

**SINGULARITIES OF PLANAR SYMMETRIC
FOUR-BODY PROBLEMS**

by

FLORIN N. DIACU

DMS-591-IR

November 1991

SINGULARITIES OF PLANAR SYMMETRIC FOUR-BODY PROBLEMS

Florin N. Diacu

Department of Mathematics and Statistics
University of Victoria
Victoria, B.C. V8W 3P4
CANADA

Abstract. We first show that trapezoidal and rhomboidal solutions of the four-body problem with equal masses, do not lead to non-collision singularities and that any orbit encounters a collision, forwards or backwards in time. The existence of square homothetic solutions on negative energy levels, as connecting orbits between equilibria, is further proved and a transversality theorem, interpreted as a structural stability result, follows. Using McGehee transformations the total collapse singularity is blown up and the binary collisions are regularized. Finally we discuss the block-regularization of quadruple collision orbits as it follows from the qualitative behavior of the flow on the collision manifold.

1. INTRODUCTION

In this paper we deal with collision and non-collision singularities of trapezoidal and rhomboidal four-body problems. We first prove that if all masses are equal, then trapezoidal solutions do not encounter non-collision singularities but always lead to a collision which may appear forwards or backwards in time. A similar result can be proved for rhomboidal orbits.

We consider further McGehee transformations in order to study quadruple collision and near-collision solutions of the rhomboidal problem. The blow-up technique gives the possibility to delete the singularity and to paste instead a *collision manifold*, to determine the equilibria and to perform a qualitative study of the flow. The existence of square homothetic heteroclinic solutions on negative energy levels is further pointed out and a transversality result, similar to Devaney, 1979, theorem for collinear orbits is proved. More precisely we see that the above described orbits lie in the transverse intersection of the unstable manifold of one equilibrium point with the stable manifold of the other one. In a certain sense which will be discussed, this can be interpreted as a structural stability result.

Regularizing binary collisions one can see that the collision manifold becomes a sphere without four points like in the classical McGehee case of triple collision in the collinear three-body problem. A study of the flow on this manifold enables us to discuss the block-regularization problem of the total collapse solutions.

It is important to note the analogy of the behavior of the flow on the collision manifold with that of the flow of the three-body problem, as it can be seen in McGehee, 1974 or Simó, 1980. However there are some works treating different kinds of four-body problems like Lacomba, 1981, Simó and Lacomba, 1982, Shelton, 1979 for the Newtonian potential or Casasayas and Nunes, 1990 for a charged

restricted case. While writing down this research I found out that in a recent Ph.D. thesis (Pérez-Chavela, 1991), some new transformations are introduced which regularize binary collisions in the rhomboidal four-body problem such that the collision manifold is compact. However, details on this work are not available to me at the moment of finishing this paper.

2. THE FOUR-BODY PROBLEM

Consider four particles (joint masses, bodies) in the plane \mathbb{R}^2 , identified by their constant masses $m_i > 0$, $i = 1,2,3,4$, having the position vectors $\mathbf{q}_i = (q_i^1, q_i^2)$, $i = 1,2,3,4$, in an arbitrary fixed frame. Denote by $\mathbf{q} = (q_1, q_2, q_3, q_4)$ the *configuration* of the particle system and define $\mathbf{p} = \mathbf{A}\dot{\mathbf{q}}$ to be the *momentum*, where \mathbf{A} is the nonsingular matrix:

$$\mathbf{A} = \text{diag}(m_1, m_1, m_2, m_2, m_3, m_3, m_4, m_4)$$

and the dot denotes differentiation with respect to time.

The equations of motion of the planar four-body problem are given by the second order system:

$$(2.1) \quad \ddot{\mathbf{q}} = \mathbf{A}^{-1} \nabla U(\mathbf{q}),$$

or the equivalent first order system

$$(2.2) \quad \begin{cases} \dot{\mathbf{q}} = \mathbf{A}^{-1} \mathbf{p} \\ \dot{\mathbf{p}} = -\nabla U(\mathbf{q}), \end{cases}$$

where $U : \mathbb{R}^8 \setminus \Delta \rightarrow \mathbb{R}_+$, $U(\mathbf{q}) = \sum_{i < j} m_i m_j |\mathbf{q}_i - \mathbf{q}_j|^{-1}$ is the *potential function* of the particle system ($-U$ being the potential energy),

$$\Delta := \bigcup_{i < j} \{\mathbf{q} \mid \mathbf{q}_i = \mathbf{q}_j\}$$

represents the collision set, $\nabla = (\partial_1, \partial_2, \partial_3, \partial_4)$ is the gradient operator and $|\cdot|$ denotes the Euclidean norm.

The integral of energy of the Eqs. (2.2) is given by

$$(2.3) \quad H(\mathbf{q}, \mathbf{p}) = h,$$

where h is an integration constant and $H(\mathbf{q}, \mathbf{p}) := T(\mathbf{p}) - U(\mathbf{q})$ is the Hamiltonian of the system (indeed the Eqs. (2.2) are Hamiltonian), T being the *kinetic energy*:

$$T : \mathbb{R}^8 \rightarrow [0, \infty), \quad T(\mathbf{p}) = (1/2) \langle \mathbf{A}^{-1} \mathbf{p}, \mathbf{A}^{-1} \mathbf{p} \rangle.$$

We have denoted here by $\langle \cdot, \cdot \rangle$ the scalar product $\langle \mathbf{a}, \mathbf{b} \rangle = \sum_i m_i \mathbf{a}_i^T \mathbf{b}_i$, where $\mathbf{a}, \mathbf{b} \in \mathbb{R}^8$ and the upper index T means transposition.

