Ring Structures on the K-Theory of C^*-Algebras Associated to Smale Spaces

by

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B.Sc., University of Victoria, 2004
M.Sc., University of Toronto, 2005

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ABSTRACT

We study the hyperbolic dynamical systems known as Smale spaces. More specifically we investigate the C^*-algebras constructed from these systems. The K_* group of one of these algebras has a natural ring structure arising from an asymptotically abelian property. The K_* groups of the other algebras are then modules over this ring. In the case of a shift of finite type we compute these structures explicitly and show that the stable and unstable algebras exhibit a certain type of duality as modules. We also investigate the Bowen measure and its stable and unstable components with respect to resolving factor maps, and prove several results about the traces that arise as integration against these measures. Specifically we show that the trace is a ring/module homomorphism into $\mathbb{R}$ and prove a result relating these integration traces to an asymptotic of the usual trace of an operator on a Hilbert space.

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Chapter 1

Introduction

1.1 Introduction

In the document that follows we study the class of hyperbolic dynamical systems known as Smale spaces. More specifically we study the C^*-algebras constructed from a given Smale space and the K-theory of these algebras.

Smale spaces were defined by Ruelle in [25], based on the Axiom A systems studied by Smale in [28]. We defer the precise definition until section 2.2, and instead begin with a non-technical description. Roughly speaking, a Smale space is a topological dynamical system (X, φ) in which X is a compact metric space with distance function d, and φ is a homeomorphism. The structure of (X, φ) is such that each point $x \in X$ has two local sets associated to it, a set, $V^s(x, \epsilon)$, on which the map φ is (exponentially) contracting, and a set, $V^u(x, \epsilon)$, on which the map φ^{-1} is contracting. We call these sets the local stable and unstable sets for x. Furthermore, x has a neighbourhood, U_x that is isomorphic to $V^u(x, \epsilon) \times V^s(x, \epsilon)$, see figure 1.1. In other words, the sets $V^u(x, \epsilon)$ and $V^s(x, \epsilon)$ provide a coordinate system for U_x such that, under application of the map φ, one coordinate contracts, and the other expands. We denote this homeomorphism by $[\cdot, \cdot] : V^u(x, \epsilon) \times V^s(x, \epsilon) \to U_x$.

We now define three equivalence relations on X.

Definition 1.1. *Let (X, d, φ) be a Smale space, and let $x, y \in X$. We say x and y are* stably equivalent *and write $x \overset{s}{\sim} y$ if*

$$\lim_{n \to +\infty} d(\varphi^n(x), \varphi^n(y)) = 0.$$

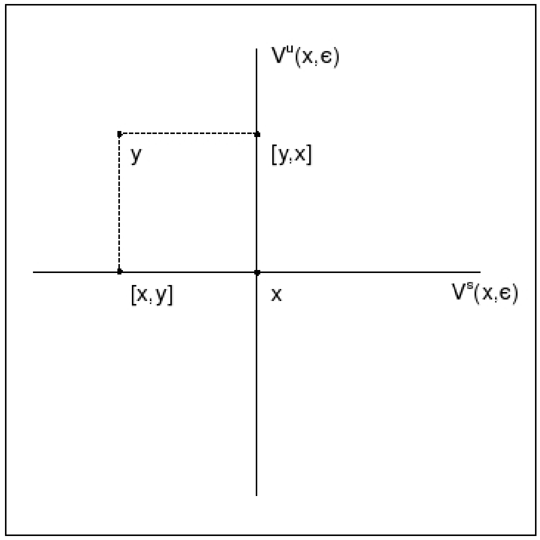

Figure 1.1: $V^u(x,\epsilon) \times V^s(x,\epsilon)$ is homeomorphic to a neighbourhood of x.

We say x and y are unstably equivalent *and write $x \overset{u}{\sim} y$ if*

$$\lim_{n \to -\infty} d(\varphi^n(x), \varphi^n(y)) = 0.$$

Finally, we say x and y are homoclinic *and write $x \overset{h}{\sim} y$ if $x \overset{s}{\sim} y$ and $x \overset{u}{\sim} y$.*

We then construct groupoid C^*-algebras from the groupoids of stable, unstable, and homoclinic equivalence(G^s,G^u, and G^h). The construction of these C^*-algebras is also originally due to Ruelle ([26]). In the case of a Shift of Finite type these are the algebras studied by Cuntz and Krieger in [8], [15]. In the more general Smale space setting, these algebras have been studied extensively by Putnam, eg. [19], [20], [21]. The details of the construction of the C^*-algebras will be presented in more detail in section 2.3, however the general idea is as follows.

For each groupoid, G, we find a Haar system and consider the convolution algebra of continuous functions with compact support, $C_c(G)$. The groupoids in question are amenable, so when completing in norm, the full and reduced C^*-algebras are isomorphic. This yields 3 C^*-algebras, which for now we call $S(X,\varphi)$, $U(X,\varphi)$, and $H(X,\varphi)$ for the algebras of stable, unstable, and homoclinic equivalence. In practice, the stable and unstable algebras that we deal with are defined in a slightly different way. We first fix a finite φ-invariant subset of X, P (the obvious choice is a single periodic orbit). We then consider the set of all points in X that are unstably equivalent to a point in P, call this $V^u(P)$. The groupoid that we actually use to construct our stable algebra is then the groupoid of stable equivalence restricted to the set $V^u(P)$. This set is an abstract transversal for the groupoid G^s so this new groupoid is equivalent to G^s in the sense of [17], and the groupoid C^*-algebras are Morita equivalent. We call these equivalent algebras $S(X, \varphi, P)$, $U(X, \varphi, P)$. One of the advantages of this approach is that the new restricted groupoids are r-discrete (the groupoid G^h is already r-discrete). Moreover, if we consider the set $V^h(P) = V^s(P) \cap V^u(P)$, all 3 of our C^*-algebras can be faithfully represented on $l^2(V^h(P))$, and H is contained in the multiplier algebra of both S and U.

The homeomorphism φ yields a $*$-automorphism on each of the 3 algebras associated to (X, φ). For $f \in C_c(G)$ we define $\alpha(f)$ by $\alpha(f)(x, y) = f(\varphi^1(x), \varphi^1(y))$. As shown in [19] there are several asymptotic commutation results that arise from α. We state several in the following theorem. These results appear again, with proof, later in this document.

Theorem 1.2. *Let $a \in S(X, \varphi, P)$, $b \in U(X, \varphi, P)$, and $f, g \in H(X, \varphi)$. Then*

1. $||[\alpha^n(f), g]|| \to 0$ *as* $n \to \pm\infty$,
2. $||[\alpha^n(f), a]|| \to 0$ *as* $n \to -\infty$,
3. $||[\alpha^n(f), b]|| \to 0$ *as* $n \to +\infty$,
4. $||[\alpha^n(a), b]|| \to 0$ *as* $n \to +\infty$.

Moreover, ab and ba are compact operators.

This asymptotically abelian structure on $H(X, \varphi)$ suggests a product on K-theory as in [7]. However, to define the product on K-theory, we need an asymptotic morphism from $H \otimes H \to H$. To achieve this we need a version of α which yields a family α_t parametrized by a real number t, instead of the discrete α^n. The algebra on which we can do this is the mapping cylinder of $H(X, \varphi)$, $C(H, \alpha) = \{f \in C(\mathbb{R}, H(X, \varphi)) \mid f(t+1) = \alpha(f(t))\}$. We then define the family of automorphisms $\alpha_t(f)(s) = f(t+s)$. The asymptotically abelian structure of $H(X, \varphi)$ is inherited by $C(H, \alpha)$ and thus have an asymptotic morphism $C(H, \alpha) \otimes C(H, \alpha) \to C(H, \alpha)$ given by $f \otimes g \mapsto \alpha_t(f)\alpha_{-t}(g)$ (see for example 25.2.3 in [1]). This yields a map $K_*(C(H, \alpha) \otimes C(H, \alpha)) \to K_*(C(H, \alpha))$. Combining this with a Kunneth Theorem (eg. Theorem 23.1.3 in [1]) gives a map $K_*(C(H,\alpha)) \otimes K_*(C(H,\alpha)) \to K_*(C(H,\alpha))$. This is the desired product on K-theory. This is described in [19] and will be covered in more detail in section 3.2.

In the case of a SFT we describe $K_*(C(H,\alpha))$ as an inductive limit of groups of integer matrices. We are then able to write the product in terms of matrix algebras. We state the results in the following theorem. This result is proved in several parts in chapter 3.

Theorem 1.3. *Let (Σ,σ) be a SFT with $n \times n$ adjacency matrix A, and $C(H,\alpha)$ the mapping cylinder of the associated homoclinic algebra. Then*

$$K_0(C(H,\alpha)) \cong \big(C(A) \times \mathbb{N}\big)/\sim$$

where $C(A) = \{X \in M_n(\mathbb{Z}) \mid XA = AX\}$, and for $m \geq k$, $(X,k) \sim (Y,m)$ if and only if there exists l such that $A^{m-k+l}XA^{m-k+l} = A^lYA^l$. Furthermore,

$$K_1(C(H,\alpha)) \cong \big(M_n(\mathbb{Z})/B(A) \ \times \mathbb{N}\big)/\sim$$

where $B(A) = \{X \in M_n(\mathbb{Z}) \mid X = AY - YA \text{ for some } Y \in M_n(\mathbb{Z})\}$, and $\sim$ is the same equivalence relation as above. Finally, the product on $K_*(C(H,\alpha))$ is given by

$$([X_1,k]+[Y_1+B(A),k])*([X_2,k]+[Y_2+B(A),k])=[X_1X_2,2k]+[X_1Y_2+Y_1X_2+B(A),2k].$$

It is worth noting that the above described ring is, in general, non-commutative. In the SFT case this happens when there exist $X_1, X_2 \in C(A)$ such that $X_1X_2 \neq X_2X_1$ (see section 4.4).

Similarly, there are asymptotic morphisms from $S(X,\varphi,P)\otimes C(H,\alpha) \to S(X,\varphi,P)$ and $C(H,\alpha) \otimes U(X,\varphi,P) \to U(X,\varphi,P)$ given by $a \otimes f \to a\alpha_{-t}(f)$ and $f \otimes b \to \alpha_t(f)b$ respectively. These give rise to right and left $C(H,\alpha)$-module structures for $S(X,\varphi,P)$ and $U(X,\varphi,P)$.

In the case of a SFT we once again compute these structures concretely. This result is proved in section 4.1.

Theorem 1.4. *Let (Σ,σ) be a SFT with $n \times n$ adjacency matrix A, and $C(H,\alpha)$ the mapping cylinder of the associated homoclinic algebra. Then*

$$K_0(S(\Sigma,\sigma,P)) \cong \big(\mathbb{Z}^n \times \mathbb{N}\big)/\sim$$

where, for $m \geq k$*,* $(v,k) \sim (w,m)$ *if and only if there exists* l *such that* $vA^{m-k+l} = wA^l$*.*

$$K_0(U(\Sigma,\sigma,P)) \cong \big(\mathbb{Z}^n \times \mathbb{N}\big)/\sim$$

where, for $m \geq k$*,* $(v,k) \sim (w,m)$ *if and only if there exists* l *such that* $A^{m-k+l}v = A^l w$*. Moreover the module structures are given by*

$$[v,n] * [X,n] = [vX, 3n],$$

and

$$[X,n] * [w,n] = [Xw, 3n].$$

We further investigate the module structures in the SFT case by considering $K_0(S(\Sigma,\sigma,P))$, $K_0(U(\Sigma,\sigma,P))$ as modules over a certain subring R of $K_0(C(H,\alpha))$. The goal was to prove a duality type result of the form $Hom_R(K_0(S(\Sigma,\sigma,P)),R) \cong K_0(U(\Sigma,\sigma,P))$. As $K_0(C(H,\alpha))$ is, in general, non-commutative, the subring R for which we first attempted to prove the result was the center of the ring $K_0(C(H,\alpha))$. While in many cases the result does hold when R is the center, it is not true in general. To obtain this duality result in general we must restrict to a smaller (in terms of containment, though not in terms of rank) subring.

The induced maps α_* and α_*^{-1} on $K_0(S(\Sigma,\sigma,P))$ can be realized by multiplication by an element of $K_0C(H,\alpha)$. Specifically, $\alpha_*[v,k] = [v,k] * [A,0]$, $\alpha_*^{-1}[v,k] = [v,k] * [A,1]$. If we let R be the subgroup generated by $[A,0]$ and $[A,1]$, then the duality result holds. This result is proved in section 4.2.

Theorem 1.5. *Let* (Σ,σ) *be an irreducible SFT, and* R *the subring of* $K_0(C(H,\alpha))$ *generated by the elements which realize the maps* α_* *and* α_*^{-1} *on* $K_0(S(\Sigma,\sigma,P))$*. Then*

$$Hom_R(K_0(S(\Sigma,\sigma,P)),R) \cong K_0(U(\Sigma,\sigma,P))$$

as left R*-modules.*

Roughly speaking, the difference between R and the center of $K_0(C(H,\alpha))$, Z, is that R consists of all integer polynomials in A, whereas Z consists of all integer matrices which are rational polynomials in A, see section 4.4 for examples.

We also prove that two SFTs with shift equivalent adjacency matrices have isomorphic K-theory ring/module structures (section 4.3).

In chapter 5 we turn our attention to measures on Smale space, and the traces that arise from integration against these measures.

A Smale space (X, φ) has a unique φ-invariant, entropy maximizing probability measure, μ, called the Bowen measure. Moreover, this Bowen measure can be written locally as a product measure $\mu^u \times \mu^s$, where μ^u and μ^s are supported on $V^u(x, \epsilon)$, $V^s(x, \epsilon)$ respectively. There are several characterizations of the Bowen measure. In addition to the above characterization, it can be constructed as a limit of measures supported on periodic points and in the case of a SFT it can be written down explicitly in terms of the Perron eigenvalue/eigenvector of the adjacency matrix, in this case it is often called the Parry measure. As every irreducible Smale space (X, φ) is the image of a SFT, (Σ, σ) under an almost one-to-one factor map. The Bowen measure on X is then the push-forward of the Bowen(Parry) measure on Σ.

We provide a new construction of the Bowen measure as the limit of measures supported on homoclinic points. See section 5.1.3 for the statement and proof of this theorem. We do not state it here as we would have to introduce some cumbersome notation to do so. We also show that the stable and unstable components of the Bowen measure can be obtained as the push-forward of the corresponding measures on a SFT. We do this by writing the almost one-to-one factor map as the composition of a u-resolving and an s-resolving map (as in [21]).

The Bowen measure and its stable and unstable components lead to traces on $H(X, \varphi)$, $S(X, \varphi, P)$, and $U(X, \varphi, P)$ given as follows. For $f \in C_c(G^h(X, \varphi))$, $a \in C_c(G^s(X, \varphi, P))$ and $b \in C_c(G^u(X, \varphi, P))$ the traces are

$$\begin{aligned} \tau^h(f) &= \int_X f(x,x)d\mu, \\ \tau^s(a) &= \int_{V^u(P)} a(x,x)d\mu^u, \text{ and} \\ \tau^u(b) &= \int_{V^s(P)} b(x,x)d\mu^s. \end{aligned}$$

τ^h extends to a bounded trace on $H(X, \varphi)$, while τ^s and τ^u extend to (unbounded) traces on $S(X, \varphi, P)$ and $U(X, \varphi, P)$. We can also define a trace on $C(H, \alpha)$. For $g \in C(H, \alpha)$ define $\tau^{CH}(g) = \int_0^1 \tau^h(g(t))dt$, and notice that for a projection $p \in C(H, \alpha)$ we have $\tau^{CH}(p) = \tau^h(p(0))$.

In the case of a mixing Smale space, Putnam ([19]) proved that τ^h is multiplicative, and hence τ_*^{CH} is a ring homomorphism from $K_0(C(H, \alpha))$ to $\mathbb{R}$. In section 5.2, we extend Putnam's result to the case of an irreducible Smale space, and use a similar argument to prove that the traces also respect the module structures. We state the result here as follows.

Theorem 1.6. *Let* $[a]_0 \in K_0(S(X,\varphi,P))$, $[b]_0 \in K_0(U(X,\varphi,P))$ *and let* $[p]_0$, $[q]_0 \in K_0(C(H,\alpha))$, *then*

1. $\tau_*^{CH}([p]_0 * [q]_0) = \tau_*^{CH}([p]_0)\tau_*^{CH}([q]_0)$,
2. $\tau_*^{s}([a]_0 * [p]_0) = \tau_*^{s}([a]_0)\tau_*^{CH}([p]_0)$,
3. $\tau_*^{u}([p]_0 * [b]_0) = \tau_*^{CH}([p]_0)\tau_*^{u}([b]_0)$.

Once again, in the case of a SFT we work out these traces explicitly in terms of the Perron eigenvector/eigenvalue of the adjacency matrix.

Finally, in section 5.2.3, we relate the traces τ^s and τ^u to the usual trace of an operator on a Hilbert space by an asymptotic result. Let (X,φ) be a mixing Smale space and let $a \in S(X,\varphi,P)$, $b \in U(X,\varphi,P)$ be projections. Recall that ab and ba are compact operators. If $Tr(\cdot)$ is the usual trace on $\mathfrak{B}(l^2(V^h(P)))$, we have the following result.

Theorem 1.7. *Let* (X,φ) *be a mixing Smale space with topological entropy* $\log(\lambda)$, *and let* $a \in S(X,\varphi,P)$, $b \in U(X,\varphi,P)$ *be projections. Then*

$$\lim_{k\to+\infty} \lambda^{-2k} Tr(\alpha^k(a)\alpha^{-k}(b)) = \tau^s(a)\tau^u(b).$$

Chapter 2

Background

In this chapter we provide some background material from topological dynamics, the construction of the C^*-algebras, and K-theory. This background material is not intended to be self contained. In many places we simply refer the reader to other sources for more a detailed foundation. In the cases that we do provide proofs of the results stated in this chapter, we do so because the result, or its proof is a key ingredient in a future chapter.

2.1 Topological Dynamics

We work in the context of a topological dynamical system (TDS). For more background on TDS see for example [5], [12]. In fact, our topological spaces will be compact metric spaces. Let (X, d) be a compact metric space, and $\varphi : X \to X$ a homeomorphism. We start by defining several different notions of recurrence in a TDS.

Definition 2.1. *Let (X, d, φ) be as above, and let $x \in X$. We say that x is a* fixed point *of φ if $\varphi(x) = x$. We say that x is a* periodic point *of φ if there is a positive integer n such that $\varphi^n(x) = x$. If n is the least integer such that $\varphi^n(x) = x$, we say x has period n. For $m \in \mathbb{N}$ we denote the set of all periodic points with period m by $Per_m(X, \varphi)$. We also define the set of all periodic points in X:*

$$Per(X, \varphi) = \bigcup_{m \geq 1} Per_m(X, \varphi).$$

We now define a weaker notion of recurrence on X.

Definition 2.2. *Let (X, d, φ) be as above, and let $x \in X$. We say that x is a* non-wandering point *if, for every open set U, containing x, there is a positive integer n such that $\varphi^n(U) \cap U$ is non-empty. We denote the set of all non-wandering points in X by $NW(X, \varphi)$. Notice that $NW(X, \varphi)$ is closed and φ-invariant.*

Definition 2.3. *We say (X, d, φ) is* irreducible *if, for every (ordered) pair of non-empty open sets, U,V, there exists a positive integer n such that $\varphi^n(U) \cap V$ is non-empty.*

Definition 2.4. *We say (X, d, φ) is* mixing *if, for every (ordered) pair of non-empty open sets, U,V, there exists a positive integer N such that $\varphi^n(U) \cap V$ is non-empty for all $n \geq N$.*

Maps between TDS will also be of significant importance to us. The natural class of maps to consider are those which intertwine the dynamics: *factor maps*.

Definition 2.5. *Let (Y, ψ) and (X, φ) be TDS. We say that the continuous map $\pi : Y \to X$ is a* factor map *if π is surjective, and $\varphi \circ \pi = \pi \circ \psi$. In this case we write*

$$\pi : (Y, \psi) \to (X, \varphi).$$

In addition we say that π is finite-to-one *if there exists $M \in \mathbb{N}$ such that, for all $x \in X$, $\#(\pi^{-1}\{x\}) \leq M$, and we say π is* almost one-to-one *if there exists $x \in X$ such that $\#(\pi^{-1}\{x\}) = 1$. Finally, if π is injective, we say that (Y, ψ) and (X, φ) are topologically conjugate.*

Topological conjugacy is the natural notion of equivalence or isomorphism for TDS.

2.1.1 Topological Entropy

For a more thorough explanation of topological entropy, see for example [5] or [12]. The brief description given below follows [5].

Topological entropy is a conjugacy invariant which gives a measure of the complexity of the orbit structure of the system. It describes the growth rate of the number of orbit segments which are 'essentially different' in that they can be distinguished with an arbitrarily fine but finite mesh.

Let (X, d) be a compact metric space, and $\varphi : X \to X$ a homeomorphism. For each N define

$$d_N(x, y) = \max_{0 \leq i < N} d(\varphi^i(x), \varphi^i(y)).$$

For $\epsilon > 0$ we say a set $A \subset X$ is (n, ϵ)-spanning if for each $x \in X$ there is a $y \in A$ such that $d_n(x, y) < \epsilon$. Ie. each orbit segment of length n is ϵ-close to an orbit segment from A. As X is compact, there are (n, ϵ)-spanning sets which are finite. Let

$$span(n, \epsilon, \varphi) = \min\{\#A | A \text{ is } (n, \epsilon)\text{-spanning}\}.$$

Similarly, $A \subset X$ is (n, ϵ)-separated if for all $x, y \in A$, $d_n(x, y) > \epsilon$ and

$$sep(n, \epsilon, \varphi) = \min\{\#A | A \text{ is } (n, \epsilon)\text{-separated}\}.$$

Finally, we say the collection of sets $\mathcal{A}$ is an (n, ϵ)-cover if $X \subset \bigcup_{A_\alpha \in \mathcal{A}} A_\alpha$, and for each $A_\alpha \in \mathcal{A}$, the d_n diameter of A_α is less than ϵ. Compactness of X implies that there are (n, ϵ)-covers with finitely many sets.

$$cov(n, \epsilon, \varphi) = \min\{\#\mathcal{A} | \mathcal{A} \text{ is an } (n, \epsilon)\text{-cover}\}.$$

These three numbers count the number of different length-n orbit segments which are ϵ-distinguishable. The topological entropy is then defined to be

$$h(\varphi) = \lim_{\epsilon \to 0^+} \left(\limsup_{n \to \infty} \frac{1}{n} \log(cov(n, \epsilon, \varphi)) \right).$$

As in [5] this limit exists and is either $+\infty$ or a non-negative real number. Moreover, in the above definition, $cov(n, \epsilon, \varphi)$ can be replaced by either $sep(n, \epsilon, \varphi)$ or $span(n, \epsilon, \varphi)$ and the $\limsup$ can be replaced by $\liminf$.

2.2 Smale Space

The material in this section comes primarily from [19], [25], [26], and follows the development presented by I.F. Putnam in a course on Smale spaces delivered in the spring of 2006 at the University of Victoria. Let (X, d) be a compact metric space, and $\varphi : X \to X$ a homeomorphism. Assume that there is a constant ϵ_X and a map

$[\cdot,\cdot] : \Delta_{\epsilon_X} \to X$ where

$$\Delta_{\epsilon_X} = \{(x,y)|d(x,y) \leq \epsilon_X\}$$

which satisfies the following axioms:

B1. $[x,x] = x$,

B2. $[x,[y,z]] = [x,z]$, whenever both sides are defined,

B3. $[[x,y],z] = [x,z]$, whenever both sides are defined,

B4. $[\varphi(x),\varphi(y)] = \varphi([x,y])$, whenever both sides are defined.

In addition, also assume that there is a constant $0 < c_X < 1$ such that, for all $x \in X$, the following two conditions are satisfied:

C5. For $y,\ z \in X$ such that $d(x,y), d(x,z) \leq \epsilon_X$ and $[y,x] = x = [z,x]$, we have

$$d(\varphi(y),\varphi(z)) \leq c_X d(y,z).$$

C6. For $y,\ z \in X$ such that $d(x,y), d(x,z) \leq \epsilon_X$ and $[x,y] = x = [x,z]$, we have

$$d(\varphi^{-1}(y),\varphi^{-1}(z)) \leq c_X d(y,z).$$

Definition 2.6. *Any quadruple* $(X,d,\varphi,[\cdot,\cdot])$ *satisfying the above 6 axioms is a* Smale space*.*

For $x \in X$ and $0 < \epsilon \leq \epsilon_X$ we define the following two sets.

$$\begin{aligned} V^s(x,\epsilon) &= \{y|d(x,y) \leq \epsilon, [y,x] = x\} \\ V^u(x,\epsilon) &= \{y|d(x,y) \leq \epsilon, [x,y] = x\} \end{aligned}$$

These sets are called, respectively, the *local stable set* and *local unstable set* at x. The following lemma shows that these sets provide a local coordinate system for the X, this result can be found in, for example, [19]. Also, see figure 1.1.

Lemma 2.7. *For* $\epsilon_1, \epsilon_2 < \epsilon_X/2$, $x \in X$ *the map* $[\cdot,\cdot] : V^u(x,\epsilon_1) \times V^s(x,\epsilon_2) \to X$ *is a homeomorphism onto its range, which is a neighbourhood of* x*.*

We now define three equivalence relations on X.

Definition 2.8. *Let (X, d, φ) be a Smale space, and let $x, y \in X$. We say x and y are* stably equivalent *and write $x \overset{s}{\sim} y$ if*

$$\lim_{n \to +\infty} d(\varphi^n(x), \varphi^n(y)) = 0.$$

We say x and y are unstably equivalent *and write $x \overset{u}{\sim} y$ if*

$$\lim_{n \to -\infty} d(\varphi^n(x), \varphi^n(y)) = 0.$$

Finally, we say x and y are homoclinic *and write $x \overset{h}{\sim} y$ if $x \overset{s}{\sim} y$ and $x \overset{u}{\sim} y$.*

Proposition 2.9. *Let $x \in X$ and $0 < \epsilon \leq \epsilon_X$. The equivalence class of x under $\overset{s}{\sim}$ is*

$$\bigcup_{n \geq 0} \varphi^{-n} \left(V^s(\varphi^n(x), \epsilon) \right),$$

and the equivalence class of x under $\overset{u}{\sim}$ is

$$\bigcup_{n \geq 0} \varphi^{n} \left(V^u(\varphi^{-n}(x), \epsilon) \right).$$

Definition 2.10. *We denote the equivalence class of $x \in X$ under $\overset{s}{\sim}$ by $V^s(x)$. Similarly, the equivalence class under $\overset{u}{\sim}$ is denoted $V^u(x)$, and the equivalence class under $\overset{h}{\sim}$ is $V^h(x)$.*

We wish to endow these sets with a topology. The topology inherited as subsets of X, is not the most natural topology to consider. The idea is that, for $V^s(x)$, "locally" φ should be contracting. In the relative topology of X, this need not be the case. The more natural topology comes from the characterization of $V^s(x)$ as

$$V^s(x) = \bigcup_{n \geq 0} \varphi^{-n} \left(V^s(\varphi^n(x), \epsilon) \right).$$

We first notice that

$$\varphi^{-n}(V^s(\varphi^n(x), \epsilon)) \subset \varphi^{-n-1}(V^s(\varphi^{n+1}(x), \epsilon)).$$

Each set $\varphi^{-n}(V^s(\varphi^n(x), \epsilon))$ is thus given the relative topology of X and $V^s(x)$ is given the inductive limit topology. A subset U of $V^s(x)$ is thus open if and only if its intersection with $\varphi^{-n}(V^s(\varphi^n(x), \epsilon))$ is open (in $\varphi^{-n}(V^s(\varphi^n(x), \epsilon))$) for all but finitely many n. We topologize $V^u(x)$ in a completely analogous way. The topology obtained in this manner has a number of nice properties, summarized in the following theorem.

Theorem 2.11. *Let $x \in (X, d, \varphi)$ and let $V^s(x)$ and $V^u(x)$ be endowed with the inductive limit topology as above.*

1. *$V^s(x)$ and $V^s(u)$ are locally compact and Hausdorff.*

2. *$\{y_n\}$ (in $V^s(x)$) converges to y (in $V^s(x)$) if and only if y_n converges to y in the topology on X and $[y_n, y] = y$ for all n sufficiently large.*

3. *$\{y_n\}$ (in $V^u(x)$) converges to y (in $V^u(x)$) if and only if y_n converges to y in the topology on X and $[y, y_n] = y$ for all n sufficiently large.*

4. *Sets of the form $V^s(y, \epsilon)$ where $y \in V^s(x)$ and $0 < \epsilon < \epsilon_X$ form a neighborhood base for the topology on $V^s(x)$.*

5. *Sets of the form $V^u(y, \epsilon)$ where $y \in V^u(x)$ and $0 < \epsilon < \epsilon_X$ form a neighborhood base for the topology on $V^u(x)$.*

We now state some results about the structure of Smale spaces. These are known as Smale's spectral decomposition, see [25], [28]. We first see that a non-wandering Smale space may be decomposed into a finite number of irreducible Smale spaces.

Proposition 2.12. *Let (X, d, φ) be a non-wandering Smale space. Then there exists a positive integer N, and subsets $X_1, \ldots, X_N$ of X which are open, closed, pairwise disjoint, and φ-invariant. Furthermore, $\cup_1^N X_i = X$, and $(X_i, d, \varphi|_{X_i})$ is irreducible for each i. The sets X_i are unique up to relabeling.*

We now see that an irreducible Smale space can be decomposed into finitely many components, each of which is mixing.

Proposition 2.13. *Let (X, d, φ) be an irreducible Smale space. Then there exists a positive integer N and subsets $X_1, \ldots, X_N$ of X which are open, closed, pairwise disjoint, and whose union is X. These sets are cyclicly permuted by φ, and $\varphi^N|_{X_i}$ is mixing for each i. These sets are unique up to (cyclic) relabeling.*

The preceding Proposition can be rewritten in the following, seemingly stronger, version.

Proposition 2.14. *Let (X, φ) be an irreducible Smale space, then there exists a mixing Smale space (Y, ψ) and a positive integer N such that $X \cong Y \times \{1, \ldots, N\}$ and*

$$\varphi(y, i) = \begin{cases} (y, i+1) & \text{if } 1 \leq i \leq N-1 \\ (\psi(y), 1) & \text{if } i = N \end{cases}$$

Proof: Let $X_1, \ldots, X_N$ be as in Prop. 2.13. It suffices to show that for $1 \leq i \leq N-1$, $(X_i, \varphi^N) \cong (X_{i+1}, \varphi^N)$ with the topological conjugacy realized by the map φ.

As φ is a homeomorphism, it suffices to show that, for all $x \in X_i$

$$\varphi \circ \varphi^N(x) = \varphi^N \circ \varphi(x),$$

which is obvious. Now setting $Y = X_1$, $\psi = \varphi^N$ we have $X \cong Y \times \{1, \ldots, N\}$ and $\varphi(y, i) = (y, i+1)$ for $1 \leq i \leq N-1$. Finally, for all $1 \leq i \leq N$, we have $\varphi^N(y, i) = (\varphi^N(y), i) = (\psi(y), i)$ which implies $\varphi(y, N) = (\psi(y), 1)$. $\square$

2.2.1 Shifts of Finite Type

For the general definition of a shift of finite type (SFT), we refer the reader to [5] or [16], wherein it is shown that every shift of finite type is topologically conjugate to the following edge shift description.

Let G be a directed graph. We think of G as consisting of a vertex set V, an edge set E, and two maps $i, t : E \to V$ where $i(e)$ is the *initial* vertex for the edge e, and $t(e)$ is the *terminal* vertex. We then define

$$\Sigma_G = \{(e_n)_{n \in \mathbb{Z}} \mid e_n \in E,\ t(e_n) = i(e_{n+1}),\ \forall n\}.$$

In other words, Σ_G is the space of all doubly infinite paths in G. We define the metric on Σ_G as follows. For $e, f \in \Sigma_G$

$$d(e, f) = \inf\{2^{-n} \mid n \geq 0,\ e_i = f_i\ \forall |i| < n\}.$$

We now define the map σ_G on Σ_G to be the left shift. In other words, for $e \in \Sigma$, $n \in \mathbb{Z}$

$$(\sigma(e))_n = e_{n+1}.$$

We call the resulting space the *edge shift* on G.

Given a SFT, with graph G, we may enumerate the vertex set so that we have $V = \{v_1, \ldots, v_n\}$. We then consider the $n \times n$ matrix A_G defined entry-wise by

$$A_G(i,j) = \#\{e \in E \mid i(e) = v_i,\ t(e) = v_j\}.$$

We call A_G the adjacency matrix for the graph G. Similarly, if we start with an $n \times n$ matrix, A, with non-negative integer entries. We can construct a graph G_A by setting $V = \{v_1, \ldots, v_n\}$ and for each pair (i,j) with $1 \le i, j \le n$ creating $A(i,j)$ edges from v_i to v_j. Clearly $A_{G_A} = A$, and $G_{A_G} = G$. We construct a SFT from the matrix A by setting $\Sigma_A = \Sigma_{G_A}$. From this point forward, whenever we talk about a SFT, we will freely talk about its associated graph, G, and adjacency matrix, A.

