

***A MODEL WITH CORRECTION TERM
FOR THE HELIUM ATOM***

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Abstract. We propose a mechanical model for atomic physics. The potential function defining it is formed by the sum of the negative classical Coulomb potential plus a negative homogeneous function of degree -2, which involves the mutual distances between particles, having the role of a correction term. We study the isosceles problem and apply it to the Helium atom. The goal of the paper is to study solutions coming close to collisions. We prove that binary collisions are not possible and restrict our study to triple- and near-triple-collision orbits. We blow-up the triple-collision singularity and paste instead of it, to the phase space, a collision manifold. This is shown to be topologically equivalent to a two-dimensional sphere. The flow on the sphere has two equilibria and is foliated by periodic orbits. We prove the existence of solutions reaching asymptotically these orbits. The set of connecting orbits is finally analysed. We compare the results obtained here with those of a previous paper on Maneff's gravitational law.

I. INTRODUCTION

In a previous paper [1] we have studied the gravitational law of Maneff in the case of the isosceles 3-body problem within the framework of classical mechanics, discussing also possible applications of this law in atomic physics. This article comes therefore as a natural continuation of [1], by applying the previous ideas to the study of the Helium atom. We do not offer here a direct translation of Maneff's potential to the Coulombic one, since such a model would exclude the occurrence of any kind of collisions between particles. We propose instead a slightly modified model which allows the occurrence of triple collisions but excludes the collisions between electrons. Moreover, this model is not restricted to the Helium atom only, and may be further applied in atomic physics. The advantage of studying the isosceles problem is that of having only two degrees of freedom, exactly as in the rectilinear problem. These are the simplest cases among all possible n -body problems, so they offer a starting point for future research.

We present here the ideas independently on the previous paper, such that a reader interested only in atomic physics doesn't have to learn the classical mechanics literature. Nevertheless, we will not repeat the technical details that can be taken, without effort, from [1], but we give all the details for the new mathematical aspects. The mathematical technique we use is completely new for the atomic physics, so it can present also an interest in itself. For the reader interested in both models, we compare the results obtained in these two papers.

The model we offer, described by the equations (1) in Section II, is given by a potential defined through the sum of the negative classical Coulomb potential plus a homogeneous function of degree -2, which depends only on the mutual distances between particles. This second function can be viewed as a correction term. In the case of Maneff's law this correction term is of relativistic type, in the sense that it is multiplied by a constant involving c , and the whole law provides a first approximation of the relativistic model, without leaving the field of classical mechanics. In any case, we use in our theoretical endeavors only the fact that the constant is positive. We leave to experimental physicists the determination of its value, such that to fit practical or other theoretical purposes. The value of this constant doesn't play any role in this paper.

There is a good reason to propose such a model at the atomic physics level. First of all problems of this type lead to a better understanding of the connections between classical and quantum mechanics. Research in this sense has been already performed by Gutzwiller [2], through a careful analysis of the anisotropic Kepler problem, as well as in the paper [1] mentioned above. The mathematical field of chaos theory tells us that nonlinear deterministic systems can behave extremely wild, being hard to predict the state of the system at a certain moment of time. Therefore the dynamical behavior in a chaotic system looks rather closer to that described by quantum mechanics than by the 19-th century mechanics.

Chaotic behavior is known to exist in the classical 3-body problem. Even special types of chaos, like the phenomenon called *Arnold diffusion*, have been put into the evidence for this problem [3]. It is expected that such behavior also exists in the model we will describe

below. If this is true, the resemblance between quantum and classical mechanics might be better understood than we expect today. The reason is that there are many similarities between the chaotic behavior of dynamical systems and the quantum mechanical description of the microcosmos. This article is a small step towards the understanding of such connections.

The paper is divided into sections. In Section II we derive the equations of motion of the planar 3-body problem with the defined potential, and then obtain the equations of the isosceles problem. The main result of this section proves that binary collisions between electrons are not possible. We also obtain a result on syzygy solutions. Section III deals with the process of transforming the equations of motion to regular ones. We prove that the new equations of motion are globally defined. In Section IV we use the collision sphere to understand the motion in the neighborhood of triple collisions. Some final conclusions together with further perspectives are presented in Section VI.

II. NONEXISTENCE OF BINARY COLLISIONS

Consider a system formed by three particles of masses m_i and charges $e_i, i = 1, 2, 3$, moving in the Euclidean plane \mathbb{R}^2 . Define the potential function $\tilde{W}: \mathbb{R}^6 \setminus \Delta \rightarrow \mathbb{R}_+$, $\tilde{W} = \tilde{U} + \tilde{V}$, with

$$\begin{aligned} \tilde{U}: \mathbb{R}^6 \setminus \Delta \rightarrow \mathbb{R}_+, \quad \tilde{U}(\mathbf{q}) &= - \sum_{1 \leq i < j \leq 3} \frac{e_i e_j}{|\mathbf{q}_i - \mathbf{q}_j|}, \\ \tilde{V}: \mathbb{R}^6 \setminus \Delta \rightarrow \mathbb{R}_+, \quad \tilde{V}(\mathbf{q}) &= -\gamma \sum_{1 \leq i < j \leq 3} \frac{e_i e_j}{|\mathbf{q}_i - \mathbf{q}_j|^2}, \end{aligned}$$