The center of mass and the momentum integrals allow us to conclude that the set

$$I = \left\{ (\mathbf{q}, \mathbf{p}) \mid \langle \mathbf{q}, \mathbf{e}^i \rangle = \langle \mathbf{A}^{-1} \mathbf{p}, \mathbf{e}^i \rangle = 0, \quad i = 1, 2 \right\}$$

is invariant for the Eqs. (2.2), where

$$\mathbf{e}^i = (e_i, e_i, e_i, e_i), \quad \mathbf{e}_1 = (1, 0), \quad \mathbf{e}_2 = (0, 1).$$

By restricting the Eqs. (2.2) to the invariant set \mathbf{I} means, without loss of generality, that the motion is considered relative to a frame having the origin at the center of mass of the particle system.

For any choice of the initial conditions $(\mathbf{q}, \mathbf{p})(0) \in \mathbf{I}$, $\mathbf{q}(0) \notin \Delta$, standard results of differential equations theory, ensure the existence and uniqueness of an analytic solution of the Eqs. (2.2), defined on a maximal interval (t^-, t^+) , $-\infty \leq t^- < 0 < t^+ \leq +\infty$. If t^- or t^+ is finite then the solution is said to experience a *singularity* which may be a *collision* or a *pseudocollision* (non-collision singularity), *i.e.* a motion becoming unbounded in finite time (see *e.g.* Von Zeipel, 1908; Sperling 1970; McGehee, 1986).

3. TRAPEZOIDAL SOLUTIONS

In a previous note (Diacu, 1989) we have seen that if a noncollinear (not necessarily planar) solution of the four-body problem has a symmetry axis, then the center of mass lies on this axis and the symmetric masses are equal. This result can be generalized for n bodies (Diacu, 1990). The converse is also true in the sense that equal symmetric masses with symmetric initial data give rise to a symmetric solution (Robinson and Saari, 1983). A different proof of this fact, adapted here for trapezoidal solutions but which also works in general, is given below. For this consider first the following:

Definition. Let (\mathbf{q}, \mathbf{p}) be a solution of the Eqs. (2.2) and define the function

$$F(\mathbf{q}, \mathbf{p}) = (G(\mathbf{q}), G(\mathbf{A}^{-1}\mathbf{p})),$$

where $\mathbf{G}(\mathbf{q}) = (q_1^1 + q_2^1, q_3^1 + q_4^1, q_1^2 - q_2^2, q_3^2 - q_4^2)$. The solution (\mathbf{q}, \mathbf{p}) is called *trapezoidal* if $\mathbf{F}(\mathbf{q}(t), \mathbf{p}(t)) = \mathbf{0}$ for all t where the solution is defined.

Proposition 1. *Suppose $m_1 = m_2$ and $m_3 = m_4$. Then the set $\mathcal{S} = \{(\mathbf{q}, \mathbf{p}) | \mathbf{F}(\mathbf{q}, \mathbf{p}) = \mathbf{0}\}$ is invariant for the Eqs. (2.2).*

Proof. Denote by $\mathbf{f}(\mathbf{q}, \mathbf{p}) = (\mathbf{A}^{-1} \mathbf{p}, \nabla U(\mathbf{q}))$, the vector field defining Eqs. (2.2). A straightforward computation shows that

$$(3.1) \quad \mathbb{D}\mathbf{F}(\mathbf{q}, \mathbf{p}) \mathbf{f}(\mathbf{q}, \mathbf{p}) = \Psi(\mathbf{F}(\mathbf{q}, \mathbf{p})),$$

where $\Psi(\mathbf{0}) = \mathbf{0}$. The Eq. (3.1) can be written as

$$(3.2) \quad \dot{\Phi} = \Psi(\Phi),$$

where $\Phi = \mathbf{F}(\mathbf{q}, \mathbf{p})$. Considering the initial condition $\Psi(\mathbf{0}) = \mathbf{0}$, the Eq. (3.2) has the solution $\Phi \equiv \mathbf{0}$ and consequently \mathcal{S} is invariant for the Eqs. (2.2)

We will now transform Eqs. (2.2) into a form suitable for our purposes.

Consider first the analytic diffeomorphism:

$$(3.3) \quad \begin{aligned} \mathbf{x}_i &= \mathbf{q}_i - (1/2)(\mathbf{q}_1 + \mathbf{q}_2), \quad i = 1, 2, 3, 4 \\ \mathbf{y}_i &= \mathbf{p}_i - (1/2)(\mathbf{p}_1 + \mathbf{p}_2), \quad i = 1, 2 \\ \mathbf{y}_j &= \mathbf{p}_j - (m/2M)(\mathbf{p}_1 + \mathbf{p}_2), \quad j = 3, 4, \end{aligned}$$

where $M := m_1 = m_2$ and $m := m_3 = m_4$.

Under the transformations (3.3), the Eqs. (2.2) become:

$$(3.4) \quad \begin{cases} \dot{\mathbf{x}} = \mathbf{A}^{-1} \mathbf{y} \\ \dot{\mathbf{y}} = \nabla U(\mathbf{x}) - (1/2) \mathbf{M}^{-1} \mathbf{A} (\mathbf{B} \cdot \nabla U(\mathbf{x})), \end{cases}$$

where

$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix},$$

$1, 0$ being four-dimensional quadratic matrices having all elements equal to 1, respectively 0. The invariant set \mathbf{I} is transformed into \mathbf{J} ,

$$\mathbf{J} := \{(\mathbf{x}, \mathbf{y}) \mid \mathbf{x}_1 + \mathbf{x}_2 = \mathbf{y}_1 + \mathbf{y}_2 = \mathbf{0}\},$$

which allows us to define the following analytic diffeomorphism:

$$(3.5) \quad \begin{aligned} \mathbf{z} &= \mathbf{x}_1 = -\mathbf{x}_2, & \mathbf{v} &= (1/2)(\mathbf{x}_4 - \mathbf{x}_3), & \mathbf{u} &= (1/2)(\mathbf{x}_4 + \mathbf{x}_3), \\ \mathbf{w} &= \mathbf{y}_1 = -\mathbf{y}_2, & \mathbf{s} &= (1/2)(\mathbf{y}_4 - \mathbf{y}_3), & \mathbf{r} &= (1/2)(\mathbf{y}_4 + \mathbf{y}_3). \end{aligned}$$