We have yet to show that a SFT is in fact a Smale Space. Let X be a SFT with directed graph G, let $\epsilon_X = 1/2$ and for $x, y \in X$ with $d(x,y) \le 1/2$ we define $[\cdot,\cdot]$ by

$$[x,y]_n = \begin{cases} x_n & \text{if } n \le 0, \text{ and} \\ y_n & \text{if } n \ge 0. \end{cases}$$

Notice that $d(x,y) \le 1/2$ implies that $x_0 = y_0$, so the above definition makes sense. Moreover, $[x,y] \in X$ because $t([x,y]_{-1}) = t(x_{-1}) = i(x_0) = i([x,y]_0)$ and similarly $t([x,y]_0) = t(y_0) = i(y_1) = i([x,y]_1)$. It is straightforward to verify that X, with $[\cdot,\cdot]$ defined in this way is a Smale space.

We now characterize the three equivalence relations on X. For $x, y \in X$, the following statements are all straightforward applications of the definitions.

$$\begin{aligned}
x \overset{s}{\sim} y &\iff \exists\, k \in \mathbb{Z} \text{ such that } x_n = y_n\ \forall\, n > k \text{ (right tail equivalence)} \\
x \overset{u}{\sim} y &\iff \exists\, k \in \mathbb{Z} \text{ such that } x_n = y_n\ \forall\, n < k \text{ (left tail equivalence)} \\
x \overset{h}{\sim} y &\iff x_n = y_n \text{ for all but finitely many } n \text{ (right and left tail equivalence)}
\end{aligned}$$

We leave further description of the equivalence relations to section 2.3.

We conclude this section with a result characterizing the topological entropy of a SFT. This appears as Theorem 4.3.1 in [16].

Theorem 2.15. *Let (Σ, σ) be an irreducible SFT with adjacency matrix A. There exists a positive eigenvalue λ of A such that $\lambda \ge |\lambda'|$ for all eigenvalues λ' of A. Moreover the topological entropy of (Σ, σ) is*

$$h(\sigma) = \log(\lambda).$$

2.2.2 Markov Partitions and Resolving Maps

We will not go into the details of Markov partitions here, see for example [12], [5] for an introduction to the subject. In [2] the existence of Markov partitions for irreducible Smale space is proved. The general idea is as follows. We divide our Smale space into a finite number of sets called 'rectangles', say $\{R_1, \ldots, R_N\}$, which satisfy certain conditions. We can then consider the SFT (Σ, σ) with graph consisting of N vertices in which there is an edge from vertex i to j when $\varphi(R_i) \cap R_j$ is non-empty. Moreover there is a factor map $\pi : (\Sigma, \sigma) \to (X, \varphi)$ defined as follows. For $(a)_{-\infty}^{\infty} \in \Sigma$ we have

$$\pi(a) = \bigcap_{-\infty}^{\infty} \varphi^i(R_{a_i}).$$

It can also be shown that the factor map is almost one-to-one, and that if (X, φ) is irreducible (resp. mixing), then so is (Σ, σ).

We now consider a special class of factor maps called *resolving* maps (see [9], [20], [21], [4]).

Definition 2.16. *A factor map $\pi : (Y, \psi) \to (X, \varphi)$ is said to be s-resolving (u-resolving) if for each $y \in Y$, $\pi|_{V^s(y)}$ ($\pi|_{V^u(y)}$) is injective.*

In [20] it is shown that resolving maps are the natural maps to consider in the context of constructing C^*-algebras from Smale spaces in the sense that the construction of the stable and unstable algebras (see section 2.3) is functorial for these maps. Furthermore, in [21] it is shown that the almost-one-to-one factor maps between the SFT (Σ, σ) and (X, φ) can be realized as the composition of an s-resolving map and a u-resolving map. In other words, Cor. 1.4 of [21] shows that for any irreducible (resp. mixing) Smale space (X, φ) we can find a Smale space (Y, ψ), a SFT (Σ, σ) and almost one-to-one factor maps $\pi_1 : (\Sigma, \sigma) \to (Y, \psi)$, $\pi_2 : (Y, \psi) \to (X, \varphi)$ such that

1. (Σ, σ) and (Y, ψ) are irreducible (resp. mixing),

2. π_1, π_2 are almost one-to-one, and

3. π_1 is s-resolving, π_2 is u-resolving.

2.2.3 Measures on Smale Space

In section 2.1 we discussed the topological entropy of the map φ. There is also a notion of measure-theoretic entropy of φ with respect to a given φ-invariant probability measure on X. We refer the reader to [29], [12], [5] for more on this topic.

For (X, φ) a Smale space, there is a unique φ-invariant probability measure maximizing the entropy of φ, see [27] [12]. We call this the Bowen measure and denote it by μ_X, or when the space is obvious, simply μ. In [3], Bowen constructed this measure as a limit of measures supported on periodic points. In [27] it is proved that if $\pi : (Y, \psi) \to (X, \varphi)$ is an almost one-to-one factor map, then the Bowen measure on X is the 'push forward' of the Bowen measure on Y. In other words for $E \subset X$, $\mu_X(E) = \mu_Y(\pi^{-1}(E))$.

As in Lemma 2.7, the map $[\cdot, \cdot]$ defines a homeomorphism from $V^u(x, \epsilon_1) \times V^s(x, \epsilon_2)$ to a neighbourhood U of x in X. Identifying U with $V^u(x, \epsilon) \times V^s(x, \epsilon)$, μ restricted to U is a product measure $\mu^{u,x} \times \mu^{s,x}$. Where $\mu^{u,x}$ and $\mu^{s,x}$ are measures on $V^u(x, \epsilon)$ and $V^s(x, \epsilon)$ respectively. These measures depend on x, however as in [27] they may be chosen so that:

1. For x, y close; ϵ, ϵ' small, the map $z \mapsto [y, z]$ defines a homeomorphism from $V^s(x, \epsilon)$ to $V^s(y, \epsilon')$ which takes $\mu^{s,x}$ to $\mu^{s,y}$. Also, $z \mapsto [z, y]$ is a homeomorphism from $V^u(x, \epsilon)$ to $V^u(y, \epsilon')$ taking $\mu^{u,x}$ to $\mu^{u,y}$.

2. $\mu^{s,\varphi(x)} \circ \varphi = \lambda^{-1} \mu^{s,x}$.

3. $\mu^{u,\varphi(x)} \circ \varphi = \lambda \mu^{u,x}$.

Where $\log(\lambda)$ is the topological entropy of (X, φ).

2.2.4 Local Homeomorphisms

For the rest of this section we let (X, φ) be an irreducible Smale space, and P a finite invariant set ($\varphi(P) = P$) (the obvious choice for P is a single periodic orbit, but everything that follows is true in more generality). We define

$$V^s(P) = \bigcup_{p \in P} V^s(p), \; V^u(P) = \bigcup_{p \in P} V^u(p), \text{ and } V^h(P) = V^s(P) \cap V^u(P).$$

Notice that this definition of $V^h(P)$ allows a point to be stably equivalent to some point $p \in P$ and unstably equivalent to a different point $q \in P$.

We define three equivalence relations on (X, φ) as follows:

$$\begin{aligned} G^s(X,\varphi,P) &= \{(y,z) \mid y,z \in V^u(P),\ y \stackrel{s}{\sim} z\}, \\ G^u(X,\varphi,P) &= \{(y,z) \mid y,z \in V^s(P),\ y \stackrel{u}{\sim} z\},\ \text{and} \\ G^h(X,\varphi) &= \{(y,z) \mid y \stackrel{h}{\sim} z\}. \end{aligned}$$

The relation $G^s(X,\varphi,P)$ ($G^u(X,\varphi,P)$) is just stable (unstable) equivalence restricted to $V^u(P)$ ($V^s(P)$). We make these restrictions so that the equivalence classes are countable (see [22]). $G^h(X,\varphi)$ is defined as the entire relation of homoclinic equivalence, since under $\stackrel{h}{\sim}$ equivalence classes are countable ([26], [19]).

Now suppose $x \stackrel{s}{\sim} y$, then there exists $N \in \mathbb{N}$ such that

$$\varphi^N(x) \in V^s(\varphi^N(x), \epsilon_X/2).$$

We can also find $0 < \delta \leq \epsilon_X/2$ such that, for all $0 \leq n \leq N$ we have

$$\begin{aligned} \varphi^n(B(x,\delta)) &\subset B(\varphi^n(x), \epsilon_X/2) \\ \varphi^n(B(y,\delta)) &\subset B(\varphi^n(y), \epsilon_X/2). \end{aligned}$$

We can then define two maps $h^u_x : V^u(x,\delta) \to V^u(y,\epsilon_X)$ and $h^u_y : V^u(y,\delta) \to V^u(x,\epsilon_X)$ by

$$\begin{aligned} h^u_x(z) &= \varphi^{-N}([\varphi^N(z), \varphi^N(y)]),\ z \in V^u(x,\delta), \\ h^u_y(z) &= \varphi^{-N}([\varphi^N(z), \varphi^N(x)]),\ z \in V^u(y,\delta) \end{aligned}$$

Similarly, if $x \stackrel{u}{\sim} y$ we can define maps $h^s_x : V^s(x,\delta) \to V^s(y,\epsilon_X)$ and $h^s_y : V^s(y,\delta) \to V^s(x,\epsilon_X)$ by

$$\begin{aligned} h^s_x(z) &= \varphi^{N}([\varphi^{-N}(z), \varphi^{-N}(y)]),\ z \in V^s(x,\delta), \\ h^s_y(z) &= \varphi^{N}([\varphi^{-N}(z), \varphi^{-N}(x)]),\ z \in V^s(y,\delta) \end{aligned}$$

Finally, for $x \stackrel{h}{\sim} y$, we can find appropriate N, δ and define maps $h_x : B(x,\delta) \to B(y,\epsilon_X)$ and $h_y : B(y,\delta) \to B(x,\epsilon_X)$ by

$$\begin{aligned} h_x(z) &= \left[h^u_x([z,x]), h^s_x([x,z])\right],\ z \in B(x,\delta), \\ h_y(z) &= \left[h^u_y([z,y]), h^s_y([x,y])\right],\ z \in B(y,\delta). \end{aligned}$$

The following figure shows these local homeomorphisms.

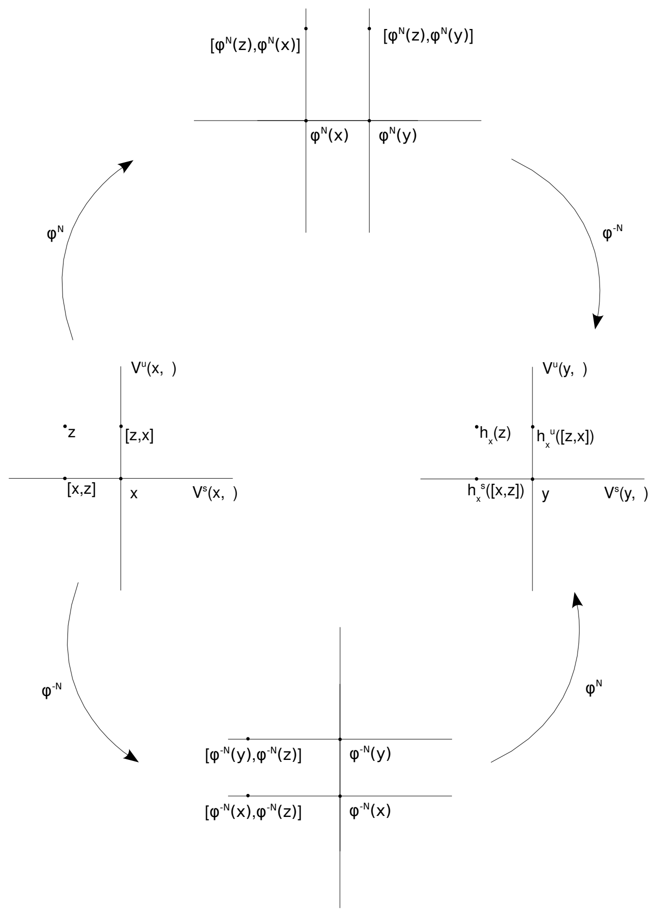

Figure 2.1: The map h_x maps a neighbourhood of x homeomorphically onto a neighbourhood of y.

Now, for $x \overset{s}{\sim} y,\ x, y \in V^u(P)$ and N, δ, h_x^u, h_y^u as above, consider the following subset of $G^s(X, \varphi, P)$.

$$V(x, y, h_y^u, \delta) = \{(h_y^u(z), z) |\ z \in V^u(y, \delta), h_y^u(z) \in V^u(x, \delta)\}.$$

Theorem 2.17. *The collection of sets $V(x, y, h_y^u, \delta)$ forms a neighbourhood base for a topology on $G^s(X, \varphi, P)$. In this topology, the canonical projection maps to $V^u(P)$ map basic sets homeomorphically to open sets. Furthermore, $G^s(X, \varphi, P)$ is second countable, locally compact, and Hausdorff.*

We can of course do the same thing for subsets $V(x, y, h^s_y, \delta)$ of $G^u(X, \varphi, P)$. For $x \overset{h}{\sim} y$ and N, δ, h_x, h_y as above, we also consider the following subsets of $G^h(X, \varphi)$.

$$V(x, y, h_y, \delta) = \{(h_y(z), z) |\ z \in B(y, \delta), h_y(z) \in B(x, \delta)\}.$$

Theorem 2.18. *The collection of sets $V(x, y, h_y, \delta)$ forms a neighbourhood base for a topology on $G^h(X, \varphi)$. In this topology, the two canonical projections to X map basic sets homeomorphically to open sets. Furthermore, $G^h(X, \varphi)$ is second countable, locally compact, and Hausdorff.*

2.3 C^*-Algebras from Smale Space

In this section we describe the construction of three C^*-algebras from a given Smale space, one algebra for each of the three equivalence relations described in section 2.2.4 above. In fact, we will only describe the construction of the stable algebra, and the homoclinic algebra. Since unstable equivalence on the Smale space (X, φ) is exactly stable equivalence on the Smale space (X, φ^{-1}), it suffices to only construct the stable algebra.

Let (X, φ) be an irreducible Smale space, and P a finite invariant set. Recall the relations

$$\begin{aligned} G^h(X, \varphi) &= \{(y, z) \mid y \overset{h}{\sim} z\} \\ G^s(X, \varphi, P) &= \{(y, z) \mid y, z \in V^u(P), \quad y \overset{s}{\sim} z\} \end{aligned}$$

Recall that both $\overset{h}{\sim}$ and $\overset{s}{\sim}$ restricted to $V^u(P)$ have countable equivalence classes.

The stable and homoclinic algebras, are now constructed as the groupoid C^*-algebras of the groupoids $G^h(X, \varphi)$, and $G^s(X, \varphi, P)$ respectively. See [24], [17] for more on groupoid C^*-algebras.

We begin with the homoclinic algebra. Consider the space of complex-valued, compactly supported continuous functions on $G^h(X, \varphi)$, $C_c(G^h(X, \varphi))$. This space is a $*$-algebra with product and involution defined as follows. For $f, g \in C_c(G^h(X, \varphi))$, $(x, y) \in G^h(X, \varphi)$ we have

$$(f * g)(x, y) = \sum_{z \overset{h}{\sim} x} f(x, z) g(z, y),$$

and

$$f^*(x,y) = \overline{f(y,x)}.$$

To obtain the C^*-algebra, we now complete in some norm. As the groupoid $G^h(X,\varphi)$ is amenable in the sense of Renault, the full and reduced C^*-algebras are isomorphic [24]. To see that $G^h(X,\varphi)$ is in fact amenable, notice that $G^h(X,\varphi)$ and $H = G^u(X,\varphi,P) \times G^s(X,\varphi,P)$ are equivalent groupoids in the sense of [17] (see [19]), and from [22] $G^u(X,\varphi,P)$ and $G^s(X,\varphi,P)$ are amenable. Let us describe the reduced C^*-algebra. The idea is to take, for each equivalence class $[x]$ in $G^h(X,\varphi)$, the representation $\pi_{[x]} : C_c(G^h(X,\varphi)) \to \mathfrak{B}(l^2(V^h(x)))$ defined by

$$\pi_{[x]}(f)\xi(x) = \sum_{y \overset{h}{\sim} x} f(x,y)\xi(y).$$

If we denote by $||\pi_{[x]}f||$ the operator norm on $\mathfrak{B}(l^2(V^h(x)))$, the norm we wish to complete in is

$$||f|| = \sup_{[x]} ||\pi_{[x]}f||.$$

In our situation we can do something a little simpler. Since we have an irreducible Smale space, with P as above, $V^h(P)$ is dense in X and the representation $\pi : C_c(G^h(X,\varphi)) \to \mathfrak{B}(l^2(V^h(P)))$ defined by

$$\pi(f)\xi(x) = \sum_{x \overset{h}{\sim} y} f(x,y)\xi(y)$$

is faithful (ie. injective), so $||\pi(f)|| = ||f||$ and we can write $H(X,\varphi) = \overline{\pi(C_c(G^h))}$.

Remark 2.19. *For the duration of this document we will consider $C_c(G^h(X,\varphi)) \subset H(X,\varphi) \subset \mathfrak{B}(l^2(V^h(P)))$ and omit the use of π. I.e. we write*

$$(f\xi)(x) = \sum_{x \overset{h}{\sim} y} f(x,y)\xi(y).$$

We now briefly describe the construction of the stable algebra, $S(X,\varphi,P)$. Consider the set $C_c(G^s(X,\varphi,P))$ with product and involution defined similarly to the above.

$$\begin{aligned}(f*g)(x,y) &= \sum_{(x,z)\in G^s(X,\varphi,P)} f(x,z)g(z,y),\\ f^*(x,y) &= \overline{f(y,x)}.\end{aligned}$$

As in the case of $G^h(X,\varphi)$, $G^s(X,\varphi,P)$ is amenable, so the full and reduced C^*-algebras are isomorphic (see [22]). Furthermore, the fact that (X,φ) is irreducible once again provides us with a faithful representation, π_s, on $\mathfrak{B}(l^2(V^h(P)))$ where

$$(\pi_s(f)\xi)(x) = \sum_{(x,y)\in G^s(X,\varphi,P)} f(x,y)\xi(y).$$

We then write $S(X,\varphi,P) = \overline{\pi_s(C_c(G^s(X,\varphi,P)))}$ where the closure is taken in the operator norm. The definition of $U(X,\varphi,P)$ is completely analogous with the groupoid $G^s(X,\varphi,P)$ replaced by $G^u(X,\varphi,P)$. As in the case of $H(X,\varphi)$, for the rest of this document we consider $S(X,\varphi,P)$, $U(X,\varphi,P)$ to be subalgebras of $\mathfrak{B}(l^2(V^h(P)))$ and omit the use of π_s, π_u in our notation.

Remark 2.20. *When proving results for $H(X,\varphi)$ it will suffice to prove them for the dense $*$-subalgebra $C_c(G^h(X,\varphi))$. Moreover, it will suffice to prove results for functions supported on sets of the form $V(x,y,h_y,\delta)$. This is seen by noting that for $f \in C_c(G^h(X,\varphi))$, supp$(f)$ is compact, and sets of the form $V(x,y,h_y,\delta)$ cover. A partition of unity (see [18]) then allows us to write f as a finite sum of functions supported on sets of the form $V(x,y,h_y,\delta)$. Similarly, for results about $S(X,\varphi,P)$ ($U(X,\varphi,P)$), we will be left to prove in the case of functions supported on sets of the form $V(x,y,h^u_y,\delta)$ ($V(x,y,h^s_y,\delta)$).*

The homeomorphism φ on the Smale space naturally leads to a $*$-automorphism on each of the three algebras described above. For $f \in C_c(G)$, where $G = G^h(X,\varphi)$, $G^s(X,\varphi,P)$, or $G^u(X,\varphi,P)$ let $\alpha(f)(x,y) = f(\varphi^{-1}(x),\varphi^{-1}(y))$. α then extends to a $*$-automorphism on $H(X,\varphi)$, $S(X,\varphi,P)$, and $U(X,\varphi,P)$.

We now list a few results about the algebras $H(X,\varphi)$, $S(X,\varphi,P)$, $U(X,\varphi,P)$. These results appear in [19], [11]. We include some proofs here for completeness.

Proposition 2.21. *Let $a \in S(X,\varphi,P)$, $b \in U(X,\varphi,P)$, then $ab,\ ba \in \mathcal{K}$.*

Proof: First suppose a is supported on a set of the form $V_a = V(x_a, y_a, h^u_{y_a}, \delta_a)$, b supported on $V_b = V(x_b, y_b, h^u_{y_b}, \delta_b)$, and consider the orthonormal basis $\{\delta_z\}_{z \in V^h(P)}$ for $l^2(V^h(P))$.

$$(a\delta_z)(x) = \sum_{(x,y)\in G^s(X,\varphi,P)} a(x,y)\delta_z(y).$$

Each term in the sum is zero, except if $y = z$ and $x = h^u_{y_a}(z)$ (and $(h^u_{y_a}(z), z) \in V_a$). If these conditions are satisfied, then

$$(a\delta_z)(x) = a(h^u_{y_a}(z), z).$$

Thus we can write

$$a\delta_z = a(h^u_{y_a}(z), z)\delta_{h^u_{y_a}(z)}.$$

Similarly

$$b\delta_z = b(h^s_{y_b}(z), z)\delta_{h^s_{y_a}(z)}.$$

So we have

$$ba\delta_z = b(h^s_{y_b} \circ h^u_{y_a}(z), h^u_{y_a}(z))a(h^u_{y_a}(z), z)\delta_{h^s_{y_b} \circ h^u_{y_a}(z)}.$$

This is non-zero only if $h^u_{y_a}(z)$ is in both $range(V_b) \subset V^s(y_b, \delta_b)$ and $source(V_a) \subset V^u(h^u_{y_a}(y_a), \delta_a)$. These two sets intersect in at most one point, so if ab is not the zero operator, then ab has rank one. Similarly, ba is either the zero operator, or has rank one.

Now suppose $a \in C_c(G^s(X, \varphi, P))$, $b \in C_c(G^u(X, \varphi, P))$. Using a partition of unity argument, we can write a (and b) as a finite sum of functions supported on basic sets as above. We thus have that ab and ba are finite rank operators.

Finally, if $a \in S(X, \varphi, P)$, $b \in U(X, \varphi, P)$ then ab and ba are the limit of finite rank operators and hence are compact. □

Proposition 2.22. *Let $a \in S(X, \varphi, P)$, $b \in U(X, \varphi, P)$ then*

$$\lim_{n\to+\infty} ||\alpha^n(a)\alpha^{-n}(b) - \alpha^{-n}(b)\alpha^n(a)|| = 0.$$

Proof: We first assume that (X, φ) is mixing. It suffices to prove the result for a supported on $V_a = V(x_a, y_a, h^u_{y_a}, \delta_a)$, b supported on $V_b = V(x_b, y_b, h^s_{y_b}, \delta_b)$. Fix $\epsilon > 0$. A similar calculation to those in the proof of Prop. 2.21 shows that

$$\alpha^n(a)\delta_z = a(h^u_{y_a} \circ \varphi^{-n}(z), \varphi^{-n}(z))\delta_{\varphi^n \circ h^u_{y_a} \circ \varphi^{-n}(z)}$$

and

$$\alpha^{-n}(b)\alpha^n(a)\delta_z = \\ b(h^s_{y_b}\varphi^{2n}h^u_{y_a}\varphi^{-n}(z), \varphi^{2n}h^u_{y_a}\varphi^{-n}(z))a(h^u_{y_a}\varphi^{-n}(z), \varphi^{-n}(z))\delta_{\varphi^{-n}h^s_{y_b}\varphi^{2n}h^u_{y_a}\varphi^{-n}(z)}.$$

Similarly

$$\alpha^n(a)\alpha^{-n}(b)\delta_z = \\ a(h^u_{y_a}\varphi^{-2n}h^s_{y_b}\varphi^n(z), \varphi^{-2n}h^s_{y_b}\varphi^n(z))b(h^s_{y_b}\varphi^n(z), \varphi^n(z))\delta_{\varphi^n h^u_{y_a}\varphi^{-2n}h^s_{y_b}\varphi^n(z)}.$$

Now, as in Lemma 2.2 in [19] for n sufficiently large

$$\varphi^n h^u_{y_a}\varphi^{-2n}h^s_{y_b}\varphi^n(z) = \varphi^{-n}h^s_{y_b}\varphi^{2n}h^u_{y_a}\varphi^{-n}(z).$$

Denote this by x_3 and let

$$\begin{aligned} x_1 &= z \\ x_2 &= \varphi^n h^u_{y_a}\varphi^{-n}(z) \\ x_4 &= \varphi^{-n}h^s_{y_b}\varphi^n(z) \end{aligned}$$

we can then write

$$||\alpha^n(a)\alpha^{-n}(b) - \alpha^{-n}(b)\alpha^n(a)|| = \\ \sup_z |a(\varphi^{-n}(x_3), \varphi^{-n}(x_4))b(\varphi^n(x_4), \varphi^n(x_1)) - b(\varphi^n(x_3), \varphi^n(x_2))a(\varphi^{-n}(x_2), \varphi^{-n}(x_1))|.$$

Now

$$\begin{aligned} x_1 &\overset{u}{\sim} x_4 \\ x_1 &\overset{s}{\sim} x_2 \\ x_2 &\overset{u}{\sim} x_3 \\ x_3 &\overset{s}{\sim} x_4 \end{aligned}$$

So, by uniform continuity of a, b we can choose n large enough so that

$$|a(\varphi^{-n}(x_2), \varphi^{-n}(x_1)) - a(\varphi^{-n}(x_3), \varphi^{-n}(x_4))| < \epsilon/(2||b||)$$

and

$$|b(\varphi^{n}(x_3), \varphi^{n}(x_2)) - b(\varphi^{n}(x_4), \varphi^{n}(x_1))| < \epsilon/(2||a||).$$

Now

$$\begin{aligned}
&|a(\varphi^{-n}(x_2), \varphi^{-n}(x_1))b(\varphi^{n}(x_3), \varphi^{n}(x_2)) - a(\varphi^{-n}(x_3), \varphi^{-n}(x_4))b(\varphi^{n}(x_4), \varphi^{n}(x_1))| \\
&= |a(\varphi^{-n}(x_2), \varphi^{-n}(x_1))b(\varphi^{n}(x_3), \varphi^{n}(x_2)) - a(\varphi^{-n}(x_2), \varphi^{-n}(x_1))b(\varphi^{n}(x_4), \varphi^{n}(x_1)) + \\
&\quad a(\varphi^{-n}(x_2), \varphi^{-n}(x_1))b(\varphi^{n}(x_4), \varphi^{n}(x_1)) - a(\varphi^{-n}(x_3), \varphi^{-n}(x_4))b(\varphi^{n}(x_4), \varphi^{n}(x_1))| \\
&= |a(\varphi^{-n}(x_2), \varphi^{-n}(x_1))\big(b(\varphi^{n}(x_3), \varphi^{n}(x_2)) - b(\varphi^{n}(x_4), \varphi^{n}(x_1))\big) + \\
&\qquad\qquad + \big(a(\varphi^{-n}(x_2), \varphi^{-n}(x_1)) - a(\varphi^{-n}(x_3), \varphi^{-n}(x_4))\big)b(\varphi^{n}(x_4), \varphi^{n}(x_1))| \\
&\leq |a(\varphi^{-n}(x_2), \varphi^{-n}(x_1))\big(b(\varphi^{n}(x_3), \varphi^{n}(x_2)) - b(\varphi^{n}(x_4), \varphi^{n}(x_1))\big)| + \\
&\qquad\qquad + |\big(a(\varphi^{-n}(x_2), \varphi^{-n}(x_1)) - a(\varphi^{-n}(x_3), \varphi^{-n}(x_4))\big)b(\varphi^{n}(x_4), \varphi^{n}(x_1))| \\
&\leq ||a||\epsilon/(2||a||) + ||b||(\epsilon/(2||b||) = \epsilon.
\end{aligned}$$

So

$$\lim_{n\to\infty} ||\alpha^n(a)\alpha^{-n}(b) - \alpha^{-n}(b)\alpha^n(a)|| = 0$$

Now suppose (X, φ) is irreducible and not mixing. Then as in section 2.5 there exists a mixing Smale space (Y, ψ) and natural number N such that $S(X, \varphi, P) \cong \oplus_1^N S(Y, \psi, \tilde{P})$ and $U(X, \varphi, P) \cong \oplus_1^N U(Y, \psi, \tilde{P})$. It suffices to prove the result for a an element of the i^{th} summand of $\oplus_1^N S(Y, \psi, \tilde{P})$, b an element of the j^{th} summand of $\oplus_1^N U(Y, \psi, \tilde{P})$. Recalling that α_φ permutes the summands we have that

$$\alpha^n(a)\alpha^{-n}(b) = \alpha^{-n}(b)\alpha^n(a) = 0$$

for all n such that $i + n \neq j - n (mod N)$. If we then consider the subsequence of n such that $i + n \equiv j - n (mod N)$ the result follows from the mixing case.

□

Proposition 2.23. *Let $a \in S(X, \varphi, P)$, $b \in U(X, \varphi, P)$, $f \in H(X, \varphi)$. Then $af,\ fa \in S(X, \varphi, P)$, $bf,\ fb \in U(X, \varphi, P)$.*

Proof: We prove the result in the stable case, the unstable case is completely analogous. It suffices to consider $a \in C_c(G^s(X,\varphi,P))$ supported on $V_a = V(x_a, y_a, h^u_{y_a}, \delta_a)$, $f \in C_c(G^h(X,\varphi))$ supported on $V_f = V(x_f, y_f, h_{y_f}, \delta_f)$.

$$(af)(x,y) = \sum_{\substack{(x,z)\in G^s(X,\varphi,P)\\(z,y)\in G^h(X,\varphi)}} a(x,z)f(z,y),$$

and each summand is zero unless $z = h_{y_f}(y)$ and $x = h^u_{y_a}(z) = h^u_{y_a} \circ h_{y_f}(y)$. Thus af is supported on the set

$$\{(h^u_{y_a} \circ h_{y_f}(y), y) \mid (h_{y_f}(y), y) \in V_f,\ (h^u_{y_a} \circ h_{y_f}(y), h_{y_f}(y)) \in V_a\},$$

which is non-empty only if $source(V_f) \cap range(V_a)$ is non empty. In this case we can write

$$supp(af) = \{(h^u_{y_a}(z), h_{x_f}(z)) \mid z \in source(V_f) \cap range(V_a)\} \subset G^s(X,\varphi,P)$$

so $af \in C_c(G^s(X,\varphi,P)) \subset S(X,\varphi,P)$. Similarly

$$supp(fa) = \{(h_{y_f}(z), h^u_{x_a}(z)) \mid z \in range(V_f) \cap source(V_a)\} \subset G^s(X,\varphi,P).$$

□

The following proposition shows that α gives $H(X,\varphi)$ an asymptotically abelian structure. This result is key to defining a product structure on K-theory groups (chapter 3).

Proposition 2.24. *Let (X,φ) be an irreducible Smale space, $H(X,\varphi)$ the associated homoclinic algebra. If $a,\ b \in H(X,\varphi)$, then*

$$\lim_{|n|\to\infty} ||\alpha^n(a)\alpha^{-n}(b) - \alpha^{-n}(b)\alpha^n(a)|| = 0$$

Proof: We prove the result for $n \to +\infty$, the $n \to -\infty$ case is completely analogous. Furthermore, we prove the result in the case that (X,φ) is *mixing*, the general irreducible case then follows easily using the results of section 2.5, as in the proof of 2.22. The proof for the mixing case is in [19], we include it here for completeness. The proof is very similar to the proof of Prop. 2.22.