where $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$ is the *configuration* of the system, \mathbf{q}_i is the position vector of the i -th particle, $\Delta = \cup_{1 \leq i < j \leq 3} \{\mathbf{q} | \mathbf{q}_i = \mathbf{q}_j\}$ represents the collision set, and γ is a positive constant. The motion of these particles in the plane is described by the equations

$$\ddot{\mathbf{q}} = A^{-1} \nabla \tilde{W}(\mathbf{q}), \tag{1}$$

where $A = \text{diag}(m_1, m_1, m_2, m_2, m_3, m_3)$. Equations (1) can be written in Hamiltonian form but we will not use this characteristic anywhere. In fact the McGehee transformations we perform later will destroy the Hamiltonian character of these equations.

Notice that the potential \tilde{W} is formed by the classical Coulomb potential \tilde{U} given by the inverse power of the distance, but considered with opposite sign, plus a perturbation \tilde{V} , given by the inverse square power of the distance between particles. The perturbation is taken with negative sign and is multiplied with a positive constant γ which doesn't play any role in our theoretical considerations. Possible future experiments may attach therefore a suitable value to this constant. This would probably be a small value, such that for large distances between particles, the influence of the perturbation is negligible. Nevertheless, notice that independently on how small this constant is, the perturbation becomes the main force when the particles are close to collisions. \tilde{V} was defined to take

negative values in order to eliminate the possibility of binary collisions in the isosceles problem, as we will see later in this section.

Classical results of the differential equations theory assure, for given initial data outside the collision set, the existence and uniqueness of an analytic solution of the equations (1), solution defined on a maximal interval (t^-, t^+) , interval which contains the initial time 0. If t^- or t^+ is finite, the solution is said to experience a *singularity* in the past or in the future. In particular the singularity can be a collision. For the Newtonian potential it is known that for 5 or more particles, there can appear singularities which are not collisions, when the motion becomes unbounded in finite time [4].

The above formulation shows the general character of the model. It can be obviously extended to n particles of any masses and charges. In the present paper we are interested only in a special case of the equations (1), namely to a restriction of them to a certain invariant set. This invariant set is given by the isosceles solutions of the equations (1). Since we would like to use this as a model of the Helium atom, consider the units such that $m_1 = m_2 = 1$, $e_1 = e_2 = -1$, for the electrons, and take $e_3 = 2$, $m_3 = m$, for the nucleus. We do not assign to m the real mass-value of the Helium nucleus since we would like to see how certain quantities do, or do not, depend on this value. We keep in mind, however, that the order of magnitude of m is 10^4 .

To make the analysis easier we will express the equations of motion in terms of the coordinates

$$\mathbf{x} = (1/2)(\mathbf{q}_1 - \mathbf{q}_2), \quad \mathbf{y} = \mathbf{q}_3 - (1/2)(\mathbf{q}_1 + \mathbf{q}_2). \quad (2)$$

The new coordinate \mathbf{x} is the position vector based at the half distance between m_1 and m_2 , ending at m_2 , while \mathbf{y} is the position vector having the same base point as \mathbf{x} and ending at m_3 . These variables are a special case of the so-called Jacobi coordinates used in celestial mechanics. This means $\mathbf{x} = (x, 0)$ and $\mathbf{y} = (0, y)$. Taking into account the symmetries involved in an isosceles triangle, the computations show that the equations (1) restricted to the invariant set of isosceles triangles take, in the new variables x, y , the form

$$\begin{cases} \ddot{x} = \frac{1}{4x^2} - \frac{2x}{(x^2+y^2)^{3/2}} + \frac{\gamma}{2x^3} + \frac{4\gamma x}{(x^2+y^2)^2} \\ \ddot{y} = -\frac{2(2+m)}{m} \left[\frac{y}{(x^2+y^2)^{3/2}} + \frac{2\gamma y}{(x^2+y^2)^2} \right]. \end{cases} \quad (3)$$

We have reached now the point where we can prove the main result of this section.

Theorem 1. *There are no solutions of the planar isosceles problem, described by the equations (3) above, leading to binary collisions.*

Proof. Due to the symmetries the only binary collision could occur between m_1 and m_2 . Suppose that such a collision may indeed take place. If t^* is the time moment when it occurs, then $x \rightarrow 0$ when $t \rightarrow t^*$. A simple computation shows that the integral of energy of the equations (3) is given by the relation

$$\dot{x}^2 + \frac{m}{m+2} \dot{y}^2 + \frac{1}{2x} - \frac{4}{(x^2+y^2)^{1/2}} + \frac{\gamma}{4x^2} - \frac{4\gamma}{x^2+y^2} = h, \quad (4)$$

where h is the energy constant. It is easy to see that when $x \rightarrow 0$ and $y > 0$, the left hand side of the energy integral goes to infinity while the right one is constant. This is a

contradiction, consequently binary collisions do not occur. Notice that, from the physical point of view, this happens because near the binary collision, the repelling forces become stronger than the attractive ones. The theorem is thus proved.

