Energy Localization in General Relativity and Kerr-Newman Fields

by

Stephen Alan Richardson
B.A.Sc., University of British Columbia, 1983

A Thesis Submitted in Partial Fulfillment
of the Requirements for the Degree of

MASTER OF SCIENCE

in the Department of Physics and Astronomy

We accept this thesis as conforming
to the required standard

[redacted]

Dr. F. I. Cooperstock, Supervisor (Dept. of Physics and Astronomy)

[redacted]

Dr. C. E. Picciotto, Departmental Member (Dept. of Physics and Astronomy)

[redacted]

Dr. G. G. Miller, Outside Member (Dept. of Mathematics)

[redacted]

Dr. J. Pratt, External Examiner (Dept. of Physics, Camosun College)

Supervisor: Dr. F. I. Cooperstock

Abstract

In the theory of general relativity, a meaningful generally covariant expression for energy density has not been found. It has been argued, however, that energy should be localizable, if not in general, then at least for spacetimes with specific symmetry properties. For example, in spherically symmetric spacetimes physically meaningful energy localizations have been constructed. This thesis explores the conjecture that meaningful localizations can be constructed for certain classes of axially symmetric spacetimes, in particular, for Kerr-Newman spacetimes.

The existing definitions of energy in general relativity are first reviewed. This review includes the canonical energy-momentum complexes and pseudotensors, various isolated system integrals, the symmetry method as well as several quasi-local approaches. Next, the properties of Kerr-Newman spacetimes are summarized. With this background established, various localization schemes are then applied to the Kerr-Newman fields. As a result, a new exact expression for localized Kerr-Newman energy is successfully derived.

Examiners:

[redacted]

Dr. F. I. Cooperstock, Supervisor (Dept. of Physics and Astronomy)

[redacted]

Dr. C. E. Picciotto, Departmental Member (Dept. of Physics and Astronomy)

[redacted]

Dr. G. G. Miller, Outside Member (Dept. of Mathematics)

[redacted]

Dr. J. Pratt, External Examiner (Dept. of Physics, Camosun College)

Table of Contents

	Abstract	ii
	Table of Contents	iii
	List of Figures	v
	Acknowledgements	vi
	Dedication	vii
1	**Introduction**	**1**
2	**Canonical Energy-Momentum**	**7**
	2.1 Overview	8
	2.2 General Covariance	10
	2.3 Gravitational Pseudotensors	13
3	**Alternatives**	**21**
	3.1 Total Energy Integrals	21
	3.2 The Symmetry Interpretation	25
	3.3 Localized Energy	28
4	**Kerr-Newman Geometry**	**32**
	4.1 Introduction	32
	4.2 Metric Structure	33
	4.3 Singularities and Horizons	37
	4.4 Electromagnetic Field	39
5	**Localized Kerr-Newman Energy**	**42**
	5.1 Reissner-Nordström Energy	43
	5.2 Kerr-Newman Energy	46
6	**Summary and Discussion**	**51**
	References	**53**
A	**Integral Conservation Laws**	**58**

B First Order Gravitational Lagrangian **60**

C The General Covariance Identities **62**

C.1 Lagrangian Invariance 62

C.2 Transformation of Scalar Densities 65

C.3 Identities 66

List of Figures

4.3 Kerr-Newman Ergosphere 38

A Integral Conservation Laws 58

Acknowledgements

Sincere thanks go to Dr. Fred Cooperstock for his patience and encouragement. I would also like to acknowledge the support of my family, the University of Victoria and all donors, by means or spirit, to the Richardson Castle fund.

Dedication

How many times in your life,
Can you say you did something,
Just for the hell of it.

S. Dzioba, 1989

1

Introduction*

The theory of general relativity contains an intriguing paradox. There is no known, unambiguous definition of gravitational energy. The paradox arises because in Einstein's field equations, $G_\mu{}^\nu = 8\pi T_\mu{}^\nu$, the energy-momentum tensor $T_\mu{}^\nu$ refers to the energy of all matter fields but does not directly attribute any energy to the gravitational field itself. Rather, the influence of the gravitational field on the system energy is indirect. It originates from the curvature of the spacetime on which $T_\mu{}^\nu$ is defined. This nonlinear interaction between spacetime and matter has prevented the separate, unambiguous definition of gravitational field energy.

The energy paradox has intrigued and frustrated physicists from the very beginnings of general relativity. Physical intuition suggests that the total energy of a gravitating system must be lower when the constituent masses are closer together. Thus even though gravitational energy is not well defined, gravitational fields must some how contribute to the total energy of physical systems. Enormous effort has been spent trying to resolve this issue. To what extent is gravitational energy, or energy itself, a meaningful concept? Just from the field equations themselves, two different conjectures spring to mind. The first would identify $G_\mu{}^\nu/8\pi$ as the energy-momentum tensor of the gravitational field. The second would assign no energy to the gravitational field at all. Either case can be argued. In the hope of clarifying the issue, physicists have turned to Newtonian and special relativistic limits. These efforts have been inconclusive. The total energy of isolated systems seems to

*Throughout this thesis, geometrical units are used where $c = G = 1$. Greek indices run from 0-3; Latin indices run from 1-3. Semi-colons denote covariant derivatives; commas denote ordinary derivatives. Sign conventions follow Landau and Lifshitz[32].

be well-defined but attempts to separate the gravitational contribution or to localize the system energy have had mixed success.

The nonlinear influence of gravitation on $T_\mu{}^\nu$ has another important consequence. The curvature of spacetime disrupts the formulation of integral conservation laws for $T_\mu{}^\nu$. To see this, first recall the situation in the Minkowski spacetime of special relativity. In special relativity, the energy-momentum tensor is divergence free, $T_\mu{}^\nu{}_{,\nu} = 0$. Consequently, provided $T_\mu{}^\nu$ vanishes at spatial infinity, one can construct a conserved energy-momentum four vector,

$$P_\mu = \int_\Sigma T_\mu{}^\nu \, d\Sigma_\nu, \tag{1.1}$$

where Σ is any spacelike hypersurface that includes all of three space.[†] In the curved spacetime of general relativity, however, it is the covariant divergence that vanishes, $T_\mu{}^\nu{}_{;\nu} = 0$. The difference between the two situations is elegantly summarized by expanding the covariant divergence in terms of Christoffel symbols. One finds,

$$\sqrt{-g}\, T^{\mu\nu}{}_{;\nu} = (\sqrt{-g}\, T^{\mu\nu})_{,\nu} + \sqrt{-g}\, \Gamma^{\mu}_{\sigma\nu}\, T^{\sigma\nu} = 0. \tag{1.2}$$

The first term is an ordinary divergence. Thus if for some reason the second term was absent, an integral conservation law is again obtained,

$$P_\mu = \int_\Sigma T_\mu{}^\nu \sqrt{-g}\, d\Sigma_\nu. \tag{1.3}$$

Note that although the integrand forms a tensor density, the quantity P_μ is not a generally covariant object. This is characteristic of integral conservation laws in general relativity. The covariance properties of conserved quantities will usually be restricted in some manner. In general, of course, the second

---

[†]See Appendix A.

term in (1.2) is not zero. In addition, due to the presence of the Christoffel symbol, this "nonconservative" term is not generally covariant. For these reasons, a completely general, generally covariant integral conservation law cannot be formed.

Despite the above difficulties, expressions for energy in general relativity can still be defined. The previous discussion indicates, however, that a particular approach can not be completely general. Some restrictions, either in interpretation or application, will necessarily apply. Roughly speaking, each energy definition can be categorized on the basis of its interpretation of the nonconservative term. Four basic categories appear: special coordinate frames, pseudotensors, global integrals, and quasi-local integrals.

In the first category, energy is considered meaningful only in special coordinate systems. For example, in one interpretation[46], the vanishing of the nonconservative term $\mathcal{F}^{\mu}$,

$$\mathcal{F}^{\mu} \equiv \sqrt{-g}\,\Gamma^{\mu}_{\sigma\nu}\,T^{\sigma\nu} = 0, \tag{1.4}$$

is believed to define a preferred class of coordinate systems. The system energy is then defined by the integral (1.3) and no separate definition of gravitational energy exists. In another interpretation[43], the deDonder condition for harmonic coordinates,

$$(\sqrt{-g}\,g^{\mu\nu})_{,\nu} = 0, \tag{1.5}$$

is taken to define the preferred coordinates. This choice is motivated by quantum gravity where the deDonder gauge condition is often adopted. Given the peculiar status of energy in general relativity, the use of preferred frames to express conservation laws is not unusual. As we will see, most other energy definitions also require some form of coordinate conditions.

In the pseudotensor approach, the nonconservative term is considered to represent the actual energy content of the gravitational field. Thus instead of discarding the second term, the field equations are used to transform equation (1.2) into the form of an ordinary divergence,

$$\sqrt{-g}\,T^{\mu\nu}{}_{;\nu} = [\sqrt{-g}(T_{\mu}{}^{\nu} + t_{\mu}{}^{\nu})]_{,\nu} = 0. \tag{1.6}$$

The quantities $t_{\mu}{}^{\nu}$ are called the energy-momentum pseudotensors.‡ Since they are defined by a divergence condition, the pseudotensors are not unique. In addition, due to the Christoffel symbols in the original nonconservative term, the pseudotensors can be made to vanish at any particular point by choosing appropriate geodesic coordinates. This undermines their interpretation as a gravitational energy density. Nevertheless, in certain situations, the conserved quantities P_{μ} derived from the pseudotensors provide useful results.

The intrinsic noncovariance of the nonconservative term has led many researchers to completely abandon the idea of localized energy. In this global energy approach, only the total energy of isolated systems is considered to be physically meaningful. Under these schemes, energy is defined by integrals that include spatial or null infinity. In addition, systems are usually required to be asymptotically flat. The global energy philosophy is appealing because it is consistent with the principle of general covariance. It can be viewed, however, as being overly pessimistic.

The global energy requirement of asymptotic flatness also appears in the pseudotensor and quasi-local approaches. An intuitive understanding of why the flatness requirement frequently appears can be obtained from the following argument.[27] The Einstein field equations can be expressed symbolically in the

‡Here the usage of the name pseudotensor is historical and simply means 'not a tensor'. It does not refer to an object that transforms as a tensor except under reflections.

form

$$\partial^2 g = T + (\partial g)^2, \tag{1.7}$$

where g represents the metric of the gravitational field and T represents the energy-momentum tensor. Compare this symbolic form to the Newtonian field equations,

$$\nabla^2 \phi = 4\pi\rho, \tag{1.8}$$

where ϕ is the gravitational potential and ρ is the mass density. In the limit of asymptotic flatness, one can imagine that $(\partial g)^2$ becomes negligible. Thus in this limit, a Newtonian definition of total field energy would apply. The analogy is even stronger if one considers the linearized equations of general relativity.[49]

The last category of energy definitions accepts the global energy principle but only as a worst case scenario. Rather these quasi-local approaches contend that the presence of symmetry in a particular spacetime permits a more specific localization of the energy. They do not insist that a unique energy density exist; only that the energy can be localized to subregions of the field. For example, quasi-local expressions that apply to spherically symmetric spacetimes tend to localize energy within spherical shells centered about the origin of symmetry.

In this thesis, the fundamental paradox of energy in general relativity is not addressed directly. Rather, a more investigative approach is taken. As we have just seen, there are many approaches to the definition of energy within general relativity. In contrast, there are surprisingly few actual calculations using the various definitions. For a particular energy expression, the spherically symmetric Schwarzschild energy is often the only solution cited. It is therefore important to calculate and compare the predicted energies for other spacetimes. In this way, the validity of the various approaches can be tested. In addition, such comparisons may indirectly lead to an improved understanding of the meaning of energy in general relativity.

Given the symmetry requirements of the quasi-local energy definitions, axially symmetric spacetimes are a logical choice for study. It is also desirable to choose a spacetime for which the energy-momentum tensor is well-known. The most important family of axially symmetric spacetimes are the charged, rotating Kerr-Newman fields. Since the Kerr-Newman source is charged, electromagnetic fields extend throughout the exterior metric. As a result, the energy-momentum tensor of the Kerr-Newman field is both nonzero and explicitly known. In addition, these metrics are asymptotically flat.

This thesis investigates energy localization in Kerr-Newman spacetime. In subsequent chapters, the various approaches to energy in general relativity are reviewed in greater detail. Next the properties of the Kerr-Newman fields are summarized. Various energy localization schemes are then applied to the Kerr-Newman fields. As a result, a new, exact expression for Kerr-Newman energy localization is successfully derived.

2

Canonical Energy-Momentum

In the introduction, the various approaches to energy in general relativity were categorized according to their interpretation of the covariant conservation law,

$$T_{\mu}{}^{\nu}{}_{;\nu} = 0. \tag{2.1}$$

Each category used a different method for constructing a vanishing ordinary divergence from the covariant identity (2.1); the motivation being that ordinary divergences permit integral conservation laws. General relativity, however, can also be formulated as an action principle. In field theories described by an action principle, the Noether theorems provide a prescription for constructing ordinary conservation laws directly from symmetries in the Lagrangian. This Noether or canonical approach to conservation laws is widely used in many branches of physics. In general relativity, the canonical method also has a long history, originally considered by Einstein[25] in 1916. Thus it is desirable to review the canonical conservation laws of general relativity for two reasons: first, for comparison with the energy categories identified earlier from the field equations and, second, to establish the background perspective for the alternative approaches considered in chapter 3.

In special relativity, it is Lagrangian invariance under coordinate translations that leads under the Noether theorems to the usual conservation laws of energy-momentum. In general relativity, the Lagrangian is invariant under the wider class of general coordinate transformations. This general covariance also leads to canonical conservation laws but, as we will see, the analysis is more complicated. In the end, the canonical methods lead to a unified treatment of the pseudotensors. Thus section 2.1 begins by first reviewing the pseudotensor approach. The related concept of a superpotential is also discussed. Then, in section 2.2, the Noether theorems for generally covariant theories are presented. Not until section 2.3 is the specific case of general relativity considered. This treatment will cover the pseudotensors of Einstein[25], Landau-Lifshitz[32], Tolman[61], and Møller[41] as well as the superpotentials of von Freud[26], Bergmann[9] and Komar[35].

2.1 Overview

Before looking at the canonical analysis, first recall what is meant by a gravitational pseudotensor. As shown earlier, the vanishing covariant divergence of the energy-momentum tensor $T_\mu{}^\nu$ can be split into an ordinary divergence and a nonconservative term (1.2). In the pseudotensor approach, the nonconservative term $\mathcal{F}_\mu$ is redefined in terms of an ordinary divergence, $\mathcal{F}_\mu \equiv (\sqrt{-g}\, t_\mu{}^\nu)_{,\nu}$. We then write,

$$\sqrt{-g}\, T_\mu{}^\nu{}_{;\nu} = [\sqrt{-g}(T_\mu{}^\nu + t_\mu{}^\nu)]_{,\nu} = 0. \tag{2.2}$$

The quantity $t_\mu{}^\nu$ is called the energy-momentum pseudotensor of the gravitational field. As mentioned previously, $t_\mu{}^\nu$ is neither unique nor a tensor.

Even though the pseudotensors are not unique, it is tempting to interpret $t_\mu{}^\nu$ as the energy-momentum density of the gravitational field. However, the pseudotensors can be made to vanish at any particular point by choosing appropriate coordinates. Thus instead of treating the pseudotensors on their own, it is often convenient to consider just the total complex,

$$\Theta_\mu{}^\nu \equiv \sqrt{-g}(T_\mu{}^\nu + t_\mu{}^\nu), \tag{2.3}$$

where now $\Theta_\mu{}^\nu{}_{,\nu} = 0$. Expressions for the total energy-momentum complexes $\Theta_\mu{}^\nu$ are usually simpler than those for the corresponding pseudotensors. In addition, the energy-momentum complexes, rather than the pseudotensors, are the natural outcome of the canonical analysis.

It is useful to determine those properties of the energy-momentum complexes that can be deduced just from their definition. Since the complexes $\Theta_\mu{}^\nu$ are divergence free, they must be expressible as the divergence of a quantity $U_\mu{}^{\nu\sigma}$ which is antisymmetric in ν and σ,

$$\Theta_\mu{}^\nu = U_\mu{}^{\nu\sigma}{}_{,\sigma}. \tag{2.4}$$

The quantities $U_\mu{}^{\nu\sigma}$ are called superpotentials. Note that the superpotentials are not uniquely determined by the complexes. If $V_\mu{}^{\nu\sigma\tau}{}_{,\tau}$ is any quantity antisymmetric in σ and τ, then it can be added to the superpotential without changing the original complex.

