The Asymptotic Existence of Graph Decompositions with Loops

by

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Supervisory Committee

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Abstract

Let $v \geq k \geq 1$ and $\lambda \geq 0$ be integers and G be a graph with n vertices, m edges, and no multiple edges. A (v, k, λ) *block design* is a collection $\mathcal{B}$ of k-subsets of a v-set X in which every unordered pair of elements in X is contained in exactly λ of the subsets in $\mathcal{B}$. A *G-decomposition*, or (v, G, λ) *graph design*, is a collection $H_1, H_2, ..., H_t$ of subgraphs of K_v (the complete graph on v vertices) such that each edge of K_v is an edge of exactly λ of the subgraphs H_i and each of the subgraphs H_i is isomorphic to G. A famous result by Wilson says that for a fixed graph G and integer λ, there exists a (v, G, λ) graph design for all sufficiently large integers v satisfying certain necessary conditions. In this thesis, we extend this result to include the case of loops in G. As a consequence, one obtains asymptotic existence of equireplicate graph designs for values of v satisfying certain necessary conditions, where a graph design is called *equireplicate* if each vertex of K_v occurs in the same number of subgraphs H_i of the decomposition.

Contents

List of Figures

Chapter 1

Introduction

A *graph* G is a set of vertices V together with a set of edges E, and an incidence relation $\iota \subseteq V \times E$ such that each edge $e \in E$ is incident with one or two vertices of V. When e is incident with only one vertex $x \in V$ we call e a *loop* and x a *looped vertex*. On the other hand, when e is incident with vertices $u \neq v \in V$ we say that e is an *ordinary* edge and that u and v are *adjacent* in G. Rather than referencing ι, it is standard to simply consider E as a multiset of singletons (representing loops) and pairsets (representing ordinary edges). Also, we use $V(G)$ and $E(G)$ when we wish to specify which graph we are referring to and we will only be considering undirected graphs.

A *graph decomposition* is a collection $H_1, H_2, \ldots, H_t$ of subgraphs of K_v, the *complete graph* on v vertices such that every edge $\{i, j\} \in E(K_v)$ appears in the exactly λ of the subgraphs. This can also be thought of as decomposing K_v^λ, the complete graph with v vertices and λ edges between every pair of vertices, such that every edge of K_v^λ occurs in some subgraph. If we also have H_i isomorphic to some graph G for every $i \in \{1, 2, \ldots, t\}$ then we have a *G-decomposition* and each of the subgraphs H_i in the decomposition is called a *G-block*. Graph decompositions have been studied extensively in the field of combinatorics because they can be used to solve many other combinatorial problems. See, for instance, Section 1.3 and the introduction of [6].

Necessary and asymptotically sufficient conditions in v for the existence of G-decompositions with $\lambda = 1$ were given by Wilson in [14] and are stated in Section 2.2. In this thesis, we prove a generalization of this famous result by considering all positive 'admissible' values of λ and graphs G with loops at certain vertices; we call v and λ *admissible* for a given graph G if they satisfy the conditions stated in (1.6). Due to the extra loop constraint we are forced to modify the idea of graph decomposition. Instead of decomposing K_v^λ we need to decompose a complete graph with (several) loops at each vertex. Thus, before we can state our main theorem we must first determine the necessary conditions that the complete graph we want to decompose must satisfy. This is discussed in the following section.

1.1 Necessary Conditions

Let G be a fixed graph, with $|V(G)| = n$, $|E(G)| = m$, and degree sequence $d_1, d_2, \ldots, d_n$. Let $V_L(G) \subseteq V(G)$ represent the set of vertices which have loops in G, where $|V_L(G)| = p$. We assume there are no multiple edges (or multiple loops) in G and that loops at vertex i do not contribute to the degree d_i. To determine the conditions that we require to ensure a G-decomposition is possible, we will use techniques similar to that presented in [14] by Wilson. First, we note that the number of ordinary edges in G must divide the number of ordinary edges of the complete graph we plan to decompose. For otherwise, it is not possible to partition the edges of the complete graph into subgraphs isomorphic to G. From this observation we can easily recognize that we need

$$m \mid \lambda \binom{v}{2},$$

since the complete graph we wish to decompose has λ ordinary edges between every pair of vertices; thereby yielding the first necessary condition, namely

$$\lambda v(v-1) \equiv 0 \pmod{2m}. \tag{1.1}$$

Secondly, we need to calculate the number of loops, μ, required at each vertex of the complete graph we intend to decompose, which we will denote $K_v^{[\mu,\lambda]}$. If we let b equal the number of G-blocks in a decomposition then we know that we must have

$$\begin{aligned} bm &= \lambda \binom{v}{2} \\ bp &= \mu v. \end{aligned}$$

The first equation ensures we have in fact partitioned the number of edges in the complete graph into b copies of G, while the second verifies that the number of loops in the decomposition equals the number of loops in the complete graph. Combining these equations we obtain $\frac{m}{p} = \frac{\lambda(v-1)}{2\mu}$; and consequently, we solve for μ and realize we need

$$\mu = \frac{\lambda p(v-1)}{2m} \tag{1.2}$$

loops at each vertex in $K_v^{[\mu,\lambda]}$.

Now, we need to ensure we are able to find a positive integer combination of the degrees of the vertices in G such that

$$\sum_{i \in V(G)} s_i d_i = \lambda(v-1), \quad \text{and} \tag{1.3}$$

$$\sum_{i \in V_L(G)} s_i = \mu, \tag{1.4}$$

where, as before, $V_L(G)$ is the set of looped vertices in G. The first sum guarantees it is possible to take copies of G containing a particular vertex of $K_v^{[\mu,\lambda]}$ and exhaust all the non-loop edges incident with that vertex, while the second sum ensures the number of loops that occur at each vertex in $V(K_v^{[\mu,\lambda]})$ is equal to μ. The condition stated in (1.3) is the same as that found by Wilson in [14], from which he obtained

$$\lambda(v-1) \equiv 0 \pmod{g},$$

where g is the greatest common divisor of the degrees in G.

Remark 1.1. Because we require (1.3) and (1.4) to be satisfied concurrently, our second necessary condition is slightly more complicated and yields fewer admissible values for v and λ than its counterpart given in [14].

By combining (1.3), (1.4), and the fact we need $\mu \in \mathbb{Z}$, we deduce that in order to ensure a G-decomposition for a graph G with loops is possible we need

$$\lambda(v-1)\begin{bmatrix} 1 \\ \frac{p}{2m} \end{bmatrix} \in \operatorname{span}_{\mathbb{Z}}\left\{\begin{bmatrix} d_i \\ e_i \end{bmatrix}\right\} \tag{1.5}$$

where d_i is the degree of vertex $i \in V(G)$ and $e_i = 0$ or 1 representing the number of loops at vertex i. Now, combining (1.1) with a reduced form of (1.5), we finally state our necessary conditions:

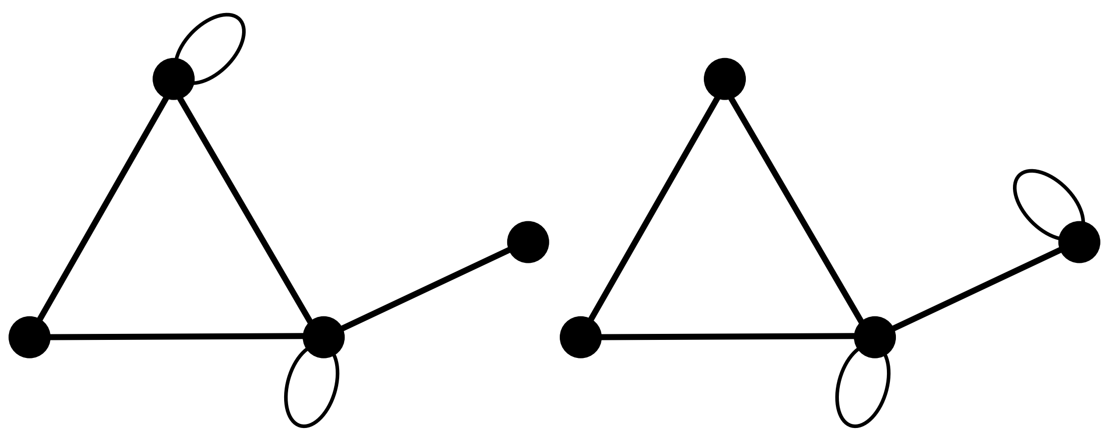
$$\begin{aligned} \lambda v(v-1) &\equiv 0 \pmod{2m} \\ \lambda(v-1) &\equiv 0 \pmod{\gamma} \end{aligned} \tag{1.6}$$

where γ is the least positive integer satisfying

$$\gamma\begin{bmatrix} 1 \\ \frac{p}{2m} \end{bmatrix} \in \operatorname{span}_{\mathbb{Z}}\left\{\begin{bmatrix} d_i \\ e_i \end{bmatrix}\right\}. \tag{1.7}$$

Remark 1.2. We are able to reduce (1.5) and attain the second congruence in (1.6) because γ generates the ideal of solutions to (1.7); since $\lambda(v-1)$ is also a solution, we must have $\gamma \mid \lambda(v-1)$. Also, when all coefficients in the linear combination are taken to be 1 we find $2m$ is a solution (since the sum of all the degrees in G is $2m$); and so, $\gamma \mid 2m$.

As previously mentioned in Remark 1.1, these new necessary conditions bear fewer admissible values for v and λ. Moreover, they yield differing allowable values depending on the placement of the loops in the graph; see Example 1.3.



(a) Graph G (b) Graph H

Figure 1.1: Example Graphs for Computing γ

Example 1.3. Consider the two graphs given in Figure 1.1. Both have $n = 4, m = 4$, and $p = 2$; however, the loops are incident with different vertices.