This shows that the only possible collision is the simultaneous one between all particles. In Section IV we will also show that such solutions exist.

The next result concerns *syzygy* solutions, which are orbits encountering at least one syzygy configuration. We say that a nonrectilinear orbit has a *syzygy configuration* if at some instant of time all the particles lie on the same line. Recall that a solution is called *rectilinear* if all particles move on a straight line for all time the solution exists. In fact the set of rectilinear solutions is invariant. This means that if the particles move on a straight line for a short interval of time, then they will forever move on that line. From the physical point of view, by saying that a solution of the isosceles problem is *syzygy*, we understand that, in its motion, the nucleus intersects at least once the imaginary line connecting the electrons.

Proposition 2. *Every nonrectilinear solution of the planar isosceles problem, described by the equations (3) above, is a syzygy solution.*

Proof. From the second equation in (3) notice that $\ddot{y} < 0$ for every $y > 0$. Nevertheless, there are no functions $y: \mathfrak{R} \rightarrow \mathfrak{R}$ having this property, there are, consequently, two possibilities. Either y is not defined on all \mathfrak{R} , which means that a singularity is encountered, or there is an instant of time t_0 , such that $y(t_0) = 0$. Since in the case of the 3-body problem the only singularities are collisions (see the proof of Lemma 3), and by Theorem 1 above the only collisions are triple, it follows that a solutions either encounters a triple collision or it has a syzygy configuration. In fact, the two conditions above do not completely exclude each other. As we will see in Theorem 4, the only triple-collision orbits free of syzygy configurations are the rectilinear ones. This completes the proof.

III. McGEHEE TRANSFORMATIONS

As proceeded in [1] we will further consider the powerful technique of McGehee transformations suited to analyse the dynamical behavior of particle systems near collisions. For this we first notice that the equations (3) can be written in the form

$$\begin{cases} \dot{\mathbf{z}} = M^{-1}\zeta \\ \dot{\zeta} = \nabla \hat{W}(\mathbf{z}), \end{cases} \quad (5)$$

where $\mathbf{z} = (x, y)$, $\zeta = (2\dot{x}, \frac{2m}{m+2}\dot{y})$, $M = \begin{pmatrix} 2 & 0 \\ 0 & \frac{2m}{m+2} \end{pmatrix}$, $\hat{W} = \hat{U} + \hat{V}$,

$$\hat{U}(x, y) = -\frac{1}{2x} + \frac{4}{(x^2 + y^2)^{1/2}},$$

$$\hat{V}(x, y) = -\frac{\gamma}{2x^2} - \frac{4\gamma}{x^2 + y^2}.$$

The McGehee transformations are given by the analytic diffeomorphism

$$\begin{cases} r = (\mathbf{z}^T M \mathbf{z})^{1/2} \\ \mathbf{s} = r^{-1} \mathbf{z} \\ v = r \zeta^T \mathbf{s} \\ \mathbf{u} = r \zeta - v M \mathbf{s}, \end{cases} \quad (6)$$

where there exist the constrains $\mathbf{s}^T M \mathbf{s} = 1$ and $\mathbf{s}^T \mathbf{u} = 0$ between the variables. We do not further explain their physical meaning since this was done in detail in [1]. Composing (6) with the time transformation

$$d\tau = r^{-2} dt, \quad (7)$$

and applying the new transformation (which is again an analytic diffeomorphism) to the equations (5), we obtain the system

$$\begin{cases} r' = rv \\ v' = v^2 + \mathbf{u}^T M^{-1} \mathbf{u} - r \hat{U}(\mathbf{s}) - 2\hat{V}(\mathbf{s}) \\ \mathbf{s}' = M^{-1} \mathbf{u} \\ \mathbf{u}' = -(\mathbf{u}^T M^{-1} \mathbf{u}) M \mathbf{s} + r(\nabla \hat{U}(\mathbf{s}) + \hat{U}(\mathbf{s}) M \mathbf{s}) + \nabla \hat{V}(\mathbf{s}) + 2\hat{V}(\mathbf{s}) M \mathbf{s}. \end{cases} \quad (8)$$

Notice also that, by abuse, we have denoted the coordinates by the same letters. Prime means differentiation with respect to the new fictitious time variable τ . The integral of energy gets transformed into the relation

$$(1/2)(\mathbf{u}^T M^{-1} \mathbf{u} + v^2) - r \hat{U}(\mathbf{s}) - \hat{V}(\mathbf{s}) = 2r^2 h, \quad (9)$$

which is not a first integral anymore. A closer look to the equations (8) shows them to be free of triple collision singularities. Indeed, from the way transformations (6) were defined, the triple collision takes place if and only if $r = 0$.

Notice that though looking similar to the corresponding equations in the gravitational case treated in [1], these equations are different since the functions \hat{U} and \hat{V} are different from those in [1]. We have arranged the equations to look similar such that the computations are easier to follow and the comparison is easier to make.

