

# Components and Colourings of Singly- and Doubly-periodic Graphs

by

Bethany Joy Smith  
B.Sc., University of Victoria, 2001

A Thesis Submitted in Partial Fulfillment of the  
Requirements for the Degree of

MASTER OF SCIENCE

in the Department of Mathematics and Statistics

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## Abstract

Singly-periodic (SP) and doubly-periodic (DP) graphs are infinite graphs which have translational symmetries in one and two dimensions, respectively. The problem of counting the number of connected components in such graphs is investigated. A method for determining whether or not an SP graph is  $k$ -colourable for a given positive integer  $k$  is given, and the question of deciding  $k$ -colourability of DP graphs is discussed. Colourings of SP and DP graphs can themselves be either periodic or aperiodic, and properties which determine the symmetries of their colourings are also explored.

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# Chapter 1

## Introduction

Periodic graphs are infinite graphs which have translational symmetries in a certain number of dimensions. In this thesis we investigate properties of periodic graphs with symmetries in one and two dimensions.

A Singly-Periodic (SP) graph can be visualized as an infinite strip of cells with isomorphic copies of the same finite graph in each cell, and the same edges between corresponding vertices in adjacent cells. Such a graph can be defined by giving the cell graph and listing the neighbours in the next cell of each vertex.

Similarly, a Doubly-Periodic (DP) graph can be viewed as an infinite 2-dimensional grid, with isomorphic copies of the same finite graph in each cell of the grid, and the same edges between corresponding vertices in cells which are adjacent either horizontally or vertically. A DP graph can be defined by giving a cell graph and listing the neighbours of each vertex in the cell

directly above and in the cell to the right.

The broad question we consider is: given a periodic graph defined in this way, how can we determine properties of the entire graph? We focus our attention on two main problems, looking at both SP and DP graphs in both cases: how to determine the properties of the components of an SP or DP graph, and how to decide whether or not an SP or DP graph is  $k$ -colourable, for some integer  $k \geq 2$ . Chapter 2 presents the graph theoretic definitions and notations that are used, and Chapter 3 surveys some known results which are relevant to the questions under consideration here. Chapter 4 deals with methods for determining the number and type(s) of components in a given SP or DP graph, and presents results showing how to determine, for any given vertex in an SP or DP graph, the exact properties of the component in which it appears. Chapter 5 looks at ways to determine whether or not an SP or DP graph is  $k$ -colourable, and if so what types of colourings it allows. For SP graphs, we can determine whether or not it is  $k$ -colourable, and if so whether or not it allows aperiodic colourings. For DP graphs, the question of  $k$ -colourability for  $k \geq 3$  is more complex; we discuss how this problem is related to tiling problems, and investigate two special cases of DP graphs for which  $k$ -colourability is decidable.

# Chapter 2

## Preliminaries

This chapter introduces the notation and definitions that will be used throughout this thesis. It is assumed that the reader has some knowledge of basic graph theory; for an introduction to graph theory and discussion of any terms and concepts not included here, the reader is referred to West [8]. It is also assumed that the reader has a basic understanding of decidability; for a classic reference, see Hopcroft and Ullman [6].

All graphs in this thesis are assumed to be undirected, unless they are referred to specifically as directed graphs or digraphs. Directed and undirected graphs are not assumed to be simple; digraphs in particular may contain loops.

The number of connected components of a graph  $G$  or digraph  $D$  will be denoted by  $\#c(G)$  or  $\#c(D)$ , respectively. From now on we will omit the qualifier “connected” when discussing components.

For a positive integer  $k$ , a *proper vertex  $k$ -colouring*, or simply  *$k$ -colouring*, of a graph  $G$  is a mapping  $f$  from the vertex set  $V(G)$  to the set  $\{1, 2, \dots, k\}$  such that for all  $u, v \in V(G)$ , if  $uv \in E(G)$ , then  $f(u) \neq f(v)$ . From now on we will omit the qualifier “proper” when discussing colourings.

## 2.1 Singly-Periodic Graphs

A *singly-periodic graph (SP graph)*  $\Gamma$  is an infinite graph whose vertex set can be partitioned into sets  $V_i = \{v_i^1, v_i^2, \dots, v_i^n\}$ ,  $i \in \mathbb{Z}$  such that

1. a vertex in  $V_i$  is adjacent to vertices in  $V_j$  only if  $|i - j| \leq 1$ , and
2. if  $v_i^k$  is adjacent to  $v_j^l$  then  $v_{i+a}^k$  is adjacent to  $v_{j+a}^l$  for all  $a \in \mathbb{Z}$ .

The subgraphs of  $\Gamma$  induced by  $V_i$  are pairwise isomorphic; they are denoted by  $G_i$  and are referred to as the *cells* or *cell graphs* of  $\Gamma$ . For all  $i, j \in \mathbb{Z}$  and  $k \in \{1, 2, \dots, n\}$ , an isomorphism from  $G_i$  to  $G_j$  is given by  $f(v_i^k) = v_j^k$ . Vertices with the same superscript are said to be of the same *type*. In formal terms, an SP graph can be defined by giving the cell graph  $G_0$  and listing the neighbours in  $G_1$  of each vertex  $v_0^k$ , for  $k = \{1, 2, \dots, n\}$ . Figure 2.1 shows an SP graph; edges within cells are drawn in black, while edges between cells are drawn in red.

For an SP graph  $\Gamma$ , the function  $T_a : V(\Gamma) \rightarrow V(\Gamma)$  defined by  $T_a(v_i^k) = v_{i+a}^k$  is an automorphism. The mapping  $T_a$  can be thought of as a translation of  $\Gamma$  by  $a$  cells along the strip, and  $a$  is called the *period* of  $T_a$ . The inverse

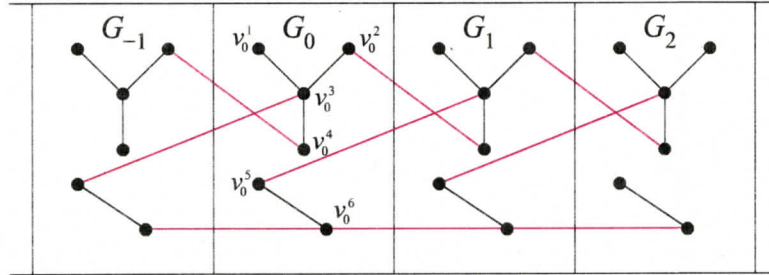


Figure 2.1: Four cells of an SP graph  $\Gamma$ .

of  $T_a$  is  $T_a^{-1} = T_{-a}$ . The composition of two translations  $T_a$  and  $T_b$  is the translation  $T_a \circ T_b = T_{a+b}$ .

For a vertex  $v$  of an SP graph, there is a unique  $i$  such that  $v \in G_i$ ; this  $i$  is denoted by  $x(v)$ . If  $H$  is a finite subgraph of an SP graph  $\Gamma$ , the *width* of  $H$  is defined as  $\max_{u,v \in V(H)} \{|x(u) - x(v)|\}$ . Thus the width of  $H$  is the maximum distance between cells that intersect with  $H$ .

## 2.2 Doubly-Periodic Graphs

A *doubly-periodic graph* (*DP graph*)  $\Gamma$  is an infinite graph whose vertex set can be partitioned into sets  $V_{ij} = \{v_{ij}^1, v_{ij}^2, \dots, v_{ij}^n\}$ ,  $i, j \in \mathbb{Z}$  such that

1. a vertex in  $V_{ij}$  is adjacent to vertices in  $V_{pq}$  only if  $|i - p| \leq 1$  and  $q = j$ , or  $i = p$  and  $|j - q| \leq 1$ , and
2. if  $v_{ij}^k$  is adjacent to  $v_{pq}^l$  then  $v_{(i+a)(j+b)}^k$  is adjacent to  $v_{(p+a)(q+b)}^l$  for all  $a, b \in \mathbb{Z}$ .

The subgraphs of  $\Gamma$  induced by  $V_{ij}$  are pairwise isomorphic; they are denoted by  $G_{ij}$  and are referred to as the *cells* or *cell graphs* of  $\Gamma$ . For all  $i, j, p, q \in \mathbb{Z}$  and  $k \in \{1, 2, \dots, n\}$ , an isomorphism from  $G_{ij}$  to  $G_{pq}$  is given by  $f(v_{ij}^k) = v_{pq}^k$ . Vertices with the same superscript are said to be of the same *type*. A DP graph can be defined by giving a cell graph  $G_{00}$  and listing the neighbours in  $G_{01}$  and  $G_{10}$  of each  $v_{00}^k$ , for  $k = \{1, 2, \dots, n\}$ . Figure 2.2 shows a DP graph; edges within cells are drawn in black, while horizontal edges between cells are drawn in red and vertical edges in blue.

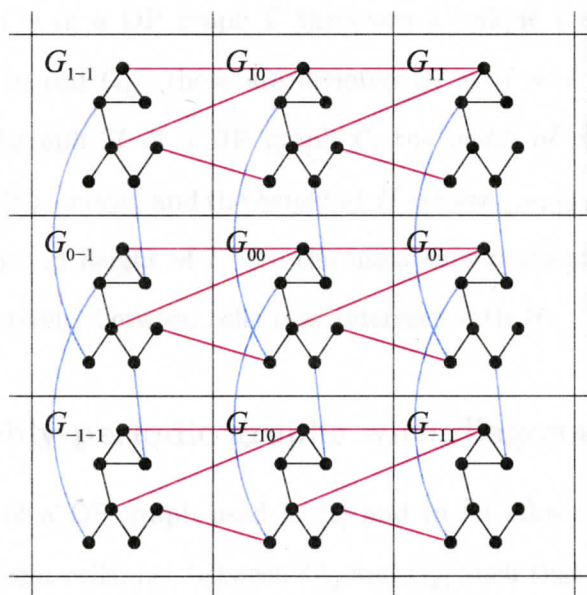


Figure 2.2: Nine cells of a DP graph  $\Gamma$ .

The  $i^{\text{th}}$  row of  $\Gamma$  is the subgraph induced by  $\bigcup_{j=-\infty}^{\infty} V_{ij}$  and the  $j^{\text{th}}$  column is the subgraph induced by  $\bigcup_{i=-\infty}^{\infty} V_{ij}$ . In fact, a DP graph  $\Gamma$  can be thought of as being formed from two SP graphs defined on the same cell graph: each

row of  $\Gamma$  is a copy of one, and each column a copy of the other.

For a DP graph  $\Gamma$ , the function  $T_{ab} : V(\Gamma) \rightarrow V(\Gamma)$  defined by  $T_{ab}(v_{ij}^k) = v_{(i+a)(j+b)}^k$  is an automorphism. The mapping  $T_{ab}$  can be thought of as a translation of  $\Gamma$  by  $a$  cells horizontally and  $b$  cells vertically;  $a$  is called the *horizontal period* and  $b$  the *vertical period*. For any two vertices  $u$  and  $v$  of the same type, the unique translation that maps  $u$  to  $v$  is denoted by  $t_{uv}$ . The inverse of  $T_{ab}$  is  $T_{ab}^{-1} = T_{(-a)(-b)}$ . The composition of two translations  $T_{ab}$  and  $T_{cd}$  is the translation  $T_{ab} \circ T_{cd} = T_{(a+c)(b+d)}$ .

For a vertex  $v$  in a DP graph  $\Gamma$  there are a unique  $i$  and a unique  $j$  such that  $v$  is in cell  $G_{ij}$ ; these are denoted by  $x(v) = i$  and  $y(v) = j$ . For a finite subgraph  $H$  of a DP graph  $\Gamma$ , the *width of  $H$*  is defined as  $\max_{u,v \in V(H)} \{|x(u) - x(v)|\}$  and the *height of  $H$*  is  $\max_{u,v \in V(H)} \{|y(u) - y(v)|\}$ . Thus the width and height of  $H$  are the maximum horizontal and vertical distance, respectively, between cells that intersect with  $H$ .

### 2.2.1 Doubly-periodic graphs with diagonal edges

The definition of a DP graph used in [1] and in [4] allows edges between diagonally-adjacent cells, i.e. between  $G_{ij}$  and  $G_{pq}$  such that  $|i - p| \leq 1$  and  $|j - q| \leq 1$ . For such a DP graph  $\Gamma$ , we can construct a new DP graph  $\Gamma'$  as follows: if there is an edge  $x_0y$  in  $\Gamma$ , with  $x_0 \in V(G_{ij})$  and  $y \in V(G_{(i+1)(j+1)})$ , we remove this edge and add new vertices  $x_1, x_2, \dots, x_{k-1} \in V(G_{ij})$ , and  $y' \in V(G_{(i+1)j})$ . We then add new edges  $x_i x_j$  for  $0 \leq i, j \leq k-1$ ,  $x_i y'$  for  $i \neq 0$ , and  $y' y$ . The vertices  $v_0, v_1, \dots, v_{k-1}$  form a  $k$ -clique, and  $y'$  is adjacent

to all of them except  $x_0$ . This construction for the case of  $k = 3$  is illustrated in Figure 2.3.

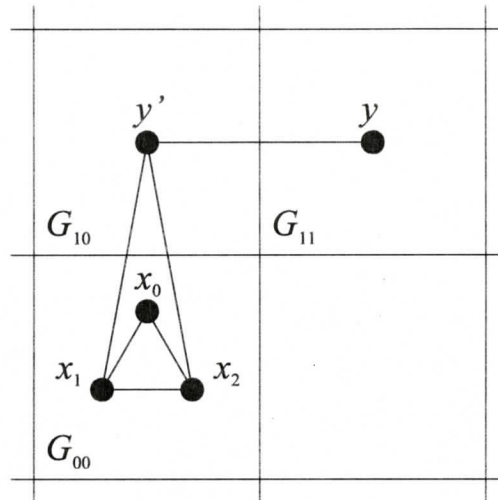


Figure 2.3: The construction for  $\Gamma'$ , with  $k = 3$ .

**Theorem 2.1** *A DP graph  $\Gamma$  with diagonal edges is  $k$ -colourable if and only if  $\Gamma'$  is  $k$ -colourable.*

**Proof.** In a  $k$ -colouring of  $\Gamma'$ ,  $x_0$  and  $y'$  must receive the same colour and  $y$  must receive a different colour, which is exactly what must happen in a  $k$ -colouring of  $\Gamma$ . Thus  $\Gamma'$  is  $k$ -colourable if and only if  $\Gamma$  is  $k$ -colourable. ■

Because of this theorem, results on colourings of DP graphs with diagonal edges also apply to the DP graphs under consideration in this thesis, and vice versa.

## 2.3 Digraphs

We will make extensive use of auxiliary digraphs in this thesis, and this section states some necessary definitions and preliminary results for digraphs.

For concepts not defined here, the reader may refer to West [8].

An *oriented walk*  $W$  in a digraph  $D$  is an alternating sequence  $v_0e_1v_1e_2v_2 \cdots v_{k-1}e_kv_k$  of vertices and arcs such that  $e_i = v_{i-1}v_i$  or  $e_i = v_iv_{i-1}$ , for  $1 \leq i \leq k$ . If  $e_i = v_{i-1}v_i$ , then  $e_i$  is a *forward arc*, and if  $e_i = v_iv_{i-1}$ , then  $e_i$  is a *backward arc*. Provided there is no ambiguity, we will refer to a walk by the sequence of vertices  $v_0v_1v_2 \cdots v_{k-1}v_k$ , omitting the arcs from the listing. The integer  $k$  is the *length* of  $W$ . The walk  $W$  is *closed* if  $v_0 = v_k$ . If all arcs in  $W$  are forward arcs, then  $W$  is a *directed walk* from  $v_0$  to  $v_k$ . Oriented and directed paths and cycles are defined analogously to paths and cycles of a graph. In general, we will drop the qualifier “oriented” and abbreviate “oriented walk” to “walk”, and similarly for paths and cycles. If a walk is directed, then this will be stated explicitly.

The *net length* of a walk  $W$  is the number of forward arcs minus the number of backward arcs in  $W$ . An oriented walk is said to be *balanced* if its net length is zero. A component  $Q$  of a digraph is said to be *balanced* if every oriented cycle in  $Q$  is balanced.

If  $W = v_0e_1v_1e_2v_2 \cdots v_{k-1}e_kv_0$  is a closed walk in a digraph  $D$ , then a *cyclic sub-walk* of  $W$  is a sub-walk  $v_cv_{c+1} \cdots v_{d-1}v_d$  of  $W$  such that  $v_c = v_d$  and  $v_c, v_{c+1}, v_{c+2}, \dots, v_{d-1}$  are all distinct.

**Lemma 2.2** *The net length of a closed walk  $W$  is equal to the sum of the net lengths of the cyclic sub-walks in  $W$ .*

**Proof.** A closed walk  $W$  can be broken down into cycles and/or paths which are traversed in both directions. Any such path contributes exactly 0 to the net length of  $W$ , since each arc is counted once in either direction. Thus the net length of  $W$  is simply the sum of the net lengths of the cycles. ■

A 2-edge-coloured digraph  $D$  consists of a vertex set  $V(D)$ , a set  $E_r(D)$  of red arcs and a set  $E_b(D)$  of blue arcs. The two arc sets are not necessarily disjoint, so there can be both a red arc and a blue arc between some pairs of vertices.

In a digraph  $D$ , the red [blue] net length of an oriented walk  $W$ , which we will denote by  $rnl(W)$  [ $bnl(W)$ ] is the number of forward red [blue] arcs minus the number of backward red [blue] arcs in  $W$ . The walk  $W$  is said to be red-balanced [blue-balanced] if its red [blue] net length is zero, and doubly-balanced if its red net length and blue net length are both zero. A component  $Q$  of a 2-edge-coloured digraph is said to be red-balanced [blue-balanced] if every oriented cycle in  $Q$  is red-balanced [blue-balanced], and doubly-balanced if every oriented cycle in  $Q$  is doubly-balanced.

A digraph homomorphism from  $D$  to  $D'$  is a mapping  $f : V(D) \rightarrow V(D')$  that preserves adjacency; i.e. if  $uv \in E(D)$  then  $f(u)f(v) \in E(D')$ . In the case of 2-edge-coloured digraphs,  $f$  is a homomorphism as long as  $f(u)f(v) \in E_r(D')$  whenever  $uv \in E_r(D)$ , and  $f(w)f(z) \in E_b(D')$  whenever

$wz \in E_b(D)$ . A digraph  $D$  is said to contain a *homomorphic image* of a digraph  $D'$  if there exists a digraph homomorphism from  $D'$  to  $D$ .

# Chapter 3

## Previous Results

The purpose of this chapter is to review previous work which relates to the results of this thesis.

### 3.1 $k$ -Colourings of Infinite Graphs

The results of Chapter 5 depend on the following theorem about  $k$ -colourings of infinite graphs.