Fix $\epsilon > 0$. It suffices to prove the result for $a, b \in C_c(G^h)$. In fact, it suffices to prove it for a supported on $V_a = V(x_a, y_a, h_{x_a}, \delta_a)$ and b supported on $V_b = V(x_b, y_b, h_{x_b}, \delta_b)$. So

$$a(x,y) \neq 0 \Longrightarrow x \in B(x_a, \delta),\ y = h_{x_a}(x) \in h_{x_a}(B(x_a, \delta))$$

so

$$\begin{aligned}\alpha^n(a)(x,y) = a(\varphi^{-n}(x), \varphi^{-n}(y)) \neq 0 \\ \Longrightarrow \varphi^{-n}(x) \in B(x_a,\delta),\ \varphi^{-n}(y) = h_{x_a}(\varphi^{-n}(x)) \in h_{x_a}(B(x_a,\delta))\end{aligned}$$

or

$$x \in \varphi^n(B(x_a,\delta)), \quad \varphi^n \circ h_{x_a} \circ \varphi^{-n}(x) \in \varphi^n \circ h_{x_a}(B(x_a,\delta))$$

similarly

$$\alpha^{-n}(b)(x,y) \neq 0 \Longrightarrow x \in \varphi^{-n}(B(x_b,\delta)),\ \varphi^{-n} \circ h_{x_b} \circ \varphi^n(x) \in \varphi^{-n} \circ h_{x_b}(B(x_b), \delta')).$$

Now

$$\begin{aligned}\alpha^n(a)\alpha^{-n}(b)(x,y) &= \sum_{x\sim z} \alpha^n(a)(x,z)\alpha^{-n}(b)(z,y) \\ &= \sum_{x\sim z} a(\varphi^{-n}(x), \varphi^{-n}(z)) b(\varphi^n(z), \varphi^n(y)) \\ &= a(\varphi^{-n}(x), h_{x_a}\varphi^{-n}(x)) b(\varphi^{2n} h_{x_a}\varphi^{-n}(x), h_{x_b}\varphi^{2n} h_{x_a}\varphi^{-n}(x)) \\ &= 0,\ \text{unless } x \in \varphi^n(B(x_a,\delta)),\ \varphi^n h_{x_a}\varphi^{-n}(x) \in \varphi^n h_{x_a}(B(x_a,\delta)), \\ &\qquad \varphi^n h_{x_a}\varphi^{-n}(x) \in \varphi^{-n}(B(x_b,\delta)),\ \text{and} \\ &\qquad \varphi^{-n} h_{x_b}\varphi^{2n} h_{x_a}\varphi^{-n}(x) \in \varphi^{-n} h_{x_b}(B(x_b,\delta))\end{aligned}$$

similarly

$$\begin{aligned}\alpha^{-n}(b)\alpha^n(a)(x,y) &= b(\varphi^n(x), h_{x_b}\varphi^n(x)) a(\varphi^{-2n} h_{x_b}\varphi^n(x), h_{x_a}\varphi^{-2n} h_{x_b}\varphi^n(x)) \\ &= 0,\ \text{unless } x \in \varphi^{-n}(B(x_b,\delta)), \varphi^{-n} h_{x_b}\varphi^n(x) \in \varphi^{-n} h_{x_b}(B(x_a,\delta)), \\ &\qquad \varphi^{-n} h_{x_b}\varphi^n(x) \in \varphi^n(B(x_a,\delta)),\ \text{and} \\ &\qquad \varphi^n h_{x_a}\varphi^{-2n} h_{x_b}\varphi^n(x) \in \varphi^n h_{x_a}(B(x_a,\delta)).\end{aligned}$$

So

$$\alpha^n(a)\alpha^{-n}(b)(x,y) - \alpha^{-n}(b)\alpha^n(a)(x,y) = 0,$$

unless

$$\begin{aligned}
x_1 = x &\in \varphi^n(B(x_a,\delta)) \cap \varphi^{-n}(B(x_b,\delta)),\\
x_2 = \varphi^n \circ h_{x_a} \circ \varphi^{-n}(x) &\in \varphi^n \circ h_{x_a}(B(x_a,\delta)) \cap \varphi^{-n}(B(x_b,\delta))\\
x_3 &\in \varphi^{-n} \circ h_{x_b}(B(x_b,\delta)) \cap \varphi^n \circ h_{x_a}(B(x_a,\delta))\\
x_4 = \varphi^{-n} \circ h_{x_b} \circ \varphi^n(x) &\in \varphi^{-n} \circ h_{x_b}(B(x_a,\delta)) \cap \varphi^n(B(x_a,\delta))
\end{aligned}$$

where

$$x_3 = \varphi^{-n} \circ h_{x_b} \circ \varphi^{2n} \circ h_{x_a} \circ \varphi^{-n}(x) = \varphi^n \circ h_{x_a} \circ \varphi^{-2n} \circ h_{x_b} \circ \varphi^n(x)$$

(see Lemma 2.2 in [19] for this last equality.) In which case we have

$$a(\varphi^{-n}(x_1),\varphi^{-n}(x_2))b(\varphi^n(x_2),\varphi^n(x_3)) - a(\varphi^{-n}(x_4),\varphi^{-n}(x_3))b(\varphi^n(x_1),\varphi^n(x_4)).$$

Now we notice that

$$\begin{aligned}
x_1 &\overset{u}{\sim} x_4,\\
x_2 &\overset{u}{\sim} x_3,\\
x_1 &\overset{s}{\sim} x_2, \text{ and}\\
x_3 &\overset{s}{\sim} x_4.
\end{aligned}$$

By continuity of a and b, we can choose n large enough so that

$$|a(\varphi^{-n}(x_1),\varphi^{-n}(x_2)) - a(\varphi^{-n}(x_4),\varphi^{-n}(x_3))| < \epsilon/(2||b||)$$

and

$$|b(\varphi^n(x_2),\varphi^n(x_3)) - b(\varphi^n(x_1),\varphi^n(x_4))| < \epsilon/(2||a||).$$

Now

$$\begin{aligned}
&|a(\varphi^{-n}(x_1),\varphi^{-n}(x_2))b(\varphi^{n}(x_2),\varphi^{n}(x_3)) - a(\varphi^{-n}(x_4),\varphi^{-n}(x_3))b(\varphi^{n}(x_1),\varphi^{n}(x_4))| \\
&= |a(\varphi^{-n}(x_1),\varphi^{-n}(x_2))b(\varphi^{n}(x_2),\varphi^{n}(x_3)) - a(\varphi^{-n}(x_1),\varphi^{-n}(x_2))b(\varphi^{n}(x_1),\varphi^{n}(x_4)) + \\
&\quad a(\varphi^{-n}(x_1),\varphi^{-n}(x_2))b(\varphi^{n}(x_1),\varphi^{n}(x_4)) - a(\varphi^{-n}(x_4),\varphi^{-n}(x_3))b(\varphi^{n}(x_1),\varphi^{n}(x_4))| \\
&= |a(\varphi^{-n}(x_1),\varphi^{-n}(x_2))\big(b(\varphi^{n}(x_2),\varphi^{n}(x_3)) - b(\varphi^{n}(x_1),\varphi^{n}(x_4))\big) + \\
&\qquad\qquad + \big(a(\varphi^{-n}(x_1),\varphi^{-n}(x_2)) - a(\varphi^{-n}(x_4),\varphi^{-n}(x_3))\big)b(\varphi^{n}(x_1),\varphi^{n}(x_4))| \\
&\leq |a(\varphi^{-n}(x_1),\varphi^{-n}(x_2))\big(b(\varphi^{n}(x_2),\varphi^{n}(x_3)) - b(\varphi^{n}(x_1),\varphi^{n}(x_4))\big)| + \\
&\qquad\qquad + |\big(a(\varphi^{-n}(x_1),\varphi^{-n}(x_2)) - a(\varphi^{-n}(x_4),\varphi^{-n}(x_3))\big)b(\varphi^{n}(x_1),\varphi^{n}(x_4))| \\
&\leq ||a||\epsilon/(2||a||) + ||b||(\epsilon/(2||b||) = \epsilon.
\end{aligned}$$

So

$$\lim_{n\to\infty} ||\alpha^n(a)\alpha^{-n}(b) - \alpha^{-n}(b)\alpha^n(a)|| = 0$$

□

Remark 2.25. *There are results similar to Prop.'s 2.22 and 2.24 which show that $H(X,\varphi)$ commutes asymptotically with both $U(X,\varphi,P)$ and $S(X,\varphi,P)$. We leave this until chapter 4 where we use the result.*

2.3.1 C^*-Algebras from SFT

In this section we construct the algebras $H(\Sigma,\sigma)$ and $S(\Sigma,\sigma,P)$ for a mixing SFT (Σ,σ), the irreducible case then follows immediately from the results of section 2.5. We begin with the homoclinic algebra.

Let (Σ,σ) be a mixing SFT with corresponding graph G and adjacency matrix A. As (Σ,σ) is mixing, A is primitive, so there exists M such that A^N is strictly positive for all $N \geq M$. Fix $N \geq M$, $v_i, v_j \in V(G)$. Define

$$\Xi_{N,v_i,v_j} = \{\xi = (\xi_{-N+1},\cdots,\xi_N) \;\;|\;\; t(\xi_N) = v_j,\; i(\xi_{-N+1}) = v_i\}.$$

Notice that Ξ_{N,v_i,v_j} consists of all paths of length $2N$ in G which originate at v_i and terminate at v_j, so $\#\Xi_{N,v_i,v_j} = A^{2N}_{ij} > 0$ For $\xi \in \Xi_{N,v_i,v_j}$ define

$$V_{N,v_i,v_j}(\xi) = \{x \in \Sigma \;\;|\;\; x_n = \xi_n \quad \forall -N+1 \leq n \leq N\}.$$

Note that for fixed N, $V_{N,v_i,v_j}(\xi)$ and $V_{N,v_i',v_j'}(\eta)$ intersect only if $\xi = \eta$, $v_i = v_i'$, and $v_j = v_j'$. Now let $\xi\ ,\eta \in \Xi_{N,v_i,v_j}$. Define

$$E_{N,v_i,v_j}(\xi,\eta)=\{(x,y) \mid \{x_n\}_{-N+1}^{N} = \xi,\ \{y_n\}_{-N+1}^{N} = \eta,\ x_n = y_n\ \forall n > N,\ n < -N+1\}.$$

Then

1. $E_{N,v_i,v_j}(\xi,\eta) \subseteq G^h(\Sigma,\sigma)$.
2. $E_{N,v_i,v_j}(\xi,\eta)$ and $E_{N,v_i',v_j'}(\xi',\eta')$ intersect only if $\xi = \xi'$, $\eta = \eta'$, $v_i = v_i'$, $v_j = v_j'$.
3. $E_{N,v_i,v_j}(\xi,\eta)$ is compact and open in $G^h(\Sigma,\sigma)$.
4. The sets $E_{N,v_i,v_j}(\xi,\eta)$ for $N \geq 1$, $v_i,\ v_j \in V(G)$, $\xi,\ \eta \in \Xi_{N,v_i,v_j}$ form a neighbourhood base for the topology on $G^h(\Sigma,\sigma)$.

Now let

$$e_{N,v_i,v_j}(\xi,\eta) = \chi_{E_{N,v_i,v_j}(\xi,\eta)} \in C_c(G^h(\Sigma,\sigma)).$$

Note that $span\overline{(e_{N,v_i,v_j}(\xi,\eta))} = C_c(G^h(\Sigma,\sigma))$. Consider the product of two such functions.

$$e_{N,v_i,v_j}(\xi,\eta) * e_{N,v_i',v_j'}(\xi',\eta')(x,y) = \sum_{x \overset{h}{\sim} z} e_{N,v_i,v_j}(\xi,\eta)(x,z) e_{N,v_i',v_j'}(\xi',\eta')(z,y).$$

This product is 0 unless

1. $x_n = \xi_n\ \forall -N+1 \leq n \leq N$,
2. $x_n = z_n\ \forall n > N$ and $n < -N+1$,
3. $z_n = \eta_n\ \forall -N+1 \leq n \leq N$,
4. $z_n = \xi_n'\ \forall n > N$ and $n < -N+1$,
5. $y_n = z_n\ \forall n > N$ and $n < -N+1$,
6. $y_n = \eta_n'\ \forall -N+1 \leq n \leq N$.

Or equivalently

1. $\eta = \xi'$, $v_i = v_i'$, $v_j = v_j'$,

2. $x_n = y_n \; \forall n > N$ and $n < -N+1$,

3. $x_n = \xi_n \; \forall -N+1 \le n \le N$,

4. $y_n = \eta' \; \forall -N+1 \le n \le N$.

If the above 4 conditions hold, there is exactly one z for which the product is non-zero, namely $z_n = \eta_n \; (= \xi'_n)$ for $-N+1 \le n \le N$, $z_n = x_n = y_n$ for $n > N$ and $n < -N+1$. In other words, the sum contains only one non-zero term, hence if the 4 conditions above hold, the product is 1. So

$$e_{N,v_i,v_j}(\xi,\eta) * e_{N,v'_i,v'_j}(\xi',\eta') = \begin{cases} e_{N,v_i,v_j}(\xi,\eta') & \text{if } \eta = \xi' \\ 0 & \text{otherwise} \end{cases}$$

Now let

$$H_{N,v_i,v_j} = span\{e_{N,v_i,v_j}(\xi,\eta) \mid \xi,\, \eta \in \Xi_{N,v_i,v_j}\}.$$

We also notice that $H_{N,v_i,v_j} \cong M_{k(N,v_i,v_j)}(\mathbb{C})$ where $k(N,v_i,v_j) = \#\Xi_{N,v_i,v_j} \; (= A^{2N}_{v_1,v_2})$. Note that for $N \ge M$, $k(N,v_i,v_j) \ne 0$, so H_{N,v_1,v_2} is not the zero algebra. Now we define

$$H_N = span(\{e_{N,v_i,v_j}(\xi,\eta) \mid \xi,\, \eta \in \Xi_{N,v_i,v_j};\; v_i,\, v_j \in V(G)\}),$$

and notice that

$$H_N = \bigoplus_{v_i \in V(G)} \bigoplus_{v_j \in V(G)} H_{N,v_i,v_j} = \bigoplus_{(v_i,v_j) \in V(G)\times V(G)} H_{N,v_i,v_j} \cong \bigoplus_{(v_i,v_j)} M_{k(N,v_i,v_j)}(\mathbb{C}),$$

Notice now that $H_N \subset H_{N+1}$ and $H(\Sigma,\sigma)$ is the direct limit of the H_N's. To see how H_N is imbedded in H_{N+1} consider the following.

$$e_{N,v_i,v_j}(\xi,\eta) = \sum_{y_1 \in E_i} \sum_{y_2 \in E_j} e_{N+1,v_l,v_k}(y_1\xi y_2, y_1\eta y_2),$$

where $i(y_1) = v_l$, $t(y_2) = v_k$, $E_i = \{y \in E(G) \mid t(y) = v_i\}$, and $E_j = \{y \in E(G) \mid i(y) = v_j\}$. In particular, H_{N+1,v_l,v_k} contains $A_{li}A_{jk}$ copies of H_{N,v_l,v_k}.

We now describe the action of α on $H(\Sigma,\sigma)$.

$$\alpha(e_{N,v_i,v_j}(\xi,\eta)) = \sum_k \sum_{\xi' \in \Xi_{1,v_j,v_k}} e_{N+1,v_i,v_k}(\xi\xi', \eta\xi'),$$

and

$$\alpha^{-1}(e_{N,v_i,v_j}(\xi,\eta)) = \sum_l \sum_{\xi' \in \Xi_{1,v_l,v_i}} e_{N+1,v_l,v_j}(\xi'\xi, \xi'\eta).$$

In particular α and α^{-1} map H_N into H_{N+1}.

The construction of $S(\Sigma, \sigma, P)$ is very similar. We briefly outline the details. Fix a finite σ-invariant set $P \subset \Sigma$. Fix $N \geq M$, $v_i \in V(G)$. Define

$$\Xi_{N,v_i} = \{\xi = (\xi_{-N+1}, \cdots, \xi_N) \ \mid \ t(\xi_N) = v_i, \ i(\xi_{-N+1}) = i(p_{-N}) \text{ for some } p \in P\}.$$

Again we mention that Ξ_{N,v_i} is non-empty, as A^{2N} is strictly positive. For $\xi \in \Xi_{N,v_i}$ we can extend ξ backwards by setting $\xi_{-n} = p_{-n}$ for $n > N-1$. Now for $\xi \in \Xi_{N,v_i}$ we define

$$V_{N,v_i}(\xi) = \{x \in \Sigma \ \mid \ x_n = \xi_n \quad \forall n \leq N\}.$$

Note that for fixed N, $V_{N,v_i}(\xi)$ and $V_{N,v_j}(\eta)$ intersect only if $\xi = \eta$, and $v_i = v_j$. Now let $\xi\ ,\eta \in \Xi_{N,v_i}$. Define

$$E_{N,v_i}(\xi,\eta) = \{(x,y) \mid x_n = \xi_n, \ y_n = \eta_n \ \forall n \leq N, \ x_n = y_n \ \forall n > N\}.$$

The collection of sets $\{E_{N,v_i}(\xi,\eta)\}$ forms a clopen base for the topology on $G^s(\Sigma, \sigma, P)$, and we are left to consider functions of the form

$$e_{N,v_i}(\xi,\eta) = \chi_{E_{N,v_i}(\xi,\eta)}.$$

Proceeding as we did above for $H(\Sigma,\sigma)$, we see that for fixed N and i

$$e_{N,v_i}(\xi,\eta) * e_{N,v_i}(\xi',\eta') = \begin{cases} e_{N,v_i}(\xi,\eta') & \text{if } \eta = \xi', \\ 0 & \text{if } \eta \neq \xi'. \end{cases}$$

As above, we let $S_{N,v_i} = span\{e_{N,v_i}(\xi,\eta) \mid \xi,\eta \in \Xi_{N,v_i}\}$ and notice that

$$S_{N,v_i} \cong M_{k(N,v_i)}(\mathbb{C}),$$

where $k(N,v_i)$ is the number of paths of length $2N$ starting at a vertex of $p \in P$ and ending at v_i.

$$S_N = \bigoplus_{v_i \in V(G)} S_{N,v_i} \cong \bigoplus M_{k(N,v_i)}(\mathbb{C}).$$

Finally we notice that $S_N \subset S_{N+1}$ and let $S(\Sigma, \sigma, P)$ be the direct limit of the S_N's. Similar to the above,

$$e_{N,v_i}(\xi, \eta) = \sum_{y \in S} e_{N+1,v_k}(\xi y, \eta y),$$

where $t(y) = v_k$ and $S = \{y \in E(G) \mid i(y) = v_i\}$. So we see that S_{N,v_k} contains A_{ik} copies of S_{N,v_i}.

Similar to the $H(\Sigma, \sigma)$ case we see that

$$\alpha(e_{N,v_i}(\xi, \eta)) = \sum_k \sum_{\xi' \in \Xi_{1,v_i,v_k}} e_{N+1,v_k}(\xi\xi', \eta\xi'),$$

and

$$\alpha^{-1}(e_{N,v_i,v_j}(\xi, \eta)) = e_{N+1,v_i,v_j}(\xi, \eta).$$

2.4 K-theory

For a proper introduction to K-theory for C^*-algebras we refer the reader to [23], or for a more advanced treatment, [1]. In this section we state, without proof, a few of the basic definitions and results.

We begin by defining the K_0 group for a unital C^*-algebra. Let A be a unital C^*-algebra, $M_n(A)$ the $n \times n$ matrices with entries from A $(= M_n(\mathbb{C}) \otimes A)$, and $P_n(A)$ the projections in $M_n(A)$. Let $P_\infty(A) = \cup_1^\infty P_n(A)$. For $p, q \in P_\infty(A)$ let

$$p \otimes q = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}.$$

We say p and q are homotopic and write $p \overset{hom}{\sim} q$ if there exists a continuous path of projections in $P_\infty(A)$ from p to q. See [23] for more of homotopy equivalence and its relation to Murray-von Neumann equivalence and unitary equivalence. We write the equivalence class of p under $\overset{hom}{\sim}$ as $[p]_0$. Taking $P_\infty(A)/\overset{hom}{\sim}$ gives an abelian semi-group with $[p]_0 + [q]_0 = [p \oplus q]_0$. The group $K_0(A)$ is then defined to be the Grothendieck group of this semi-group. Ie. the group of all formal differences $[p]_0 - [q]_0$.

We now describe K_0 for a non-unital C^*-algebra. Let A be a non-unital C^*-algebra.

Define the unitization of A, $\bar{A}$ as follows.

$$\bar{A} = \{(a, z) \mid a \in A,\ z \in \mathbb{C}\}$$

with multiplication and involution

$$(a, z)(b, w) = (ab + wa + zb, zw), \quad (a, z)^* = (a^*, \bar{z}).$$

$\bar{A}$ is then a unital C^*-algebra with unit $(0, 1)$. Now consider the following split exact sequence.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\iota"] & \bar{A} \arrow[r, bend left, "\pi"] & \mathbb{C} \arrow[l, bend left, "\lambda"] \arrow[r] & 0 \end{tikzcd}$$

where $\pi(a, z) = z$ and $\lambda(w) = (0, w)$. The map $s(a, z) = \lambda \circ \pi(a, z) = (0, z)$ is called the scalar map. We then define $K_0(A)$ so that the above split exact sequence is preserved under K_0, so we have $K_0(A) = K_0(\bar{A})/K_0(\mathbb{C}) = K_0(\bar{A})/\mathbb{Z}$. Moreover, we have that $K_0(A)$ is generated by elements of the form $[p]_0 - [s(p)]_0$ where $p \in P_\infty(\bar{A})$.

We define the positive cone of $K_0(A)$ to be

$$K_0(A)^+ = \{[p]_0 \mid p \in P_\infty(A)\}.$$

Furthermore, if A is unital with unit 1, then $[1]_0$ is an order unit for $K_0(A)$ and if in addition A is stably finite, then

$$\big(K_0(A), K_0(A)^+, [1]_0\big)$$

is an ordered abelian group with distinguished order unit $[1]_0$. If (G, G^+, g) and (H, H^+, h) are ordered abelian groups, a group homomorphism $\phi : G \to H$ is said to be *positive* if $\phi(G^+) \subset H^+$, and is said to be *order unit preserving* if $\phi(g) = h$.

We now briefly describe the group $K_1(A)$. We begin by defining the suspension of A, SA.

$$SA = \{f : [0, 1] \to A \mid f(0) = f(1) = 0\}.$$

We can then define $K_1(A) = K_0(SA)$. There is an alternative picture of $K_1(A)$ in terms of unitaries which we now describe. Let $U_n(\bar{A})$ be the set of unitaries in $M_n(\bar{A})$, and $U_\infty(\bar{A}) = \cup_1^\infty U_n(\bar{A})$. As in the case above with the projections, $U_\infty(\bar{A})/\overset{hom}{\sim}$ gives a semi-group, and $K_1(A)$ is its Grothendieck group. See [23] for a proof that these two descriptions are equivalent.

Finally, the higher K groups are defined by $K_{n+1}(A) = K_n(SA)$. However, $K_2(A) = K_1(SA) \cong K_0(A)$ (Bott periodicity, see [23]), so we only ever need consider K_0 and K_1. We write $K_*(A) = K_0(A) \oplus K_1(A)$.

Finally, we mention that if A is the inductive limit of the following sequence

$$A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} A_3 \xrightarrow{\phi_3} \cdots$$

then $K_*(A)$ is the limit of

$$K_*(A_1) \xrightarrow{K_*(\phi_1)} K_*(A_2) \xrightarrow{K_*(\phi_2)} K_*(A_3) \xrightarrow{K_*(\phi_3)} \cdots$$

2.4.1 K-theory for SFT

We now compute the K-theory for $H(\Sigma, \sigma)$ and $S(\Sigma, \sigma, P)$ in the case that (Σ, σ) is *mixing*. The irreducible case is handled in section 2.5. We begin with $H(\Sigma, \sigma)$.

$H(\Sigma, \sigma)$ is an AF algebra, the direct limit of the finite dimensional algebras H_N, hence $K_*(H(\Sigma, \sigma))$ is the direct limit of $K_*(H_N)$. Since H_N is finite dimensional, $K_1(H_N) = 0$ and hence $K_1(H) = 0$.

As (Σ, σ) is *mixing*, A is primitive and hence there exists M such that for all $n > M$, A^n is strictly positive. Thus, for N large enough so that $2N > M$, $k(N, v_i, v_j) \neq 0$ and we have

$$K_0(H_N) \cong K_0\left(\bigoplus_{(v_i, v_j)} M_{k(N, v_i, v_j)}(\mathbb{C})\right) \cong \mathbb{Z}^{(\#V(G))^2}.$$

For our purposes it will be more convenient to regard $\mathbb{Z}^{(\#V(G))^2}$ as $M_{\#V(G)}(\mathbb{Z})$. We can thus describe $K_0(H)$ as the inductive limit of the following system.

$$M_{\#V(G)}(\mathbb{Z}) \xrightarrow{\iota} M_{\#V(G)}(\mathbb{Z}) \xrightarrow{\iota} M_{\#V(G)}(\mathbb{Z}) \xrightarrow{\iota} \cdots$$

We now must describe the connecting maps. As $K_0(H_N)$ is generated by the rank one projections in H_N it suffices to consider elements of the form $[e_{N, v_i, v_j}(\xi, \xi)]_0$. We also remark that $e_{N, v_i, v_j}(\xi, \xi)$ is homotopic to $e_{N, v_i, v_j}(\eta, \eta)$, so they give the same element of K_0. From section 2.3.1 we know that

$$\iota(e_{N, v_i, v_j}(\xi, \xi)) = \sum_{y_1 \in S_i} \sum_{y_2 \in S_j} e_{N+1, v_l, v_k}(y_1 \xi y_2, y_1 \eta y_2)$$

where, for fixed i, k the number of summands is $A_{li}A_{jk}$. Hence

$$\iota_*[e_{N,v_i,v_j}(\xi,\xi)]_0 = \sum_{l,k} A_{li}A_{jk}[e_{N+1,v_l,v_k}(y_1\xi y_2, y_1\xi y_2)]_0.$$

The isomorphism $K_0(H_N) \to M_{\#V(G)}(\mathbb{Z})$ sends $[e_{N,v_i,v_j}(\xi,\xi)]_0$ to (e_{ij}, N), so we can write

$$\iota_*(e_{ij}, N) = \sum_{l,\ k} A_{li}A_{jk}(e_{lk}, N+1) = (Ae_{ij}A, N+1).$$

So by linearity, for any $X \in M_{\#V(G)}(\mathbb{Z})$, the inclusion map is given by $\iota_*(X,N) = (AXA, N+1)$. We can thus describe $K_0(H) = \lim K_0(H_N)$ as follows.

$$K_0(H) \cong (M_{\#V(G)}(\mathbb{Z}) \times \mathbb{N})/\sim .$$

Where, for $n \leq k$, $(X,n) \sim (Y,k)$ if and only if $A^{k-n+l}XA^{k-n+l} = A^lYA^l$ for some $l \in \mathbb{N}$. We denote the equivalence class of (X,N) under $\sim$ by $[X,N]$.

Recall the automorphism $\alpha : H \to H$. We now wish to describe $\alpha_* : K_0(H) \to K_0(H)$. Again, by linearity it suffices to consider $[e_{N,v_i,v_j}(\xi,\xi)]_0 \in K_0(H_N)$. Referring back to section 2.3.1, we see that

$$\alpha_*[e_{N,v_i,v_j}(\xi,\xi)]_0 = \sum_k \sum_{\xi' \in \Xi_{1,v_j,v_k}} [e_{N+1,v_i,v_k}(\xi\xi',\xi\xi')]_0.$$

Under the isomorphism with $M_{\#V(G)}(\mathbb{Z})$ this becomes

$$\alpha_*[e_{ij}, N] = \sum_k A^2_{jk}[e_{ik}, N+1] = [e_{ij}A^2, N+1].$$

So for $[X,N] \in K_0(H)$ we have $\alpha([X,N]) = [XA^2, N+1]$. Similarly, $\alpha^{-1}([X,N]) = [A^2X, N+1]$. Notice that $\alpha^{-1}(\alpha([X,N])) = \alpha^{-1}([XA^2, N+1]) = [A^2XA^2, N+2] = [X,N]$.

We now briefly outline the computation of $K_*(S(\Sigma,\sigma,P))$. As in the case of $H(\Sigma,\sigma)$, $S(\Sigma,\sigma,P)$ is AF and hence $K_1(S(\Sigma,\sigma,P)) = 0$. For each N such that $2N+1 > M$, $K_0(S_N) = \mathbb{Z}^{\#V(G)}$ so $K_0(S(\Sigma,\sigma,P))$ is the direct limit of the following system

$$\mathbb{Z}^{\#V(G)} \xrightarrow{\iota} \mathbb{Z}^{\#V(G)} \xrightarrow{\iota} \mathbb{Z}^{\#V(G)} \xrightarrow{\iota} \cdots .$$

We need only determine the connecting maps. As above, it suffices to consider the rank one projections $e_{N,v_i}(\xi,\xi)$. Under the inclusion of $K_0(S_N)$ into $K_0(S_{N+1})$ we have

$$\iota_*[e_{N,v_i}(\xi,\xi)]_0 = \sum_{y\in\tilde{E}}[e_{N,t(y)}(\xi y,\xi y)]_0,$$

where $\tilde{E} = \{y \in E(G) \mid i(y) = v_i\}$. So the number of summands is the number of edges in G originating at v_i, or $\sum_j A_{ij}$. Under the isomorphism $K_0(S_N) \cong \mathbb{Z}^{\#V(G)}$ (thinking of $\mathbb{Z}^{\#V(G)}$ as row vectors) this becomes

$$\iota_*(e_i, N) = \sum_j A_{ij}(e_j, N+1) = (e_i A, N+1).$$

By linearity we have that the connecting maps are $i(v,N) = (vA, N+1)$. We can therefore write

$$K_0(S(\Sigma,\sigma,P)) \cong (\mathbb{Z}^{\#V(G)} \times \mathbb{N})/\sim,$$

where, for $n \leq m$, $(v,n) \sim (w,m)$ if and only if there exists $k \in \mathbb{N}$ such that $vA^{k+m-n} = wA^k$. We write $[v,n]$ for the equivalence class under $\sim$.

Once again proceeding as in the case of $H(\Sigma,\sigma)$ we can show that $\alpha_*[v,N] = [vA^2, N+1]$ and $\alpha_*^{-1}[v,N] = [v, N+1]$.

2.5 C^*-Algebras from Irreducible Smale space

In this chapter we describe the C^*-algebras associated with an irreducible Smale space as direct sums of algebras associated to a mixing Smale space. As we will see, this follows easily from the spectral decomposition result, Prop. 2.14. We will use this fact to extend many of the results in later chapters from the mixing case to the irreducible case.

Let (X,φ) be a Smale space and fix $n \in \mathbb{N}$. It is easy to see that (X,φ^n) is also a Smale space with the same bracket function $[\cdot,\cdot]$ (recall the axioms for a Smale space from section 2.2). The only condition that may pose a problem is the condition that requires

$$\varphi^n([x,y]) = [\varphi^n(x), \varphi^n(y)],$$

whenever both sides are defined, ie whenever $d(x,y) < \epsilon_X$ and $d(\varphi^n(x),\varphi^n(y)) < \epsilon_X$. In general this need not be true, however if $d(x,y) < \epsilon_X$ and $d(\varphi^n(x),\varphi^n(y)) < \epsilon_X$ implies $d(\varphi^i(x),\varphi^i(y)) < \epsilon_X$ for all $0 < i \leq n$, then the above is true. By replacing

ϵ_X with a smaller constant, say ϵ'_X we can ensure the above holds.

It is also easy to see that the 3 equivalence relations are unchanged by switching from φ to φ^n. For example, for $x \in X$ the set $V^s(x)$ is the same whether we consider the map φ or φ^n. In particular, for a finite φ-invariant (also φ^n-invariant) set $P \subset X$ the groupoids $G^s(X, \varphi, P)$ and $G^s(X, \varphi^n, P)$ are the same. Similarly for $G^u(X, \varphi, P)$ and $G^u(X, \varphi^n, P)$, and $G^h(X, \varphi)$ and $G^h(X, \varphi^n)$. It then follows that $S(X, \varphi, P) = S(X, \varphi^n, P)$ and similarly for the unstable and homoclinic algebras. It should be noted that while $S(X, \varphi, P) = S(X, \varphi^n, P)$, the automorphisms α_φ and α_{φ^n} are not equal.

Now suppose (X, φ) is an irreducible Smale space and (Y, ψ), $n \in \mathbb{N}$ are as in Prop. 2.14. So (Y, ψ) is mixing, $X \cong Y \times \{1, 2, \ldots, n\}$ and $\varphi(x, i) = (x, i + 1)$ if $1 \leq i \leq n - 1$, $\varphi(x, n) = (\psi(x), 1)$. If we consider the Smale space (X, φ^n) we still have $X \cong Y \times \{1, 2, \ldots, n\}$, and now $\varphi^n(x, i) = (\psi(x), i)$. So (X, φ^n) is a disjoint union of n copies of the mixing Smale space (Y, φ).

If we now fix a finite φ-invariant set $P \subset X \cong Y \times \{1, 2, \ldots, n\}$, and let $\tilde{P}$ be $P \cap Y \times \{1\}$ we immediately see that

$$\begin{aligned} S(X, \varphi, P) = S(X, \varphi^n, P) &\cong \bigoplus_i^n S(Y, \psi, \tilde{P}), \\ U(X, \varphi, P) = U(X, \varphi^n, P) &\cong \bigoplus_i^n U(Y, \psi, \tilde{P}), \text{ and} \\ H(X, \varphi) = H(X, \varphi^n, P) &\cong \bigoplus_i^n H(Y, \psi). \end{aligned}$$

Denote by α_φ and α_ψ the $*$-automorphisms on $S(X, \varphi, P)$ and $S(Y, \psi, \tilde{P})$ respectively. It is then straightforward to see that α_φ permutes the summands of $\bigoplus_i^n S(Y, \psi, \tilde{P})$. In particular, for $a \in S(Y, \psi, \tilde{P})$ we have

$$\alpha_\varphi(a, i) = \begin{cases} (a, i+1) & 1 \leq i \leq n-1 \\ (\alpha_\psi(a), 1) & i = n. \end{cases}$$

The corresponding results hold for $U(X, \varphi, P)$ and $H(X, \varphi)$ similarly.

2.6 K-Theory of a Commutative C^*-Algebra

In this section we discuss how, in the case of a commutative C^*-Algebra, A, $K_*(A)$, can be given a product structure.

Let A be a commutative C^*-algebra. For projections p and q in $P_1(A)$, $pq = qp$ is also in $P_1(A)$. So we can define $[p]_0[q]_0 = [pq]_0$. For $p \in P_m(A)$ and $q \in P_n(A)$ things are slightly more complicated. We define $p \times q \in P_{mn}(A)$ entry-wise by $(p \times q)_{(i,j)(k,l)} = p_{ik}q_{jl}$ where we have indexed the rows and columns of $P_{mn}(A)$ by dictionary ordering on ordered pairs (i, j) with $1 \leq i \leq m$, $1 \leq j \leq n$. Writing $[p]_0[q]_0 = [p \times q]_0$ now describes a ring structure on $K_0(A)$.

Remark 2.26. *By noticing that A commutative implies that $C(S^1, A)$ is commutative, and that $K_0(C(S^1, A)) \cong K_0(A) \oplus K_1(A)$, we can use the above idea to define a ring structure on $K_0(A) \oplus K_1(A)$. We will discuss this in further detail in a later section.*

Unfortunately, the C^*-algebras that we are interested in here (from Smale spaces), are not commutative, so we cannot define a ring structure as above. However, proposition 2.24 provides some hope that we may be able to do something similar. For $p,\ q \in P_1(H(X, \varphi))$, n large we define

$$a = \frac{\alpha^n(p)q + q\alpha^n(p)}{2},$$

; so $a = a^*$, and $||a^2 - a||$ is small. Then $\chi_{(1/2,\infty)}(a)$ is a projection. We would like to define $[p]_0[q]_0 = [\chi_{(1/2,\infty)}(a)]_0$. The problem is that a as defined above depends on n, ie

$$\lim_{n\to\infty} \frac{\alpha^n(p)q + q\alpha^n(p)}{2}$$

need not exist. Thus, our efforts to define a product structure on $K_0(H(X, \varphi))$ fail. We can, however, define a product structure on the K-theory of a very closely related algebra, the mapping cylinder (see Chapter 3).