In order to further use the symmetries offered by the isosceles problem, define the transformations given by the analytic diffeomorphism

$$\begin{cases} \mathbf{s} = (\frac{1}{\sqrt{2}} \cos \theta, \sqrt{\frac{m+2}{2m}} \sin \theta) \\ \mathbf{u} = (-\sqrt{2}u \sin \theta, \sqrt{\frac{2m}{m+2}} u \cos \theta), \end{cases} \quad (10)$$

which are compatible with the above mentioned properties of McGehee's coordinates. In other words, computing $\mathbf{s}^T M \mathbf{s}$ and $\mathbf{s}^T \mathbf{u}$ in terms of θ and u , we obtain 1 and 0, respectively.

The energy relation (9) gets transformed into

$$u^2 + v^2 - 2V(\theta) = 2r(rh + U(\theta)). \quad (11)$$

Making use of (11), the equations of motion (8) become

$$\begin{cases} r' = rv \\ v' = 2hr^2 + rU(\theta) \\ \theta' = u \\ u' = r \frac{d}{d\theta} U(\theta) + \frac{d}{d\theta} V(\theta), \end{cases} \quad (12)$$

where

$$U(\theta) = -\frac{1}{\sqrt{2} \cos \theta} + \frac{4\sqrt{2m}}{(m + 2 \sin^2 \theta)^{1/2}},$$

$$V(\theta) = \gamma \frac{(16m + 2) \cos^2 \theta - (m + 2)}{2 \cos^2 \theta (m + 2 \sin^2 \theta)}.$$

Let us take now a better look to the equations (12). Apparently they also look similar to the ones obtained in the gravitational case. This time there do not occur only different functions U and V , but these equations are regular because binary collisions are impossible. So, the complicated technique applied in [1] in order to regularize binary collisions, is not necessary anymore. Now we can prove the following important property

Lemma 3. *The equations (12) are globally defined.*

Proof. To prove the above statement we notice first that the only possible singularities are collisions. This follows exactly like in the case of the Newtonian potential. A detailed proof of this fact is given in [5]. Since all regular solutions are obviously globally defined, the only question is whether triple-collision solutions have the same property. The rest of this proof will show that this is indeed the case.

Triple-collision solutions have the property that $r \rightarrow 0$. It is natural then to study the flow on the set $\{r = 0\}$ and to see how triple-collision solutions behave asymptotically towards this set. We can further simplify this study by foliating the phase space with all energy levels and intersect the above set with every energy level. For this denote

$$C = \{(r, v, \theta, u) | r = 0 \text{ and } u^2 + v^2 = 2V(\theta)\}.$$

The second condition comes from taking $r = 0$ in the energy relation (11). Notice that C doesn't depend on the energy constant h , consequently the intersection of every energy level with the set $\{r = 0\}$ give rise to a single set C . From the first equation in (12) we have that $\{r = 0\}$ is an invariant set. The same thing can be said about every energy level, and therefore C is also an invariant set for the equations (12). Let us further determine the topological structure of C . For this we need to understand first the qualitative behavior of the function V . Its graph in (θ, V) coordinates is like the one in Figure 1. A simple computation shows that V is nonnegative in the interval

$$I = [-\sqrt{(m+2)/(16m+2)}, \sqrt{(m+2)/(16m+2)}], \quad (13)$$

and it is positive and concave down on this interval. Therefore C is clearly homeomorphic with the 2-dimensional sphere S^2 . We call C the *collision sphere*.

The equations (12) restricted to the collision sphere C take the form

$$\begin{cases} v' = 0 \\ \theta' = u \\ u' = \frac{d}{d\theta}V(\theta). \end{cases} \quad (14)$$

The only equilibrium solutions of the equations (12) are given by $u = 0$ and $\frac{d}{d\theta}V(\theta) = 0$. This means that we have exactly two equilibria and they are

$$(r, v, \theta, u) = (0, \pm\sqrt{2V(0)}, 0, 0) = (0, \pm\sqrt{15\gamma}, 0, 0).$$

They obviously belong to the collision sphere C and we will call them the north pole N and the south pole S . From the first equation in (14) and from the fact that N and S are the only equilibria, it follows that the flow on C is given by periodic orbits. Therefore the flow on C is globally defined (see Figure 2). Now, since C is compact, invariant, and the flow on it is globally defined, it follows that independently on how a solution reaches C , it needs an infinite amount of (fictitious) time τ to do that. Thus, for a triple-collision solution, $|\tau| \rightarrow \infty$. This completes the proof.

IV. THE FLOW NEAR THE COLLISION SPHERE

The goal of this section is to prove the existence of triple-collision (ejection) solutions of the equations (12) and to see how they reach the collision sphere C . The main theorem is given below.

Theorem 4. *For every orbit belonging to the southern hemisphere of C , there exists at least a solution tending to it. For every orbit belonging to the northern hemisphere of C , there is at least a solution ejecting from it. The equator is the only orbit of C for which both types of solutions exist.*

Proof. The proof works, in principle, similar to that given for the gravitational Maneff potential. There are, however, some difficulties which need to be carefully treated. Let us perform the proof for orbits of the northern hemisphere. The existence of solutions tending to orbits belonging to the southern hemisphere, follows then by symmetry.