First, we compute γ for graph G in Figure 1.1; so, we need to find the least positive integer γ such that

$$\gamma \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} \in \text{span}_{\mathbb{Z}} \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}.$$

Obviously we must have $\gamma \geq 4$ and a multiple of 4 so that the second row has an integer value on the left side and, in fact, $\gamma = 4$ since

$$\begin{bmatrix} 4 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Now consider graph H, shown in Figure 1.1. Again we must have $\gamma \geq 4$ as the least positive integer such that

$$\gamma \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} \in \text{span}_{\mathbb{Z}} \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\};$$

however, $\gamma \neq 4$ since

$$\begin{bmatrix} 4 \\ 1 \end{bmatrix} \notin \text{span}_{\mathbb{Z}} \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

Thus, $\gamma \geq 8$, and so $\gamma = 8$ since

$$\begin{bmatrix} 8 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Also notice, Wilson's necessary conditions in [14] would give

$$\begin{aligned} \lambda v(v-1) &\equiv 0 \pmod 8 \\ \lambda(v-1) &\equiv 0 \pmod 1 \end{aligned}$$

since $\gcd\{1,2,3\} = 1$; thus, yielding many more admissible values for v and λ.

Now that we have established the necessary conditions we can present our main theorem and its immediate corollary.

1.2 Main Results

Using the necessary conditions found in Section 1.1, the goal of this thesis is to prove that there exists a point, when v is 'large' enough, so that the necessary conditions of Section 1.1 are also sufficient; this is accomplished in Theorem 1.4, stated below.

The proof of Theorem 1.4 will be done in several steps using four propositions. In Section 3.1, we construct some examples using cyclotomy in finite fields which, in Section 3.2, we extend to obtain more examples using pairwise balanced designs. Next, in Section 4.1, we show using a number theoretic argument that although the necessary conditions given in (1.6) are not in general sufficient for a G-decomposition they are sufficient for the existence of a 'signed' G-decomposition. In other words, (1.6) guarantees a G-decomposition in which we imagine negative copies of G are permitted. Using this result and finite vector spaces, in Section 4.2, we inflate a decomposition with high λ and 'fill in the holes' that result from Wilson's construction to obtain a G-decomposition for a large enough v_0 satisfying the congruences in (1.6). The explanation of how Theorem 1.4 follows from these four propositions will be given in Section 5.1.

Theorem 1.4. *Let $\lambda \in \mathbb{Z}$, $\lambda \geq 0$. Suppose G is an undirected graph with n vertices, m edges, and a loop at $p \leq n$ vertices with no multiple edges. Then there exists a G-decomposition of $K_v^{[\mu,\lambda]}$ for all sufficiently large integers v satisfying the necessary conditions given in* (1.6).

Remark 1.5. For simplicity we assume G has at most one loop at any particular vertex, but the proof can be modified to allow for multiple loops by letting e_i be the number of loops at vertex $i \in V(G)$.

Notice that each vertex in K_v^λ must occur in the decomposition as a looped vertex exactly μ times in the decomposition, leading us to consider similar 'balanced' conditions that could have practical applications. One very important special case is to require each vertex in K_v^λ to occur an equal number of times in the decomposition. When this happens we call the decomposition *equireplicate*, which is formally defined below.

Definition 1.6. A G-decomposition of K_v^λ is called *equireplicate* when every vertex of K_v^λ appears in the same number r of G-blocks of the decomposition.

Examples of equireplicate decompositions are not hard to find; for instance, a K_k-decomposition of K_v is always equireplicate because it is equivalent to a balanced incomplete block design with parameters v, k and λ (denoted BIBD(v, k, λ)), which will be discussed in more depth in Section 2.1. In fact, G-decompositions are always equireplicate whenever G is a d-regular graph because the degree of each vertex in K_v^λ must be the same multiple of d. Also, the cyclotomic construction used in Section 3.1 produces an equireplicate decomposition because of its cyclic nature.

Notice that when G is a reflexive graph (that is, $p = n$) it is easy to see this forces the vertices of $K_v^{[\mu,\lambda]}$ to occur in the same number of blocks of the decomposition; and so, in view of Definition 1.6, the decomposition must be equireplicate. Every G-block in which a vertex x appears must account for one loop at x. Also, as a result of (1.2) for any graph G with $p = n$, we must have

$$\mu = \frac{\lambda n(v-1)}{2m};$$

hence, each vertex must appear in exactly $r = \mu$ blocks of the decomposition.

Corollary 1.7. *Let $\lambda \in \mathbb{Z}$, $\lambda \geq 0$. Suppose G is a graph with n vertices, m edges, with no multiple edges and degrees $d_1, d_2, \ldots, d_n$. Then there exists an equireplicate G-decomposition of K_v^λ for all sufficiently large v satisfying* (1.6) *with $p = n$* (*and all $e_i = 1$*).

1.3 Application

As previously mentioned, graph decompositions are very important because of their relationship with many other combinatorial problems, including balanced bipartite designs, directed triple systems and especially optical networks. One application we found particularly interesting is that of grooming in optical networks. This usually refers to grouping nodes in a given network and assigning a specific wavelength having a fixed capacity in such a way to satisfy other constraints. Research in this area began in the 1990s and concentrated on minimizing the number of wavelengths used because, for each different wavelength, costly hardware is required at each node where traffic is added, dropped or converted to a different wavelength. Thus, another important objective is to minimize the 'drop cost'; that is, the cost of the required hardware. It was noted in [2] that in general these two constraints cannot be achieved simultaneously. However, G-decompositions, where edges of each G-block correspond to transmission opportunities at a given frequency, balance the drop cost across all nodes precisely when the equireplicate condition holds. In other words, this would result in an equal amount of hardware at each node. Groomings satisfying this constraints are called *balanced groomings*. Minimizing the drop cost sometimes leads to groomings that are quite unbalanced. The amount of hardware that can be placed at a node is often limited and this may impede the implementation of a grooming that satisfies all the initial constraints. Due to this hindrance, researchers have recently become interested in the study of balanced groomings, which is in its infancy.

Also, equireplicate decompositions would ensure ‘fairness’ in a network communication schedule, so that each node would be used (or turned on for transmission/listening) an equal number of times. For instance, the blocks might represent time slots (or transmission opportunities) and edges in G represent allowed transmissions.

The asymptotic existence of equireplicate graph decompositions was used in [3] by Dukes and Ling, in which they prove the asymptotic existence of resolvable graph decompositions, discussed in Section 2.2. Even though they used Corollary 1.7 (stated as Theorem 1.1 in [3]), a proof could not be found among the literature.

Chapter 2

Background

2.1 Basic Designs

A *balanced incomplete block design* with parameters v, k, λ, such that $v > k \geq 2$ and $\lambda \geq 0$, is a pair $(V, \mathcal{B})$, where V is a v-set of *points*, $\mathcal{B}$ is a collection of k-subsets of V, called *blocks*, and every pair of distinct points occur together in exactly λ blocks of $\mathcal{B}$. We denote this as BIBD(v, k, λ). Notice that we refer to $\mathcal{B}$ as a collection of blocks rather than a set because blocks may be repeated. More generally, we define *t-designs* (or *t-(v, k, λ) designs*) on v points with block size k where every set of t points occur together in exactly λ blocks; then a BIBD(v, k, λ) is a 2-design. Many of the proofs in this section can be found in [10], namely those for Lemma $2.3, 2.4, 2.6, 2.9$ and Theorem 2.11, so the reader is directed there for more information.

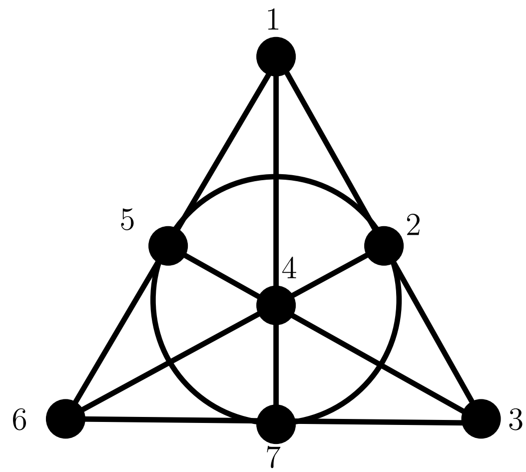


Figure 2.1: The Fano Plane: A BIBD$(7, 3, 1)$

Example 2.1.

$$\begin{aligned} V &= \{1, 2, 3, 4, 5, 6, 7\} \quad \text{and} \\ \mathcal{B} &= \{\{1, 2, 3\}, \{1, 4, 7\}, \{1, 5, 6\}, \{2, 4, 6\}, \{2, 5, 7\}, \{3, 4, 5\}, \{3, 6, 7\}\} \end{aligned}$$

is a BIBD$(7, 3, 1)$; the smallest, non-trivial design. There is a useful representation of this BIBD using the Fano Plane; see Figure 2.1. The blocks are the lines (including the circle) of the diagram.

Remark 2.2. We can interpret BIBDs as a partition of the edge set of K_v^λ into k-cliques. In other words, the existence of a BIBD(v, k, λ) ensures that a K_k-decomposition of K_v^λ is possible.

The following lemma, which supplies two necessary conditions for the existence of a BIBD(v, k, λ), is important when decompositions of K_v^λ are desired.

Lemma 2.3. *If a BIBD(v, k, λ) exists, then* $\lambda(v-1) \equiv 0 \pmod{k-1}$ *and* $\lambda v(v-1) \equiv 0 \pmod{k(k-1)}$.

The first congruence in Lemma 2.3 follows from the fact that every point of the design must appear in a common number, r, of blocks giving us $r(k-1) = \lambda(v-1)$, where r must be an integer. A double-counting argument is used. The second congruence arises since the number of edges in each block times the number of blocks, b, must equal the total number of edges giving us

$$b\binom{k}{2} = \lambda\binom{v}{2};$$

and so, we must have $bk(k-1) = \lambda v(v-1)$.

Another important necessary condition for the existence of a BIBD(v, k, λ) is Fisher's Inequality, stated below in Lemma 2.4.