2.7 Shift Equivalence

In this section we briefly discuss two equivalence relations on non-negative, square integer matrices, and their importance as sources of invariants for SFTs. Much of the work along this vein was done by Williams (eg [30]) and Kim and Roush (eg [13], [14]). Chapter 7 of [16] presents a nice treatment of this material.

Definition 2.27. *The non-negative $n\times n$ integer matrix A and the non-negative $m\times m$ integer matrix B are said to be* **elementary strong shift equivalent** *(ESSE) if there exist non negative integer matrices U $(n \times m)$ and V $(m \times n)$ such that $UV = A$, $VU = B$.*

We let **strong shift equivalence** *be the equivalence relation generated by ESSE. We denote strong shift equivalence by $\sim_{SSE}$.*

Definition 2.28. *The non-negative $n \times n$ integer matrix A and the non-negative $m \times m$ integer matrix B are said to be* **shift equivalent** *if there exist non negative integer matrices R $(n \times m)$ and S $(m \times n)$ and a positive integer k such that:*

- $RS = A^k$
- $SR = B^k$
- $AR = RB$
- $SA = BS$.

In this case we call k the **lag**. *We denote shift equivalence by $\sim_{SE}$*

Strong shift equivalence implies shift equivalence, but as shown by Kim and Roush ([13], [14]) the converse is not true. The following proposition is due to Williams ([30]).

Proposition 2.29. *Let (Σ_A, σ_A) and (Σ_B, σ_B) be SFTs with adjacency matrices A and B respectively. Then (Σ_A, σ_A) and (Σ_B, σ_B) are topologically conjugate if and only if $A \sim_{SSE} B$.*

The next proposition shows that shift equivalence classifies $K_0(S(\Sigma, \sigma, P))$. It appears as Theorem 7.5.8 in [16].

Proposition 2.30. *Let (Σ_A, σ_A) and (Σ_B, σ_B) be SFTs with adjacency matrices A and B respectively. Then the dimension triples*

$$\big(K_0(S(\Sigma_A, \sigma_A, P_A)), K_0(S(\Sigma_A, \sigma_A, P_A))^+, \alpha_{\sigma_A}\big)$$

and

$$\big(K_0(S(\Sigma_B, \sigma_B, P_B)), K_0(S(\Sigma_B, \sigma_B, P_B))^+, \alpha_{\sigma_B}\big)$$

are isomorphic if and only if $A \sim_{SE} B$.

In section 4.3 we show that shift equivalence of adjacency matrices implies isomorphism of the ring and module structures that we develop in chapters 3 and 4.

Chapter 3

$K_*(C(H,\alpha))$

Let (X, φ) be an irreducible Smale space with homoclinic algebra $H(X, \varphi)$, and recall the $*$-homomorphism α defined in section 2.3. In this chapter we introduce the mapping cylinder for the algebra $H(X, \varphi)$ (with respect to α) and show that the K-theory of this algebra has a natural ring structure. In the case of a mixing Smale space, the existence of this ring structure was shown in [19]. We extend this result to irreducible Smale space, and in the case of a SFT, we explicitly compute the product in terms of matrix multiplication.

3.1 The Mapping Cylinder

Definition 3.1. *We define the mapping cylinder, $C(H, \alpha)$, of $H(X, \varphi)$ (with respect to α) as follows.*

$$C(H, \alpha) = \{f : \mathbb{R} \to H(X, \varphi) \,|\, f(t+1) = \alpha(f(t)) \;\forall\, t\}$$

We then define $\alpha_s(f)(t) = f(t+s)$ for $s \in \mathbb{R}$ so that $\alpha_n(f)(t) = \alpha^n(f(t))$ for $n \in \mathbb{Z}$.

The following result is analogous to Prop. 2.24, and shows that the asymptotically abelian structure of $H(X, \varphi)$ is inherited by $C(H, \alpha)$. This result and its proof appear in [19].

Proposition 3.2. *For $f, g \in C(H, \alpha)$,*

$$\lim_{t\to\infty} ||\alpha_t(f)\alpha_{-t}(g) - \alpha_{-t}(g)\alpha_t(f)|| = 0$$

Proof: Fix $\varepsilon > 0$. Since f, g are uniformly continuous we can partition $[0,1]$, $0 = x_1 < x_2 < \cdots < x_m = 1$ such that for $x \in [x_i, x_{i+1}]$ we have

$$\begin{aligned} ||f(x) - f(x_i)|| &< \varepsilon/(5||g||), and \\ ||g(x) - g(x_i)|| &< \varepsilon/(5||f||). \end{aligned}$$

Now from Prop. 2.24 we can choose $N \in \mathbb{N}$ such that for $|n| > N$, and for all $i, j \in \{1, 2, \ldots, m\}$ we have

$$||\alpha^n(f(x_i))\alpha^{-n}(g(x_j)) - \alpha^{-n}(g(x_j))\alpha^n(f(x_i))|| < \varepsilon/5.$$

Now, for any $|t| > N+1$ and $x \in [0,1]$ let $k = \lfloor t+x \rfloor$, $y = t+x-k$, and $z = x-t+k$. So $|k| > N$ and there exist i, j such that $z \in [x_i, x_{i+1}]$, $y \in [x_j, x_{j+1}]$. We now have

$$\begin{aligned} &||\alpha_t(f(x))\alpha_{-t}(g(x)) - \alpha_{-t}(g(x))\alpha_t(f(x))|| \\ &\quad = ||\alpha^k(f(y))\alpha^{-k}(g(z)) - \alpha^{-k}(g(z))\alpha^k(f(y))|| \\ &\quad \leq ||\alpha^k(f(y))\alpha^{-k}(g(z)) - \alpha^k(f(y))\alpha^{-k}(g(x_i))|| \\ &\qquad + ||\alpha^k(f(y))\alpha^{-k}(g(x_i)) - \alpha^k(f(x_j))\alpha^{-k}(g(x_i))|| \\ &\qquad + ||\alpha^k(f(x_j))\alpha^{-k}(g(x_i)) - \alpha^{-k}(g(x_i))\alpha^k(f(x_j))|| \\ &\qquad + ||\alpha^{-k}(g(x_i))\alpha^k(f(x_j)) - \alpha^{-k}(g(x_i))\alpha^k(f(y))|| \\ &\qquad + ||\alpha^{-k}(g(x_i))\alpha^k(f(y)) - \alpha^{-k}(g(z))\alpha^k(f(y))|| \\ &\quad < 2||f||\varepsilon/(5||f||) + 2||g||\varepsilon/(5||g||) + \varepsilon/5 \\ &\quad = \varepsilon. \end{aligned}$$

So

$$\lim_{|t|\to\infty} ||\alpha_t(f)\alpha_{-t}(g) - \alpha_{-t}(g)\alpha_t(f)|| = 0$$

□

This asymptotically abelian structure yields a ring structure on $K_*(C(H,\alpha))$ as in ([19],[7]). We say more about this in section 3.2. First we describe the group $K_*(C(H,\alpha))$ in the case of a SFT.

3.1.1 $K_0(C(H,\alpha))$ for a SFT

Let (Σ,σ) be a mixing SFT with $n\times n$ adjacency matrix A, and $H(\Sigma,\sigma)$ the associated homoclinic algebra (see section 2.3.1). Recall the suspension of the C^*-algebra $H(\Sigma,\sigma)$,

$$SH = \{f : [0,1] \to H(\Sigma,\sigma) \,|\, f(0) = f(1) = 0\}.$$

The following sequence is exact.

$$0 \longrightarrow SH \xrightarrow{\iota} C(H,\alpha) \xrightarrow{e_0} H(\Sigma,\sigma) \longrightarrow 0$$

The map e_0 is evaluation at 0, and $\iota(f)(s) = \alpha^k(f(s-k))$ for $k \le s \le k+1$. We thus get the following 6-term exact sequence of K groups.

$$\begin{tikzcd} K_0(SH) \arrow[r, "\iota_*"] & K_0(C(H,\alpha)) \arrow[r, "(e_0)_*"] & K_0(H(\Sigma,\sigma)) \arrow[d] \\ K_1(H(\Sigma,\sigma)) \arrow[u] & K_1(C(H,\alpha)) \arrow[l, "(e_0)_*"] & K_1(SH) \arrow[l, "\iota_*"] \end{tikzcd}$$

and recalling that $K_0(SH) \cong K_1(H(\Sigma,\sigma))$, $K_1(SH) \cong K_0(H(\Sigma,\sigma))$, and that for a SFT $K_1(H(\Sigma,\sigma)) = 0$ (as $H(\Sigma,\sigma)$ is AF) this becomes

$$\begin{tikzcd} 0 \arrow[r] & K_0(C(H,\alpha)) \arrow[r, "(e_0)_*"] & K_0(H(\Sigma,\sigma)) \arrow[d, "id-\alpha_*"] \\ 0 & K_1(C(H,\alpha)) \arrow[l] & K_0(H(\Sigma,\sigma)) \arrow[l] \end{tikzcd}$$

where the map $id-\alpha_*$ is as in [19], [6].We see that $K_0(C(H,\alpha)) \cong ker(id-\alpha_*)$ and $K_1(C(H,\alpha)) \cong K_0(H(\Sigma,\sigma))/im(id-\alpha_*)$. Recall from section 2.4.1 that

$$K_0(H(\Sigma,\sigma)) = \lim_{\rightarrow} M_n(\mathbb{Z}) = (M_n(\mathbb{Z})\times\mathbb{N})/\sim,$$

where for $k < m$, $(X,k) \sim (Y,m)$ iff $A^{m-k+l}XA^{m-k+l} = A^lYA^l$ for some $l \in \mathbb{N}$. We denote the equivalence class of (X,k) under $\sim$ by $[X,k]$. As $K_0(C(H,\alpha)) \cong ker(id-\alpha_*) \subset K_0(H(\Sigma,\sigma))$, we wish to describe $K_0(C(H,\alpha))$ as a direct limit of subsets of $M_n(\mathbb{Z})$.

Consider the set $C(A) = \{X \in M_n(\mathbb{Z}) \,|\, AX = XA\}$ and the inductive system

$$C(A) \overset{X \mapsto AXA}{\longrightarrow} C(A) \longrightarrow \cdots .$$

As above, we can write the limit group as

$$\varinjlim C(A) = (C(A) \times \mathbb{N})/\sim .$$

Where $\sim$ is the same equivalence relation as above. The following theorem gives us a useful characterization of $K_0(C(H, \alpha))$.

Theorem 3.3.

$$\varinjlim C(A) \cong ker(id - \alpha_*) \;\; (\cong K_0(C(H, \alpha))).$$

Proof: Consider the following diagram:

$$\begin{tikzcd}
C(A) \arrow[r, "X \mapsto AXA"] \arrow[d, "\iota"] & C(A) \arrow[r] \arrow[d, "\iota"] & \cdots \\
M_n(\mathbb{Z}) \arrow[r, "X \mapsto AXA"] & M_n(\mathbb{Z}) \arrow[r] & \cdots
\end{tikzcd}$$

Where the vertical maps are given by $\iota(X) = X$. We must show that ι determines a well defined group homomorphism on the inductive limit groups, and then show that it is injective and the image of ι is $ker(id - \alpha_*) \subset K_0(H(\Sigma, \sigma))$.

That ι induces a well defined group homomorphism $\iota : \lim C(A) \to K_0(H)$ is obvious. It should also be obvious that $im(\iota) \subset ker(id - \alpha_*)$.

Let's now show that ι is injective. Let $[X, m] \in \lim C(A)$ be such that $\iota([X, m]) = [0, m+k] \in K_0(H(\Sigma, \sigma))$ for some k, then $A^{k+l}XA^{k+l} = A^l 0 A^l$ for some l, ie $A^j X A^j = 0$ for some $j \geq n$. But then $[X, m] = [0, j + m] \in \lim C(A)$, and ι is injective.

Now suppose $[X, k] \in ker(id - \alpha_*) \subset K_0(H)$. Then $(id - \alpha*)[X, k] = [0, m] \in K_0(H)$ for some $m > k$, ie $[AXA - XA^2, k + 1] = [0, k]$ or $A^{m-k+l}XA^{m-k+l} - A^{m-k+l-1}XA^{m-k+l+1} = 0$ for some l. Letting $j = m - k + l$ we get $A^j X A^j = A^{j-1}XA^{j+1}$ or after multiplying on the left by A, $A^{j+1}XA^j = A^j XA^{j+1}$. So $Y = A^j XA^j \in C(A)$ and $\iota([Y, j + k]) = [X, k]$. So $\iota : \lim C(A) \to ker(id - \alpha_*)$ is an isomorphism. $\square$

3.1.2 $K_1(C(H,\alpha))$

From above we know that $K_1(C(H,\alpha)) \cong K_0(H(\Sigma,\sigma))/im(id-\alpha_*)$, but since we have a characterization of $K_0(H(\Sigma,\sigma))$ as a direct limit of groups of matrices, we would like to find a corresponding characterization of $K_1(C(H,\alpha))$. We start by defining a certain subgroup of $M_n(\mathbb{Z})$. Let

$$B(A) = \{X \mid X = YA - AY, \text{ for some } Y \in M_n(\mathbb{Z})\},$$

and notice that, for $X = YA - AY$, $AXA = (AYA)A - A(AYA)$. We can therefore consider the inductive system

$$B(A) \overset{X\mapsto AXA}{\longrightarrow} B(A) \longrightarrow \cdots .$$

As before we write

$$\varinjlim B(A) = (B(A) \times \mathbb{N})/\!\sim .$$

The following proposition characterizes $im(id-\alpha_*) \subset K_0(H(\Sigma,\sigma))$.

Proposition 3.4.

$$\varinjlim B(A)) \cong im(id-\alpha_*) \subset K_0(H)$$

Proof: Consider the following diagram of inductive systems

$$\begin{tikzcd} B(A) \arrow[r, "X\mapsto AXA"] \arrow[d, "\iota"] & B(A) \arrow[r] \arrow[d, "\iota"] & \cdots \\ M_n(\mathbb{Z}) \arrow[r, "X\mapsto AXA"] & M_n(\mathbb{Z}) \arrow[r] & \cdots \end{tikzcd}$$

where the vertical maps, ι, are given by inclusion $\iota(X) = (X)$. Clearly the above diagram commutes, so ι extends to a well defined map on the inductive limit groups:

$$\iota : \varinjlim B(A) \longrightarrow \varinjlim M_n(\mathbb{Z}) \quad (\cong K_0(H)).$$

We first show that $im(\iota) \subset im(id-\alpha_*)$. Suppose $[X,k] \in (B(A)\times\mathbb{N})/\sim$, then there exists Y such that $X = YA - AY$, so

$$\iota[X,k] = \iota[YA - AY, k] = [YA - AY, k] \in (M_{\#V(G)}(\mathbb{Z}) \times \mathbb{N})/\sim .$$

Now

$$[YA-AY,k]=[A(YA-AY)A,k+1]=[(AY)A^2-A(AY)A,k+1]=(id-\alpha_*)[-YA,k],$$

so

$$\iota(\lim_{\rightarrow} B(A)) \subset im(id-\alpha_*).$$

Now suppose we have $[X,k] \in B(A) \times \mathbb{N})/\sim$ such that $\iota[X,k] = [0,0]$. Now $[X,k] = [0,0] \in (M_n(\mathbb{Z}) \times \mathbb{N})/\sim$ if and only if there exists an l such that $A^l X A^l = 0$. So

$$[X,k] = [A^l X A^l, k+l] = [0, k+l] = [0,0] \in (B(A) \times \mathbb{N})/\sim .$$

so $[X,k] = [0,0] \in (B(A) \times \mathbb{N})/\sim$, and ι is one-to-one.

Now take $[X,k] \in im(id-\alpha_*)$, then there exists Y, l, such that

$$A^l X A^l = AYA - YA^2 = A(YA) - (YA)A.$$

So $A^l X A^l \in B(A)$. Now

$$\iota[A^l X A^l, l+k] = [A^l X A^l, l+k] = [X,k] \in (M_n(\mathbb{Z}) \times \mathbb{N})/\sim,$$

so ι is onto. Therefore,

$$\iota : \lim_{\rightarrow} B(A) \rightarrow im(id-\alpha_*)$$

is an isomorphism. □

We now know that

$$\lim_{\rightarrow} B(A) \cong im(id-\alpha_*),$$

and

$$\lim_{\rightarrow} M_n(\mathbb{Z}) \cong K_0(H(\Sigma,\sigma)).$$

So we have

$$K_1(C(H,\alpha)) \cong K_0(H)/im(id-\alpha_*) \cong (\lim_{\rightarrow} M_n(\mathbb{Z}))/(\lim_{\rightarrow} B(A)).$$

However, we would like to know if

$$K_1(C(H,\alpha)) \cong \lim_{\rightarrow} \left(M_{\#V(G)}(\mathbb{Z})/B(A)\right).$$

The following well known lemma provides the answer.

Lemma 3.5. *Let G be an abelian group, $\psi : G \to G$ an endomorphism, and $H < G$ a ψ-invariant subgroup, and consider the following diagram*

$$\begin{tikzcd}
H \arrow[r, "\psi"] \arrow[d, "\iota"] & H \arrow[r, "\psi"] \arrow[d, "\iota"] & \cdots \\
G \arrow[r, "\psi"] & G \arrow[r, "\psi"] & \cdots
\end{tikzcd}$$

where the vertical maps ι are given by inclusion, $\iota(x) = x$. Then $\lim_{\to} H < \lim_{\to} G$ and

$$\lim_{\to} G / \lim_{\to} H \cong \lim_{\to}(G/H)$$

Proof: We know that $\lim_{\to} G \cong (G \times \mathbb{N})/\sim$ where, for $n < m$ we say $(x, n) \sim (y, m)$ if there exists a $k \geq m-n$ such that $\psi^{m-n+k}(x) = \psi^k(y)$. Similarly, we have $\lim_{\to} H \cong (H \times \mathbb{N})/\sim$, and $\lim_{\to} G/H \cong (G/H \times \mathbb{N})/\sim$. The diagram in the statement of the Lemma clearly commutes, so ι extends to a map $\iota : \lim_{\to} H \to \lim_{\to} G$. It is also clear that $\lim_{\to} H < \lim_{\to} G$, so it suffices to show that the following sequence is exact:

$$0 \to \lim_{\to} H \xrightarrow{\iota} \lim_{\to} G \xrightarrow{\pi} \lim_{\to}(G/H) \to 0.$$

Where $\pi[x, n] = [x + H, n]$. We must show that π is onto, and that $im(\iota) = ker(\pi)$ (exactness at H is clear).

First suppose $[x + H, n] \in (G/H \times \mathbb{N})/\sim$, then $[x, n] \in (G \times \mathbb{N})/\sim$ and $\pi[x, n] = [x + H, n]$, so π is onto.

Now suppose $[h, n] \in (H \times \mathbb{N})/\sim$, then

$$\pi(\iota[h, n]) = \pi[h, n] = [h+H, n] = [e+H, n] = [e+H, 0] \in (G/H \times \mathbb{N})/\sim \ \text{(since } h \in H\text{)}.$$

so $im(\iota) \subset ker(\pi)$.

Now suppose $[x, n] \in ker(\pi) \subset (G \times \mathbb{N})/\sim$. So $[x + H, n] = [e + H, 0]$, and there exists k such that $\psi^k(x) + H = e + H$, ie $\psi^k(x) \in H$. So $[\psi^k(x), n + k] \in im(\iota) \subset (G \times \mathbb{N})/\sim$, but $[x, n] = [\psi^k(x), n + k] \in (G \times \mathbb{N})/\sim$, so $[x, n] \in im(\iota)$. Ie $ker(\pi) \subset im(\iota)$. Thus $ker(\pi) = im(\iota)$, and the sequence is exact. Therefore

$$\lim_{\to} G / \lim_{\to} H \cong \lim_{\to}(G/H).$$

□

The following theorem is now an immediate corollary.

Theorem 3.6.

$$K_1(C(H,\alpha)) \cong \varinjlim \left(M_{\#V(G)}(\mathbb{Z})/B(A)\right).$$

Proof: Follows immediately from Lemma 3.5 and the comments following Prop. 3.4. □

3.2 The Ring Structure on $K_0(C(H,\alpha)) \oplus K_1(C(H,\alpha))$

That $K_*(C(H,\alpha))$ has a $\mathbb{Z}_2$-graded ring structure follows from the asymptotically abelian structure on $C(H,\alpha)$ (Prop. 3.2) as in ([19], [7]). However, we wish to explicitly describe the product. We begin with the product on the subring $K_0(C(H,\alpha))$ and proceed as in [19]. For $f,\ g \in C(H,\alpha)$ let

$$(f \times g)_t = \left(\frac{\alpha_t(f)\alpha_{-t}(g) + \alpha_{-t}(g)\alpha_t(f)}{2}\right).$$

Similarly, for $f \in M_n(C(H,\alpha))$, $g \in M_m(C(H,\alpha))$ we define $(f \times g)_t \in M_{nm}(C(H,\alpha))$ componentwise by

$$((f \times g)_t)_{(ij)(i'j')} = (f_{ii'} \times g_{jj'})_t.$$

For $p \in P_n(C(H,\alpha))$, $q \in P_m(C(H,\alpha))$, $(p \times q)_t$ is self adjoint, and

$$\lim_{t\to\infty} ||(p \times q)_t^2 - (p \times q)_t|| = 0$$

so for $\epsilon > 0$ there exists T such that for $t \geq T$, the spectrum of $(p \times q)_t$ is contained in $(-\epsilon, \epsilon) \cup (1-\epsilon, 1+\epsilon)$. In particular, for $\epsilon < \frac{1}{2}$, $\chi_{(1/2,\infty)}$ is a continuous function on the spectrum. In other words, there exists a $T > 0$ such that

$$\chi_{(1/2,\infty)}(p \times q)_t \in P_{nm}(C(H,\alpha)) \text{ for } t \geq T.$$

The function $t \mapsto \chi_{(1/2,\infty)}(p \times q)_t$ is continuous, so for $t \geq T$, $\chi_{(1/2,\infty)}(p \times q)_t$ forms a continuous path of projections in $P_{nm}(CH)$. Thus, for $t_1, t_2 \geq T$

$$\left[\chi_{(1/2,\infty)}(p \times q)_{t_1}\right]_0 = \left[\chi_{(1/2,\infty)}(p \times q)_{t_2}\right]_0$$

and we can concretely define the product on $K_0(C(H,\alpha))$ as follows.

Definition 3.7. *For $p, q \in P_\infty(C(H,\alpha))$ we define the product $[p]_0[q]_0 \in K_0(C(H,\alpha))$ by*

$$[p]_0[q]_0 = \lim_{t\to\infty} \left[\chi_{(1/2,\infty)}(p \times q)_t\right]_0$$

We now show that we have a product structure on all of $K_*(C(H,\alpha))$, not just $K_0(C(H,\alpha))$. The key here is to consider the algebra of continuous functions on the circle S^1 taking values in $C(H,\alpha)$, $C(S^1, C(H,\alpha))$. Also recall the suspension $S(C(H,\alpha)) = \{f \in C(S^1, C(H,\alpha)) \mid f(1) = 0\}$. We then have the following split exact sequence:

$$0 \to S(C(H,\alpha)) \hookrightarrow C(S^1, C(H,\alpha)) \overset{f\mapsto f(1)}{\longrightarrow} C(H,\alpha) \to 0$$

where the splitting is given by $a \mapsto (f(t) = a \forall t \in S^1)$. Applying the functor K_0 we get the corresponding split exact sequence of K_0 groups.

$$0 \to K_0(S(C(H,\alpha))) \hookrightarrow K_0(C(S^1, C(H,\alpha))) \longrightarrow K_0(C(H,\alpha)) \to 0.$$

Recalling that $K_0(S(C(H,\alpha))) \cong K_1(C(H,\alpha))$ and using the fact that the above sequence is split-exact, we have

$$K_0(C(S^1, C(H,\alpha))) \cong K_0(C(H,\alpha)) \oplus K_1(C(H,\alpha)).$$

So we need to show that there is a well defined ring structure on $K_0(C(S^1, C(H,\alpha)))$. We again use the asymptotically abelian structure of $C(H,\alpha)$. We first define a family of $*$-automorphisms of $C(S^1, C(H,\alpha))$, α_t. For $f \in C(S^1, C(H,\alpha))$ we define

$$(\alpha_t(f))(s) = \alpha_t(f(s))$$

where the α_t on the right hand side is the automorphism of $C(H,\alpha)$. In other words

$$(\alpha_t(f))(s)(r) = \alpha_t(f(s))(r) = f(s)(r+t).$$

We now prove a result which is analogous to Lemma 3.2:

Lemma 3.8. *For $f, g \in C(S^1, C(H,\alpha))$,*

$$\lim_{|t|\to\infty} ||\alpha_t(f)\alpha_{-t}(g) - \alpha_{-t}(g)\alpha_t(f)|| = 0$$

Proof: Fix $\varepsilon > 0$. Partition S^1 by $\{s_1, s_2, \ldots, s_m\}$ where $s_i = e^{2\pi x_i i}$ and

$$0 = x_1 < x_2 < \cdots < x_m = 1$$

and such that for all i, and $s \in [s_i, s_{i+1}]$ we have

$$\begin{aligned} ||f(s) - f(s_i)|| &< \varepsilon/(5||g||), \text{ and} \\ ||g(s) - g(s_i)|| &< \varepsilon/(5||f||). \end{aligned}$$

Now using Lemma 3.2 we can find $T \in \mathbb{R}^+$ such that, for $|t| > T$, $i \in \{1, 2, \ldots, m\}$ we have

$$||\alpha_t(f(s_i))\alpha_{-t}(g(s_i)) - \alpha_{-t}(g(s_i))\alpha_t(f(s_i))|| < \varepsilon/5.$$

Now, for $s \in S^1$, there exists i such that $s \in [s_i, s_{i+1}]$. So that

$$\begin{aligned} ||\alpha_t(f)\alpha_{-t}(g)(s) - \alpha_{-t}(g)\alpha_t(f)(s)|| &= ||\alpha_t(f(s))\alpha_{-t}(g(s)) - \alpha_{-t}(g(s))\alpha_t(f(s))|| \\ &\leq ||\alpha_t(f(s))\alpha_{-t}(g(s)) - \alpha_t(f(s))\alpha_{-t}(g(s_i))|| \\ &\quad + ||\alpha_t(f(s))\alpha_{-t}(g(s_i)) - \alpha_t(f(s_i))\alpha_{-t}(g(s_i))|| \\ &\quad + ||\alpha_t(f(s_i))\alpha_{-t}(g(s_i)) - \alpha_{-t}(g(s_i))\alpha_t(f(s_i))|| \\ &\quad + ||\alpha_{-t}(g(s_i))\alpha_t(f(s_i)) - \alpha_{-t}(g(s))\alpha_t(f(s_i))|| \\ &\quad + ||\alpha_{-t}(g(s))\alpha_t(f(s_i)) - \alpha_{-t}(g(s))\alpha_t(f(s))|| \\ &< 2||f||\varepsilon/(5||f||) + 2||g||\varepsilon/(5||g||) + \varepsilon/5 \\ &= \varepsilon \end{aligned}$$

so

$$\lim_{|t|\to\infty} ||\alpha_t(f)\alpha_{-t}(g) - \alpha_{-t}(g)\alpha_t(f)|| = 0$$

□

For $f,\ g \in C(S^1, C(H, \alpha))$ let

$$(f \times g)_t = \left(\frac{\alpha_t(f)\alpha_{-t}(g) + \alpha_{-t}(g)\alpha_t(f)}{2}\right).$$

Similarly, for $f \in M_n(C(S^1, C(H, \alpha)))$, $g \in M_m(C(S^1, C(H, \alpha)))$ we define $(f \times g)_t \in M_{nm}(C(S^1, C(H, \alpha)))$ componentwise by

$$((f \times g)_t)_{(ij)(i'j')} = (f_{ii'} \times g_{jj'})_t.$$

For $p \in P_n(C(S^1, C(H,\alpha)))$, $q \in P_m(C(S^1, C(H,\alpha)))$ we know there exists a $T > 0$ such that

$$\chi_{(1/2,\infty)}(p \times q)_t \in P_{nm}(C(S^1, C(H,\alpha))) \text{ for } t \geq T.$$

The function $t \mapsto \chi_{(1/2,\infty)}(p \times q)_t$ is continuous, so for $t > T$, $\chi_{(1/2,\infty)}(p \times q)_t$ forms a path of projections in $P_{nm}(C(S^1, C(H,\alpha)))$. Thus, for $t_1, t_2 \geq T$

$$\left[\chi_{(1/2,\infty)}(p \times q)_{t_1}\right]_0 = \left[\chi_{(1/2,\infty)}(p \times q)_{t_2}\right]_0$$

and we can once again concretely define the product on

$$K_0(C(S^1, C(H,\alpha))) \cong K_0(C(H,\alpha)) \oplus K_1(C(H,\alpha))$$

as follows.

Definition 3.9. *For $p, q \in P_\infty(C(S^1, C(H,\alpha)))$ we define the product $[p]_0[q]_0 \in K_0(C(S^1, C(H,\alpha)))$ by*

$$[p]_0[q]_0 = \lim_{t\to\infty} \left[\chi_{(1/2,\infty)}(p \times q)_t\right]_0$$

Remark 3.10. *In the case that $p, q \in P_\infty(C(S^1, C(H,\alpha)))$ are constant functions, ie $p, q \in P_\infty(C(H,\alpha))$ in a natural way, the above definition of the product coincides with our previous definition for the product on $K_0(C(H,\alpha))$*

3.2.1 $K_0(C(H,\alpha))$ for SFT

Let (Σ, σ) be an irreducible SFT with adjacency matrix A. We wish to describe the ring structure on $K_*(C(H,\alpha))$ in terms of the matrix characterization of $K_*(C(H,\alpha))$ obtained in section 3.1.1. We start by determining the ring structure on the subring $K_0(C(H,\alpha))$, and leave the full ring structure until the next section.

We begin with a couple of observations.

1. The real parameter t in Defn. 3.9 can clearly be replaced with the integer parameter n. This will be helpful as in the SFT case $K_0(C(H,\alpha)) \subset K_0(H(\Sigma,\sigma))$, and on $K_0(H(\Sigma,\sigma))$ α_*^n is defined, but $(\alpha_t)_*$ is not.

2. In the case of a SFT, we can use a slightly simpler, though equivalent, definition for the product on $K_0(C(H,\alpha))$. See Prop. 3.12.

Consider the following $*$-subalgebra of $H(\Sigma,\sigma)$.

$$\mathcal{H}(\Sigma,\sigma) = span\{e_{Nv_iv_j}(\xi,\eta)\}.$$

Notice that $\mathcal{H}(\Sigma,\sigma)$ is dense in $H(\Sigma,\sigma)$ and that for each $p \in M_\infty(H(\Sigma,\sigma))$ there exists $q \in M_\infty(\mathcal{H}(\Sigma,\sigma))$ such that $[p]_0 = [q]_0$. We also consider the following $*$-subalgebra of $C(H,\alpha)$. $\mathcal{CH} = \{f \in C(H,\alpha) \mid f(0) \in \mathcal{H}\}$. We again notice that $\mathcal{CH}$ is dense in $C(H,\alpha)$ and for each $p \in M_\infty(C(H,\alpha))$ there exists $q \in M_\infty(\mathcal{CH})$ such that $[p]_0 = [q]_0$.

The following lemma will be useful in computing the product on $K_0(C(H,\alpha))$.