Analysing the qualitative behavior of the function U on the interval I (defined by (13)), since m is large for the Helium atom (of the order of magnitude 10^4 in comparison with the mass of the electrons), we can compute that U is positive on this interval and has a positive inferior limit. Notice, however, that this is not true anymore for small values of the constant m , and such a case would imply the impossibility of triple collision solutions, making the model unrealistic with respect to the Helium atom. Coming back to our large m case, and looking at the second equation in (12), we can admit the existence of a constant $K > 0$, such that $v' \geq Kr$. Multiplying this inequality by v (which is positive since we are in the northern hemisphere) we obtain $vv' \leq Krv = Kr'$, the equality following from the first equation in (12). Integrating this last relation we get $r \leq (1/2K)v^2 + c$, where c

is a constant of integration. Consider first the cases when $c \leq 0$. Then for $v = \sqrt{-2Kc}$, r is forced to take the value 0. Therefore when v tends from above to that constant, the solution tends to the corresponding periodic orbit of the collision sphere C . This proves the existence of ejecting orbits from periodic ones, for the values of c for which the periodic orbits exist. It is clear that for positive values of the integration constant, such orbits do not exist.

The special case of the north and south poles, needs a remark. Though the existence of solutions ejecting/tending from/to N and S respectively, is proved by the above procedure, there is a better way to see what happens in their neighborhood. This classical method involves the study of the linearized system in the neighborhood of the equilibria. Proceeding in this way we see that the eigenvalues obtained at the equilibria are $\lambda_r = \pm\sqrt{2V(0)}$, $\lambda_v = 0$ and $\lambda_{\theta,u} = \pm(\frac{d^2}{d\theta^2}V(0))^{1/2}$. This gives rise to the picture in Figure 3, which was already clear from the above analysis. This completes the proof.

We still do not know if the picture in Figure 3 is complete. Due to the fact that N and S respectively, are not hyperbolic equilibria, it is not clear what dimension their unstable and stable set respectively, has. There might exist other orbits tending to S (or ejecting from N) than the ones in the picture. The partial answer we can give in this sense is that the only *analytic* orbits with this property are the ones in Figure 3. The precise mathematical statement and the proof, work exactly as the ones described in Section V (on center manifolds) in [1]. In the next section we will see that there are also other orbits tending/ejecting to/from N and S respectively.

Let us describe further the physical interpretation of the triple-collision/ejection solutions we have just proved to exist. We talk about those reaching orbits of the southern hemisphere. They correspond to collisions. The ejection ones have the same interpretation but with reversed time.

We start with the south pole S . The orbit in Figure 3 reaching S can be described as follows. The nucleus of the Helium atom stays at the center of mass of the system while the electrons are equidistantly situated from it, on both sides of a straight line, as in Figure 4. The electrons will move symmetrically with equal and increasing velocities towards the center of mass until the triple collision takes place.

The physical interpretation of an orbit reaching any of the periodic orbits of the southern hemisphere is as follows. Suppose the particles have an initial position like the one in Figure 4. Then the nucleus oscillates up and down on an axis perpendicular to that of the electrons, which move, like before, towards the point where the nucleus was initially situated. The amplitude of the oscillations of the nucleus is smaller and smaller, and the particle moves faster and faster. After infinitely many oscillations the triple collision occurs. The difference between a solution tending to a periodic orbit and one tending to another periodic orbit, is the amplitude of the oscillations as well as the velocity of the particles.

It is easy to see that these triple collision solutions have no natural rotation. This is due to the fact that the motion is constrained to remain isosceles. It is not clear at all what happens if the angular momentum remains 0 in a problem where the restriction of being isosceles is lifted. It might happen that things are different than in the classical Newtonian case, where the angular momentum always vanishes.

Having now the picture in Figure 3, as well as the physical interpretation of the triple-collision orbits, one also has information on orbits passing close to a triple collision. Due to the property of continuity of the solutions with respect to the initial data, a solution coming close to a triple collision will follow closely, in phase space, a solution leading to a triple collision. Figure 5 shows how one such solution can behave. This doesn't tell us what the final picture of this solution is, but says what happens in the neighborhood of the triple collision.

The physical interpretation of an orbit like that in Figure 5 is first similar to the one described above. The nucleus oscillates while the electrons move towards one another. The three particles, however, miss the triple collision, so the repelling force between electrons makes them move away from each other after they have come close enough together. The nucleus continues to oscillate in the mean time. It is, however, hard to predict what happens with a particular orbit, long after the particles have passed close to the triple collision.