The superpotentials provide an alternative way to evaluate integral conservation laws constructed from energy-momentum complexes. Consider a spatial region $\mathcal{R}$ bounded by a 2-surface $\mathcal{S}$. Then provided $\Theta_\mu{}^\nu$ vanishes outside of $\mathcal{R}$, the quantity,

$$Q_\mu = \int_{\mathcal{R}} \Theta_\mu{}^0 \, d^3x = \int_{\mathcal{R}} U_\mu{}^{0\sigma}{}_{,\sigma} \, d^3x, \tag{2.5}$$

is conserved. Since the superpotential is antisymmetric, the components $U_\mu{}^{00}$ are zero. Thus Gauss's Law in three dimensions can be applied to the σ divergence. Hence,

$$Q_\mu = \oint_{\mathcal{S}} U_\mu{}^{0i} \, dS_i. \tag{2.6}$$

Thus the amount of "charge" Q_μ in a region $\mathcal{R}$ can be determined by evaluating the components $U_\mu{}^{0i}$ on the bounding surface $\mathcal{S}$.

In many applications, the region $\mathcal{R}$ includes all of 3-space. In this case, in order for the surface integral to converge, the superpotentials $U_\mu{}^{0i}$ must behave appropriately at spatial infinity. Since the superpotentials are not generally covariant, this requirement restricts the types of coordinate systems that can be used to evaluate Q_μ. At the same time, however, two coordinate systems that are asymptotically equal will give the same value for Q_μ.

2.2 General Covariance

In field theories described by a Lagrangian, the system dynamics are determined by an action integral,

$$A = \int_\Omega \mathcal{L}(\phi_J,\ \phi_{J,\mu},\ \phi_{J,\mu\nu};\ x)\, d^4x. \tag{2.7}$$

The field variables $\phi_J(x)$, $J = 1 \ldots N$, represent all of the tensor components of all of the system fields, including the metric tensor $g_{\mu\nu}$. The Lagrangian is assumed to contain only first and second derivatives of the fields.

The Noether theorems consider transformations of the coordinates and field variables that form a continuous group $G(\epsilon)$, where $\epsilon = 0$ denotes the identity transformation.* Since the group is continuous, it is sufficient to consider the infinitesimal transformations produced by small variations in ϵ about $\epsilon = 0$. If $\bar{\phi_J}$ and $\bar{x}$ denote the transformed fields and coordinates, then one can define two different variations,

$$\delta A \equiv A(\bar{\phi_J}, \bar{x}) - A(\phi_J, x) \quad \text{and} \quad \bar{\delta} A \equiv A(\bar{\phi_J}, x) - A(\phi_J, x). \tag{2.8}$$

The variation δ measures the change at a fixed point in spacetime. In contrast, the variation $\bar{\delta}$ measures the change at a fixed coordinate value. For transformations that involve the coordinates, only the operator $\bar{\delta}$ commutes with differentiation.

---

*A general familiarity with the Noether theorems is assumed. For reference, Appendix C treats the topics covered in this section in greater detail.

An action is said to be invariant under a transformation group $G(\epsilon)$ if $\bar{\delta}A = 0$ for all ϵ and for all fields ϕ_J. In other words, A is invariant if the variation of A with respect to variations in ϵ is zero. Invariance of an action, however, does not require invariance of the Lagrangian. Rather, the Lagrangian need only transform as the divergence of a functional $\bar{\delta}B^\mu$ which vanishes on the boundary of Ω. That is, $\bar{\delta}\mathcal{L} = \bar{\delta}B^\mu{}_{,\mu}$. The Noether conservation theorems assume that this Lagrangian divergence condition holds for the group of transformations $G(\epsilon)$. This assumed form for $\bar{\delta}\mathcal{L}$ is then compared to the variation calculated directly from the corresponding variations $\bar{\delta}\phi_J$ in the fields. One obtains the well-known result,

$$L^J\,\bar{\delta}\phi_J + J^\mu{}_{,\mu} = 0, \tag{2.9}$$

where L^J denotes the Euler-Lagrange field equations and J^μ denotes the Noether current. Specifically,

$$J^\mu \equiv \left(\frac{\partial\mathcal{L}}{\partial\phi_{J,\mu}} - \partial_\nu\frac{\partial\mathcal{L}}{\partial\phi_{J,\mu\nu}}\right)\bar{\delta}\phi_J + \frac{\partial\mathcal{L}}{\partial\phi_{J,\mu\nu}}\bar{\delta}\phi_{J,\nu} - \bar{\delta}B^\mu. \tag{2.10}$$

If the field equations are satisfied, that is $L^J = 0$, then the Noether current has a vanishing ordinary divergence and thus determines an integral conservation law.†

In generally covariant field theories, the action is invariant under general coordinate transformations. Thus we consider the set of transformations,

$$\bar{x}^\mu = x^\mu + \epsilon(x)\xi^\mu(x). \tag{2.11}$$

By letting ϵ depend on x, this form accounts for the infinity of possible coordinate transformation groups, each group with 4 coordinate degrees of freedom. In other words, we are allowing for local gauge invariance.

†Such laws are sometimes called *weak* conservation laws; in contrast, *strong* conservation laws hold even when the field equations are not satisfied.

In general relativity, it is convenient to split the system Lagrangian into a term for the matter fields and their interactions and a term for the free gravitational field,

$$\mathcal{L} = \mathcal{L}_M + \mathcal{L}_G. \tag{2.12}$$

The matter Lagrangian is usually a scalar density because this ensures that the corresponding action is a scalar under general coordinate transformations. It is also reasonable to assume that $\mathcal{L}_M$ is first order in the derivatives of the fields. The gravitational Lagrangian, however, is not necessarily a first order scalar density. As discussed in appendix B, the Hilbert gravitational Lagrangian is a scalar density but contains second order derivatives of the field components $g_{\mu\nu}$. In contrast, the alternative Lagrangian $\mathcal{L}'_G$ is first order in the field derivatives but has a more complicated transformation law.

If the first order gravitational Lagrangian $\mathcal{L}'_G$ is used, then it can be shown that the Noether current (2.10) takes the form,

$$J^\nu = \frac{\partial \mathcal{L}}{\partial \phi_{J,\nu}} \left[F^{I\tau}_{J\lambda}\, \xi^\lambda{}_{,\tau} \phi_I - \phi_{J,\lambda} \xi^\lambda \right] + \mathcal{L}\xi^\nu + Q_\lambda{}^{\nu\sigma} \xi^\lambda{}_{,\sigma}. \tag{2.13}$$

The coefficients $F^{I\tau}_{J\lambda}$ are just constants that characterize the particular tensor or spinor order of the field variables. For example, the coefficients for an nth order covariant tensor field $\phi_{\beta_1 \ldots \beta_n}$ are,

$$F^{\beta_1 \ldots \beta_n \tau}_{\alpha_1 \ldots \alpha_n \lambda} = \sum_i \delta^\tau_{\alpha_i} \delta^{\beta_i}_\lambda \prod_{j \neq i} \delta^{\beta_j}_{\alpha_j}. \tag{2.14}$$

The $Q_\lambda{}^{\nu\sigma}$ are functions of the metric tensor and arise from the divergence condition of the Lagrangian $\mathcal{L}'_G$. They are described in appendix B.

If the action was invariant only under the group of global transformations with constant ϵ then nothing more could be learned about the system. However, the local gauge invariance leads to several further results. In solving for the Noether current, one also finds that,

$$L^J\, F^{I\nu}_{J\lambda}\, \xi^\lambda \phi_I + J^\nu + U^{\nu\sigma}{}_{,\sigma} = 0, \tag{2.15}$$

where the quantities $U^{\nu\sigma}$ are antisymmetric and take the form,

$$U^{\nu\sigma} = (\frac{\partial \mathcal{L}}{\partial \phi_{J,\sigma}} F^{I\nu}_{J\lambda} \phi_I - Q_\lambda{}^{\sigma\nu})\xi^\lambda. \tag{2.16}$$

Thus provided the field equations are satisfied, the Noether current J^ν can be expressed in terms of the superpotentials $U^{\nu\sigma}$. Then either the Noether current or the superpotential can be integrated to generate a conserved quantity Q.

In conclusion, the Noether theorems for generally covariant field theories associate a conserved quantity Q with each of the infinity of coordinate transformations $\xi^\lambda(x)$. Note, however, that due to the freedom in choice of the Lagrangian, this association is not unique. Choosing the Hilbert Lagrangian to represent the gravitational field instead of the first order Lagrangian $\mathcal{L}'_G$ produces a quite different relationship. Thus it is the set of conservation laws themselves that are important, not the particular prescription $\xi^\lambda \to Q$.

2.3 Gravitational Pseudotensors

The Noether currents and superpotentials can be used to construct the energy-momentum complexes and the gravitational pseudotensors that are described in the overview. Expressions due to Einstein[25], Tolman[59], Freud[26], Landau-Lifshitz[32], and Bergmann[7-9] all follow from the first order Noether analysis of the previous section. It must be remembered, however, that this analysis depends on the choice of the gravitational Lagrangian. Thus following the description of the first order results, the consequences of choosing the second order Hilbert Lagrangian will be presented. This will lead to the conservation laws of Møller[41,42] and Komar[35].

Consider just the free gravitational contribution to the system Lagrangian. Then the Noether identity (2.15) gives,

$$U_G^{\nu\sigma}{}_{,\sigma} = -J_G^\nu - L_G^{\alpha\beta} F_{\alpha\beta\lambda}^{\rho\tau\nu} \xi^\lambda g_{\rho\tau}, \tag{2.17}$$

where $L_G^{\alpha\beta}$ are the Euler-Lagrange equations for the free gravitational field. The coefficients $F_{\alpha\beta\lambda}^{\rho\tau\nu}$ for the metric tensor components $g_{\rho\tau}$ are given by equation (2.14). If these are inserted into (2.17), one obtains,

$$U_G^{\nu\sigma}{}_{,\sigma} = -J_G^\nu + \sqrt{-g}\frac{G^{\alpha\nu}}{8\pi}\xi_\alpha, \tag{2.18}$$

where $G^{\alpha\nu}$ is the Einstein tensor. When the Einstein field equations are satisfied, $G^{\alpha\beta} = 8\pi T^{\alpha\beta}$. Thus if we associate the pseudotensors with the gravitational Noether current and the complexes Θ^ν with the superpotential divergences $U_G^{\nu\sigma}{}_{,\sigma}$, then we can write,

$$\Theta^\nu \equiv U_G^{\nu\sigma}{}_{,\sigma} = \sqrt{-g}(t^\nu + T^{\alpha\nu}\xi_\alpha) \quad \text{where} \quad t^\nu \equiv \frac{-J_G^\nu}{\sqrt{-g}}. \tag{2.19}$$

Since the superpotentials $U_G^{\nu\sigma}$ are antisymmetric, $\Theta^\nu{}_{,\nu} = 0$, as desired. Thus even though the complexes Θ^ν characterize the entire system, they can be determined from the gravitational Lagrangian alone.

The gravitational superpotentials $U_G^{\nu\sigma}$ are obtained from the general expression (2.16) by again keeping only those terms due to the gravitational Lagrangian. Thus we have,

$$U_G^{\nu\sigma} = \left(\frac{\partial \mathcal{L}'_G}{\partial g_{\alpha\beta,\sigma}} F_{\alpha\beta\lambda}^{\rho\tau\nu} g_{\rho\tau} - Q_\lambda{}^{\sigma\nu}\right)\xi^\lambda. \tag{2.20}$$

A straight-forward substitution of the Lagrangian $\mathcal{L}'_G$, the coefficients $F_{\alpha\beta\lambda}^{\rho\tau\nu}$, and the transformation term $Q_\lambda{}^{\sigma\nu}$, yields the following compact expression[56],

$${}^{(B)}U^{\nu\sigma} = \frac{1}{16\pi\sqrt{-g}}\left[-g(g^{\alpha\nu}g^{\omega\sigma} - g^{\alpha\sigma}g^{\omega\nu})\right]_{,\omega}\xi_\alpha. \tag{2.21}$$

This expression, known as the Bergmann superpotential,[9] has played a pivotal role in the development of the canonical conservation laws in general relativity.

Up to this point, the coordinate transformation functions ξ^λ have remained completely arbitrary. The relationship (2.19), however, suggests a preferred status for the four choices $\xi^\lambda = \delta^\lambda_{(\mu)}$. For in this way, the originally postulated form for the energy-momentum complex is recovered. Namely,

$$\Theta_\mu{}^\nu = \sqrt{-g}(t_\mu{}^\nu + T_\mu{}^\nu). \tag{2.22}$$

It is interesting to calculate the corresponding forms for the pseudotensors and the total complex. The free gravitational Noether current J^ν_G follows from the general current (2.13). Since the transformation functions ξ^λ are now constants, the two terms containing the derivatives of ξ^λ drop out. This leaves

$$\sqrt{-g}\,{}^{(E)}t_\mu{}^\nu = \frac{\partial \mathcal{L}'_G}{\partial g_{\alpha\beta,\nu}} g_{\alpha\beta,\mu} - \delta^\nu_\mu \mathcal{L}'_G. \tag{2.23}$$

This form for the pseudotensor was first proposed by Einstein[25] in 1916. Note that it has the form of the canonical energy-momentum for the first order Lagrangian $\mathcal{L}'_G$. The corresponding total energy-momentum complex follows immediately from the Bergmann superpotential (2.21). One finds,

$${}^{(E)}\Theta_\mu{}^\nu = {}^{(F)}U_\mu{}^{\nu\sigma}{}_{,\sigma} \quad \text{where} \quad {}^{(F)}U_\mu{}^{\nu\sigma} \equiv \frac{1}{16\pi}\frac{g_{\mu\alpha}}{\sqrt{-g}}\left[-g(g^{\alpha\nu}g^{\omega\sigma} - g^{\alpha\sigma}g^{\omega\nu})\right]_{,\omega}. \tag{2.24}$$

The superpotential ${}^{(F)}U_\mu{}^{\nu\sigma}$ was originally discovered by von Freud in 1939.[26]

As discussed in section 2.1, the complexes Θ^ν can be integrated over spatial regions to generate conserved quantities Q. Usually, the regions cover all of 3-space to allow for matter distributions that extend out to spatial infinity. It was mentioned that convergence requirements may restrict the possible coordinate systems that can be used. If the Einstein complex is considered, it can be shown that the resulting conserved quantities,

$$ {}^{(E)}P_\mu = \int_{\mathcal{R}} {}^{(E)}\Theta_\mu{}^0\, d^3x = \oint_{S} {}^{(F)}U_\mu{}^{0i}\, dS_i, \tag{2.25} $$

are consistently defined only for coordinate systems that asymptotically approach the Minkowski metric at spatial infinity.‡ This implies that the matter distribution is sufficiently isolated that the spacetime at spatial infinity is flat.

In asymptotically Minkowskian coordinates, the quantity ${}^{(E)}P_\mu$ forms a proper 4 vector with respect to linear transformations[29]. It can also be shown that for Schwarzschild spacetime in Cartesian coordinates, one obtains the desired result,

$$ {}^{(E)}P_0 = m \quad \text{and} \quad {}^{(E)}P_i = 0, \tag{2.26} $$

where m is the Schwarzschild mass.

Since the conserved vector ${}^{(E)}P_\mu$ is related to a superpotential, other choices for the transformation functions ξ^λ that are asymptotically equivalent will still give the same four values ${}^{(E)}P_\mu$. For example, consider the choice, $\xi^\lambda = \sqrt{-g}\, g^{\lambda(\mu)}$, where the metric is required to be asymptotically Minkowskian. In this case, the Bergmann superpotential reduces to give,

$$ {}^{(L)}\Theta^{\mu\nu} = \frac{1}{16\pi}\left[-g(g^{\mu\nu}g^{\omega\sigma} - g^{\mu\sigma}g^{\omega\nu})\right]_{,\omega\sigma}. \tag{2.27} $$

This form for the energy-momentum complex is due to Landau and Lifshitz[32]. Unlike the Einstein complex, the Landau-Lifshitz complex is symmetric. However, the functions ξ^λ also appear in the relationship (2.19) that relates the total complex to the energy-momentum tensor. For the Landau-Lifshitz case, this becomes§

$$ {}^{(L)}\Theta^{\mu\nu} = \sqrt{-g}({}^{(L)}t^{\mu\nu} + \sqrt{-g}T^{\mu\nu}). \tag{2.28} $$

‡Such spacetimes are called asymptotically Minkowskian or quasi-Galilean.

