classes we first take the partial σ classes guaranteed by the σ-frame. We pair each of the $\frac{\lambda g_i}{\sigma(k-1)}$ partial σ-classes missing the group G_i (for $1 \leq i \leq u-1$) with one of the $\frac{\lambda g_i}{\sigma(k-1)}$ complete σ-parallel classes of the PBD on $G_i \cup H$, which we can do arbitrarily. We also pair each of the $\frac{\lambda g_u}{\sigma(k-1)}$ partial σ classes missing the group G_u with a σ-parallel class from the PBD on $G_u \cup H$. This we can do completely arbitrarily and will leave $\frac{\lambda(h-1)}{\sigma(k-1)}$ σ-parallel classes unpaired. Finally, we pair each of these leftover classes on $G_u \cup H$ with one of the $\frac{\lambda(h-1)}{\sigma(k-1)}$ reduced classes from each of the ingredient PBDs. Each of these pairings will form a σ-parallel class on X'. Thus, we have formed exactly

$$\frac{\lambda g_1}{\sigma(k-1)} + \frac{\lambda g_2}{\sigma(k-1)} + \cdots + \frac{\lambda g_u}{\sigma(k-1)} + \frac{\lambda(h-1)}{\sigma(k-1)} = \frac{\lambda(v+h-1)}{\sigma(k-1)}$$

σ-parallel classes using the blocks of $\mathcal{B}'$. □

Chapter 5

Main Proof

The proof of Theorem 1.19 will be accomplished by first proving the necessary tools in each of three broad steps. In Section 5.1 we will make use of the existence of combinatorial configurations and apply a trick to construct one example of a thickly-resolvable design. Then, in Section 5.2, we will make use of the existence of σ-frames, proved in Chapter 3, to construct instances of thickly-resolvable designs (with the required parameters) in each admissible congruence class, but modulo a larger period than the desired period ab, where, as defined in Chapter 1, $a = \frac{\sigma(k-1)}{\gcd(\sigma(k-1),\lambda)}$ and $b = \frac{k}{\gcd(k,\sigma)}$. We will then, in Section 5.3, make use of non-uniform σ-frames with a recursion similar to that in [6] to close out each congruence class and obtain eventual periodicity modulo ab. Finally, we will put all these pieces together and complete the proof in Section 5.4.

5.1 A First Example

We begin by using combinatorial configurations together with resolvable graph designs to construct our first example of a general thickly-resolvable design with parameters σ, k, and λ.

We first take an (n_r, m_k) configuration with $r = \sigma$ and $n = \frac{p_{k,\sigma}k}{\gcd(k,\sigma)} = p_{k,\sigma}b$, where $p_{k,\sigma}$ is a prime chosen greater than a and b (guaranteeing that $\gcd(p_{k,\sigma}, b) = 1$) and large enough to satisfy Theorem 2.14. We then construct a graph, which we will denote as $G_{k,\sigma}$, by replacing each line of this configuration with a clique K_k on the same set of points. Notice that this graph will be regular of degree $\sigma(k-1)$.
Now, using Theorem 2.12, we know that there exists a resolvable $\mathrm{PBD}_\lambda(z, G_{k,\sigma})$ for

all sufficiently large integers z satisfying

$$z \equiv 0 \pmod{p_{k,\sigma} b} \quad \text{and} \quad z \equiv 1 \pmod{a}. \tag{5.1}$$

This follows since $G_{k,\sigma}$ has $p_{k,\sigma}b$ vertices and the regularity of $G_{k,\sigma}$ implies that the α^* of Theorem 2.12 is $\sigma(k-1)$, yielding the necessary condition $\lambda(z-1) \equiv 0 \pmod{\sigma(k-1)}$, which simplifies to the second congruence in (5.1).
Once we have this resolvable $G_{k,\sigma}$-decomposition, we can break up the $G_{k,\sigma}$-blocks into K_k-blocks; this yields a K_k-decomposition of K_z, or alternatively, a $\text{PBD}_\lambda(z,k)$. In fact, this design is also σ-resolvable since a parallel class in the $G_{k,\sigma}$-decomposition becomes a σ-parallel class of cliques K_k. Each point of K_z occurs in exactly one $G_{k,\sigma}$-block in each parallel class and hence it appears in exactly σ K_k cliques (since we took $r = \sigma$ in the configuration). Thus, it follows that there exists a σ-resolvable $\text{PBD}_\lambda(z,k)$ when z is sufficiently large and satisfies the congruences given in (5.1).

Using this method, we have obtained an example of a σ-resolvable $\text{PBD}_\lambda(z,k)$, our first thickly-resolvable design with the required parameters.

Figure 5.1 shows an example of a $(15_2, 10_3)$-configuration, drawn on a cylinder. This configuration can be turned into a graph $G_{3,2}$ by replacing lines with triangles, or by extending adjacencies vertically to the torus. Figure 5.2 shows the configuration given in Figure 5.1 as a graph with the lines replaced by triangles.

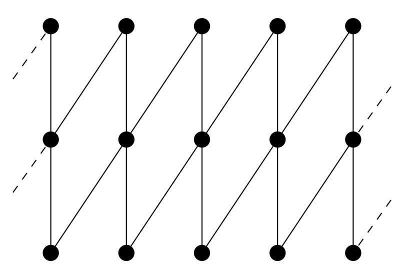

Figure 5.1: A $(15_2, 10_3)$-Configuration

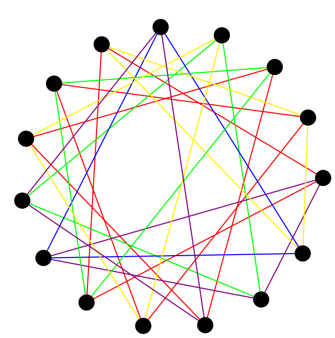

Figure 5.2: $G_{3,2}$

5.2 Constructions in Each Congruence Class

Now that we have an example of a σ-resolvable design, we use σ-frames to construct instances of σ-resolvable designs in each admissible congruence class modulo arbitrarily large periods. This result is stated as Proposition 5.2.