Lemma 2.4. *In any BIBD(v, k, λ) we must have $b \geq v$, where $b = \frac{\lambda v(v-1)}{k(k-1)}$ is the number of blocks in the design.*

We would like to note that it is possible for parameters v, k, and λ to satisfy the congruences of Lemma 2.3, while failing the one given in Lemma 2.4. For example see Example 2.5.

Example 2.5. Consider the parameters $(v, k, \lambda) = (16, 6, 1)$. According to Lemma 2.3, it is possible for a BIBD$(16, 6, 1)$ to exist since

$$\begin{aligned} \lambda(v-1) = 15 &\equiv 0 \pmod 5 \\ \lambda v(v-1) = 240 &\equiv 0 \pmod{30}; \end{aligned}$$

however,

$$b = \frac{\lambda v(v-1)}{k(k-1)} = \frac{240}{30} = 8.$$

Thus, Lemma 2.4 is not satisfied so there does not exist a BIBD$(16, 6, 1)$.

In general, the necessary conditions given in Lemma 2.3 and Lemma 2.4 are not also sufficient (although Fisher's inequality is always satisfied when v is sufficiently large). More is known for *Steiner triple systems*, which are BIBD$(v, 3, 1)$s and denoted STS(v), where a necessary and sufficient condition is given below in Lemma 2.6. Steiner triple systems are the most commonly studied type of BIBD and we have already seen an example of a STS(7) in Example 2.1 and will soon present an example of an STS(9) in Example 2.7.

Lemma 2.6. *An STS*(v) *exists if and only if* $v \equiv 1$ *or* $3 \pmod 6, v \geq 1$.

Example 2.7. Below is an example of an STS(9), which we know exists by Lemma 2.6.

$$\begin{aligned} V &= \{1,2,3,4,5,6,7,8,9\} \quad \text{and} \\ \mathcal{B} &= \left\{ \begin{array}{cccccc} \{1,2,3\}, & \{1,4,7\}, & \{1,5,9\}, & \{1,6,8\}, & \{2,4,9\}, & \{2,5,8\}, \\ \{2,6,7\}, & \{3,4,8\}, & \{3,5,7\}, & \{3,6,9\}, & \{4,5,6\}, & \{7,8,9\} \end{array} \right\}. \end{aligned}$$

As in Example 2.1, this BIBD can also be represented diagrammatically; see Figure 2.2. As done in Lemma 2.6, Hanani proved that the congruences

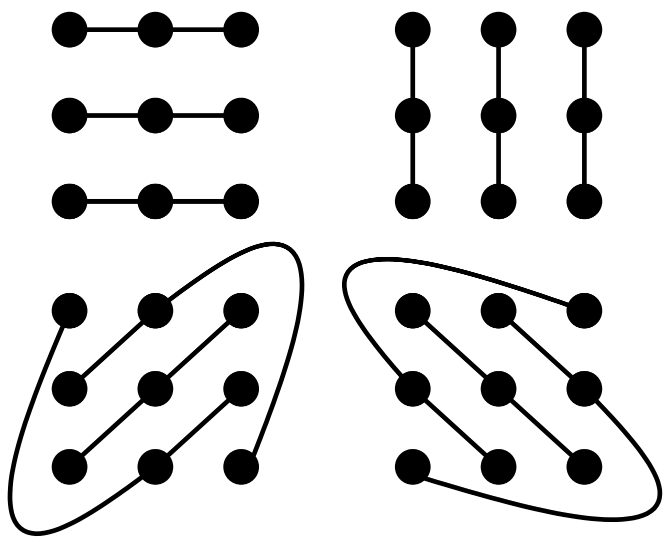


Figure 2.2: An affine plane: A BIBD$(9,3,1)$

given in Lemma 2.3 are not only necessary, but also sufficient for $k = 3$ and 4. He was also able to prove the congruences are sufficient for $k = 5$ for all admissible values except $(v, k, \lambda) = (15, 5, 2)$. He showed a design with these parameters could not exist.

2.1.1 Resolvabilty

As shown in Figure 2.2, the blocks in Example 2.7 can be partitioned into 4 sets of 3 blocks where each element of V occurs exactly once in each set. When it is possible to group the blocks so that each group partitions the points we call the design *resolvable*. The design shown in Figure 2.2 is a resolvable STS, also called a *Kirkman triple system*.

Definition 2.8. Suppose $(V, \mathcal{B})$ is a BIBD(v, k, λ). A *resolution class* in $(V, \mathcal{B})$ is a subset of disjoint blocks from $\mathcal{B}$ such that every $x \in V$ occurs exactly once. Then $(V, \mathcal{B})$ is a *resolvable BIBD* if $\mathcal{B}$ can be partitioned into resolution classes.

In order for a BIBD to be resolvable, in addition to the necessary conditions given in Lemma 2.3, we must also have

$$v \equiv 0 \pmod{k}. \tag{2.1}$$

This congruence ensures that the elements in V can be partitioned into $\frac{v}{k}$ blocks within each resolution class. Another necessary condition, which is similar to Fisher's Inequality (Lemma 2.4), was found in 1942 by Bose.

Lemma 2.9. *If there exists a resolvable BIBD(v, k, λ), then $b \geq v + r - 1$.*

We again note that these necessary conditions for the existence of a resolvable BIBD are not in general also sufficient. However, Theorem 2.11

gives a necessary and sufficient condition for the existence of a resolvable BIBD$(v, 2, 1)$.

Remark 2.10. In terms of Remark 2.2, a BIBD$(v, 2, 1)$ can be thought of as decomposing K_v into K_2s, which exists trivially. However, the extra resolvability condition results in a more difficult and interesting concept, namely that of a 1-factorization.

Theorem 2.11. *A resolvable BIBD*$(v, 2, 1)$ *exists if and only if* v *is an even integer and* $v \geq 2$.

It was also found by Ray-Chaudhuri and Wilson in [8] that resolvable BIBD$(v, 3, 1)$ exist whenever $v \equiv 3 \pmod 6$. Later in [5] Hanani *et al.* showed resolvable BIBD$(v, 4, 1)$ exist whenever $v \equiv 4 \pmod{12}$.

2.1.2 PBD

Another type of design that will be crucial in the proof of Proposition 3.5 is pairwise balanced designs, which are similar to BIBDs except that they do not require that all blocks have the same size. We now give the formal definition.

Definition 2.12. For $v \geq 2$, $\lambda \geq 1$, and $K \subseteq \{n \in \mathbb{Z} \mid n \geq 2\}$, a (v, K, λ) *pairwise balanced design*, which we abbreviate to PBD(v, K, λ), is a set system $(X, \mathcal{A})$ such that $|X| = v$, $|A| \in K$ for all $A \in \mathcal{A}$, and every pair of distinct points in X is contained in exactly λ blocks of $\mathcal{A}$.

If $K = \{k\}$ then a PBD(v, K, λ) is equivalent to a BIBD(v, k, λ). It is common practice to denote a PBD$(v, K, 1)$ simply as a PBD(v, K). We say that K is *PBD-closed* if $\mathbf{B}(K) = K$, where

$$\mathbf{B}(K) = \{v \mid \text{there exists a PBD}(v, K)\};$$

so, in other words, K is PBD-closed if $v \in K$ whenever there exists a PBD(v, K).

2.2 Asymptotic Existence

There are many instances in which the necessary conditions for different types of designs have been shown to be sufficient for all admissible values (with perhaps a finite number of exceptions). When this is possible we know there exist a constant v_0 for which there exists designs for all values of v satisfying the necessary conditions with $v \geq v_0$ and we say the necessary conditions are *asymptotically sufficient*.

Richard M. Wilson was the first to carefully consider the question of asymptotic existence of designs and has since settled asymptotic existence for many types of designs. It had been long conjectured that for any choice of k and λ, BIBD(v, k, λ)s exist for all sufficiently large values of v satisfying the congruences given in Lemma 2.3. This was settled as a consequence of Wilson's Theorem 1 in [12] on PBD closure.

Theorem 2.13. [12] *Given a set K of positive integers and a positive integer λ, there exists a PBD(v, K, λ) for all sufficiently large integers satisfying*

$$\begin{aligned} \lambda(v-1) &\equiv 0 \pmod{\alpha(K)} \\ \lambda v(v-1) &\equiv 0 \pmod{\beta(K)} \end{aligned}$$

where $\alpha(K) = \gcd\{k-1 \mid k \in K\}$ and $\beta(K) = \gcd\{k(k-1) \mid k \in K\}$.

Later in [8], Ray-Chaudhuri and Wilson proved the asymptotic existence of resolvable BIBDs for $\lambda = 1$.

Theorem 2.14. [8] *Given $k \geq 2$, there exists a constant v_0 such that if $v \geq v_0$ and $v \equiv k \pmod{k(k-1)}$, then a resolvable BIBD$(v, k, 1)$ exists.*

Wilson later considered the connection between designs and graph decompositions and wondered if it was possible to prove asymptotic existence of graph decompositions for any graph G. It did not take him long to prove Theorem 2.15, and settle the asymptotic existence of G-decompositions with $\lambda = 1$. This is among Wilson's most famous results.

Theorem 2.15. [14] *Given a graph G with n vertices and m edges, K_v can be G-decomposed for all sufficiently large integers v satisfying*

$$\begin{aligned} v(v-1) &\equiv 0 \pmod{2m} \\ v-1 &\equiv 0 \pmod{g} \end{aligned}$$

where g is the greatest common divisor of the degrees in G.

The next advancement was made in 2000 by Lamken and Wilson in [6] when they looked for a coloured decomposition into a family of graphs. In other words, they wanted to decompose the *edge-r-coloured complete digraph*, $K_v^{(r)}$, (the directed complete graph on v vertices that has all possible edges for each of r distinct colours) into a family of edge-r-coloured subgraphs. All the previous results focussed on decomposing into a single graph. These two variations require more complicated necessary conditions than Wilson needed in Theorem 2.15.