§The pseudotensor ${}^{(L)}t^{\mu\nu}$ defined here is consistent with the general formalism expressed by (2.19). It differs by a factor of $\sqrt{-g}$ from the definition of Landau-Lifshitz[32] who prefer to write $\Theta^{\mu\nu} = -g(t^{\mu\nu} + T^{\mu\nu})$.

Thus, not surprisingly, the energy-momentum vector calculated from ${}^{(L)}\Theta^{\mu\nu}$ does not transform as a vector under linear transformations. Rather it transforms as a vector density[29].

Recall that any quantity with a vanishing divergence can be added to a superpotential without changing the corresponding complex. For example, an alternative superpotential for the Einstein energy-momentum complex, attributed to Tolman,¶ is obtained by adding the functions $Q_\mu{}^{\sigma\nu}$ defined in appendix B to the Freud superpotential,

$$ {}^{(T)}U_\mu{}^{\nu\sigma} = {}^{(F)}U_\mu{}^{\nu\sigma} + Q_\mu{}^{\sigma\nu}. \tag{2.29}$$

Since the divergence $Q_\mu{}^{\sigma\nu}{}_{,\sigma}$ is zero, the corresponding complex is unchanged. The significance of considering the functions $Q_\mu{}^{\sigma\nu}$ becomes clear when one looks back at the general expression (2.20). In the Tolman superpotential, the functions $Q_\mu{}^{\sigma\nu}$ cancel out. Thus when the coefficients $F^{\rho\tau\nu}_{\alpha\beta\lambda}$ are inserted, one obtains the elegant result,

$$ {}^{(T)}U_\mu{}^{\nu\sigma} = 2\frac{\partial \mathcal{L}'_G}{\partial g_{\alpha\nu,\sigma}} g_{\alpha\mu}. \tag{2.30}$$

However, when the Tolman superpotential is evaluated, the result is more complicated than the superpotential of Freud. It is also no longer antisymmetric in ν and σ. For these reasons, the Tolman superpotential is rarely used.

When restricted to asymptotically Minkowskian coordinate systems, the Einstein and Landau-Lifshitz complexes give a consistent definition of the total energy of isolated systems. They do not, however, provide a reasonable definition of localized energy because their "energy densities" $\Theta_0{}^0$ are not scalars, even under spatial transformations. This lack of covariance led Møller

¶Virbhadra[61] attributes the superpotential (2.29) to Tolman, citing Møller's 1958 paper. Møller, however, does not reference Tolman in this regard.

in 1958 to construct yet another 'canonical' complex[41]. By adding various antisymmetric terms to the Einstein superpotential, Møller deduced the following superpotential,

$$ {}^{(M)}U_{\mu}{}^{\nu\sigma} = \frac{\sqrt{-g}}{8\pi} g^{\nu\kappa} g^{\sigma\lambda} (g_{\mu\lambda,\kappa} - g_{\mu\kappa,\lambda}). \tag{2.31} $$

For isolated systems and Cartesian coordinates, the corresponding Møller complex ${}^{(M)}\Theta_{\mu}{}^{\nu}$ still yields the desired conserved values ${}^{(E)}P_{\mu}$. In addition, the four quantities ${}^{(M)}\Theta_{\mu}{}^{0}$ form a four vector density with respect to the group of spatial transformations. Therefore the energy ${}^{(M)}P_0$, even for finite regions $\mathcal{R}$, is a scalar under spatial transformations. This invariance makes the Møller complex more desirable as a description of localized energy. Unfortunately, as Møller showed three years later[42], although in a given frame the quantities ${}^{(M)}P_{\mu}$ equal the Einstein values, they no longer transform as a vector under linear transformations.

It should be noted that since the Møller complex was not constructed from the Bergmann superpotential, the relationship (2.19) between the canonical complexes and the energy-momentum tensor is not valid for the Møller complex. It is not clear how the full Møller complex relates to $T_{\mu}{}^{\nu}$. However, for static systems the Møller time component ${}^{(M)}\Theta_0{}^0$ reduces to $(T_0{}^0 - T_i{}^i)\sqrt{-g}$. This is the integrand for the Tolman total integral approach, discussed further in section 3.1.

One year after Møller's original derivation, Komar[35] was able to generalize Møller's complex into a family of covariant conservation laws. It was subsequently discovered that the Komar superpotentials follow naturally from the Noether theorems and the Hilbert Lagrangian just as the Bergmann superpotentials follow from the first order Lagrangian.[23] Specifically, the Komar superpotentials are

$$ {}^{(K)}U^{\nu\sigma} = \frac{\sqrt{-g}}{8\pi}(\xi^{\sigma;\nu} - \xi^{\nu;\sigma}). \tag{2.32} $$

The functions ξ^{μ} again characterize the arbitrary infinitesimal coordinate transformations. Notice that the covariant divergence of the Komar superpotentials reduces to an ordinary divergence. Thus the Komar complex is covariantly defined,

$$ {}^{(K)}\Theta^{\nu} = {}^{(K)}U^{\nu\sigma}{}_{,\sigma} = \frac{\sqrt{-g}}{8\pi}(\xi^{\sigma;\nu} - \xi^{\nu;\sigma})_{;\sigma}. \tag{2.33} $$

The Møller superpotential is obtained from the Komar superpotentials by choosing $\xi^{\lambda} = \delta^{\lambda}_{(\mu)}$. Coincidently, this is the same set of translations that relates the Einstein superpotential to the Bergmann conservation laws.

In summary, the canonical energy-momentum complexes of general relativity are related to the Noether theorems for generally covariant field theories. The Noether analysis associates a conserved quantity with each infinitesimal transformation function ξ^{μ}. The association depends on the form of the gravitational Lagrangian. The Bergmann family of conservation laws follows from the first order Lagrangian $\mathcal{L}'_G$ while the Komar family follows from the Hilbert Lagrangian. All of the energy-momentum complexes and pseudotensors proposed over the years can be derived from these two canonical families by choosing particular sets of transformation functions ξ^{μ}.

While the various complexes do give a reasonable definition for the total energy of isolated systems, they do not provide a satisfactory description of localized energy. Also, within a given family of conservation laws, the Noether analysis does not provide a physical basis for choosing one set of ξ^{μ} over another. These ambiguities have led to two different interpretations. In the first approach, localized energy is considered undefinable. Only the total energy of isolated systems is considered to be physically meaningful. In the second approach, localized energy is definable but only in the presence of spacetime symmetries. Both approaches will be discussed further in the next chapter.

3

Alternative Methods

The presence of ambiguities in the canonical analysis has inspired physicists to search for alternative ways to describe energy in general relativity. Numerous alternative methods have been proposed. A complete discussion of all the proposals is well beyond the scope of this thesis. Nevertheless, a survey of the most significant alternative methods is useful. In the first section, the main approaches within the total integral interpretation are reviewed. These are the Tolman[59], Arnowitt-Deser-Misner[2–5], and Bondi[10] energy integrals. Next, in section 3.2, the relevance of spacetime symmetry is considered. There the symmetry interpretation of Davis and Moss[20] is presented. In addition, a class of conservation laws constructed from Killing vectors is discussed. Finally, in section 3.3, quasi-local approaches are discussed. This includes the localizations of Misner-Sharp[39], Cooperstock-Sarracino[17,18], and Penrose[50].

3.1 Total Energy Integrals

In 1930, Tolman considered the total energy of an isolated system for which the gravitational field was constant. Starting with the Einstein pseudotensor and assuming asymptotically Minkowskian spacetime, he found that the total system energy could be written as,

$$E = \int (T_0{}^0 - T_i{}^i)\sqrt{-g}\, d^3x, \tag{3.1}$$

where the integral is over all of 3-space. Tolman noted that his integral need 'be extended only over the portion of space actually occupied by matter or radiation', that is, where the energy-momentum tensor is nonzero. This observation appears to support the conjecture that energy in general relativity is localized in the non-vacuum regions of spacetime. The Tolman integral, however, was derived to give the total system energy. Thus interpreting the Tolman integrand as an energy localization may not be justified. This question will be revisited in section 5.1.

The integrand in the Tolman integral can easily be written in terms of the gravitational field. One finds that

$$T_0{}^0 - T_i{}^i = \frac{R_0{}^0}{4\pi}. \tag{3.2}$$

Since the Tolman integrand is equal to the time component of a tensor density, it transforms as a scalar density under purely spatial transformations. This property is shared by the energy component of the Møller complex. It can be shown that the Tolman integrand is equivalent to the Møller energy density for time-independent fields[63].

Beginning in 1959, Arnowitt, Deser and Misner (ADM) made a serious attempt to quantize the gravitational field. Their analysis considered Hamiltonian formulations of general relativity for various sets of canonical variables. In Hamiltonian systems, sets of initial data defined on initial spacelike hypersurfaces evolve together in time, according to the dynamics determined by the Hamiltonian. Since the Hamiltonian is a measure of system energy, the ADM quantization program provided a definition for the total energy of isolated systems. This ADM energy is

$$^{(ADM)}E = \frac{1}{16\pi} \int_S (g^{ij}{}_{,j} - g_j{}^{j,i})\, dS_i, \tag{3.3}$$

where the integral is over a 2-surface that encloses all of 3-space. The striking feature of the ADM energy is that the integrand contains only spatial components of the metric and their spatial derivatives. In other words, the ADM energy is intrinsic to the spacelike hypersurface on which it is defined.

There is an elegant derivation of the ADM energy, due to Weinberg, that follows from a partial linearization of the Einstein field equations.[48,64] The metric tensor for an isolated system is split into a Minkowski part $\eta_{\mu\nu}$ and a remainder $h_{\mu\nu}$,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{3.4}$$

The components $h_{\mu\nu}$ vanish appropriately at spatial infinity but are not necessarily small within the interior of the system. The field equations can then be written as,

$$\tilde{G}^{\mu\nu} = 8\pi(T^{\mu\nu} + \tilde{t}^{\mu\nu}), \tag{3.5}$$

where $\tilde{G}^{\mu\nu}$ contains only those terms in the Einstein tensor that are linear in $h_{\mu\nu}$. All the nonlinear terms have been moved to the right-hand side of the equation and are represented by the quantity $\tilde{t}^{\mu\nu}$. The linearized Einstein tensor has a vanishing ordinary divergence. Thus $\tilde{t}^{\mu\nu}$ defines yet another gravitational pseudotensor which by construction must be symmetric.*

The linearized tensor $\tilde{G}^{\mu\nu}/8\pi$ defines the energy-momentum complex corresponding to the pseudotensor $\tilde{t}^{\mu\nu}$. Furthermore, $\tilde{G}^{\mu\nu}$ can be written as an ordinary divergence and thus defines a superpotential. One finds,

$$\tilde{U}_{\mu}{}^{\nu\sigma} = \frac{1}{16\pi}\left[h_{\tau}{}^{\tau,\nu}\delta^{\sigma}_{\mu} - h_{\tau}{}^{\tau,\sigma}\delta^{\nu}_{\mu} + h^{\tau\sigma}{}_{,\tau}\delta^{\nu}_{\mu} - h^{\tau\nu}{}_{,\tau}\delta^{\sigma}_{\mu} + h_{\mu}{}^{\nu,\sigma} - h_{\mu}{}^{\sigma,\nu}\right]. \tag{3.6}$$

In the usual manner, the superpotential can be used to express the total energy $\tilde{P}_0$ in terms of a surface integral at spatial infinity. Remarkably, the resulting expression is the ADM energy. That is,

$$\tilde{P}_0 = \int_S \tilde{U}_0{}^{0i}\, dS_i = \frac{1}{16\pi}\int_S (h^{ij}{}_{,j} - h_j{}^{j,i})\, dS_i = {}^{(ADM)}E. \tag{3.7}$$

*Note that the relationship (3.5) between the 'complex' $\tilde{G}^{\mu\nu}$ and the energy-momentum tensor $T^{\mu\nu}$ differs from the earlier definition (2.3) by a factor of $\sqrt{-g}$. Thus one expects the Weinberg conserved quantities ${}^{(W)}P_{\mu}$ to transform under linear transformations as a vector density of weight -1. In contrast, the Einstein ${}^{(E)}P_{\mu}$ transform as a vector while the Landau-Lifshitz ${}^{(L)}P_{\mu}$ transforms as a vector density of weight +1.

Thus for isolated systems, the total energy determined by the ADM Hamiltonian formulation of general relativity is equal to that determined by the nonlinear terms of the Einstein tensor.

All the total energy integrals considered so far are defined on spacelike hypersurfaces that are asymptotically flat at spatial infinity. In order to study radiating systems, a more appropriate choice is spacelike hypersurfaces that are asymptotically null at null infinity. In this way, any differences in energy between surfaces with different asymptotic limits can be attributed to the radiated energy flux emitted between the two asymptotically null surfaces. This null infinity approach defines what is known as the Bondi energy.[10,52]

A proper definition of the Bondi energy requires a rigorous definition of asymptotic flatness using the principle of conformal infinity. A full discussion is beyond the level of this survey. Formally, the Bondi energy definition appears similar to the canonical Komar definition.[63] In particular,

$$
{}^{(B)}E = \frac{1}{8\pi} \int_{S} \sqrt{-g}(\xi^{i;0} - \xi^{0;i})\, dS_i. \tag{3.8}
$$

However, the functions ξ^{μ} are now 'asymptotic time translations at null infinity' and the surfaces S are now 'asymptotically null cross-sections.' Even without further explanation, these descriptions provide at least an intuitive understanding of the Bondi energy expression.

Several important global theorems relate the Bondi and ADM energies. First, provided the energy-momentum tensor satisfies the dominant energy condition, both the Bondi and ADM energies are positive.[31,54,66] Second, the radiated energy flux determined by the Bondi prescription is always positive. In other words, radiation always carries positive energy away from a radiating system. Finally, the ADM and Bondi definitions are consistent in the following sense. In a given isolated system, the difference between the ADM energy and the Bondi energy associated with a given 'retarded instant of time' is equal to the radiated energy emitted between the infinite past and the given retarded instant.[6]

For asymptotically Minkowskian systems, the ADM energy equals the total energy determined by the canonical complexes. If the spacetime is also constant, then there is no radiation and the Bondi energy also equals the ADM energy. Moreover, as already mentioned, the Møller energy for constant spacetimes is equivalent to the Tolman energy. Thus for isolated constant spacetimes, the total energy determined by all the total integral methods is the same. The methods differ only in the relative transformation properties of their energy-momentum four 'vectors' P_μ.

In the total energy interpretation, localized energy is not considered meaningful. As a result, differences in the integrands are not considered physically meaningful. There are, however, alternative interpretations that do attribute physical significance to energy localizations. These quasi-local methods usually require some form of spacetime symmetry. Thus the relationship between symmetries in spacetime and conservation laws is the subject of the next section.

3.2 The Symmetry Interpretation

In special relativity, invariance under the ten parameter group of Lorentz transformations leads via Noether's theorems to the conservation of ten quantities: energy, three components of linear momentum and six components of angular momentum. This relationship between coordinate invariance and conservation laws inspired the canonical analysis presented in chapter 2. There it was shown that invariance under the group of general coordinate transformations leads to an infinite number of conserved quantities. Each arbitrary function ξ^μ determines a one parameter group of transformations $\epsilon\xi^\mu$ that determines a conserved quantity Q. In chapter 2, the transformation functions were chosen so that the resulting conserved quantities matched previously known energy-momentum expressions. As a result, the functions ξ^μ were interpreted as merely a mechanism to generate possible conservation laws; they were not given any physical significance.

In the symmetry interpretation of Davis and Moss[20−22], the above interpretation of the canonical analysis is considered incorrect. In special relativity, there is also a one-to-one correspondence between the group of Lorentz transformations and the symmetry of Minkowski spacetime. Thus the existence of the ten conservation laws can also be attributed to the existence of the ten symmetries of flat spacetime. Thus Davis and Moss argue that the relationship between the Lorentz transformations and the conservation laws is a coincidence, unique to the particular case of flat spacetime. Consequently, they argue that physically meaningful conservation laws exist in more general spacetimes only if corresponding symmetries exist in the gravitational field. In other words, the interpretation of Davis and Moss attributes physical meaning to only a few of the infinite number of vanishing divergences that can be formulated using the canonical methods. For a spacetime with no symmetries, no conservation laws exist.