In order to prove Proposition 5.2, we will use a uniform σ-frame with group size $z-1$ and make use of Lemma 4.3 to fill in the groups with the thickly-resolvable design obtained in the previous section on z points. We will also need the following fact to complete the number theory involved.

Fact 5.1. $\gcd(k, \sigma(z-1)) = \gcd(k, \sigma)$

Proof. Clearly $\gcd(k, \sigma) \leq \gcd(k, \sigma(z-1))$ since any number dividing σ must also divide $\sigma(z-1)$. Now suppose $g = \gcd(k, \sigma(z-1))$. Then $g \mid k$ and $g \mid \sigma(z-1)$ and g is the largest such integer. First we notice that here we have a σ-resolvable $\mathrm{PBD}_\lambda(z, k)$, thus we must have that $k \mid \sigma z$ (since the necessary conditions must be satisfied) and therefore $g \mid \sigma z$. Now, writing $\sigma = \sigma z - \sigma(z-1)$ we can easily see that we must have $g \mid \sigma$. Therefore, $\gcd(k, \sigma(z-1)) = \gcd(k, \sigma)$. □

The following proposition guarantees that for any admissible value x (satisfying both necessary conditions), there is a thickly-resolvable design on some number of points that is congruent to x modulo some larger period (that is divisible by the desired period).

Proposition 5.2. *Suppose $x \equiv 1 \pmod{a}$ and $x \equiv 0 \pmod{b}$. Let P be given with $ab \mid P$. There exists a σ-resolvable $PBD_\lambda(v, k)$ for some integer $v \equiv x \pmod{P}$.*

Proof. Write $P = AB$, where all prime factors of b occur in B and all other prime factors occur in A, so that $\gcd(A, B) = \gcd(b, A) = 1$.

Using the method in Section 5.1, we start with a σ-resolvable $\mathrm{PBD}_\lambda(z, k)$ where z satisfies both conditions in (5.1). We now take a uniform σ-frame of type $(z-1)^u$ and index λ for u satisfying the frame condition given in (3.6) with $g = z-1$. Notice that $z \equiv 1 \pmod{a}$ is equivalent to the condition $\lambda(z-1) \equiv 0 \pmod{\sigma(k-1)}$, so the group size satisfies the necessary condition on the parameters given in (3.6). Now we make use of Lemma 4.3 to construct a σ-resolvable $\mathrm{PBD}_\lambda(v, k)$ for $v = u(z-1)+1$. To the σ-frame we add one point and fill each group together with the extra point, using Lemma 4.3, with the σ-resolvable PBD of order z, identifying subdesigns of order 1. This procedure results in a σ-resolvable $\mathrm{PBD}_\lambda(v, k)$, so all that remains is to analyze the constructible values of v, which we do separately modulo A and B.

Recall that the only condition we have on our choice of u is $\sigma(z-1)(u-1) \equiv 0 \pmod{k}$ or equivalently $u \equiv 1 \pmod{\frac{k}{\gcd(k,\sigma(z-1))}}$, which, given Fact 5.1, reduces to $u \equiv 1 \pmod{\frac{k}{\gcd(k,\sigma)}}$. Thus, constructible values of u are $u \equiv 1 \pmod{b}$. Also, as given in (5.1), constructible values of z are precisely the values that are $0 \pmod{p_{k,\sigma}b}$ and $1 \pmod{a}$.

Now, since constructible values of u only depend on b and $\gcd(A, b) = 1$, they must exhaust all possibilities modulo A. Notice, if we choose u and z so that $u \equiv 1 \pmod{A}$ and $z \equiv x \pmod{A}$, we guarantee that $u \equiv 1 \pmod{a}$ and $z \equiv x \equiv 1 \pmod{a}$. Using this selection of u and z we have

$$v = uz - u + 1 \equiv (1)(x) - 1 + 1 \equiv x \pmod{A}.$$

Modulo B, we can take $u \equiv 1 \pmod{B}$ and by the selection of $p_{k,\sigma}$ we know $\gcd(p_{k,\sigma}, B) = 1$ so we can take $z \equiv x \pmod{B}$. Therefore, we have

$$v = uz - u + 1 \equiv (1)(x) - 1 + 1 \equiv x \pmod{B}.$$

Thus, using the Chinese Remainder Theorem, we know there exists a selection for u and z with $v \equiv x \pmod{P}$. □

5.3 Recursion

In this section we give two constructions that will enable us to close out each admissible congruence class and obtain eventual periodicity modulo ab, where remember $a = \frac{\sigma(k-1)}{\gcd(\sigma(k-1),\lambda)}$ and $b = \frac{k}{\gcd(k,\sigma)}$.

Lemma 5.3. *Suppose that there exists a σ-resolvable PBD$(P + z, k, \lambda)$ containing a σ-resolvable PBD$_\lambda(z, k)$ as a subdesign. Then for some s_0 and all integers $s \geq s_0$, there exists a σ-resolvable PBD$(sP + z, k, \lambda)$ containing a σ-resolvable PBD(z, k, λ) as a subdesign.*

Proof. To prove this lemma we will apply Lemma 4.3. We first note that since $P = AB$ and $a = \frac{\sigma(k-1)}{\gcd(\sigma(k-1),\lambda)} \mid A$ we know $\lambda P \equiv 0 \pmod{\sigma(k-1)}$. Now since P satisfies the necessary condition for the group size, we can appeal to Theorem 3.4 and conclude that there exists a σ-frame of type P^s for sufficiently large s satisfying

$\sigma P(s-1) \equiv 0 \pmod{k}$ and since k necessarily divides σP already (since $k \mid \sigma b$) the equivalence is satisfied for all values of s.

