A NOTE ON A GENERALIZATION OF A q-SERIES TRANSFORMATION OF RAMANUJAN

By

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ABSTRACT

It is shown how readily a recent generalization of a q-series transformation of Srinivasa Ramanujan would follow as a limiting case of Heine's transformation for basic hypergeometric series. Several interesting consequences of this general result are also deduced.

For real or complex q, $|q| < 1$, let

$$(1) \qquad (\lambda;q)_\mu = \prod_{j=0}^{\infty} \left(\frac{1-\lambda q^j}{1-\lambda q^{\mu+j}} \right)$$

for arbitrary λ and μ, so that

$$(2) \qquad (\lambda;q)_n = \begin{cases} 1, \text{ if } n = 0, \\ (1-\lambda)(1-\lambda q) \ldots (1-\lambda q^{n-1}), \ \forall n \in \{1,2,3,\ldots\}, \end{cases}$$

and

$$(3) \qquad (\lambda;q)_\infty = \prod_{j=0}^{\infty} (1-\lambda q^j).$$

The q-series transformation

$$(4) \qquad (-bq;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-bq;q)_n} \frac{\lambda^n}{(q;q)_n} = \sum_{n=0}^{\infty} q^{\frac{1}{2}n(n+1)} \left(-\frac{\lambda}{b};q \right)_n \frac{b^n}{(q;q)_n}$$

is stated in Chapter 16 of the Second Notebook of Srinivasa Ramanujan [9, Vol. II, p. 194, Entry 9]. A special case of Ramanujan's identity (4) when $b = 1$ was posed as an Advanced Problem by Carlitz [5, p. 440, Equation (1)] who, in fact, proved the general case (4) by using Euler's expansion for $(\lambda;q)_n$ as a polynomial in λ (*cf.* [6, p. 917]). The identity (4) has received considerable attention in several subsequent works (see, for example, [1], [2], and [8]). In particular, in their excellent memoir [1, pp. 9-10] Adiga *et al.* have presented two interesting proofs of (4). It should be remarked in passing that one of their proofs using Heine's transformation [7, p. 306, Equation (79)] iteratively is essentially equivalent to the earlier proof by Andrews [2, p. 105] who deduced (4) as a limiting case of a result attributed to Rogers.

An interesting generalization of Ramanujan's q-series transformation (4) was given recently by Bhargava and Adiga in the form (cf. [4, p. 339, Equation (3)]; see also [3, p. 14, Equation (4*)]):

$$(5)\qquad (-bq;q)_\infty \sum_{n=0}^{\infty} q^{\frac{1}{2}n(n+1)} \frac{(-\lambda/a;q)_n}{(-bq;q)_n} \frac{a^n}{(q;q)_n}$$

$$= (-aq;q)_\infty \sum_{n=0}^{\infty} q^{\frac{1}{2}n(n+1)} \frac{(-\lambda/b;q)_n}{(-aq;q)_n} \frac{b^n}{(q;q)_n},$$

which would obviously reduce to (4) in the limiting case when $a \to 0$. Replacing λ by λ/q, and setting

$$a = -x/q \quad \text{and} \quad b = -y/q,$$

the identity (5) becomes

$$(6)\qquad \sum_{n=0}^{\infty} q^{\frac{1}{2}n(n-1)} \frac{(\lambda/x;q)_n}{(y;q)_n} \frac{(-x)^n}{(q;q)_n}$$

$$= \frac{(x;q)_\infty}{(y;q)_\infty} \sum_{n=0}^{\infty} q^{\frac{1}{2}n(n-1)} \frac{(\lambda/y;q)_n}{(x;q)_n} \frac{(-y)^n}{(q;q)_n}$$

or, equivalently,

$$(7)\qquad {}_1\Phi_1\left[\begin{matrix}\lambda/x; \\ y;\end{matrix}\; q,\; x\right] = \frac{(x;q)_\infty}{(y;q)_\infty}\, {}_1\Phi_1\left[\begin{matrix}\lambda/y; \\ x;\end{matrix}\; q,\; y\right],$$

where ${}_{p+1}\Phi_{p+j}$ $(p, j = 0,1,2,\ldots)$ denotes a generalized basic (or q-) hypergeometric series defined by

(8)
$$
{}_{p+1}\Phi_{p+j}\left[\begin{matrix}\alpha_1, \ldots, \alpha_{p+1};\\ \beta_1, \ldots, \beta_{p+j};\end{matrix}\; q, x\right] = \sum_{n=0}^{\infty} (-1)^{jn} q^{\frac{1}{2}jn(n-1)} \frac{(\alpha_1;q)_n \cdots (\alpha_{p+1};q)_n}{(\beta_1;q)_n \cdots (\beta_{p+j};q)_n} \frac{x^n}{(q;q)_n},
$$

$(|x| < \infty$ when $j = 1,2,3, \ldots,$ or $|x| < 1$ when $j = 0)$.