In general relativity, spacetime symmetries are determined by the Killing equation,

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0. \tag{3.9}$$

Solutions are called Killing vectors and determine groups of coordinate transformations for which the metric tensor is unchanged. In the symmetry interpretation, each Killing vector ξ^μ must determine a physically meaningful conservation law. In chapter 2, the canonical superpotentials were derived by considering arbitrary coordinate transformations. Thus these expressions are still valid when the functions ξ^μ are Killing vectors. Insisting on general covariance, Davis and Moss thus identify the Komar superpotential as the correct generator of physically meaningful conservation laws. Note, however, that the interpretation given the Komar expression is quite different. In a given spacetime, and for a particular set of coordinates, the Killing vectors are first determined. For each Killing vector field, the Komar expression then determines the corresponding covariant differential conservation law. If the Komar expression is integrated to determine an integrally conserved quantity, then the expression is no longer covariant. However, if the coordinate system is changed, the conserved quantity is uniquely prescribed by determining the Killing vector in the new coordinates.

The interpretation of Davis and Moss creates conservation laws by combining the Killing vectors with the canonical energy-momentum complexes. The Killing vectors, however, can also generate conservation laws directly from the energy-momentum tensor. It is simple to show that the contraction $T_\mu{}^\nu \xi^\mu$ is a vector field with a vanishing covariant divergence. The corresponding vector density thus has a vanishing ordinary divergence,

$$(\sqrt{-g}T_\mu{}^\nu \xi^\mu)_{,\nu} = 0. \tag{3.10}$$

Since a single conserved quantity is generated from each Killing vector, this method of constructing conservation laws is compatible with the symmetry interpretation of Davis and Moss, but it avoids the ambiguity of the canonical complexes.

In the symmetry method, energy conservation is associated with the existence of a time-like Killing vector. Consider a constant gravitational field. Then a time-like Killing vector exists and one can always choose the coordinates so that the Killing vector has the simple form $\xi^\mu = \delta^\mu_0$. Then for a constant time hypersurface, the conservation law (3.10) leads in the usual manner to the conserved quantity,

$$\int T_0{}^0 \sqrt{-g}\, d^3x. \tag{3.11}$$

For constant isolated systems, all the total integral approaches, including the pseudotensors, give the same value for the total system energy. Thus at first glance, the expression (3.11) seems incorrect because no pseudotensor appears in the integrand. However, the fundamental expansion (1.2) can also be written as,

$$\sqrt{-g}\, T_\mu{}^\nu{}_{;\nu} = (\sqrt{-g}\, T_\mu{}^\nu)_{,\nu} - \frac{1}{2}\sqrt{-g}\, g_{\alpha\beta,\mu} T^{\alpha\beta}. \tag{3.12}$$

For constant fields, the derivatives $g_{\alpha\beta,0}$ are zero. As a result, the pseudotensor divergences $(\sqrt{-g} t_0{}^\nu)_{,\nu}$ must also vanish. In this case, the integrated energy determined by the canonical complexes reduces to exactly expression (3.11). In other words, if m denotes the total system energy then we have

$$m = \int (T_0{}^0 + t_0{}^0)\sqrt{-g}\, d^3x = \int T_0{}^0 \sqrt{-g}\, d^3x. \tag{3.13}$$

Thus the direct Killing vector approach to energy conservation also yields the expected total energy for constant isolated systems.

3.3 Localized Energy

Despite the ambiguous nature of energy in general relativity, many researchers have continued to look for definitions of localized energy. The majority of proposed energy localizations are quasi-local, that is, they only localize energy to within finite regions of spacetime, not to a point. The simplest solutions of the Einstein field equations govern spherically symmetric spacetimes. In addition, spherical symmetry is a reasonable starting assumption for stellar models. Thus the dynamics and energy content of spherically symmetric systems have been extensively studied. Not surprisingly, most quasi-local definitions of energy apply specifically to spherically symmetric systems.

In 1964, Misner and Sharp[39] considered the spherically symmetric gravitational collapse of an ideal fluid. As part of their analysis, they were led to define the mass-energy within a sphere of Schwarzschild radius r as

$$m(r) = \int_0^r {T_0}^0 \, 4\pi r^2 \, dr. \tag{3.14}$$

Misner and Sharp justified their quasi-local definition by showing that $m(r)$ can change in time only to the extent that locally measurable fluid fluxes can be detected at the boundary of the sphere. In addition, spherically symmetric gravitational waves do not exist. Thus uncertainty in gravitational wave energy density is avoided.

The Misner-Sharp localization has provoked considerable discussion. In their influential textbook[40], Misner, Thorne and Wheeler are strong proponents of the total integral interpretation. They concede the validity of the localization (3.14) but dismiss it as an isolated case, unique to spherically symmetric spacetimes. Many authors find this viewpoint contradictory. For example, Cooperstock and Sarracino[17] contend that if energy localization is meaningful for spherically symmetric spacetimes then it is surely meaningful in in spacetimes which are static and hence also free from the effects of gravitational radiation.

It is important to note that the Misner-Sharp localization attributes mass-energy only to those regions where ${T_0}^0$ is non-zero. This interpretation follows from the physical arguments used to justify the localization in the first place. The Misner-Sharp integrand can also be given a completely local interpretation. The line element for a general Schwarzschild spacetime is

$$ds^2 = e^{\nu}\, dt^2 - e^{\lambda}\, dr^2 - r^2\, d\Omega^2, \tag{3.15}$$

where ν and λ are functions of r and t. Thus if the Misner-Sharp localization is written as an integral over proper volume,

$$m(r) = \int_0^r \mathcal{E}\, e^{\lambda/2} 4\pi r^2\, dr, \tag{3.16}$$

then the local energy density $\mathcal{E}$ can be identified as ${T_0}^0 e^{-\lambda/2}$. This interpretation is due to Cooperstock and Sarracino.[17,18].

In 1985, Lynden-Bell and Katz[37] (LBK) suggested another expression for quasi-local energy in spherically symmetric spacetimes. Their approach is based upon the Killing vector conservation laws of the previous section. They postulate, however, that the integrated quantity,

$$\int {T_0}^0 \sqrt{-g}\, d^3x, \tag{3.17}$$

represents just the matter energy in a given spacetime. Thus LBK argue that gravitational field energy is some difference between the quantity (3.17) and the total isolated system energy m. Energy localization is achieved by considering spheres of Schwarzschild radius r, replacing the sphere with a stressed shell and then comparing the integrated energies exterior and interior to the shell. The basic LBK postulate, however, contradicts the agreement between the canonical complexes and the Killing complex (3.17) for constant isolated systems. Moreover, LBK give no arguments against the more reasonable definition of matter energy as the integral of ${T_0}^0$ over proper volume. For these reasons the approach of LBK is suspect.

A survey of quasi-local energy would not be complete without mentioning the twistor based definition of Penrose.[50] Using the spinor formulation of general relativity, Penrose is able to define an energy E for every topological 2-sphere constructed in a given spacetime. Regrettably, an exposition of the Penrose method is far beyond the level of this thesis. Nevertheless, it is useful to compare the predictions of the Penrose method with other localizations. For example, it can be shown that the Penrose definition agrees in the appropriate limits with the Bondi and ADM energies for isolated systems. Also, as we will see in chapter 5, the Penrose approach gives a physically reasonable localization for the charged Reissner-Nordstrom solution.[58]

Given the multitude of approaches to energy in general relativity, it is clear that the paradox presented in the introduction has intrigued and continues to challenge many physicists. In the next chapter, the basic properties of Kerr-Newman spacetimes are presented. Then in chapter 5, armed with the above methods, energy localization in Kerr-Newman spacetimes is addressed.

4

Kerr-Newman Geometry

4.1 Introduction

The Kerr-Newman spacetimes describe the exterior fields of axially symmetric, charged, rotating sources. Since the sources are charged, the exterior spacetime is not pure vacuum. Electromagnetic fields extend out from the source, vanishing only as they approach spatial infinity. In addition, as expected for isolated sources, the Kerr-Newman metrics are asymptotically flat.

The Kerr-Newman spacetimes are characterized by three parameters: m, q, and a. By studying the asymptotic behavior of the fields at spatial infinity, these parameters can be given physically meaningful interpretations. One finds that m is the total isolated system energy, q is the total charge, and a is the angular momentum per unit mass. If one or more of these parameters is set to zero, the Kerr-Newman fields simplify to other well-known solutions. If the angular momentum vanishes, then the Kerr-Newman metric reduces to the spherically symmetric, charged Reissner-Nordström field. If the charge vanishes, then the Kerr-Newman metric reduces to the axially symmetric, neutral Kerr field. Finally, if both the charge and rotation vanish, then the spherically symmetric Schwarzschild solution is recovered.

Despite their important physical interpretation, historically the Kerr and Kerr-Newman solutions were discovered by mathematical conjecture. It was known that the Schwarzschild metric possesses a null congruence that is hypersurface orthogonal. Kerr discovered his uncharged solution by considering generalizations that were not hypersurface orthogonal.[33] Newman then discovered an algebraic trick by which the Kerr metric could be quickly obtained from the Schwarzschild metric.[44] The charged Kerr-Newman fields were found by applying the same trick to the Reissner-Nordström field.[45]

Since their discovery, the geometric and algebraic properties of the Kerr-Newman fields have been extensively studied.[24,34,53] In addition, the metric's global characteristics have been determined.[11,12] The scope of this chapter is limited to establishing those properties of the Kerr-Newman fields that are required for the energy analysis in chapter 5. The remaining sections are organized as follows. In section 4.2, the basic structure of the Kerr-Newman metric is presented. The commonly used coordinate sets are discussed as well as a number of general algebraic identities. Section 4.3 briefly examines singularities and horizons. Then section 4.4 considers the Kerr-Newman electromagnetic field. Computer-aided algebra is used to calculate the energy-momentum tensor for several different coordinate systems. As a final note, it is worth mentioning that the global properties satisfied by the Kerr-Newman solutions are very restrictive. Provided a simple event horizon is assumed, it can be shown that the Kerr-Newman fields are the only stationary, axially symmetric, asymptotically flat, electrovacuum solutions to the coupled Einstein-Maxwell equations.[38]

4.2 Metric Structure

In the Petrov-Pirani classification scheme,* spacetimes are categorized by the degree of degeneracy of their principle null vectors. A principle null vector k^α satisfies the Debever-Penrose equation,

$$k_{[\rho}C_{\alpha]\beta\gamma[\delta}k_{\sigma]}k^\beta k^\gamma = 0, \tag{4.1}$$

where $C_{\alpha\beta\gamma\delta}$ is the Weyl tensor for the particular spacetime.† In general,

*See for example Papapetrou.

†The brackets $[\mu\nu]$ denote the antisymmetric pairing $\mu\nu - \nu\mu$.

a given Weyl tensor has four independent solutions for k^α. If there is any degeneracy in the four vectors, then the corresponding spacetime is said to be algebraically special and the spacetime is further classified according to the degeneracy. A double root is class II, two double roots is class II-D, a triple root is class III, and all four roots equal is class III-N.

The Kerr-Newman fields belong to the class of algebraically special metrics that can be expressed in the form,

$$g_{\mu\nu} = \eta_{\mu\nu} - l_\mu l_\nu, \tag{4.2}$$

where $\eta_{\mu\nu}$ is the Minkowski metric and l_μ is a null vector field. Kerr's original discovery of his uncharged solution came from studying metrics of this special form, perhaps motivated by the observation that the Schwarzschild solution in Eddington form fits this structure. In these metrics, the vector l_μ also defines the degenerate principle null direction. The Kerr-Newman solutions are actually of Petrov-Pirani class II-D. Thus their representation in the metric form (4.2) is not unique, since either of the two doubly degenerate Debever-Penrose directions could serve as l_μ.

The class of special metrics (4.2) has many useful properties. First of all, the contravariant components of the metric tensor also have a simple relationship to the null vector. One finds that

$$g^{\mu\nu} = \eta^{\mu\nu} + l^\mu l^\nu \quad \text{where} \quad l^\mu = g^{\mu\nu} l_\nu = \eta^{\mu\nu} l_\nu. \tag{4.3}$$

Notice that the indices of the null vector can be raised and lowered with just the Minkowski metric. One can also show that the determinant of the metric tensor must satisfy $\sqrt{-g} = 1$. Thus contractions of the form $\Gamma^\nu_{\tau\nu}$ are identically zero and the Ricci tensor simplifies from four terms to two.

There are a number of useful identities involving contractions of the null vector and the Christoffel symbols. To begin, notice that differentiation of the null condition, $l^\mu l_\mu = 0$, generates a second identity, $l^\mu l_{\mu,\nu} = 0$. If this identity is contracted with l^ν, then the vector field $v^\mu \equiv l^\mu{}_{,\nu}\, l^\nu = l^\mu{}_{;\nu}\, l^\nu$ must be orthogonal to l^μ. A vector which is orthogonal to a null vector is either space-like or a multiple of the null vector. It can be shown that for vacuum spacetimes v^μ must be null. Thus in vacuum there must exist a scalar field k such that $v^\mu = kl^\mu$. Note that this condition implies that the null vector field l^μ is geodesic. It also follows that

$$v_\mu l^\mu{}_{,\alpha} = v_{\mu,\alpha} l^\mu = l_{\mu,\sigma\alpha} l^\sigma l^\mu = 0. \tag{4.4}$$

For non-vacuum spacetimes, it is useful to assume that the geodesic condition holds. In particular, it is satisfied by the Kerr-Newman fields.

When the Christoffel symbols are expanded in terms of the metric tensor, the null vector identities help to simplify the expression. The result is

$$\Gamma^\sigma_{\alpha\beta} = -\frac{1}{2}\left[(l^\sigma l_\alpha)_{,\beta} + (l^\sigma l_\beta)_{,\alpha} - \eta^{\sigma\tau}(l_\alpha l_\beta)_{,\tau} - l^\sigma l^\tau (l_\alpha l_\beta)_{,\tau}\right]. \tag{4.5}$$

Now consider the contraction $l^\alpha \Gamma^\sigma_{\alpha\beta}$. Due to the null identities, the factor of l^α will cancel all terms in $\Gamma^\sigma_{\alpha\beta}$ that contain factors of l_α or its first derivative. By inspection, only the second term in (4.5) will remain. Hence,

$$l^\alpha \Gamma^\sigma_{\alpha\beta} = -\frac{1}{2} l^\alpha (l^\sigma l_\beta)_{,\alpha} = -kl^\sigma l_\beta \tag{4.6}$$

We will also be interested in the quantity $l^\beta{}_{,\mu} l^\alpha{}_{,\sigma} \Gamma^\sigma_{\alpha\beta}$. This time the factors of $l^\alpha{}_{,\sigma}$ and $l^\beta{}_{,\mu}$ will cancel all terms in $\Gamma^\sigma_{\alpha\beta}$ with factors of l_α or l_β. Referring to (4.5) we find,

$$l^\beta{}_{,\mu} l^\alpha{}_{,\sigma} \Gamma^\sigma_{\alpha\beta} = -\frac{1}{2} l^\beta{}_{,\mu} l^\alpha{}_{,\sigma}\left[l^\sigma l_{\alpha,\beta} + l^\sigma l_{\beta,\alpha}\right]. \tag{4.7}$$

If the geodesic condition is satisfied, then $l^\alpha{}_{,\sigma}$ can be combined with l^σ to give kl^α. The first term in (4.7) is then zero due to the null identity. In the second term, l^α combines with $l_{\beta,\alpha}$ to give kl_β which in turn cancels with $l^\beta{}_{,\mu}$. Thus provided the null vector is geodesic,

$$l^\beta{}_{,\mu} l^\alpha{}_{,\sigma} \Gamma^\sigma_{\alpha\beta} = 0. \tag{4.8}$$

This relationship, as well as equation (4.6), will prove useful in section 5.2.