We now take this σ-frame, add z additional points and apply Lemma 4.3 using our ingredient PBD as the s ingredient PBDs of Lemma 4.3. □

Lemma 5.4. *For some positive integer m, there exists a σ-frame of type $g^m h_1^1 h_2^1$ for all sufficiently large integers g and any integers $h_1, h_2 \leq g$ with $\pi \mid g, h_1, h_2$.*

Proof. We begin the proof by selecting an m such that there exist σ-frames of types π^m, π^{m+1}, and π^{m+2}. We know such a selection exists because $\sigma\pi \equiv 0 \pmod{k}$; thus the necessary condition in Theorem 3.4 is satisfied for any number of groups. So if we select m large enough Theorem 3.4 guarantees the existence of σ-frames of type π^m and if m is large enough $m+1$ and $m+2$ must also be large enough.

We also take a TD$(m+2, \frac{g}{\pi})$, which exists by Theorem 2.17. We truncate the last two groups of the TD down to sizes $\frac{h_1}{\pi}$ and $\frac{h_2}{\pi}$. This will truncate some of the blocks of the TD and so we now have blocks of sizes m, $m+1$, and $m+2$. Now we apply the fundamental construction, Lemma 4.2, by giving a weight of π to all the remaining points of the TD and use the appropriate σ-frame (depending on the block size). This yields the desired σ-frame. □

5.4 Proof of Theorem 1.19

We are now ready to prove our main result, Theorem 1.19, which is restated below for convenience. Suppose in what follows that we are given parameters k, σ, and λ for thickly resolvable designs.

Theorem 1.19. *Let $k \geq 2$, $\sigma \geq 1$, and $\lambda \geq 0$ be integers. There exists a σ-resolvable PBD$_\lambda(v,k)$ for all sufficiently large v satisfying* (1.4) *and* (1.5)*; that is, $\sigma v \equiv 0 \pmod{k}$ and $\lambda(v-1) \equiv 0 \pmod{\sigma(k-1)}$.*

Proof. We begin by using the method described in Section 5.1 with the graph $G_{k,\sigma}$ to construct a σ-resolvable PBD$_\lambda(z,k)$. In doing this, we know that z must satisfy the two congruences given in (5.1). Now we take t large enough so that there exists a σ-frame of type $(z-1)^t$. Notice that the second congruence in (5.1) guarantees that $z-1$ satisfies the necessary condition on the groups of the frame given in (3.6) and we will need to take a t satisfying $\sigma(z-1)(t-1) \equiv 0 \pmod{k}$, the necessary condition in (3.1). We take this σ-frame, add a new point and use Lemma 4.3 to fill each group (with the added point) with the σ-resolvable $\mathrm{PBD}_\lambda(z,k)$ identifying subdesigns of order 1. This yields a σ-resolvable $\mathrm{PBD}_\lambda(P+z,k)$, where $P=(z-1)(t-1)$, that contains a subdesign of order z.

Now we consider an admissible integer x, so suppose $x \equiv 1 \pmod{a}$ and $x \equiv 0 \pmod{b}$. Notice that since we chose z satisfying (5.1), we must have $a \mid (z-1)$ and our selection of t guarantees that $b \mid (z-1)(t-1)$. Both of these facts together with the assumption that $\gcd(a,b)=1$ imply that $ab \mid P$. Therefore, given Proposition 5.2, there must exist a σ-resolvable $\mathrm{PBD}_\lambda(w,k)$ for some $w \equiv x \pmod{P}$. We may assume that $w \gg x$. Thus, it suffices to close the congruence class $x \pmod{P}$; for this, we construct a σ-resolvable $\mathrm{PBD}_\lambda(nP+w,k)$ for all sufficiently large integers n.

This gives us asymptotic existence modulo P and since we can do this for any admissible x and take the greatest n that occurs, we in fact get asymptotic existence modulo π.

We apply Lemma 5.3, to get σ-resolvable PBDs of order $sP+z$ for all integers $s \geq s_0$. We select one of order $sP+z$ and $tP+z$ where $s \geq t \geq s_0$ and $sP \geq w-z$. We need s and t larger than s_0 in order to apply Lemma 5.3 for both, but we also require sP to be larger than $w-z$ in order to utilize a non-uniform frame given in Lemma 5.4.

Notice, by the definition of P, that we must have $\pi \mid P$, so $\pi \mid sP$ and $\pi \mid tP$. Also, since z and x are both admissible orders for σ-resolvable PBDs, we must have $z \equiv x \pmod{\pi}$ and we have that x and w are congruent modulo P (and hence modulo π); therefore, we conclude that $z \equiv w \pmod{\pi}$. So, we are able to apply Lemma 5.4 and for a large enough selection of m we obtain a σ-frame of type $(sP)^m(tP)^1(w-z)^1$.

To this frame, we add z new points and apply Lemma 4.3, filling all but the last group with the examples of order $sP+z$ or $tP+z$ (each of which contains a subdesign of order z that we identify) and on the last group we include the example of order w. The resultant design is a σ-resolvable $\mathrm{PBD}_\lambda((sm+t)P+w,k)$. □

Chapter 6

Applications, Extensions and Conclusion

6.1 Designs with Holes

We first discuss an application of our main result in Theorem 1.19. We consider designs in which some pairs of points do not occur together in any blocks of the design.

Definition 6.1. An *incomplete pairwise balanced design*, $\mathrm{IPBD}_\lambda((v;w),\{k\})$, is a triple $(V, W, \mathcal{B})$ where

- V is a set of v *points* and $W \subset V$ is a *hole* of size w;
- $\mathcal{B}$ is a collection of k-subsets of V called blocks;
- no two distinct points of W appear together in a block of $\mathcal{B}$; and
- any two distinct points, not both in W, appear together in exactly λ blocks of $\mathcal{B}$.