The greatest common divisor of the degrees in G (or the family of subgraphs $\mathcal{G}$) must be calculated using the in and out degrees at each vertex for each colour, so let $\deg_i^-(x)$ be the in-degree and $\deg_i^+(x)$ the out-degree of colour i at vertex x. We will also need

$$\tau(x) = (\deg_1^+(x), \deg_1^-(x), \deg_2^+(x), \deg_2^-(x), \ldots, \deg_r^+(x), \deg_r^-(x))$$

to represent the degree vector for vertex x. Then we define $\alpha(\mathcal{G})$ to be the greatest common divisor of the integers t such that the $2r$-vector $(t, t, \ldots, t)$ is in the integer span of the degree vectors $\tau(x)$ for all $x \in V(K_v^{(r)})$. This gives the second congruence in Theorem 2.16 and ensures $v-1$ can be written as a linear combination of the degrees of the vertices in $\mathcal{G}$ for each of the r colours.

Now, akin to what Wilson did with one colour in [14], we need the number of ordinary edges of each colour in $E(\mathcal{G})$ to divide the total number of ordinary edges of each colour in $K_v^{(r)}$, namely $v(v-1)$. So, define $\kappa(G) = (m_1, m_2, \ldots, m_r)$ for each $G \in \mathcal{G}$, where m_i is the number of edges of colour i in G, and let $\beta(\mathcal{G})$ be the greatest common divisor of the integers m such that $(m, m, \ldots, m)$ is in the integer span of the vectors $\kappa(G)$ for $G \in \mathcal{G}$.

Now we have established the necessary conditions for the extra edge-colour constraint, we also need to make sure that the family $\mathcal{G}$ of subgraphs is 'admissible'. Lamken and Wilson call $\mathcal{G}$ an *admissible* family if there exists a positive rational linear relation

$$(1, 1, \ldots, 1) = \sum_{G \in \mathcal{G}} c_G \kappa(G) \qquad \text{with all} \quad c_G > 0.$$

We are now able to state the theorem that gives asymptotic existence of $\mathcal{G}$-decompositions of $K_v^{(r)}$.

Theorem 2.16. [6] *Let $\mathcal{G}$ be an admissible family of edge-r-coloured digraphs with no multiple edges in any of the r colours. Then there exists a constant v_0 such that $\mathcal{G}$-decompositions of $K_v^{(r)}$ exist for all $v \geq v_0$ satisfying the congruences*

$$\begin{aligned} v(v-1) &\equiv 0 \pmod{\beta(\mathcal{G})} \\ v-1 &\equiv 0 \pmod{\alpha(\mathcal{G})}. \end{aligned}$$

Remark 2.17. Theorem 2.16 is stated here for $\lambda = 1$, but was proved for $\lambda \geq 1$ in [6] using this result and an extension involving only elementary techniques.

Dukes and Ling were the next to contribute to the question of asymptotic existence of designs with their result on resolvable graph designs; stated below.

Theorem 2.18. [3] *Let $\lambda \in \mathbb{Z}, \lambda \geq 0$. Suppose G is a graph with n vertices, m edges with no multiple edges and degree sequence $d_1, d_2, \ldots, d_n$. Then there exists v_0 such that there exists a resolvable G-decomposition for all $v \geq v_0$ satisfying* (2.1) *and the second congruence in* (1.6).

Chapter 3

Constructions

3.1 Cyclotomy

Infinitely many examples of G-decompositions of $K_q^{[\mu,\lambda]}$ can be created using cyclotomy in finite fields when q is a prime-power. The technique is used often in design theory, and we exploit the algebraic structure to balance loop coverage. Specifically, Proposition 3.1 gives necessary and sufficient conditions for G-decompositions of $K_q^{[\mu,\lambda]}$ where q is a prime-power. Its proof, given in Section 3.1.2, describes how to construct such decompositions.

Proposition 3.1. *Consider a graph G with n vertices, m edges, with m even, no multiple edges and a loop at p vertices. Then $K_q^{[\mu,\lambda]}$ can be G-decomposed whenever q is a prime-power with $q \equiv 1 \pmod{2m}$ and $q > m^{n^2}$.*

Remark 3.2. Observe the single congruence given on q implies that q satisfies (1.6), since $\gamma \mid 2m$.

Remark 3.3. Even though Proposition 3.1 is stated for graphs G with an even number of edges, it can be applied to two disjoint copies of G and the conclusion of infinitely many prime-power examples still follows. We note that μ is unchanged if G is replaced by $G \cup G$.

3.1.1 Example

Because of the constructive nature of the proof of Proposition 3.1, we will demonstrate the idea of the proof with the following example before we continue with the general proof which will follow in Section 3.1.2.

We begin by choosing a graph G and a prime-power satisfying the necessary conditions (1.6) for G. Consider using graph G in Figure 1.1 and $q = 17$ for our prime-power. We know from Example 1.3 that $\gamma = 4$ for this graph, and so $q = 17$ satisfies (1.6) with $\lambda = 1$ since $17(16) \equiv 0 \pmod 8$ and $16 \equiv 0 \pmod 4$. Notice that for this selection of G and λ we require that $\mu = 4$; thus, our goal is to decompose $K_{17}^{[4,1]}$ into copies of G. Now,

$$\mathbb{Z}_{17}^* = \langle 3 \rangle = \{1, 3, 9, 10, 13, 5, 15, 11, 16, 14, 8, 7, 4, 12, 2, 6\}.$$

In other words, since 3 is a generator for $\mathbb{Z}_{17}^*$ all non-zero elements of the group can be written as a power of 3 and $\mathbb{Z}_{17}^*$ has multiplicative cosets of index 4, namely

$$\begin{aligned} C_0 &= \{1, 13, 16, 4\} \\ C_1 &= \{3, 5, 14, 12\} \\ C_2 &= \{9, 15, 8, 2\} \\ C_3 &= \{10, 11, 7, 6\}. \end{aligned}$$

Notice that C_0 is a subgroup of $\mathbb{Z}_{17}^*$ since $13^2 \equiv -1 \pmod{17}$. Now, we label the vertices of $K_{17}^{[4,1]}$ with the set $\{0, 1, \ldots, 16\}$ and place $V(G) = \{a_1, a_2, a_3, a_4\}$ on $V(K_{17}^{[4,1]})$ such that the $m = 4$ differences $a_j - a_i$, for $\{i, j\} \in E(G)$ and $j > i$, lie in different cosets. One way to do this is by labeling

$$V(G) = \{0, 1, 3, 6\}$$

with which we obtain the differences

$$\begin{aligned} 1 - 0 &= 1 \\ 3 - 0 &= 3 \\ 3 - 1 &= 2 \\ 6 - 0 &= 6. \end{aligned}$$

Notice that we can rewrite $C_0 = \{1, 13, -1, -13\}$ since $16 \equiv -1 \pmod{17}$ and $4 \equiv -13 \pmod{17}$. This revision allows us to think only about the "positive half" of C_0. Multiply each $a_i \in V(G)$ by $\{1, 13\}$, giving us the sets $\{0, 1, 3, 6\}$ and $\{0, 13, 5, 10\}$ which represent two placements of G on $K_{17}^{[4,1]}$ that we will regard as our *base blocks*. In each case, 0 is the vertex of degree 3. We know there will be $\frac{|C_0|}{2} = \frac{q-1}{2m} = 2$ base blocks and we develop each of them additively modulo 17 to obtain 34 different sets (or placements of G) representing the blocks of our design. When developed, each vertex of $K_{17}^{[4,1]}$ will appear as a vertex in one of the blocks exactly $n = 4$ times for each base block (once for each vertex in the block). Thus, each vertex appears as a looped vertex exactly $p = 2$ times for each base block, and so will appear precisely $\mu = 4$ times in the design, as required.

Notice in the statement of Proposition 3.1 that we require $q > m^{n^2}$. In our example, a G-decomposition of $K_q^{[\mu,\lambda]}$ was guaranteed for $q > 4^{16}$; however, we were able to decompose K_{17}!

3.1.2 Proof of Proposition 3.1

Let G be a graph with n vertices, m edges with m even and a loop at p vertices. Suppose that q is a prime-power such that $q \equiv 1 \pmod{2m}$ and $q > m^{n^2}$. First, notice that $q \equiv 1 \pmod{2m}$ implies that $q-1 = km$ for some even $k \in \mathbb{Z}^+$. Consider $\mathbb{F}_q$, the Galois field of order q, it is well known (see [7] for proofs) that the multiplicative subgroup of $\mathbb{F}_q$ is cyclic of order $q-1$ when q is a prime-power and if $q \equiv 1 \pmod{m}$, then $\mathbb{F}_q$ contains a multiplicative subgroup C_0 of index m. Also, notice that the cosets $C_0, C_1, ..., C_{m-1}$ of C_0 are assumed to be indexed such that $x \in C_a$ and $y \in C_b$ implies that

$xy \in C_{a+b \pmod m}$. Within our proof we use the following lemma, proved in [13].

Lemma 3.4. [13] *Let $m, n \geq 2$ be integers. Let π be any mapping from the unordered pairs of distinct elements in $\{1, 2, ..., n\}$ to $\{0, 1, ..., m-1\}$. Suppose q is a prime-power with $q > m^{n^2}$ and $q \equiv 1 \pmod m$. Then there exist $a_1, a_2, ..., a_n \in \mathbb{F}_q$ such that for $1 \leq i < j \leq n$, $a_j - a_i \in C_{\pi(\{i,j\})}$, where $C_0, C_1, ..., C_{m-1}$ in $\mathbb{F}_q$ are indexed such that $x \in C_a$ and $y \in C_b$ implies that $xy \in C_{a+b \pmod m}$.*

Now, a G-decomposition of K_q can be obtained by letting the elements of $\mathbb{F}_q$ be the vertices of K_q. Choose $\pi(\{i, j\})$ for each $\{i, j\} \in E(G)$ such that the set $\{0, 1, ..., m-1\}$ of coset labels is exhausted. The labeling of other pairs is not important. Now, place $V(G) = \{a_1, a_2, ..., a_n\} \subseteq \mathbb{F}_q$ on $V(K_q)$ such that for $1 \leq i < j \leq n$, we have $a_j - a_i \in C_{\pi(\{i,j\})}$, as guaranteed by Lemma 3.4. We require $|C_0| = \frac{q-1}{m} = k$ to be even to ensure that $-1 \in C_0$, which will always be the case since $2m \mid (q-1)$.