Lemma 3.11. *For $a = e_{Nv_iv_j}(\xi,\xi)$, $b = e_{Nv_lv_k}(\eta,\eta)$, and $n \geq N$ we have*

$$\alpha^n(a)\alpha^{-n}(b) = \sum_{\xi_1 \in S} e_{N+n,v_i,v_l}(\xi\xi_1\eta, \xi\xi_1\eta),$$

where

$$S = \{\xi_1 \mid |\xi_1| = 2n - 2N,\ i(\xi_1) = t(\xi),\ t(\xi_1) = i(\eta).\}$$

Proof:

$$\alpha^n(a) = \sum_{v \in V(G)} \sum_{\xi' \in \Xi_{2n,v_j,v}} e_{N+n,v_i,v}(\xi\xi', \xi\xi')$$

and

$$\alpha^{-n}(b) = \sum_{v \in V(G)} \sum_{\eta' \in \Xi_{2n,v,v_k}} e_{N+n,v,v_k}(\eta'\eta, \eta'\eta)$$

so

$$\alpha^n(a)\alpha^{-n}(b) = \sum_{v \in V(G)} \sum_{\xi'} \sum_{\tilde{v} \in V(G)} \sum_{\eta'} e_{N+n,v_i,v}(\xi\xi', \xi\xi') e_{N+n,v,v_k}(\eta'\eta, \eta'\eta).$$

Where the sum is over ξ' such that $i\xi' = t\xi$ and η' such that $t(\eta') = i(\eta)$. Furthermore, each summand is 0 unless $\xi\xi' = \eta'\eta$. Write $\xi' = \xi_1\xi_2$, $\eta' = \eta_1\eta_2$ where $|\xi_2| = |\eta| = |\xi| = |\eta_1| = 2N$ and $|\xi_1| = |\eta_2| = 2n - 2N$. Now $\xi\xi_1\xi_2 = \eta_1\eta_2\eta$ implies $\xi = \eta_1$, $\xi_2 = \eta$, and $\xi_1 = \eta_2$, which in turn imply $v_i = \tilde{v}$, $v = v_l$, $i(\xi_1) = i(\eta_2) = t(\xi) = v_j$, and $t(\xi_1) = t(\eta_2) = i(\eta) = v_k$. So the sum becomes

$$\alpha^n(a)\alpha^{-n}(b) = \sum_{\xi_1 \in S} e_{N+n,v_i,v_l}(\xi\xi_1\eta),$$

where

$$S = \{\xi_1 \mid |\xi_1| = 2n - 2N,\ i(\xi_1) = t(\xi),\ t(\xi_1) = i(\eta).\}$$

$\square$

Proposition 3.12. *Let (Σ, σ) be an irreducible SFT, $p, q \in P_\infty(\mathcal{CH})$. By identifying $K_0(C(H, \alpha))$ with a subgroup of $K_0(H(\Sigma, \sigma))$, the product can be written*

$$[p(0)]_0[q(0)]_0 = \lim_{n\to\infty}[\alpha^n(p(0))\alpha^{-n}(q(0))]_0.$$

Proof: Now from section 3.1.1 we know that $K_0(C(H, \alpha)) \cong ker(id - \alpha_*) \subset K_0(H(\Sigma, \sigma))$ where the isomorphism is given by evaluation at 0. So we have

$$\begin{aligned} [p(0)]_0[q(0)]_0 &= \lim_{t\to\infty}\left[\chi_{(1/2,\infty)}(p \times q)_t(0)\right]_0 \\ &= \lim_{t\to\infty}\left[\chi_{(1/2,\infty)}(p(0) \times q(0))_t\right]_0 \\ &= \lim_{n\to\infty}\left[\chi_{(1/2,\infty)}(p(0) \times q(0))_n\right]_0. \end{aligned}$$

We now look more closely at

$$(p(0) \times q(0))_n = \frac{\alpha^n(p(0))\alpha^{-n}(q(0)) + \alpha^{-n}(q(0))\alpha^n(p(0))}{2}.$$

Since $p(0), q(0) \in M_\infty(\mathcal{H}(\Sigma, \sigma))$, it will suffice to consider the product $\alpha^n(a)\alpha^{-n}(b)$, where $a = e_{Nv_iv_j}(\xi, \xi)$ and $b = e_{Nv_lv_k}(\eta, \eta)$. We then have $[a]_0 = [e_{ij}, N], [b]_0 = [e_{kl}, N] \in K_0(H(\Sigma, \sigma))$. Fix $n \geq N$, then from Lemma 3.11

$$\alpha^n(a)\alpha^{-n}(b) = \sum_{\xi_1 \in S} e_{N+n,v_i,v_l}(\xi\xi_1\eta).$$

A similar computation yields

$$\alpha^{-n}(b)\alpha^n(a) = \sum_{\eta_2 \in S} e_{N+n,v_i,v_l}(\xi\eta_2\eta),$$

so these two sums are equal. This implies

$$\alpha^n(a)\alpha^{-n}(b) = \alpha^{-n}(b)\alpha^n(a)$$

is a projection. Now for p, q as above, there exists N such that for $n > N$

$$\alpha^n(p(0))\alpha^{-n}(q(0)) = \alpha^n(q(0))\alpha^{-n}(p(0)).$$

This implies

$$(p(0) \times q(0))_n = \frac{\alpha^n(p(0))\alpha^{-n}(q(0)) + \alpha^{-n}(q(0))\alpha^n(p(0))}{2} = \alpha^n(p(0))\alpha^{-n}(q(0))$$

is a projection. Thus

$$\chi_{(1/2,\infty)}(p(0) \times q(0))_n = \alpha^n(p(0))\alpha^{-n}(q(0))$$

and we have

$$[p(0)]_0[q(0)]_0 = \lim_{n\to\infty}[\alpha^n(p(0))\alpha^{-n}(q(0))]_0.$$

□

Under the isomorphism of Prop. 3.3, every element of $K_0(C(H,\alpha))$ is equal to some $[X,N] \in \lim C(A)$. As each such X is a linear combination of matrices of the form e_{ij}, we start with two matrices of this form, their corresponding projections in $H(\Sigma,\sigma)$ and multiply according to Prop. 3.12.

Remark 3.13. *As $[e_{ij}, N] \in K_0(H(\Sigma,\sigma))$ need not be an element of the subgroup $K_0(C(H,\alpha))$, the formula we derive for the product of two such elements will not be well defined in general. Ie. the element $K_0(H(\Sigma,\sigma))$, $[\alpha^n(a)\alpha^n(b)]_0$ in the following Lemma depends (in general) on the integer n and thus $\lim_{n\to\infty}[\alpha^n(a)\alpha^n(b)]_0$ need not exist. However, if we apply the formula to linear combinations of such elements, $[X,N]$, $[Y,M]$ which are in $K_0(C(H,\alpha))$ the product is well defined.*

Lemma 3.14. *Let $a, b \in P_\infty(H(\Sigma,\sigma))$. If $[a]_0 = [X,N]$, $[b]_0 = [Y,N]$, then for $n \geq N$*

$$[\alpha^n(a)\alpha^n(b)]_0 = [XA^{2n-2N}Y, N+n].$$

Proof: For $a, b \in P_\infty(H(\Sigma,\sigma))$, we can find $\bar{a}, \bar{b} \in P_\infty(\mathcal{H}(\Sigma,\sigma))$ such that $[a]_0 = [\bar{a}]_0$, $[b]_0 = [\bar{b}]_0$. It therefore suffices to prove the result for rank one projections $\bar{a} = e_{Nv_iv_j}(\xi,\xi)$ and $\bar{b} = e_{Nv_lv_k}(\eta,\eta)$. Then $[\bar{a}]_0 = [e_{ij}, N], [\bar{b}]_0 = [e_{kl}, N] \in K_0(H(\Sigma,\sigma))$.

Fix $n \geq N$, then using Lemmas 3.11 and 3.12 we see that

$$\alpha^n(a)\alpha^{-n}(b) = \sum_{\xi_1} e_{N+n,v_i,v_l}(\xi\xi_1\eta)$$

is a projection. The number of summands is thus the number of paths ξ_1 of length $2n-2N$ from v_j to v_k, ie A^{2n-2N}_{jk}. Noticing that $A^{2n-2N}_{jk}e_{il} = e_{ij}A^{2n-2N}e_{kl}$, we have

$$[\alpha^n(a)\alpha^{-n}(b)]_0 = [e_{ij}A^{2n-2N}e_{kl}, N+n].$$

$\square$

Proposition 3.15. *Let* $[X,N]$, $[Y,M] \in (C(A)\times\mathbb{N})/\sim\ \cong K_0(C(H,\alpha))$. *The product on* $K_0(C(H,\alpha))$ *is given by*

$$[X,N]*[Y,M] = [XY, N+M].$$

Proof: Let p, $q \in P_\infty(\mathcal{CH})$, with $[p(0)]_0 = [X,N]$, $[q(0)]_0 = [Z,N]$ (under the isomorphism in Prop. 3.3). Note here that we have chosen representatives of each equivalence class such that the second coordinate is equal. From Prop. 3.12 we know the product is given by

$$[p(0)]_0[q(0)]_0 = \lim_{n\to+\infty}[\alpha^n(p(0))\alpha^{-n}(q(0))]_0.$$

However, we know that for $n \geq N$ the sequence $[\alpha^n(p(0))\alpha^{-n}(q(0))]_0$ is constant, so we can write

$$[p]_0[q]_0 = \lim_{n\to+\infty}[\alpha^n(p(0))\alpha^{-n}(q(0))]_0 = [\alpha^N(p(0))\alpha^{-N}(q(0))]_0$$

By Lemma 3.14 this is

$$[X,N]*[Z,N] = \lim_{n\to+\infty}[XA^{2n-2N}Z, N+n] = [XZ, 2N].$$

Now suppose $[Y,M] \in K_0(C(H,\alpha))$. Let $M_1 = \max\{N,M\}$.

$$\begin{aligned}
[X,N]*[Y,M] &= [A^{M_1-N}XA^{M_1-N},M_1]*[A^{M_1-M}YA^{M_1-M},M_1] \\
&= [A^{M_1-N}XA^{M_1-N}A^{M_1-M}YA^{M_1-M},2M_1] \\
&= [A^{M_1-N}A^{M_1-M}XYA^{M_1-M}A^{M_1-N},N+M+|M-N|] \\
&= [A^{|M-N|}XYA^{|M-N|},N+M+|M-N|] \\
&= [XY,N+M]
\end{aligned}$$

□

3.2.2 $K_*(C(H,\alpha))$ for SFT

We now turn our attention to describing the full ring structure on $K_*(C(H,\alpha))$ in terms of the matrix characterizations. Recall from Propositions 3.3 and 3.6:

$$\begin{aligned}
K_0(C(H,\alpha)) &\cong (C(A)\times\mathbb{N})/\sim, \\
K_1(C(H,\alpha)) &\cong \big(M_{\#V(G)}(\mathbb{Z})/B(A)\times\mathbb{N}\big)/\sim .
\end{aligned}$$

To determine the product structure on $K_*(C(H,\alpha)) = K_0(C(H,\alpha))\oplus K_1(C(H,\alpha))$ we embed $K_0(C(H,\alpha))$ and $K_1(C(H,\alpha))$ in $K_0(C(S^1,C(H,\alpha))) \cong K_0(C(H,\alpha))\oplus K_1(C(H,\alpha))$ and compute the product as in Defn. 3.9. In fact, since we are considering the case of a SFT, we can use a simpler form of the product, analogous to Prop. 3.12. Prop. 3.15 tells us how to multiply two elements of $K_0(C(H,\alpha))$, so we need only calculate the product of two elements of $K_1(C(H,\alpha))$, and the product of an element of $K_0(C(H,\alpha))$ with an element of $K_1(C(H,\alpha))$. We begin by computing the product of an element of $K_0(C(H,\alpha))$ with an element of $K_1(C(H,\alpha))$. First recall the 6-term exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & K_0(C(H,\alpha)) \arrow[r, "(e_0)_*"] & K_0(H(\Sigma,\sigma)) \arrow[drr, "id-\alpha_*"] & & \\
0 & K_1(C(H,\alpha)) \arrow[l] & K_1(SH) \arrow[l, "\iota_*"'] & \cong & K_0(H(\Sigma,\sigma))
\end{tikzcd}$$

Hence every element of $K_1(C(H,\alpha))$ is the image under ι_* of some element in $K_1(SH) \cong K_0(H(\Sigma,\sigma))$. Also recall that $K_0(H)$ is generated by rank one projections in $P_1(H(\Sigma,\sigma))$. We will proceed as follows. Starting with a $p \in P_1(H(\Sigma,\sigma))$, and $q \in P_m(C(H,\alpha))$. We find $u_p \in U_1(\widetilde{SH})$ (corresponding to the isomorphism $K_1(SH) \cong K_0(H(\Sigma,\sigma)))$. We then find $\iota_*(u_p) \in U_1(\widetilde{C(H,\alpha)})$ and $\tilde{p} \in P_2(\widetilde{SC(H,\alpha)})$ (corresponding to the isomorphism $K_1(C(H,\alpha)) \cong K_0(SC(H,\alpha)))$. We then simply embed $\tilde{p}$ and q in $P_\infty(C(S^1, C(H,\alpha)))$ and multiply according to Lemma 3.23.

The following Lemmas are standard results in K-theory, see for example Theorems 10.1.3 and 11.1.2 in [23].

Lemma 3.16. *Let $p \in P_m(H)$, then under the isomorphism $K_0(H) \cong K_1(SH)$, $p \mapsto u_p \in U_m(\widetilde{SH})$ where*

$$u_p(s) = e^{2\pi i s} p + (1-p), \text{ for } 0 \leq s \leq 1$$

Lemma 3.17. *Let $u_p \in U_m(\widetilde{C(H,\alpha)})$, then under the isomorphism $K_1(C(H,\alpha)) \cong K_0(\widetilde{SC(H,\alpha)})$, $u_p \mapsto \tilde{p} \in P_{2m}(\widetilde{SC(H,\alpha)})$ where*

$$\tilde{p} = v_p \begin{bmatrix} I_m & 0_m \\ 0_m & 0_m \end{bmatrix} v_p^*$$

and

$$v_p(t) = \overline{R_t} \begin{bmatrix} u_p & 0 \\ 0 & I_m \end{bmatrix} \overline{R_t}^* \begin{bmatrix} I_m & 0 \\ 0 & u_p^* \end{bmatrix}$$

where

$$\overline{R_t} = \begin{bmatrix} \cos(\frac{\pi}{2}t) I_m & -\sin(\frac{\pi}{2}t) I_m \\ \sin(\frac{\pi}{2}t) I_m & \cos(\frac{\pi}{2}t) I_m \end{bmatrix},$$

so that

$$v_p(0) = \begin{bmatrix} u_p & 0 \\ 0 & u_p^* \end{bmatrix}, \quad v_p(1) = \begin{bmatrix} I_m & 0 \\ 0 & I_m \end{bmatrix}.$$

We state and prove one more Lemma before we compute the product of an element of $K_0(C(H,\alpha))$ with an element of $K_1(C(H,\alpha))$.

Lemma 3.18. *Let (Σ,σ) be a SFT with corresponding algebras $H(\Sigma,\sigma)$, $C(H,\alpha)$. For $p \in P_1(\mathcal{H}(\Sigma,\sigma))$, $q \in P_\infty(\mathcal{CH})$, there exists $N \in \mathbb{N}$ such that, for $n,\ m \geq N$, the matrix $(p \times q)_n$ with (i,j) entry given by $\alpha^n(p)\alpha^{-n}(q_{ij}(0))$ is in $P_\infty(H(\Sigma,\sigma))$ and*

$$[(p \times q)_n]_0 - [(p \times q)_m]_0 \in Im(id - \alpha_*).$$

Proof: The existence of N such that $(p \times q)_n \in P_\infty(H)$ for all $n \geq N$ follows immediately from 3.12. Now fix $m > n \geq N$ and consider

$$\begin{aligned}\alpha_*[\alpha^n(p)\alpha^{-n}(q_{ij}(0))]_0 &= [\alpha^{n+1}(p)\alpha^{-n+1}(q_{ij}(0))]_0 \\ &= [\alpha^{n+1}(p)\alpha^{-n+1}(q_{ij}(-2))]_0 \text{ by homotopy invariance} \\ &= [\alpha^{n+1}(p)\alpha^{-n-1}(q_{ij}(0))]_0.\end{aligned}$$

So

$$(id - \alpha_*)[\alpha^n(p)\alpha^{-n}(q_{ij}(0))]_0 = [\alpha^n(p)\alpha^{-n}(q_{ij}(0))]_0 - [\alpha^{n+1}(p)\alpha^{-n-1}(q_{ij}(0))]_0,$$

and hence, by induction

$$[\alpha^n(p)\alpha^{-n}(q_{ij}(0))]_0 - [\alpha^m(p)\alpha^{-m}(q_{ij}(0))]_0 \in Im(id - \alpha_*).$$

$\square$

Remark 3.19. *The same result clearly holds for the product $(q \times p)_n$ with entries $\alpha^n(q_{ij}(0))\alpha^{-n}(p)$.*

Definition 3.20. *For $p \in P_1(\mathcal{H}(\Sigma, \sigma))$, $q \in P_\infty(\mathcal{CH})$, $n \geq N$ define $(p \times q)_n$ as in Lemma 3.18. With a slight abuse of notation, we will often drop the n and write $(p \times q)$.*

We are now ready to state a main result.

Proposition 3.21. *Let $[X, N] \in K_0(C(H, \alpha))$, $[Y + B(A), M] \in K_1(C(H, \alpha))$. The product of these two is*

$$[X, N] * [Y + B(A), M] = [XY + B(A), N + M] \in K_1(C(H, \alpha)).$$

Similarly,

$$[Y + B(A), M] * [X, N]_0 = [YX + B(A), N + M] \in K_1(C(H, \alpha)).$$

The proof of this Proposition is quite long, so we will break the proof down into a series of Lemmas.

Lemma 3.22. *Let $p \in P_1(\mathcal{H}(\Sigma,\sigma))$, $q \in P_m(\mathcal{CH})$ for some m. Let $u_p \in U_1(\widetilde{C(H,\alpha)})$ be*

$$u_p(s) = e^{2\pi is}\alpha^n(p) + (1-\alpha^n(p)) \quad \text{for } n \le s \le n+1,$$

Then for n large enough (as in Lemma 3.18), the matrix $(u_p \times q)_n$ is given by

$$((u_p \times q)_n)_{ij} = \alpha_n(u_p)\alpha_{-n}(q_{ij}) = (u_{p\times q})_{ij} - (I_m - \alpha_{-n}(q_{ij})).$$

in other words

$$(u_p \times q)_n = u_{p\times q} - (I_m - \alpha_{-n}(q))$$

Proof:

$$\begin{aligned}
(\alpha_n(u_p)\alpha_{-n}(q_{ij}))(s) &= (\alpha_n(u_p)(s))(\alpha_{-n}(q_{ij})(s)) \\
&= u_p(s+n)q_{ij}(s-n) \\
&= \alpha_s((u_p(n))(q_{ij}(-n))) \\
&= \alpha_s\left(\left(e^{2\pi in}\alpha^n(p) + (1-\alpha^n(p))\right)\alpha^{-n}(q_{ij}(0))\right) \\
&= \alpha_s\left(e^{2\pi in}\alpha^n(p)\alpha^{-n}(q_{ij}(0)) - \alpha^n(p)\alpha^{-n}(q_{ij}(0)) + \alpha^{-n}(q_{ij}(0))\right) \\
&= \alpha_s\left(e^{2\pi in}((p\times q)_n)_{ij} - ((p\times q)_n)_{ij} + \alpha^{-n}(q_{ij}(0))\right) \\
&= \alpha_s\left(e^{2\pi in}(p\times q)_n + (I_m - (p\times q)_n) - (I_m - \alpha^{-n}(q(0)))\right)_{ij} \\
&= \alpha_s\left(u_{(p\times q)}(0) - (I_m - \alpha^{-n}(q(0)))\right)_{ij} \\
&= \left(u_{(p\times q)}(s) - (I_m - \alpha^{-n}(q(s)))\right)_{ij}
\end{aligned}$$

so

$$(u_p \times q)_n = u_{p\times q} - (I_m - \alpha_{-n}(q)).$$

□

The following Lemma allows us to use a slightly simplified form of the product, as in Prop. 3.12.

Lemma 3.23. *Let $p \in P_1(\mathcal{H}(\Sigma,\sigma))$, $q \in P_m(\mathcal{CH})$. Let $\tilde{p} \in P_2(C(S^1, C(H,\alpha)))$, then there exists $T \in \mathbb{R}$ such that for $t > T$*

$$\alpha_t(\tilde{p})_{(ij)}\alpha_{-t}(q_{kl}) = \alpha_{-t}(q_{kl})\alpha_t(\tilde{p}_{(ij)}).$$

Moreover, if we define $(\tilde{p}\times q)_t$ componentwise by

$$((\tilde{p}\times q)_t)_{(ik)(jl)} = \alpha_t(\tilde{p}_{(ij)})\alpha_{-t}(q_{kl})$$

then for $t > T$ $(\tilde{p}\times q)_t$ is a projection in $P_{2m}(C(S^1, C(H,\alpha)))$ and

$$[\tilde{p}]_0[q]_0 = \lim_{t\to\infty}[(\tilde{p}\times q)_t]_0$$

Proof: The product $(\tilde{p}\times q)_n$ defined in the statement of the Lemma is the same as regular matrix multiplication between the following two $2m\times 2m$ matrices.

$$(\tilde{p}\times q)_n = \begin{bmatrix} \alpha_n\tilde{p}_{11}I_m & \alpha_n\tilde{p}_{12}I_m \\ \alpha_n\tilde{p}_{21}I_m & \alpha_n\tilde{p}_{22}I_m \end{bmatrix}\begin{bmatrix} \alpha_{-n}(q) & 0 \\ 0 & \alpha_{-n}(q) \end{bmatrix}.$$

For the 2×2 matrix Y we will denote by $\bar{Y}$ the $2m\times 2m$ with $m\times m$ block entries $Y_{ij}I_m$. Also notice that $\overline{XY} = \bar{X}\bar{Y}$. We now have

$$\begin{aligned}(\tilde{p}\times q)_n &= \alpha_n\Big(\bar{v_p}\begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}\bar{v_p^*}\Big)\alpha_{-n}\begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix} \\ &= \overline{\alpha_n(v_p)}\begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}\overline{\alpha_n(v_p^*)}\begin{bmatrix} \alpha_{-n}(q) & 0 \\ 0 & \alpha_{-n}(q) \end{bmatrix}.\end{aligned}$$

Lets first consider the last part of this product.

$$\begin{aligned}&\overline{\alpha_n(v_p^*)}\begin{bmatrix} \alpha_{-n}(q) & 0 \\ 0 & \alpha_{-n}(q) \end{bmatrix} \\ &= \left(\bar{R}_t\begin{bmatrix} \alpha_n(u_p)I_m & 0 \\ 0 & I_m \end{bmatrix}\bar{R}_t^*\begin{bmatrix} I_m & 0 \\ 0 & \alpha_n(u_p^*)I_m \end{bmatrix}\right)^*\begin{bmatrix} \alpha_{-n}(q) & 0 \\ 0 & \alpha_{-n}(q) \end{bmatrix} \\ &= \begin{bmatrix} I_m & 0 \\ 0 & \alpha_n(u_p)I_m \end{bmatrix}\bar{R}_t\begin{bmatrix} \alpha_n(u_p^*)I_m & 0 \\ 0 & I_m \end{bmatrix}\bar{R}_t^*\begin{bmatrix} \alpha_{-n}(q) & 0 \\ 0 & \alpha_{-n}(q) \end{bmatrix} \\ &= \begin{bmatrix} I_m & 0 \\ 0 & \alpha_n(u_p)I_m \end{bmatrix}\bar{R}_t\begin{bmatrix} \alpha_n(u_p^*)\alpha_{-n}(q) & 0 \\ 0 & \alpha_{-n}(q) \end{bmatrix}\bar{R}_t^*\end{aligned}$$

Now, from the proof of Lemma 3.22 and Prop. 3.12 we see that

$$\alpha_n(u_p^*)\alpha_{-n}(q) = \alpha_{-n}(q)\alpha_n(u_p^*)$$

so we have

$$
\begin{aligned}
&\overline{\alpha_n(v_p^*)} \begin{bmatrix} \alpha_{-n}(q) & 0 \\ 0 & \alpha_{-n}(q) \end{bmatrix} \\
&\quad = \begin{bmatrix} I_m & 0 \\ 0 & \alpha_n(u_p) I_m \end{bmatrix} \bar{R}_t \begin{bmatrix} \alpha_n(u_p^*)\alpha_{-n}(q) & 0 \\ 0 & \alpha_{-n}(q) \end{bmatrix} \bar{R}_t^* \\
&\quad = \begin{bmatrix} I_m & 0 \\ 0 & \alpha_n(u_p) I_m \end{bmatrix} \bar{R}_t \begin{bmatrix} \alpha_{-n}(q)\alpha_n(u_p^*) & 0 \\ 0 & \alpha_{-n}(q) \end{bmatrix} \bar{R}_t^* \\
&\quad = \begin{bmatrix} \alpha_{-n}(q) & 0 \\ 0 & \alpha_n(u_p)\alpha_{-n}(q) \end{bmatrix} \bar{R}_t \begin{bmatrix} \alpha_n(u_p^*) I_m & 0 \\ 0 & I_m \end{bmatrix} \bar{R}_t^* \\
&\quad = \begin{bmatrix} \alpha_{-n}(q) & 0 \\ 0 & \alpha_{-n}(q)\alpha_n(u_p) \end{bmatrix} \bar{R}_t \begin{bmatrix} \alpha_n(u_p^*) I_m & 0 \\ 0 & I_m \end{bmatrix} \bar{R}_t^* \\
&\quad = \begin{bmatrix} \alpha_{-n}(q) & 0 \\ 0 & \alpha_{-n}(q) \end{bmatrix} \begin{bmatrix} I_m & 0 \\ 0 & \alpha_n(u_p) I_m \end{bmatrix} \bar{R}_t \begin{bmatrix} \alpha_n(u_p^*) I_m & 0 \\ 0 & I_m \end{bmatrix} \bar{R}_t^* \\
&\quad = \begin{bmatrix} \alpha_{-n}(q) & 0 \\ 0 & \alpha_{-n}(q) \end{bmatrix} \overline{\alpha_n(v_p^*)}.
\end{aligned}
$$

Similarly

$$
\overline{\alpha_n(v_p)} \begin{bmatrix} \alpha_{-n}(q) & 0 \\ 0 & \alpha_{-n}(q) \end{bmatrix} = \begin{bmatrix} \alpha_{-n}(q) & 0 \\ 0 & \alpha_{-n}(q) \end{bmatrix} \overline{\alpha_n(v_p)}.
$$

So

$$
\begin{aligned}
(\tilde{p} \times q)_n &= \overline{\alpha_n(v_p)} \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix} \overline{\alpha_n(v_p^*)} \begin{bmatrix} \alpha_{-n}(q) & 0 \\ 0 & \alpha_{-n}(q) \end{bmatrix} \\
&= \begin{bmatrix} \alpha_{-n}(q) & 0 \\ 0 & \alpha_{-n}(q) \end{bmatrix} \overline{\alpha_n(v_p)} \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix} \overline{\alpha_n(v_p^*)}.
\end{aligned}
$$

Hence for sufficiently large n $(\tilde{p} \times q)_n \in P_{2m}(C(S^1, C(H, \alpha)))$ and

$$
[\tilde{p}]_0 [q]_0 = \lim_{n \to \infty} [(\tilde{p} \times q)_n]_0
$$

$\square$

The following Lemma contains the bulk of the calculation involved in proving Prop. 3.21.

Lemma 3.24. *Let* $p \in P_1(\mathcal{H}(\Sigma, \sigma))$, $q \in P_m(\mathcal{CH})$. *Let* $u_p \in U_1(\widetilde{C(H, \alpha)})$ *be*

$$u_p(s) = e^{2\pi i s}\alpha^n(p) + (1 - \alpha^n(p)) \quad \textit{for } n \leq s \leq n+1.$$

Let $\tilde{p} \in P_2(S\widetilde{C(H, \alpha)})$ *be as in Lemma 3.17. Then*

$$[\tilde{p}]_0[q]_0 = \left[v_{p\times q} \begin{bmatrix} \alpha_{-n}(q) & 0 \\ 0 & 0 \end{bmatrix} v^*_{p\times q} \right]_0$$

Proof: As $\tilde{p} \in P_2(C(\widetilde{S^1, CH}))$, $q \in P_m(C(S^1\tilde{,} CH))$, by Lemma 3.23 the product is (for sufficiently large n)

$$[\tilde{p}]_0[q]_0 = [(\tilde{p} \times q)_n]_0.$$

Where $(\tilde{p} \times q)_n \in P_{2m}(C(\widetilde{S^1, CH}))$ is as defined in Lemma 3.23. Now as in the proof of Lemma 3.23,

$$[(\tilde{p} \times q)_n]_0 = \left[\overline{\alpha_n(v_p)} \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix} \overline{\alpha_n(v_p^*)} \begin{bmatrix} \alpha_{-n}(q) & 0 \\ 0 & \alpha_{-n}(q) \end{bmatrix} \right]_0,$$

and

$$\overline{\alpha_n(v_p^*)} \begin{bmatrix} \alpha_{-n}(q) & 0 \\ 0 & \alpha_{-n}(q) \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & \alpha_n(u_p)I_m \end{bmatrix} \bar{R}_t \begin{bmatrix} \alpha_n(u_p^*)\alpha_{-n}(q) & 0 \\ 0 & \alpha_{-n}(q) \end{bmatrix} \bar{R}_t^*$$

Now, as in Lemma 3.22,

$$\alpha_n(u_p^*)\alpha_{-n}(q) = \alpha_{-n}(q)\alpha_n(u_p^*) = u_{p\times q} - (I_m - \alpha_{-n}(q)).$$

So we can write

$$
\begin{aligned}
\alpha_n(\bar{v}_p^*) & \begin{bmatrix} \alpha_{-n}(q) & 0 \\ 0 & \alpha_{-n}(q) \end{bmatrix} \\
&= \alpha_n(\bar{v}_p^*) \begin{bmatrix} \alpha_{-n}(q)^2 & 0 \\ 0 & \alpha_{-n}(q)^2 \end{bmatrix} \\
&= \begin{bmatrix} I_m & 0 \\ 0 & \alpha_n(u_p) I_m \end{bmatrix} \bar{R}_t \begin{bmatrix} \alpha_{-n}(q)\alpha_n(u_p^*)\alpha_{-n}(q) & 0 \\ 0 & \alpha_{-n}(q)^2 \end{bmatrix} \bar{R}_t^* \\
&= \begin{bmatrix} \alpha_{-n}(q) & 0 \\ 0 & \alpha_n(u_p)\alpha_{-n}(q) \end{bmatrix} \bar{R}_t \begin{bmatrix} \alpha_n(u_p^*)\alpha_{-n}(q) & 0 \\ 0 & \alpha_{-n}(q) \end{bmatrix} \bar{R}_t^*
\end{aligned}
$$

or

$$
\begin{aligned}
&\overline{\alpha_n(v_p^*)}\begin{bmatrix} \alpha_{-n}(q)^2 & 0 \\ 0 & \alpha_{-n}(q)^2 \end{bmatrix} \\
&\quad= \begin{bmatrix} \alpha_{-n}(q) & 0 \\ 0 & \alpha_n(u_p)\alpha_{-n}(q) \end{bmatrix} \bar{R}_t \begin{bmatrix} \alpha_n(u_p^*)\alpha_{-n}(q) & 0 \\ 0 & \alpha_{-n}(q) \end{bmatrix} \bar{R}_t^* \\
&\quad= \begin{bmatrix} I_m - (I_m - \alpha_{-n}(q)) & 0 \\ 0 & u_{p\times q} - (I_m - \alpha_{-n}(q)) \end{bmatrix} \bar{R}_t \\
&\qquad\quad \begin{bmatrix} u_{p\times q}^* - (I_m - \alpha_{-n}(q)) & 0 \\ 0 & I_m - (I_m - \alpha_{-n}(q)) \end{bmatrix} \bar{R}_t^* \\
&\quad= \begin{bmatrix} I_m & 0 \\ 0 & u_{p\times q} \end{bmatrix} \bar{R}_t \begin{bmatrix} u_{p\times q}^* & 0 \\ 0 & I_m \end{bmatrix} \bar{R}_t^* \\
&\qquad - \begin{bmatrix} I_m & 0 \\ 0 & u_{p\times q} \end{bmatrix} \bar{R}_t \begin{bmatrix} (I_m - \alpha_{-n}(q)) & 0 \\ 0 & (I_m - \alpha_{-n}(q)) \end{bmatrix} \bar{R}_t^* \\
&\qquad - \begin{bmatrix} (I_m - \alpha_{-n}(q)) & 0 \\ 0 & (I_m - \alpha_{-n}(q)) \end{bmatrix} \bar{R}_t \begin{bmatrix} u_{p\times q}^* & 0 \\ 0 & I_m \end{bmatrix} \bar{R}_t^* \\
&\qquad + \begin{bmatrix} (I_m - \alpha_{-n}(q)) & 0 \\ 0 & (I_m - \alpha_{-n}(q)) \end{bmatrix} \bar{R}_t \begin{bmatrix} (I_m - \alpha_{-n}(q)) & 0 \\ 0 & (I_m - \alpha_{-n}(q)) \end{bmatrix} \bar{R}_t^* \\
&\quad= v_{p\times q}^* - \begin{bmatrix} (I_m - \alpha_{-n}(q)) & 0 \\ 0 & (I_m - \alpha_{-n}(q)) \end{bmatrix} \begin{bmatrix} I_m & 0 \\ 0 & u_{p\times q} \end{bmatrix} \\
&\qquad - \begin{bmatrix} (I_m - \alpha_{-n}(q)) & 0 \\ 0 & (I_m - \alpha_{-n}(q)) \end{bmatrix} \bar{R}_t \begin{bmatrix} u_{p\times q}^* & 0 \\ 0 & I_m \end{bmatrix} \bar{R}_t^* \\
&\qquad + \begin{bmatrix} (I_m - \alpha_{-n}(q)) & 0 \\ 0 & (I_m - \alpha_{-n}(q)) \end{bmatrix} \\
&\quad= v_{p\times q}^* + \begin{bmatrix} (I_m - \alpha_{-n}(q)) & 0 \\ 0 & (I_m - \alpha_{-n}(q)) \end{bmatrix} \\
&\qquad\quad \left(\begin{bmatrix} I_m & 0 \\ 0 & I_m \end{bmatrix} - \bar{R}_t \begin{bmatrix} u_{p\times q}^* & 0 \\ 0 & I_m \end{bmatrix} \bar{R}_t^* - \begin{bmatrix} I_m & 0 \\ 0 & u_{p\times q} \end{bmatrix} \right).
\end{aligned}
$$