**Theorem 3.1 (Erdős-De Bruijn)** [3] *Every finite subgraph of an infinite graph  $G$  is  $k$ -colourable if and only if  $G$  is  $k$ -colourable.*

Any finite subgraph of an SP graph  $\Gamma$  is contained in a finite, contiguous set of cells. Therefore this theorem implies that  $\Gamma$  is  $k$ -colourable if and only if every finite “substrip” of cells  $\bigcup_{i=a}^b G_i$ , for some integers  $a$  and  $b$ , is  $k$ -colourable. Similarly, if  $\Gamma$  is a DP graph, then any finite subgraph is

contained in some rectangular set of cells,  $\bigcup_{i=a}^b \bigcup_{j=c}^d G_{ij}$ , for some integers  $a, b, c$  and  $d$ , and so  $\Gamma$  is  $k$ -colourable if and only if every finite rectangular set of cells is  $k$ -colourable. In particular, if some  $4n \times 16n^2$ -cell subgraph of  $\Gamma$  is not 2-colourable, then  $\Gamma$  itself is not bipartite. As we will see in the next section, the contrapositive of Theorem 3.3 states that if no  $4n \times 16n^2$ -cell subgraph of  $\Gamma$  contains an odd cycle, then  $\Gamma$  does not contain an odd cycle, and is therefore bipartite.

## 3.2 2-Colourability of Doubly-Periodic Graphs

This section states a useful lemma, along with the result that it is possible to determine whether or not a DP graph  $\Gamma$  is bipartite simply by looking at a finite subgraph of  $\Gamma$ . The DP graphs under consideration in [1] can have diagonal edges; see Theorem 2.1 for a discussion of such graphs.

**Lemma 3.2** [1] *If there is a path from a vertex  $v$  to a vertex  $w$  in  $\Gamma$ , then there is a path of length less than  $n$  from  $v$  to a vertex of the same type as  $w$ , where  $n = |V(G_{ij})|$ .*

**Theorem 3.3** [1] *If a DP graph  $\Gamma$  contains an odd cycle, then the subgraph of  $\Gamma$  induced by the cells  $G_{ij}$ ,  $0 \leq i < 16n^2$ ,  $0 \leq j < 4n$  contains an odd cycle.*

As a consequence of this result, bipartiteness of a DP graph can be determined from the cell graph and inter-cell connections by constructing a

$4n \times 16n^2$ -cell piece of the graph and checking to see whether or not it contains an odd cycle, which is a finite problem.

### 3.3 Extendable $k$ -Colouring of Doubly-Periodic Graphs

The problem of extending some pre-colouring of a subset of the vertices of a graph to a colouring of the entire graph is stated formally as follows:

#### EXTENDABLE VERTEX $k$ -COLOURING

*Instance:* A graph  $G$ , a positive integer  $k$ , and a  $k$ -colouring  $c$  of some finite subset of the vertices of  $G$ .

*Question:* Can  $c$  be extended to a  $k$ -colouring of all the vertices of  $G$ ?

**Theorem 3.4** [4] *For each fixed  $k \geq 3$ , EXTENDABLE VERTEX  $k$ -COLOURING of DP graphs is undecidable.*

As in the case of the previous section, the DP graphs studied in [4] may contain diagonal edges.

The corresponding problem for SP graphs is decidable, as we will explain in Section 5.2.3.

### 3.4 The Undecidability of the Domino Problem

Colourings of DP graphs are related to tilings of the plane that use a special kind of square tile; this relationship will be discussed more explicitly in Section 5.3.. A *domino set* is a finite set of square tiles, called *dominoes*, which are all the same size and whose edges are marked in different ways with symbols from some finite symbol set. The goal is to use copies of these tiles to cover an infinite plane which is divided into domino-sized squares, according to the following rules:

1. the dominos may not be rotated or reflected, only translated;
2. the symbols on adjacent domino edges must match;
3. one domino must be placed exactly over each of the squares.

A domino set is *solvable* if it is possible to tile the plane in this manner, and the *Domino Problem* consists of determining whether any given domino set is solvable. In [2], it was proved that the Domino Problem is undecidable, and thus there is no general algorithm for determining whether an arbitrary domino set is solvable.

# Chapter 4

## Components

This chapter presents results pertaining to the components of SP and DP graphs, including methods for determining properties of the components.

When studying the components of an SP or DP graph  $\Gamma$ , it suffices to consider the graph  $\Gamma^*$  which is obtained by collapsing each component of  $G_i$  or  $G_{ij}$  down to a single vertex and then replacing multiple edges with a single edge. The graph  $\Gamma^*$  is itself an SP or DP graph, with the property that no two vertices in the same cell are adjacent. There is an obvious one-to-one correspondence between the components of  $\Gamma^*$  and the components of  $\Gamma$ , with each component of  $\Gamma^*$  being a simplified version of the corresponding component of  $\Gamma$ .

If  $\Gamma$  is an SP graph, then the cell graph of  $\Gamma^*$  is denoted by  $G_i^*$ . Its vertex set  $V(G_i^*)$  is the set of components of  $G_i$  (so  $|V(G_i^*)| = \#c(G_i)$ ), and there is an edge in  $\Gamma^*$  between  $v_i^k$  and  $v_{i+1}^l$  if and only if there is an edge between

some vertex in the component corresponding to  $v^k$  in cell  $G_i$  and some vertex in the component corresponding to  $v^l$  in cell  $G_{i+1}$ .

If  $\Gamma$  is a DP graph, then the cell graph of  $\Gamma^*$  is denoted by  $G_{ij}^*$ , and its vertex set  $V(G_{ij}^*)$  is the set of components of  $G_{ij}$  (so  $|V(G_{ij}^*)| = \#c(G_{ij})$ ). There is an edge in  $\Gamma^*$  between  $v_{ij}^k$  and  $v_{i(j+1)}^l$  if and only if there is an edge between some vertex in the component corresponding to  $v^k$  in cell  $G_{ij}$  and some vertex in the component corresponding to  $v^l$  in cell  $G_{i(j+1)}$ . Similarly, there is an edge in  $\Gamma^*$  between  $v_{ij}^k$  and  $v_{(i+1)j}^l$  if and only if there is an edge between some vertex in the component corresponding to  $v^k$  in cell  $G_{ij}$  and some vertex in the component corresponding to  $v^l$  in cell  $G_{(i+1)j}$ .

## 4.1 The Connection Digraph

For an SP graph  $\Gamma$ , we define an auxiliary digraph  $D(\Gamma)$ , called the *connection digraph*. The connection digraph provides a useful representation of the connections between cells of  $\Gamma$ , and allows us to determine the exact nature of the components of  $\Gamma$ .

The vertex set  $V(D(\Gamma))$  is the set of components of  $G_i$  (or vertices of  $G_i^*$ ), and there is an arc  $uv$  from  $u$  to  $v$  in  $E(D(\Gamma))$  if and only if there is an edge between component  $u$  in  $G_i$  and component  $v$  in  $G_{i+1}$ . For any SP graph,  $|V(D(\Gamma))| = \#c(G_i)$ . For the sake of simplicity, the number of components of  $G_i$  will be denoted by  $\#c$ . The components of  $\Gamma$  can be completely described based on properties of the connection digraph. Figure 4.1 shows an SP graph

$\Gamma$  and its connection digraph  $D(\Gamma)$ .

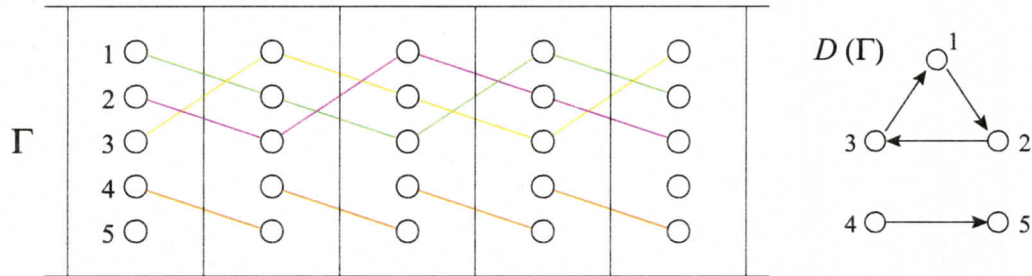


Figure 4.1: An SP graph and its connection digraph.

The connection digraph of a DP graph  $\Gamma$  is a 2-edge-coloured digraph in which the two edge colours represent connections horizontally and vertically. The vertex set  $V(D(\Gamma))$  is the set of components of  $G_{ij}$  (or vertices of  $G_{ij}^*$ ). There is a red arc  $uv$  in  $D(\Gamma)$  (i.e.  $uv \in E_r(D(\Gamma))$ ) if and only if there is an edge between component  $u$  in  $G_{ij}$  and component  $v$  in  $G_{i(j+1)}$ , and a blue arc  $uv$  in  $D(\Gamma)$  (i.e.  $uv \in E_b(D(\Gamma))$ ) if and only if there is an edge between component  $u$  in  $G_{ij}$  and component  $v$  in  $G_{(i+1)j}$ . Thus the red arcs in  $D(\Gamma)$  indicate horizontal connections between components, and the blue arcs indicate vertical connections. For a DP graph,  $|V(D(\Gamma))| = \#c(G_{ij}) = \#c$ .

Figures 4.2 and 4.3 show an example DP graph and its connection digraph.

## 4.2 Singly-Periodic Graphs

In this section we describe how the number of components in an SP graph and the size of each component can be determined by looking at the cell graph

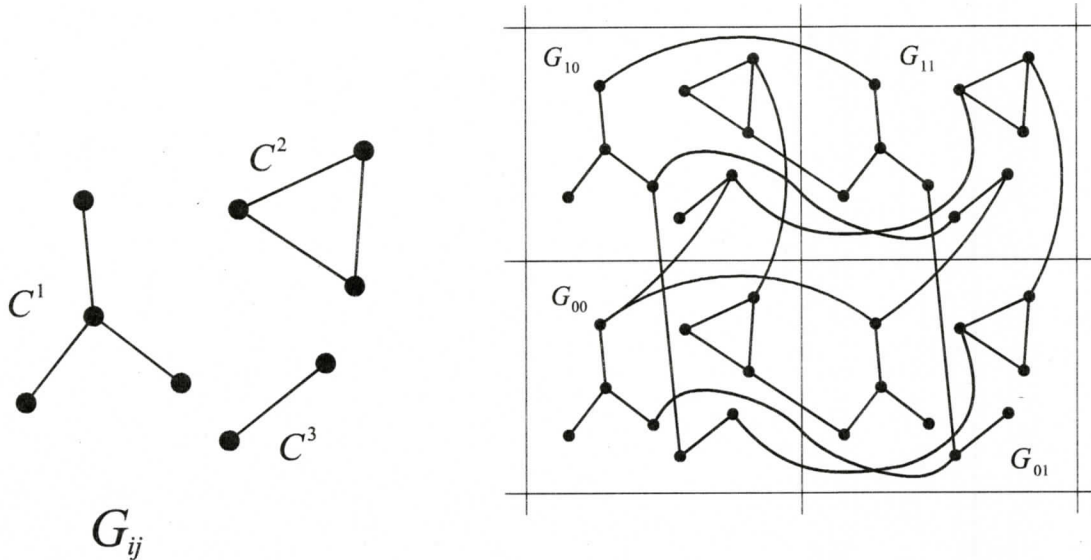


Figure 4.2: The cell graph  $G_{ij}$  and four cells of  $\Gamma$ .

and the connections between cells. Throughout this section, all graphs  $\Gamma$  are SP graphs.

**Lemma 4.1** *For each component  $\Theta$  of  $\Gamma$ , either  $\Theta$  is finite and has width at most  $\#c$ , or  $V(\Theta) \cap V(G_i) \neq \emptyset$ , for every integer  $i$ .*

**Proof.** Let  $\Theta$  be a component of  $\Gamma$ , and  $\Theta^*$  the corresponding component of  $\Gamma^*$ . Clearly  $\Theta^*$  can only contain vertices from consecutive cells, since there are no edges between non-consecutive cells. If  $\Theta^*$  contains more than one vertex of the same type, say  $v_i^k$  and  $v_{i+a}^k$ ,  $a \neq 0$ , then there is a  $v_i^k v_{i+a}^k$  path in  $\Gamma^*$ . Because of the translational symmetry of  $\Gamma^*$ , there must also be a  $v_i^k v_{i+2a}^k$  path in  $\Gamma^*$ , and in fact there is a  $v_i^k v_{i+ma}^k$  path for every  $m \in \mathbb{Z}$ . Thus  $\Theta^*$ , and therefore  $\Theta$ , must be infinite, and since there are no edges between

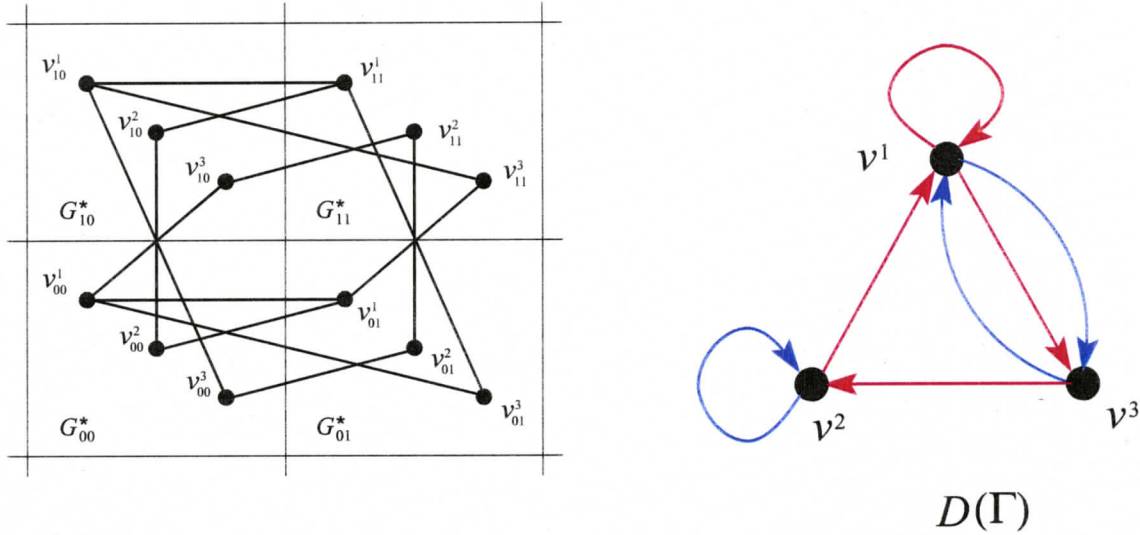


Figure 4.3: Four cells of  $\Gamma^*$  and the connection digraph  $D(\Gamma)$ .

non-adjacent cells,  $\Theta$  must contain at least one vertex from every cell of  $\Gamma$ . On the other hand, a finite component of  $\Gamma^*$  can contain at most one vertex of each type, and therefore  $|V(\Theta^*)| \leq \#c$ ; neither  $\Theta^*$  nor its corresponding component  $\Theta$  in  $\Gamma$  can contain vertices from more than  $\#c$  cells. ■

Intuitively, traversing a path in  $D(\Gamma)$  corresponds to moving one cell to the right in  $\Gamma^*$  for every forward arc, and one cell to the left for every backward arc.

**Lemma 4.2** *A walk  $W = v^0v^1 \dots v^m$  with net length  $b$  in a connection digraph  $D(\Gamma)$  corresponds to a walk in  $\Gamma^*$  from vertex  $v_i^0$  in  $G_i^*$  to vertex  $v_{i+b}^m$  in  $G_{i+b}^*$ , where  $i \in \mathbb{Z}$ .*

**Proof.** Every forward arc  $v^s v^t$  in  $W$  represents an edge between vertex  $v^s$  in  $G_i^*$  and vertex  $v^t$  in  $G_{i+1}^*$ , and every backward arc  $v^s v^t$  represents an edge

between vertex  $v^s$  in  $G_i^*$  and vertex  $v^t$  in  $G_{i-1}^*$ . ■

**Theorem 4.3** *An SP graph  $\Gamma$  is connected if and only if any two vertices of  $D(\Gamma)$  are joined by a balanced walk.*

**Proof.** If  $\Gamma$  is connected, then  $\Gamma^*$  is also connected, and thus there is a path in  $\Gamma^*$  between any two vertices  $v_i^k$  and  $v_i^l$  in the same cell. By Lemma 4.2, this path corresponds to a walk in  $D(\Gamma)$  of net length zero, i.e. a balanced walk.

Now suppose that any two vertices  $v^k$  and  $v^l$  of  $D(\Gamma)$  are joined by a balanced walk. This corresponds to a walk in  $\Gamma^*$  between vertices  $v_i^k$  and  $v_i^l$ , for all integers  $i$ . Thus all of the vertices in any particular cell belong to the same component of  $\Gamma^*$ . Since any two vertices of  $D(\Gamma)$  are joined by a balanced walk,  $D(\Gamma)$  must contain arcs between different vertices, which correspond to edges between adjacent cells. Therefore the entire graph must be connected. ■

**Lemma 4.4** *For an SP graph  $\Gamma$ , a component  $Q$  of its connection digraph  $D(\Gamma)$  corresponds either to a single component of  $\Gamma$ , or to a set of pairwise isomorphic components of  $\Gamma$ .*

**Proof.** Clearly there can only be a path in  $\Gamma^*$  between  $v_i^k$  and  $v_j^l$  if there is a walk in  $D(\Gamma)$  between  $v^k$  and  $v^l$ , so if two vertices are in different components of  $D(\Gamma)$  then no two vertices of those types respectively can be in the same component of  $\Gamma^*$ . On the other hand, for any two vertices  $v^k$  and  $v^l$  in a

component  $Q$  of  $D(\Gamma)$ , there is a path in  $\Gamma^*$  between some vertex  $v_i^k$  and some  $v_j^l$ , so there is a component of  $\Gamma^*$  which contains vertices of those types. Because of the translational symmetry of  $\Gamma^*$ , any translation  $T_a$  of  $\Gamma^*$  must map a component  $Q$  either to itself or to a component isomorphic to itself, so  $Q$  corresponds either to a single component of  $\Gamma^*$ , or to a set of pairwise isomorphic components of  $\Gamma^*$ , and likewise for  $\Gamma$ . ■

The components of  $\Gamma$  or  $\Gamma^*$  corresponding to a component  $Q$  of  $D(\Gamma)$ , will be referred to as the  $Q$ -components. A translation that fixes the  $Q$ -components of  $\Gamma$  maps the vertices of each  $Q$ -component to vertices of the same component; such a mapping is called a  $Q$ -component-fixing, or  $Q$ -cf, translation.