Also, since the multiplicative group of $\mathbb{F}_q$ is isomorphic to $\mathbb{Z}_{q-1}$ when q is a prime-power, we have for some generator $g \in \mathbb{F}_q$,

$$C_0 = \{g^0 = 1, g^m, g^{2m}, ...g^{\frac{(k-1)m}{2}}, g^{\frac{k}{2}m} \equiv -1, -g^m, ..., -g^{\frac{(k-1)m}{2}}\}.$$

Multiply each element $a_i \in V(G)$ by $\{1, g^m, g^{2m}, ..., g^{\frac{(k-1)m}{2}}\}$; this can be thought of as multiplying each element in $V(G)$ by the "positive half" of C_0. This gives $t = \frac{q-1}{2m}$ base blocks, each of which we develop additively in $\mathbb{F}_q$.

As a result, we acquire graphs isomorphic to G such that every edge $\{i, j\}$ of K_q is contained in exactly one copy of the graph G. Forgetting about loops in G for a moment, we can easily see that we have a G-decomposition of K_q. Now, for each base block orbit, each vertex in G with a loop will be placed at each vertex of K_q exactly once. This is due to the transitive automorphism on $V(K_q)$ arising from addition in $\mathbb{F}_q$. Thus, every vertex in K_q is in $\frac{p(q-1)}{2m}$ blocks as a looped vertex. If we repeat each of these G-blocks λ times, it is clear that we will obtain a G-decomposition of $K_q^{[\mu,\lambda]}$. □

3.2 PBD Closure

After using Proposition 3.1 to find some prime-power values of v where we have a G-decompositions of K_v, we would like to somehow obtain more examples. This is accomplished by Proposition 3.5 which uses pairwise balanced designs, defined in Section 2.1.2, to eventually cover residue classes that satisfy the necessary conditions given in (1.6).

Proposition 3.5. *Let G be a graph with n vertices, m edges with no multiple edges and a loop at p vertices. There exists a positive integer n_G (divisible by m) such that if $K_{v_0}^{[\mu_0,\lambda]}$ can be G-decomposed for some positive integer v_0, then $K_v^{[\mu,\lambda]}$ can be G-decomposed for all sufficiently large integers $v \equiv v_0 \pmod{n_G}$.*

3.2.1 Proof of Proposition 3.5

Suppose, for some positive integer v_0, that $K_{v_0}^{[\mu_0,\lambda]}$ can be G-decomposed. Let $S_G = \{u \in \mathbb{Z} \mid K_u^{[\mu',\lambda]}$ can be G-decomposed $\}$, where for each $u \in S_G$,

$$\mu' = \frac{\lambda p(u-1)}{2m}.$$

From the definition of PBD, Definition 2.12, stated in Section 2.1.2, we conclude that if a PBD$(v, S_G) = (V, \mathcal{B})$ exists with $\mathcal{B} = \{B_1, B_2, \ldots, B_t\}$, then K_v can be decomposed into subgraphs, each of which is a K_{u_i} (with vertex set B_i) for some $u_i \in S_G$. Likewise, K_v^λ can be decomposed into subgraphs, each of which is a $K_u^{[\mu',\lambda]}$ for some $u \in S_G$, by taking λ copies of the subgraphs in the decompositions of K_v. Now, since $K_u^{[\mu',\lambda]}$ can be G-decomposed for each $u \in S_G$, we can attach μ' loops to each vertex of B_i. Therefore, by composing these decompositions we get a G-decomposition of $K_v^{[\mu,\lambda]}$ as long as for each vertex x in K_v we have

$$\sum_{\substack{B_i \ni x,\\ |B_i| = u_i}} \frac{\lambda p(u_i - 1)}{2m} = \frac{\lambda p(v-1)}{2m}. \tag{3.1}$$

In other words, we need to ensure that the number of loops at each vertex x sum to μ. We must also have the sum of the degrees of the non-loop edges in the subgraphs K_u^λ containing x equaling the degree of x in K_v^λ; hence, we know

$$\sum_{B_i \ni x} \lambda(u_i - 1) = \lambda(v - 1). \tag{3.2}$$

Now, (3.2) implies (3.1), and so we conclude that there is a G-decomposition of $K_v^{[\mu,\lambda]}$. What we have just shown is that $v \in S_G$ whenever there exists a PBD(v, S_G); thus, the set S_G is PBD-closed. In the terminology of [11], Proposition 3.5 asserts that the set S_G is eventually periodic with period n_G, for some nonnegative integer n_G. Actually, the main result in [11], given below in Theorem 3.6, found that this n_G can be taken as

$$n_G = \beta(S_G) = \gcd\{u(u-1) \mid u \in S_G\}. \tag{3.3}$$

Theorem 3.6. [11] *Every PBD-closed set K is eventually periodic with period $\beta(K) = \gcd\{k(k-1) \mid k \in K\}$. That is, there exists a constant C such that, for every $k \in K$, $\{v \mid v \geq C, v \equiv k \pmod{\beta(K)}\} \subseteq K$.*

But, by Proposition 3.1 combined with Dirichlet's Theorem on the existence of primes in arithmetic progressions, stated below as Theorem 3.7 (see [1] for a proof), we know that S_G is nonempty, containing integers greater than one; and so, $\beta(S_G) > 0$.

Theorem 3.7. *The arithmetic progression $a + tb$ where $t = 1, 2, 3, \cdots$, contains infinitely many prime numbers if and only if $\gcd\{a, b\} = 1$.*

In summary, S_G is eventually periodic with period $n_G = \beta(S_G)$; and so, $K_v^{[\mu,\lambda]}$ can be G-decomposed for all sufficiently large integers $v \equiv v_0 \pmod{n_G}$. □

What remains is to construct a single example of a G-decomposition of $K_{v_0}^{[\mu,\lambda]}$ for each admissible residue class $v_0 \pmod{n_G}$. This is the topic of the next chapter.

Chapter 4

Holey Constructions

4.1 Integral Solutions

While the necessary conditions given in (1.6) are not in general sufficient for the existence of a G-decomposition, Proposition 4.1 below proves they ensure the existence of a 'signed' decomposition, which is one in which we imagine it possible to allow negative copies of G in the decomposition. In other words, Proposition 4.1 implies the congruences in (1.6) ensure an integer solution to a particular system of linear equations; so, if each of the integers in the solution vector is nonnegative we have a G-decomposition.

Proposition 4.1. *Let G be an n-vertex graph with no multiple edges and $\mathcal{D}_u$ be the set of all subgraphs of K_u that are isomorphic to G. If $u \geq n+2$ and u satisfies the congruences given in* (1.6)*, then there exist integers x_H for each $H \in \mathcal{D}_u$ such that for each edge $\{i, j\} \in E(K_u)$, the sum of the integers*

x_H over those subgraphs $H \in \mathcal{D}_u$ that contain the edge $\{i,j\}$ is always λ. Moreover, for each vertex $i \in V(K_u)$ the sum of the integers x_H over the subgraphs $H \in \mathcal{D}_u$ that contain the vertex i as a looped vertex is always $\mu = \frac{\lambda p(u-1)}{2m}$.

4.1.1 Proof of Proposition 4.1

Let $\mathcal{D}_u$ be the set of all subgraphs of $K_u^{[\mu,\lambda]}$ that are isomorphic to G and suppose that $u \geq n+2$ and satisfies the congruences in (1.6). For the proof we use the following well-known lemma; refer to [9] for a proof.

Lemma 4.2. *Given an $m \times n$ rational matrix M and some $\mathbf{f} \in \mathbb{Q}^m$, the equation $M\mathbf{x} = \mathbf{f}$ has an integral solution $\mathbf{x}$ if and only if $\mathbf{y}^\top \mathbf{f}$ is integral whenever $\mathbf{y} \in \mathbb{Q}^m$ is such that $\mathbf{y}^\top M$ is integral.*

In order to prove Proposition 4.1 we need to show that there exist integers x_H such that

$$\begin{aligned} \sum_{H:\{i,j\} \text{ is an edge of } H} x_H &= \lambda \\ \sum_{H:\, i \text{ is a looped vertex of } H} x_H &= \mu, \end{aligned} \tag{4.1}$$

where we have one variable x_H for each $H \in \mathcal{D}_u$, one equation for each edge $\{i,j\}$ and one equation for each vertex i of K_u. In context of Lemma 4.2, we need to show that whenever integers $\beta_{\{i,j\}}$ are assigned to the edges $\{i,j\}$ of K_u and integers β_i are assigned to the vertices of K_u in such a manner that for each subgraph H the sum

$$\sigma_H = \sum_{\{i,j\}\in E(H)} \beta_{\{i,j\}} + \sum_{i\in V_L(H)} \beta_i$$

is divisible by some integer d, where $V_L(H)$ is the set of looped vertices of H, then the sum

$$\sigma = \lambda \sum_{\{i,j\}\in E(K_u)} \beta_{\{i,j\}} + \mu \sum_{i\in V(K_u)} \beta_i$$

is also divisible by d. Here, $\beta_{\{i,j\}}$ and β_i are the entries of $\mathbf{y}$ in Lemma 4.2. So, suppose that integers $\beta_{\{i,j\}}$ and β_i have been assigned to the edges and vertices (respectively) of K_u such that $\sigma_H \equiv 0 \pmod{d}$ for each $H \in \mathcal{D}_u$ and let i and j be any distinct vertices of K_u. Now, since we assumed $u \geq n+2$, there will be a subgraph $H \in \mathcal{D}_u$ that contains the vertex i but not the vertex j. Let $H' \in \mathcal{D}_u$ be obtained from H by applying the permutation $\pi_1 = (ij)$, where vertex i is replaced by vertex j and the edges, $\{i, a_1\}, \{i, a_2\}, ..., \{i, a_s\}$, that are incident with i in H, are replaced by $\{j, a_1\}, \{j, a_2\}, ..., \{j, a_s\}$, a loop at i becomes a loop at j, and everything else in H remains the same in H'. Clearly, we must have

$$\begin{aligned}\beta_{\{i,a_1\}} + \beta_{\{i,a_2\}} + ... + \beta_{\{i,a_s\}} + e_i\beta_i \\ \equiv \beta_{\{j,a_1\}} + \beta_{\{j,a_2\}} + ... + \beta_{\{j,a_s\}} + e_i\beta_j \pmod{d}.\end{aligned} \tag{4.2}$$