Similarly,

$$\overline{\alpha_n(v_p)}\begin{bmatrix} \alpha_{-n}(q)^2 & 0 \\ 0 & \alpha_{-n}(q)^2 \end{bmatrix}$$
$$= v_{p\times q} + \begin{bmatrix} (I_m - \alpha_{-n}(q)) & 0 \\ 0 & (I_m - \alpha_{-n}(q)) \end{bmatrix}$$
$$\left(\begin{bmatrix} I_m & 0 \\ 0 & I_m \end{bmatrix} - \bar{R}_t \begin{bmatrix} u_{p\times q} & 0 \\ 0 & I_m \end{bmatrix} \bar{R}_t^* - \begin{bmatrix} I_m & 0 \\ 0 & u_{p\times q}^* \end{bmatrix} \right).$$

Now

$$[\tilde{p}]_0[q]_0 = \left[\overline{\alpha_n(v_p)} \begin{bmatrix} \alpha_{-n}(\bar{q})^2 & 0 \\ 0 & \alpha_{-n}(\bar{q})^2 \end{bmatrix} \right] \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix} \overline{\alpha_n(v_p^*)} \begin{bmatrix} \alpha_{-n}(\bar{q})^2 & 0 \\ 0 & \alpha_{-n}(\bar{q})^2 \end{bmatrix} \Bigg]_0 .$$

Recalling that

$$[\tilde{p}]_0[q]_0 = [\tilde{p} \times q]_0.$$

So with the above expressions for the first and last parts of the product, we can expand the product and cancel many of the terms. After the canceling we are left with

$$\begin{aligned}
\tilde{p} \times q &= v_{p\times q} \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix} v_{p\times q}^* \\
&\quad - \bar{R}_t \begin{bmatrix} u_{p\times q} & 0 \\ 0 & I_m \end{bmatrix} \bar{R}_t^* \begin{bmatrix} I_m - \alpha_{-n}(q) & 0 \\ 0 & 0 \end{bmatrix} \bar{R}_t \begin{bmatrix} u_{p\times q}^* & 0 \\ 0 & I_m \end{bmatrix} \bar{R}_t^* \\
&= v_{p\times q} \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix} v_{p\times q}^* \\
&\quad - \bar{R}_t \begin{bmatrix} u_{p\times q} & 0 \\ 0 & I_m \end{bmatrix} \bar{R}_t^* \begin{bmatrix} I_m & 0 \\ 0 & u_{p\times q}^* \end{bmatrix} \\
&\qquad \begin{bmatrix} I_m - \alpha_{-n}(q) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_m & 0 \\ 0 & u_{p\times q} \end{bmatrix} \bar{R}_t \begin{bmatrix} u_{p\times q}^* & 0 \\ 0 & I_m \end{bmatrix} \bar{R}_t^* \\
&= v_{p\times q} \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix} v_{p\times q}^* - v_{p\times q} \begin{bmatrix} I_m - \alpha_{-n}(q) & 0 \\ 0 & 0 \end{bmatrix} v_{p\times q}^* \\
&= v_{p\times q} \begin{bmatrix} \alpha_{-n}(q) & 0 \\ 0 & 0 \end{bmatrix} v_{p\times q}^*
\end{aligned}$$

□

Lemma 3.25. *For $(p\times q)_{ij}=\alpha^n(p)\alpha^{-n}(q_{ij}(0))$ with n sufficiently large as in Lemma 3.18, and $\widetilde{p\times q}$ as in Lemma 3.17 we can write*

$$[\tilde{p}]_0[q]_0=[\widetilde{p\times q}]_0-[I_m-q]_0.$$

Proof: From the proof of Lemma 3.24 we have

$$\begin{aligned}
[\tilde{p}\times q]_0 &= \left[v_{p\times q}\begin{bmatrix} I_m & 0\\ 0 & 0\end{bmatrix}v^*_{p\times q}-v_{p\times q}\begin{bmatrix} I_m-\alpha_{-n}(q) & 0\\ 0 & 0\end{bmatrix}v^*_{p\times q}\right]_0\\
&= \left[\widetilde{p\times q}-v_{p\times q}\begin{bmatrix} I_m-\alpha_{-n}(q) & 0\\ 0 & I_m-\alpha_{-n}(q)\end{bmatrix}\begin{bmatrix} I_m & 0\\ 0 & 0\end{bmatrix}\right.\\
&\qquad\qquad \left.\begin{bmatrix} I_m-\alpha_{-n}(q) & 0\\ 0 & I_m-\alpha_{-n}(q)\end{bmatrix}v^*_{p\times q}\right]_0
\end{aligned}$$

Now consider the product

$$\begin{aligned}
&v_{p\times q}\begin{bmatrix} I_m-\alpha_{-n}(q) & 0\\ 0 & I_m-\alpha_{-n}(q)\end{bmatrix}\\
&\qquad = \bar{R}_t\begin{bmatrix} u_{p\times q} & 0\\ 0 & I_m\end{bmatrix}\bar{R}^*_t\begin{bmatrix} I_m & 0\\ 0 & u^*_{p\times q}\end{bmatrix}\begin{bmatrix} I_m-\alpha_{-n}(q) & 0\\ 0 & I_m-\alpha_{-n}(q)\end{bmatrix}
\end{aligned}$$

Now

$$\begin{aligned}
(u^*_{p\times q}\alpha_{-n}(q))(s) &= \alpha_s\left(e^{-2\pi i0}(p\times q)_n\alpha_{-n}(q(0))+(I_m-(p\times q)_n)\alpha_{-n}(q(0))\right)\\
&= \alpha_s\left(e^{-2\pi i0}\alpha^n(p)(\alpha^{-n}(q(0)))^2+\alpha_{-n}(q(0))-\alpha^n(p)q)(\alpha^{-n}(q(0)))^2\right)\\
&= \alpha_s\left(e^{-2\pi i0}(p\times q)_n+\alpha_{-n}(q(0))-(p\times q)_n\right)\\
&= \alpha_s\left(e^{-2\pi i0}(p\times q)_n+(I_m-(p\times q)_n)-(I_m-\alpha_{-n}(q(0)))\right)\\
&= \alpha_s\left(u_{p\times q}(0)-(I_m-\alpha_{-n}(q(0)))\right)\\
&= u^*_{p\times q}(s)-(I_m-\alpha_{-n}(q(s)))
\end{aligned}$$

so we have

$$\begin{aligned}
v_{p\times q}&\begin{bmatrix} I_m-\alpha_{-n}(q) & 0 \\ 0 & I_m-\alpha_{-n}(q)\end{bmatrix}\\
&=\bar{R}_t\begin{bmatrix} u_{p\times q} & 0 \\ 0 & I_m\end{bmatrix}\bar{R}_t^*\begin{bmatrix} I_m-\alpha_{-n}(q) & 0 \\ 0 & u_{p\times q}^*-u_{p\times q}^*+I_m-\alpha_{-n}(q)\end{bmatrix}\\
&=\bar{R}_t\begin{bmatrix} u_{p\times q} & 0 \\ 0 & I_m\end{bmatrix}\begin{bmatrix} I_m-\alpha_{-n}(q) & 0 \\ 0 & I_m-\alpha_{-n}(q)\end{bmatrix}\bar{R}_t^*\\
&=\bar{R}_t\begin{bmatrix} I_m-\alpha_{-n}(q) & 0 \\ 0 & I_m-\alpha_{-n}(q)\end{bmatrix}\bar{R}_t^*\\
&=\begin{bmatrix} I_m-\alpha_{-n}(q) & 0 \\ 0 & I_m-\alpha_{-n}(q)\end{bmatrix}
\end{aligned}$$

similarly

$$\begin{bmatrix} I_m-\alpha_{-n}(q) & 0 \\ 0 & I_m-\alpha_{-n}(q)\end{bmatrix}v_{p\times q}^*=\begin{bmatrix} I_m-\alpha_{-n}(q) & 0 \\ 0 & I_m-\alpha_{-n}(q)\end{bmatrix}.$$

We now have

$$\begin{aligned}
[\tilde{p}\times q]_0&=\left[\widetilde{p\times q}-\begin{bmatrix} I_m-\alpha_{-n}(q) & 0 \\ 0 & I_m-\alpha_{-n}(q)\end{bmatrix}\begin{bmatrix} I_m & 0 \\ 0 & 0\end{bmatrix}\right.\\
&\qquad\qquad\left.\begin{bmatrix} I_m-\alpha_{-n}(q) & 0 \\ 0 & I_m-\alpha_{-n}(q)\end{bmatrix}\right]_0\\
&=\left[\widetilde{p\times q}-\begin{bmatrix} I_m-\alpha_{-n}(q) & 0 \\ 0 & 0\end{bmatrix}\right]_0\\
&=[\widetilde{p\times q}]_0-[I_m-\alpha_{-n}(q)]_0\\
&=[\widetilde{p\times q}]_0-[I_m-q]_0
\end{aligned}$$

□

In the above, $[\tilde{p}]_0$ is an element of $K_0(\widetilde{SC(H,\alpha)})$. We wish to consider $K_0(SC(H,\alpha))$, for which it suffices to consider elements of the form $[\tilde{p}]_0-[s(\tilde{p})]_0$.

Lemma 3.26. *Let* $p\in P_1(\mathcal{H}(\Sigma,\sigma))$, $q\in P_m(\mathcal{CH})$ *so that* $[\tilde{p}]_0-[s(\tilde{p})]_0\in K_0(SC(H,\alpha))$,

$[q]_0 \in K_0(C(H,\alpha))$. *Then*

$$([\tilde{p}]_0 - [s(\tilde{p})]_0)[q]_0 = [\widetilde{p \times q}]_0 - [s(\widetilde{p \times q})]_0.$$

Proof:

$$\begin{aligned} ([\tilde{p}]_0 - [s(\tilde{p})]_0)[q]_0 &= ([\tilde{p}]_0 - [I_1]_0)[q]_0 \\ &= [\tilde{p} \times q]_0 - [q]_0 \\ &= [\widetilde{p \times q}]_0 - [I_m - q]_0 - [q]_0 \quad \text{Lemma 3.25} \\ &= [\widetilde{p \times q}]_0 - [I_m]_0 \\ &= [\widetilde{p \times q}]_0 - [s(\widetilde{p \times q})]_0. \end{aligned}$$

□

Remark 3.27. *The map* $\phi : K_0(H) \cong K_1(SH) \hookrightarrow K_1(C(H,\alpha)) \cong K_0(SC(H,\alpha))$ *is given by*

$$\phi([p]_0) = [\tilde{p}]_0 - [s(\tilde{p})]_0.$$

Where, for $\tilde{p} \in P_{2n}(SC(H,\alpha))$, $s(\tilde{p}) = I_n$. *So Lemma 3.26 simply says that to multiply* $\phi([p]_0)$ *by* $[q]_0$*, we can first multiply* $[p]_0$ *and* $[q]_0$ *in* $K_0(H)$*, which by Lemma 3.18 is well defined modulo* $Im(id - \alpha_*)$*, and then apply* ϕ*. As* $Im(id - \alpha_*)$ *is exactly the kernel of the map* $K_1(SH) \hookrightarrow K_1(C(H,\alpha))$ *everything is well defined.*

Remark 3.28. *Completely analogous calculations to those above show that*

$$[q]_0([\tilde{p}]_0 - [s(\tilde{p})]_0) = [\widetilde{q \times p}]_0 - [s(\widetilde{q \times p})]_0.$$

The proof of Prop. 3.21 is now immediate.

Proof of Prop. 3.21 Let $[X, N] \in K_0(C(H,\alpha))$, $[Y + B(A), M] \in K_1(C(H,\alpha))$. By Lemma 3.26 and the remarks following it, the product

$$[X, N] * [Y + B(A), M] \in K_1(C(H,\alpha)).$$

Is obtained by multiplying $[X, N]$ and $[Y, M]$ in $K_0(H)$, and projecting into

$$K_0(H)/Im(id - \alpha_*) \cong K_1(C(H,\alpha)).$$

Now the product of $[X, N]$ and $[Y, M]$ is computed in the same way as in the proof of Prop. 3.15, so we have

$$[X, N] * [Y + B(A), M] = [XY + B(A), N + M],$$

and similarly

$$[Y + B(A), M] * [X, N] = [YX + B(A), N + M].$$

□

To determine the full ring structure on $K_0(C(H, \alpha)) \oplus K_1(C(H, \alpha))$ we now only need to compute the product of two elements of $K_1(C(H, \alpha))$. We prove a slightly more general result, which is due to Ian Putnam.

Proposition 3.29. *Let $\mathcal{A}$ be a unital C^*-algebra with α_t an asymptotically abelian action. Denote the product on $K_*(\mathcal{A}) \cong K_0(C(S^1, \mathcal{A}))$ by $[p]_0[q]_0 = [(p \times q)]_0$. Then for $p \in P_n(C(S^1, \mathcal{A}))$, $q \in P_m(C(S^1, \mathcal{A}))$ we have*

$$([p]_0 - [p(0)]_0)([q]_0 - [q(0)]_0) = 0.$$

In other words, the product of two elements of $K_1(\mathcal{A})$ is zero.

Proof: Let $p \in M_n(C(S^1, \mathcal{A}))$, $q \in M_m(C(S^1, \mathcal{A}))$. For notational convenience, we consider the domain of p, q to be $[0, 1]$ instead of S^1 (with the obvious condition that $p(0) = p(1)$, $q(0) = q(1)$. Recall that $(p \times q) \in M_{mn}(C(S^1, \mathcal{A}))$. Define

$$p_r(t) = \begin{cases} p(rt) & 0 \le t \le \frac{1}{r}, \\ p(1) & \frac{1}{r} < t \le 1. \end{cases}$$

Similarly, we define

$$q_r(t) = \begin{cases} q(0) & 0 \le t \le \frac{1}{2} \\ q(r(t - 1/2)) & \frac{1}{2} < t \le \frac{1}{2} + \frac{1}{r}, \\ q(1) & \frac{1}{2} + \frac{1}{r} < t \le 1. \end{cases}$$

p_r and q_r are continuous paths of projections, so if we set $p' = p_4$, $q' = q_4$ we have $[p]_0 = [p']_0$ and $[q]_0 = [q']_0$. Now let U_t be the unitary defined as follows. For

$0 \leq t \leq \frac{1}{4}$,

$$U_t = \begin{bmatrix} I_{mn} & 0 \\ 0 & I_{mn} \end{bmatrix}.$$

For $\frac{1}{2} \leq t \leq \frac{3}{4}$

$$U_t = \begin{bmatrix} 0 & -I_{mn} \\ I_{mn} & 0 \end{bmatrix}.$$

For $\frac{1}{4} < t < \frac{1}{2}$, $U_t = R_{4(t-1/4)}$, and for $\frac{3}{4} < t \leq 1$, $U_t = R^*_{4(t-3/4)}$, where R_t is as in Lemma 3.17. We now have

$$U_t\left((p' \times q') \oplus (p(0) \times q(0))\right) U_t^* = \begin{cases} ((p' \times q(0)) \oplus (p(0) \times q(0))) & 1 \leq t \leq \frac{1}{4}, \\ ((p(0) \times q(0)) \oplus (p(0) \times q(0))) & \frac{1}{4} < t \leq \frac{1}{2}, \\ ((p(0) \times q(0)) \oplus (p(0) \times q')) & \frac{1}{2} <\leq \frac{3}{4}, \\ ((p(0) \times q(0)) \oplus (p(0) \times q(0))) & \frac{3}{4} < t \leq 1. \end{cases}$$

In other words

$$U_t\left((p' \times q') \oplus (p(0) \times q(0))\right) U_t^* = \left((p' \times q(0)) \oplus (p(0) \times q')\right).$$

We can now write

$$\begin{aligned} [(p' \times q')]_0 + [(p(0) \times q(0))]_0 &= [(p' \times q') \oplus (p(0) \times q(0))]_0 \\ &= [U_t\left((p' \times q') \oplus (p(0) \times q(0))\right) U_t^*]_0 \\ &= [p' \times q(0)]_0 + [p(0) \times q']_0, \end{aligned}$$

so

$$[(p' \times q')]_0 + [(p(0) \times q(0))]_0 - [p' \times q(0)]_0 + [p(0) \times q']_0 = 0,$$

or, after factoring and using $[p]_0 = [p']_0$,

$$([p]_0 - [p(0)]_0)([q]_0 - [q(0)]_0) = 0.$$

□

We now return to the specific case of a SFT.

Corollary 3.30. *Let $p, q \in P_\infty(S(\widetilde{C(H,\alpha)}))$, then $([p]_0 - [s(p)])([q]_0 - [s(q)]) = [0]_0$.*

Proof: Follows immediately from Prop. 3.29. □

Using our characterization of $K_*(C(H,\alpha))$ in terms of matrices, we now summarize the ring structure in the following theorem.

Theorem 3.31. *The product structure on the ring*

$$K_*(C(H,\alpha)) \cong (M_{\#V(G)}(\mathbb{Z})/B(A) \times \mathbb{N})/\!\sim \oplus (C(A) \times \mathbb{N})/\!\sim$$

is given by

$$\begin{aligned}([X_1 + B(A), N] + [X_2, N]) * ([Y_1 + B(A), M] + [Y_2, M])\\ = [X_1Y_2 + X_2Y_1 + B(A), N + M] + [X_2Y_2, N + M].\end{aligned}$$

Proof: Follows immediately from Propositions 3.15 and 3.21, and Corollary 3.30. □

3.3 Irreducible Smale Space

We briefly describe the mapping cylider and its K-theory ring in the case that (X,φ) is an irreducible Smale space.

Lemma 3.32. *Let $(X,\varphi) \cong (Y,\psi) \times \{1,2,\ldots,n\}$ be an irreducible Smale space as in Prop. 2.14. Then for $f \in C(H(X,\varphi),\alpha^\varphi)$ we can write*

$$f(t) = (f_1(t), f_1(t-1), \ldots, f_1(t-n+1))$$

for some $f_1 \in C(\mathbb{R}, H(Y,\psi))$ such that $\alpha^\psi(f_1(t)) = f_1(t+n)$. In other words $f_1(t) = \tilde{f}_1(\frac{t}{n})$ for some $\tilde{f}_1 \in C(H(Y,\psi),\alpha^\psi)$.

Proof: We can write $f \in C(H(X,\varphi),\alpha^\varphi)$ as

$$f(t) = (f_1(t), f_2(t), \ldots, f_n(t))$$

where each $f_i \in C(\mathbb{R}, H(Y,\psi))$ and recall that $\alpha^\varphi(f(t)) = f(t+1)$. Now

$$\alpha^\varphi(f(t)) = (\alpha^\psi(f_n(t)), f_1(t), f_2(t), \ldots, f_{n-1}(t))$$

so, for $1 \leq i \leq n-1$ we have

$$f_i(t) = f_{i+1}(t+1), \text{ or } f_{i+1}(t) = f_i(t-1),$$

so for $0 \leq k \leq n-1$

$$f_{1+k}(t) = f_1(t-k).$$

Also,

$$f_1(t+1) = \alpha^{\psi}(f_n(t)) = \alpha^{\psi}(f_1(t-n+1)), \text{ or } f_1(t+n) = \alpha^{\psi}(f_1(t)).$$

$\square$

Proposition 3.33. *Let $(X, \varphi) \cong (Y, \psi) \times \{1, 2, \ldots, n\}$ be an irreducible Smale space, then the rings $C(H_X, \alpha_{\varphi}) \cong C(H_Y, \alpha_{\psi})$.*

Proof: For $f \in C(H(X, \varphi), \alpha^{\varphi})$ the map from Lemma 3.32 which sends f to $\tilde{f}_1 \in C(H(Y, \psi), \alpha^{\psi})$. Has inverse

$$g(t) \mapsto (g(nt), g(nt-1), \ldots, g(nt-n+1)).$$

$\square$

Corollary 3.34. $K_*(C(H_X, \alpha_{\varphi})) \cong K_*(C(H_Y, \alpha_{\psi}))$ *as rings.*

Chapter 4

$K_*(C(H,\alpha))$-Modules

In this chapter we show that $K_0(S(X,\varphi,P))$, $K_0(U(X,\varphi,P))$, and $K_0(H(X,\varphi))$ have natural module structures over the ring $K_*(C(H,\alpha))$. In the case of a SFT we compute the module product in terms of the characterization of $K_*(C(H,\alpha))$ given in the previous chapter and show that $K_0(S(X,\varphi,P))$ and $K_0(U(X,\varphi,P))$ exhibit a certain type of duality as modules over a specific subring. We then show that shift equivalence on adjacency matrices is a complete invariant for the ring/module structures.

Let (X,φ) be an irreducible Smale space, P a finite φ-invariant set of periodic points. Recall that $S(X,\varphi,P)$ and $H(X,\varphi)$ are subalgebras of $B(l^2(V^h(P)))$ (section 2.3).

Proposition 4.1. *Let $a \in H(X,\varphi)$, $b \in S(X,\varphi,P)$ then ab and $ba \in S(X,\varphi,P)$, and* $\lim_{n\to+\infty} ||b\alpha^{-n}(a) - \alpha^{-n}(a)b|| = 0$.

Proof: We prove the result in the case that (X,φ) is mixing. The irreducible case then follows from the mixing case as in the proof of Prop. 2.24. It suffices to prove the result for $a \in C_c(G^h)$ supported on a set of the form $V(x_a,y_a,h_a,\delta_a)$ and $b \in C_c(G^s)$ of the form $V(x_b,y_b,h_b^u,\delta_b)$. Recall that

$$V(x_a,y_a,h_a,\delta_a) = \{(h_a(y),y) |\ d(h_a(y),x_a) < \delta_a,\ d(y,y_a) < \delta_a\},\ \text{and}\ h_a(y_a) = x_a\}.$$

The first part of the statement is simply Prop. 2.23, recall from the proof that

$$(ba)(x,y) = \sum_{z\sim_h y} b(x,z)a(z,y) \in C_c(G^s).$$

Now, considering the sets on which a, b are supported, the sum reduces to a single term and we are left with

$$(ba)(x,y) = \begin{cases} b(h_b^u \circ h_a(y), h_a(y))a(h_a(y), y) & \text{if } d(y, y_a) < \delta_a \text{ and } x = h_b^u \circ h_a(y), \\ 0 & \text{otherwise.} \end{cases}$$

Similarly

$$(ab)(x,y) = \begin{cases} a(h_a \circ h_b^u(y), h_b^u(y))b(h_b^u(y), y) & \text{if } d(y, y_b) < \delta_b \text{ and } x = h_a \circ h_b^u(y) \\ 0 & \text{otherwise.} \end{cases}$$

The proof of the second statement now follows closely the proof of Prop.2.24. Fix $\epsilon > 0$

$$\alpha^{-n}a(x,y) = a(\varphi^n(x), \varphi^n(y))$$

so

$$\begin{aligned} (b\alpha^{-n}a)(x,y) &= \sum_{z \sim_h y} b(x,z)a(\varphi^n(z), \varphi^n(y)) \\ &= \begin{cases} b(h_b^u \circ \varphi^{-n} \circ h_a \circ \varphi^n(y), \varphi^{-n} \circ h_a \circ \varphi^n(y))a(h_a \circ \varphi^n(y), \varphi^n(y)) \\ 0 \end{cases} \end{aligned}$$

and

$$\begin{aligned} (\alpha^{-n}(a)b)(x,y) &= \sum_{z \sim_h y} a(\varphi^n(x), \varphi^n(z))b(z,y) \\ &= \begin{cases} a(h_a \circ \varphi^n \circ h_b^u(y), \varphi^n \circ h_b^u(y))b(h_b^u(y), y) \\ 0 \end{cases} \end{aligned}$$

Now

$$|(b\alpha^{-n}a)(x,y) - (\alpha^{-n}(a)b)(x,y)| = 0$$

unless

$$\begin{aligned} x_4 &= y \in \varphi^{-n}B(y_a, \delta_a) \cap B(y_b, \delta_b) \\ x_3 &= \varphi^{-n} \circ h_a \circ \varphi^n(y) \in \varphi^{-n} \circ h_a B(y_a, \delta_a) \cap B(y_b, \delta_b) \\ x_2 &= h_b^u \circ \varphi^{-n} \circ h_a \circ \varphi^n(y) \in h_b^u B(y_b, \delta_b) \cap \varphi^{-n} \circ h_a B(y_a, \delta_a) \\ x_1 &= h_b^u(y) \in h_b^u B(y_b, \delta_b) \cap \varphi^{-n} B(y_a, \delta_a). \end{aligned}$$

notice that $x_2 = h_b^u \circ \varphi^{-n} \circ h_a \circ \varphi^n(y) = \varphi^{-n} \circ h_a \circ \varphi^n \circ h_b^u(y)$ (see Lemma 2.2 in [19]). In which case it is

$$|(b\alpha^{-n}(a) - \alpha^{-n}(a)b)(x,y)| = |b(x_2,x_3)a(\varphi^n(x_3),\varphi^n(x_4)) - a(\varphi^n(x_2),\varphi^n(x_1))b(x_1,x_4)|.$$

Now let $d = \sup\{d(x,y) \mid (x,y) \in V(x_b, y_b, h_b^u, \delta_b)\}$, and notice that $x_3 \sim_s x_4$, $x_1 \sim_s x_2$. Using the continuity of a, we can find $N \in \mathbb{N}$ such that for all $n > N$

$$\begin{aligned} |a(\varphi^n(x_2),\varphi^n(x_1))| &< \frac{\epsilon}{2d} \\ |a(\varphi^n(x_3),\varphi^n(x_4))| &< \frac{\epsilon}{2d}. \end{aligned}$$

We then have

$$\begin{aligned} |(b\alpha^{-n}(a) - \alpha^{-n}(a)b)(x,y)| &\leq |b(x_2,x_3)a(\varphi^n(x_3),\varphi^n(x_4))| + |a(\varphi^n(x_2),\varphi^n(x_1))b(x_1,x_4)| \\ &\leq |b(x_2,x_3)|\frac{\epsilon}{2d} + |b(x_1,x_4)|\frac{\epsilon}{2d} \\ &\leq d\frac{\epsilon}{2d} + d\frac{\epsilon}{2d} \\ &= \epsilon. \end{aligned}$$

We therefore have

$$\lim_{n\to+\infty} ||b\alpha^{-n}(a) - \alpha^{-n}(a)b|| = 0.$$

□

Corollary 4.2. *Let $f \in C(H,\alpha)$ and $b \in S$, then for each $t \in \mathbb{R}$,*

$$\lim_{s\to-\infty} ||\alpha_s(f(t))b - b\alpha_s(f(t))|| = 0$$

Proof: Fix $\varepsilon > 0$. Since f is uniformly continuous, we can partition $[0,1]$ by $0 = x_1 < x_2 < \cdots < x_m = 1$ such that for each $1 \leq i \leq m-1$ and $x \in [x_i, x_{i+l}]$ we have

$$||\alpha^k(f(x)) - \alpha^k(f(x_i))|| < \frac{\varepsilon}{3||b||}.$$

Now for from Prop. 4.1 we can find $N \in \mathbb{Z}^+$ such that, for $n \geq N$, and for all $0 \leq i \leq m$

$$||\alpha^{-n}(f(x_i))b - b\alpha^{-n}(f(x_i))|| < \frac{\varepsilon}{3}.$$

Let $s < -(N+t+1)$, and set $k = \lfloor t+s \rfloor$, and $x = t+s-k$. Then $x \in [0,1]$, so $x \in [x_i, x_{i+1}]$ for some i. Also $k < t+s+1 < -N$. Now

$$\begin{aligned}
||\alpha_s(f(t))b - b\alpha_s(f(t))|| &= ||f(t+s)b - bf(t+s)|| \\
&= ||\alpha^k(f(x))b - b\alpha^k(f(x))|| \\
&\leq ||\alpha^k(f(x))b - \alpha^k(f(x_i))b|| + ||\alpha^k(f(x_i))b - b\alpha^k(f(x_i))|| \\
&\quad + ||b\alpha^k(f(x_i)) - b\alpha^k(f(x_i))|| \\
&< \frac{\varepsilon||b||}{3||b||} + \frac{\varepsilon}{3} + \frac{\varepsilon||b||}{3||b||} = \varepsilon.
\end{aligned}$$

So

$$\lim_{s \to -\infty} ||\alpha_s(f(t))b - b\alpha_s(f(t))|| = 0.$$

$\square$

Definition 4.3. *For $p \in P_m(C(H,\alpha))$, and $a \in P_m(S(X,\varphi,P))$ define $(a \times p)_t \in P_{mn}(S(X,\varphi,P))$ componentwise by*

$$((a \times p)_t)_{(ij)(i'j')} = \frac{a_{ii'}\alpha_t(p_{jj'}(0)) + \alpha_s(p_{jj'}(0))a_{ii'}}{2}$$

Corollary 4.4. *Let $p \in P_m(C(H,\alpha))$, and $a \in P_m(S(X,\varphi,P))$. There exists $T \in \mathbb{R}$ such that for $t < T$*

$$\chi_{(1/2,\infty)}\left((a \times p)_t\right)$$

is a projection in S.

Proof: This is immediate from the previous corollary 4.2 and from the comments at the start of section 3.2. $\square$

Now, for $t < T$

$$\chi_{(1/2,\infty)}\left((a \times p)_t\right)$$

is a continuous path of projections in S, so for $t_1, t_2 > T$

$$\left[\chi_{(1/2,\infty)}\left((a \times p)_{t_1}\right)\right]_0 = \left[\chi_{(1/2,\infty)}\left((a \times p)_{t_2}\right)\right]_0.$$

We are now ready to define a module structure of $K_0(S(X,\varphi,P))$ over $K_0(C(H,\alpha))$.

Definition 4.5. *For $p \in P_m(C(H,\alpha))$, $a \in P_n(S(X,\varphi,P))$ we define the product*

$$[a]_0[p]_0 = \lim_{t\to-\infty} \left[\chi_{(1/2,\infty)}\left((a\times p)_t\right)\right]_0 \in K_0(S(X,\varphi,P))$$

We proceed as in section 3.2 to describe the module structure of $K_*(S(X,\varphi,P))$ over $K_*(C(H,\alpha))$.

Proposition 4.6. *Let $f \in C(S^1, S(X,\varphi,P))$, $g \in C(S^1, C(H,\alpha))$. Then*

$$\lim_{t\to-\infty} ||f\alpha_t(g) - \alpha_t(g)f|| = 0.$$

Proof: This follows from Cor. 4.2 in the same way Lemma 3.8 follows from Lemma 3.2. □

For $f \in M_m(C(S^1, S(X,\varphi,P)))$, $g \in M_n(C(S^1, C(H,\alpha)))$ we then define $(f\times g)_t$ componentwise by

$$((f\times g)_t)_{(ij)(kl)} = \frac{f_{ik}\alpha_t(g_{jl}) + \alpha_t(g_{jl})f_{ik}}{2}.$$

We can now define the module structure of $K_*(S(X,\varphi,P))$ over $K_*(C(H,\alpha))$.

Definition 4.7. *For $a \in P_m(C(S^1, S(X,\varphi,P)))$, $p \in P_n(C(S^1, C(H,\alpha)))$ we define the product*

$$[a]_0[p]_0 = \lim_{t\to-\infty} \left[\chi_{(1/2,\infty)}\left((a\times p)_t\right)\right]_0 \in K_0(C(S^1, S(X,\varphi,P))) = K_*(S(X,\varphi,P)).$$

Remark 4.8. *The module structure of $K_*(U(X,\varphi,P))$ over $K_*(C(H,\alpha))$ is completely analogous, so we simply record the result here.*

For $b \in P_m(C(S^1, U(X,\varphi,P)))$, $p \in P_n(C(S^1, C(H,\alpha)))$ we define the product

$$[p]_0[b]_0 = \lim_{t\to+\infty} \left[\chi_{(1/2,\infty)}\left((b\times p)_t\right)\right]_0 \in K_0(C(S^1, U(X,\varphi,P))) = K_*(U(X,\varphi,P)).$$

The module structure of $K_*(H(X,\varphi))$ over $K_*(C(H,\alpha))$ is arrived at in a similar manner, however $K_*(H(X,\varphi))$ has both a left and right $K_*(C(H,\alpha))$-module structure. This follows from the following Lemma.

Lemma 4.9. *Let $f \in C(S^1, H(X,\varphi))$, $g \in C(S^1, C(H,\alpha))$. Then*

$$\lim_{t\to\pm\infty} ||f\alpha_t(g) - \alpha_t(g)f|| = 0.$$

Proof: This is an easy consequence of Lemma 3.2. $\square$

We then proceed as above to arrive at the following definition of the module structure of $K_*(H(X,\varphi))$ over $K_*(C(H,\alpha))$.

Definition 4.10. *For $a \in P_m(C(S^1, H(X,\varphi)))$, $p \in P_n(C(S^1, C(H,\alpha)))$ the left module structure is given by*

$$[p]_0[a]_0 = \lim_{t\to+\infty} \left[\chi_{(1/2,\infty)}\left((p\times a)_t\right)\right]_0 \in K_0(C(S^1, H(X,\varphi))) = K_*(H(X,\varphi)),$$

and the right module structure is

$$[a]_0[p]_0 = \lim_{t\to-\infty} \left[\chi_{(1/2,\infty)}\left((a\times p)_t\right)\right]_0 \in K_0(C(S^1, H(X,\varphi))) = K_*(H(X,\varphi)).$$

Remark 4.11. *The results of sections 2.5 and 3.3 alow us to write the module structures of this chapter for an irreducible Smale space. In particular, if (X,φ) is an irreducible Smale space then*

$$S(X,\varphi,P) \cong \bigoplus_{1}^{n} S(Y,\psi,\tilde{P})$$

where $X \cong Y \times \{1,2,\ldots,n\}$, and

$$C(H_X,\alpha_\varphi) \cong C(H_Y,\alpha_\psi).$$

So for $[b] \in K_*C(H_Y,\alpha_\psi)$, $([a_1],[a_2],\ldots,[a_n]) \in S(X,\varphi,P)$ we have

$$([a_1],[a_2],\ldots,[a_n])[b] = ([a_1][b],[a_2][b],\ldots,[a_n][b]).$$

The corresponding results for the module structures on $U(X,\varphi,P)$ and $H(X,\varphi)$ also hold.