**Theorem 4.5** *For an SP graph  $\Gamma$ , a component  $Q$  of its connection digraph  $D(\Gamma)$  is balanced if and only if  $Q$  corresponds to infinitely many pairwise isomorphic finite components of  $\Gamma$ , each with width at most  $|V(Q)|$ .*

**Proof.** Suppose every cycle in  $Q$  is balanced, and note that this implies that every closed walk is balanced. If  $v$  is a vertex in  $Q$ , then any closed walk from  $v$  back to itself must have net length zero, and so by Lemma 4.2 it represents a path in  $\Gamma^*$  from vertex  $v_i$  to vertex  $v_{i+0} = v_i$ , i.e. a closed walk from  $v_i$  back to itself. Thus for any vertex type in  $Q$ , there are no paths in  $\Gamma^*$  between vertices of that type in different cells. Every component of  $\Gamma^*$  with vertex types from  $Q$  must therefore contain exactly one vertex of each type from  $Q$ , and can span at most  $|V(Q)|$  cells, and likewise for the  $Q$ -components of  $\Gamma$ .

Now suppose that  $Q$  corresponds to infinitely many pairwise isomorphic finite components of  $\Gamma$ , and that there is a cycle in  $Q$  with non-zero net length. This cycle corresponds to a path in  $\Gamma^*$  between two vertices of the same type in different cells, implying that there must be infinite  $Q$ -components and leading to a contradiction. ■

**Theorem 4.6** *Let  $\Gamma$  be an SP graph and  $Q$  be a component of its connection digraph  $D(\Gamma)$ . If there is a cycle in  $Q$  that is not balanced, then  $Q$  corresponds to  $\psi$  pairwise isomorphic infinite components of  $\Gamma$ , where  $\psi$  is the gcd of the net lengths of all cycles in  $Q$ .*

**Proof.** Suppose there is a cycle  $C = v^0v^1 \dots v^mv^0$  with net length  $a \neq 0$  in  $Q$ . For any vertex  $u$  in  $Q$ , let  $P$  be a shortest path from  $u$  to a vertex  $v^k$  on  $C$ . (If  $u$  is on  $Q$ , then  $P = \emptyset$ .) The closed walk  $P \cup C \cup P^{-1}$ , where  $P^{-1}$  is  $P$  backwards, is a closed walk of net length  $a$  from  $u$  back to itself. Thus the translation  $T_a$  fixes the  $Q$ -components in  $\Gamma$ , since for a vertex  $v^k$  in  $Q$ , there will be a path in  $\Gamma^*$  between  $v_i^k$  and  $v_{i+a}^k$ , for all  $i \in \mathbb{Z}$ . Every  $a^{\text{th}}$  vertex of type  $k$  will be in the same component of  $\Gamma^*$ , and  $T_a$  maps each one to the next.

If  $Q$  contains another cycle  $C'$  with net length  $b$ , then the translation  $T_b$  also fixes the  $Q$ -components in  $\Gamma^*$ . Clearly the set of  $Q$ -cf translations is closed under composition, so any linear combination  $c = ax + by$  will yield a  $Q$ -cf translation with period  $c$ . If  $d$  is the greatest common divisor (gcd) of  $a$  and  $b$ , there exist integers  $x$  and  $y$  such that  $ax + by = d$ . Thus the

translation  $T_d$  fixes components and has period  $d \leq a, b$ .

If  $\psi$  is the gcd of the net lengths of all cycles in  $Q$ , then using the above argument inductively,  $T_\psi$  is a  $Q$ -cf translation. It must also be the smallest, for if there is another  $Q$ -cf translation  $T_x$  with  $x < \psi$  then there must be a closed walk  $W$  of net length  $x$  in  $Q$ . Let  $C_1, C_2, \dots, C_t$  be the cycles in  $W$ , and let  $x_i$  be the net length of  $C_i$ , for  $1 \leq i \leq t$ . Thus by Lemma 2.1,  $x = \sum_{i=1}^t x_i$ . If  $g$  is the gcd of the net lengths of the non-balanced cycles in  $W$ , then  $g$  divides each  $x_i$ , and therefore  $g$  divides  $x$ . But since each of the cycles in  $W$  are in  $Q$ ,  $\psi$  divides  $g$  and therefore  $x$ , and this contradicts the assumption that  $x < \psi$ .

Thus for a vertex  $v^k$  in  $Q$ , the vertices of type  $k$  in those and only those cells whose indices are congruent modulo  $\psi$  will be in the same component of  $\Gamma^*$ . Because of the symmetry of the SP graph, this means that there are  $\psi$  pairwise isomorphic components of  $\Gamma^*$  that correspond to  $Q$ . ■

As a consequence of the results of this section, it is possible to determine the number and shapes of all of the components of an SP graph  $\Gamma$ . This can be done by first constructing  $D(\Gamma)$  and finding its components. If a component  $Q$  of  $D(\Gamma)$  is balanced, then  $\Gamma$  contains infinitely many finite components, which contain the vertices found in the components of  $G_i$  which correspond to the vertices of  $Q$ . If  $Q$  is not balanced and  $\psi$  is the gcd of the net lengths of all cycles in  $Q$ , then  $\Gamma$  contains  $\psi$  pairwise isomorphic infinite components, each of which again contain the vertices found in the components of  $G_i$  which correspond to the vertices of  $Q$ .

### 4.3 Doubly-Periodic Graphs

This section deals with questions analogous to those covered in the previous section, for the case of DP graphs. While many of the problems are similar, the increase in complexity leads to richer and more interesting results. Much like in the case of SP graphs, components of DP graphs can be either finite, with width and height bounded by a known constant, or infinite. The added dimension of the DP case allows for infinite components which may have either finite width or finite height and which appear infinitely many times in  $\Gamma$ , as well as components which have infinite height and width and appear only finitely many times.

All graphs  $\Gamma$  in this section are assumed to be DP graphs.

**Lemma 4.7** *A walk  $W = v^0 v^1 \dots v^m$  with  $rnl(W) = r$  and  $bnl(W) = b$  in a connection digraph  $D(\Gamma)$  corresponds to a walk in  $\Gamma^*$  from vertex  $v^0$  in  $G_{ij}$  to vertex  $v^m$  in  $G_{(i+r)(j+b)}$ , where  $i, j \in \mathbb{Z}$ .*

**Proof.** Every forward red arc  $v^s v^t$  in  $W$  represents an edge between vertex  $v^s$  in  $G_{ij}^*$  and vertex  $v^t$  in  $G_{i(j+1)}^*$ , and every backward red arc  $v^s v^t$  represents an edge between vertex  $v^s$  in  $G_{ij}^*$  and vertex  $v^t$  in  $G_{i(j-1)}^*$ ; similarly every forward blue arc  $v^s v^t$  in  $W$  represents an edge between vertex  $v^s$  in  $G_{ij}^*$  and vertex  $v^t$  in  $G_{(i+1)j}^*$ , and every backward blue arc  $v^s v^t$  represents an edge between vertex  $v^s$  in  $G_{ij}^*$  and vertex  $v^t$  in  $G_{(i-1)j}^*$ . ■

The following theorem provides a way to check for connectedness by looking only at the cell graph and the connections between cells, as represented

by  $D(\Gamma)$ .

**Theorem 4.8** *A DP graph  $\Gamma$  is connected if and only if  $D(\Gamma)$  contains both red and blue edges and any two vertices of  $D(\Gamma)$  are joined by a doubly-balanced walk.*

**Proof.** If  $\Gamma$  is connected, then clearly  $\Gamma^*$  is also connected, and thus there is a path in  $\Gamma^*$  between any two vertices  $v_{ij}^k$  and  $v_{ij}^l$  in the same cell. This path corresponds to a walk in  $D(\Gamma)$  of red net length zero and blue net length zero, i.e. a doubly-balanced walk. If  $D(\Gamma)$  contains only red edges then there are no edges between adjacent rows of  $\Gamma$ , and if  $D(\Gamma)$  contains only blue edges then there are no edges between adjacent columns of  $\Gamma$ , so in either case  $\Gamma$  cannot be connected.

Conversely, suppose that any two vertices  $v^k$  and  $v^l$  of  $D(\Gamma)$  are joined by a doubly-balanced walk. This corresponds to a walk in  $\Gamma^*$  between vertices  $v_{ij}^k$  and  $v_{ij}^l$ , for all integers  $i$  and  $j$ . Thus all of the vertices in any particular cell belong to the same component of  $\Gamma^*$ . Since  $D(\Gamma)$  contains both red and blue edges, there are edges between cells both horizontally and vertically, and therefore the entire graph must be connected. ■

**Lemma 4.9** *A component  $Q$  of a connection digraph  $D(\Gamma)$  corresponds either to a single component of  $\Gamma$ , or to a set of pairwise isomorphic components of  $\Gamma$ .*

**Proof.** Clearly there can only be a path in  $\Gamma^*$  between  $v_{ij}^k$  and  $v_{pq}^l$  if there is a walk in  $D(\Gamma)$  between  $v^k$  and  $v^l$ , so if two vertices are in different components

of  $D(\Gamma)$  then no two vertices of those types respectively can be in the same component of  $\Gamma^*$ . On the other hand, for any two vertices  $v^k$  and  $v^l$  in a component  $Q$  of  $D(\Gamma)$ , there is a path in  $\Gamma^*$  between some vertex  $v_{ij}^k$  and some  $v_{pq}^l$ , so there is a component of  $\Gamma^*$  which contains vertices of those types. Because of the translational symmetry of  $\Gamma^*$ , any translation  $T_{ab}$  of  $\Gamma^*$  must map a component  $Q$  either to itself or to a component isomorphic to itself, so  $Q$  corresponds either to a single component of  $\Gamma^*$ , or to a set of pairwise isomorphic components of  $\Gamma^*$ , and likewise for  $\Gamma$ . ■

**Theorem 4.10** *A component  $Q$  of a connection digraph  $D(\Gamma)$  is doubly-balanced if and only if  $Q$  corresponds to infinitely many pairwise isomorphic finite components of  $\Gamma$ , each with width and height at most  $|V(Q)|$ .*

**Proof.** Suppose every cycle in  $Q$  is doubly-balanced, and let  $v$  be a vertex in  $Q$ . Note that this implies that every closed walk is doubly-balanced. Thus any closed walk from  $v$  back to itself must have net length zero, and therefore by Lemma 4.7 it represents a path in  $\Gamma^*$  from vertex  $v_{ij}$  to vertex  $v_{(i+0)(j+0)} = v_{ij}$ , i.e. a closed walk from  $v_{ij}$  back to itself. Thus for any vertex type in  $Q$ , there are no paths in  $\Gamma^*$  between vertices of that type in different cells. Every component of  $\Gamma^*$  with vertex types from  $Q$  must therefore contain exactly one vertex of each type from  $Q$ , and can intersect with at most  $|V(Q)|$  cells.

The proof of the converse is identical to the second part of the proof of Theorem 4.5. ■

**Lemma 4.11** *If  $Q$  is a component of  $D(\Gamma)$ , and  $r$  and  $b$  are integers, then the translation  $T_{rb}$  fixes the  $Q$ -components in  $\Gamma$  if and only if  $Q$  contains a closed walk with red net length  $r$  and blue net length  $b$ .*

**Proof.** Suppose there is a closed walk  $C = v^0v^1 \dots v^mv^0$  in  $Q$  with  $rnl(C) = r$  and  $bnl(C) = b$ . If  $r = b = 0$ , then clearly  $T_{rb} = T_{00}$  fixes the  $Q$ -components, so assume  $r$  and  $b$  are not both equal to zero. For any vertex  $u$  in  $Q$ , let  $P$  be a shortest path from  $u$  to a vertex  $v^k$  on  $C$ . (If  $u$  is on  $Q$ , then  $P = \emptyset$ .) The closed walk  $P \cup C \cup P^{-1}$  is a closed walk from  $u$  back to itself with red and blue net lengths  $r$  and  $b$  respectively. Thus the translation  $T_{rb}$  fixes the  $Q$ -components in  $\Gamma$ , since for a vertex  $v^k$  in  $Q$ , there will be a path in  $\Gamma^*$  between  $v_{ij}^k$  and  $v_{(i+r)(j+b)}^k$ , for all  $i, j \in \mathbb{Z}$ .

Now let  $T_{rb}$  be a  $Q$ -cf translation, and let  $v_{ij}^k \in V(\Gamma)$ . The vertices  $v_{ij}^k$  and  $v_{(i+r)(j+b)}^k$  are in the same component of  $\Gamma$ , so there must be a path in  $\Gamma$  between them, and by Lemma 4.7 this path corresponds to a closed walk containing the vertex  $v^k$  in  $D(\Gamma)$ , with red net length  $r$  and blue net length  $b$ . ■

Note that as in the SP case, if two translations  $T$  and  $T'$  fix the same set of components of a DP graph  $\Gamma$ , then their composition  $T \circ T'$  also fixes this set of components. By induction, composing any finite number of such translations yields another translation that fixes these components.

**Theorem 4.12** *A component  $Q$  of a connection digraph  $D(\Gamma)$  corresponds to finitely many infinite components of  $\Gamma$  if and only if it contains a closed*

walk which is red-balanced but not blue-balanced, and a closed walk which is blue-balanced but not red-balanced.

**Proof.** By Lemma 4.11, if  $Q$  contains a closed walk with red net length  $r > 0$  and blue net length 0, then the translation  $T_{r0}$  fixes the  $Q$ -components in  $\Gamma$ . Similarly, if  $Q$  contains a closed walk with red net length 0 and blue net length  $b > 0$ , then the translation  $T_{0b}$  fixes the  $Q$ -components. In fact all translations of the form  $T_{(\rho r)(\beta b)}$ , with  $\rho, \beta \in \mathbb{Z}$ , fix these components.

Suppose that  $Q$  corresponds to finitely many infinite components of  $\Gamma$ . Let  $\Theta$  be one of these components, with corresponding component  $\Theta^*$  in  $\Gamma^*$ . Let  $v_{ij}^k$  be a vertex in  $\Theta^*$ . All other vertices of the same type as  $v_{ij}^k$  must be in  $Q$ -components in  $\Gamma^*$  (which are isomorphic to  $\Theta^*$ ). Since there is a finite number of these components, there must be vertices  $v_{i(j+y)}^k$  and  $v_{(i+x)j}^k$  in, respectively, the same row and column as  $v_{ij}^k$ , which are also in  $\Theta^*$ . The translations  $T_{x0}$  and  $T_{0y}$  map  $v_{ij}^k$  to  $v_{(i+x)j}^k$  and  $v_{i(j+y)}^k$ , respectively, and hence fix the  $Q$ -components.

Conversely, suppose there exist translations  $T_{x0}$  and  $T_{0y}$  that fix the  $Q$ -components. As above, let  $\Theta^*$  be a  $Q$ -component in  $\Gamma^*$  and  $v_{ij}^k$  a vertex in  $\Theta^*$ . Since  $T_{x0}$  and  $T_{0y}$  fix the  $Q$ -components, any translation of the form  $T_{(px)(qy)}$  also fixes the  $Q$ -components, and so all vertices of the form  $v_{(i+px)(j+qy)}^k$ , where  $p$  and  $q$  are integers, are in the same component as  $v_{ij}^k$ . If  $x = y = 1$ , then all vertices of type  $k$  are in the same component, so  $Q$  corresponds to a single component of  $\Gamma^*$ . Assume without loss of generality that  $x > 1$ . For  $1 \leq n < x$  and  $1 \leq m < y$ , all vertices of the form  $v_{(i+px+n)(j+qy+m)}^k$  must be

in a single component, by a similar argument. Thus there are at most  $nm$  different components containing vertices of type  $k$ . Since all such vertices must appear in exactly all of the  $Q$ -components, there is a finite number of  $Q$ -components, each of which must be infinite in size. ■

In other words, a component  $Q$  of  $D(\Gamma)$  corresponds to finitely many infinite components of  $\Gamma$  if and only if there are horizontal and vertical translations that fix the  $Q$ -components. Such a component of  $D(\Gamma)$  will be called an *h-v component* from now on. A component of  $\Gamma$  is either finite, with width and height at most  $\#c(G_{ij})$ , or infinite. An infinite component with infinitely many isomorphic copies in  $\Gamma$  is called a *2-way component*. An infinite component with finitely many isomorphic copies is called a *4-way component*. Theorem 4.12 says that if  $Q$  is an h-v component of  $D(\Gamma)$ , then the  $Q$ -components in  $\Gamma$  and  $\Gamma^*$  are 4-way components.

**Theorem 4.13** *A component  $Q$  of a connection digraph  $D(\Gamma)$  corresponds to 2-way components of  $\Gamma$  if and only if  $Q$  is not balanced and is not an h-v component. One of the following three cases will apply to such a component:*

1. *if  $Q$  contains a closed walk which is blue-balanced but not red-balanced, but no closed walk which is red-balanced but not blue-balanced, then the  $Q$ -components in  $\Gamma$  have finite height and infinite width;*
2. *if  $Q$  contains a closed walk which is red-balanced but not blue-balanced, but no closed walk which is blue-balanced but not red-balanced, then the  $Q$ -components in  $\Gamma$  have finite width and infinite height;*

3. if  $Q$  contains no closed walk which is balanced in one colour but not the other, then the  $Q$ -components in  $\Gamma$  have infinite width and infinite height;

**Proof.** If  $Q$  does not satisfy the hypotheses of either Theorem 4.10 or Theorem 4.12, then the  $Q$  components of  $\Gamma$  must be 2-way components. Suppose  $Q$  meets the requirements of Case 1; the closed walk which is blue-balanced but not red-balanced corresponds to a horizontal  $Q$ -cf translation, so the  $Q$ -components have infinite width. Since  $Q$  does not contain a closed walk which is red-balanced but not blue-balanced, there are no vertical  $Q$ -cf translations, so the  $Q$ -components have finite height. The proof of Case 2 is similar. If  $Q$  falls under Case 3, then there are no horizontal or vertical  $Q$ -cf translations, but since  $Q$  is not balanced the  $Q$ -components must be infinite, and the only remaining possibility is that they cut across the rows and columns of  $\Gamma$  in a diagonal fashion. ■

**Definition 4.14** For an  $h$ - $v$  component  $Q$  of a connection digraph  $D(\Gamma)$ ,  $\mu(Q)$  is the number of  $Q$ -components in  $\Gamma$ . The total number of 4-way components in  $\Gamma$  is denoted by  $\mu(\Gamma)$ .

In Figure 4.4, the components are coloured according to their types. A 4-way component is shown in yellow, and the brown and green components are examples from Cases 1 and 3 of Theorem 4.13, respectively. In Figure 4.5, all of the components are isomorphic copies of a single finite component; the four copies shown in their entirety are coloured dark blue, yellow, red,

and green, while the copies that are only partially drawn are all coloured light blue.