Now, let i, j, x, y be any four distinct vertices of $K_u^{[\mu,\lambda]}$ and find $H_1 \in \mathcal{D}_u$ such that the edge $\{x,y\} \in E(H_1)$ and $i,j \notin V(H_1)$, which again we can do since $u \geq n+2$. Let H_2, H_3, and H_4 be the images of H_1 under the permutations $\pi_2 = (ix), \pi_3 = (jy)$, and $\pi_4 = (ix)(jy)$, respectively. By our assumption that $\sigma_H \equiv 0 \pmod{d}$ for every $H \in \mathcal{D}_u$, we have $\sigma_{H_1} + \sigma_{H_4} \equiv \sigma_{H_2} + \sigma_{H_3} \pmod{d}$, which reduces to

$$\beta_{\{i,j\}} + \beta_{\{x,y\}} \equiv \beta_{\{i,y\}} + \beta_{\{x,j\}} \pmod{d}. \tag{4.3}$$

This implies that there exist integers ϵ and α_i such that

$$\beta_{\{i,j\}} \equiv \alpha_i + \alpha_j + \epsilon \pmod{d}. \tag{4.4}$$

To prove (4.4) fix three distinct vertices i, j, and y of K_u and choose integers $\alpha_i, \alpha_j, \alpha_y$, and ϵ satisfying

$$\begin{cases} \beta_{\{i,j\}} = \alpha_i + \alpha_j + \epsilon \\ \beta_{\{i,y\}} = \alpha_i + \alpha_y + \epsilon \\ \beta_{\{j,y\}} = \alpha_j + \alpha_y + \epsilon. \end{cases}$$

There exist many solutions to the above system of equations since as an augmented matrix over the ring $\mathbb{Z}_d$, we have

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & \beta_{\{i,j\}} \\ 1 & 0 & 1 & 1 & \beta_{\{i,y\}} \\ 0 & 1 & 1 & 1 & \beta_{\{j,y\}} \end{array}\right]$$

and remembering operations are performed modulo d, we can reduce to

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & \beta_{\{i,j\}} \\ 0 & 1 & 1 & 1 & \beta_{\{j,y\}} \\ 0 & 0 & 2 & 1 & \beta_{\{i,y\}} + \beta_{\{j,y\}} - \beta_{\{i,j\}} \end{array}\right]. \tag{4.5}$$

Therefore, guaranteeing infinitely many solutions. Now, pick α_x for each $x \neq i, j, y \in V$ such that $\beta_{\{i,x\}} = \alpha_i + \alpha_x + \epsilon$. Using this and (4.3) we can solve for $\beta_{\{j,x\}}$ since we must have $\beta_{\{j,x\}} + \beta_{\{i,y\}} \equiv \beta_{\{i,x\}} + \beta_{\{j,y\}} \pmod{d}$. We are now ready to solve for $\beta_{\{w,x\}}$ for any vertices w and x with $|\{w,x\} \cap \{i,j,y\}| = \emptyset$, since we must have $\beta_{\{w,x\}} \equiv \beta_{\{i,x\}} + \beta_{\{j,w\}} - \beta_{\{i,j\}} \pmod{d}$. This gives $\beta_{\{w,x\}} = \alpha_w + \alpha_x + \epsilon$ as required.

We would like to note that (4.4) can be simplified using Claim 4.3, given below, but this is unnecessary for our purposes.

Claim 4.3. *In* (4.4)*, we can take* $\epsilon = 0$ *when* d *is odd or* $\epsilon = 0$ *or* 1 *when* d *is even.*

Proof. It is clear (4.5) ensures the system of linear equations resulting from (4.4) has infinitely many solutions, but in particular when d is odd, 2 has a multiplicative inverse in $\mathbb{Z}_d$ giving us

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & \beta_{\{i,j\}} \\ 0 & 1 & 1 & 1 & \beta_{\{j,y\}} \\ 0 & 0 & 1 & 2^{-1} & 2^{-1}(\beta_{\{i,y\}} + \beta_{\{j,y\}} - \beta_{\{i,j\}}) \end{array}\right],$$

leaving ϵ as a free variable we can take to be zero. On the other hand, when d is even, the last row in the matrix in (4.5) gives us $2\alpha_y + \epsilon \equiv \beta_{\{i,y\}} + \beta_{\{j,y\}} - \beta_{\{i,j\}} \pmod{d}$; and so, we must choose $\epsilon = 0$ or 1 depending on whether $\beta_{\{i,y\}} + \beta_{\{j,y\}} - \beta_{\{i,j\}}$ is even or odd. □

Using (4.4) we can reduce (4.2) to

$$d_i\alpha_i + e_i\beta_i \equiv d_i\alpha_j + e_i\beta_j \pmod{d}, \tag{4.6}$$

which then yields $d_i(\alpha_i - \alpha_j) \equiv e_i(\beta_j - \beta_i) \pmod{d}$. And since this is true for any degree in H, we obtain

$$\gamma(\alpha_i - \alpha_j) \equiv \frac{p\gamma}{2m}(\beta_j - \beta_i) \pmod{d},$$

which gives us

$$\gamma\alpha_i + \frac{p\gamma}{2m}\beta_i \equiv \gamma\alpha_j + \frac{p\gamma}{2m}\beta_j \pmod{d}. \tag{4.7}$$

Recall that i and j were selected arbitrarily, making (4.7) true for any choice of vertices i and j of K_u. Therefore,

$$\gamma\alpha_i + \frac{p\gamma}{2m}\beta_i \equiv \gamma\alpha_0 + \frac{p\gamma}{2m}\beta_0 \pmod{d}, \tag{4.8}$$

where α_0 and β_0 are constants. By (4.4), we have

$$\begin{aligned}\sigma &= \lambda \sum_{\{i,j\}\in E(K_u)} \beta_{\{i,j\}} + \mu \sum_{i\in V(K_u)} \beta_i \\ &\equiv \lambda \sum_{\{i,j\}\in E(K_u)} (\alpha_i + \alpha_j + \epsilon) + \mu \sum_{i\in V(K_u)} \beta_i \pmod{d}.\end{aligned}$$

Notice that each α_i appears in the sum $u-1$ times, since each vertex i is adjacent to all other $u-1$ vertices of K_u, and ϵ occurs within the sum $\frac{u(u-1)}{2}$ times since the sum is taken over all $\frac{u(u-1)}{2}$ edges of K_u. This yields

$$\begin{aligned}\sigma &\equiv \lambda(u-1) \sum_{i\in V(K_u)} \alpha_i + \frac{\lambda u(u-1)}{2}\epsilon + \mu \sum_{i\in V(K_u)} \beta_i \\ &\equiv \sum_{i\in V(K_u)} (\lambda(u-1)\alpha_i + \mu\beta_i) + \frac{\lambda u(u-1)}{2}\epsilon \\ &\equiv \sum_{i\in V(K_u)} \left(\lambda(u-1)\alpha_i + \frac{\lambda p(u-1)}{2m}\beta_i\right) + \frac{\lambda u(u-1)}{2}\epsilon\end{aligned}$$

Since γ divides $\lambda(u-1)$ we can let $\lambda(u-1) = k\gamma$ for some $k \in \mathbb{Z}$ and obtain

$$\sigma \equiv k \sum_{i\in V(K_u)} \left(\gamma\alpha_i + \frac{p\gamma}{2m}\beta_i\right) + \frac{\lambda u(u-1)}{2}\epsilon \pmod{d}.$$

Then, as a result of (4.8), we have

$$\sigma \equiv k \sum_{i \in V(K_u)} \left(\gamma\alpha_0 + \frac{p\gamma}{2m}\beta_0\right) + \frac{\lambda u(u-1)}{2}\epsilon \pmod{d}.$$

The above sum is taken over all vertices $i \in V(K_u)$, thus

$$\begin{aligned}
\sigma &\equiv k\left(\gamma u\alpha_0 + \frac{\gamma u p}{2m}\beta_0\right) + \frac{\lambda u(u-1)}{2}\epsilon \\
&\equiv \lambda u(u-1)\alpha_0 + \frac{\lambda u(u-1)p}{2m}\beta_0 + \frac{\lambda u(u-1)}{2}\epsilon \\
&\equiv \frac{\lambda u(u-1)}{2m}\left(2m\alpha_0 + p\beta_0 + m\epsilon\right) \pmod{d}. \qquad (4.9)
\end{aligned}$$

Also, using a similar argument we obtain

$$\begin{aligned}
\sigma_H &= \sum_{\{i,j\} \in E(H)} \beta_{\{i,j\}} + \sum_{i \in V_L(H)} \beta_i \\
&\equiv \left(\sum_{i \in V(H)} d_i\alpha_i + e_i\beta_i\right) + m\epsilon \\
&\equiv \left(\sum_{i \in V(H)} d_i\alpha_0 + e_i\beta_0\right) + m\epsilon \qquad (4.10) \\
&= 2m\alpha_0 + p\beta_0 + m\epsilon \pmod{d},
\end{aligned}$$

where d_i and e_i are as in (1.5). Observe, (4.10) follows from (4.6) and (4.8). And so, as a consequence of our initial assumption $\sigma_H \equiv 0 \pmod{d}$ we obtain

$$2m\alpha_0 + p\beta_0 + m\epsilon \equiv 0 \pmod{d}. \qquad (4.11)$$

Therefore using (4.9) and (4.11), we can complete the proof by concluding that $\sigma \equiv 0 \pmod{d}$; hence, d must also divide σ. $\square$

4.2 Vector Spaces

Using Wilson's construction, Proposition 4.4 asserts that for each u satisfying (1.6) we can stretch the 'signed' decomposition with high λ found in Proposition 4.1 to construct a complete graph on a larger number, v_0, of vertices that can be G-decomposed such that v_0 is congruent to u modulo the period n_G, found in Proposition 3.5.