As before, we will omit the subscript whenever $\lambda = 1$.

We would like to note when $w \neq k$, an $\mathrm{IPBD}((v;w),\{k\})$ is equivalent to a $\mathrm{PBD}(v,\{k,w^*\})$, which is a PBD that has all blocks of size k except one of size w (when $w = k$ it is equivalent to a $\mathrm{PBD}(v,k)$ with one block removed). An $\mathrm{IPBD}_\lambda((v;w),\{k\})$ can be thought of as a $\mathrm{PBD}_\lambda(v,k)$ with the blocks of a $\mathrm{PBD}_\lambda(w,k)$ removed for some w-set that is a subset of the initial v-set; however, the existence of the $\text{IPBD}_\lambda((v;w),\{k\})$ does not imply the existence of either PBD of order v or w.

In the language of graph decompositions, an $\text{IPBD}_\lambda((v;w),\{k\})$ is equivalent to a K_k-(edge) decomposition of $K_v^\lambda - K_w^\lambda$ and to a K_k-(edge) decomposition of the complete multipartite graph with $\frac{v-w}{k-1}$ partite sets of size $k-1$ and one of size $w-1$.

The necessary conditions for an IPBD are obtained in a similar fashion as other combinatorial objects discussed so far. First, the $\lambda(\binom{v}{2}-\binom{w}{2})$ pairs need to be covered by blocks, each of which covers $\binom{k}{2}$ pairs; hence, we must have

$$\lambda(v(v-1)-w(w-1)) \equiv 0 \pmod{k(k-1)}. \tag{6.1}$$

Also, all $v-w$ non-hole points need to occur in exactly λ blocks with all $v-1$ other points, so we need $\lambda(v-1) \equiv 0 \pmod{k-1}$. Each of the w hole points is covered exactly λ times with the $v-w$ non-hole points (and never with the other $w-1$ hole points), so $\lambda(v-w) \equiv 0 \pmod{k-1}$; which is equivalent to $\lambda(w-1) \equiv 0 \pmod{k-1}$, given the previous necessary condition on v. Therefore, we write our local condition as the dual equivalence

$$\lambda(v-1) \equiv \lambda(w-1) \equiv 0 \pmod{k-1}. \tag{6.2}$$

For IPBDs we get a third necessary condition, which can be thought of as a generalized version of Fisher's Inequality (stated in Chapter 2 as Theorem 2.19). This extra condition gives a lower bound on the number of points, v, in terms of the hole size, w.

Each non-hole point x must be in exactly $\frac{\lambda(v-1)}{k-1}$ blocks of the design and each of these blocks covers at most one hole point (since no two hole points can occur together). Since x must occur within exacly λ blocks with each hole point, we must have that $\frac{\lambda(v-1)}{k-1} \geq \lambda w$; and hence,

$$v \geq (k-1)w+1, \tag{6.3}$$

where equality in the bound is achieved precisely when every block of the design contains exactly one hole point.

It is interesting to note that (6.1) and (6.2) depend on λ whereas the inequality in (6.3) does not weaken for λ values larger than one.

Something approximating an asymptotic existence theory for $\lambda = 1$ can be found in [12], with the main theorem stated below.

Theorem 6.2. [12] *Let $k \geq 2$ be a positive integer. For every real number $\epsilon \geq 0$, there exists an IPBD$((v;w),\{k\})$ for all sufficiently large v and w satisfying* (6.1) *and* (6.2) *with $v \geq (k-1+\epsilon)w$.*

An important starting point for the proof of Theorem 6.2 is the existence of examples with equality in (6.3).

Proposition 6.3. *Let $k \geq 3$ and $w \geq 1$ with $v = (k-1)w+1$. There exists an IPBD$((v;w),\{k\})$ if and only if there exists a resolvable PBD$(v-w,k-1)$.*

We omit the proof of Proposition 6.3 as the proof is similar to the one we give for Proposition 6.4 for higher λ values. We would like to comment that this is reminiscent of the equivalence between projective and affine planes and reveals the difficulty in constructing IPBDs with mixed block sizes K, since resolvable designs in this general context are presently not well understood.

Our main purpose here is to motivate σ-resolvable designs as the needed generalization in Proposition 6.3 to construct IPBDs with arbitrary index λ, so we now give the generalized equivalence.

Proposition 6.4. *Let $k \geq 3$ and $w \geq 1$ with $v = (k-1)w+1$. There exists an IPBD$_\lambda((v;w),\{k\})$ if and only if there exists a λ-resolvable PBD$_\lambda(v-w,k-1)$.*

Proof. First consider such an IPBD. Recall since we have equality in the bound given in (6.3) we know that every point of the hole touches exactly λ blocks and every block intersects the hole in exactly one point. Thus, it follows that truncating all hole points will reduce each block in size by one and induce parallel classes of thickness λ. The result of this truncation is then a λ-resolvable PBD$_\lambda(v-w,k-1)$. Conversely, given such a PBD, we compute that it has $\frac{\lambda(v-w-1)}{\lambda(k-2)} = \frac{v-w-1}{k-2}$ parallel classes of thickness λ. When $v = (k-1)w+1$, we see there are precisely w parallel classes of thickness λ. We extend each class with the addition of a new point, one new point per class. Now, every pair of points, old with new, appears in exactly λ blocks, as desired. So we obtain an IPBD$_\lambda((v;w),\{k\})$ for $v = (k-1)w+1$. □

We can now apply our main result, Theorem 1.19, together with Proposition 6.4 to construct an example of an IPBD$_\lambda((v;w),k)$ where w, k and λ are given with $\lambda(w-1) \equiv 0 \pmod{k-1}$. First, we put $n = (k-2)w+1$ and observe that $n \equiv 1 \pmod{k-2}$ and also $\lambda n \equiv \lambda(1-w) \equiv 0 \pmod{k-1}$. Thus, n satisfies the necessary conditions (1.4) and (1.5) of our main result; and so, Theorem 1.19 guarantees that we have a λ-resolvable PBD$_\lambda(n,k-1)$ for such n, provided it is large enough.