In the special form (4.2), the traditional choice of coordinates for the Kerr-Newman solutions are called Cartesian or Kerr-Schild coordinates.[12,61] Specifically, letting $x^\mu = (t, x, y, z)$, the Kerr-Newman line element is

$$ds^2 = d\eta^2 - \frac{(2mr - q^2)r^2}{r^4 + a^2 z^2} \left[dt + \frac{(rx - ay)dx + (ry + ax)dy}{r^2 + a^2} + \frac{z\,dz}{r} \right]^2, \tag{4.9}$$

where $d\eta^2$ denotes the Minkowski line element. Note that r is an intermediate radial parameter, related to the usual Cartesian radius $R^2 = x^2 + y^2 + z^2$ by

$$r^4 - (R^2 - a^2)r^2 - a^2 z^2 = 0. \tag{4.10}$$

Using the relation (4.10), it is easy to verify that the vector field

$$l_\mu = \left[1, \frac{rx - ay}{r^2 + a^2}, \frac{ry + ax}{r^2 + a^2}, \frac{z}{r} \right] \tag{4.11}$$

is indeed null. Recall that if the rotation is removed, the Kerr-Newman fields reduce to spherically symmetric solutions. For example, setting $a = 0$ in (4.9) yields the Cartesian form of the Reissner-Nordström field,

$$ds^2 = d\eta^2 - \frac{2}{R}(m - \frac{q^2}{2R})\,[dt + dR]^2. \tag{4.12}$$

Note that as $a \to 0$, the radial parameter $r \to R$. If the charge q is also set to zero, (4.12) reduces to the Schwarzschild solution expressed in Eddington-Cartesian coordinates.

While the Kerr-Schild coordinates display the algebraically special character of the Kerr-Newman metrics, they do not reflect the axial and stationary symmetries of the fields. An alternative coordinate set, used originally by Kerr, is based upon the radial parameter r. In these "Kerr-Newman" coordinates (u, r, θ, ϕ) the line element is[45]

$$\begin{aligned} ds^2 &= (1 - \rho^{-2}M)\, du^2 + 2\, dr\, du + 2a\rho^{-2}M \sin^2\theta\, d\phi\, du - \rho^2 d\theta^2 \\ &\quad -2a\sin^2\theta\, dr\, d\phi - \sin^2\theta(r^2 + a^2 + a^2\rho^{-2}M\sin^2\theta)\, d\phi^2, \end{aligned} \tag{4.13}$$

where $\rho^2 = r^2 + a^2\cos^2\theta$ and $M = 2mr - q^2$. For reference, the transformation equations between the Kerr-Schild and Kerr-Newman coordinates are

$$t = -u - r, \quad x + iy = (r - ia)e^{i\phi}\sin\theta, \quad \text{and} \quad z = r\cos\theta. \tag{4.14}$$

Notice that the time coordinate u and azimuthal angle ϕ do not appear in the metric (4.13). Thus in these coordinates the stationary and axially symmetric character of the Kerr-Newman fields is self-evident.

As we've seen, if the $a = q = 0$ limit is taken using Kerr-Schild coordinates, then the Schwarzschild metric is obtained in Eddington-Cartesian coordinates. If Kerr-Newman coordinates are used, then Eddington-Polar coordinates result. Not surprisingly, there is another coordinate set, due to Boyer and Lindquist[11], that in the same limit yields the standard Schwarzschild coordinates. The Boyer-Lindquist coordinates are closely related to the Kerr-Newman set, differing only in the time and azimuthal coordinates. In this thesis, however, they are not used.

4.3 Singularities and Horizons

The Kerr-Newman metric in Kerr-Newman coordinates (4.13) apparently has a singularity at $\rho = 0$. This occurs when both

$$r = 0 \quad \text{and} \quad \theta = \pi/2. \tag{4.15}$$

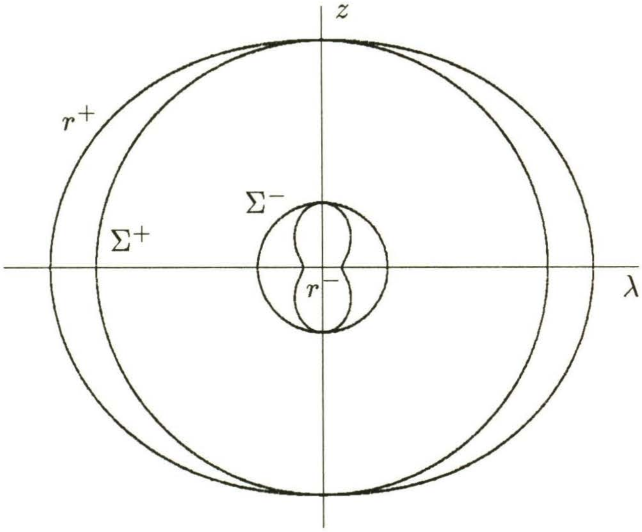

Figure 4.3: Kerr-Newman Ergosphere. The hypersurface $\phi = 0$ is shown in Kerr-Newman coordinate space. The parameter values are $m = 1$, $a = 2/3$, and $q = 1/2$. The horizontal axis $\lambda = r \sin\theta$ should not be confused with the Kerr-Schild coordinate x.

At these points, the Riemann tensor is indeed infinite. Thus the condition $\rho = 0$ is a true curvature singularity in the Kerr-Newman metric. The geometry of this singularity is clearer in Kerr-Schild coordinates. Using the transformation equations (4.14), the relation (4.10) can be rewritten as

$$R^2 = r^2 + a^2 \sin^2\theta. \tag{4.16}$$

Thus the singularity conditions (4.15) in Kerr-Schild coordinates are $z = 0$ and $R = a$, that is, a ring of radius a in the equatorial plane. Consequently $\rho = 0$ is called the ring singularity of the Kerr-Newman metric.

In standard Schwarzschild coordinates, the Schwarzschild black hole horizon can be obtained by setting $g_{00} = 0$. This condition corresponds to the criterion for infinite redshift. It is instructive to apply the same condition to the Kerr-Newman metric (4.13). One obtains two different hypersurfaces,

$$r^{\pm} = m \pm \sqrt{m^2 - a^2\cos^2\theta - q^2}, \tag{4.17}$$

provided $m^2 \geq a^2 + q^2$. The normal vectors to $r^{\pm}$, however, are not null. Thus these hypersurfaces are not characteristic and do not represent horizons. The actual horizons for the Kerr-Newman fields can be derived from first principles by looking for stationary, axially symmetric, characteristic surfaces. Two horizons exist, again provided $m^2 \geq a^2 + q^2$. They are

$$\Sigma^{\pm} = m \pm \sqrt{m^2 - a^2 - q^2}. \tag{4.18}$$

Figure (4.3) illustrates the relationships among the various hypersurfaces. The region between r^+ and Σ^+ is called the Kerr-Newman ergosphere. Observers in this region cannot remain at rest with respect to spatial infinity. Rather all time-like trajectories are necessarily swept around the source in the direction of rotation. The ergosphere plays an important role in the astrophysics of Kerr and Kerr-Newman black holes.

The inner hypersurfaces Σ^- and r^- along with the ring singularity lead to very complicated behavior. For example, closed time-like trajectories can be constructed.[12] The implications of such phenomena, however, will not be addressed in this thesis. Rather it will be assumed that some interior matter distribution excludes these regions from the exterior fields.

4.4 Electromagnetic Field

Every charged Kerr-Newman metric has a companion electromagnetic field tensor. Together they solve the coupled Einstein-Maxwell equations for electrovacuum. In Kerr-Newman coordinates the electromagnetic field tensor is[12]

$$\begin{aligned} F &= \frac{q}{\rho^4}\big[(r^2 - a^2\cos^2\theta)\,du \wedge dr + a\sin^2\theta(r^2 - a^2\cos^2\theta)\,dr \wedge d\phi \\ &\quad + 2a^2 r\cos\theta\sin\theta\,d\theta \wedge du + 2ar(r^2 + a^2)\cos\theta\sin\theta\,d\phi \wedge d\theta\big]. \end{aligned} \tag{4.19}$$

In accordance with the metric symmetry properties, the coordinates u and ϕ do not appear in F. Notice also that the electromagnetic field behaves appropriately in the spherically symmetric limit $a \to 0$. In this case, the field tensor (4.19) reduces to

$$F = \frac{q}{R^2}\, du \wedge dR, \tag{4.20}$$

which is the electrostatic field for a point charge q.

For energy expressions the energy-momentum tensor $T_\mu{}^\nu$ is of more interest than the electromagnetic field tensor. Surprisingly, a search of the literature failed to locate any previous calculations for the components of $T_\mu{}^\nu$. One way to determine $T_\mu{}^\nu$ would be to compute the Einstein tensor from the metric and its derivatives. It is easier, however, to construct the energy-momentum tensor directly from the field tensor. For a general electromagnetic field,

$$T_\mu{}^\nu = \frac{1}{4\pi} F_{\mu\alpha} F^{\alpha\nu} + \frac{1}{16\pi} \delta^\nu_\mu F_{\alpha\beta} F^{\alpha\beta}. \tag{4.21}$$

Notice that no derivatives are required. It is a straightforward but tedious operation to compute $T_\mu{}^\nu$ using (4.21). Thus computer-aided algebra was used to verify the calculations.‡ In order to generate the contravariant components of the field tensor, the inverse metric components are required. These are[45]

$$g^{\mu\nu} = \frac{1}{\rho^2} \begin{bmatrix} -a^2 \sin^2\theta & r^2 + a^2 & 0 & -a \\ r^2 + a^2 & M - r^2 - a^2 & 0 & a \\ 0 & 0 & -1 & 0 \\ -a & a & 0 & -\sin^{-2}\theta \end{bmatrix}. \tag{4.22}$$

Then substitution of the Maxwell tensor (4.19) and the Kerr-Newman metric components into Eq. 4.21 yields the components of $T_\mu{}^\nu$. One finds

$$T_\mu{}^\nu = \frac{q^2}{8\pi\rho^6} \begin{bmatrix} r^2 + a^2 + a^2\sin^2\theta & 0 & 0 & 2a \\ 0 & \rho^2 & 0 & 0 \\ 0 & 0 & -\rho^2 & 0 \\ -2a(r^2 + a^2)\sin^2\theta & 0 & 0 & -(r^2 + a^2 + a^2\sin^2\theta) \end{bmatrix}. \tag{4.23}$$

‡Maple version 4.3.

For most definitions of energy in general relativity, the metric must be asymptotically Minkowskian. Thus the energy-momentum components (4.23) are also required in Kerr-Schild coordinates. These could be obtained by starting with the field tensor $F_\mu{}^\nu$ in Kerr-Schild form and then repeating the above procedure. It is easier, however, to transform the tensor components (4.23) from Kerr-Newman to Kerr-Schild coordinates. Again the calculation is straightforward but lengthly. The result is

$$T_\mu{}^\nu = \frac{q^2}{8\pi\rho^6}\begin{bmatrix} R^2+a^2 & 2ay & -2ax & 0 \\ -2ay & -(R^2+a^2-2x^2) & 2xy & 2xz \\ 2ax & 2xy & -(R^2+a^2-2y^2) & 2yz \\ 0 & 2zx & 2yz & -(R^2-a^2-2z^2) \end{bmatrix} \tag{4.24}$$

where now

$$\rho^4 = (R^2-a^2)^2 + 4a^2z^2. \tag{4.25}$$

In the next chapter, this Kerr-Schild form for the energy-momentum tensor will be used to determine a new expression for the localization of energy in Kerr-Newman spacetimes.

5

Localized Kerr-Newman Energy

The paradox of localized energy in general relativity remains unresolved. For arbitrary spacetimes, a meaningful, generally covariant, expression for energy density has not been found. Yet, in the specific case of spherically symmetric spacetimes, reasonable energy localizations have been constructed. Thus the question arises. Is there any middle ground between the general and spherically symmetric cases? As discussed in section 3.3, Cooperstock and Sarracino[17] argue that if localized energy is meaningful in spherically symmetric spacetimes then it should also be meaningful in less symmetric but constant spacetimes. This conjecture is the motivation behind the hypothesis of this chapter. Can one construct a physically meaningful localization of energy for the axially symmetric, stationary, Kerr-Newman spacetimes.

The energy content of Kerr-Newman spacetimes has been examined previously by several authors. In order to correctly interpret their efforts, it is necessary to first establish some background information. Recall that the Kerr-Newman metric reduces to the spherically symmetric Reissner-Nordström metric when the rotation parameter tends to zero. Thus Reissner-Nordström energy localizations provide a limiting check for Kerr-Newman localizations. Section 5.1 begins with a review of the existing definitions of energy in Reissner-Nordström spacetimes. In addition, the closely related concept of effective Newtonian mass is discussed. Then, in section 5.2, Kerr-Newman energy localization is finally addressed. The approach of Cohen and de Felice[15] and the recent approximations of Virbhadra[61,62] are analyzed in detail. Several problems are identified. As a result, a new exact expression is developed for localized energy in Kerr-Newman spacetimes.

5.1 Reissner-Nordström Energy

The Reissner-Nordström solutions are natural candidates for the study of localized energy. Since the metrics are asymptotically flat and spherically symmetric, the majority of energy definitions apply. In addition, the presence of the electric field means that ${T_0}^0$ is nonzero throughout the spacetime. Thus by constructing 2-surfaces that enclose the source region, and determining the energy contained outside rather than inside the surface, energy localization can be studied without introducing interior solutions. In other words, if m denotes the total energy, and $E(\mathcal{S})$ denotes the energy exterior to the surface $\mathcal{S}$, then the energy interior to the surface is just,

$$m(\mathcal{S}) = m - E(\mathcal{S}). \tag{5.1}$$

This localization procedure is also used for Kerr-Newman localizations.

There is a sequence of heuristic arguments associated with the energy of Reissner-Nordström spacetimes. Far from its source, the metric of an asymptotically flat spacetime is approximately Schwarzschild. In this limit, the time component of the metric tensor can be related to an equivalent Newtonian gravitational potential, that is, $g_{00} \simeq 1 + 2\phi$. In Schwarzschild coordinates, the Reissner-Nordström metric is,

$$ds^2 = e^{\nu}\, dt^2 - e^{-\nu}\, dR^2 - R^2\, d\Omega^2 \quad \text{where} \quad e^{\nu} = 1 - \frac{2m}{R} + \frac{q^2}{R^2}. \tag{5.2}$$

Thus in the Newtonian limit one is led to the association,

$$\phi = -\frac{\mu}{R} \quad \text{where} \quad \mu = m - \frac{q^2}{2R}. \tag{5.3}$$

Since $q^2/2R$ is the classical self-energy of an isolated charge, it seems reasonable to identify μ as the limiting energy content of the Reissner-Nordström field.

There is an alternative interpretation, however, of the preceding argument. If ϕ is the Newtonian potential, then $-\nabla\phi$ is the corresponding field intensity,

$$|\nabla\phi| = \frac{\mathcal{M}}{R^2} \quad \text{where} \quad \mathcal{M} = m - \frac{q^2}{R}. \tag{5.4}$$

The quantity $\mathcal{M}$, which differs from μ by only a factor of two in the second term, determines the numerator in an inverse square law. It is thus called the effective Newtonian mass. If the relativistic equivalence of mass and energy is used, then $\mathcal{M}$ could also be construed as the limiting energy content of the Reissner-Nordström field.

Physicists rely on Newtonian limits to guide their investigations. In the case of Reissner-Nordström energy, however, the difference between μ and $\mathcal{M}$ has led to some confusion. Since μ contains the correct electrostatic energy and is derived from the Newtonian potential energy, it would appear to be the correct limiting energy. Associating the effective mass $\mathcal{M}$ with the energy must be dismissed as an invalid introduction of a relativistic concept in a Newtonian limit. However, there are a number of related calculations that blur this distinction and a number of researchers have confused $\mathcal{M}$ with the limiting energy of Reissner-Nordström spacetimes.

In 1967, de la Cruz and Israel[19] analyzed the motion of charged test particles in a Reissner-Nordström field. For neutral particles and radial trajectories, the geodesic equation reduces to,

$$\frac{d^2R}{ds^2} = -\frac{\mathcal{M}}{R^2} \quad \text{where now} \quad \mathcal{M} = m - \frac{q^2}{R}. \tag{5.5}$$

Thus a particle, at a particular radius R, accelerates *as if* it were moving in a Newtonian gravitational field of effective mass $\mathcal{M}$. Since this is a relativistic result, the use of mass-energy equivalence to interpret the effective mass as the system energy seems less objectionable. However, since $\mathcal{M}$ depends on R, the extraction of the inverse square law is quite arbitrary and de la Cruz and Israel were careful to maintain this distinction.