Consequently the equations of motion become:

$$(3.6) \quad \begin{cases} \dot{\mathbf{z}} = \mathbf{M}^{-1} \mathbf{w} \\ \dot{\mathbf{v}} = \mathbf{m}^{-1} \mathbf{s} \\ \dot{\mathbf{u}} = \mathbf{m}^{-1} \mathbf{r} \\ \dot{\mathbf{w}} = -\mathbf{M} \left[\frac{\mathbf{M}}{4} \cdot \frac{\mathbf{z}}{|\mathbf{z}|^3} + \mathbf{m} \left[\frac{\mathbf{z} + \mathbf{v}}{|\mathbf{u} + \mathbf{z} + \mathbf{v}|^3} + \frac{\mathbf{z} - \mathbf{v}}{|\mathbf{u} + \mathbf{z} - \mathbf{v}|^3} \right] \right] \\ \dot{\mathbf{s}} = -\mathbf{m} \left[\frac{\mathbf{m}}{4} \cdot \frac{\mathbf{v}}{|\mathbf{v}|^3} + \mathbf{M} \left[\frac{\mathbf{z} + \mathbf{v}}{|\mathbf{u} + \mathbf{z} + \mathbf{v}|^3} + \frac{\mathbf{v} - \mathbf{z}}{|\mathbf{u} + \mathbf{z} - \mathbf{v}|^3} \right] \right] \\ \dot{\mathbf{r}} = -\mathbf{m}(\mathbf{m} + \mathbf{M}) \mathbf{u} (|\mathbf{u} + \mathbf{z} + \mathbf{v}|^3 + |\mathbf{u} + \mathbf{z} - \mathbf{v}|^3). \end{cases}$$

For trapezoidal solutions, the connection between the variables and their geometrical meaning can be better understood, by using Fig. 1.

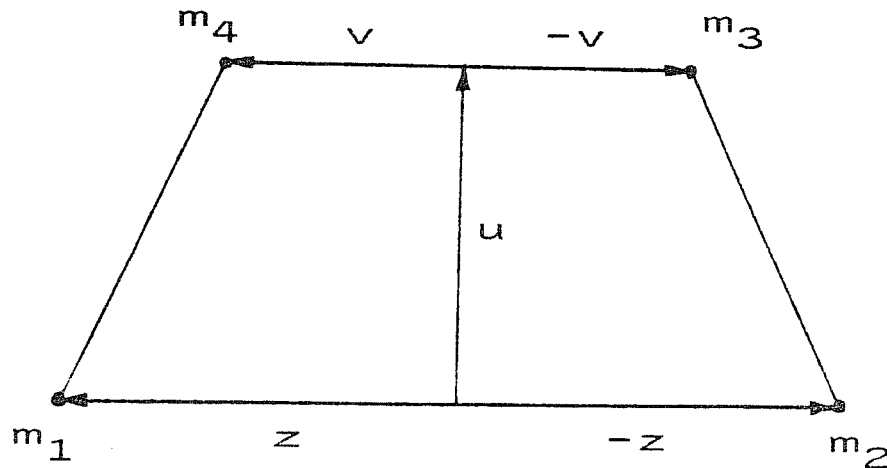


Figure 1.

A trapezoidal configuration

It is known that for collinear solutions (Saari, 1973) and for the three-body problem (Painlevé, 1896), pseudocollisions do not appear. It is also known that in the four-body problem the set of initial data leading to singularities is negligible (of measure zero and of the first Baire category) (see Saari, 1984). The main result of this section is the following:

Theorem 2. *If all masses are equal (i.e. $m = M$), the following two properties take place:*

- (a) *no trapezoidal solutions of the Eqs. (3.6) encounter pseudocollisions,*
- (b) *every trapezoidal solution of the Eqs. (3.6) leads to a collision, which may occur forwards or backwards in time.*

Proof. We will use in the proof the following well known fact:

Convexity Lemma. There do not exist functions $\alpha : \mathbb{R} \rightarrow \mathbb{R}$, twice differentiable, such that $\alpha > 0$ and $\ddot{\alpha} < 0$ on \mathbb{R} (or $\alpha < 0$ and $\ddot{\alpha} > 0$ on \mathbb{R}).

(a) Let's start by noticing (Painlevé, 1896) that t^* is a singularity of the solution if and only if the minimum distance between particles tends to 0 when $t \rightarrow t^*$ (of course t^* may be t^+ or t^- as described in Section 2 and then by $t \rightarrow t^*$ we understand $t \rightarrow t^*$, $t < t^*$ or $t > t^*$, respectively). In the language of Eqs. (3.6) this means:

$$\rho(t) = \min\{2z(t), 2v(t), |\mathbf{u}(t) + \mathbf{z}(t) - \mathbf{v}(t)|\} \rightarrow 0, \quad t \rightarrow t^*,$$

where $z = |\mathbf{z}|$, $v = |\mathbf{v}|$.

Observe that $|\mathbf{u} + \mathbf{z} - \mathbf{v}| \geq |\mathbf{u}|$, which implies that $\tilde{\rho}(t) \rightarrow 0$ when $\rho(t) \rightarrow 0$, where

$$\tilde{\rho}(t) = \min\{2z(t), 2v(t), u(t)\} \geq 0.$$

Let's analyze the situations that may arise.