Proposition 4.4. *Let G be a graph with n vertices, m edges, p loops and no multiple edges. Then for every integer u satisfying*

$$\begin{aligned} \lambda u(u-1) &\equiv 0 \pmod{2m} \quad \textit{and} \\ \lambda(u-1) &\equiv 0 \pmod{\gamma}, \end{aligned}$$

where γ is defined in (1.7), there exists an integer $v_0 \equiv u \pmod{n_G}$, such that $K_{v_0}^{[\mu,\lambda]}$ can be G-decomposed.

4.2.1 Proof of Proposition 4.4

Let u satisfy the congruences of Proposition 4.4 for some graph G and suppose that $u \geq n+2$ so that Proposition 4.1 applies. Let $V(K_u) = \{1, 2, \ldots, u\}$. Now, the proof of Proposition 4.1 finds an integral solution of the system given in Lemma 4.2, namely $\{x_H \mid H \in \mathcal{D}_u\}$. If we set $x'_H = x_H + c$ for each x_H and some integer c, then, as a result of (4.1) and the fact that every ordinary edge occurs λ times, we have

$$\sum_{H\,:\,\{i,j\}\text{ is an edge of }H} x'_H = \lambda + c\lambda_0 = \lambda\left(1 + c\frac{\lambda_0}{\lambda}\right),$$

where $\lambda_0 = \frac{2m|\mathcal{D}_u|}{u(u-1)}$ is the number of graphs $H \in \mathcal{D}_u$ that contain a given edge. We may choose c such that:

- each $x'_H > 0$
- $\lambda \mid c$ and
- $1+c\frac{\lambda_0}{\lambda}$ is a prime congruent to 1 modulo n_G (the period found in (3.3)).

The first condition on c ensures our 'signed' decomposition becomes a 'positive' decomposition, giving us a list of not necessarily distinct subgraphs $H_1, H_2, \cdots, H_k$ in $\mathcal{D}_u$ such that each $\{i, j\}$ occurs in exactly $\lambda(1 + c\frac{\lambda_0}{\lambda})$ subgraphs H_i. We know we can choose c large enough to simultaneously satisfy all three conditions because of Dirichlet's Theorem. Let $q = 1 + c\frac{\lambda_0}{\lambda}$. This gives us

$$q\lambda \equiv \lambda \pmod{\lambda_0};$$

which is the multiplicity of every edge arising from the 'positive' decomposition. In other words, $H_1, H_2, \ldots, H_k$ gives a G-decomposition of $K_u^{[\mu',\lambda q]}$; and so we must have

$$k = \frac{(\lambda q)u(u-1)}{2m} \tag{4.12}$$

since there are $\lambda q\binom{u}{2}$ edges in $K_u^{[\mu',\lambda q]}$ and we have k G-blocks in the decomposition. Also, because the padding of the signed decomposition just adds an equal number of copies of each H_i we must have

$$\mu' = \frac{(\lambda q)p(u-1)}{2m}. \tag{4.13}$$

Now, what we wish to do is stretch the edges of the decomposition in q sets of λ at a time into λ edges for some larger $v_0 \equiv u \pmod{n_G}$. We now choose $t \geq u^2$ large enough to ensure that Proposition 3.1 applies, so that $K_{q^t}^{[\mu_t,\lambda]}$ can be G-decomposed. Let $v_0 = uq^t$; so we have

$$v_0 \equiv u \pmod{n_G}.$$

Notice that the proof of Proposition 4.4 will be complete if we can show $K_{v_0}^{[\mu,\lambda]}$ can be G-decomposed, where $\mu = \frac{\lambda p(v_0-1)}{2m}$; this is done using vector space transformations. So, let $\mathcal{W}$ be a t-dimensional vector space over $\mathbb{F}_q$, the Galois field of order q, and let $f : \mathcal{W} \mapsto \mathbb{F}_q$ be any nonzero linear functional with kernel $\mathcal{K}$ of dimension $t-1$. Wilson discovers in [12] that when $t \geq u^2$ there exist linear mappings $T_1, T_2, \ldots, T_u$ from $\mathcal{W}$ to itself such that $S_{ij} = (T_j - T_i)^{-1}$ exists for all $1 \leq i < j \leq u$ and there exist vectors

$\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_u$ such that

$$f(S_{ij}(\mathbf{x}_j - \mathbf{x}_i)) = \alpha_{ij} \tag{4.14}$$

for any choice of $\frac{u(u-1)}{2}$ scalars α_{ij} with $1 \leq i < j \leq u$.

Now consider the complete graph, $K_{v_0}^{[\mu,\lambda]}$, with vertex-set $\mathcal{W} \times \{1, 2, \ldots, u\}$ portrayed in Figure 4.1. Recall that we chose t large enough so that Proposition 3.1 ensures a G-decomposition of $K_{q^t}^{[\mu_t,\lambda]}$; hence, we can G-decompose the complete graphs with vertex-sets $\mathcal{W} \times \{1\}, \mathcal{W} \times \{2\}, \ldots, \mathcal{W} \times \{u\}$ which are the u subgraphs of $K_{v_0}^{[\mu,\lambda]}$ that are enclosed in ovals in Figure 4.1 and each of which has cardinality q^t. Because we can decompose each of these subgraphs, it will suffice to construct a decomposition of the remaining subgraph. In other words, we wish to decompose the complete multipartite graph M, with vertex set $V(K_{v_0}^{[\mu,\lambda]})$, and with ordinary edges (forgetting loops for the moment) from $E(K_{v_0}^{[\mu,\lambda]})$ that do not join two vertices both of which are from any set $\mathcal{W} \times \{i\}$ for $i \in \{1, 2, \ldots, u\}$. On account of Figure 4.1, we can label vertices of M as ordered pairs $(\mathbf{w}, i)$ where $\mathbf{w} \in \mathcal{W}$ and $1 \leq i \leq u$, so we can see we now wish to decompose the multipartite subgraph of $K_{v_0}^{[\mu,\lambda]}$ induced by all the edges of the form $\{(\mathbf{w}, i), (\mathbf{w}', j)\}$ for $i \neq j$.

Now, for each of the subgraphs $H_1, H_2, \ldots, H_k$ that arise from our initial choice of c (ie. the 'padding' of the decomposition) we wish to construct q^{2t-1} mappings giving subgraphs of M isomorphic to G. We do this by first assigning scalars $\alpha_h(\{i, j\}) \in \mathbb{F}_q$ to the edges $\{i, j\}$ of each subgraph H_h (for

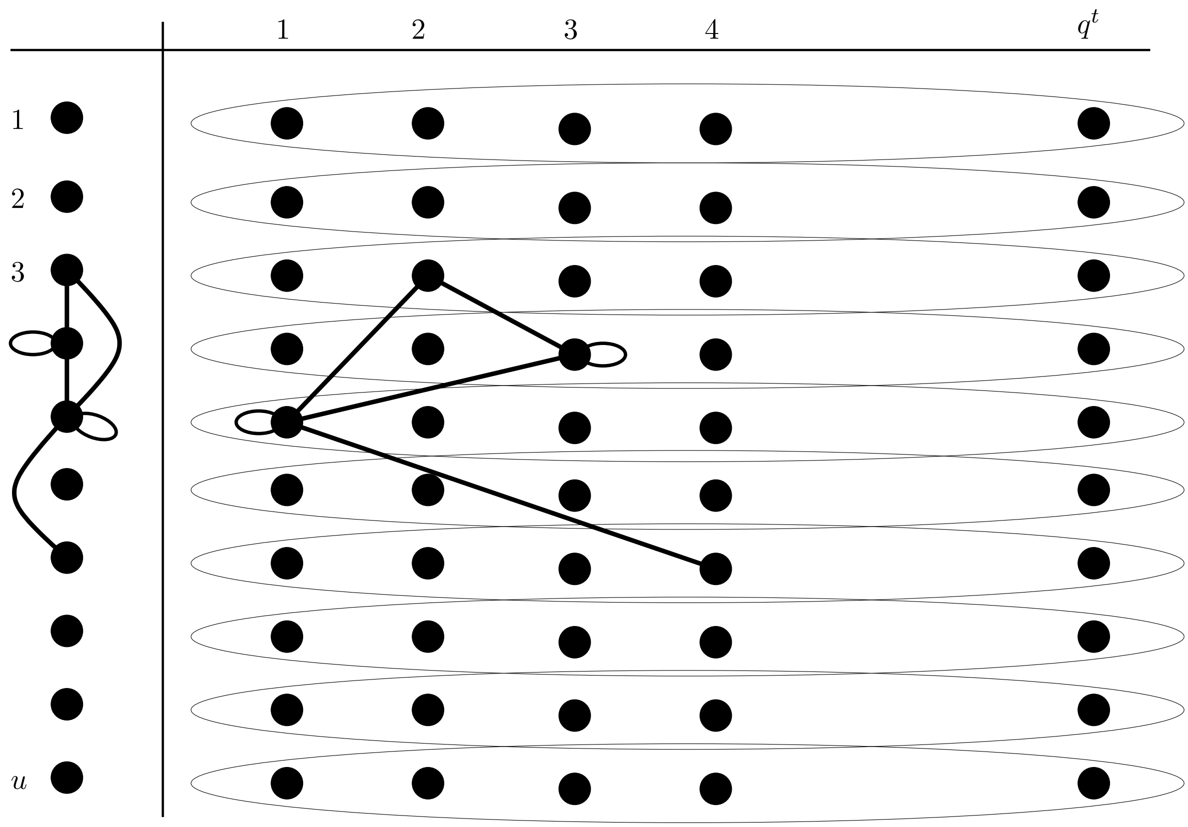


Figure 4.1: Wilson's Construction

$h \in \{1, 2, \ldots, k\}$) such that for each edge $\{i, j\}$ of $K_{v_0}^{[\mu,\lambda]}$, every element of $\mathbb{F}_q$ occurs exactly λ times among the $q\lambda$ scalars $\alpha_h(\{i, j\})$. Then, assign vectors $\mathbf{x}_i^h \in \mathcal{W}$ to the vertices i of H_h such that for each edge $\{i, j\}$ of H_h, we have

$$f(S_{ij}(\mathbf{x}_j^h - \mathbf{x}_i^h)) = \alpha_h(\{i, j\}), \tag{4.15}$$

which we know is possible to do because of (4.14).