Formula (5) was proven by Bhargava and Adiga [4, pp. 340-341] by making use of Ramanujan's identity (4) and of certain functional relations which they had derived earlier [3, p. 14, Lemma 1] for the left side of (5). With a view to presenting a much shorter and direct proof of the equivalent result (6) or (7), we now recall the aforementioned Heine's transformation [7, p. 306, Equation (79)]

(9)
$$
{}_2\Phi_1\left[\begin{matrix}a, b;\\ c;\end{matrix}\; q, x\right] = \frac{(b;q)_\infty (ax;q)_\infty}{(c;q)_\infty (x;q)_\infty}\, {}_2\Phi_1\left[\begin{matrix}x, c/b;\\ ax;\end{matrix}\; q, b\right],
$$

which, upon repeated application, yields

(10)
$$
{}_2\Phi_1\left[\begin{matrix}a, b;\\ c;\end{matrix}\; q, x\right] = \frac{(c/b;q)_\infty (bx;q)_\infty}{(c;q)_\infty (x;q)_\infty}\, {}_2\Phi_1\left[\begin{matrix}abx/c, b;\\ bx;\end{matrix}\; q, \frac{c}{b}\right].
$$

It is the transformation (10) which was, in fact, employed by Andrews [2, p. 105] as well as Adiga *et al.* [1, pp. 9-10] to deduce Ramanujan's identity (4) as a limiting case. Indeed, as we indicated above, Andrews [2, p. 98, Equation (4.6)] attributed (10) to Rogers, although Heine did give (9) *and* a relatively more familiar ${}_2\Phi_1$ transformation (*cf.* [7, p. 325, Theorem XVIII]) which follows readily upon merely iterating Heine's result (9) one more step beyond the transformation (10).

Replacing x by x/b and letting $b \to \infty$ in (10), we have

$$(11)\qquad \sum_{n=0}^{\infty} q^{\frac{1}{2}n(n-1)} \frac{(a;q)_n}{(c;q)_n} \frac{(-x)^n}{(q;q)_n}$$

$$= \frac{(x;q)_\infty}{(c;q)_\infty} \sum_{n=0}^{\infty} q^{\frac{1}{2}n(n-1)} \frac{(ax/c;q)_n}{(x;q)_n} \frac{(-c)^n}{(q;q)_n} .$$

Now set $a = \lambda/x$ and $c = y$ in (11), and we are led immediately to the q-series identity (6) which, in turn, yields the q-hypergeometric form (7) by virtue of the definition (8).

Several consequences of the q-series identity (6) are worthy of note. First of all, if in (6) we set $\lambda = y$, we immediately obtain

$$(12)\qquad \sum_{n=0}^{\infty} q^{\frac{1}{2}n(n-1)} \frac{(y/x;q)_n}{(y;q)_n} \frac{(-x)^n}{(q;q)_n} = \frac{(x;q)_\infty}{(y;q)_\infty}$$

or, equivalently,

$$(13)\qquad {}_1\Phi_1\left[\begin{matrix} y/x; \\ y; \end{matrix}\; q,\; x\right] = \frac{(x;q)_\infty}{(y;q)_\infty} .$$

Formula (12) reduces, when $x \to 0$ and $y = aq$, to Entry 3 in Chapter 16 of Ramanujan's Second Notebook (cf., e.g., [1, p. 5, Equation (5.1)]).

Next we put $y = q$ and $\lambda = q^2$ in (6), and replace x by qx. We thus find that

$$(14)\qquad \sum_{n=0}^{\infty} q^{\frac{1}{2}n(n+1)} \frac{(q/x;q)_n}{\{(q;q)_n\}^2} (-x)^n$$

$$= \frac{(qx;q)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n+1)}}{(qx;q)_n} .$$

Finally, we replace q by q^2 in (6), and then set $y = q$ and $\lambda = q^3$. Writing q^2x for x in the resulting identity, we have

(15) $$\sum_{n=0}^{\infty} q^{n(n+1)} \frac{(q/x;q^2)_n}{(q;q)_{2n}} (-x)^n$$

$$= \frac{(q^2x;q^2)_\infty}{(q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(q^2x;q^2)_n} .$$

Each of these last q-series identities (14) and (15) reduces, when $x \to 0$, to a corresponding result given earlier by Adiga et al. [1, p. 10, Corollaries (i) and (ii)].

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