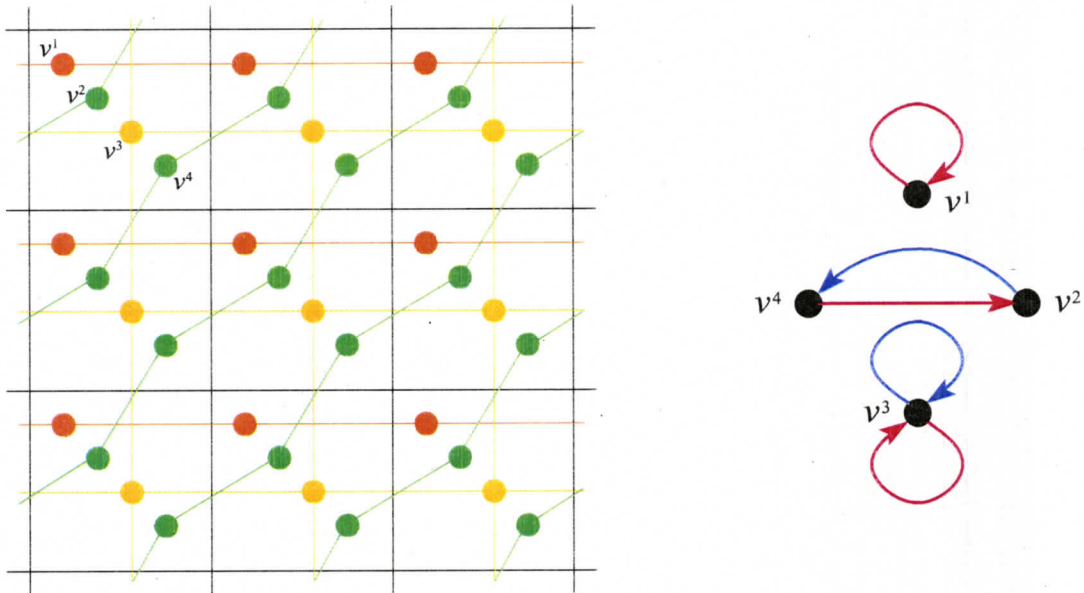


Figure 4.4: Examples of 2-way and 4-way components.

**Definition 4.15** A minimal closed walk in a digraph  $D$  is a closed walk  $W$  such that there is no cycle in  $D$  whose vertices appear as a subsequence in  $W$  more than once, i.e. it does not repeat any cycle.

Clearly, every closed walk  $W$  in a digraph  $D$  contains a unique minimal closed walk, obtained by removing all but one occurrence of each cycle whose vertices appear as subsequences in  $W$ . We will denote this minimal closed walk by  $W^{min}$ .

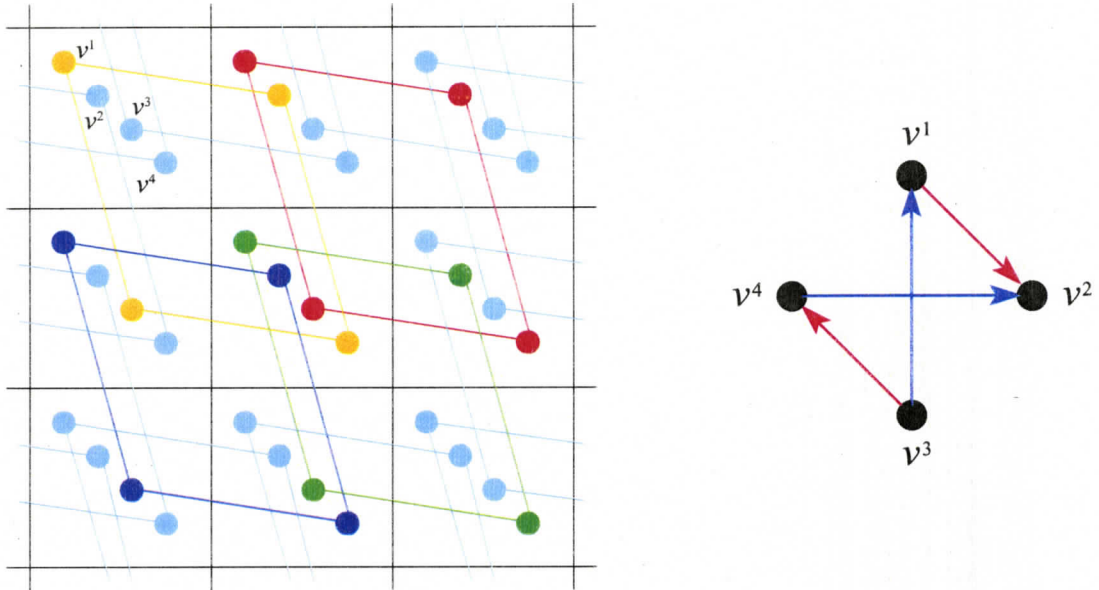


Figure 4.5: Examples of finite components.

**Definition 4.16** Let  $Q$  be an  $h$ - $v$  component of a connection digraph  $D(\Gamma)$ . Define  $r_0(Q)$  to be the gcd of the red net lengths of all minimal closed walks in  $Q$ , and define  $b_0(Q)$  to be the gcd of the blue net lengths of all minimal closed walks in  $Q$ .

**Definition 4.17** Let  $Q$  be an  $h$ - $v$  component of a connection digraph  $D(\Gamma)$ . Define  $r(Q)$  to be the gcd of the red net lengths of all blue-balanced minimal closed walks in  $Q$ , and define  $b(Q)$  to be the gcd of the blue net lengths of all red-balanced minimal closed walks in  $Q$ .

Note that since there is a finite number of cycles in  $Q$ , there is a finite number of ways to combine them into closed walks without repetition, and so

there cannot be infinitely many minimal closed walks in  $Q$ . Thus the values defined above are finite.

**Theorem 4.18** *Let  $Q$  be an  $h$ - $v$  component of a connection digraph  $D(\Gamma)$ . The translations  $T_{r(Q)0}$  and  $T_{0b(Q)}$  fix the  $Q$ -components in  $\Gamma$ . Furthermore,  $r(Q)$  and  $b(Q)$  are the smallest positive integers such that  $T_{r(Q)0}$  and  $T_{0b(Q)}$  are  $Q$ -cf translations.*

**Proof.** Let  $B_1, B_2, \dots, B_m$  be the minimal blue-balanced closed walks in  $Q$ , and let  $r_k = rnl(B_k)$ , for  $k = 1, 2, \dots, m$ . By Lemma 4.11, the translations  $T_{r_1 0}, T_{r_2 0}, \dots, T_{r_m 0}$  are  $Q$ -cf translations. Since  $r(Q)$  is the gcd of  $r_1, r_2, \dots, r_m$ , there exist integers  $e_1, e_2, \dots, e_m$  such that  $e_1 r_1 + e_2 r_2 + \dots + e_m r_m = r(Q)$ . Thus the translation  $T_{r(Q)0}$  is a composition of  $Q$ -cf translations, and therefore also fixes the  $Q$ -components. Similarly,  $T_{0b(Q)}$  is a  $Q$ -cf translation.

Suppose there exists some  $0 < x < r(Q)$  such that  $T_{x0}$  is a  $Q$ -cf translation. This implies the existence of a blue-balanced closed walk of red net length  $x$  in  $Q$ , and so  $r(Q)$  must divide  $x$ , contradicting the condition that  $x < r(Q)$ . Similarly,  $b(Q)$  is the smallest positive integer such that  $T_{0b(Q)}$  fixes the  $Q$ -components. ■

**Definition 4.19** *Let  $r(\Gamma)$  and  $b(\Gamma)$  be the smallest positive integers such that  $T_{r(\Gamma)0}$  and  $T_{0b(\Gamma)}$  fix all 4-way components of  $\Gamma$ .*

**Theorem 4.20** *For a DP graph  $\Gamma$ ,  $r(\Gamma)$  is the least common multiple of the values of  $r(Q)$  over all h-v components  $Q$  of  $D(\Gamma)$ , and  $b(\Gamma)$  is the least common multiple of the values of  $b(Q)$  over all h-v components  $Q$  of  $D(\Gamma)$ .*

**Proof.** Let  $Q_1, Q_2, \dots, Q_M$  be the h-v components of  $D(\Gamma)$ , and let  $R = \text{lcm}\{r(Q_1), r(Q_2), \dots, r(Q_M)\}$ . The translation  $T_{r(Q_m)0}$  is a  $Q_m$ -cf translation, for  $m = 1, 2, \dots, M$ , as are all translations of the form  $T_{cr(Q_m)0}$ , where  $c$  is an integer. Since each  $r(Q_m)$  divides  $R$ , the translation  $T_{R0}$  fixes all components of  $\Gamma$  corresponding to h-v components of  $D(\Gamma)$ , which are all of the 4-way components of  $\Gamma$ . Suppose there is a translation  $T_{R'0}$ , with  $R' < R$ , which fixes all of the 4-way components of  $\Gamma$ . For  $1 \leq m \leq M$ ,  $T_{R'0}$  fixes the  $Q_m$ -components, so there must be a closed walk  $W$  in  $Q_m$  with  $rnl(W) = R'$  and  $bnl(W) = 0$ . Since  $r(Q_m)$  is the gcd of the red net lengths of the blue-balanced closed walks in  $Q_m$ , we have  $r(Q_m) | R'$  and so  $R'$  is a common multiple of  $r(Q_m)$ ,  $m = 1, 2, \dots, M$ . Hence  $R \leq R'$ , contradicting the assumption that  $R' < R$ . Thus  $R = r(\Gamma)$ , and by a similar argument  $b(\Gamma) = \text{lcm}\{b(Q_1), b(Q_2), \dots, b(Q_M)\}$ . ■

**Theorem 4.21** *If  $Q$  is an h-v component of a connection digraph  $D(\Gamma)$ , then  $\mu(Q) = r(Q)b_0(Q) = b(Q)r_0(Q)$ .*

**Proof.** Let  $Q$  be an h-v component of  $D(\Gamma)$ , and let  $v^k$  be a vertex in  $Q$ . Since  $r(Q)$  is the smallest integer such that  $T_{r(Q)0}$  is a  $Q$ -cf translation, the translations  $T_{x0}$ , where  $0 < x < r(Q)$ , map each  $Q$ -component to a different one. Thus the vertices  $v_{ij}^k$  and  $v_{i(j+x)}^k$  are all in different components of  $\Gamma^*$ ,

and so the vertices of type  $k$  in row  $i$  of  $\Gamma^*$  are in  $r(Q)$  different components of  $\Gamma^*$ . Since this is true for each vertex  $v^k$  in  $Q$ ,  $\mu(Q)$  must be a multiple of  $r(Q)$ . If  $v^k$  is on a minimal closed walk  $W$  in  $Q$  with  $rnl(W) = r$  and  $bnl(W) = b \neq 0$ , then the translation  $T_{rb}$  is a  $Q$ -cf translation, so the vertices of type  $k$  in row  $i + b$  must be in the same  $r(Q)$  components of  $\Gamma^*$  as those in row  $i$ . Since  $b_0(Q)$  is the gcd of the blue net lengths of the minimal closed walks in  $Q$ ,  $b_0(Q)$  is the smallest such  $b$ . For  $0 < y < b_0(Q)$ , any translations with vertical period  $y$  are not  $Q$ -cf fixing translations, so the vertices of type  $k$  in row  $i + y$  are in  $r(Q)$  components which are all different from each other and from those in row  $i$ , so the vertices of type  $k$  appear in exactly  $r(Q)b_0(Q)$   $Q$ -components of  $\Gamma^*$ . For each vertex  $v^k$  in  $Q$ , every  $Q$ -component in  $\Gamma^*$  contains vertices of type  $k$ , since  $Q$  is a connected component of  $D(\Gamma)$ . Thus all of the  $Q$ -components have been accounted for, and  $\mu(Q) = r(Q)b_0(Q)$ . The  $Q$ -components can also be counted in a similar manner using the columns of  $\Gamma^*$ , and the result follows. ■

**Theorem 4.22** *For a DP graph  $\Gamma$ ,  $\mu(\Gamma) = \sum_Q \mu(Q)$ , where  $Q$  is an h-v component of  $D(\Gamma)$ .*

**Proof.** This follows from the fact that if  $Q_i$  and  $Q_j$  are distinct h-v components of  $D(\Gamma)$ , then the set of  $Q_i$ -components of  $\Gamma$  and the set of  $Q_j$ -components of  $\Gamma$  are disjoint. ■

As in the SP case, the results of this section allow us to completely determine the number and properties of the components of  $\Gamma$  simply by analysing

the connection digraph  $D(\Gamma)$ . If a component  $Q$  of  $D(\Gamma)$  is doubly-balanced, then  $\Gamma$  contains infinitely many finite components, the vertices of which are determined by the vertices of  $Q$ . If  $Q$  is an h-v component, then Theorem 4.21 allows us to determine the number of 4-way infinite components. Finally, if  $Q$  is neither balanced nor an h-v component, then  $\Gamma$  contains infinitely many copies of 2-way infinite components, which can be horizontal (Case 1 of Theorem 4.13), vertical (Case 2), or diagonal (Case 3).

# Chapter 5

## Colourings

In this chapter, we discuss vertex-colourings of SP and DP graphs. Clearly an SP or DP graph  $\Gamma$  is  $k$ -colourable only if its cell graph is. By Theorem 3.1,  $\Gamma$  is  $k$ -colourable if and only if the subgraph induced by any finite subset of cells is  $k$ -colourable. We will make use of an auxiliary digraph that relates the  $k$ -colourings of the cell graph to each other.

A  $k$ -colouring of an SP graph is said to be *periodic* if there is a translation  $T_x$  that fixes the colour of each vertex, or equivalently the colouring used on each cell. Similarly a  $k$ -colouring of a DP graph is periodic if there is a translation  $T_{xy}$  that fixes its colours. A  $k$ -colouring is *aperiodic* if there exists no such translation. For a colouring  $Z$  of an SP or DP graph, a translation that fixes its colours is called a  $Z$ -fixing translation.

The problem of colouring a DP graph is related to the problem of tiling the plane using square tiles which are compatible with each other in a specific

way. Each of the  $k$ -colourings of the cell graph  $G_{ij}$  can be represented by a tile, and a  $k$ -colouring of  $\Gamma$  is a tiling of the plane using copies of these tiles obtained by translations only, subject to compatibility restrictions determined by the cell colourings; there exists a  $k$ -colouring of  $\Gamma$  if and only if this set of tiles can be used to tile the entire plane.

## 5.1 The $k$ -Colouring Digraph

In this section we define an auxiliary digraph that provides a convenient way to represent the compatibility relationships between the  $k$ -colourings of the cell graph of an SP or DP graph, and thus can be used to determine whether or not the SP or DP graph is  $k$ -colourable.

For an SP graph  $\Gamma$  and positive integer  $k$ , the  $k$ -colouring digraph, or simply *colouring digraph*,  $C_k(\Gamma)$  is defined as follows:  $V(C_k(\Gamma))$  is the set of  $k$ -colourings of  $G_i$ , and for any  $u, v \in V(C_k(\Gamma))$  the arc  $uv$  is in  $E(C_k(\Gamma))$  if and only if colouring  $v$  is permissible on  $G_{i+1}$  whenever colouring  $u$  is used on  $G_i$ , meaning that this results in a  $k$ -colouring of the subgraph of  $\Gamma$  induced by  $G_i \cup G_{i+1}$ .

Figures 5.1 and 5.2 show examples of SP graphs and their colouring digraphs. Each figure shows two cells of an SP graph, along with its colouring digraph, in which the vertex labelled  $xyz$  represents the colouring of  $G_i$  in which vertices  $v_i^1$ ,  $v_i^2$ , and  $v_i^3$  receive colours  $x$ ,  $y$ , and  $z$ , respectively.

The colouring digraph of a DP graph is a 2-edge-coloured digraph defined

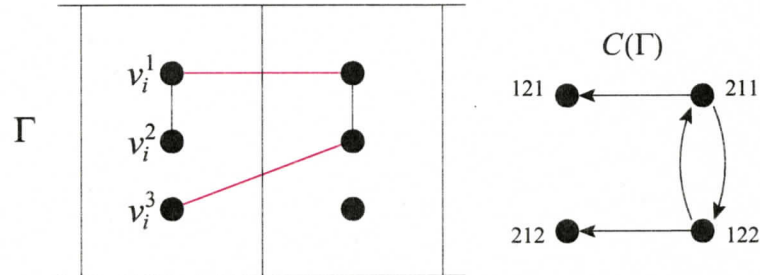


Figure 5.1: An SP graph and its 2-colouring digraph.

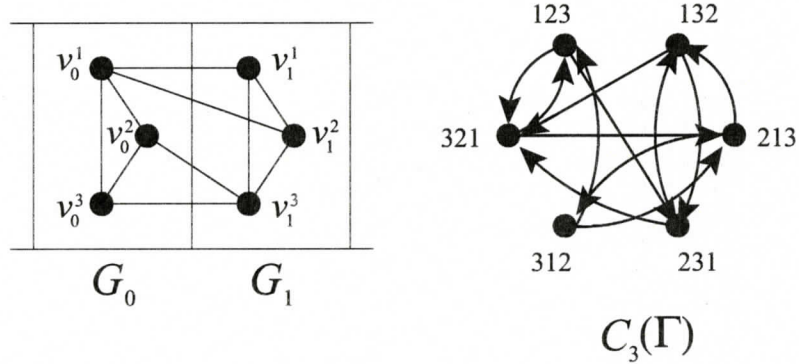


Figure 5.2: An SP graph and its 3-colouring digraph.

analogously to the connection digraph:  $V(C_k(\Gamma))$  is the set of  $k$ -colourings of  $G_{ij}$ , and for any  $u, v \in V(C_k(\Gamma))$  the arc  $uv$  is in  $E_r(C_k(\Gamma))$  if and only if colouring  $v$  is permissible on  $G_{i(j+1)}$  whenever colouring  $u$  is used on  $G_{ij}$  and in  $E_b(C_k(\Gamma))$  if and only if colouring  $v$  is permissible on  $G_{(i+1)j}$  whenever colouring  $u$  is used on  $G_{ij}$ . Figure 5.3 shows a DP graph and its 2-colouring digraph.