In 1979, Cohen and Gautreau[14] also analyzed trajectories in Reissner-Nordström spacetime. As part of their analysis, they considered the general localization procedure (5.1) for spheres of radius R. Using the Tolman integral (3.1) for $E(\mathcal{S})$, they found

$$E(R) = \frac{q^2}{R}. \tag{5.6}$$

which agrees with the effective mass $\mathcal{M}$. Since the Tolman integral determines the total system energy, one might conclude that the effective mass represents localized energy. This is incorrect. As Cohen and Gautreau noted, the agreement between the localized Tolman integral and the enclosed effective mass is entirely consistent with a theorem derived by Whittaker[65] in 1935. Whittaker showed that in static, asymptotically flat spacetimes, the effective mass within a 2-surface $\mathcal{S}$ is equal to the Tolman integral taken over the interior of $\mathcal{S}$. In other words, the localized Tolman integral yields effective mass, not energy.[18,51]

In contrast, the quasi-local definitions of Penrose[58], Misner-Sharp[16], and the Killing vector approach all yield

$$m(R) = m - \frac{q^2}{2R}, \tag{5.7}$$

for Reissner-Nordström spacetime. Moreover, in special relativity the integral of ${T_0}^0$ for a static charge gives exactly $q^2/2R$ for the energy outside a sphere of radius R. We conclude that (5.7) is the correct energy localization for the Reissner-Nordström field.

5.2 Kerr-Newman Energy

The effective mass of Kerr-Newman spacetimes was first analyzed in 1984 by Cohen and de Felice[15]. Their analysis was based upon the following observations. Within the symmetry interpretation of Davis and Moss, if a spacetime is constant and asymptotically flat, then the Komar complex reduces to the Møller canonical energy. The Møller energy is in turn equal to the Tolman energy. For static spacetimes, Whittaker's theorem asserts that the localized Tolman integral gives the effective mass within a closed 2-surface. Since the Komar complex is a generally covariant quantity, Cohen and de Felice proposed that for any constant spacetime, the quasi-localized Komar complex still yields the local effective mass. Under this assumption, Cohen and de Felice applied the Komar complex to the Kerr-Newman field. Using Boyer-Lindquist coordinates, they derived the following expression,

$$\mathcal{M} = m - \frac{q^2}{2r} - \frac{q^2(r^2+a^2)}{2ar^2} \arctan\frac{a}{r}, \tag{5.8}$$

In the limit $a \to 0$, this expression does indeed reduce to the Reissner-Nordström effective mass. Thus Kerr-Newman spacetime is consistent with the conjecture that the localized Komar and hence Møller complexes yield effective mass, not energy.

The energy content of Kerr-Newman spacetimes was first considered only recently. In 1990, Virbhadra constructed an energy localization by using the canonical complexes. Since asymptotically Minkowskian coordinates were required, Virbhadra simplified his calculation by first expanding the Kerr-Newman metric in Kerr-Schild coordinates to third order in the rotation parameter a. He then applied the general localization procedure (5.1) to spheres of Kerr-Schild radius R, that is,

$$m(r) = m - E(R) \quad \text{where} \quad E(R) = \int_R^\infty \Theta_0{}^0 \, d^3x. \tag{5.9}$$

Virbhadra found that the complexes of Einstein and Landau-Lifshitz both gave

$$E(R) = \frac{q^2}{2R}\left[1 + \frac{2}{3}\left(\frac{a}{R}\right)^2\right], \tag{5.10}$$

while the complex of Møller gave exactly twice this value.* In the limit $a \to 0$, the localization (5.10) reduces to $q^2/2R$. Thus the localized Einstein and Landau-Lifshitz complexes do appear to yield energy, at least for Kerr-Schild coordinates. In contrast, the localized Møller energy again appears to yield effective mass.

The Virbhadra approximations suggest that the canonical complexes of Einstein and Landau-Lifshitz determine the same Kerr-Newman energy localization. To check this result, computer-aided algebra† was used to extend the calculations to seventh order in the rotation parameter. The higher order terms confirm the previous relationships. The localized Einstein and Landau-Lifshitz energies agreed, both giving

$$E(R) = \frac{q^2}{2R}\left[1 + \frac{2}{3}\left(\frac{a}{R}\right)^2 + \frac{3}{5}\left(\frac{a}{R}\right)^4 + \frac{4}{7}\left(\frac{a}{R}\right)^6\right], \tag{5.11}$$

while that of Møller was exactly twice this value.

In hindsight, this agreement in the canonical localizations is easily explained. It was shown earlier in section 3.2 that the canonical pseudotensor divergences $(\sqrt{-g}t_0{}^{\nu})_{,\nu}$ vanish in constant spacetimes. Furthermore, in Kerr-Schild coordinates $\sqrt{-g} = 1$. Under these conditions, both the localized Einstein and Landau-Lifshitz energy, as well as the Killing vector energy, all reduce to the same expression. That is,

$$E(R) = \int_R^{\infty} T_0{}^0\, d^3x. \tag{5.12}$$

*Virbhadra also explicitly calculated the Tolman complex. He was apparently unaware that it is identical to the Einstein complex.

†Maple, version 4.3

In contrast, the Møller energy is related to the energy-momentum tensor through the Tolman integral. Furthermore, the energy-momentum tensor of electromagnetic fields is traceless. Thus the localized Tolman integral also simplifies,

$$\int_R^\infty ({T_0}^0 - {T_i}^i)\sqrt{-g}\, d^3x = 2\int_R^\infty {T_0}^0\, d^3x, \tag{5.13}$$

which, as expected, is exactly twice the localization (5.12).

The preceding analysis suggests that the correct energy localization for the Kerr-Newman field is just the integral of ${T_0}^0$ over coordinate volume, using Kerr-Schild coordinates. The Kerr-Schild form of ${T_0}^0$ was determined earlier in section 4.4. Combining equations (4.24) and (4.25) gives

$${T_0}^0 = \frac{q^2}{8\pi}\frac{R^2+a^2}{[(R^2-a^2)^2+4a^2z^2]^{\frac{3}{2}}}. \tag{5.14}$$

Thus all that remains is the integration itself. Since the localization is over spheres of Kerr-Schild radius R, it is easier to use the spherical-polar variables R, θ, ϕ instead of the Kerr-Schild coordinates x, y, z to perform the integration. In this case, the localization (5.12) becomes,

$$E(R) = \frac{q^2}{8\pi}\int_R^\infty \int_0^\pi \int_0^{2\pi} \frac{R^2+a^2}{[(R^2-a^2)^2+4a^2R^2\cos^2\theta]^{\frac{3}{2}}} R^2 \sin\theta\, d\phi\, d\theta\, dR. \tag{5.15}$$

Despite its appearance, this integral can be determined exactly. The integration over ϕ contributes a factor of 2π. The integration over θ requires only simple trigonometric substitutions and the integration over R needs only a single partial fraction expansion. The final result is,

$$E(R) = \frac{q^2}{8}\left(\frac{2R}{R^2-a^2} + \frac{1}{a}\ln\frac{R+a}{R-a}\right), \qquad R > a. \tag{5.16}$$

This is our new exact expression for localized Kerr-Newman energy.

The Virbhadra approximations were based upon expanding the Kerr-Schild coordinates in powers of a/R. If the localization (5.16) is expanded, one obtains

$$E(R) = \frac{q^2}{2R}\left[1 + \sum_{n=1}^{\infty} \frac{n+1}{2n+1}\left(\frac{a}{R}\right)^{2n}\right]. \tag{5.17}$$

This series agrees with the previously determined expansions (5.11). Note also that this new localization clearly gives the correct Reissner-Nordström energy in the limit as $a \to 0$.

As we have seen, the vanishing of the nonconservative term $\mathcal{F}_\mu$ plays a significant role in many of the approaches to energy in general relativity. For example, the vanishing of $\mathcal{F}_0$ implies that the pseudotensor divergences $(\sqrt{-g}t_0{}^\nu)_{,\nu}$ also vanish. In constant spacetimes, it is this condition that ties the Killing vector energy to the Einstein canonical complex and led to our choice of expression (5.12) for the localized Kerr-Newman energy. It is thus important to determine under what conditions the various components of $\mathcal{F}_\mu$ are zero. To this end, we now evaluate $\mathcal{F}_\mu$ for the general class of algebraically special metrics presented in section 4.2.

To begin the calculation, we choose the form of the nonconservative term given in section 3.2,

$$\mathcal{F}_\mu = -\frac{1}{2}\sqrt{-g}\, g_{\alpha\beta,\mu} T^{\alpha\beta}. \tag{5.18}$$

The Einstein field equations are now used to express the energy-momentum tensor in terms of the Ricci tensor. At the same time, the metric tensor is replaced with the degenerate metric form (4.2). Noting that $\sqrt{-g} = 1$ we find that,

$$\mathcal{F}_\mu = \frac{1}{16\pi}(l_\alpha l_\beta)_{,\mu}\left[R^{\alpha\beta} - \frac{1}{2}(\eta^{\alpha\beta} - l^\alpha l^\beta)R\right]. \tag{5.19}$$

The terms containing the Ricci scalar drop out because of the null vector identity, $l^\alpha l_{\alpha,\mu} = 0$. Next recall that since $\sqrt{-g} = 1$, contractions of the form $\Gamma^\nu_{\tau\nu}$ are identically zero. Thus when the Ricci tensor $R_{\alpha\beta}$ is expanded in terms of Christoffel symbols, only two of the four terms remain. Hence,

$$\mathcal{F}_\mu = \frac{1}{16\pi}(l^\alpha l^\beta)_{,\mu}\left[\Gamma^\sigma_{\alpha\beta,\sigma} - \Gamma^\rho_{\alpha\sigma}\Gamma^\sigma_{\beta\rho}\right]. \tag{5.20}$$

At this point, the contraction (4.6) is particularly useful, namely $l^\alpha\Gamma^\sigma_{\alpha\beta} = kl^\sigma l_\beta$, where k is a scalar field. Note that this contraction assumes that the null vector l^μ is geodesic. Consider the term in (5.20) containing the product of Christoffel symbols. Using the symmetry in α and β we have,

$$(l^\alpha l^\beta)_{,\mu}\Gamma^\rho_{\alpha\sigma}\Gamma^\sigma_{\beta\rho} = 2l^\alpha l^\beta{}_{,\mu}\Gamma^\rho_{\alpha\sigma}\Gamma^\sigma_{\beta\rho} = 2kl^\beta{}_{,\mu}l^\rho l_\sigma\Gamma^\sigma_{\beta\rho} = 2k^2 l^\beta{}_{,\mu}l_\sigma l^\sigma l_\beta = 0. \tag{5.21}$$

Thus only the first term in (5.20) remains. Taking the σ derivative outside and again using the symmetry in α and β leaves,

$$\mathcal{F}_\mu = \frac{1}{8\pi}(l^\alpha l^\beta{}_{,\mu}\Gamma^\sigma_{\alpha\beta})_{,\sigma} - \frac{1}{8\pi}l^\alpha l^\beta{}_{,\mu\sigma}\Gamma^\sigma_{\alpha\beta} - \frac{1}{8\pi}l^\alpha{}_{,\sigma}l^\beta{}_{,\mu}\Gamma^\sigma_{\alpha\beta}. \tag{5.22}$$

The third term vanishes immediately due to the identity (4.8). In the first two terms, the contraction (4.6) is applied once more. Then the first term vanishes due to the null identity and the second vanishes due to the geodesic condition (4.4). Thus, we have the remarkable result,

$$\mathcal{F}_\mu \equiv 0. \tag{5.23}$$

That is, the nonconservative term for all algebraically special metrics of the form (4.2), with geodesic null vectors, is identically zero.

The algebraically special class of metrics considered above includes the Schwarzschild, Kerr, Reissner-Nordström, and Kerr-Newman spacetimes, all in asymptotically Minkowskian coordinates. These spacetimes are the test cases for all existing definitions of energy in general relativity. The vanishing of the nonconservative term poses an intriguing question. Do any of the existing energy definitions rely on this coincidence to justify their validity?

6

Summary and Discussion

If localized energy is meaningful in spherically symmetric spacetimes then it should also be meaningful in less symmetric but constant spacetimes. Working from this hypothesis, the localization of energy in the axially symmetric Kerr-Newman spacetimes was investigated. In order to guide the investigation, existing localizations in the related but spherically symmetric Reissner-Nordström spacetimes were analyzed. It was observed that the total energy definitions of Komar, Møller and Tolman do not generate the expected localized energy for the Reissner-Nordström fields. Rather, they generate the Reissner-Nordström effective mass. In contrast, the total energy definitions of Einstein, Landau-Lifshitz and the Killing vector approach as well as the quasi-local definitions of Misner-Sharp and Penrose do produce the expected localized energy.

Following the Reissner-Nordström analysis, the Kerr-Newman spacetimes were considered. First the effective mass approach of Cohen and de Felice and the approximate energy calculations of Virbhadra were examined. Both methods confirmed the general localization behavior displayed by the Reissner-Nordström fields. The localized Einstein and Landau-Lifshitz complexes appear to yield localized energy while the localized Møller and Komar complexes appear to yield effective mass. It was then shown that this pattern in localizations can be explained by straightforward analysis of the various complexes. Furthermore, this analysis suggests that the correct energy localization for the Kerr-Newman field is just the integral of ${T_0}^0$ over coordinate volume, using

Kerr-Schild coordinates. The resulting localization is

$$E(R) = \frac{q^2}{8}\left(\frac{2R}{R^2 - a^2} + \frac{1}{a}\ln\frac{R+a}{R-a}\right), \qquad R > a. \tag{6.1}$$

This new exact expression for the energy content of Kerr-Newman fields was shown to be in complete agreement with the approximate calculations of Virbhadra. In addition, the localization procedure used appears to be valid for all algebraically special metrics of the form

$$g_{\mu\nu} = \eta_{\mu\nu} - l_\mu l_\nu, \tag{6.2}$$

where $\eta_{\mu\nu}$ is the Minkowski metric and l_μ is a null geodesic vector field.

As as final note, it should not be forgotten that the fundamental paradox remains. Is the concept of energy truly meaningful in general relativity. The efforts described in this thesis suggest that reasonable energy localizations can be constructed in sufficiently well-behaved spacetimes. However, an assertion due to Synge comes to mind[57],

> " ... thou shalt not confuse two entirely different sets of concepts even though they lead to similar conclusions."

Given the ambiguous nature of energy in general relativity, Synge's warning must be acknowledged. Applying the special relativistic label 'energy' to quantities that behave appropriately only in special spacetimes may not be justified. Nevertheless, the quest is certainly admirable. After all, the foundations of gravitational radiation and quantum gravity may well depend on its resolution.

References

[1] R. Adler, M. Bazin and M. Schiffer, *Introduction To General Relativity*, (McGraw-Hill Book Company, Toronto, 1975).

[2] R. Arnowitt and S. Deser, "Quantum Theory of Gravitation: General Formulation and Linearized Theory", *Physical Review* **113**, 745 (1959).

[3] R. Arnowitt, S. Deser and C. W. Misner, "Dynamical Structure and Definition of Energy in General Relativity", *Physical Review* **116**, 1322 (1959).

[4] R. Arnowitt, S. Deser and C. W. Misner, "Canonical Variables for General Relativity", *Physical Review* **117**, 1595 (1960).

[5] R. Arnowitt, S. Deser and C. W. Misner, "Energy and the Criteria for Radiation in General Relativity", *Physical Review* **118**, 1100 (1960).

[6] A. Ashtekar and A. Magnon-Ashtekar, "Energy-Momentum in General Relativity", *Physical Review Letters* **43**, 181 (1979).

[7] P. G. Bergmann, "NonLinear Field Theories", *Physical Review* **75**, 680 (1949).

[8] P. G. Bergmann and R. Schiller, "Classical and Quantum Field Theories in the Lagrangian Formalism", *Physical Review* **89**, 4 (1953).

[9] P. G. Bergmann, "Conservation Laws in General Relativity as the Generators of Coordinate Transformations", *Physical Review* **112**, 287 (1958).

[10] H. Bondi, M. G. J. van der Burg and A. W. K. Metzner, "Gravitational waves in general relativity VII: Waves from axi-symmetric isolated systems", *Proceedings of the Royal Society of London A* **269**, 21 (1962).

[11] R. H. Boyer and R. W. Lindquist, "Maximal Analytic Extension of the Kerr Metric", *Journal of Mathematical Physics* **8**, 265 (1967).

[12] B. Carter, "Global Structure of the Kerr Family of Gravitational Fields", *Physical Review* **174**, 1559 (1968).

[13] J. M. Cohen, "Angular Momentum and the Kerr Metric", *Journal of Mathematical Physics* **9**, 905 (1968).