In case $z(t) \rightarrow 0$ or $v(t) \rightarrow 0$, $t \rightarrow t^*$, it is clear that we have a collision and no pseudocollision occurs. Suppose therefore

$$\limsup_{t \rightarrow t^*} z(t) > 0 \quad \text{and} \quad \limsup_{t \rightarrow t^*} v(t) > 0.$$

Observe that since $m = M$, from the Eqs. (3.6) we obtain:

$$\bar{z} + \bar{v} = - \left[\frac{m}{4} \left[\frac{z}{|z|^3} + \frac{v}{|v|^3} \right] + 2m \frac{z + v}{|\mathbf{u} + \mathbf{z} + \mathbf{v}|^3} \right].$$

This means that on the first component (the second component of the parallel vectors \mathbf{z} and \mathbf{v} being actually 0) we have:

$$(3.7) \quad \ddot{z} + \ddot{v} < 0 \quad \text{for } z, v > 0,$$

i.e. $z + v$ has a limit when $t \rightarrow t^*$ and consequently, by the Convexity Lemma, v cannot become unbounded in finite time.

In case $u(t) \rightarrow 0$, $t \rightarrow t^*$, it follows that $I(t)$, the sum of the square of the distances between particles, tends to a finite limit. It is a result due to von Zeipel, 1908, which states that if this limit is finite then the singularity is due to a collision. Thus we necessarily have

$$\limsup_{t \rightarrow t^*} u(t) > 0.$$

Since $\tilde{\rho}(t) \rightarrow 0$, $t \rightarrow t^*$, at least two of the functions z , v and u interchange infinitely often the role of being the minimum. Let's discuss all possibilities:

- (1) If only z and v change roles then $u(t) > \tilde{\rho}(t) > 0$ (make the choice $u(t) > 0$ on the second component) and consequently $\ddot{u} < 0$ on the second component, thus u cannot become unbounded in finite time. Therefore $I(t)$ has a finite limit, consequently the singularity is a collision.
- (2) If u and z change roles, let $(\tau_n)_n$ be the sequence of all time moments where the roles of u and z are changed. Thus $\tau_n \rightarrow t^*$ and $u(\tau_n) = z(\tau_n)$. Since $\tilde{\rho}(t) \rightarrow 0$ it follows that $z(\tau_n) \rightarrow 0$. Thus $\liminf I(t)$ is finite and since it is well known that $I(t)$ always has a limit, the result of Von Zeipel implies again that the singularity cannot be a pseudocollision.
- (3) In case u and v change roles, the discussion is the same like in (2).

- (4) Suppose all functions z, v, u change roles in being $\tilde{\rho}$ for different values of t and let (τ_n^j) , $j = 1, 2, 3$ be the sequences of time where the corresponding roles are changed, let's say such that

$$z(\tau_n^1) = v(\tau_n^1), \quad v(\tau_n^2) = u(\tau_n^2), \quad z(\tau_n^3) = u(\tau_n^3).$$

Then $(\tau_n^j) \rightarrow t^*$, $n \rightarrow \infty$, $j = 1, 2, 3$ and since $\tilde{\rho}(t) \rightarrow 0$ we obtain, like in (2), that I has a finite limit.

The proof of (a) is thus complete.

(b) Suppose the trapezoidal solutions do not encounter simple/double binary or total collisions. This means $z, v > 0$ on their first components, all along the motion. Then, relation (3.7) takes place and since no pseudocollisions occur, it follows by the Convexity Lemma that $z + v$ has at least one zero, a contradiction.

The theorem is thus proved.

The following obvious consequence comes to close this section.

Corollary 3. *Rectangular solutions of the four-body problem have no pseudocollisions but always encounter a double binary or a quadruple collision.*

4. RHOMBOIDAL SOLUTIONS

While trapezoidal solutions were defined such that to have a symmetry axis, rhomboidal orbits can be considered as having two symmetry axes like in Fig. 2. Obviously the results in Diacu, 1990 can be used, implying that symmetric masses

are equal. The converse, *i.e.* a result similar to Prop. 1, can be analogously obtained.

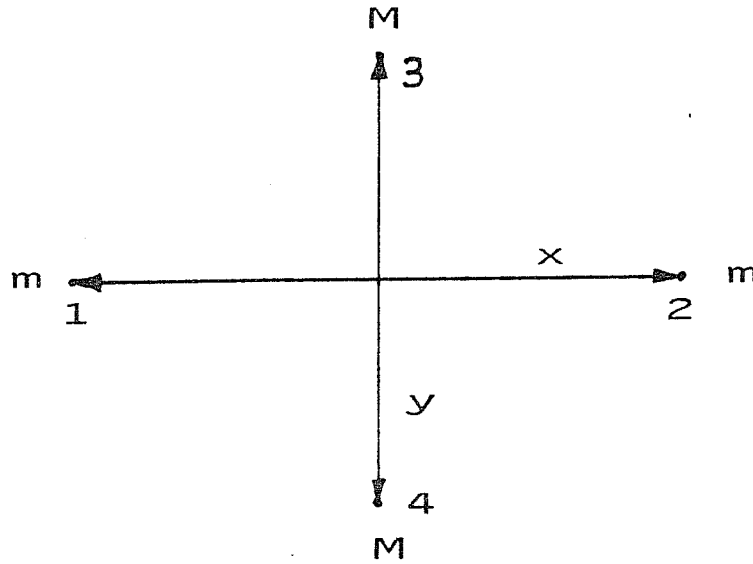


Figure 2.