Our use of the colouring digraph to study the colourability of SP and DP graphs relies implicitly on Theorem 3.1, due to Erdős and DeBruijn. An SP or DP graph  $\Gamma$  is  $k$ -colourable if (and only if) every finite subgraph

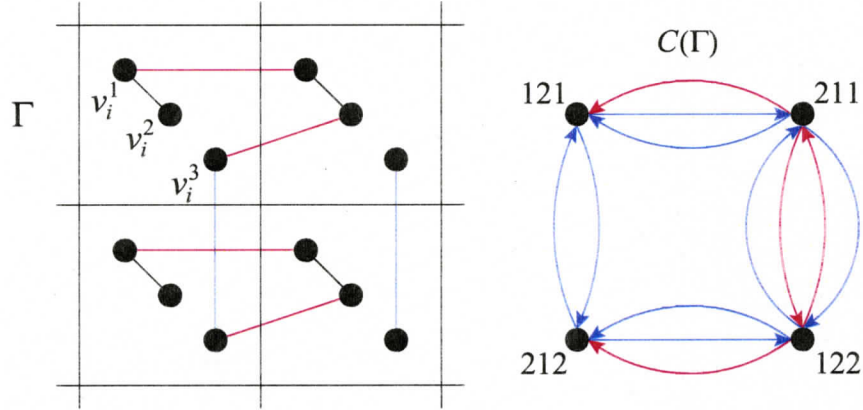


Figure 5.3: A DP graph and its 2-colouring digraph.

of  $\Gamma$  is  $k$ -colourable. Every finite subgraph  $H$  of an SP graph  $\Gamma$  has finite width  $\omega$ , and therefore must be contained in some subset  $S(H) = \{G_t \mid t_0 \leq t \leq t_0 + \omega\}$  of the cells of  $\Gamma$ , where  $t_0 = \min\{x(v) \mid v \in V(H)\}$ . Similarly, every finite subgraph  $H$  of a DP graph  $\Gamma$  has finite width  $\omega$  and height  $\lambda$ , and therefore must be contained in some subset  $S(H) = \{G_{st} \mid s_0 \leq s \leq s_0 + \lambda, t_0 \leq t \leq t_0 + \omega\}$  of the cells of  $\Gamma$ , where  $s_0 = \min\{y(v) \mid v \in V(H)\}$  and  $t_0 = \min\{x(v) \mid v \in V(H)\}$ . Because it codes the compatibility relationships between cell colourings, the colouring digraph can be used to determine whether or not the subgraph of  $\Gamma$  induced by  $\bigcup_{G_t \in S(H)} V(G_t)$  has a  $k$ -colouring, for any finite subgraph  $H$  of  $\Gamma$ . If there is a set of cell colourings that can be used to colour the cells of  $S(H)$  so that the resultant colouring is proper, then  $H$  is  $k$ -colourable.

Under a given  $k$ -colouring  $Z$  of an SP or DP graph  $\Gamma$ , the vertex of  $C_k(\Gamma)$  corresponding to the colouring used on cell  $G_i$  or  $G_{ij}$  will be denoted by

$Z(G_i)$  or  $Z(G_{i_j})$ , respectively.

## 5.2 Singly-Periodic Graphs

This section deals with topics related to colourings of SP graphs, including determining colourability, periodic and aperiodic colourings, and symmetry groups of colourings. All graphs  $\Gamma$  in this section are SP graphs.

### 5.2.1 $k$ -Colourability and Periodic Colourings

This section describes the  $k$ -colouring digraph  $C_k(\Gamma)$  of a  $k$ -colourable SP graph  $\Gamma$ . In the case of SP graphs, the existence of  $k$ -colourings implies the existence of periodic  $k$ -colourings.

**Theorem 5.1** *For an SP graph  $\Gamma$  and an integer  $k \geq 2$ , there is a  $k$ -colouring of  $\Gamma$  if and only if there is a directed cycle in  $C_k(\Gamma)$ .*

**Proof.** Let  $D = v_0v_1 \cdots v_{m-1}v_0$  be a directed cycle in  $C_k(\Gamma)$ . Each vertex on  $D$  represents a  $k$ -colouring of  $G_i$ , and the sequence of colourings given by  $D$  can be repeated infinitely often to give a  $k$ -colouring of  $\Gamma$ . We can colour  $G_0$  with colouring  $v_0$ ,  $G_1$  with colouring  $v_1$ , and so on up to  $G_m$ , which will be coloured with colouring  $v_0$ . We can then colour  $G_{m+1}$  using colouring  $v_1$ , and continue around the cycle indefinitely. Similarly,  $G_{-1}$  receives colouring  $v_{m-1}$ ,  $G_{-2}$  receives colouring  $v_{m-2}$ , and so on going around  $D$  in the backwards

direction. In general  $G_x$  is coloured using colouring  $v_{[x]_m}$ , where  $[x]_m$  is the residue of  $x$  modulo  $m$ .

Conversely, suppose that  $\Gamma$  is  $k$ -colourable and consider a  $k$ -colouring of  $\Gamma$ . Since  $G_i$  has a finite number of  $k$ -colourings, any  $k$ -colouring  $Z$  of  $\Gamma$  must use at least one of them infinitely often. Thus there exist integers  $i$  and  $j$  such that  $G_i$  and  $G_j$  are coloured the same; that is,  $Z(G_i) = Z(G_j)$ . For  $i \leq t \leq j$ , let  $v_t = Z(G_t)$ , and then  $v_i = v_j$  and  $v_i v_{i+1} v_{i+2} \cdots v_{j-1} v_j$  is a directed closed walk in  $C_k(\Gamma)$ , and therefore  $C_k(\Gamma)$  contains a directed cycle.

■

Figure 5.4 illustrates this situation; the directed cycle 12345 appears in  $C_k(\Gamma)$ , so these five colourings can be used to periodically colour  $\Gamma$ .

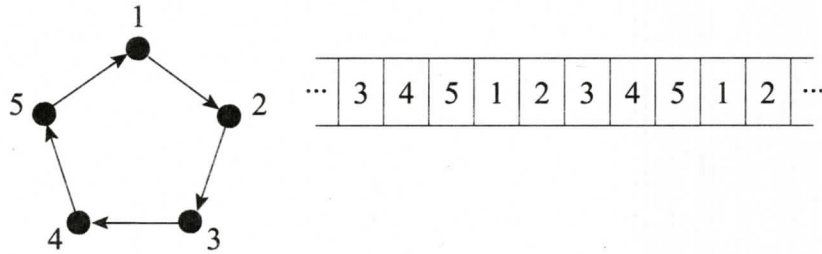


Figure 5.4: A directed cycle in a colouring digraph and the corresponding periodic SP graph colouring.

**Corollary 5.2** *An SP graph  $\Gamma$  has a  $k$ -colouring if and only if it has a periodic  $k$ -colouring.*

**Proof.** If  $\Gamma$  has a  $k$ -colouring, then the cycle described in the previous proof gives a sequence of colourings of the cell graph; this sequence of colourings

can be repeated infinitely often to colour all of the cells of  $\Gamma$ . Specifically, for  $i \leq t \leq j$ , colouring  $v_t$  can be used to colour cell  $G_{i+m(j-i)+t}$  for every integer  $m$ , and the translation  $T_{j-i}$  fixes colours.

The converse is true trivially. ■

**Corollary 5.3** *There exists a  $k$ -colouring of an SP graph  $\Gamma$  if and only if there is a homomorphism from an infinite directed path to  $C_k(\Gamma)$ .*

**Proof.** Clearly, a directed cycle is a homomorphic image of an infinite directed path; the result then follows from Theorem 5.1. ■

### 5.2.2 Aperiodic Colourings

In this section we present two results on aperiodic colourings of SP graphs.

**Theorem 5.4** *For an SP graph  $\Gamma$  and an integer  $k \geq 2$ , there exist aperiodic  $k$ -colourings of  $\Gamma$  if and only if  $C_k(\Gamma)$  contains intersecting directed cycles, or directed cycles that are joined by a directed path.*

Figure 5.5 illustrates these possibilities.

**Proof.** If  $C_k(\Gamma)$  contains intersecting directed cycles, then an aperiodic  $k$ -colouring of  $\Gamma$  can be found as follows: take two intersecting directed cycles in  $C_k(\Gamma)$ ,  $C_1 = v_0v_1 \cdots v_{r-1}v_0$  and  $C_2 = u_0u_1 \cdots u_{s-1}u_0$ . Without loss of generality, we assume that  $v_0 = u_0$ . Colour  $G_0$  using colouring  $v_0$ , and then colour the cells to the right of  $G_0$  with the colourings on cycle 1, in order: colour  $G_1$  using colouring  $v_1$ ,  $G_2$  using  $v_2$ , and so on. Cell  $G_{r-1}$  is coloured

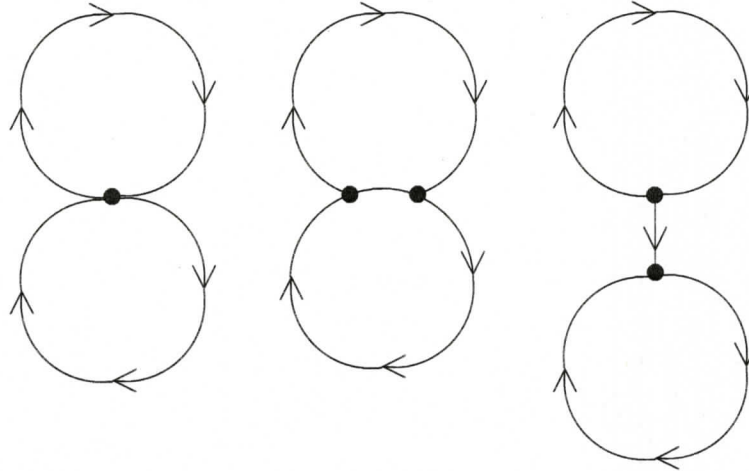


Figure 5.5: Subgraphs of  $C_k(\Gamma)$  that lead to aperiodic colourings.

with colouring  $v_{r-1}$ , and  $G_r$  with  $v_0$ . Continuing from left to right along  $\Gamma$ , the next  $s$  cells are coloured using the colourings on  $C_2$ , as follows: colour  $G_{r+1}$  with colouring  $u_1$ ,  $G_{r+2}$  with  $u_2$ , and so on up to  $G_{r+s-1}$  and  $G_{r+s}$ , which receive colourings  $u_{s-1}$  and  $u_0$ , respectively. We then continue colouring the cells from left to right, as above, with the colourings on  $C_1$ , followed by  $C_1$  again, then  $C_2$ ,  $C_1$  three times,  $C_2$ ,  $C_1$  four times, and so on. For the cells to the left of  $G_0$ , we follow a similar pattern, proceeding around the directed cycles backwards. The sequence of traversals of the two cycles is as follows:

$$\dots, C_2, C_1, C_1, C_1, C_2, C_1, C_1, C_2, C_1, C_2, C_1, C_1, C_2, C_1, C_1, C_1, C_2, \dots \quad (5.1)$$

Clearly if this pattern is continued in both directions, the resulting sequence of  $C_1$ 's and  $C_2$ 's will not have any translational symmetry, and therefore the

corresponding  $k$ -colouring of  $\Gamma$  will be aperiodic.

If  $C_k(\Gamma)$  contains two directed cycles joined by a directed path  $P$ , then again we call the cycles  $C_1$  and  $C_2$ , so that the directed path goes from  $C_1$  to  $C_2$ . We colour  $G_0$  with the colouring that lies on both  $C_1$  and  $P$ , and all cells to the left of  $G_0$  with the sequence of colourings obtained by travelling repeatedly around  $C_1$  backwards. The cells immediately to the right of  $G_0$  - specifically  $G_1$  through  $G_p$ , where  $p$  is the length of  $P$  - are assigned the colourings that appear on the path, in order. We then colour the remaining cells to the right of  $G_p$  with the sequence of colourings given by  $C_2$ . The sequence of traversals is as follows:

$$\dots, C_1, C_1, C_1, C_1, C_1, P, C_2, C_2, C_2, C_2, \dots \quad (5.2)$$

Conversely, suppose  $\Gamma$  has an aperiodic  $k$ -colouring  $F$ . By Corollary 5.3, there exists a homomorphism  $f$  from an infinite directed path to  $C_k(\Gamma)$ . Since  $V(C_k(\Gamma))$  is finite, there must be a repeated vertex, and hence a directed cycle. The colouring  $F$  is aperiodic, so the image of  $f$  cannot be simply a directed cycle, and must contain more than one directed cycle. If two such cycles intersect, then we are done. If not, then the image of  $f$  contains at least two directed cycles which are joined by a directed walk; if we take the two cycles which are joined by the shortest directed walk, this walk must be a directed path. ■

**Theorem 5.5** *A bipartite SP graph  $\Gamma$  has aperiodic 2-colourings if and only*

*if it has finite components.*

**Proof.** If  $\Gamma$  has a 2-colouring, then assigning a colour to any one vertex in a connected component of  $\Gamma$  uniquely determines the colour of every other vertex in that component. Thus for each component of  $\Gamma$  there is, up to permutation of the colour classes, only one 2-colouring. By Corollary 5.2,  $\Gamma$  must have a periodic 2-colouring; its restriction to any infinite component of  $\Gamma$  must also be periodic, so the unique (up to permuting the colour classes) 2-colouring of each infinite component must be periodic. The cell graph of  $\Gamma$  is finite and by Lemma 4.1 every infinite component of  $\Gamma$  intersects every cell, so  $\Gamma$  can contain only a finite number of infinite components. Each of these infinite components has only periodic colourings, so the 2-colourings of  $\Gamma$  must also be periodic; i.e. if  $\Gamma$  has an aperiodic colouring then it must contain finite components.

Conversely, suppose  $\Gamma$  has finite components. Then by Theorem 4.3,  $\Gamma$  contains infinitely many pairwise isomorphic finite components, each of width at most  $\#c$ . Each of these components has exactly two 2-colourings, which we will label 1 and 2. Clearly each of these finite components can be assigned one of these colourings independently of the others. An aperiodic 2-colouring of  $\Gamma$  can be created by a similar method to that used in the proof of Theorem 5.3. We choose one of these components, call it  $C_0$ , and colour it using colouring 1. We then assign colouring 1 or 2 to each isomorphic copy of  $C_0$  according to the sequence of indices in (5.1). For example, the first copy of  $C_0$  appearing to the right and the first copy on the left both receive

colouring 2, and then next two copies to the right and left of these receive colouring 1, and so on. ■

### 5.2.3 Extendable Vertex $k$ -Colouring

Recall from Section 3.3 that the problem of determining whether a given  $k$ -colouring of some finite subset of the vertices of a DP graph can be extended to a  $k$ -colouring of the entire DP graph is undecidable [4]. On the other hand, the corresponding problem for SP graphs is decidable.

To see this, consider an SP graph  $\Gamma$  in which a finite set of vertices has been pre-coloured. Redefine the cells of  $\Gamma$  so that all of the pre-coloured vertices are in the same cell. Let  $X$  be the set of colourings of the expanded cell graph that are extensions of the pre-colouring. The pre-colouring can be extended to the entire SP graph if and only if some vertex in  $X$  is on a directed cycle or a directed path between directed cycles.

### 5.2.4 Symmetry Groups of $k$ -Colourings

This section expands on the idea of colour-fixing translations by describing automorphism groups of  $k$ -colourings of SP graphs.

An automorphism  $\phi$  of a  $k$ -colouring of an SP graph  $\Gamma$  maps the cells of  $\Gamma$  to each other so that the cells  $G_i$  and  $\phi(G_i)$  are coloured with the same cell colouring, and  $\phi(G_i)$  and  $\phi(G_{i+1})$  are adjacent cells of  $\Gamma$ . In other words, if  $\phi(G_i) = G_j$ , then either  $\phi(G_{i+1}) = G_{j+1}$  or  $\phi(G_{i+1}) = G_{j-1}$ . Such a mapping

can be either a translation by a finite number of cells or a flip over a certain cell or the line between a certain pair of cells.

Under composition of functions, the set of all automorphisms, or symmetries, of a colouring  $Z$  of  $\Gamma$  forms a group, called the *symmetry group of  $Z$* . The symmetry group of a periodic colouring of  $\Gamma$  will contain translations, and may or may not include flips. If it does contain a flip, it will contain infinitely many flips. An aperiodic colouring of  $\Gamma$  can have only a single flip in its symmetry group, otherwise there would be a translation and the colouring would be periodic.

The first aperiodic colouring example in the proof of Theorem 5.4 has a symmetry group consisting of a single flip. Figure 5.6 shows a periodic colouring whose symmetry group contains flips and translations.

...	1	2	2	1	1	2	2	1	1	2	...
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Figure 5.6: A colouring of  $\Gamma$ , using two colourings of  $G_{ij}$ , labelled 1 and 2.

### 5.3 Doubly-Periodic Graphs

The  $k$ -colourability decision problem is considerably more complex for DP graphs than for SP graphs. This section presents a general discussion of this problem, and provides a description of the colouring digraphs of  $k$ -colourable DP graphs. It also outlines some special cases for which it is relatively sim-

ple to determine  $k$ -colourability, and discusses automorphisms of DP graph colourings. In this section, all graphs  $\Gamma$  are DP graphs.

### 5.3.1 $k$ -Colourability

According to Theorem 3.3, if a DP graph  $\Gamma$  contains an odd cycle, then a subgraph of  $\Gamma$  of bounded size contains an odd cycle. It is therefore possible to determine if any given DP graph is bipartite in polynomial time. For  $k \geq 3$ , the question of whether or not  $k$ -colourability of DP graphs is decidable is still open. It is known to be impossible to decide if a pre-colouring of a subset of the vertices of any given DP graph can be extended to a colouring of the entire graph [4]. It is also known that the related Domino Problem is undecidable [2], so it seems likely that  $k$ -colourability of DP graphs is also undecidable, but no proof of this has yet been found.

**Theorem 5.6** *There exists a  $k$ -colouring of a DP graph  $\Gamma$  if and only if there is a homomorphism to  $C_k(\Gamma)$  from the Cartesian product of two infinite 2-edge-coloured directed paths, one of which has all red edges and the other all blue edges.*

Figure 5.7 shows this Cartesian product.