4.1 $K_0(C(H,\alpha))$-Modules for SFT

We now return our attention to the specific case of a SFT. For a SFT (Σ,σ), the algebras $S(\Sigma,\sigma,P)$, $U(\Sigma,\sigma,P)$, and $H(\Sigma,\sigma)$ are all AF and hence have trivial K_1 groups. This means that in the SFT case the K_1 parts of the module structure described above are all 0 and we are left to consider only the module structures of

$K_0(\cdot)$ over $K_0(C(H,\alpha))$.

We begin by considering $K_0(S(\Sigma,\sigma,P))$. We can once again use a simpler version of the product (similar to Prop. 3.12). First recall the $*$-subalgebra $\mathcal{H}(\Sigma,\sigma)$ of $H(\Sigma,\sigma)$, and similarly define the $*$-subalgebra $\mathcal{S}(\Sigma,\sigma,P)$ of $S(\Sigma,\sigma,P)$ by

$$\mathcal{S}(\Sigma,\sigma,P) = span\{e_{N,v_i}(\xi,\eta)\}.$$

Lemma 4.12. *Let $a \in P_l(\mathcal{S}(\Sigma,\sigma,P))$, $b \in P_m(\mathcal{CH})$. The module structure can be written*

$$[a]_0[b]_0 = \lim_{n\to+\infty}[(a\times b)_n)]_0$$

where $(a\times b)_n \in M_{lm}(S(\Sigma,\sigma,P))$ has entries

$$((a\times b)_n)_{(ij),(i'j')} = a_{ii'}\alpha^{-n}(b(0)_{jj'}).$$

Proof: It suffices to prove the result for $a = e_{N,v_i}(\xi,\xi)$, $b(0) = e_{N,v_j,v_k}(\eta,\eta)$. As in the proof of Prop. 3.12, it suffices to show that there exists a $K \in \mathbb{N}$ such that, for $n > K$

$$a\alpha^{-n}(b(0)) = \alpha^{-n}(b(0))a.$$

Fix $n \geq 2N$. Then

$$\alpha^{-n}(b(0)) = \alpha^{-n}e_{N,v_j,v_k}(\eta,\eta) = \sum_{v'}\sum_{\eta'} e_{N+n,v',v_k}(\eta'\eta,\eta'\eta)$$

where $|\eta'| = 2n$, $i(\eta') = v'$, and $t(\eta') = i(\eta) = v_j$. Also

$$a = id^n(e_{N,v_i}(\xi,\xi)) = \sum_{\tilde{v}}\sum_{\xi'} e_{N+n,\tilde{v}}(\xi\xi',\xi\xi')$$

where $|\xi'| = n$, $i(\xi') = t(\xi) = v_i$, and $t(\xi') = \tilde{v}$. So the product is

$$id^n(e_{N,v_i}(\xi,\xi))\alpha^{-n}e_{N,v_j,v_k}(\eta,\eta) = \sum_{\tilde{v}}\sum_{v'}\sum_{\xi'}\sum_{\eta'} e_{N+n,\tilde{v}}(\xi\xi',\xi\xi')e_{N+n,v',v_k}(\eta'\eta,\eta'\eta).$$

Where the sum is over ξ' such that $i(\xi') = t(\xi)$ and $t(\xi') \in P$, and η' such that $i(\eta') \in P$ and $t(\eta') = i(\eta)$. Furthermore, each summand is non-zero only if $\xi\xi' = \eta'\eta$. Write $\xi' = \xi_1\xi_2$, and $\eta' = \eta_1\eta_2$ where $\xi_1 = \eta_2$, $\xi = \eta_1$, and $\xi_2 = \eta$. So $|\xi_1| = |\eta_2| = n - 2N$, $|\xi| = |\eta_1| = 2N + n$, and $|\xi_2| = |\eta| = 2N$. Also $i(\xi_1) = i(\eta_2) = t(\eta_1) = t(\xi) = v_i$,

$t(\xi_1) = t(\eta_2) = i(\eta) = i(\xi_2) = v_j$, $t(\xi_2) = t(\eta) = v_k$, and $i(\eta_1) = i(\xi) \in F$. The product then reduces to a single sum:

$$id^n(e_{N,v_i}(\xi,\xi))\alpha^{-n}e_{N,v_j,v_k}(\eta,\eta) = \sum_{\xi_1 \in S} e_{N+n,v_k}(\xi\xi_1\eta, \xi\xi_1\eta)$$

where,

$$S = \{\xi_1 \mid |\xi_1| = n - 2N,\ i(\xi_1) = t(\xi),\ t(\xi_1) = i(\eta)\}.$$

Computing the product in the other order we similarly obtain

$$\alpha^{-n}e_{N,v_j,v_k}(\eta,\eta)id^n(e_{N,v_i}(\xi,\xi)) = \sum_{\xi_1 \in S} e_{N+n,v_k}(\xi\xi_1\eta, \xi\xi_1\eta),$$

so we have

$$a\alpha^{-n}(b(0)) = \alpha^{-n}(b(0))a$$

and hence

$$[a]_0[b]_0 = \lim_{n\to\infty}[a\alpha^{-n}(b(0))]_0.$$

□

Lemma 4.13. *Let $a \in P_\infty(\mathcal{S}(\Sigma,\sigma,P))$, $b \in P_\infty(\mathcal{H}(\Sigma,\sigma))$. If $[a]_0 = [v,N]$, $[b]_0 = [X,N]$, then for $n \geq 2N$*

$$[a\alpha^{-n}(b)]_0 = [vA^{n-2N}X, n+N].$$

Proof: It suffices to consider rank one projections $a = e_{N,v_i}(\xi,\xi)$, $b = e_{N,v_j,v_k}(\eta,\eta)$. Fix $n \geq 2N$. Then from the proof of Lemma 4.12 we have

$$id^n(e_{N,v_i}(\xi,\xi))\alpha^{-n}e_{N,v_j,v_k}(\eta,\eta) = \sum_{\xi_1 \in S} e_{N+n,v_k}(\xi\xi_1\eta, \xi\xi_1\eta).$$

The number of summands is the number of paths of length $n-2N$ starting at $t(\xi) = v_i$ and ending at $i(\eta) = v_j$. So if we look at the induced product on the K theory, we get:

$$[a\alpha^{-n}(b)]_0 = [A_{ij}^{n-2N}e_k, N+n] = [e_iA^{n-2N}e_{jk}, N+n].$$

□

We are now ready to write the $K_0(C(H,\alpha))$-module structure on $K_0(S(\Sigma,\sigma,P))$.

Theorem 4.14. *Let* $[v, N] \in K_0(S(\Sigma, \sigma, P))$*,* $[X, M] \in K_0(C(H, \alpha))$*. The module structure is given by*

$$[v, N] * [X, M] = [vX, N + 2M].$$

Proof: It suffices to prove the result for $a \in P_\infty(S(\Sigma, \sigma, P))$, $b \in P_\infty(C(H, \alpha))$, with $[a]_0 = [v, N]$, $[b]_0 = [X, N]$. Furthermore, in this case there exist $\bar{a} \in P_\infty(\mathcal{S}(\Sigma, \sigma.P))$, $\bar{b} \in P_\infty(\mathcal{CH})$, with $[\bar{a}]_0 = [v, N]$, $[\bar{b}]_0 = [X, N]$. Then from Lemma 4.12 the module structure is given by

$$[\bar{a}]_0[\bar{b}]_0 = \lim_{t \to -\infty} [(a \times b(0))_t]_0,$$

where

$$((a \times b(0))_t)_{(ij),(i'j')} = (a_{ii'} \times b(0)_{jj'})_t = a_{ii'}\alpha_t(b(0)_{jj'}).$$

Now $b(0) \in P_\infty(\mathcal{H}(\Sigma, \sigma))$ so by lemma 4.13 we have

$$\begin{aligned} [v, N] * [X, N] &= [vA^{n-2N}X, N + n] \\ &= [vXA^{n-2N}, N + (n - 2N) + 2N] \\ &= [vX, 3N]. \end{aligned}$$

We now derive the formula for $[v, N] * [X, M]$ in the case $N \neq M$. First suppose $N \leq M$, then

$$\begin{aligned} [v, N] * [X, M] &= [vA^{M-N}, M] * [X, M] \\ &= [vA^{M-N}X, 3M] \\ &= [vXA^{M-N}, 2M + (M - N) + N] \\ &= [vX, N + 2M]. \end{aligned}$$

Finally if $N \leq M$, then

$$\begin{aligned} [v, N] * [X, M] &= [v, N] * [A^{N-M}XA^{N-M}, N] \\ &= [vA^{N-M}XA^{N-M}, 3N] \\ &= [vXA^{2(N-M)}, N + 2(N - M) + 2M] \\ &= [vX, N + 2M]. \end{aligned}$$

$\square$

Remark 4.15. *The $K_0(C(H,\alpha))$-Module structure on $K_0(U(\Sigma,\sigma,P))$ is completely analogous and is given as follows. Let $[w,N]\in K_0(U(\Sigma,\sigma,P))$, $[X,M]\in K_0(C(H,\alpha))$, then*

$$[X,M]*[w,N]=[Xw,2M+N].$$

We can also consider $K_0(H(\Sigma,\sigma))$ as a $K_0(C(H,\alpha))$-module. For $a\in P_\infty(H(\Sigma,\sigma))$, $b\in P_\infty(C(H,\alpha))$

$$[a]_0[b]_0=\lim_{t\to-\infty}[(a\times b(0))_t]_0$$

where

$$((a\times b(0))_t)_{(ij),(i'j')}=(a_{ii'}\times b(0)_{jj'})_t=a_{ii'}\alpha_t(b(0)_{jj'})$$

gives a right $K_0(C(H,\alpha))$-module structure, and

$$[b]_0[a]_0=\lim_{t\to\infty}[(b(0)\times a)_t]_0$$

where

$$((b(0)\times a)_t)_{(ij),(i'j')}=(b(0)_{jj'}\times a_{ii'})_t=\alpha_t(b(0)_{jj'})a_{ii'}$$

gives a left $K_0(C(H,\alpha))$-module structure. To determine the form of the module product in terms of the inductive systems, we proceed as we did for $K_0(S(\Sigma,\sigma,P))$. We omit the details here as they are very similar and simply record the result here.

Remark 4.16. *Let $[X,N]\in K_0(C(H,\alpha))$, $[Y,M]\in K_0(H(\Sigma,\sigma))$. The left (resp. right) $K_0(C(H,\alpha))$-Module structure on $K_0(H(\Sigma,\sigma))$ is given by*

$$[X,N]*[Y,M]=[XYA^{2N},M+2N],$$

$$[Y,M]*[X,N]=[A^{2N}YX,M+2N].$$

4.2 Module Homomorphisms

Let (Σ,σ) be a mixing SFT, and $S(\Sigma,\sigma,P)$, $U(\Sigma,\sigma,P)$, $H(\Sigma,\sigma)$, $C(H,\alpha)$ the corresponding algebras. In this section we consider homomorphisms which respect the module structures described in the previous section. We begin by defining a certain subring of $K_*(C(H,\alpha))$.

Recall the automorphism α_* on $K_0(S(\Sigma,\sigma,P))$, $K_0(U(\Sigma,\sigma,P))$. The following proposition shows that α_* is multiplication by an element of $K_0(C(H,\alpha))$.

Proposition 4.17. *Let $[v, N] \in K_0(S(\Sigma, \sigma, P))$, $[w, M] \in K_0(U(\Sigma, \sigma, P))$. Then*

$$\alpha_*[v, N] = [v, N] * [A, 0], \qquad \alpha_*^{-1}[v, N] = [v, N] * [A, 1]$$
$$\alpha_*[w, M] = [A, 1] * [w, M], \qquad \alpha_*^{-1}[w, M] = [A, 0] * [w, M]$$

Proof:

$$\alpha_*[v, N] = [vA^2, N + 1] = [vA, N] = [v, N] * [A, 0].$$

The other statements are equally straightforward. □

Remark 4.18. $[A, 0]*[A, 1] = [A^2, 1] = [I, 0]$, *the identity in* $K_0(C(H, \alpha))$. *Moreover, in the case that A is invertible and A^{-1} is an integer matrix (ie* $det(A) = \pm 1$*),* $[A, 1] = [A^{-1}, 0]$.

Definition 4.19. *Let R be the subring of $K_0(C(H, \alpha))$ generated by $[A, 0]$, $[A, 1]$. In other words, generated by the elements which realize the automorphism α_* on $K_0(S(\Sigma, \sigma, P))$, $K_0(U(\Sigma, \sigma, P))$.*

We are now ready to state the main result of this section, which says that in a certain sense, $K_0(S(\Sigma, \sigma, P))$ and $K_0(U(\Sigma, \sigma, P))$ are dual.

Theorem 4.20. *Let $Hom_R(K_0(S(\Sigma, \sigma, P)), R)$ be the set of all (right) R-module homomorphisms from $K_0(S(\Sigma, \sigma, P))$ to R. Then $Hom_R(K_0(S(\Sigma, \sigma, P)), R)$ has a natural left R-module structure, and*

$$Hom_R(K_0(S(\Sigma, \sigma, P)), R) \cong K_0(U(\Sigma, \sigma, P))$$

as left R-modules.

Remark 4.21. *Similarly $Hom_R(K_0(U(\Sigma, \sigma, P)), R) \cong K_0(S(\Sigma, \sigma, P))$, as right R-modules.*

The proof of proposition 4.20 is fairly long, so we have broken it down into a number of Lemmas.

Lemma 4.22.

$$R = (\mathbb{Z}[A] \times \mathbb{N}) / \sim$$

where $\mathbb{Z}[A]$ is the set of all polynomials in A with integer coefficients, and $\sim$ is the restriction of the equivalence relation on $K_0(C(H, \alpha))$.

Proof: As $(\mathbb{Z}[A] \times \mathbb{N})/\sim$ is clearly a subring of $K_0(C(H,\alpha))$ containing $[A,0]$ and $[A,1]$, it suffices to show that $[A,0]$, $[A,1]$ generate it.

First, $[A,0] * [A,1] = [I,0]$, and $[A,0]^n = [A^n,0]$ so closure under addition then shows that, for any $p \in \mathbb{Z}[x]$, $[p(A),0] \in R$. Now notice that

$$[A,1] * [A,1] = [A^2,2] = [I,1],$$

and, for any $p(A) \in \mathbb{Z}[A]$, $N \in \mathbb{N}$

$$[I,1] * [p(A),N] = [p(A),N+1].$$

So, by induction, $[p(A),N] \in R$ for all $p(A) \in \mathbb{Z}[A]$, $N \in \mathbb{N}$. $\square$

Remark 4.23. *From the description of R in Lemma 4.22 it is clear that R is contained in the center of the ring $K_0(C(H,\alpha))$, $Z(K_0(C(H,\alpha)))$. In fact, in many, but not all, cases $R = Z(K_0(C(H,\alpha)))$. See section 4.4 for further description of $Z(K_0(C(H,\alpha)))$ and examples where these two subrings are not equal.*

The following Lemma sharpens our description of R

Lemma 4.24. *Let A have minimal polynomial $p(x) = x^l(x^k + a_{k-1}x^{k-1} + \cdots + a_0)$ (so p has degree $k+l$, and l is the multiplicity of 0 as a root), and let $S(A) = span_{\mathbb{Z}}\{I, A, \ldots, A^{k-1}\}$ then*

$$R = (S(A) \times \mathbb{N})/\sim .$$

Proof: First notice that the minimal polynomial of A has integer coefficients. That the characteristic polynomial is monic and has integer coefficients is immediate. We then notice that the minimal polynomial is a factor of the characteristic polynomial and recall that a monic polynomial over the integers that factors over the rationals must factor over the integers. In light of Lemma 4.22 it suffices to show $(S(A) \times \mathbb{N})/\sim = (\mathbb{Z}[A] \times \mathbb{N})/\sim$. Clearly $(S(A) \times \mathbb{N})/\sim \subset (\mathbb{Z}[A] \times \mathbb{N})/\sim$. Now suppose $[X,N] \in (\mathbb{Z}[A] \times \mathbb{N})/\sim$, we show that $[X,N] \in (S(A) \times \mathbb{N})/\sim$. Since $X \in \mathbb{Z}[A]$ and since the degree of the minimal polynomial of A is $k+l$, we know $X = c_{k+l-1}A^{k+l-1} + c_{k+l-2}A^{k+l-2} + \cdots + c_0 I$. By linearity, it suffices to prove the the result for $X = A^i$ for $0 \le i \le k+l-1$. Now if $i \le k-1$, then $X = A^i \in S(A)$ so $[X,N] \in (S(A) \times \mathbb{N})/\sim$. We now complete the proof by induction. Fix j such that $k \le j \le k+l-1$ and suppose the result holds for all $i \leq j$. If $X = A^j$ then

$$\begin{aligned}
[X, N] &= [A^{k+l-j}A^jA^{k+l-j}, N+k+l-j] \\
&= [A^{k+l}A^{k+l-j}, N+k+l-j] \\
&= [A^l(-a_{k-1}A^{k-1} - \cdots - a_0)A^{k+l-j}, N+k+l-j] \\
&= [A^{k+l-j}(-a_{k-1}A^{j-1} - \cdots - a_0A^{j-k})A^{k+l-j}, N+k+l-j] \\
&= [-a_{k-1}A^{j-1} - \cdots - a_0A^{j-k}, N]
\end{aligned}$$

so we have reduced the $X = A^j$ case to the case where all powers are strictly less than j. Thus by induction, the result holds for all powers of A. In other words $[X, N] \in (S(A) \times \mathbb{N})/\sim$ and

$$(S(A) \times \mathbb{N})/\sim = (\mathbb{Z}[A] \times \mathbb{N})/\sim = R.$$

$\square$

The next Lemma shows that, in a certain sense, the description of R in Lemma 4.24 is as good as we can do.

Lemma 4.25. $[c_{k-1}A^{k-1} + \cdots + c_0I, N] = [0, 0]$ *if and only if* $c_i = 0$ *for all* $0 \leq i \leq k-1$.

Proof: If $c_i = 0$ for all i, then clearly $[c_{k-1}A^{k-1} + \cdots + c_0I, N] = [0, N] = [0, 0]$. Now suppose $[c_{k-1}A^{k-1} + \cdots + c_0I, N] = [0, 0]$. In other words, there exists m such that $(c_{k-1}A^{k-1} + \cdots + c_0I)A^{2m} = 0$. But since l is the multiplicity of 0 as a root of the minimal polynomial, it must be true that $(c_{k-1}A^{k-1} + \cdots + c_0I)A^l = 0$. So we have $c_{k-l-1}A^{k+l-1} + \cdots + c_0A^l = 0$. We recall that the minimal polynomial of A has degree $k+l$, so $\{A^{k+l-1}, A^{k+l-2}, \ldots, A^l\}$ is a linearly independent set, and we see that all the c_i's equal 0. $\square$

Lemma 4.26. *Let* $\varphi \in Hom_R(K_0(S), R)$*, there exists* $z \in \mathbb{Z}^{\#V(G)}$ *(considered as a column vector) and* $N \in \mathbb{N}$ *such that, for each* $[v, n] \in K_0(S)$*,*

$$\begin{aligned}
\varphi[v, n] = [&vA^nzA^{k-1} + vA^n(A + a_{k-1}I)zA^{k-2} + \cdots \\
&+ vA^n(A^{k-1} + a_{k-1}A^{k-2} + \cdots + a_1I)zI, N+n].
\end{aligned}$$

We denote this homomorphism by $\varphi_{(z,N)}$*.*

Proof: Let $\{v_i\}_{i=1}^{\#V(G)}$ be the standard basis for $\mathbb{Z}^{\#V(G)}$ and fix $\varphi \in Hom_R(K_0(S), R)$. For each i consider $\varphi[v_i, 0] = [X_i, N_i]$ where N_i is the least integer in the equivalence class. That is to say, if $[Y, M] = [X_i, N_i]$ then $N_i \leq M$. Now define

$$N = \max\{N_i \mid 1 \leq i \leq \#V(G)\}.$$

So for all $v \in \mathbb{Z}^{\#V(G)}$ we can write $\varphi[v, 0] = [X_v, N]$ for some $X_v \in S(A)$. Now recall that $[v, n] * [A, 1] = [v, n+1]$, and $[X_v, N] * [A, 1] = [X_v A, N+1]$, and φ is an R-module homomorphism. So

$$\varphi[v, n] = \varphi\left([v, 0] * [A, 1]^n\right) = \varphi[v, 0] * [A, 1]^n = [X_v, N] \otimes [A, 1]^n = [X_v A^n, N + n].$$

This shows that the number N is important data in describing φ and that φ can be described completely by its restriction to $S(A) \times \{0\}$. Also, for any $[v, n] \in K_0(S)$, $\varphi[v, n] = [Y_v, N + n]$ for some $Y_v \in S(A)$.

Recalling that $S(A)$ is spanned by $\{I, A, \ldots, A^{k-1}\}$ and that φ is a group homomorphism, we see that φ must be of the form

$$\varphi[v_i, 0] = [(x_{k-1})_i A^{k-1} + (x_{k-2})_i A^{k-2} + \cdots + (x_0)_i I, N],$$

for some integers $(x_{k-1})_i, \ldots, (x_0)_i$. It then follows from linearity that for $v \in \mathbb{Z}^{\#V(G)}$ we have

$$\varphi[v, 0] = [v x_{k-1} A^{k-1} + v x_{k-2} A^{k-2} + \cdots + v x_0 I, N],$$

where $x_i \in \mathbb{Z}^{\#V(G)}$ is a column vector for each $0 \leq i \leq k-1$.

We now use the fact that φ is a module homomorphism to impose conditions on the x_n^i. First, notice that $[v, 0] * [A, 0] = [vA, 0]$, and $[X, N] * [A, 0] = [XA, N]$. Now, from above

$$\varphi[vA, 0] = [vAx_{k-1} A^{k-1} + vAx_{k-2} A^{k-2} + \cdots + vAx_0 I, N]$$

but, since $[vA, 0] = [v, 0] * [A, 0]$ and φ is a module homomorphism we have

$$\begin{aligned}
\varphi[vA, 0] &= \varphi([v, 0] * [A, 0]) \\
&= \varphi([v, 0]) * [A, 0] \\
&= [vx_{k-1}A^{k-1} + vx_{k-2}A^{k-2} + \cdots + vx_0 I, N] * [A, 0] \\
&= [vx_{k-1}A^{k} + vx_{k-2}A^{k-1} + \cdots + vx_0 A, N] \\
&= [vx_{k-1}(-a_{k-l}A^{k-1} - \cdots - a_0 I) + vx_{k-2}A^{k-1} + \cdots + vx_0 A, N] \\
&= [v(x_{k-2} - a_{k-1}x_{k-1})A^{k-1} + \cdots + v(x_0 - a_1 x_{k-1})A - a_0 v x_{k-1} I, N].
\end{aligned}$$

Comparing coefficients of like powers of A in these two expressions for $\varphi[vA, 0]$, and noting that $v \in \mathbb{Z}^{\#V(G)}$ was arbitrary, we see (in light of Lemma 4.25) that

$$\begin{aligned}
x_{k-2} - a_{k-1}x_{k-1} &= Ax_{k-1} \\
x_{k-3} - a_{k-2}x_{k-1} &= Ax_{k-2} \\
&\vdots \\
x_0 - a_1 x_{k-1} &= Ax_1 \\
-a_0 x_{k-1} &= Ax_0,
\end{aligned}$$

or

$$\begin{aligned}
x_{k-2} &= (A + a_{k-1}I)x_{k-1} \\
x_{k-3} &= (Ax_{k-2} + a_{k-2}x_{k-1}) = A^l(A^2 + a_{k-1}A + a_{k-2}I)x_{k-1} \\
&\vdots \\
x_0 &= (Ax_1 + a_1 x_{k-1}) = A^l(A^{k-1} + a_{k-1}A^{k-2} + \cdots + a_1 I)x_{k-1} \\
(-a_0 x_{k-1}) &= (A^k + a_{k-1}A^{k-1} + \cdots + a_1 A)x_{k-1}.
\end{aligned}$$

So if we let $z = x_{k-1}$ we have

$$\varphi[v, 0] = [vzA^{k-1} + v(A + a_{k-1}I)zA^{k-2} + \cdots + v(A^{k-1} + a_{k-1}A^{k-2} + \cdots + a_1 I)zI, N].$$

Now consider

$$\begin{aligned}\varphi[v,1] &= \varphi([v,0]*[A,1]) \\ &= [vzA^{k-1}+v(A+a_{k-1}I)zA^{k-2}+\cdots \\ &\quad + v(A^{k-1}+a_{k-1}A^{k-2}+\cdots+a_1 I)zI, N]*[A,1] \\ &= [vzA^{k}+v(A+a_{k-1}I)zA^{k-1}+\cdots+v(A^{k-1}+a_{k-1}A^{k-2}+\cdots+a_1 I)zA, N+1].\end{aligned}$$

Expanding this expression using $[A^K, N+1] = [-a_{k-1}A^{k-1} - \cdots - a_0 I, N+1]$ and simplifying we are left with

$$\begin{aligned}\varphi[v,1] = [&vAzA^{k-1}+vA(A+a_{k-1}I)zA^{k-2}+\cdots \\ &+ vA(A^{k-1}+a_{k-1}A^{k-2}+\cdots+a_1 I)zI, N+1],\end{aligned}$$

Similarly, we can show inductively that

$$\begin{aligned}\varphi[v,n] = [&vA^n zA^{k-1}+vA^n(A+a_{k-1}I)zA^{k-2}+\cdots \\ &+ vA^n(A^{k-1}+a_{k-1}A^{k-2}+\cdots+a_1 I)zI, N+n].\end{aligned}$$

We denote this homomorphism $\varphi_{(z,N)}$. □

Lemma 4.26 shows that each $\varphi \in Hom_R(K_0(S), R)$ is of the form $\varphi_{(z,N)}$ for some $(z,N) \in \mathbb{Z}^{\#V(G)} \times \mathbb{N}$. It is also clear that each $(z,N) \in \mathbb{Z}^{\#V(G)} \times \mathbb{N}$ gives rise to $\varphi_{(z,N)} \in Hom_R(K_0(S), R)$ and that $\varphi_{(z,N)} + \varphi_{(w,N)} = \varphi_{(z+w,N)}$. It remains to be be determined when $\varphi_{(z,N)} = \varphi_{(w,M)}$.

Lemma 4.27. *Let $z,\ w \in \mathbb{Z}^{\#V(G)}$, $N \leq M \in \mathbb{N}$. $\varphi_{(z,N)} = \varphi_{(w,M)}$ if and only if there exists $m \in \mathbb{N}$ such that $A^{2(m+M-N)}z = A^{2m}w$.*

Proof: First suppose there exists $m \in \mathbb{N}$ such that $A^{2(m+M-N)}z = A^{2m}w$. Let $k = m + M - N$, then for all $[v, n] \in K_0(S)$

$$\begin{aligned}
\varphi_{(z,N)}[v,n] &= [vA^n zA^{k-1} + \cdots + vA^n(A^{k-1} + a_{k-1}A^{k-2} + \cdots + a_1 I)zI, N+n] \\
&= [A^{2k}(vA^n zA^{k-1} + \cdots \\
&\quad + vA^n(A^{k-1} + a_{k-1}A^{k-2} + \cdots + a_1 I)zI), M+m+n] \\
&= [vA^n A^{2k} zA^{k-1} + \cdots \\
&\quad + vA^n(A^{k-1} + a_{k-1}A^{k-2} + \cdots + a_1 I)A^{2k}zI, M+m+n] \\
&= [vA^n A^{2m} wA^{k-1} + \cdots \\
&\quad + vA^n(A^{k-1} + a_{k-1}A^{k-2} + \cdots + a_1 I)A^{2m}wI, M+m+n] \\
&= [A^{2m}(vA^n wA^{k-1} + \cdots \\
&\quad + vA^n(A^{k-1} + a_{k-1}A^{k-2} + \cdots + a_1 I)wI), M+m+n] \\
&= [vA^n wA^{k-1} + \cdots + vA^n(A^{k-1} + a_{k-1}A^{k-2} + \cdots + a_1 I)wI, M+n] \\
&= \varphi_{(w,M)}[v,n].
\end{aligned}$$

Now suppose $\varphi_{(z,N)} = \varphi_{(w,M)}$, for each $[v, n] \in K_0(S)$ there exists $m \in \mathbb{N}$ such that

$$\begin{aligned}
&A^{2(m+M-N)}(vA^n zA^{k-1} + \cdots + vA^n(A^{k-1} + a_{k-1}A^{k-2} + \cdots + a_1 I)zI) \\
&\qquad = A^{2m}(vA^n wA^{k-1} + \cdots + vA^n(A^{k-1} + a_{k-1}A^{k-2} + \cdots + a_1 I)wI),
\end{aligned}$$

However, as l is the multiplicity of 0 as a root to the minimal polynomial of A, $A^{l+j}X = A^{l+j}Y$ if and only if $A^l X = A^l Y$, so we can replace m by l to get an expression which is valid for any $[v, n]$. This becomes

$$\begin{aligned}
&[vA^n A^{2(l+M-N)} zA^{k-1} + \cdots + vA^n(A^{k-1} + a_{k-1}A^{k-2} + \cdots + a_1 I)A^{2(l+M-N)}zI, M+l+n] \\
&\quad = [vA^n A^{2l} wA^{k-1} + \cdots + vA^n(A^{k-1} + a_{k-1}A^{k-2} + \cdots + a_1 I)A^{2l}wI, M+l+n].
\end{aligned}$$

Comparing the coefficients of A^{k-1}, in light of Lemma 4.25, we see that

$$vA^n A^{2(l+M-N)}z = vA^n A^{2l}w$$

for any $v \in \mathbb{Z}^{\#V(G)}$, so

$$A^n A^{2(l+M-N)}z = A^n A^{2l}w.$$

∎

Corollary 4.28. *$Hom_R(K_0(S), R)$ is equal to the limit of the following inductive system*

$$\mathbb{Z}^{\#V(G)} \xrightarrow{z\mapsto A^2z} \mathbb{Z}^{\#V(G)} \xrightarrow{z\mapsto A^2z} \mathbb{Z}^{\#V(G)} \xrightarrow{z\mapsto A^2z} \cdots$$

In other words

$$Hom_R(K_0(S), R) \cong (\mathbb{Z}^{\#V(G)} \times \mathbb{N})/\sim_2$$

where, for $N \leq M$, $(z, N) \sim_2 (w, M)$ if and only if there exists $m \in \mathbb{N}$ such that $A^{2(m+M-N)}z = A^{2m}w$.

Proof: Follows immediately from Lemma 4.27 and the comments following Lemma 4.26. ∎

We are now ready to prove our main result.

Proof of Theorem 4.20 To prove that $Hom_R(K_0(S), R) \cong K_0(U)$ as left R-modules we use the characterizations

$$Hom_R(K_0(S), R) \cong (\mathbb{Z}^{\#V(G)} \times \mathbb{N})/\sim_2$$

and

$$K_0(U) \cong (\mathbb{Z}^{\#V(G)} \times \mathbb{N})/\sim_1$$

where, for $N \leq M$, $(z, N) \sim_1 (w, M)$ if and only if there exists $m \in \mathbb{N}$ such that $A^{m+M-N}z = A^m w$. For $(z, N) \in \mathbb{Z}^{\#V(G)} \times \mathbb{N}$ we denote the equivalence class under $\sim_2$ by $[z, N]_2$, and the equivalence class under $\sim_1$ by $[z, N]_1$.

Let $[z, N]_2 \in Hom_R(K_0(S), R)$, $[w, M]_1 \in K_0(U)$. Consider the maps

$$\begin{aligned} \phi : Hom_R(K_0(S), R) &\rightarrow K_0(U), \text{ and} \\ \psi : K_0(U) &\rightarrow Hom_R(K_0(S), R) \end{aligned}$$

given by

$$\begin{aligned} \phi[z, N]_2 &= [z, 2N]_1, \\ \psi[w, M]_1 &= \begin{cases} [w, \frac{M}{2}]_2 & \text{if } M \text{ even} \\ [Aw, \frac{M+1}{2}]_2 & \text{if } M \text{ odd.} \end{cases} \end{aligned}$$

Note that the second part of the above definition of ψ is simply the first part of the definition applied to $[Aw, M+1] = [w, M]$.