A rhomboidal configuration

However, the Eqs. (3.6) used before do not fit into the study of rhomboidal solutions. Consider therefore the analytic diffeomorphism:

$$(4.1) \quad \begin{aligned} \mathbf{x} &= (1/2)(\mathbf{q}_1 - \mathbf{q}_2), & \mathbf{y} &= (1/2)(\mathbf{q}_4 - \mathbf{q}_3), \\ \tilde{\mathbf{x}} &= (1/2)(\mathbf{p}_1 - \mathbf{p}_2), & \tilde{\mathbf{y}} &= (1/2)(\mathbf{p}_4 - \mathbf{p}_3), \end{aligned}$$

The Eqs. (2.2) are transformed into:

$$(4.2) \quad \begin{cases} \dot{\mathbf{x}} = \tilde{\mathbf{x}} \\ \dot{\mathbf{y}} = \tilde{\mathbf{y}} \\ \dot{\tilde{\mathbf{x}}} = -(m\mathbf{x}/4)|\mathbf{x}|^{-3} - 2M\mathbf{x}|\mathbf{x}+\mathbf{y}|^{-3} \\ \dot{\tilde{\mathbf{y}}} = -(M\mathbf{y}/4)|\mathbf{y}|^{-3} - 2m\mathbf{y}|\mathbf{x}+\mathbf{y}|^{-3}. \end{cases}$$

The integral of energy (2.3) becomes

$$(4.3) \quad m|\tilde{\mathbf{x}}|^2 + M|\tilde{\mathbf{y}}|^2 - (m^2/2)|\mathbf{x}|^{-1} - (M^2/2)|\mathbf{y}|^{-1} - 4mM|\mathbf{x}+\mathbf{y}|^{-1} = h.$$

It is easy to see from Eqs. (4.2) that an analogous result to Th. 2 can be obtained for rhomboidal solutions. Consequently, no pseudocollisions occur and by the Convexity Lemma, every rhomboidal solution encounters a simple binary or a quadruple collision, forwards or backwards in time.

Assume further that $m = M = 1$. This restriction will simplify the computations without to change, however, the structure of the setting below. Similar results are expected in the general case.

Define the McGehee transformations (see also Diacu, 1987) given by the real analytic diffeomorphism

$$(4.4) \quad \begin{aligned} r &= (x^2+y^2)^{\frac{1}{2}} \\ v &= r^{-\frac{1}{2}}(x\tilde{x}+y\tilde{y}) \\ \theta &= \arctan(y/x) \\ u &= r^{-\frac{1}{2}}(x\tilde{y}-y\tilde{x}) \end{aligned}$$

and then associate to a quadruple collision orbit a time rescaling written formally as

$$(4.5) \quad dt = r^{3/2} d\tau.$$

By (4.4) and (4.5) the Eqs. (4.2) become

$$(4.6) \quad \begin{cases} r' = rv \\ v' = \frac{1}{2}v^2 + u^2 - \frac{1}{2}V(\theta) \\ \theta' = u \\ u' = -\frac{1}{2}vu + \frac{1}{2}(d/d\theta)V(\theta), \end{cases}$$

where prime denotes the first derivative with respect to the new (fictitious) time variable τ and

$$V(\theta) = \frac{1}{2}(\cos^{-1}\theta + \sin^{-1}\theta) + 4$$

is obtained from the potential function.

The integral of energy (4.3) gives us the relation

$$(4.7) \quad v^2 + u^2 - V(\theta) = rh.$$

Notice that the Eqs. (4.6) are regular at $r = 0$ which means that the singularity corresponding to the total collapse has been removed. Actually the quadruple collision set

$$\mathcal{E} = \{(r, v, \theta, u) \mid r = 0\}$$

is invariant for the Eqs. (4.6), being pasted to the phase space in a natural way and taking the place of the cumbersome singularity. Instead of reaching the singularity in finite time, total collision orbits now tend to \mathcal{E} , in the new flow, when $\tau \rightarrow \infty$.

On \mathcal{E} the energy relation (4.7) becomes

$$(4.8) \quad v^2 + u^2 = V(\theta),$$

which represents a cylinder like in Fig. 3, and the Eqs. (4.6) restricted to \mathcal{E} are:

$$(4.9) \quad \begin{cases} v' = \frac{1}{2}v^2 + u^2 - \frac{1}{2}V(\theta) \\ \theta' = u \\ u' = -\frac{1}{2}vu + \frac{1}{2}(d/d\theta)V(\theta). \end{cases}$$

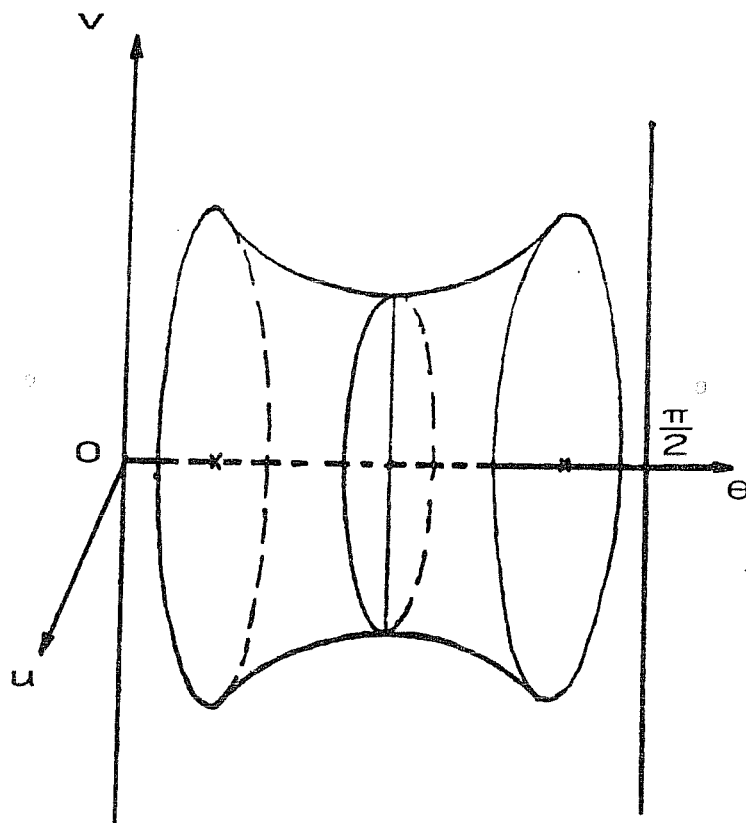


Figure 3.