Now, we put $v = n + w = (k-1)w + 1$ and obtain an $\text{IPBD}_\lambda((v;w),\{k\})$ using Proposition 6.4.

6.2 Thickly Resolvable GDDs

In this section, we will consider the extension of Theorem 1.19 to uniform GDDs with constant block size.

We begin by generalizing the definition of group divisible design given in Definition 2.15 to allow for graph blocks. We will however restrict our definition to the uniform case with only one type of graph block (instead of a family of possible graph blocks), since that is the most general instance we will need.

First, recall that in the language of graph decompositions, a $\text{GDD}_\lambda(g^u, k)$ is equivalent to an edge-decomposition of the complete λ-fold multipartite graph $K^\lambda_{g,g,\ldots,g}$ (u partite sets each of size g) into copies of K_k. We utilize this interpretation to allow for non-complete G-blocks.

We define a *group divisible G-design* of type g^u and index λ, denoted $\text{GDD}_\lambda(g^u, G)$ as an edge-decomposition, as before, but here we decompose the edges into (isomorphic) copies of G so that every pair of vertices from different groups appear in exactly λ G-blocks and pairs of points from the same group never occur together in any G-blocks.

Asymptotic existence of uniform GDDs with graph block G was established in Chan's thesis [5] in 2010 and is stated below. The requisite necessary conditions are obtained using similar counting arguments as those in Wilson's paper [48].

Theorem 6.5. [5] *Let G be a simple graph with $e > 0$ edges and* gcd *of degrees equal to d. There exists u_0 such that a GDD(g^u, G) exists for all $u \geq u_0$ satisfying $\lambda g^2 u(u-1) \equiv 0 \pmod{2e}$ and $\lambda g(u-1) \equiv 0 \pmod{d}$.*

Not surprisingly, here we are interested in $\text{GDD}_\lambda(g^u, k)$ that are *σ-resolvable*, namely, the K_k-blocks can be partitioned into σ-parallel classes each containing every vertex in exactly σ of the K_k-blocks (the graph version will be useful within the proof of Theorem 6.6, but it is not our goal here to generalize to that extent).

The necessary divisibility conditions which mirror (1.4) and (1.5) are

$$\sigma g u \equiv 0 \pmod{k} \text{ and} \tag{6.4}$$

$$\lambda g(u-1) \equiv 0 \pmod{\sigma(k-1)}. \tag{6.5}$$

For this adaptation, we let $a := \sigma(k-1)/\gcd(\sigma(k-1), \lambda g)$ and $b := k/\gcd(k, \sigma g)$ so that our necessary conditions can be written as $u \equiv 1 \pmod{a}$ and $u \equiv 0 \pmod{b}$, as before. Note that $\gcd(a, b) = 1$, otherwise there are no solutions u.

It may be of interest to consider the existence question for σ-resolvable GDDs with a large number of equal-sized groups. This extends our main result, since a GDD of type 1^v with block size k and index λ is equivalent to a $\mathrm{PBD}_\lambda(v, k)$. As before, the first work in this area has been done beginning with block size $k = 3$ by Du and Zhang in [11] in which they are able to show that the necessary conditions are also sufficient with some exceptions (all of which occur when $\sigma = 1$). We are able to show that the necessary conditions for a general k are in fact asymptotically sufficient, stated below.

Theorem 6.6. *Let $k \geq 2$, $\sigma, g \geq 1$ and $\lambda \geq 0$ be integers. There exists a σ-resolvable GDD of type g^u with block size k and index λ for all sufficiently large integers u satisfying* (6.4) *and* (6.5)*.*

To prove Theorem 6.6, we follow the same proof template as in the case $\sigma = 1$ (the resolvable case), which is the main result of [6]. Now that we have proved Theorem 1.19, the necessary steps to extend the result closely follow those in [6], so we discuss only the key differences here.

The key construction in [6] generates a resolvable GDD from a resolvable PBD with constant block size k, a resolvable TD, and an ordinary (not necessarily resolvable) uniform GDD. To extend this to a construction for σ-resolvable GDDs, some adaptation is needed. Most notably, we replace the ordinary GDD by one whose blocks are the graphs $G_{k,\sigma}$, constructed in Section 5.1.

Lemma 6.7. *Suppose there exists a σ-resolvable $PBD_\lambda(v, k)$, a resolvable $TD(k, gu)$, and a $GDD_\lambda(g^u, G_{k,\sigma})$ (with a condition on block colouring). Then there exists a σ-resolvable GDD of type g^{uv}, block size k and index λ.*

Proof Sketch. We begin constructing the resultant GDD by taking gu copies of the ingredient resolvable PBD. We then place a copy of the ingredient GDD on each of the v-levels. Now, for each block of the PBD, we place a copy of the resolvable TD (using the same placement within each copy of the PBD). The blocks of the TD covers pairs from different v-levels. We also break up the $G_{k,\sigma}$ blocks of the GDD to obtain blocks of size k that cover pairs from within the same v-levels.

To resolve the blocks obtained using the TD, we use the σ-classes of the PBD together with the parallel classes of the TD. We will need to break up some of these σ-classes in order to also resolve the $G_{k,\sigma}$ blocks of the GDD. To do this, we assume that the blocks of the TD are resolved in such a way so that one class contains only blocks in the same "level" of the GDD (we refer to these as "flat" classes). Note that there are $\frac{\lambda(v-1)}{\sigma(k-1)}$ σ-classes of the PBD and each one has one "flat" class from the TD; thus, there are $\frac{\lambda(v-1)}{\sigma(k-1)}$ "flat" σ-classes of this type.