**Proof.** Define infinite 2-edge-coloured directed paths  $P_r$  and  $P_b$  as follows:  $V(P_r) = \mathbb{Z}$ ,  $E_r(P_r) = \{j(j+1) | j \in \mathbb{Z}\}$  and  $E_b(P_r) = \emptyset$ ;  $V(P_b) = \mathbb{Z}$ ,  $E_r(P_b) = \emptyset$  and  $E_b(P_b) = \{i(i+1) | i \in \mathbb{Z}\}$ . Thus  $P_r$  has all red arcs and  $P_b$  has all blue arcs. The Cartesian product  $P_r \square P_b$  has vertex set  $\mathbb{Z} \times \mathbb{Z}$  and red

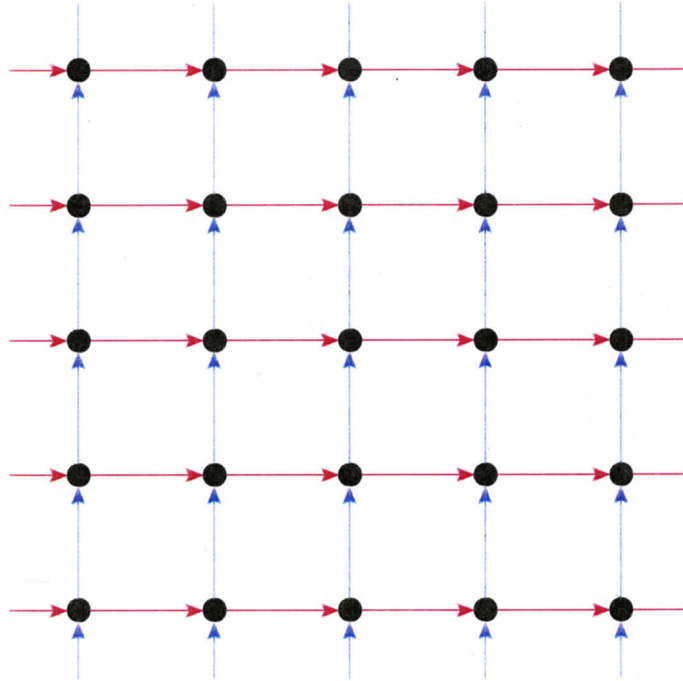


Figure 5.7: The Cartesian product of a red infinite directed path and a blue infinite directed path.

and blue arc sets  $E_r(P_r \square P_b) = \{(i, j)(i, j + 1) \mid i, j \in \mathbb{Z}\}$  and  $E_b(P_r \square P_b) = \{(i, j)(i + 1, j) \mid i, j \in \mathbb{Z}\}$ .

Let  $f$  be a homomorphism from  $P_r \square P_b$  to  $C_k(\Gamma)$ . We claim that colouring cell  $G_{ij}$  with  $f((i, j))$ , for each  $i, j \in \mathbb{Z}$ , will yield a  $k$ -colouring of  $\Gamma$ . Since  $f$  is a homomorphism,  $f((i, j))f((i, j + 1)) \in E_r(C_k(\Gamma))$  and  $f((i, j))f((i + 1, j)) \in E_b(C_k(\Gamma))$ . Thus for all integers  $i$  and  $j$  the colouring on cell  $G_{ij}$  is compatible with the colourings on the cells directly above and to the right, so every cell of  $\Gamma$  can be coloured in this manner.

Conversely, suppose we have a  $k$ -colouring  $Z$  of  $\Gamma$ . Define a mapping  $f$  from  $V(P_r \square P_b)$  to  $V(C_k(\Gamma))$  by  $f((i, j)) = Z(G_{ij})$ . If  $(i, j)(p, q) \in E_r(P_r \square P_b)$ , then  $p = i$  and  $q = j + 1$ , and

$$f((i, j))f((i, j + 1)) = Z(G_{ij})Z(G_{i(j+1)})$$

which must be in  $E_r(C_k(\Gamma))$ . Similarly for all  $(i, j)(i + 1, j) \in E_b(P_r \square P_b)$ ,  $f((i, j))f((i + 1, j)) = Z(G_{ij})Z(G_{(i+1)j})$  which must be in  $E_b(C_k(\Gamma))$ . The function  $f$  is thus a homomorphism of 2-edge-coloured digraphs. ■

Recall that DP graph colourings are related to tilings of the plane. Let each  $k$ -colouring of the cell graph  $G_{ij}$  be represented by a tile; if the arc  $uv$  is in  $E_r(C_k(\Gamma))$ , then the tile corresponding to colouring  $v$  can appear on the right of the tile corresponding to colouring  $u$ , and similarly if  $uv \in E_b(C_k(\Gamma))$  then  $v$ 's tile can appear directly above  $u$ 's tile. There exists a  $k$ -colouring of  $\Gamma$  if and only if this set of tiles can be used to tile the entire plane, subject to these compatibility relationships.

Slightly simpler (but still undecidable [7]) than the general tiling problem described in Section 5.1 is the problem involving square tiles with coloured sides, which we will refer to as *side-coloured tiles*. In this case, we want to tile the plane using translated copies of a finite set of these tiles such that abutting edges have the same colour. The *compatibility digraph*  $D$  of a set of side-coloured tiles is a 2-edge-coloured digraph that has one vertex for each tile. The arc  $uv$  appears in  $E_r(D)$  if and only if the right-hand side of tile

$u$  and the left-hand side of tile  $v$  have the same colour, and  $uv \in E_b(D)$  if and only if the top side of tile  $u$  and the bottom side of tile  $v$  have the same colour.

The following theorem characterizes the digraphs which are compatibility graphs of side-coloured tiles.

**Theorem 5.7** *A digraph  $D$  is the compatibility digraph for a set of side-coloured tiles if and only if for every pair of vertices  $u$  and  $v$  in  $V(D)$ :*

1. either  $N_r^+(u) = N_r^+(v)$  or  $N_r^+(u) \cap N_r^+(v) = \emptyset$ ,
2. either  $N_r^-(u) = N_r^-(v)$  or  $N_r^-(u) \cap N_r^-(v) = \emptyset$ ,
3. either  $N_b^+(u) = N_b^+(v)$  or  $N_b^+(u) \cap N_b^+(v) = \emptyset$ , and
4. either  $N_b^-(u) = N_b^-(v)$  or  $N_b^-(u) \cap N_b^-(v) = \emptyset$ .

**Proof.** Suppose  $T = \{\tau_1, \tau_2, \dots, \tau_n\}$  is a set of side-coloured tiles, and let  $D$  be their compatibility digraph, with  $V(D) = T$ . Take  $\tau_i$  and  $\tau_j$  in  $T$ , and suppose  $N_r^+(\tau_i) \cap N_r^+(\tau_j) \neq \emptyset$ . Thus there exists  $\tau_q \in T$  such that  $\tau_i\tau_q \in E_r(D)$  and  $\tau_j\tau_q \in E_r(D)$ . Thus the right-hand side of tile  $\tau_i$  has the same colour as the left-hand side of tile  $\tau_q$ , which has the same colour as the right-hand side of tile  $\tau_j$ . Since  $\tau_i$  and  $\tau_j$  have the same colour on their right-hand sides, any tile that can appear to the right of one of them must also be compatible with the other. Thus for all  $1 \leq p \leq n$ ,  $\tau_i\tau_p \in E_r(D)$  if and only if  $\tau_j\tau_p \in E_r(D)$ , so  $N_r^+(\tau_i) = N_r^+(\tau_j)$ . The argument for the case of

the in-neighbourhoods is similar; if  $N_r^-(\tau_i) \cap N_r^-(\tau_j) \neq \emptyset$ , then tiles  $\tau_i$  and  $\tau_j$  have the same colour on their left-hand sides. The result for the blue in- and out-neighbourhoods is proven similarly, by considering the colours of the top and bottom sides of the tiles.

Conversely, suppose we have a digraph  $D$  which satisfies the conditions on the red and blue in- and out-neighbourhoods stated in the hypothesis. We can assume that  $D$  is connected, since if it is not we can consider each component separately. Thus the red in-neighbourhoods of the vertices of  $D$  partition  $V(D)$ , and likewise for the red out-neighbourhoods, and the blue in- and out-neighbourhoods.

Define a set of side-coloured tiles as follows: for each vertex  $v$  in  $V(D)$ , Create a tile  $t(v)$ . Let  $R_1^+, R_2^+, \dots, R_l^+$  be the sets in the partition of  $D$  into red out-neighbourhoods, and  $B_1^+, B_2^+, \dots, B_j^+$  the sets in the partition of  $D$  into blue out-neighbourhoods. A different colour will be associated with each of these sets, and the sides of each tile will be coloured as follows: let  $v \in V(D)$ . If  $N_r^+(v) = R_i^+$  then the right-hand side of  $t(v)$  receives colour  $R_i^+$ , and for each vertex  $u \in N_r^+(v)$ , the left-hand side of  $t(u)$  also receives colour  $R_i^+$ . If  $N_b^+(v) = B_j^+$  then the top of  $t(v)$  receives colour  $B_j^+$  and for each  $u \in N_b^+(v)$ , the bottom of  $t(u)$  also receives colour  $B_j^+$ . If  $N_r^+(v)$  or  $N_b^+(v)$  is empty, then the right-hand side or top, respectively, of  $t(v)$  is assigned a colour which does not appear on any other tile. If there is a vertex  $u$  such that  $N_r^-(u)$  or  $N_b^-(u)$  is empty, then the left-hand side or bottom, respectively, of  $t(v)$  is assigned a unique colour.

We claim that the compatibility digraph  $D(T)$  of the set  $T = \{t(x) \mid x \in V(D)\}$  is isomorphic to  $D$ . Clearly,  $t : V(D) \rightarrow T$  is a bijection, so it suffices to show that  $uv \in E_r(D)$  if and only if  $t(u)t(v) \in E_r(D(T))$ , and  $uv \in E_b(D)$  if and only if  $t(u)t(v) \in E_b(D(T))$ . If  $uv \in E_r(D)$ , then  $v \in N_r^+(u)$  and  $u \in N_r^-(v)$  in  $D$ , so the right-hand side of  $t(u)$  and the left-hand side of  $t(v)$  have the same colour, so  $t(u)t(v) \in E_r(D(T))$ . Now suppose that  $t(u)t(v) \in E_r(D(T))$ , which means that the right-hand side of  $t(u)$  and the left-hand side of  $t(v)$  have the same colour. Because of the way that  $t(u)$  and  $t(v)$  were constructed, this implies that  $v \in N_r^+(u)$  in  $D$ , and therefore  $uv \in E_r(D)$ . Thus  $D$  is the compatibility digraph of the tiles in  $T$ . ■

Note that it may not be possible to represent the cell colourings of a DP graph  $\Gamma$  by side-coloured tiles, because  $C_k(\Gamma)$  may not meet the requirements of Theorem 5.7. Figure 5.8 presents such a DP graph  $\Gamma$ . The colourings of  $G_{ij}$  are labelled in the same way as in Figure 5.2.

The horizontal edges (between  $G_{ij}$  and  $G_{i(j+1)}$ ) are the same as the vertical edges (between  $G_{ij}$  and  $G_{(i+1)j}$ ), so the colouring compatibility relationships are the same horizontally and vertically (see Section 5.3.3 for a proof of this and further discussion of DP graphs of this type), and  $E_r(C_k(\Gamma)) = E_b(C_k(\Gamma))$ . The red edges in the subgraph of  $C_k(\Gamma)$  induced by  $\{1231, 1232, 3213\}$  are shown in the figure. Since  $N_r^-(1232) = \{1231, 3213\}$  and  $N_r^-(1231) = \{1231\}$ , we have two vertices whose red in-neighbourhoods are neither equal nor disjoint and thus  $C_k(\Gamma)$  cannot be the compatibility digraph of a set of side-coloured tiles.

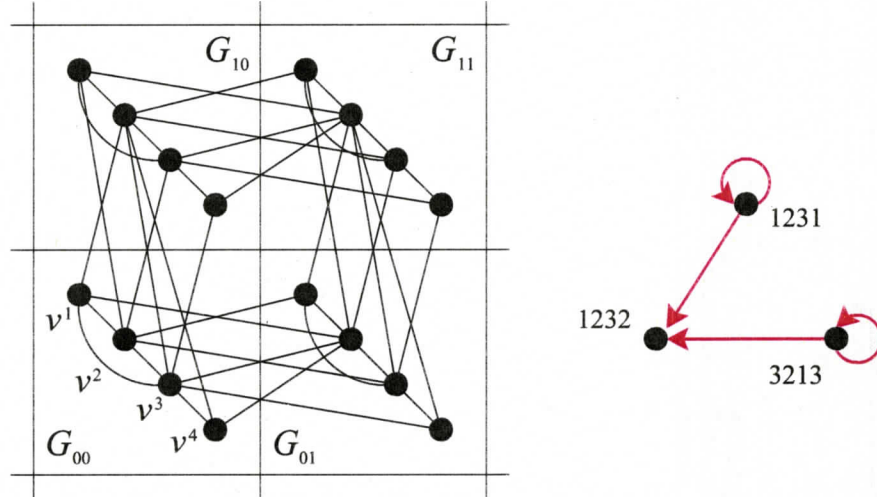


Figure 5.8: Two cells of  $\Gamma$ , and the red edges of an induced subgraph of  $C_3(\Gamma)$ .

### 5.3.2 Symmetric DP graphs

One special class of DP graphs for which  $k$ -colourability is easy to determine is the set of graphs whose inter-cell connections are the same in opposite directions, left/right and up/down.

**Definition 5.8** A DP graph  $\Gamma$  is called symmetric if it has the property that  $v_{ij}^k v_{i(j+1)}^l \in E(\Gamma)$  if and only if  $v_{ij}^l v_{i(j+1)}^k \in E(\Gamma)$  and  $v_{ij}^k v_{(i+1)j}^l \in E(\Gamma)$  if and only if  $v_{ij}^l v_{(i+1)j}^k \in E(\Gamma)$ .

The connection digraph  $D(\Gamma)$  of a symmetric DP graph can be represented by an undirected 2-edge-coloured graph (defined similarly to a 2-edge-coloured digraph), since  $v^k v^l \in E_r(D(\Gamma))$  if and only if  $v^l v^k \in E_r(D(\Gamma))$ , and similarly for the blue arcs.

**Lemma 5.9** *The colouring digraph  $C_k(\Gamma)$  of a symmetric DP graph is the equivalent digraph of an undirected 2-edge-coloured graph.*

**Proof.** It suffices to show that for any two vertices  $u$  and  $v$  in  $V(C_k(\Gamma))$ , the red arc  $uv$  is in  $E_r(C_k(\Gamma))$  if and only if the red arc  $vu$  is in  $E_r(C_k(\Gamma))$ , and similarly for blue arcs. If  $uv \in E_r(C_k(\Gamma))$ , then whenever colouring  $u$  appears on cell  $G_{ij}$ , colouring  $v$  can be used to colour cell  $G_{i(j+1)}$ . Let  $U_1, U_2, \dots, U_k$  be the colour classes associated with colouring  $u$ , and  $V_1, V_2, \dots, V_k$  be the colour classes associated with colouring  $v$ . There can be no edges between vertices in  $G_{ij}$  appearing in  $U_t$  and vertices in  $G_{i(j+1)}$  appearing in  $V_t$ , for  $1 \leq t \leq k$ . Because  $\Gamma$  is symmetric, if cell  $G_{ij}$  is coloured with colouring  $v$  and cell  $G_{i(j+1)}$  with colouring  $u$ , there will again be no edges between vertices in  $G_{ij}$  appearing in  $V_t$  and vertices in  $G_{i(j+1)}$  appearing in  $U_t$ , for  $1 \leq t \leq k$ , so the arc  $vu$  is also in  $E_r(C_k(\Gamma))$ . Similarly, if  $vu \in E_r(C_k(\Gamma))$  then  $uv \in E_r(C_k(\Gamma))$ , and  $uv \in E_b(C_k(\Gamma))$  if and only if  $vu \in E_b(C_k(\Gamma))$ . ■

**Definition 5.10** *A 2-edge-coloured undirected four-cycle with two non-adjacent edges coloured red and the other two (also non-adjacent) edges coloured blue is called  $C_4^+$ .*

**Theorem 5.11** *If  $\Gamma$  is a symmetric DP graph and  $C_k(\Gamma)$  contains a homomorphic image of  $C_4^+$ , then  $\Gamma$  has a  $k$ -colouring.*

**Proof.** Let  $V(C_4^+) = \{v_1, v_2, v_3, v_4\}$ , and let  $E_r(C_4^+) = \{v_1v_2, v_3v_4\}$  and  $E_b(C_4^+) = \{v_2v_3, v_4v_1\}$ . Let  $f : V(C_4^+) \rightarrow V(C_k(\Gamma))$  be a homomorphism.

For all  $n, m \in \mathbb{Z}$ , colour cell  $G_{(2n)(2m)}$  with colouring  $f(v_1)$ ,  $G_{(2n+1)(2m)}$  with  $f(v_2)$ ,  $G_{(2n+1)(2m+1)}$  with  $f(v_3)$ , and  $G_{(2n)(2m+1)}$  with  $f(v_4)$ . ■

**Corollary 5.12** *The  $k$ -colourability of symmetric DP graphs is decidable.*

**Proof.** By Theorem 5.11, it suffices to decide whether or not  $C_k(\Gamma)$  contains a homomorphic image of  $C_4^+$ . Since this graph has only nine homomorphic images on one, two, three, or four vertices, it is easy to check whether any of these is a subgraph of  $C_k(\Gamma)$ . ■

### 5.3.3 Happy DP graphs

This section presents another special case for which the question of  $k$ -colourability is decidable.

**Definition 5.13** *A DP graph  $\Gamma$  is called happy if it has the property that  $v_{ij}^p v_{i(j+1)}^q \in E(\Gamma)$  if and only if  $v_{ij}^p v_{(i+1)j}^q \in E(\Gamma)$ .*

The connection digraph  $D(\Gamma)$  of a happy DP graph has identical blue and red edge sets; i.e.  $E_r(D(\Gamma)) = E_b(D(\Gamma))$ . The DP graph pictured in Figure 5.8 is an example of a happy DP graph.

**Lemma 5.14** *The colouring digraph  $C_k(\Gamma)$  of a happy DP graph can be represented by a digraph  $C'_k(\Gamma)$  with  $V(C'_k(\Gamma)) = V(C_k(\Gamma))$  and  $E(C'_k(\Gamma)) = E_r(C_k(\Gamma)) = E_b(C_k(\Gamma))$ , such that the arc  $uv \in E(C'_k(\Gamma))$  if and only if colouring  $v$  can be used on cell  $G_{i(j+1)}$  and  $G_{(i+1)j}$ .*

**Proof.** It suffices to show that if  $\Gamma$  is a happy DP graph, then  $E_r(C_k(\Gamma)) = E_b(C_k(\Gamma))$ . Suppose  $uv \in E_r(C_k(\Gamma))$ . Colouring  $v$  can be used on cell  $G_{i(j+1)}$  whenever colouring  $u$  appears on cell  $G_{ij}$ . For each  $x \in \{1, 2, \dots, k\}$ , none of the vertices in  $G_{ij}$  that receive colour  $x$  under colouring  $u$  can be adjacent in  $\Gamma$  to any of the vertices in  $G_{i(j+1)}$  that receive colour  $x$  under colouring  $v$ . Since  $v_{ij}^p v_{i(j+1)}^q \in E(\Gamma)$  if and only if  $v_{ij}^p v_{(i+1)j}^q \in E(\Gamma)$ , if colouring  $v$  is used to colour cell  $G_{(i+1)j}$ , then it follows that none of the  $x$ -coloured vertices in  $G_{ij}$  will be adjacent in  $\Gamma$  to any of the  $x$ -coloured vertices in  $G_{(i+1)j}$ . Thus if colouring  $u$  appears on  $G_{ij}$  then colouring  $v$  can appear on cell  $G_{(i+1)j}$ , and so  $uv \in E_b(C_k(\Gamma))$ . By a similar argument,  $E_b(C_k(\Gamma)) \subseteq E_r(C_k(\Gamma))$ , and we are done. ■

**Theorem 5.15** *There exists a  $k$ -colouring of a happy DP graph  $\Gamma$  if and only if  $C'_k(\Gamma)$  contains a directed cycle.*

**Proof.** Suppose  $C'_k(\Gamma)$  contains a directed cycle  $C = v_0 v_1 \dots v_m v_0$ . Figure 5.9 illustrates how the cells of  $\Gamma$  can be coloured using the vertices of  $C$ .