[14] J. M. Cohen and R. Gautreau, "Naked singularities, event horizons, and charged particles", *Physical Review D* **19**, 2273 (1979).

[15] J. M. Cohen and F. de Felice, "The total effective mass of the Kerr-Newman metric", *Journal of Mathematical Physics* **25**, 992 (1984).

[16] F. I. Cooperstock and V. de la Cruz, "Sources for the Reissner-Nordström Metric", *General Relativity and Gravity* **9**, 835 (1978).

[17] F. I. Cooperstock and R. S. Sarracino, "The localization of energy in general relativity", *Journal of Physics A: Mathematical and General* **11**, 877 (1978).

[18] F. I. Cooperstock, R. S. Sarracino and S. S. Bayin, "The localization of energy in general relativity: II", *Journal of Physics A: Mathematical and General* **14**, 181 (1981).

[19] V. de la Cruz and W. Israel, "Gravitational Bounce", *Nuovo Cimento* **51 A**, 744 (1967).

[20] W. R. Davis and M. K. Moss, "On the Conservation Laws of the Theory of General Relativity", *Nuovo Cimento* **27**, 1492 (1963).

[21] W. R. Davis and M. K. Moss, "Conservation Laws of the Theory of General Relativity I", *Nuovo Cimento* **38**, 1531 (1965).

[22] W. R. Davis and M. K. Moss, "Conservation Laws of the Theory of General Relativity II", *Nuovo Cimento* **38**, 1558 (1965).

[23] W. R. Davis, *Classical Fields, Particles and the Theory of Relativity*, (Gordon and Breach, Science Publishers Inc., New York, 1970).

[24] G. C. Debney, R. P. Kerr and A. Schild, "Solutions of the Einstein and Einstein-Maxwell Equations", *Journal of Mathematical Physics* **10**, 1842 (1969).

[25] A. Einstein, "Die Grundlage der allgemeinin Relativitätstheorie", *Annalen der Physik* **49**, 769 (1916).

[26] P. Freud, "Über die Ausdrüke der Gesamtenergie und des Gesamtimpulses eines Materiellen Systems in der Allgemeinen Relativitätstheorie", *Annals of Mathematics* **40**, 417 (1939).

[27] R. Geroch, "Energy Extraction", *Annals of the New York Academy of Sciences* **224**, 108 (1973).

[28] J. N. Goldberg, "Strong Conservation Laws and Equations of Motion in Covariant Field Theories", *Physical Review* **89**, 263 (1953).

[29] J. N. Goldberg, "Conservation Laws in General Relativity", *Physical Review* **111**, 315 (1958).

[30] G. T. Horowitz and P. Tod, "A Relation between Local and Total Energy in General Relativity", *Communications in Mathematical Physics* **85**, 429 (1982).

[31] G. T. Horowitz and M. J. Perry, "Gravitational Energy Cannot Become Negative", *Physical Review Letters* **48**, 371 (1982).

[32] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, (Pergamon Press, Oxford, 1975).

[33] R. P. Kerr, "Gravitational Field of a Spinning Mass as an Example of Algebraically Special Metrics", *Physical Review Letters* **11**, 237 (1963).

[34] R. P. Kerr and A. Schild, "A New Class of Vacuum Solutions of the Einstein Field Equations", *Atti del Convegno Sulla Relativita Generale: Problemi di Energia e Onde Gravitazionali* **II**, 222 (Proceedings of the Fourth Centenary of Galileo's Birth, Firenze, 1965).

[35] A. Komar, "Covariant Conservation Laws in General Relativity", *Physical Review* **113**, 934 (1959).

[36] R. Kulkarni, V. Chellathurai and N. Dadhich, "The effective mass of the Kerr spacetime", *Classical and Quantum Gravity* **5**, 1443 (1988)

[37] D. Lynden-Bell and J. Katz, "Gravitational field density for spheres and black holes", *Monthly Notices of the Royal Astronomical Society* **213**, 21p (1985).

[38] P. O. Mazur, "Proof of uniqueness of the Kerr-Newman black hole solution", *Journal of Physics A: Mathematical and General* **15**, 3173 (1982).

[39] C . W. Misner and D. H. Sharp, "Relativistic Equations for Adiabatic, Spherically Symmetric Gravitational Collapse", *Physical Review* **136**, 571 (1964).

[40] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, (W. H. Freeman and Company, San Francisco, 1973).

[41] C. Møller, "On the Localization of the Energy of a Physical System in the General Theory of Relativity", *Annals of Physics* **4**, 347 (1958).

[42] C. Møller, "Further Remarks on the Localization of the Energy in the General Theory of Relativity", *Annals of Physics* **12**, 118 (1961).

[43] N. Nakanishi, "De Donder Condition and the Gravitational Energy-Momentum Pseudotensor in General Relativity", *Progress of Theoretical Physics* **75**, 1351 (1986).

[44] E. T. Newman and A. I. Janis, "Note on the Kerr Spinning-Particle Metric", *Journal of Mathematical Physics* **6**, 915 (1965).

[45] E. T. Newman, et. al., "Metric of a Rotating, Charged Mass", *Journal of Mathematical Physics* **6**, 918 (1965).

[46] N. Nissani and E. Leibowitz, "Global Energy-Momentum Conservation in General Relativity", *International Journal of Theoretical Physics* **28**, 235 (1989).

[47] A. Papapetrou, *Lectures On General Relativity*, (D. Reidel Publishing Company, Boston, 1974).

[48] P. C. Peters, "Gravitational Radiation and the Motion of Two Point Masses", *Physical Review* **136**, 1224 (1964).

[49] P. C. Peters, "Where is the energy stored in a gravitational field?", *American Journal of Physics* **49**, 564 (1981).

[50] R. Penrose, "Quasi-local mass and angular momentum in general relativity", *Proceedings of the Royal Society of London A* **381**, 53 (1982).

[51] R. Penrose, "Energy and Its Definition in General Relativity", *Annals of the New York Academy of Sciences* **470**, 136 (1986).

[52] R. K. Sachs, "Gravitational waves in general relativity VIII: Waves in asymptotically flat spacetime", *Proceedings of the Royal Society of London A* **270**, 103 (1962).

[53] M. M. Schiffer, R. J. Adler, J. Mark and C. Sheffield, "Kerr geometry as complexified Schwarzschild geometry," *Journal of Mathematical Physics* **14**, 52 (1973).

[54] R. Schoen and S. Yau, "Positivity of the Total Mass of a General SpaceTime", *Physical Review Letters* **12**, 1457 (1979).

[55] W. T. Shaw, "Twistor theory and the energy-momentum and angular momentum of the gravitational field at spatial infinity", *Proceedings of the Royal Society of London A* **390**, 191 (1983).

[56] D. E. Soper, *Classical Field Theory*, (John Wiley & Sons, Toronto, 1976).

[57] J. L. Synge, "What is Einstein's Theory of Gravitation?", in *Perspectives in Geometry and Relativity*, (Indiana University Press, Bloomington, 1966).

[58] K. P. Tod, "Some examples of Penrose's quasi-local mass construction", *Proceedings of the Royal Society of London A* **388**, 457 (1983).

[59] R. C. Tolman, "On the Use of the Energy-Momentum Principle in General Relativity", *Physical Review* **35**, 875 (1930).

[60] A. Trautman, "Conservation Laws in General Relativity", in *Gravitation, an Introduction to Current Research*, (Wiley, New York, 1962).

[61] K. S. Virbhadra, "Energy associated with a Kerr-Newman black hole", *Physical Review D* **41**, 1086 (1990).

[62] K. S. Virbhadra, "Energy distribution in Kerr-Newman spacetime in Einstein's as well as Møller's prescriptions", *Physical Review D* **42**, 2919 (1990).

[63] R. M. Wald, *General Relativity*, (The University of Chicago Press, Chicago, 1984).

[64] S. Weinberg, *Gravitation and Cosmology*, (John Wiley & Sons, New York, 1972).

[65] E. T. Whittaker, "On Gauss' Theorem and the concept of Mass in General Relativity", *Proceedings of the Royal Society of London A* **149**, 384 (1935).

[66] E. Witten, "A New Proof of the Positive Energy Theorem", *Communications in Mathematical Physics* **80**, 381 (1981).

A

Integral Conservation Laws

It is simple to show how a vanishing ordinary divergence leads to an integral conservation law.[32] Consider a tensor field $T_\mu{}^\nu$ that is constrained in space-time such that its non-zero regions can be enclosed in a simple 4-cylinder U. For example, see Figure A. Let $\Omega \subset U$ denote the 4-volume defined by the intersection of U with two constant time hypersurfaces t_i and t_f. Label the hypersurface bounding Ω by Σ. Finally, let $\partial_\nu T_\mu{}^\nu = 0$ denote the divergence condition.

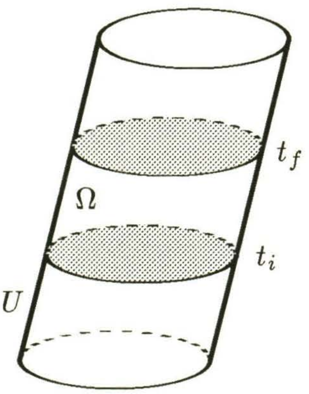

Figure A: Integral Conservation Laws. A timelike 4-cylinder U is shown with one space dimension supressed. Outside of U, $T_\mu{}^\nu$ is assumed identically zero. Two constant time hypersurfaces t_i and t_f define the 4-volume Ω.

Since the divergence is zero, the integral of $\partial_\nu T_\mu{}^\nu$ over Ω is also zero. Applying Gauss's law, this means

$$0 = \oint_\Sigma T_\mu{}^\nu \, d\Sigma_\nu = \int_{t_f} T_\mu{}^0 \, d\Sigma_0 - \int_{t_i} T_\mu{}^0 \, d\Sigma_0, \qquad \text{(A.1)}$$

since the only nonzero contributions come from the constant time hypersurfaces inside Ω. However, t_i and t_f are arbitrary. Thus one concludes that the vector

$$P_\mu \equiv \int\limits_{t=const} T_\mu{}^0 \, d^3x \tag{A.2}$$

is conserved.

The argument can be easily generalized. If the walls of U are extended to spatial infinity, then provided $T_\mu{}^\nu$ vanishes at spatial infinity, $T_\mu{}^\nu$ need not be bounded. Similarly, the integration region need not be restricted to a constant time hypersurface. Any spacelike hypersurface $\bar{\Sigma}$ that includes all of 3-space will do. In this case,

$$P_\mu \equiv \int\limits_{\bar{\Sigma}} T_\mu{}^\nu \, d\bar{\Sigma}_\nu, \tag{A.3}$$

defines the conserved four vector.

B

First Order Gravitational Lagrangian

In the formulation of generally covariant field theories, the Lagrangian is usually chosen to be a scalar density of weight one. This choice ensures that the resulting action is a scalar under coordinate transformations. Consequently, the free gravitational field is usually described by the Hilbert Lagrangian,

$$\mathcal{L}_G = \frac{1}{16\pi}\sqrt{-g}\,R, \tag{B.1}$$

where R is the Ricci scalar. In this formulation, the components of the metric tensor $g_{\mu\nu}$ serve as the field variables.* Thus the Hilbert Lagrangian is second order in the derivatives of the gravitational field.

In general, the Euler-Lagrange equations contain derivatives one order higher than those that appear in the Lagrangian. Since the Einstein field equations are already second order, it is not surprising that the Hilbert Lagrangian can be modified to eliminate the second order derivatives. By expanding the Ricci tensor in Christoffel symbols, a manipulation of terms gives the following expression[56],

$$\sqrt{-g}\,R = \left[\sqrt{-g}(g^{\alpha\beta}\Gamma^{\lambda}_{\alpha\beta} - g^{\alpha\lambda}\Gamma^{\gamma}_{\alpha\gamma})\right]_{,\lambda} + \sqrt{-g}\,g^{\alpha\beta}\left[\Gamma^{\gamma}_{\alpha\beta}\Gamma^{\delta}_{\gamma\delta} - \Gamma^{\gamma}_{\alpha\delta}\Gamma^{\delta}_{\beta\gamma}\right]. \tag{B.2}$$

The first term on the right hand side is an ordinary divergence. When inserted into the action integral, this term can be converted using Gauss's law into an integral over the bounding hypersurface. Under a variation of the field variables $\delta g_{\mu\nu}$, this surface integral vanishes. Thus the divergence term does

*Several different sets of field variables can be used with the Hilbert Lagrangian. Choices equivalent to $g_{\mu\nu}$ are $g^{\mu\nu}$, $\sqrt{-g}\,g_{\mu\nu}$, or $\sqrt{-g}\,g^{\mu\nu}$. Alternatively, in a method due to Palatini, the components of both the metric $g_{\mu\nu}$ and the connection $\Gamma^{\sigma}_{\mu\nu}$ can be treated as independent field variables.

not contribute to the action. As a result one defines the alternative Lagrangian,

$$\mathcal{L}'_G = \frac{1}{16\pi}\sqrt{-g}\, g^{\alpha\beta}\left[\Gamma^{\gamma}_{\alpha\beta}\Gamma^{\delta}_{\gamma\delta} - \Gamma^{\gamma}_{\alpha\delta}\Gamma^{\delta}_{\beta\gamma}\right]. \tag{B.3}$$

This new Lagrangian $\mathcal{L}'_G$ contains only first order derivatives of the gravitational field.

Since the original Hilbert Lagrangian is a scalar density, the gravitational action is a scalar under coordinate transformations. The alternative Lagrangian $\mathcal{L}'_G$, however, is not a scalar density. Thus in order to use the Noether theorems, the coordinate transformation properties of $\mathcal{L}'_G$ must be determined. Consider the general coordinate transformation,

$$\bar{x}^{\mu} = x^{\mu} + \epsilon(x)\xi^{\mu}(x). \tag{B.4}$$

where $\bar{x}^{\mu}$ denotes the transformed coordinates. It can be shown[56], to first order in ϵ, that

$$\bar{\delta}\mathcal{L}'_G \equiv \mathcal{L}'_G(\bar{x}) - \mathcal{L}'_G(x) = -\left[\mathcal{L}'_G\epsilon\xi^{\mu} + Q_{\lambda}{}^{\mu\sigma}(\epsilon\xi^{\lambda})_{,\sigma}\right]_{,\mu} \tag{B.5}$$

where

$$Q_{\lambda}{}^{\mu\sigma} \equiv \frac{1}{16\pi}\left[\sqrt{-g}(\delta^{\mu}_{\lambda}g^{\tau\sigma} - \delta^{\tau}_{\lambda}g^{\mu\sigma})\right]_{,\tau} \tag{B.6}$$

As required, the first order gravitational Lagrangian $\mathcal{L}'_G$ transforms as a divergence.

As a final note, observe that the functions $Q_{\lambda}{}^{\mu\sigma}$ have a vanishing divergence with respect to the second index, that is, $Q_{\lambda}{}^{\mu\sigma}{}_{,\mu} = 0$. This identity is used in chapter 2 to obtain the Tolman superpotential.

C

The General Covariance Identities

The Noether theorems are a general mathematical formalism for relating symmetries in Lagrangians to conserved quantities of the motion. In particular, the theorems show how to construct conservation laws for systems which are invariant under coordinate transformations. Thus they play an important role in the study of energy in general relativity.

In this appendix, the specific Noether results for generally covariant systems are derived.* These results are often called the general covariance identities. The appendix consists of three sections. In the first section, the necessary background notation is established. This includes a review of the relationship between the invariance of an action and the transformation properties of the Lagrangian. The general Noether current is also presented. In the second section, the specific transformation properties of scalar densities are reviewed. In the third section, the general covariance identities for general relativity are finally derived.

C.1 Lagrangian Invariance

Consider a general physical system, consisting of a number of interacting fields, including gravity. Let $\phi_J(x), J = 1 \ldots N$, represent all of the tensor components of all of the individual fields, including $g_{\mu\nu}$. Let the system dynamics be determined by an action principle,

$$A = \int\limits_{\Omega} \mathcal{L}(\phi_J, \phi_{J,\mu}, \phi_{J,\mu\nu}; x)\, d^4x, \tag{C.1}$$

*The derivations presented here are based upon treatments given by Soper[56], Davis[23], and Trautman[60].

that is, the equations of motion for the various fields can be obtained by varying the action A with respect to some or all of the ϕ_J. The Lagrangian is assumed to contain at most second order derivatives of the fields. At this point, however, no assumption is made regarding the transformation properties of the action or Lagrangian. For example, A is not necessarily a scalar with respect to coordinate transformations.