Now, we turn our attention to the $G_{k,\sigma}$ blocks from the GDD. We will be taking them v at a time (the same one from each of the v GDD). When we do this, we would like to know how many points have been missed, so we can mix these blocks with the flat σ-classes to obtain a full σ class. Notice that points that occur in the $G_{k,\sigma}$ block will appear exactly σ times (once the $G_{k,\sigma}$ block is broken up) and all other points appear 0 times. There are $\frac{\lambda g^2 u(u-1)}{\sigma n(k-1)}$ $G_{k,\sigma}$ blocks in the GDD with $\frac{\lambda g(u-1)}{\sigma(k-1)}$ occurring at a particular point; thus, each point is absent from $\frac{\lambda g^2 u(u-1)}{\sigma n(k-1)} - \frac{\lambda g(u-1)}{\sigma(k-1)} = \frac{\lambda g(u-1)}{\sigma(k-1)} \left[\frac{u}{n} - 1\right]$ of the $G_{k,\sigma}$ blocks. Since each point is missed equally often, we will need to break up this many flat classes, so we just need to ensure that we have enough classes to do so. Since we can take v to be large (much larger than u), we can guarantee that $\frac{\lambda(v-1)}{\sigma(k-1)}$, the number of flat σ classes, is larger than $\frac{\lambda g(u-1)}{\sigma(k-1)} \left[\frac{u}{n} - 1\right]$, the number of flat classes we need to break up. □

Once these first examples are obtained, the next step is to combine them with frames. Here, these frames are of course σ-frames.

The following construction fills the groups of a σ-frame by σ-resolvable designs. The block sizes in what follows are all equal to k.

Lemma 6.8. *Suppose there exists a σ-resolvable GDD of type g^w and index λ and a σ-frame of type $[g(w-1)]^t$ and index λ. Then there exists a σ-resolvable GDD of type $g^{t(w-1)+1}$ and index λ which contains as a subdesign a σ-resolvable GDD of type g^w and index λ.*

Remark. The proof is almost identical to [6, Lemma 2.3], except that parallel classes are replaced by σ-parallel classes (and λ is introduced).

As in [6], we show that, given a period P (divisible by ab) and an admissible congruence class x, there exists a σ-resolvable GDD in which the number of groups is congruent to x modulo P.

Lemma 6.9. *Suppose P and x are given with $ab \mid P$ and $x \equiv 1 \pmod{a}$, $x \equiv 0$ mod b. Then there exists a σ-resolvable GDD of type g^U and index λ, where $U \equiv x \pmod{P}$.*

Proof Sketch. As we did within the proof of Theorem 1.19, we sort the prime factors of $P = AB$ so that all primes factors common with b occur in B and all others occur in A. This guarantees that $\gcd(A, B) = 1$, thus we can use the Chinese Remainder Theorem on A and B separately. Below is a table summarizing our various selections modulo A and B for t, u, and v. We need to ensure that a selection is possible that gives $t(uv-1)+1 \equiv x \pmod{P}$ (which is true whenever there is a selection so that $t(uv-1)+1$ is x mod A and mod B).

	mod A	mod B	mod $P = AB$
t	1	$1-x$	$\exists!$ choice
u	x	$p_{k,\sigma}x$	$\exists!$ choice
v	1	0	$\exists!$ choice

Notice that these choices satisfy the required necessary conditions.

Now, given the above selection, we examine $U = t(uv-1)+1$ modulo A and B.

$$U = t(uv-1)+1 \equiv 1((x)(1)-1)+1 \equiv x \pmod{A}$$
$$U = t(uv-1)+1 \equiv (1-x)((p_{k,\sigma}x)(0)-1)+1 \equiv x \pmod{B}$$

Thus, we can finally apply the Chinese Remainder Theorem and conclude that this selection for t, u, and v, constructs an example of a σ-resolvable GDD$_\lambda$ of type g^U with $U \equiv x \pmod{P}$. □

6.3 Thickly Resolvable Graph Designs

In this section, we would like to extend our main result to allow for graph blocks. We will be using a simple graph G with vertex set V having k vertices with degrees $D = \{d_1, d_2, \ldots, d_k\}$ and e edges. We begin this extension by discussing the necessary conditions for this case.

The resolvability condition is the same as in (1.4); we are required to cover each vertex exactly σ times using blocks of k vertices in each σ-class, hence we need

$$\sigma v \equiv 0 \pmod{k}. \tag{6.6}$$

The local condition becomes much more complicated than it was in (1.5). Here we need to be able to write $\lambda(v-1)$ (the degree at each vertex in the large complete graph) using degrees of G in groups of σ. In each σ-class, every vertex appears in exactly σ G-blocks and so there will be σ degrees of G used at each vertex. Since we need to make up $\lambda(v-1)$ using these σ-fold sums of degrees from D, we define $\sigma * D$ to be the set of all possible sums of degrees in D using exactly σ summands. Then we need to be able to write $\lambda(v-1)$ as a positive integer combination of the terms in $\sigma * D$. More particularly, we need to use exactly $\frac{\lambda(v-1)k}{2e\sigma}$ terms in $\sigma * D$ since that is the number of σ-classes. In order to satisfy both of these conditions simultaneously we need to ensure that

$$\lambda(v-1) \equiv 0 \pmod{\alpha^*_\sigma(G)} \tag{6.7}$$

where $\alpha^*_\sigma(G)$ is the least positive integer c such that

$$c \begin{bmatrix} 1 \\ \frac{k}{2e\sigma} \end{bmatrix} \in \mathrm{span}_{\mathbb{Z}} \left\{ \begin{bmatrix} d \\ 1 \end{bmatrix} : d \in \sigma * D \right\}. \tag{6.8}$$

First we would like to point out that $c = 2e\sigma$ is a solution to (6.8) since we can take coefficients of 1 for the k vectors with first coordinate equal to a σ multiple of a degree in D and coefficients of 0 everywhere else.