Now suppose that we have a  $k$ -colouring  $Z$  of  $\Gamma$ . The restriction of  $Z$  to row 0 of  $\Gamma$  is a  $k$ -colouring of this row, and by the same argument as the proof of Theorem 5.1,  $C'_k(\Gamma)$  must contain a directed cycle. ■

### 5.3.4 Periodic and Aperiodic Colourings

In this section we discuss properties of periodic and aperiodic colourings of DP graphs. Recall that a colouring  $Z$  of a DP graph  $\Gamma$  is periodic if there is

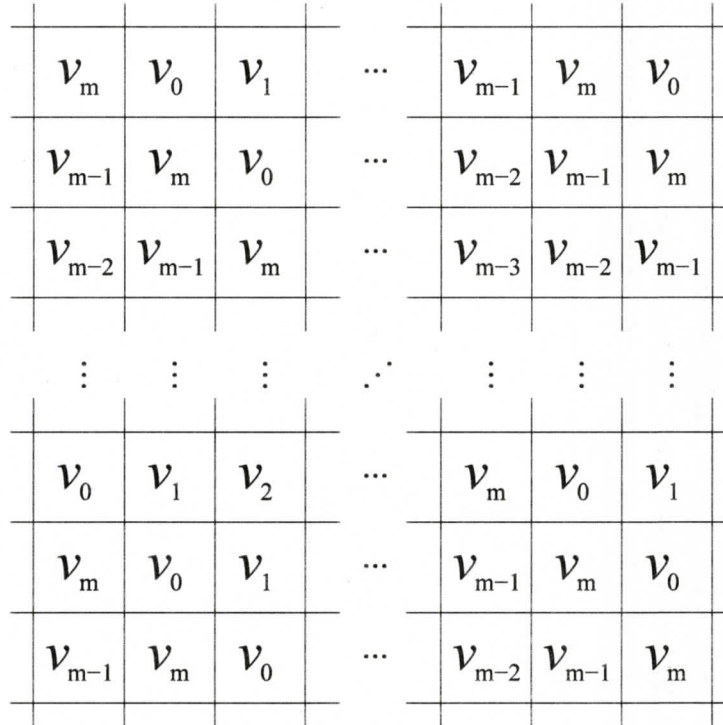


Figure 5.9: A colouring of the happy DP graph  $\Gamma$ .

a translation  $T_{xy}$  that fixes the colour of each vertex in  $\Gamma$ , and aperiodic if there exists no such translation.

**Theorem 5.16** *A bipartite DP graph  $\Gamma$  admits aperiodic 2-colourings if and only if it contains finite components.*

**Proof.** The proof of this theorem is similar to the proof of Theorem 5.5, except that in the DP case there can be infinitely many infinite 2-way components. ■

**Definition 5.17** *A  $k$ -colouring  $Z$  of a DP graph is called a domino colouring*

if there are  $\mathbb{Z}$ -fixing translations  $T_{\alpha 0}$  and  $T_{0\beta}$  with  $\alpha, \beta \neq 0$ . The block of colourings appearing on the set of cells  $\{G_{ij} | x \leq i < x + \alpha, y \leq j < y + \beta\}$  for some integers  $x$  and  $y$  is called a colouring domino of that  $k$ -colouring of  $\Gamma$ . The width of such a colouring domino is  $\alpha$ , and its height is  $\beta$ .

**Theorem 5.18** *A DP graph  $\Gamma$  has a domino  $k$ -colouring if and only if  $C_k(\Gamma)$  contains a homomorphic image of a Cartesian product of 2-edge-coloured directed cycles, one with all red arcs and one with all blue arcs.*

**Proof.** The proof of this theorem is similar to the proof of Theorem 5.6. Define 2-edge-coloured directed cycles  $C_r(n)$  and  $C_b(m)$  as follows:  $V(C_r(n)) = \mathbb{Z}_n$ ,  $E_r(C_r(n)) = \{j(j +_n 1) | 1 \leq j \leq n\}$  and  $E_b(C_r(n)) = \emptyset$ , where  $+_n$  denotes addition in  $\mathbb{Z}_n$ ;  $V(C_b(m)) = \mathbb{Z}_m$ ,  $E_r(C_b(m)) = \emptyset$  and  $E_b(C_b(m)) = \{i(i +_m 1) | 1 \leq i \leq m\}$ , with  $+_m$  denoting addition in  $\mathbb{Z}_m$ . Thus  $C_r(n)$  has  $n$  red arcs and  $C_b(m)$  has  $m$  blue arcs. The Cartesian product  $C_r(n) \square C_b(m)$  has vertex set  $\mathbb{Z}_n \times \mathbb{Z}_m$  and red and blue arc sets  $E_r(C_r(n) \square C_b(m)) = \{(i, j)(i, j +_n 1) | (i, j) \in \mathbb{Z}_n \times \mathbb{Z}_m\}$  and  $E_b(C_r(n) \square C_b(m)) = \{(i, j)(i +_m 1, j) | (i, j) \in \mathbb{Z}_n \times \mathbb{Z}_m\}$ .

If  $f$  is a homomorphism from  $C_r(n) \square C_b(m)$  to  $C_k(\Gamma)$ , define a colouring  $Z$  of  $\Gamma$  by letting  $Z(G_{ij}) = f([i]_n, [j]_m)$ , where  $[i]_n$  and  $[j]_m$  are the least residues modulo  $n$  and  $m$  of  $i, j \in \mathbb{Z}$ , respectively. Since  $f$  is a homomorphism,  $Z(G_{ij})Z(G_{i(j+1)}) = f([i]_n, [j]_m)f([i]_n, [j]_m +_m 1) \in E_r(C_k(\Gamma))$  and  $Z(G_{ij})Z(G_{(i+1)j}) = f([i]_n, [j]_m)f([i]_n +_n 1, [j]_m) \in E_b(C_k(\Gamma))$ , so  $Z$  is a  $k$ -colouring of  $\Gamma$ . Furthermore, for any  $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ ,  $Z(G_{ij}) = f([i]_n, [j]_m) =$

$f([i+n]_n, [j]_m) = Z(G_{(i+n)_j})$  and  $Z(G_{ij}) = f([i]_n, [j+m]_m) = Z(G_{i(j+m)})$ , so the translations  $T_{n0}$  and  $T_{0m}$  are  $Z$ -fixing translations, and  $Z$  is a domino colouring.

Conversely, suppose we have a domino  $k$ -colouring  $Z$  of  $\Gamma$ , with dominoes of width  $\alpha$  and height  $\beta$ . Define a mapping  $f$  from  $V(C_r(\alpha) \square C_b(\beta))$  to  $V(C_k(\Gamma))$  by  $f((i, j)) = Z(G_{i_0 j_0})$ , where  $i_0$  and  $j_0$  are the smallest positive representatives of the residue classes  $i \in \mathbb{Z}_\alpha$  and  $j \in \mathbb{Z}_\beta$ . If  $(i, j)(p, q) \in E_r(C_r(\alpha) \square C_b(\beta))$ , then  $p =_\alpha i$  and  $q =_\beta j + 1$ , where  $=_\alpha$  and  $=_\beta$  denote equality in  $\mathbb{Z}_\alpha$  and  $\mathbb{Z}_\beta$ , respectively. Thus  $f((i, j))f((i, j +_\beta 1)) = Z(G_{i_0 j_0})Z(G_{i_0(j_0+1)})$  which must be in  $E_r(C_k(\Gamma))$ . Similarly for all  $(i, j)(i +_\alpha 1, j) \in E_b(P_r \square P_b)$ ,  $f((i, j))f((i +_\alpha 1, j)) = Z(G_{i_0 j_0})Z(G_{(i_0+1)j_0})$  which must be in  $E_b(C_k(\Gamma))$ . Therefore  $f$  is a homomorphism of 2-edge-coloured digraphs. ■

**Theorem 5.19** *Let  $\Gamma$  be a DP graph that has a periodic  $k$ -colouring  $Z$ , and let  $T_{ab}$  be the  $Z$ -fixing translation with minimal positive  $a$  and  $b$ . Then  $\Gamma$  has a domino colouring with colouring dominoes of width and height no larger than  $\frac{a(\xi!)}{(ab)!}$  by  $\frac{b(\xi!)}{(ab)!}$ , where  $\xi = |V(C_k(\Gamma))|$ .*

**Proof.** The basic idea of this proof is found in the Introduction of [?, rr] Consider the subgraph of  $\Gamma$  induced by cells  $G_{ij}$ , for  $i \in \mathbb{Z}$  and  $j \in \{1, 2, \dots, b\}$ . We can divide this subgraph into blocks which are  $a$  cells wide and  $b$  cells high, which we will call  $a \times b$ -blocks. The number of possible ways to colour the cells of an  $a \times b$ -block is  $\frac{\xi!}{(ab)!}$ , and therefore finite. Since there are infinitely

many  $a \times b$ -blocks in this subgraph, at least two of them must be coloured in exactly the same way. We choose two of these blocks such that there are no repeated blocks between them, and call the configuration of colourings that appears on them  $CB_1$ . We also label from left to right each of the  $a \times b$ -blocks between these two occurrences of  $CB_1$  as  $CB_2, CB_3$ , and so on up to  $CB_t$ , which is followed by  $CB_1$ . Let  $i_0$  be the index of the leftmost column of the leftmost occurrence of  $CB_1$ . Figure 5.10 illustrates this situation.

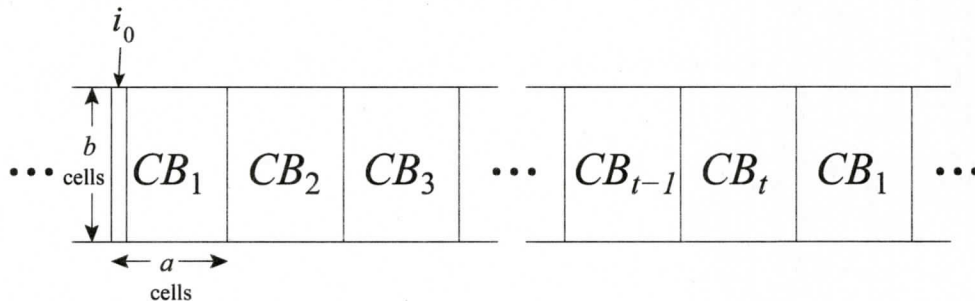


Figure 5.10: The subgraph of  $\Gamma$  induced by  $G_{ij}$ , for  $i \in \mathbb{Z}$  and  $j \in \{1, 2, \dots, b\}$ .

We now consider the subgraph of  $\Gamma$  induced by cells  $G_{ij}$  where  $i_0 \leq i < i_0 + ta$  and  $1 \leq j \leq b$ . We claim that the subgraph of  $\Gamma$  induced by  $G_{ij}$  where  $i_0 \leq i < i_0 + ta$  and  $1 \leq j \leq tb$  can be coloured by a  $t$  by  $t$  array of  $a \times b$ -blocks, whose rows are successive cyclic shifts of the sequence  $CB_1, CB_2, CB_3, \dots, CB_t$ , and that the colourings on this subgraph constitute a colouring domino for  $\Gamma$ . Since  $T_{ab}$  is a  $Z$ -fixing translation, if  $CB_s$  appears on some  $a \times b$ -block, then it must also appear on the one directly above and to the right. Thus for each  $s \in \{1, 2, \dots, t\}$ ,  $CB_s$  can appear on the  $a \times b$ -block

directly above  $CB_{s+1}$ , and  $CB_t$  can appear above  $CB_1$ . Figure 5.11 shows how the colouring domino can be constructed.

2	3	4			$t-1$	$t$	1
3	4	5	...		$t$	1	2
4	5	6			1	2	3
	⋮		⋮			⋮	
$t-1$	$t$	1			$t-4$	$t-3$	$t-2$
$t$	1	2	...		$t-3$	$t-2$	$t-1$
1	2	3			$t-2$	$t-1$	$t$

Figure 5.11: The colouring domino for  $\Gamma$ , with  $CB_s$  indicated by its index  $s$ .

Since there are  $\frac{\xi!}{(ab)!}$  different  $a \times b$ -blocks, we have  $t \leq \frac{\xi!}{(ab)!}$ , and the maximum dimensions of a domino are  $\frac{a(\xi!)}{(ab)!}$  cells wide by  $\frac{b(\xi!)}{(ab)!}$  cells high. ■

Note that this theorem implies that if a DP graph has a periodic  $k$ -colouring then the problem of finding it is finite, so  $k$ -colourability of DP graphs can only be undecidable if there exist DP graphs which admit only aperiodic colourings. In fact, Berger uses this idea in his proof of the undecidability of the domino problem in [2], by constructing a domino set which forces an aperiodic solution.

If a  $k$ -colouring of a DP graph  $\Gamma$  is a domino colouring, then the restriction of the colouring to any row or column of  $\Gamma$  will be a periodic colouring of the SP subgraph induced by that row or column. The converse is not necessarily true: it is possible to construct an aperiodic colouring of a DP graph using only periodic colourings of the horizontal and vertical SP subgraphs. Figure 5.12 illustrates a DP graph  $\Gamma$  which provides an example of this.

The restriction of this colouring to each each row and column of the DP graph is a periodic SP colouring, but each one has a different period. We assume that the colourings of row 0 and column 0 have periods of one and two, respectively. For  $i, j \in \mathbb{Z}^+$ , the colourings of rows  $i$  and  $-i$  have periods  $4i - 1$  and  $4i + 1$ , respectively, and the colourings of columns  $j$  and  $-j$  have periods  $4j$  and  $4i + 2$ , respectively. The periods of the rows and columns shown in the figure are indicated at the left and the top.

### 5.3.5 Symmetry Groups of $k$ -Colourings

The automorphism groups of DP graph colourings comprise a subset of the set of automorphism groups of tilings of the plane, which are commonly known as “wallpaper groups” [9]. Not all of the patterns represented in the wallpaper groups can be realized by DP graph colourings, as there are five which include rotations by angles other than multiples of  $90^\circ$ .

There are many DP graphs whose cell  $k$ -colourings are all compatible with each other both horizontally and vertically. Let  $G_{ij}$  be any  $k$ -colourable simple graph, and partition the vertices of each cell into the same  $k$  colour

classes. Let  $f$  and  $g$  be bijections from the set of colour classes of  $G_{ij}$  onto the sets of colour classes of  $G_{i(j+1)}$  and  $G_{(i+1)j}$ , respectively. For each colour class  $X$  in  $G_{ij}$ , there must be no edges between vertices in  $X$  and vertices in  $f(X)$ , and no edges between vertices in  $X$  and vertices in  $g(X)$ . These graphs can be quite well-connected; for example, take  $G_{ij}$  to be a complete  $k$ -partite graph, and for each of the  $k$  partite sets in  $G_{ij}$ , add all edges between vertices in that set and vertices in the other partite sets in  $G_{i(j+1)}$  and  $G_{(i+1)j}$ .

Appendix A contains figures that illustrate examples of DP graph colourings whose symmetry groups are each of the 12 wallpaper groups that contain either no rotations or only rotations of multiples of  $90^\circ$ .



# Chapter 6

## Conclusion

### 6.1 Summary of Results

#### 6.1.1 Components

For both SP and DP graphs we have presented methods for determining the exact nature of their components; if  $v$  is a vertex in an SP or DP graph  $\Gamma$ , then it is possible to determine which other vertices are in the same component, the size of the component, and how many isomorphic copies of this component appear in  $\Gamma$ . Finite components of  $\Gamma$  correspond to balanced or doubly-balanced components of the connection digraph; the number of infinite components can be found by taking gcds of net lengths of cycles in the connection digraph.

### 6.1.2 Colourings

For an SP graph  $\Gamma$ , the colouring digraph provides a simple way to decide whether or not  $\Gamma$  is  $k$ -colourable, and if so whether or not it admits aperiodic  $k$ -colourings. For two special cases of DP graphs, symmetric DP graphs and happy DP graphs, the colouring digraph can be used to determine  $k$ -colourability. If a DP graph  $\Gamma$  has a domino colouring, then the dimensions of the domino are bounded.

## 6.2 Further Research

The main question that remains open is the decidability of  $k$ -colouring DP graphs. If a DP graph that admits only aperiodic colourings could be found, then it would be possible to prove using a method similar to the one presented in [2] that  $k$ -colourability of DP graphs is undecidable. Other potential avenues for further work including looking for other special cases for which  $k$ -colourability is decidable, or finding a method to determine if a DP graph admits colourings with a particular symmetry group.

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<[http://en.wikipedia.org/w/index.php?title=Wallpaper\\_group&oldid=41464661](http://en.wikipedia.org/w/index.php?title=Wallpaper_group&oldid=41464661)>  
[accessed 8 March 2006].

# Appendix A

## Wallpaper groups

In the following figures, the DP graphs are coloured using at most four cell colourings, which are labelled 1, 2, 3, and 4. The colouring digraph of the DP graphs is assumed to be complete. The symmetries of each colouring are indicated using the symbols listed in Figure A.1, and the group is named using two standard notational systems: Crystallographic and Conway. See [9] for a description of these notations.

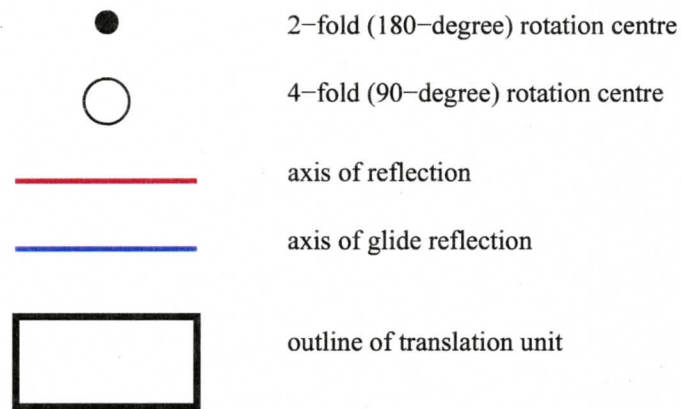


Figure A.1: Symbols used in the wallpaper group diagrams.