The energy surface for $r = 0$

5. THE TRANSVERSALITY RESULT

Let's first make a qualitative prospect of the Eqs. (4.6)-(4.9).

In order to try and get a global description of the flow we first look for rest points of it. The result below follows by straightforward computation.

Proposition 4. *The Eqs. (4.6) have two equilibrium solutions:*

$$e^+ = (0, \sqrt{4+\sqrt{2}}, \pi/4, 0), \quad e^- = (0, -\sqrt{4+\sqrt{2}}, \pi/4, 0),$$

both hyperbolic, the corresponding sign of their eigenvalues being respectively:

$$e^+: (+, +, -, +), \quad e^-: (-, -, +, -).$$

On the energy surface in \mathcal{E} , the two equilibria are saddles.

An important property, of great help in the description of flows whenever it appears is the following:

Proposition 5. *The flow on \mathcal{E} is gradient-like relative to v , i.e. v increases along any solution which is not an equilibrium.*

Proof. Introduce (4.8) into the first equation in (4.9).

Observe further that certain *square solutions* have the property of being homothetic. This means that if the four equal masses are arranged at the vertices of a square and suitable initial velocities are given, then the particles will move without rotation, having at every moment a square configuration.

The following result proves the existence of homothetic square solutions as heteroclinic orbits between e^+ and e^- , on negative energy levels.

Proposition 6. *For any $h < 0$ there exists a homothetic square solution which begins and ends in a quadruple collision.*

Proof. Observe that the plane

$$P = \{(r, v, \theta, u) \mid \theta = \pi/4, u = 0\}$$

is invariant for the Eqs. (4.6). These equations restricted to P become:

$$\begin{cases} r' = rv \\ v' = \frac{1}{2}v^2 - (4 + \sqrt{2})/2, \end{cases}$$

while the energy relation (4.7) is:

$$v^2 - 4 - \sqrt{2} = rh.$$

Using also Prop. 4, the phase portrait is easy to sketch (see Fig. 4), yielding the conclusion.

We have thus arranged the problem such that to fulfill the hypothesis under which Devaney, 1979, proves his transversality result for the collinear n -body problem. One can now follow without any difficulty his proof, step by step, concluding the following transversality result:

Theorem 7. *For any $h < 0$, the intersection of $W^u(e^-)$ with $W^s(e^+)$ is transverse and contains the corresponding homothetic square orbit of the Eqs. (4.6).*

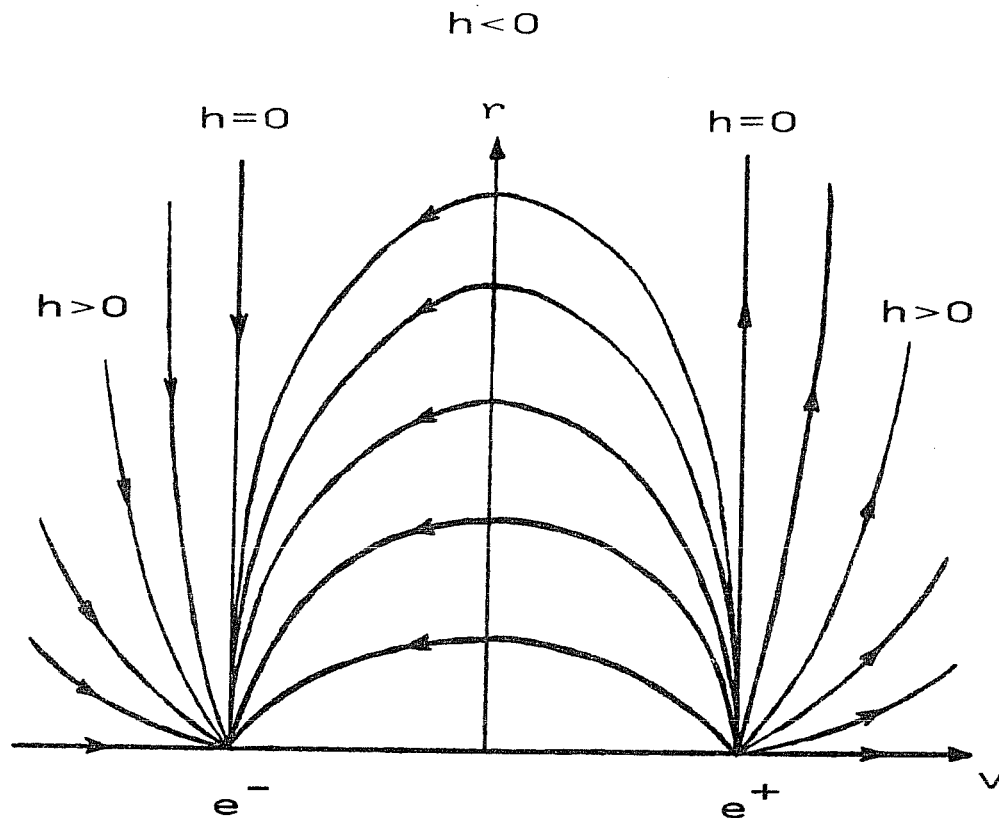


Figure 4.

The heteroclinic orbits for $h < 0$ in the (r, v) -plane

Recall that the stable manifold of an equilibrium point, denoted usually by W^s (respectively the unstable manifold, denoted by W^u) represents the set of all orbits tending forwards (respectively backwards) to that equilibrium.

Remark. Another way of getting Th. 7 is by applying the criterion of Simó and Llibre, 1981.

There would be some comment to make on the result stated in Th. 7. A transversal intersection has the remarkable property of being stable, in the sense that small perturbations will not change the fact of being transversal. This has the effect of maintaining the topological structure of the flow if small perturbations occur. Consequently Th. 7 can be interpreted as a *structural stability* result.