We wish to define the maps $\phi_{\mathbf{yz}}^h : V(H_h) \to V(M)$ by

$$\phi_{\mathbf{yz}}^h(i) = (\mathbf{x}_i^h + T_i(\mathbf{y}) + \mathbf{z}, i) \tag{4.16}$$

where $i \in V(H_h)$, $\mathbf{y} \in \mathcal{K}$, and $\mathbf{z} \in \mathcal{W}$. Because $\mathcal{K}$ has dimension $t-1$ and $\mathcal{W}$ has dimension t and both are vector spaces over $\mathbb{F}_q$, there are q^{t-1} choices for $\mathbf{y}$ and q^t choices for $\mathbf{z}$; thus, for each subgraph H_h, $h = 1, 2, \ldots, k$, there are q^{2t-1} mappings $\phi_{\mathbf{yz}}^h$. Now for $h \in \{1, 2, \ldots, k\}$, we wish to let

$$\mathcal{G}_h = \{\phi_{\mathbf{yz}}^h(V(H_h)) \mid \mathbf{y} \in \mathcal{K}, \mathbf{z} \in \mathcal{W}\}.$$

Notice we are regarding $\phi_{\mathbf{yz}}^h(V(H_h))$ not just as a set of vertices, but rather as a copy of G on M. That is, $\phi_{\mathbf{yz}}^h$ not only maps vertices of H_h to M, but also induces the edges incident with those vertices.

Ignoring loops for the moment, $\mathcal{G}_1 \cup \mathcal{G}_2 \cup \cdots \cup \mathcal{G}_k$ yields a G-decomposition of M. To prove this we consider an edge of M, $\{(\mathbf{w}, i), (\mathbf{w}', j)\}$ where $i \neq j$, and show that it is contained in exactly λ images of the edge $\{i, j\}$ of H_h

under $\phi^h_{\mathbf{yz}}$ over all choices of $h, \mathbf{y}$, and $\mathbf{z}$. We do this by first computing the scalar $f(S_{ij}(\mathbf{w}' - \mathbf{w}))$ in order to find the subgraphs containing this edge. There are exactly λ values of h such that

- H_h contains the edge $\{i, j\}$ and
- $\alpha_h(\{i, j\}) = f(S_{ij}(\mathbf{w}' - \mathbf{w}))$.

So, from (4.15) for each h we have

$$f(S_{ij}(\mathbf{x}^h_j - \mathbf{x}^h_i)) = f(S_{ij}(\mathbf{w}' - \mathbf{w}));$$

thus, we must also have

$$S_{ij}(\mathbf{x}^h_j - \mathbf{x}^h_i) + \mathbf{y} = S_{ij}(\mathbf{w}' - \mathbf{w}) \tag{4.17}$$

for some unique $\mathbf{y} \in \mathcal{K}$. Since $S_{ij} = (T_j - T_i)^{-1}$, we can apply $T_j - T_i$ to (4.17) to get

$$\mathbf{x}^h_j - \mathbf{x}^h_i + T_j(\mathbf{y}) - T_i(\mathbf{y}) = \mathbf{w}' - \mathbf{w},$$

giving us a unique $\mathbf{z} \in \mathcal{W}$ such that $\mathbf{w}' - \mathbf{x}^h_j - T_j(\mathbf{y}) = \mathbf{w} - \mathbf{x}^h_i - T_i(\mathbf{y}) = \mathbf{z}$. Or in other words,

$$\begin{aligned} \mathbf{x}^h_i + T_i(\mathbf{y}) + \mathbf{z} &= \mathbf{w} \\ \mathbf{x}^h_j + T_j(\mathbf{y}) + \mathbf{z} &= \mathbf{w}'. \end{aligned}$$

Therefore, for each these λ choices of h there are unique $\mathbf{y}$ and $\mathbf{z}$ such that the edge $\{(\mathbf{w}, i), (\mathbf{w}', j)\}$ is the image of $\{i, j\}$ of H_h under $\phi^h_{\mathbf{yz}}$. Thus, because we found λ copies of G containing $\{(\mathbf{w}, i), (\mathbf{w}', j)\}$ we know we have at least this many and now need to show there are no more that do.

Now, using a counting argument, we wish to show that no other copies of G arising from the construction contain $\{(\mathbf{w}, i), (\mathbf{w}', j)\}$. There are $\binom{u}{2}q^{2t}$ ordinary edges in M and we have just shown that each of these edge in M is in at least λ G-blocks that arise from the ϕ^h_{yz} maps, so we need to show that there are exactly $\frac{\lambda q^{2t} u(u-1)}{2}$ ordinary edges in $\mathcal{G}_1 \cup \mathcal{G}_2 \cup \cdots \cup \mathcal{G}_k$. To do this we notice that each $\mathcal{G}_i$ has q^{2t-1} different copies of H_i resulting in mq^{2t-1} edges. Therefore, from (4.12) we have

$$|E(\mathcal{G}_1 \cup \mathcal{G}_2 \cup \cdots \cup \mathcal{G}_k)| = kmq^{2t-1} = \frac{\lambda q^{2t} u(u-1)}{2};$$

and so, we conclude that every edge $\{(\mathbf{w}, i), (\mathbf{w}', j)\}$ of M is contained in exactly λ G-blocks. This gives us a G-decomposition of M.

To ensure we have decomposed $K^{[\mu,\lambda]}_{v_0}$, all that remains is to check that each vertex is used $\mu = \frac{\lambda p(v_0-1)}{2m}$ times as a looped vertex in the decomposition. We know from Proposition 3.1 that each vertex of $K^{\lambda}_{q^t}$ occurs as a looped vertex $\mu_t = \frac{\lambda p(q^t-1)}{2m}$ times. These fill the holes of M. From (4.13) we know in the original padded decomposition each $i \in V(K^{[\mu',\lambda q]}_u)$ appears as a looped vertex in exactly $\frac{\lambda pq(u-1)}{2m}$ subgraphs H_h; in each case the vertex is stretched by $\phi^h_{\mathbf{yz}}$ so that for each choice of $\mathbf{y} \in \mathcal{K}$, vertices in $\mathcal{W} \times \{i\}$ appear as a loop exactly once as $\mathbf{z}$ varies over $\mathbb{F}_q$. Therefore each vertex appears with a loop in the decomposition of M exactly $\frac{\lambda p q^t (u-1)}{2m}$ times. We already know that $E(K_{v_0}^{\lambda}) = E(K_{q^t}^{\lambda}) \cup E(M)$ and now since $v_0 = uq^t$ we also know

$$\mu = \frac{\lambda p(v_0 - 1)}{2m} = \frac{\lambda p(q^t - 1)}{2m} + \frac{\lambda p q^t (u-1)}{2m}.$$

We have now verified that our construction generates a G-decomposition of $K_{v_0}^{[\mu,\lambda]}$ □

Chapter 5

Conclusion

5.1 Summary of the Proof of Theorem 1.4

To begin the proof of Theorem 1.4 we want to construct infinitely many examples for which $K_v^{[\mu,\lambda]}$ can be G-decomposed, which we do using finite fields with prime-power order. The constructive proof of Proposition 3.1 finds infinitely many prime-power values of v that admit G-decompositions.

After some examples have been found we can use pairwise balanced designs to extend them and obtain asymptotically complete residue classes modulo the period n_G found in (3.3). However, there are only finitely many residue classes $v_0 \pmod{n_G}$ that satisfy the necessary conditions, given in (1.6), for a decomposition. So, we use Proposition 4.4 for each of these residue classes to ensure a G-decomposition can be obtained. After Proposition 3.1 finds us infinitely many examples, Proposition 3.5 together with Proposition

4.4 complete the proof of Theorem 1.4.

In order to prove Proposition 4.4 we first need to prove Proposition 4.1, which asserts that although the necessary conditions stated in (1.6) are not in general sufficient for a decomposition they are in fact sufficient for the existence of an integer solution to a particular system of linear equations. This integral solution enables us to pad by all copies of G enough times, yielding a G-decomposition with high λ. Effectively, Proposition 4.4 stretches this to obtain a new G-decomposition, with the specified value for λ, on a larger number of vertices.

5.2 Further Research

There are many ways in which to extend the research done in this thesis; for instance, as Lamken and Wilson previously did in [6], it would be an interesting problem to prove the asymptotic existence of equireplicate $\mathcal{G}$-decompositions of $K_v^{(r)}$, the complete graph on v vertices with r coloured edges between each vertex. As was the case in [6], the necessary conditions would be more difficult to determine. However, the proof techniques used in this thesis (and which were first used in [14]) would be similar, perhaps with more complications. We suspect that an extension of edge-coloured decompositions to handle loops is potentially useful for connections to other combinatorial designs.

One could also consider proving the asymptotic existence of decompositions for graphs G that are not simple. In other words, consider G with multiple edges between vertices. This problem is believed to be difficult, but perhaps extensions of the techniques here will enjoy success.

Another very interesting problem in design theory is asymptotic existence of G-decompositions of H, where H is not a complete graph. Even though this problem seems fundamentally similar to the results stated in Section 2.2, the standard proof techniques are specific to H being complete. A completely new approach is likely required. The case when H is nearly complete, say with $0.99\binom{v}{2}$ edges is of greatest interest, and initial investigations of this began in [4].

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