As before, when we discussed graph designs in Chapter 2, (6.6) and (6.7) imply the global condition that $\lambda v(v-1) \equiv 0 \pmod{2e}$. The second coordinate in the vector equation given in (6.8) implies that $2e\sigma \mid \alpha^*_\sigma k$; and thus, we must have $2e \mid \lambda v(v-1)$ since $k \mid \sigma v$ and $\alpha^*_\sigma \mid \lambda(v-1)$.

Again, first work on this began with specific small graphs. The necessary conditions have been shown to be sufficient (with no exceptions) when the graph blocks are cycles of length 4 in [33] and for graph blocks of $K_4 - e$ in [20]. As mentioned in Remark 1.15, a related concept is that of equitable specialized block colourings and in [3] Bonacini and Marino consider the case of 4-cycles.

Our goal here is to prove that there exists a σ-resolvable $\mathrm{PBD}_\lambda(v, G)$ for large enough v satisfying (6.6) and (6.7), stated formally below.

Theorem 6.10. *Let $k \geq 2$, $\sigma \geq 1$ be integers and let G be a simple graph with k vertices. There exists a σ-resolvable $PBD_\lambda(v, G)$ for all sufficiently large v satisfying* (6.6) *and* (6.7).

We follow the same proof template as we did in Chapter 5 for PBDs (with complete blocks), and again we define $a = \frac{\alpha^*_\sigma(G)}{\gcd(\alpha^*_\sigma(G),\lambda)}$ and $b = \frac{k}{\gcd(k,\sigma)}$ so that the necessary conditions can be easily stated as $v \equiv 1 \pmod{a}$ and $v \equiv 0 \pmod{b}$. However, here some extra care must be taken in the first step in which the auxiliary graph is constructed. Instead of replacing lines of a configuration by cliques, we carefully replace lines by copies of our given G so as to realize all possible σ-fold combinations of degrees. The following lemma and proof outline how we will construct the auxiliary graph, which we call G_σ, in this case.

Lemma 6.11. *Given a simple graph G with k vertices and degrees in D and an (n_σ, b_k)-configuration with $n > \sigma(k-1)\binom{\sigma+k-1}{k-1}$, there exists a graph G_σ on n vertices that can be decomposed into edge-disjoint copies of G (each vertex belonging to exactly σ of the copies of G) and having each σ-fold sum in $\sigma * D$ as a degree of at least one vertex.*

Proof. We begin our proof by noting that $|\sigma * D| \leq \binom{\sigma+k-1}{k-1}$, the number of different σ-fold sums of k summands.

Now, we greedily label the points of the configuration using $|\sigma * D|$ different labels. So, start by choosing a point x_1 at random and label it with one of the $|\sigma * D|$ labels. Now we select another point x_2 that is not one of the $\sigma(k-1)$ other points that occur on a line with x_1 and label it with a different label. Now since $n > \sigma(k-1)\binom{\sigma+k-1}{k-1}$, we can continue this process until all of the $|\sigma * D|$ labels are used up.

To construct G_σ, we replace the lines of the configuration with G-blocks beginning with the labelled points. The label on a point will tell us how to place the σ copies of G onto each line incident with that point and ensure that the appropriate σ-fold sum occurs as the degree of the labelled point. Once we have placed copies of G on the lines incident with a labelled point, we can place copies of G on the remaining lines randomly. The placements of G on the labelled points ensure that each of the σ-fold degree sums occur as a degree in G_σ at least once each. □

When constructing G_σ we take a configuration with $n > \sigma(k-1)\binom{\sigma+k-1}{k-1}$ to satisfy the hypothesis of Lemma 6.11, but in order to satisfy Theorem 2.14 and ensure the existence of the required configuration we also take $n = \frac{p_\sigma k}{\gcd(k,\sigma)}$, where p_σ is a large prime.

As we did in Section 5.1, we now employ the result of Dukes and Ling in [14], stated as Theorem 2.12, on resolvable graph decompositions to obtain a resolvable $\mathrm{PBD}_\lambda(z, G_\sigma)$ for sufficiently large z satisfying

$$z \equiv 0 \pmod{n} \quad \text{and} \quad \lambda(z-1) \equiv 0 \pmod{\alpha^*(G_\sigma)},$$

where $\alpha^*(G_\sigma)$ is the least positive integer c such that

$$c\begin{bmatrix} 1 \\ \frac{n}{2|E(G_\sigma)|} \end{bmatrix} = c\begin{bmatrix} 1 \\ \frac{k}{2e\sigma} \end{bmatrix} \in \operatorname{span}_{\mathbb{Z}}\left\{\begin{bmatrix} \deg(v_i) \\ 1 \end{bmatrix} : v_i \in V(G_\sigma), i = 1, \ldots, n\right\}.$$

Notice the necessary conditions of Theorem 2.12 are more complicated than they were when we applied this result in Section 5.1 since G_σ is not necessarily a regular graph (whereas $G_{k,\sigma}$ was regular of degree $\sigma(k-1)$). Now, once we have the resolvable graph design, we can break up the G_σ-blocks (as we did in Section 5.1) into its edge-disjoint copies of G and obtain a σ-resolvable $\text{PBD}_\lambda(z, G)$ (again, a resolution class in the resolvable PBD becomes a σ-parallel class once the G_σ blocks are broken down). Before we move on with the proof, we would like to point out that the calculation of $\alpha^*(G_\sigma)$ is actually the same as for $\alpha^*_\sigma(G)$. We have stated this as the following fact and included a brief proof below.