6. REGULARIZATION OF COLLISION ORBITS

To regularize a solution means to extend it beyond the collision singularity. This can be done in a classical way, *i.e.* analytically along the orbit (Sundman, 1912, Siegel, 1941) or with respect to initial data (Levi-Civita, 1920; Easton, 1971). The second technique, called block-regularization and which appears in its modern presentation in the work of Easton, has the good property of carrying with it the nice behavior of nearby orbits with respect to the extended solution. Anytime such a regularization fails to exist, one may expect a bad behavior of those solutions passing close to the singularity.

The goal of this section is to regularize binary collision orbits and to discuss the block-regularization of total collapse solutions.

In the first part we also use a technique of McGehee, 1974, in order to extend binary collision orbits beyond the singularity. Notice first (see Fig. 2) that if the bodies 1 and 2 collide, then $x \rightarrow 0$, from here $\cos \theta \rightarrow 0$, thus $\theta \rightarrow \pi/2$ and $r \rightarrow c_1$ (constant). Supposing that t^* is a collision singularity and using the well known fact (see Wintner, 1941) that the mutual distances (respectively velocities) behave asymptotically like $(t-t^*)^{2/3}$ (respectively $(t-t^*)^{-1/3}$) we get that $v \rightarrow c_2$ (constant) and $u \rightarrow \pm u_0$ (the sign depending on the occurrence of an ejection or collision). Analogously we obtain for the collision of the bodies 3 and 4 that $y \rightarrow 0$, thus $\sin \theta \rightarrow 0$, *i.e.* $\theta \rightarrow 0$, $r \rightarrow k_1$ (constant), $v \rightarrow k_2$ (constant) and $u \rightarrow \pm u_0$.

Denote $W(\theta) = \theta(\pi/2-\theta)V(\theta)$ and consider the analytic diffeomorphism:

$$(6.1) \quad w = \theta(\pi/2-\theta)W^{-1/2}(\theta)u.$$

Then associate the time transformation which can be formally written as

$$(6.2) \quad d\tau = \theta(\pi/2-\theta)W^{-\frac{1}{2}}(\theta) d\tilde{\tau}.$$

The Eqs. (4.6) become

$$(6.3) \quad \begin{cases} r' = \theta(\pi/2-\theta)W^{-\frac{1}{2}}(\theta)rv \\ v' = \frac{1}{2}W^{\frac{1}{2}}(\theta) - \theta(\pi/2-\theta)W^{-\frac{1}{2}}(\theta)[\frac{1}{2}v^2-rh] \\ \theta' = w \\ w' = \pi/4 - \theta - \theta(\pi/2-\theta)W^{-\frac{1}{2}}(\theta)[\frac{1}{2}vw \\ \quad - (\pi/2-2\theta)W^{-\frac{1}{2}}(\theta)(v^2-rh) - \frac{1}{2}W^{-\frac{1}{2}}(\theta) \\ \quad \cdot (d/d\theta)W(\theta)] - \frac{1}{2}W^{-1}(\theta)(d/d\theta)W(\theta)w^2, \end{cases}$$

where prime denotes the first derivative with respect to the new time variable $\tilde{\tau}$.

The new energy relation was also used to get Eqs. (6.3). It was obtained by transforming relation (4.7), getting

$$(6.4) \quad w^2 - (\pi/2)\theta + \theta^2 + \theta^2(\pi/2-\theta)^2 W^{-1}(\theta)(v^2-rh) = 0.$$

Notice that by the first equation in (6.3) the set $\{(r,v,\theta,w)|r=0\}$ is invariant for the Eqs. (6.3) and that

$$C = \{r,v,\theta,w\}|r=0, \quad (6.4) \text{ takes place}$$

forms a manifold which will be called the *collision manifold*. Since it is described by the equation:

$$(6.5) \quad w^2 - (\pi/2)\theta + \theta^2 + \theta^2(\pi/2-\theta)^2 W^{-1}(\theta)v^2 = 0,$$

the collision manifold is geometrically a sphere minus four points in the (v, θ, w) -space (see Fig. 5). It can therefore be better understood that the Eqs. (6.3) make sense for $\theta = 0$ and $\theta = \pi/2$ (see the lines $\theta = 0, w = 0$ and $\theta = \pi/2, w = 0$), where binary collisions of the point masses 1 and 2, respectively 3 and 4, appear. The picture of the flow on C also shows that binary collision orbits are continuous with respect to initial data, *i.e.* regularizable in the sense of Easton.

The picture of the flow on C is obtained by noticing that the equilibria of the Eqs. (6.3) are:

$$E^+ = (0, \sqrt{4+\sqrt{2}}, \pi/4, 0), \quad E^- = (0, -\sqrt{4+\sqrt{2}}, \pi/4, 0)$$

with the sign of the corresponding eigenvalues:

$$E^+: (+, -, -, +), \quad E^-: (-, +, +, -),$$

and observing that on C they form saddles. It is also important (and easy) to see that the vector field defining (6.3) is gradient-like with respect to v .

All these imply that the flow on C can have one of the two qualitative pictures in Fig. 5. In case the first situation occurs $W^u(E^-) = W^s(E^+)$ and the quadruple collision might be block-regularizable. However in the second case since $W^u(E^-) \neq W^s(E^+)$, there exist orbits starting in E^- that are not lifted to the same upper horn, having a completely different qualitative behavior. Consequently if such an event occurs, a block-regularisation becomes impossible.