V. CONNECTING ORBITS

The regularization result on triple-collision orbits obtained in [1] for the 3-body problem with Maneff's law can be similarly stated and proved for our case. In spite of having a simpler structure in the Helium atom case, we still cannot give a complete answer concerning the regularization. Again, this happens because we do not have information on the structure of the set of solutions leading to triple collisions. Unfortunately the Poincaré map associated to the flow in the neighborhood of a periodic orbit, has nonhyperbolic fixed points, and the known mathematical results dealing with this type of degeneracy, do not apply here. The goal of this section is to see how far the theorems on connecting orbits, obtained in [1], work in this context.

First of all notice that since $v' > 0$ in the neighborhood of the collision sphere (i.e. for $r > 0$ but small) it follows that an orbit ejecting from the northern hemisphere of C cannot return back to the northern hemisphere. The same can be said for the southern hemisphere. This means there do not exist homoclinic solutions for orbits on the sphere, excepting possibly orbits connecting the equator with itself. We have, however, no proof of the fact that such orbits would exist. Besides, there cannot exist heteroclinic solutions connecting an orbit of the northern hemisphere with another orbit of the northern hemisphere, and the same is true for the southern hemisphere.

Let us now prove the following result:

Theorem 5. *For every $h < 0$ there exists a heteroclinic orbit connecting the north and the south poles of the collision sphere. For $h \geq 0$ there do not exist such heteroclinic orbits contained in the set $D = \{(r, v, \theta, u) | \theta = u = 0\}$.*

Proof. Notice first that the set $D = \{(r, v, \theta, u) | \theta = u = 0\}$ is invariant for the equations (12). We will prove the existence of the heteroclinic orbits inside the set D .

Restricting the equations (12) to this set we obtain the system

$$\begin{cases} r' = rv \\ v' = 2hr^2 + 7\sqrt{2}r, \end{cases} \quad (15)$$

having the energy relation

$$2hr^2 + 7\sqrt{2}r + 30\gamma - v^2 = 0.$$

Analysing this last relation we see that the phase space picture of equations (15) is that of Figure 6. It is easy to see from the computations that the orbits connecting N and S are the heteroclinic orbits obtained for $h < 0$, while for $h > 0$ such connecting orbits do not exist. Interpreting this result in terms of equations (12), the picture in Figure 7 suggests how things can be visualized. This also comes to complete the question asked in Section IV on the existence of other orbits ejecting from N or tending to S , respectively. This completes the proof.

Let us give now the physical interpretation of the heteroclinic orbits connecting the north and the south poles of the collision sphere. The particles eject from a triple approach. The nucleus remains at rest while the electrons move symmetrically on a straight line in opposite directions and away from the nucleus. At some instant of time the electrons stop simultaneously, and then start to move again symmetrically towards the nucleus, until a new triple collision takes place. For different energy levels this scenario takes place with different amplitudes and velocities.

VI. CONCLUSIONS

The isosceles configuration has been a model for the Helium atom for the past 75 years. In this paper we bring a point of view which comes also from classical mechanics and is in connection with Maneff's model studied in [1]. There are many common points but also several differences between the microscopic and the macroscopic models. From the mathematical point of view, the microscopic one studied here is not as complicated as the macroscopic one, at least with respect to solutions encountering collisions. This happens because binary collisions do not occur in the Helium atom model. An essential difference here is that of not having orbits leading asymptotically to a noncollinear central configuration, as in Maneff's case.

We have studied the dynamics of this law and have now an image of the phase space picture in the neighborhood of triple collisions. Nevertheless, the important problem of proving the existence of chaotic orbits in the planar case, is still open. If achieved, this would be of interest in understanding one main aspect of the relation existing between classical and quantum mechanics. It seems that a good chance to show the existence of chaos is by understanding better what happens near a triple collision. For this, stronger mathematical tools have to be developed in the future.

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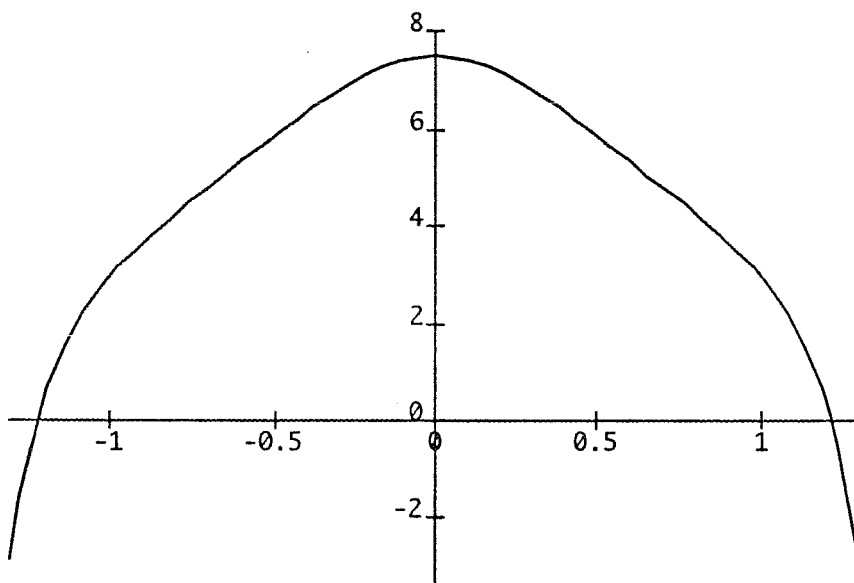