Fact 6.12. $\alpha^*(G_\sigma) = \alpha^*_\sigma(G)$.

Proof. This is true because of the careful way in which we constructed the G_σ graph. We made sure to guarantee that every possible σ-fold sum of degrees in G occurs as a degree in G_σ at least once and since every point of the configuration was in σ lines and each line was replaced with a copy of G the vertex degrees in G_σ must be σ-fold sums of degrees in G. Therefore, we must have that $\sigma * D = \{\deg(v_i) : v_i \in V(G_\sigma), i = 1, \ldots, n\}$, making the computations for $\alpha^*(G_\sigma)$ the same as those for $\alpha^*_\sigma(G)$. □

As in Section 5.1, we have constructed our first example of a general thickly resolvable design by using combinatorial configurations to construct a special auxiliary graph and taking advantage of the existence of resolvable graph designs.

Now, as we did in the proof of Theorem 1.19, we wish to use σ-frames to construct instances of σ-resolvable PBDs in each admissible congruence class modulo a large period; although, we need to use the existence of a more general σ-frame than is provided by Theorem 3.4. Here, we require the existence of σ-frame with (non-complete) graph blocks. We present the extension of Theorem 3.4 here (including the slightly more intricate necessary conditions) with a proof sketch that omits the obligatory calculations.

Lemma 6.13. *Let λ and $g > 0 \in \mathbb{Z}$ and G be a simple graph with k vertices, e edges, and degrees in D with $\lambda g \equiv 0 \pmod{2e\sigma}$. Then for sufficiently large integers* *u satisfying*

$$\sigma g(u-1) \equiv 0 \pmod{k} \quad \textit{and} \quad \lambda g(u-1) \equiv 0 \pmod{\gamma},$$

where γ is the least positive integer c such that

$$c \begin{bmatrix} 1 \\ \frac{k}{2e\sigma} \end{bmatrix} \in span_{\mathbb{Z}} \left\{ \begin{bmatrix} d_i \\ 1 \end{bmatrix} : d_i \in D, i = 1, \ldots, k \right\},$$

a σ-frame of type g^u with G-blocks and index λ exists.

Proof Sketch. As in the proof of Theorem 3.4 for complete blocks, this is an application of the main result in [30] on edge-coloured graph decompositions. Here we will avoid repeating the calculations done in the proof of Theorem 3.4 as they are simply a blend of what is given in Section 3.3 and the proof of Theorem 6.1 in [5] which handles the graph case when $\sigma = 1$.

The family of coloured graphs used in the decomposition is constructed similarly as well. We replace each (undirected) edge of G with a directed arc in each direction, add an extra vertex ∞ and arcs from each vertex of G to ∞ (so ∞ has indegree k and outdegree 0). We also still use $g^2 + mg$ colours to colour the arcs where the first g^2 colours are induced from a labelling of the vertices of G with g labels and the remaining gm labels are used on the arcs involving ∞, induced from the vertex labels and a choice of m possible labels for ∞. However, here m, the number of partial σ-classes that miss a given group, is a bit different: $m = \frac{\lambda g}{2e\sigma}$.

The colours on the arcs (resulting from the edges of G) ensure that we can "lift" blocks of the edge-coloured decomposition to a uniform GDD with group size g. The colours on the arcs to ∞ correspond to the missed group of a partial σ-class and since they occur σ times each in the decomposition, each results in a partial σ-class. □

The remainder of the proof of Theorem 6.10, is almost identical to the proof of Theorem 1.19 with K_k blocks being replaced with G-blocks.

6.4 Conclusion and Future Directions

In this dissertation we generalized the concept of resolvability in designs to allow for multiple point-coverage within thick-resolution classes and showed that the necessary divisibility conditions for the existence of these thickly-resolvable designs are in fact asymptotically sufficient. As an important corollary, we obtained an asymptotic existence theory for incomplete pairwise balanced designs with maximal holes and an arbitrary index λ. This could play an important role in obtaining a full theory for IPBDs (where maximal holes are not required). We were also able to extend this concept to group divisible designs as well as graph designs.

Here we have settled the asymptotic existence question for thickly-resolvable designs with constant block size, graph blocks and for group divisible designs for general parameters. Perhaps it would now be interesting to consider some explicit parameters. It is possible the trick of using configurations together with resolvable graph decompositions could play a role in computer search for the challenging small examples.

There are also many ways to extend our results. It would be nice to obtain a full existence theory for incomplete pairwise balanced designs, discussed in Section 6.1, for general λ without the condition of a maximal hole. It would also be satisfying to be able to handle multiple blocks sizes, a family of graph blocks, and edge-colours.

We could also consider a list Σ of positive integers summing to the replication number, where we would like the blocks to resolve into thick parallel classes whose thickness is governed by the list Σ. A related notion is that of uniformly resolvable designs and class uniformly resolvable designs when block sizes are from a set K. In a uniformly resolvable design we ask that each parallel class has only blocks of a common size; whereas, in a class uniformly resolvable design each class has the same profile of block sizes. Class uniformly resolvable designs with a family of cycle blocks could be useful in context of the full Oberwolfach Problem (discussed in Example 1.16).

Another interesting extension would be to combine the work in [16] on loopy decompositions with thick resolvability. Here we could allow h vertices within a block to have a loop (of a given colour) and we would desire that the blocks be partitioned into classes such that h-subsets of the blocks partition the point set V for each class. In other words, each class would have one loop of each colour per vertex (but vertices could appear more times as a non-looped vertex as well). We suspect that this extension could be handled using similar methods as in the proof of Theorem 1.19, but with a weaker notion of configuration for the construction of the auxiliary graph used to obtain our first example.

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