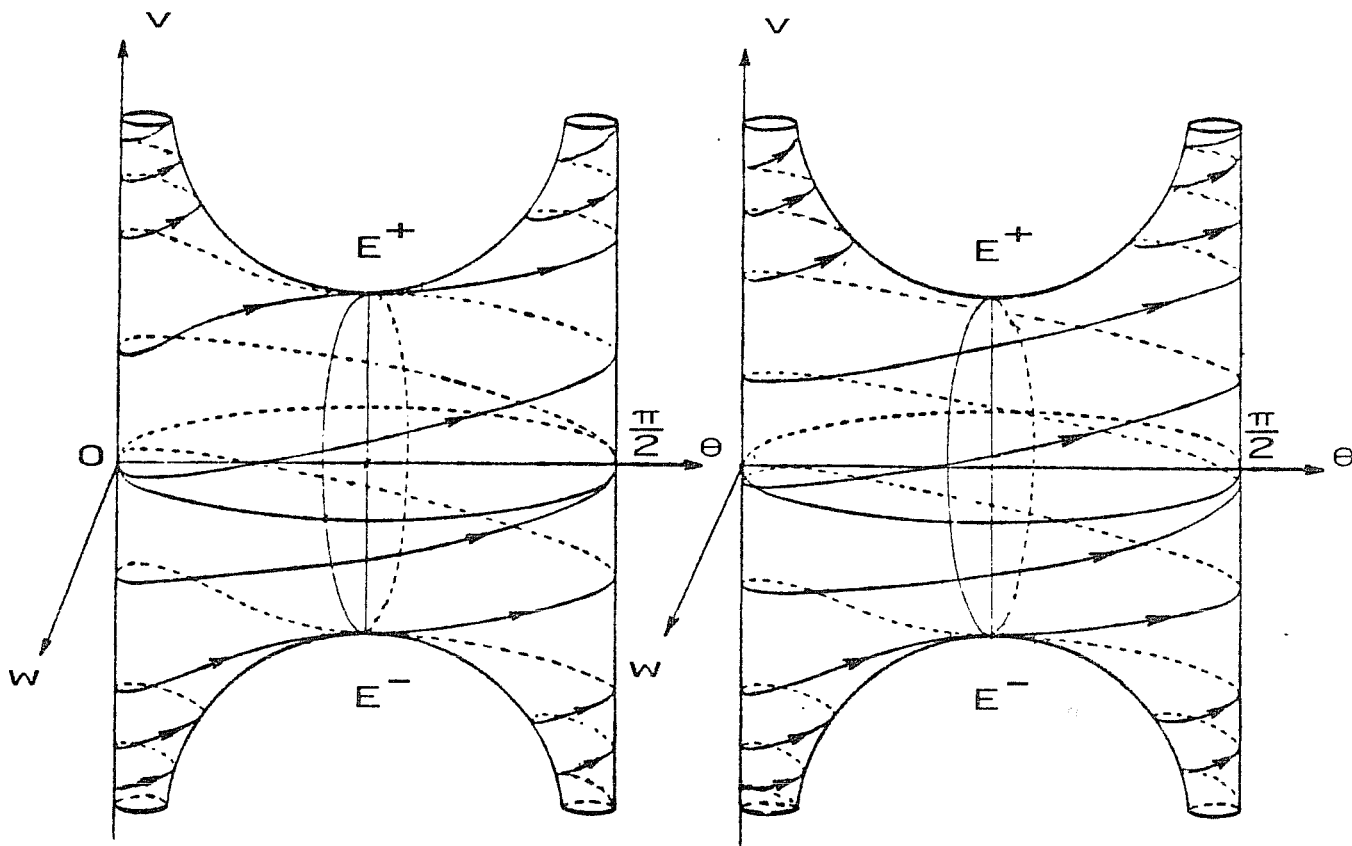


Figure 5.

The qualitative behavior of the flow on the collision manifold

The analogy with the triple collision in the collinear three-body problem is thus pointed out (see McGehee, 1974). It is anyway hard to decide which of the two situations occurs in a specific situation (here the case of equal masses). Numerical endeavors indicate that the first picture is unlikely and a chaotic behavior governs the motion in a neighborhood of total collision orbits.

REFERENCES

- Casasayas, I. and Nunes, A. 1990, *Celestial Mech.* **47**, 245-266.
- Devaney, R. 1979, *Celestial Mech.* **19**, 391-404.
- Diacu, F.N. 1987, *Astron. Nachr.* **308**, 163-168.
- Diacu, F.N. 1989, *Celestial Mech.* **46**, 27-30.
- Diacu, F.N. 1990, *Proc. Amer. Math. Soc.* **109**, 1079-1085.
- Lacomba, E. 1981, in *Classical Mech. Dyn. Syst.*, Lecture Notes Pure Appl. Math. **70**, 109-122.
- Levi-Civita, T. 1920, *Acta Math.* **42**, 99-144.
- McGehee, R. 1974, *Invent. Math.* **27**, 131-227.
- McGehee, R. 1986, *Expo. Math.* **4**, 335-345.
- Painlevé, P. 1896, *Leçons sur la théorie analytique des équations différentielles*, Hermann, Paris.
- Pérez-Chavela, E. 1991, Ph.D. Thesis, Univ. Autonoma, Ciudad de Mexico.
- Robinson, C. and Saari, D.G. 1983, *J. Diff. Eq.* **48**, 434-459.
- Saari, D.G. 1973, *Arch. Rational Mech. Anal.* **49**, 311-320.
- Saari, D.G. 1984, *J. Diff. Eq.* **55**, 300-329.
- Shelton, R.O. 1979, *Trans. Amer. Math. Soc.* **249**, 225-259.
- Siegel, K.L. 1941, *Ann. of Math.* **42**, 127-168.
- Simó, C. 1980, *Celest. Mech.* **21**, 25-36.
- Simó, C. and Lacomba, E. 1982, *Celestial Mech.* **28**, 49-62.
- Simó, C. and Llibre, J. 1981, *Arch. Rational Mech. Anal.* **77**, 189-198.
- Sperling, H.J. 1970, *J. Reine Angew. Math.* **245**, 15-40.
- Sundman, K. 1912, *Acta Math.* **36**, 105-179.
- Von Zeipel, H. 1908, *Arkiv för Mat. Astr. Fys.* **4**, 32-36.
- Wintner, A. 1941. *The Analytical Foundations of Celestial Mechanics*, Princeton Univ. Press.