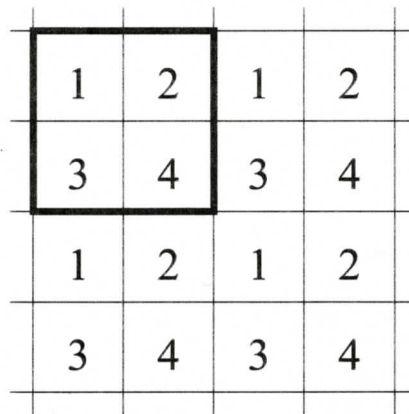


Figure A.2: Crystallographic: p1. Conway: o.

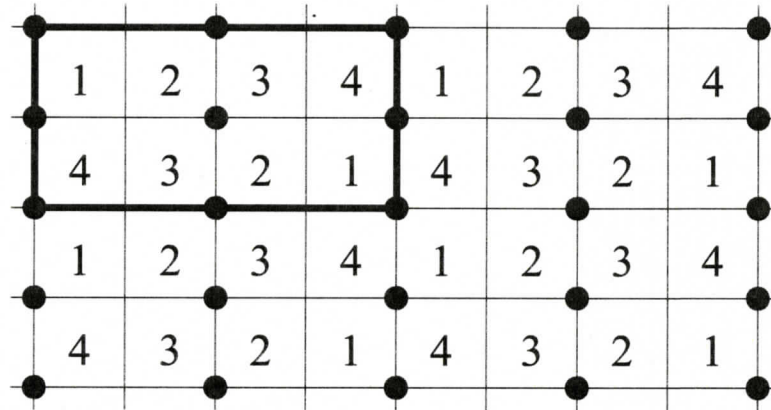


Figure A.3: Crystallographic: p2. Conway: 2222.

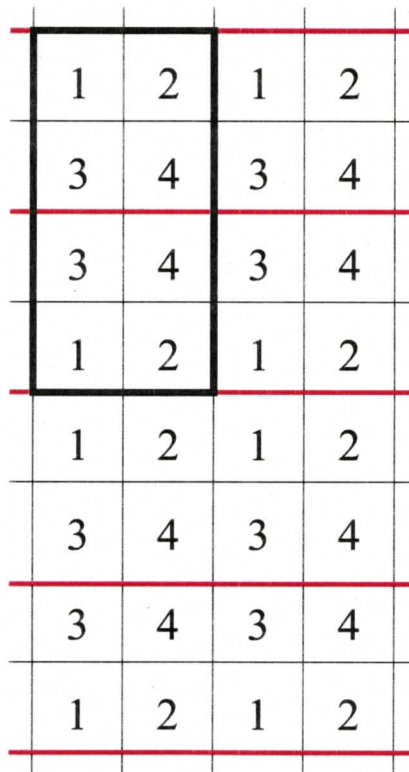


Figure A.4: Crystallographic: pm. Conway: \*\*.

1	2	3	4	1	2	3	4
3	4	1	2	3	4	1	2
1	2	3	4	1	2	3	4
3	4	1	2	3	4	1	2

Figure A.5: Crystallographic: pg. Conway: xx.

1	2	3	4	1	2	3	4
3	4	1	2	3	4	1	2
3	4	1	2	3	4	1	2
1	2	3	4	1	2	3	4
1	2	3	4	1	2	3	4
3	4	1	2	3	4	1	2
3	4	1	2	3	4	1	2
1	2	3	4	1	2	3	4

Figure A.6: Crystallographic: cm. Conway: \*x.

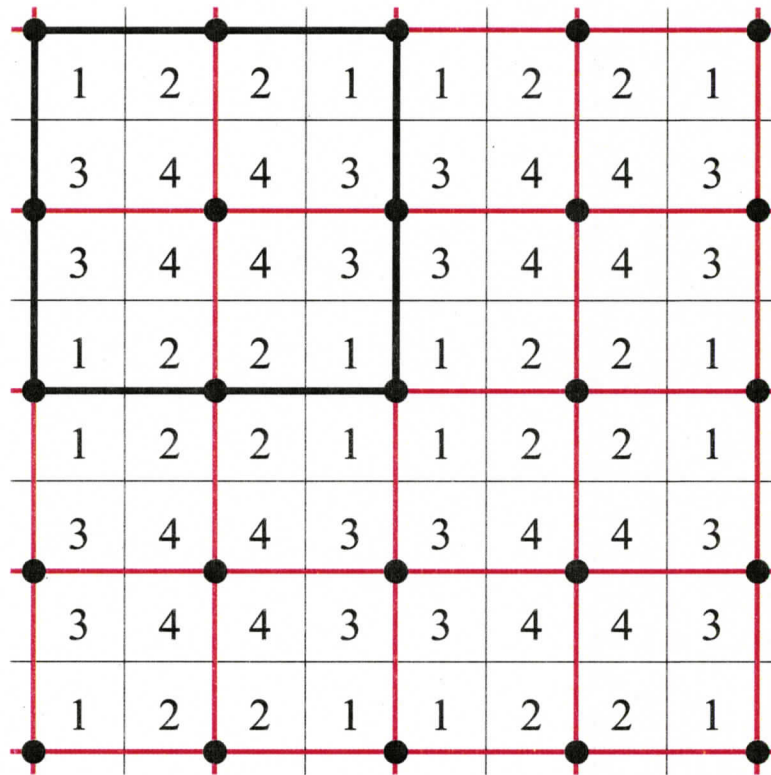


Figure A.7: Crystallographic: pmm. Conway: \*2222.

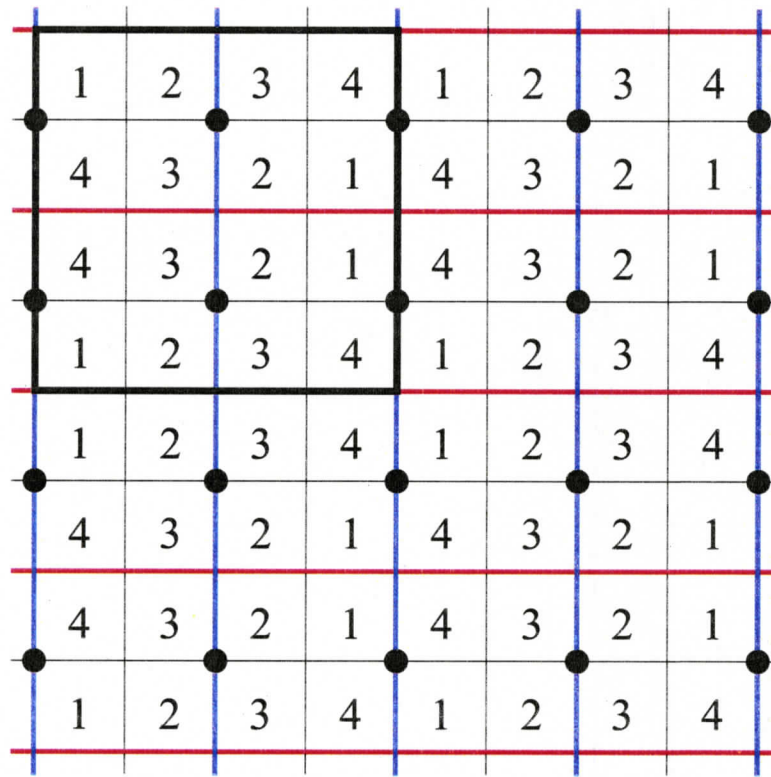


Figure A.8: Crystallographic: pmg. Conway: 22\*.

1	2	3	4	1	2	4	3	1	2	3	4
4	3	2	1	3	4	2	1	4	3	2	1
3	4	2	1	4	3	2	1	3	4	2	1
1	2	4	3	1	2	3	4	1	2	4	3
1	2	3	4	1	2	4	3	1	2	3	4
4	3	2	1	3	4	2	1	4	3	2	1
3	4	2	1	4	3	2	1	3	4	2	1
1	2	4	3	1	2	3	4	1	2	4	3

Figure A.9: Crystallographic: pgg. Conway: 22x.

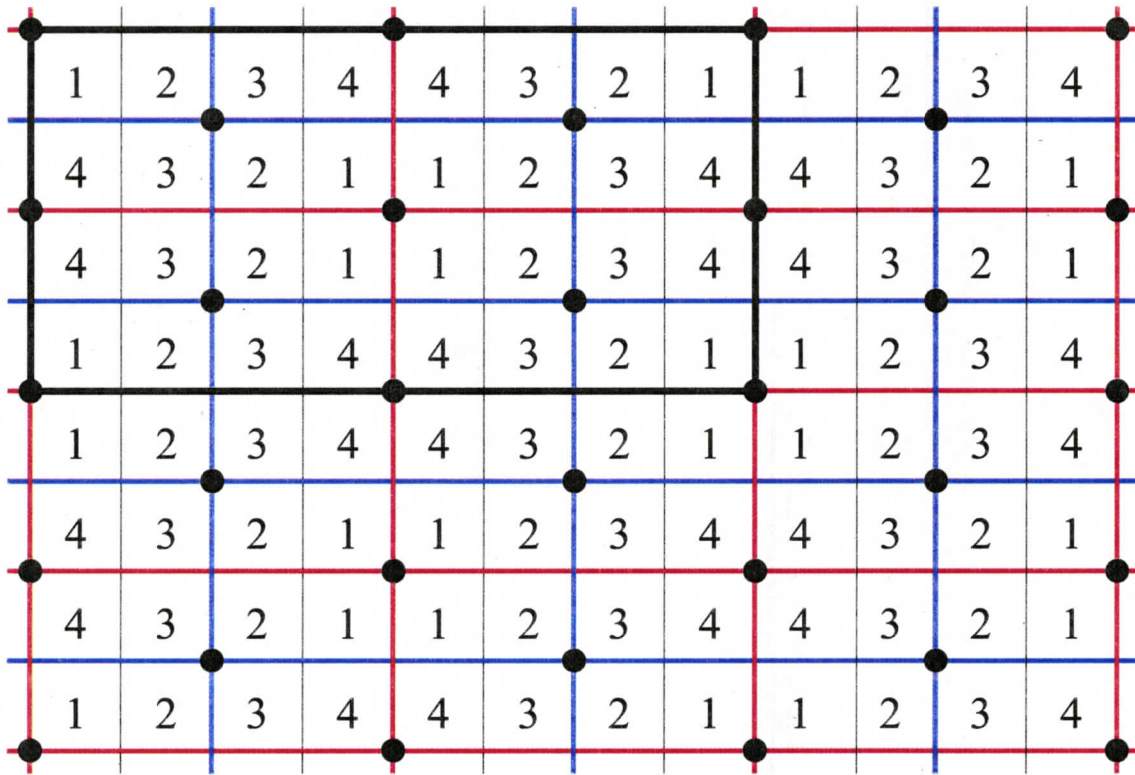


Figure A.10: Crystallographic: cmm. Conway: 2\*22.

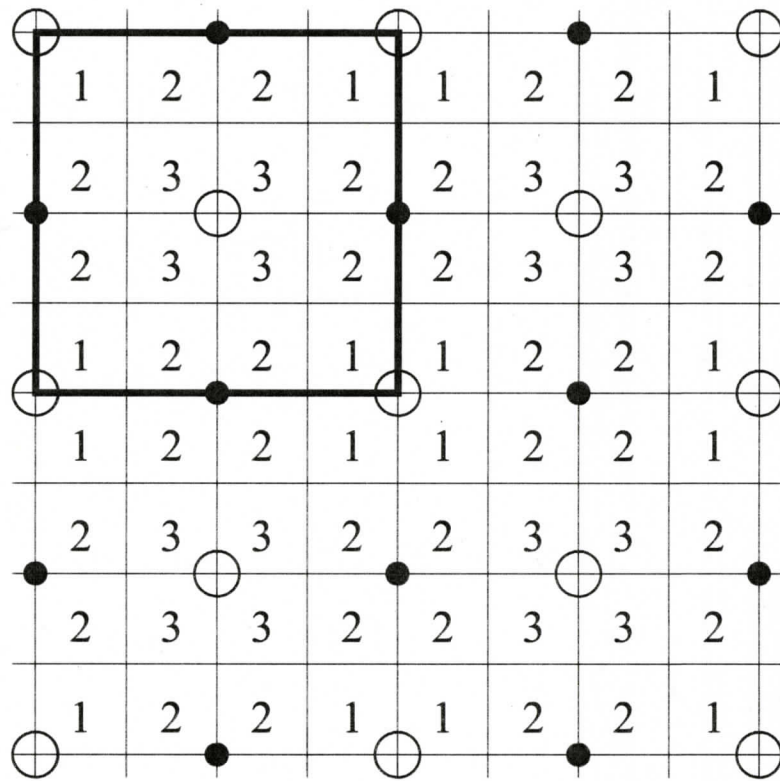


Figure A.11: Crystallographic: p4. Conway: 442.

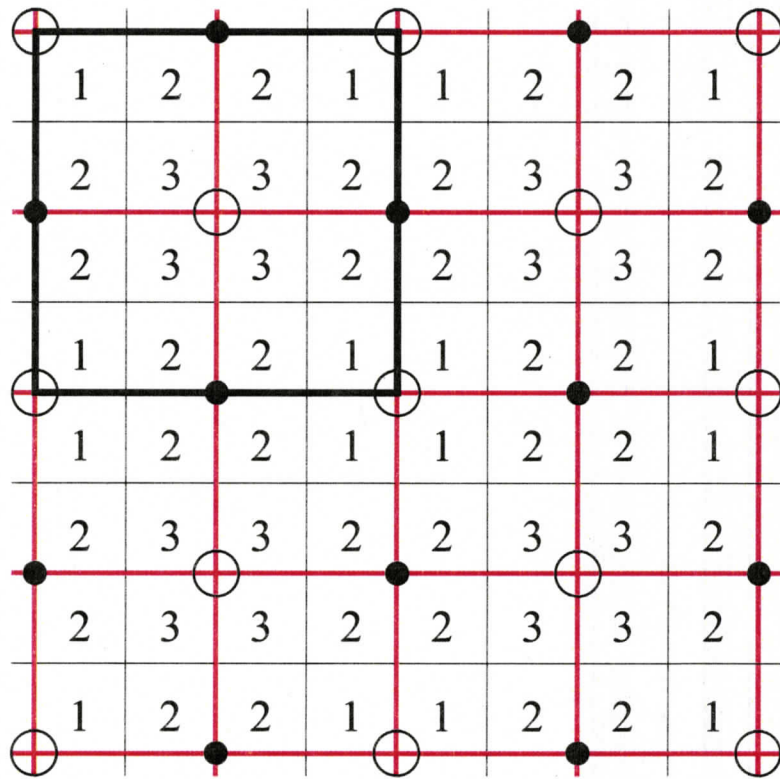


Figure A.12: Crystallographic: p4m. Conway: \*442.

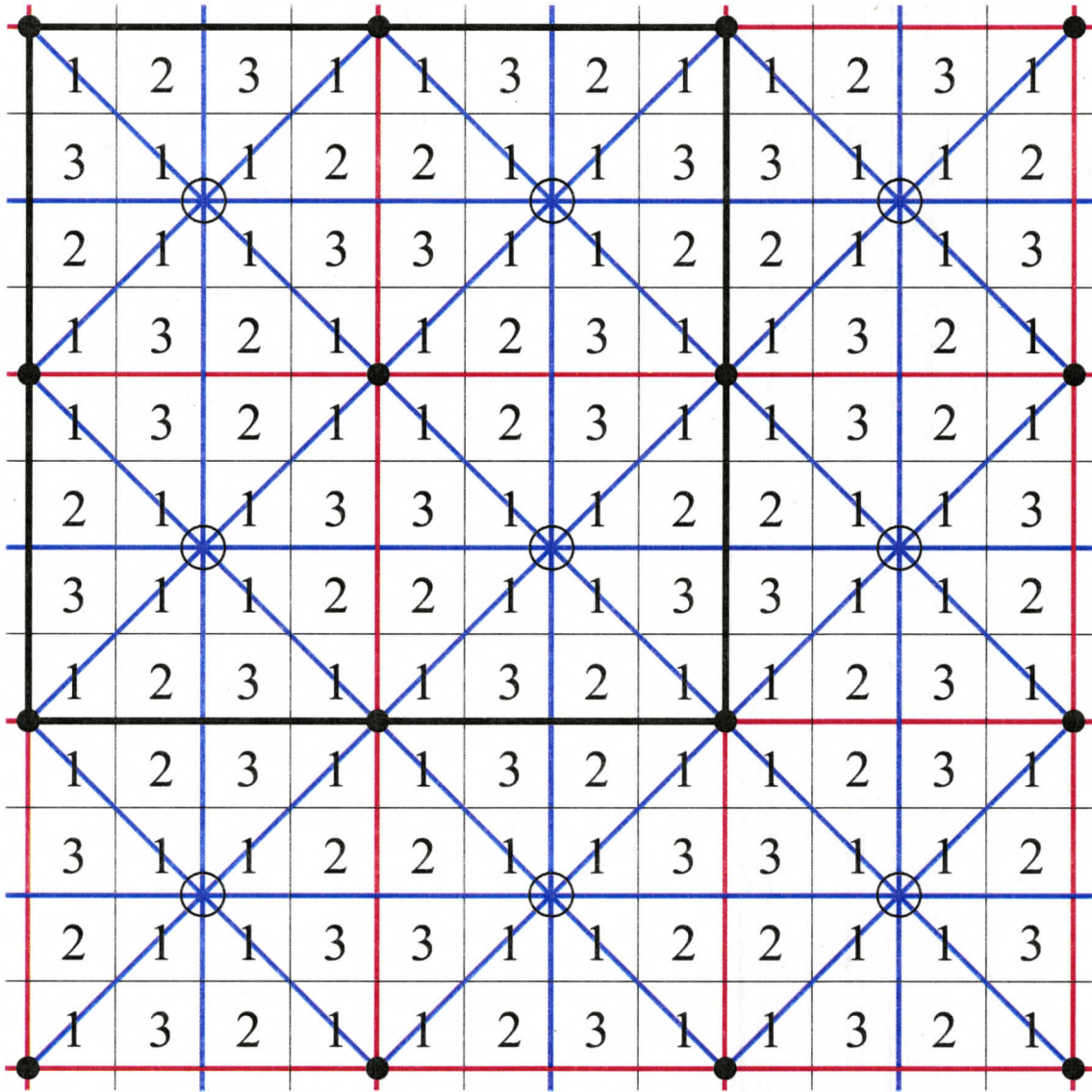


Figure A.13: Crystallographic: p4g. Conway: 4\*2.