Figure 1

The graph of V for $m=2$

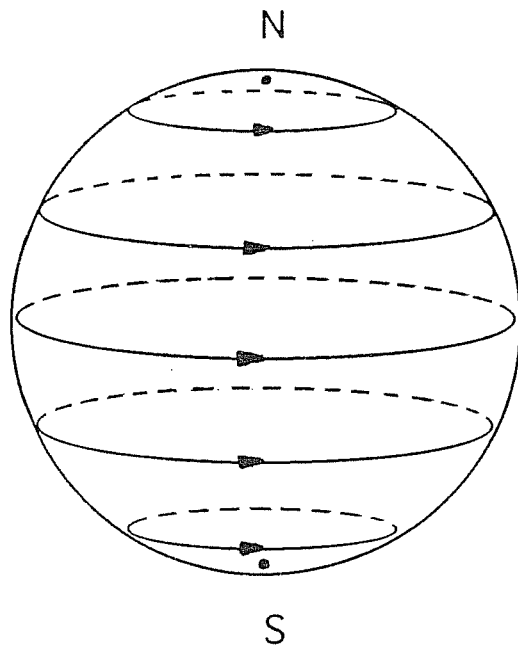


Figure 2

The flow on the collision sphere

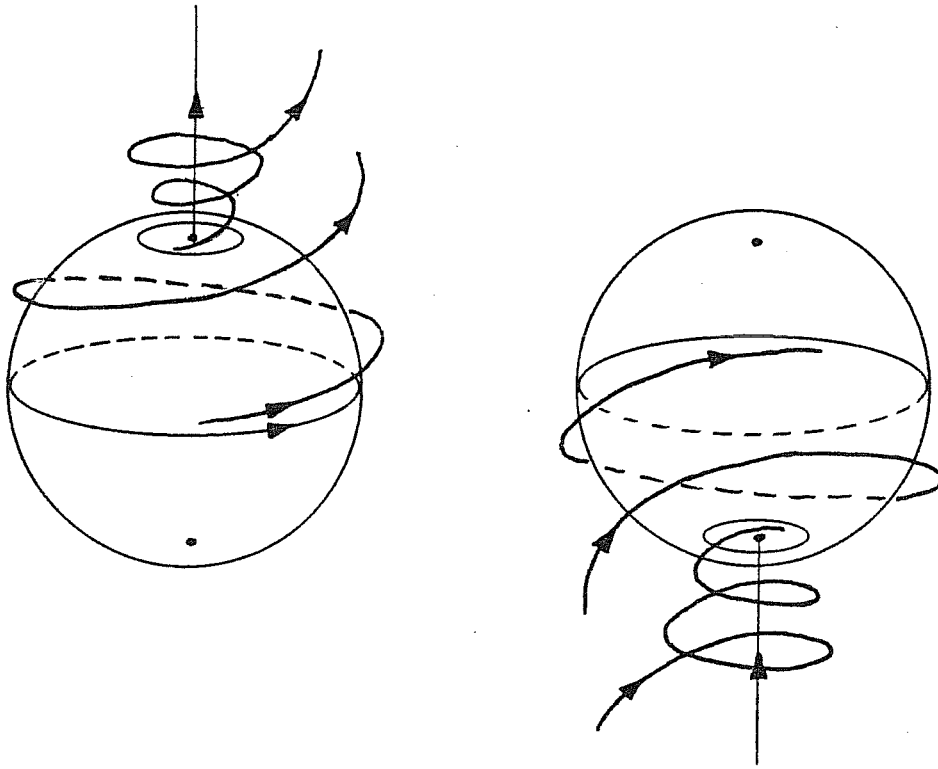


Figure 3

Ejecting and collision orbits

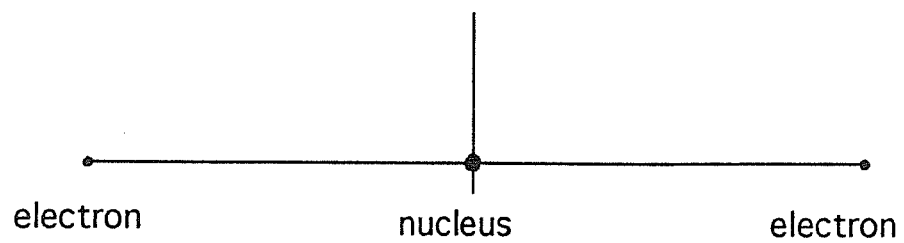


Figure 4

An initial position of the system