The Noether theorems consider transformations of the coordinates and field variables that form a continuous group $G(\epsilon)$, where ϵ labels the group members. For example, if the transformed coordinates and fields are denoted by $\bar{x}^\mu$ and $\bar{\phi}_J$, then the group of transformations can be written as,

$$\bar{x}^\mu = X^\mu(x;\epsilon) \quad \text{and} \quad \bar{\phi}_J(\bar{x}) = \Phi_J(\phi_K(x), x; \epsilon), \tag{C.2}$$

where $\epsilon = 0$ denotes the identity transformation. The functions X^μ and Φ_J are assumed to be invertible and continuously differentiable to an appropriate order.

Suppose we have some functional F which depends upon the original fields ϕ_J and coordinates x. We wish to characterize any symmetries of F that hold over the full set of transformations $G(\epsilon)$. Since the transformations form a continuous group, it is sufficient to consider the infinitesimal transformations produced by small variations in ϵ about $\epsilon = 0$. Under the infinitesimal transformations, one can consider two different variations in F,

$$\delta F \equiv F(\bar{\phi}_J, \bar{x}) - F(\phi_J, x) \quad \text{and} \quad \bar{\delta} F \equiv F(\bar{\phi}_J, x) - F(\phi_J, x). \tag{C.3}$$

The first variation δ measures the change in F at a fixed point in spacetime while the variation $\bar{\delta}$ measures the change in F at a fixed coordinate value. Note that the variation $\bar{\delta}$ commutes with differentiation.

An action A is said to be invariant under a transformation group $G(\epsilon)$ if $\bar{\delta} A = 0$ for all ϵ and for all fields ϕ_J. In other words, A is invariant if the variation of A with respect to variations in ϵ is zero. It is important to realize that invariance of an action does not require invariance of the Lagrangian. By definition,

$$\bar{\delta} A = \int_{\Omega} \bar{\delta} \mathcal{L} \, d^4 x. \tag{C.4}$$

Thus in order for $\bar{\delta} A = 0$, it is sufficient that the variation in the Lagrangian be a divergence of some functional $\bar{\delta} B^{\mu}$ which vanishes on the boundary of Ω,

$$\bar{\delta} \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \epsilon} \delta \epsilon = \bar{\delta} B^{\mu}{}_{,\mu}. \tag{C.5}$$

The Noether conservation laws result by assuming that the Lagrangian transformation condition (C.5) holds for a particular group of transformations $G(\epsilon)$. The variation in $\mathcal{L}$ is then calculated directly in terms of the induced variations in the fields. That is,

$$\bar{\delta} \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi_J} \bar{\delta} \phi_J + \frac{\partial \mathcal{L}}{\partial \phi_{J,\mu}} \bar{\delta} \phi_{J,\mu} + \frac{\partial \mathcal{L}}{\partial \phi_{J,\mu\nu}} \bar{\delta} \phi_{J,\mu\nu}. \tag{C.6}$$

If the terms corresponding to the Euler-Lagrange equations are collected together, the expansion (C.6) can be rewritten as

$$\bar{\delta} \mathcal{L} = L^J \, \bar{\delta} \phi_J + \left[\left(\frac{\partial \mathcal{L}}{\partial \phi_{J,\mu}} - \partial_\nu \frac{\partial \mathcal{L}}{\partial \phi_{J,\mu\nu}} \right) \bar{\delta} \phi_J + \frac{\partial \mathcal{L}}{\partial \phi_{J,\mu\nu}} \bar{\delta} \phi_{J,\nu} \right]_{,\mu}. \tag{C.7}$$

Here L^J denotes the field equations. The remarkable result of this procedure is that the remaining terms form a single divergence. Thus when the two expressions (C.5) and (C.7) for $\bar{\delta} \mathcal{L}$ are equated, one obtains,

$$L^J \, \bar{\delta} \phi_J + J^{\mu}{}_{,\mu} = 0, \tag{C.8}$$

where

$$J^{\mu} \equiv \left(\frac{\partial \mathcal{L}}{\partial \phi_{J,\mu}} - \partial_\nu \frac{\partial \mathcal{L}}{\partial \phi_{J,\mu\nu}} \right) \bar{\delta} \phi_J + \frac{\partial \mathcal{L}}{\partial \phi_{J,\mu\nu}} \bar{\delta} \phi_{J,\nu} - \bar{\delta} B^{\mu}. \tag{C.9}$$

If the fields satisfy the Euler-Lagrange equations, then the quantity J^{μ} has a vanishing divergence and leads to an integral conservation law.† For this reason, J^{μ} is called the conserved Noether current corresponding to the transformation group $G(\epsilon)$.

C.2 Transformation of Scalar Densities

In generally covariant field theories, the system Lagrangian is usually a scalar density. In section C.3, the general covariance identities are derived by applying the Noether theorems to coordinate transformation of the form,

$$\bar{x}^{\mu} = x^{\mu} + \epsilon(x)\xi^{\mu}(x). \tag{C.10}$$

Thus the variation of scalar densities, for variations in these transformations, must be determined.

Under coordinate transformations, a scalar density transforms as

$$\bar{\mathcal{L}}(\bar{x}) = D\mathcal{L}(x) \quad \text{where} \quad D = det\left[\frac{\partial x^{\mu}}{\partial \bar{x}^{\nu}}\right]. \tag{C.11}$$

For the particular transformations (C.10), the determinant D can be expanded in powers of ϵ. One has,

$$D = det(\delta^{\mu}_{\nu} - \frac{\partial\, \epsilon\xi^{\mu}}{\partial \bar{x}^{\nu}}) = 1 - \frac{\partial\, \epsilon\xi^{\mu}}{\partial \bar{x}^{\mu}} + \ldots \tag{C.12}$$

The transformed Lagrangian can also be expanded in a Taylor series about $\epsilon = 0$. To first order,

$$\bar{\mathcal{L}}(\bar{x}) = \bar{\mathcal{L}}(x) + \frac{\partial \bar{\mathcal{L}}(x)}{\partial x^{\mu}}\epsilon\xi^{\mu}. \tag{C.13}$$

Inserting (C.12) and (C.13) into the general transformation law (C.11) gives,

$$\bar{\mathcal{L}}(x) = \mathcal{L}(x) - \frac{\partial \bar{\mathcal{L}}(x)}{\partial x^{\mu}}\epsilon\xi^{\mu} - \mathcal{L}(x)\frac{\partial\, \epsilon\xi^{\mu}}{\partial \bar{x}^{\mu}}. \tag{C.14}$$

†Such laws are sometimes called *weak* conservation laws; in contrast, *strong* conservation laws hold even when the field equations are not satisfied.

Consider the second term on the right hand side. It is already first order in ϵ. Thus by viewing (C.14) as an iterative expansion, $\bar{\mathcal{L}}$ can be replaced by $\mathcal{L}$. The third term contains a derivative with respect to $\bar{x}$. To first order, this can be changed to a derivative with respect to x. That is,

$$\frac{\partial\,\epsilon\xi^{\mu}}{\partial\bar{x}^{\mu}} = \frac{\partial\,\epsilon\xi^{\mu}}{\partial x^{\nu}}\frac{\partial x^{\nu}}{\partial\bar{x}^{\mu}} = \frac{\partial\,\epsilon\xi^{\mu}}{\partial x^{\nu}}(\delta^{\nu}_{\mu} - \frac{\partial\,\epsilon\xi^{\nu}}{\partial\bar{x}^{\mu}}) = \frac{\partial\,\epsilon\xi^{\mu}}{\partial x^{\mu}}. \tag{C.15}$$

With these simplifications, equation C.14 becomes,

$$\bar{\delta}\mathcal{L} = -(\mathcal{L}\epsilon\xi^{\mu})_{,\mu}. \tag{C.16}$$

Thus, as required, scalar density Lagrangians transform under the transformation group (C.10) as a divergence.

C.3 Identities

The Noether current derived in section C.1 is valid for actions invariant under the general group of transformations $G(\epsilon)$. In this section, we specifically consider actions invariant under general coordinate transformations, that is, actions which are generally covariant. In particular, we consider the theory of general relativity. The specific form assumed for the general coordinate transformations is,

$$\bar{x}^{\mu} = x^{\mu} + \epsilon(x)\xi^{\mu}(x). \tag{C.17}$$

By letting ϵ depend on x, this form accounts for the infinity of possible coordinate transformation groups, each group with 4 coordinate degrees of freedom. In other words, we are allowing for local gauge invariance.

The system Lagrangian $\mathcal{L}$ is composed of terms representing the various matter fields and their interactions and a term corresponding to the free gravitational field. Matter Lagrangians are usually first order in the derivatives of the fields. Thus the analysis is simplified if the free gravitational Lagrangian is also first order. In appendix B, such a first order Lagrangian $\mathcal{L}'_G$ is derived. Thus we write,

$$\mathcal{L} = \mathcal{L}_M + \mathcal{L}'_G. \tag{C.18}$$

The matter part of the Lagrangian is assumed to transform as a scalar density. The transformation law for $\mathcal{L}'_G$ is given by equation (B.5). Combining the two results gives the transformation law for the full Lagrangian,

$$\bar{\delta}\mathcal{L} = -\left[\mathcal{L}\epsilon\xi^{\mu} + Q_{\lambda}{}^{\mu\sigma}(\epsilon\xi^{\lambda})_{,\sigma}\right]_{,\mu}. \tag{C.19}$$

As required, the system Lagrangian $\mathcal{L}$ transforms as a divergence.

In order to proceed further, the variation $\bar{\delta}\phi_J$ in the field variables due to the infinitesimal coordinate transformations (C.17) must be determined. It is reasonable to assume that the various fields contained in ϕ_J are either spinors, tensors or tensor densities. In this case, the fields ϕ_J obey linear and homogeneous transformation laws of the form,

$$\bar{\phi}_J(\bar{x}) = \Lambda_J{}^I\,\phi_I(x). \tag{C.20}$$

where the coefficients $\Lambda_J{}^I$ are functions of the derivatives $\partial\bar{x}^{\mu}/\partial x^{\nu}$. For the transformations (C.17), both $\bar{\phi}_J$ and $\Lambda_J{}^I$ can be expanded in powers of ϵ. Thus to first order in ϵ,

$$\bar{\phi}_J(x) + \bar{\phi}_{J,\mu}\epsilon\xi^{\mu} = \left[\Lambda_J{}^I(\delta^{\mu}_{\nu}) + \frac{\partial\Lambda_J{}^I}{\partial\bar{x}^{\mu}{}_{,\nu}}(\epsilon\xi^{\mu})_{,\nu}\right]\phi_I(x). \tag{C.21}$$

On the right hand side, the first term is just the identity transformation, turning ϕ_I into ϕ_J. Bringing the second term on the left over to the right leaves,

$$\bar{\phi}_J(x) = \phi_J(x) + \frac{\partial\Lambda_J{}^I}{\partial\bar{x}^{\mu}{}_{,\nu}}(\epsilon\xi^{\mu})_{,\nu}\phi_I(x) - \bar{\phi}_{J,\mu}\epsilon\xi^{\mu}. \tag{C.22}$$

The last term on the right is already first order in ϵ. By viewing equation (C.22) as an iterative expansion for $\bar{\phi}_J$, the derivatives $\bar{\phi}_{J,\mu}$ can be replaced by the derivatives of the original fields $\phi_{J,\mu}$. Hence the variation in field variables can be written as,

$$\bar{\delta}\phi_J = F^{I\nu}_{J\mu}\,(\epsilon\xi^\mu)_{,\nu}\phi_I - \phi_{J,\mu}\epsilon\xi^\mu \quad \text{where} \quad F^{I\nu}_{J\mu} \equiv \frac{\partial \Lambda_J{}^I}{\partial \bar{x}^\mu{}_{,\nu}}. \tag{C.23}$$

The coefficients $F^{I\nu}_{J\mu}$ are just constants that characterize the particular tensor or spinor transformation properties of the various field components. For example, the coefficients for an nth order covariant tensor field $\phi_{\beta_1\ldots\beta_n}$ are,

$$F^{\beta_1\ldots\beta_n\nu}_{\alpha_1\ldots\alpha_n\mu} = \sum_i \delta^\nu_{\alpha_i}\delta^{\beta_i}_\mu \prod_{j\neq i}\delta^{\beta_j}_{\alpha_j}. \tag{C.24}$$

With the field and Lagrangian variations determined, the corresponding Noether currents can be calculated. Since the Lagrangian is first order in the derivatives of the fields, the general Noether conservation law (C.9) reduces to

$$L^J\,\bar{\delta}\phi_J + \left[\frac{\partial\mathcal{L}}{\partial\phi_{J,\nu}}\bar{\delta}\phi_J - \bar{\delta}B^\nu\right]_{,\nu} = 0. \tag{C.25}$$

Since the variations $\bar{\delta}\phi_J$ and $\bar{\delta}B^\nu$ both contain derivatives of ϵ, the above conservation law (C.25) contains terms proportional to ϵ, $\epsilon_{,\nu}$ and $\epsilon_{,\nu\sigma}$. Since ϵ is arbitrary, each of the three terms must independently vanish. The three identities which result are collectively called the general covariance identities.

Not surprisingly, the term proportional to ϵ gives the usual Noether conservation law. Specifically, one finds that

$$L^J\left[F^{I\tau}_{J\lambda}\,\xi^\lambda{}_{,\tau}\phi_I - \phi_{J,\lambda}\xi^\lambda\right] + J^\nu{}_{,\nu} = 0, \tag{C.26}$$

where,

$$J^\nu = \frac{\partial\mathcal{L}}{\partial\phi_{J,\nu}}\left[F^{I\tau}_{J\lambda}\,\xi^\lambda{}_{,\tau}\phi_I - \phi_{J,\lambda}\xi^\lambda\right] + \mathcal{L}\xi^\nu + Q_\lambda{}^{\nu\sigma}\xi^\lambda{}_{,\sigma}, \tag{C.27}$$

is the conserved Noether current. If the action were invariant only under the group of global transformations with constant ϵ, nothing more could be learned about the system. However, the term proportional to $\epsilon_{,\nu}$ gives,

$$L^J F^{I\nu}_{J\lambda} \xi^\lambda \phi_I + J^\nu + U^{\nu\sigma}{}_{,\sigma} = 0, \tag{C.28}$$

where,

$$U^{\nu\sigma} = \left(\frac{\partial \mathcal{L}}{\partial \phi_{J,\sigma}} F^{I\nu}_{J\lambda} \phi_I - Q_\lambda{}^{\sigma\nu}\right)\xi^\lambda, \tag{C.29}$$

and the term proportional to $\epsilon_{,\nu\sigma}$ gives $U^{\nu\sigma}\epsilon_{,\nu\sigma} = 0$. The third identity simply asserts that the quantities $U^{\nu\sigma}$, called the superpotentials, are antisymmetric. The second identity (C.28) is more profound. It states that when the field equations are satisfied, the weakly conserved Noether current J^ν can be expressed as the divergence of a superpotential.

Vita

Surname:	Richardson
Given Names:	Stephen Alan
Place of Birth:	Prince George, British Columbia
Date of Birth:	May 5th, 1960

Educational Institutions Attended:

University of Victoria, Victoria	1989-1992
University of British Columbia, Vancouver	1978-1983

Degrees Awarded:

B. A. Sc. (Engineering Physics) University of British Columbia	1983

Honors and Awards:

Howard E. Petch Research Scholarship	1991-1992
C. S. Humphrey Graduate Student Award	1990-1991
University of Victoria Graduate Fellowship	1990-1992
University of B. C. Scholarship Fund	1979-1982
University of B. C. Entrance Scholarship	1978-1979

Publications:

F. I. Cooperstock and S. A. Richardson, "Energy Localization and the Kerr-Newman Metric", *Proceedings Fourth Canadian Conference on General Relativity and Cosmology* (World Scientific Publishing Co. PTE. Ltd, to appear).

Partial Copyright License

I hereby grant the right to lend my thesis to users of the University of Victoria Library, and to make single copies only for such users, or in response to a request from the Library of any other university or similar institution, on its behalf or for one of its users. I further agree that permission for extensive copying of this thesis for scholarly purposes may be granted by me or a member of the university designated by me. It is understood that copying or publication of this thesis for financial gain shall not be allowed without my written permission.

Title of Thesis:

Energy Localization in General Relativity and Kerr-Newman Fields

Author: Stephen Alan Richardson

[redacted]

March 25th, 1992

(Date)