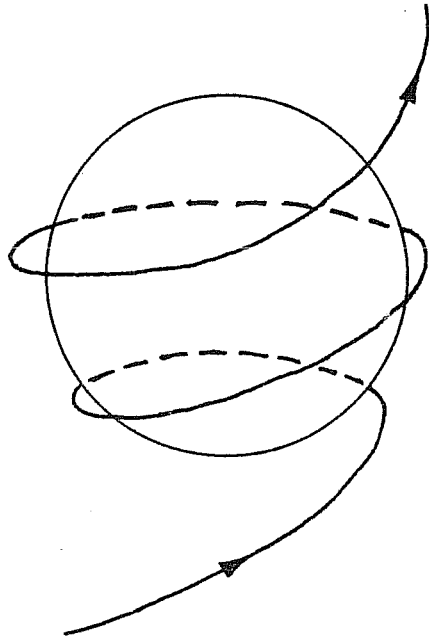


Figure 5

An orbit coming close to a triple collision

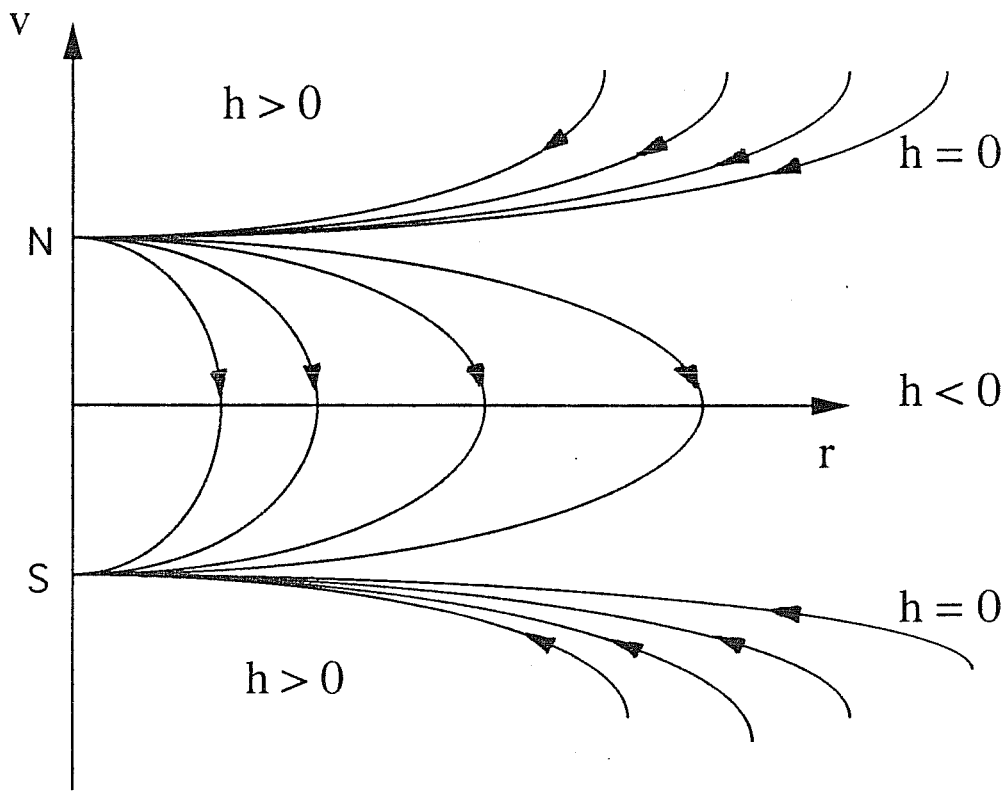


Figure 6

The phase space picture of equations (15)

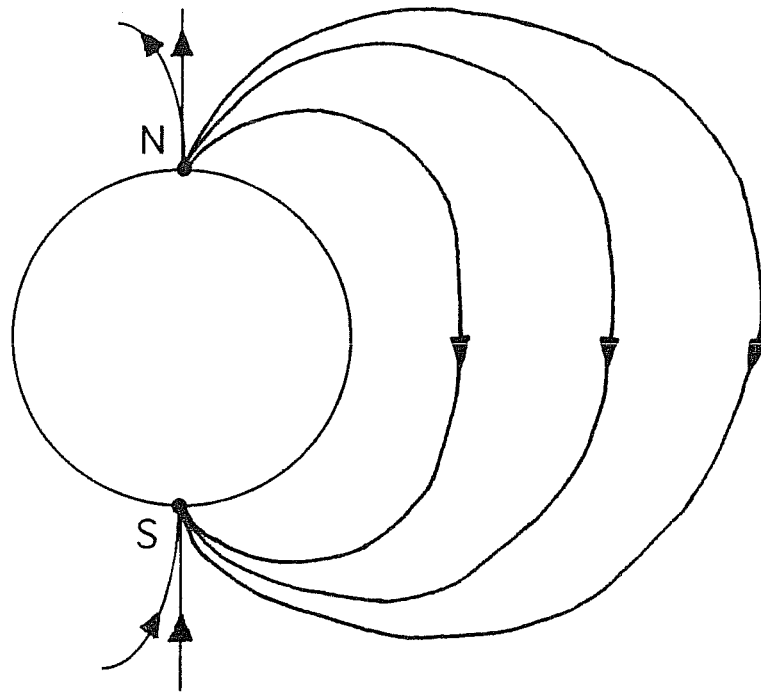


Figure 7

Heteroclinic orbits connecting the poles