We first show that ϕ is a well defined group homomorphism. Let $[z_1, N_1]_2$, $[z_2, N_2]_2$ be in $Hom_R(K_0(S), R)$. To show ϕ is well defined it is enough to consider $[z_1, N_1]_2 = [Az_1, N_1+1]_2$.

$$\phi[z_1, N_1]_2 = [z_1, 2N_1]_1 = [A^2 z_1, 2N_1 + 2] = \phi[Az_1, N_1 + 1]_2.$$

Now, WOLOG assume $N_1 \leq N_2$, then

$$\begin{aligned}
\phi[z_1, N_1]_2 + \phi[z_2, N_2]_2 &= [z_1, 2N_1]_1 + [z_2, 2N_2]_1 \\
&= [A^{2(N_2-N_1)} z_1, 2N_2]_1 + [z_2, 2N_2]_1 \\
&= [A^{2(N_2-N_1)} z_1 + z_2, 2N_2]_1 \\
&= \phi[A^{2(N_2-N_1)} z_1 + z_2, N_2]_2 \\
&= \phi([A^{2(N_2-N_1)} z_1, N_2]_2 + [z_2, N_2]_2) \\
&= \phi([z_1, N_1]_2 + [z_2, N_2]_2).
\end{aligned}$$

So ϕ is a well defined group homomorphism. We now show that ϕ is an isomorphism of groups by showing that $\psi = \phi^{-1}$.

$$\psi(\phi[z, N]_2) = \psi[z, 2N]_1 = [z, N]_2.$$

Now, for $[w, M]_1$, assume M even, otherwise consider $[Aw, M+1]_1$.

$$\phi(\psi[w, M]_1) = \phi[w, \frac{M}{2}]_2 = [w, M]_1.$$

So $\psi = \phi^{-1}$ and ϕ is a group isomorphism. All that is left to check now is that ϕ respects the R-module structure. As R is generated by $[A, 1]$ and $[A, 0]$ is suffices to check only these two elements of R. Fix $\varphi_{[z,N]_2} \in Hom_R(K_0(S), R)$, then for

$[v, n] \in K_0(S)$ we have

$$\begin{aligned}
[A,1] * \varphi_{[z,N]_2}[v,n] &= \varphi_{[z,N]_2}([v,n] * [A,1]) \\
&= \varphi_{[z,N]_2}[v,n+1] \\
&= [vA^{n+1}zA^{k-1} + \cdots \\
&\quad + vA^{n+1}(A^{k-1} + a_{k-1}A^{k-2} + \cdots + a_1 I)zI, N+n+1] \\
&= [vA^n AzA^{k-1} + \cdots \\
&\quad + vA^n(A^{k-1} + a_{k-1}A^{k-2} + \cdots + a_1 I)AzI, N+1+n] \\
&= \varphi_{[Az,N+1]_2}[v,n]
\end{aligned}$$

similarly

$$\begin{aligned}
[A,0] * \varphi_{[z,N]_2}[v,n] &= \varphi_{[z,N]_2}([v,n] * [A,0]) \\
&= \varphi_{[z,N]_2}[vA,n] \\
&= [vA^{n+1}zA^{k-1} + \cdots \\
&\quad + vA^{n+1}(A^{k-1} + a_{k-1}A^{k-2} + \cdots + a_1 I)zI, N+n] \\
&= [vA^n AzA^{k-1} + \cdots \\
&\quad + vA^n(A^{k-1} + a_{k-1}A^{k-2} + \cdots + a_1 I)AzI, N+n] \\
&= \varphi_{[Az,N]_2}[v,n].
\end{aligned}$$

So on $Hom_R(K_0(S), R) \cong (\mathbb{Z}^{\#V(G)} \times \mathbb{N})/\sim_2$ the R-module structure is given by

$$\begin{aligned}
[A,1] * [z,N]_2 &= [Az, N+1]_2, \\
[A,0] * [z,N]_2 &= [Az, N]_2.
\end{aligned}$$

Now

$$[A,1] * \phi[z,N]_2 = [A,1] * [z,2N]_1 = [Az, 2N+2]_1 = \phi[Az, N+1]_2 = \phi([A,1] * [z,N]_2),$$

and

$$[A,0] * \phi[z,N]_2 = [A,0] * [z,2N]_1 = [Az,2N]_1 = \phi[Az,N]_2 = \phi([A,0] * [z,N]_2).$$

So ϕ is an isomorphism of R-modules. $\square$

We conclude this section by presenting one more characterization of the ring R.

Proposition 4.29. *Let (Σ, σ) be a SFT with adjacency matrix A. Let the minimal polynomial of A be $m_A(x) = x^l(x^k + a_{k-1}x^{k-1} + \cdots + a_0)$. Let R be the subring of $K_*(C(H, \alpha))$ described above. Then*

$$R \cong \mathbb{Z}[x, x^{-1}]/ < p_A(x) >,$$

where $p_A(x) = x^k + a_{k-1}x^{k-1} + \cdots + a_0$.

Proof: Consider the map $\phi : \mathbb{Z}[x, x^{-1}] \to R$ given by $\phi(p) = p([A, 0], [A, 1])$. ϕ is clearly onto, so to prove the claim we need only show that $ker(\phi) =< p_A(x) >$. First notice that

$$p_A([A, 0]) = [p_A(A), 0] = [A^l p_A(A)A^l, l] = [m_A(A)A^l, l] = [0, 0].$$

So $< p_A(x) >\subset ker(\phi)$.

Now suppose $p \in ker(\phi)$, where

$$p(x, x^{-1}) = \sum_{i=-m}^{n} c_i x^i.$$

$p(x, x^{-1})$ is in $ker(\phi)$ if and only if $x^m p(x, x^{-1}) = \sum_{i=0}^{n+m} c_{i-m}x^i \in ker(\phi)$. So it suffices to consider elements $q(x) \in ker(\phi)$ of the form $q(x) = c_n x^n + \cdots + c_0$. We then have $q([A, 0]) = [0, 0]$, or

$$[(c_n A^n + \cdots + c_0 I), 0] = [0, 0].$$

Lemma 4.25 then implies that either $c_i = 0$ for all $0 \le i \le n$, in which case $q = 0 \in< p_A(x) >$, or $n \ge k$. Furthermore, we have that, for some m

$$A^m(c_n A^n + \cdots + c_0 I)A^m = A^m 0 A^m = 0,$$

but, as l is the multiplicity of 0 as a root of the minimal polynomial of A, we have

$$A^l(c_n A^n + \cdots + c_0 I) = 0.$$

So the minimal polynomial $m_A(x) = x^l(x^k+a_{k-1}x^{k-1}+\cdots+a_0)$ must divide the polynomial $x^l(c_n x^n + \cdots + c_0 I)$, and hence $p_A(x)|q(x)$. In other words $q(x) = f(x)p_A(x)$

for some $f \in \mathbb{Z}[x, x^{-1}]$ and $q(x) \in < p_A(x) >$. Therefore, $ker(\phi) = < p_A(x) >$ and

$$R \cong \mathbb{Z}[x, x^{-1}] / < p_A(x) > .$$

$\square$

4.3 Shift Equivalence

In this section we show that two SFTs with shift equivalent adjacency matrices have isomorphic ring/module structures of SFTs. Recall from Defn. 2.28, that A and B are shift equivalent with lag k if there exist R and S such that

$$\begin{aligned} RS &= A^k \\ SR &= B^k \\ AR &= RB \\ SA &= BS \end{aligned}$$

Remark 4.30. *Suppose A and B are shift equivalent with lag k, with R, S as in Defn. 2.28. Then*

$$\begin{array}{ccccc} (AR)S & = & A(RS) & = & A^{k+1} \\ S(AR) & = & SRB & = & B^{k+1} \\ A(AR) & = & A(RB) & = & (AR)B \\ SA & = & BS & & \end{array}$$

So A and B are shift equivalent with lag $k+1$, where the equivalence is implemented by AR and S (there are other choices that would also work, for example, R and BS).

Definition 4.31. *For SFTs (Σ_A, σ_A) and (Σ_B, σ_B) with adjacency matrices A, and B respectively, we say*

$$(K_*(C(H_A, \alpha_A)), K_0(S_A), K_0(U_A)) \cong (K_*(C(H_B, \alpha_B)), K_0(S_B), K_0(U_B))$$

if there exist ϕ_H, ϕ_S, ϕ_U such that

1. *$\phi_H : K_*(C(H_A, \alpha_A)) \to K_*(C(H_B, \alpha_B))$ is a ring isomorphism.*

2. *$\phi_S : K_0(S(\Sigma_A, \sigma_A)) \to K_0(S(\Sigma_B, \sigma_B))$ is an order isomorphism.*

3. $\phi_U : K_0(U(\Sigma_A, \sigma_A)) \to K_0(U(\Sigma_B, \sigma_B))$ *is an order isomorphism.*

4. For all $h \in K_*(C(H_A, \alpha_A))$, $s \in K_0(S(\Sigma_A, \sigma_A))$, $u \in K_0(U(\Sigma_A, \sigma_A))$ *we have* $\phi_S(s * h) = \phi_S(s) * \phi_H(h)$ *and* $\phi_U(h * u) = \phi_H(h) * \phi_U(u)$.

By an order isomorphism *we mean an order unit preserving, positive group isomorphism. (see section 2.4).*

Theorem 4.32. *Let* (Σ_A, σ_A) *and* (Σ_B, σ_B) *be SFTs with adjacency matrices* A*, and* B *respectively. If* A *and* B *are shift equivalent, then*

$$(K_*(C(H_A, \alpha_A)), K_0(S_A), K_0(U_A)) \cong (K_*(C(H_B, \alpha_B)), K_0(S_B), K_0(U_B))\,.$$

Proof: Suppose $A \sim_{SE} B$ with R, S and lag k as in Defn. 2.28. Without loss of generality we assume k is even (see remark above). From section 7.5 of [16] we have that ϕ_S, ϕ_U defined by

$$\phi_S[v, N] = [vR, N + \frac{k}{2}] \quad , \phi_U[w, M] = [Sw, M + \frac{k}{2}]$$

are the desired order preserving group isomorphisms. Now for $[X, N]+[Y, M+B(A)] \in K_*(C(H_A, \alpha_A))$ define

$$\phi_H([X, N] + [Y + B(A), M]) = [SXR, N + \frac{k}{2}] + [SYR + B(B), M + \frac{k}{2}].$$

Notice that if $XA = AX$ then

$$SXRB = SXAR = SAXR = BSXR$$

and if $Y = AZ - ZA \in B(A)$, then

$$SYR = SAZR - SZAR = BSZR - SZRB \in B(B),$$

so ϕ_H is well defined. It is also clear that ϕ_H is a group homomorphism. We now show that ϕ_H is a group isomorphism by exhibiting an inverse. Consider the map

$$\psi_H : K_*(C(H_B, \alpha_B)) \to K_*(C(H_A, \alpha_A))$$

defined by

$$\psi_H([X', N'] + [Y' + B(B), M']) = [RX'S, N' + \frac{k}{2}] + [RY'S + B(A), M' + \frac{k}{2}].$$

The same argument as above shows that ψ_H is a well defined group homomorphism. Now for

$$[X, N] + [Y, M + B(A)] \in (K_*(C(H_A, \alpha_A))$$

we have

$$\begin{aligned}
\psi_H \circ \phi_H([X, N] + [Y + B(A), M]) &= \psi_H([SXR, N + \frac{k}{2}] + [SYR + B(B), M + \frac{k}{2}]) \\
&= [RSXRS, N + k] + [RSYRS + B(A), M + k] \\
&= [A^k X A^k, N + k] + [A^k Y A^k + B(A), M + k] \\
&= [X, N] + [Y + B(A), M]
\end{aligned}$$

and for

$$[X', N'] + [Y' + B(B), M'] \in K_*(C(H_B, \alpha_B))$$

we have

$$\begin{aligned}
\phi_H\psi_H([X', N'] + [Y' + B(B), M']) &= \phi_H([RX'S, N' + \frac{k}{2}] + [RY'S + B(A), M' + \frac{k}{2}]) \\
&= [SRX'SR, N' + k] + [SRY'SR + B(B), M' + k] \\
&= [B^k X' B^k, N' + k] + [B^k Y B^k + B(B), M' + k] \\
&= [X', N'] + [Y' + B(B), M'].
\end{aligned}$$

Thus $\psi_H = \phi_H^{-1}$ and ϕ_H is a group isomorphism. We will now show that ϕ_H is a ring isomorphism. Fix $[X_1, N_1]$, $[X_2, N_2] \in K_0(C(H_A, \alpha_A))$, $[Y + B(A), M] \in$

$K_1(C(H_A, \alpha_A))$, then

$$\begin{aligned}
\phi_H([X_1, N_1] * [X_2, N_2]) &= \phi_H[X_1X_2, N_1 + N_2] \\
&= [SX_1X_2R, N_1 + N_2 + \frac{k}{2}] \\
&= [B^{\frac{k}{2}}SX_1X_2RB^{\frac{k}{2}}, N_1 + N_2 + k] \\
&= [SA^{\frac{k}{2}}X_1X_2A^{\frac{k}{2}}R, N_1 + N_2 + k] \\
&= [SX_1A^kX_2R, N_1 + N_2 + k] \\
&= [SX_1RSX_2R, N_1 + N_2 + k] \\
&= [SX_1R, N_1 + \frac{k}{2}] * [SX_2R, N_2 + \frac{k}{2}] \\
&= \phi_H[X_1, N_1] * \phi_H[X_2, N_2]
\end{aligned}$$

and

$$\begin{aligned}
\phi_H([X_1, N_1] * [Y + B(A), M]) &= \phi_H[X_1Y + B(A), N_1 + M] \\
&= [SX_1YR + B(B), N_1 + M + \frac{k}{2}] \\
&= [B^{\frac{k}{2}}SX_1YRB^{\frac{k}{2}} + B(B), N_1 + M + k] \\
&= [SA^{\frac{k}{2}}X_1YA^{\frac{k}{2}}R + B(B), N_1 + M + k] \\
&= [SX_1A^kYR + B(B), N_1 + M + k] \\
&= [SX_1RSYR + B(B), N_1 + M + k] \\
&= [SX_1R, N_1 + \frac{k}{2}] * [SYR + B(B), M + \frac{k}{2}] \\
&= \phi_H[X_1, N_1] * \phi_H[Y + B(A), M].
\end{aligned}$$

The calculation $\phi_H([Y + B(A), M] * [X_1, N_1]) = \phi_H[Y + B(A), M] * \phi_H[X_1, N_1]$ is completely analogous. We have thus shown that ϕ_H is a ring isomorphism.

Finally, we must show that ϕ_H, ϕ_S, and ϕ_U preserve the module structure. Fix

$[X,N] \in C(H_A, \alpha_A)$, $[v,M] \in K_0(S(\Sigma_A, \sigma_A))$, and $[w,K] \in K_0(U(\Sigma_A, \sigma_A))$, then

$$\begin{aligned}
\phi_S([v,M] * [X,N]) &= \phi_S[vX, M+2N] \\
&= [vXR, M+2N+\frac{k}{2}] \\
&= [vXRB^k, M+2N+\frac{3k}{2}] \\
&= [vXA^kR, M+2N+\frac{3k}{2}] \\
&= [vA^kXR, M+2N+\frac{3k}{2}] \\
&= [vRSXR, M+2N+\frac{3k}{2}] \\
&= [vR, M+\frac{k}{2}] * [SXR, N+\frac{k}{2}] \\
&= \phi_S[v,M] * \phi_H[X,N].
\end{aligned}$$

Similarly

$$\begin{aligned}
\phi_U([X,N] * [w,K]) &= \phi_S[Xw, K+2N] \\
&= [SXw, K+2N+\frac{k}{2}] \\
&= [B^kSXw, K+2N+\frac{3k}{2}] \\
&= [SA^kXw, K+2N+\frac{3k}{2}] \\
&= [SXA^kw, K+2N+\frac{3k}{2}] \\
&= [SXRSw, K+2N+\frac{3k}{2}] \\
&= [SXR, N+\frac{k}{2}] * [Sw, K+\frac{k}{2}] \\
&= \phi_H[X,N] * \phi_U[w,K].
\end{aligned}$$

So the module structure is preserved and

$$(K_*(C(H_A, \alpha_A)), K_0(S_A), K_0(U_A)) \cong (K_*(C(H_B, \alpha_B)), K_0(S_B), K_0(U_B)),$$

$\square$

4.4 Examples

4.4.1 Example 1

We begin with an example in which the ring $K_0(C(H,\alpha))$ is non-commutative. Consider the SFT, (Σ,σ) with adjacency matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

We can immediately compute the K-theory of the algebras $S(\Sigma,\sigma,P)$ and $U(\Sigma,\sigma,P)$.

$K_0(S(\Sigma,\sigma,P))$ is the limit of the following inductive system:

$$\mathbb{Z}^3 \xrightarrow{v\mapsto vA} \mathbb{Z}^3 \xrightarrow{v\mapsto vA} \mathbb{Z}^3 \longrightarrow \cdots$$

Fix a basis for $\mathbb{Z}^3$ as follows.

$$\begin{aligned} v_1 &= [1\ -1\ 0] \\ v_2 &= [0\ 1\ -1] \\ v_3 &= [1\ 0\ 0] \end{aligned}$$

Now notice that

$$\begin{aligned} v_1A &= [1\ -1\ 0] &= v_1 \\ v_2A &= [0\ 1\ -1] &= v_2 \\ v_3A &= [2\ 1\ 1] &= 4v_3 - 2v_1 - v_2 \end{aligned}$$

So the subgroup generated by $v_1,\ v_2$ is invariant for the automorphism $v \mapsto vA$, and moreover,

$$\lim_{\to} < v_1, v_2 > \cong < v_1, v_2 > \cong \mathbb{Z}^2,$$

and this is a subgroup of $K_0(S)$. Moreover,

$$K_0(S(\Sigma,\sigma,P))/\mathbb{Z}^2 \cong \lim_{\to}(\mathbb{Z}^3/\mathbb{Z}^2) \cong \mathbb{Z}[1/4].$$

In other words we have the following exact sequence of abelian groups

$$0 \longrightarrow \mathbb{Z}^2 \longrightarrow K_0(S(\Sigma,\sigma,P)) \longrightarrow \mathbb{Z}[1/4] \longrightarrow 0\,.$$

We can also write, as in exercise 7.5.2 of [16],

$$K_0(S(\Sigma, \sigma, P)) = \{rw + sv_1 + tv_2 \; | w = [1\ 1\ 1],\ s,\ t \in \mathbb{Z}/3,\ r+t,\ r-s \in \mathbb{Z}[1/4]\}.$$

As A is symmetric, $K_0(U(\Sigma, \sigma, P)) = K_0(S(\Sigma, \sigma, P))$.

We now compute $K_0(C(H, \alpha))$. First note that the centralizer of A, $C(A)$, is equal to the centralizer of the matrix with 1's in every entry (ie the matrix $A - I$). The centralizer of this matrix is easily seen to be the set of matrices with the property that all of the column sums and all of the row sums are equal. It is straightforward to check that the set of matrices $\{X_1, \ldots, X_5\}$ below span $C(A)$.

$$X_1 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad X_2 = \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad X_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$$

$$X_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \qquad X_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

We then consider the inductive system

$$C(A) \xrightarrow{X \mapsto AXA} C(A) \xrightarrow{X \mapsto AXA} C(A) \longrightarrow \cdots$$

We notice that

$$\begin{aligned}
AX_1A &= \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = X_1 = AX_1 = X_1A \\
AX_2A &= \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = X_2 = AX_2 = X_2A \\
AX_3A &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix} = X_3 = AX_3 = X_3A \\
AX_4A &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} = X_4 = AX_4 = X_4A \\
AIA &= \begin{bmatrix} 6 & 5 & 5 \\ 5 & 6 & 5 \\ 5 & 5 & 6 \end{bmatrix} = 16X_5 - 10X_1 - 10X_4 - 5X_2 - 5X_3.
\end{aligned}$$

So we again have an exact sequence of abelian groups:

$$0 \longrightarrow \mathbb{Z}^4 \longrightarrow K_0(C(H,\alpha)) \longrightarrow \mathbb{Z}[1/2] \longrightarrow 0$$

We now compute the product structure for $K_0(C(H,\alpha))$. As $AX_iA = X_i$ for $1 \leq i \leq 4$ we have $[X_i, N] = [X_i, 0]$. Now for $i, j \in \{1,2,3,4\}$, we compute

$$\begin{aligned}
[X_i, 0] * [X_j, 0] &= [X_iX_j, 0] \\
[I, N] * [X_i, M] &= [X_i, N+M] = [X_i, 0] \\
[I, N] * [I, M] &= [I, N+M].
\end{aligned}$$

The subgroup $< X_1, X_2, X_3, X_4 > \cong \mathbb{Z}^4$ is in fact an ideal. Computing the products

X_iX_j we get the (non-commutative) multiplication table for the ideal:

	X_1	X_2	X_3	X_4
X_1	$2X_1$	$2X_2$	$-X_1$	$-X_2$
X_2	$-X_1$	$-X_2$	$2X_1$	$2X_2$
X_3	$2X_3$	$2X_4$	$-X_3$	$-X_4$
X_4	$-X_3$	$-X_4$	$2X_3$	$2X_4$

We notice also that the quotient $K_0(C(H,\alpha))/\mathbb{Z}^4 \cong \mathbb{Z}[1/2]$ is a ring with the usual product on $\mathbb{Z}[1/2]$. This can be seen by noticing that under the isomorphism $[I,0]+\mathbb{Z}^4 \mapsto 1$. It then follows that $[I,N]+\mathbb{Z}^4 \mapsto \frac{1}{4^{2N}}$ Now from above we see that

$$([I,N]+\mathbb{Z}^4) * ([I,M]+\mathbb{Z}^4) = [X_5, N+M]+\mathbb{Z}^4 = \frac{1}{4^{2N+2M}} = \frac{1}{4^{2N}}\frac{1}{4^{2M}}.$$

We now consider $K_0(S)$ as a module over the ring R as in section 4.2. The minimal polynomial of A is $m_A(x) = x^2 - 5x + 4$, so as in Prop. 4.24, the ring R is the limit of the following system

$$S(A) \xrightarrow{X \mapsto AXA} S(A) \xrightarrow{X \mapsto AXA} S(A) \longrightarrow \cdots$$

where $S(A) = span_{\mathbb{Z}}\{I, A\}$. In this particular case, it will be more convenient to write $S(A) = span_{\mathbb{Z}}\{A-I, A\}$, this is because

$$\begin{aligned} A(A-I)A &= 16(A-I) \\ A(A)A &= 20(A-I)+A. \end{aligned}$$

Furthermore

$$[A-I,N] * [A-I,M] = [3(A-I), N+M],$$

so we have the following exact sequence.

$$0 \longrightarrow \mathbb{Z}[\tfrac{1}{2}] \longrightarrow R \longrightarrow \mathbb{Z} \longrightarrow 0$$

where multiplication on the ideal $\mathbb{Z}[\frac{1}{2}]$ is given by

$$\frac{a}{2^{4n}} * \frac{b}{2^{4m}} = \frac{3ab}{2^{4(n+m)}}.$$

We can write this ring a more familiar form by noticing that the map ϕ defined by $\phi[A-I,N] = 3\cdot 2^{-4N}$ gives an isomorphism with the ring $3\mathbb{Z}[\frac{1}{2}]$, with the usual product.

Now, for $[v,N]\in K_0(S(\Sigma,\sigma,P))$, $[z,2M]\in K_0(U(\Sigma,\sigma,P))$ we have

$$[z,2M]\mapsto \varphi_{[z,M]}\in Hom_R(K_0(S(\Sigma,\sigma,P),R)$$

and

$$\varphi_{[z,M]}[v,N] = [vA^N zA + vA^N(A-5I)zI, N+M]$$

4.4.2 Example 2

In the previous example, the ring R that is generated by the action of α is also the center of $K_0(C(H,\alpha))$. We now show that while it is always true that $R\subseteq Z(K_0(C(H,\alpha)))$, the containment can be strict. We provide two examples in which $R\neq Z(K_0(C(H,\alpha)))$, one in which $Hom_Z(K_0(S(\Sigma,\sigma,P)),Z)$ is isomorphic to $K_0(U)$, and one in which it is not.

Let us first describe the center of the ring $K_0(C(H,\alpha))$. The following is a well known result from linear algebra (eg pg. 113 of [10]).

Lemma 4.33. *Let A be an $N\times N$ matrix over a field $\mathbb{F}$, the set of matrices that commute with A, $C(A)$, is a subring of $M_N(\mathbb{F})$. The center of the ring $C(A)$ is $\mathbb{F}[A]$. In other words, the double commutant of A is exactly the set of all polynomials in A.*

We can now describe the center $Z = Z(K_0(C(H,\alpha)))$.

Lemma 4.34. *For $A\in M_N(\mathbb{Z})$, $C(A)$ as above, the center of $C(A)$ is $\mathbb{Q}[A]\cap M_N(\mathbb{Z})$. We denote this by $Z(A)$.*

Proof: We use Lemma 4.33 with the field $\mathbb{Q}$. We use $C_{\mathbb{Z}}(A)$ and $C_{\mathbb{Q}}(A)$ to distinguish between the integer matrices that commute with A, and the rational matrices.

First suppose $X\in M_N(\mathbb{Z})$ commutes with all $Y\in C_{\mathbb{Z}}(A)$. Then for each $Y'\in C_{\mathbb{Q}}(A)$ there exists $r\in\mathbb{Q}$ and $Y\in C_{\mathbb{Z}}(A)$ such that $Y'=rY$, thus X commutes with all $Y'\in C_{\mathbb{Q}}(A)$, so $X\in\mathbb{Q}[A]$.

Now suppose $X\in\mathbb{Q}[A]\cap M_N(\mathbb{Z})$. X commutes with all $Y'\in C_{\mathbb{Q}}(A)$ and hence with $Y\in C_{\mathbb{Z}}(A)$. $\square$

Proposition 4.35. *Let (Σ, σ) be a SFT with adjacency matrix A. The center of $K_0(C(H,\alpha))$, Z, is the limit of the following inductive system.*

$$Z(A) \xrightarrow{X \mapsto AXA} Z(A) \xrightarrow{X \mapsto AXA} \cdots$$

Proof: Suppose $[X,n] \in Z$, so for any $[Y,m] \in K_0(C(H,\alpha))$, $[X,n] * [Y,m] = [Y,m] * [X,n]$. In particular

$$[XY, n+m] = [YX, n+m],$$

so for some k we have $XYA^{2k} = YXA^{2k}$. Moreover, if the (algebraic) multiplicity of 0 as an eigenvalue of A is l, we can choose k above such that $k \leq l$ (can actually do $2k \leq l$). So for all $Y \in C(A)$ we have $XYA^{2l} = YXA^{2l}$. Rewriting this as $A^l XA^l Y = YA^l XA^l$ and recalling that this holds for all $Y \in C(A)$ we see that $A^l XA^l$ is in the double commutant of A. Now by Lemma 4.34 we have $A^l XA^l \in Z(A)$ and hence $[X,n] = [A^l XA^l, n+l] \in \lim_{\rightarrow} Z(A)$. So we have shown $Z \subset \lim_{\rightarrow} Z(A)$.

Now suppose $[X,n] \in \lim_{\rightarrow} Z(A)$ and let $[Y,m] \in K_0(C(H,\alpha))$. Then

$$[X,n] * [Y,m] = [XY, n+m] = [YX, n+m] = [Y,m] \otimes [X,n],$$

where the middle equality holds since $X \in Z(A)$ and $Y \in C(A)$, so $XY = YX$. Thus $[X,n] \in Z$, and $\lim_{\rightarrow} Z(A) \subset Z$. Thus $Z = \lim_{\rightarrow} Z(A)$. $\square$

We now give an example in which $Z \neq R$, but we still have an isomorphism

$$Hom_Z(K_0(S), Z) \cong K_0(U).$$

Example 2(a) Consider the SFT with adjacency matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

As in our first example, $C(A)$ consists of all matrices with equal row and column sums. $C(A)$ is spanned by I and $\frac{1}{2}(A-I)$. The minimal polynomial of A is $m_A(x) = x^2 - 2x - 3$ which is also the characteristic polynomial of A, thus $C(A) = Z(A)$. We also notice that

$$A\frac{1}{2}(A-I)A = \frac{1}{2}(A^3 - A^2) = \frac{1}{2}(5A+3I) = \frac{1}{2}(A-I) + 2A + 2I,$$

so $[\frac{1}{2}(A-I),0] \neq [X,N]$ for any $[X,N] \in R$. I.e. $Z \neq R$.

We now show that $Hom_Z(K_0(S), Z) \cong K_0(U)$.

As is the proof of Lemma 4.26, we can show that, for $\varphi \in Hom_Z(K_0(S), Z)$ there is $z \in (\mathbb{Z}/2)^2$, $N \in \mathbb{N}$ such that

$$\varphi[v,n] = [vA^n zA + vA^n(A-2I)zI, N+n],$$

with the extra condition on z that

$$vA^n z + vA^n(A-2I)z \in \mathbb{Z}$$

for all $v \in \mathbb{Z}^2$, $n \in \mathbb{N}$. However, a quick check shows that this is satisfied for any $z \in (\mathbb{Z}/2)^2$, and as in Cor. 4.28 have that $Hom_Z(K_0(S), Z)$ is the inductive limit of the following system.

$$(\mathbb{Z}/2)^2 \xrightarrow{z \mapsto zA^2} (\mathbb{Z}/2)^2 \longrightarrow \cdots$$

Now, the isomorphism $\phi : Hom_Z(K_0(S), Z) \to Hom_R(K_0(S), R)$ is simply given by

$$\phi[z,n] = [2z, n].$$

Thus

$$Hom_Z(K_0(S), Z) \cong Hom_R(K_0(S), R) \cong K_0(U).$$

In the next example $Hom_Z(K_0(S), Z) \neq Hom_R(K_0(S), R)$.

Example 2(b) Consider the SFT with adjacency matrix

$$A = \begin{bmatrix} 0 & 1 & 5 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

As in example 2(a), the characteristic polynomial is equal to the minimal polynomial,

$$m_A(x) = x^3 - 7x - 6 = (x+1)(x+2)(x-3).$$

Now

$$\frac{1}{2}(A^2+A)=\begin{bmatrix} 3 & 3 & 3 \\ 1 & 1 & 3 \\ 1 & 1 & 3 \end{bmatrix}$$

and $Z(A) = span\{I, A, \frac{1}{2}(A^2+A)\}$. Moreover

$$A\frac{1}{2}(A^2+A) = \frac{1}{2}(A^2+A) + 3A + 3I$$

so

$$Z = \lim_{\rightarrow} Z(A) \neq \lim_{\rightarrow} span\{I, A, A^2\} = R.$$

We now show that $Hom_Z(K_0(S), Z) \neq Hom_R(K_0(S), R)$. Let $\varphi \in Hom_Z(K_0(S), Z)$, there exists $z \in (\mathbb{Z}/2)^3$, $N \in \mathbb{N}$ such that

$$\varphi[v,n] = [vA^n zA^2 + vA^{n+1}zA + vA^n(A^2 - 7I)zI, N+n],$$

with the additional conditions on z that

$$vA^n z + vA^{n+1}z, \text{ and } vA^n(A^2-5I)z \text{ are in } \mathbb{Z}.$$

One readily checks that the set of all such z is

$$\mathcal{Z} = span_{\mathbb{Z}}\{\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}\} = span_{\mathbb{Z}}\{\begin{bmatrix} \frac{1}{2} \\ \frac{-1}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}\}$$

and $Hom_Z(K_0(S), Z)$ is the limit of

$$\mathcal{Z} \xrightarrow{z \mapsto zA} \mathcal{Z} \longrightarrow \cdots$$

which is not isomorphic to $K_0(U(\Sigma, \sigma, P))$ as a left Z module. To see this, note that any $[z, N] \in Hom_Z(K_0(S), Z)$ can be written as $[z, N] = [z, 0] * [A^N, N]$, so we restrict to looking at elements of the form $[z, 0]$. Suppose $\phi : Hom_Z(K_0(S), Z) \to K_0(U(\Sigma, \sigma, P))$ is a module homomorphism, we show that it cannot be an isomorphism. Let u_1, u_2, u_3 denote the three vectors in the second description of $\mathcal{Z}$ above.

$$\begin{aligned} Au_1 &= -u_1, \\ Au_2 &= 3u_2, \\ Au_3 &= 3u_2 - 2u_3. \end{aligned}$$

Now suppose $(\phi[u_i, 0]) = [v_i, n]$ (we can choose equivalence class representatives such that n does not depend on i), we know that

$$[Av_1, n] = [v_1, n] * [A, 0] = (\phi[u_1, 0]) * [A, 0] = \phi[Au_1, 0] = [-v_1, n].$$

Now, as $v_1 \in \mathbb{Z}^3$, and $Av_1 = -v_1$ we must have $v_1 = [1\ -1\ 0]^T$, or an integer multiple thereof. We similarly find that $[Av_2, n] = [3v_2, n]$, which implies $v_2 = [2\ 1\ 1]$. Finally $[Av_3, n] = [3v_2 - 2v_3, n]$ from which it follows that

$$v_3 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + k \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} \text{ for some } k \in \mathbb{Z}.$$

A quick check shows that there is no choice for k such that $[1\ 0\ 0]^T \in span_{\mathbb{Z}}\{v_1, v_2, v_3\}$, which is to say

$$\left[\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, n \right] \notin image(\phi)$$

and hence ϕ is not an isomorphism.

To summarize, we have shown that the subring R generated by the action of α need not equal the center $Z = Z(K_0(C(H, \alpha)))$, and moreover, when $R \neq Z$, $Hom_Z(K_0(S(\Sigma, \sigma, P)), Z)$ is isomorphic to $K_0(U(\Sigma, \sigma, P))$ in some, but not all cases.

Chapter 5

Measures and Traces

Recall from section 2.2.3 the Bowen measure and its description as a product of measures supported on stable and unstable sets. In this chapter we discuss how integration against these measures defines traces on the algebras $S(X,\varphi,P)$, $U(X,\varphi,P)$, $H(X,\varphi)$, and $C(H,\alpha)$. Furthermore, we show that the trace is a ring homomorphism from $K_0(C(H,\alpha))$ into $\mathbb{R}$ and that it respects the module structures of $K_0(S(X,\varphi,P))$ and $K_0(U(X,\varphi,P))$ over $K_0(C(H,\alpha))$. A key tool will be the resolving maps described in section 2.2.2. We begin with a concrete description of the measures of interest in the SFT case, and some results on how we may use resolving maps to describe the measures on a general irreducible Smale space in terms of the SFT case.

5.1 Measures

In this section we review the definitions and constructions of the Bowen measure for an irreducible Smale space (see section 2.2.3). We start with a SFT, in which case we can explicitly write down the Bowen measure μ, as well as the measures μ^s and μ^u which we defined in section 2.2.3. We then use resolving maps and the results of section 2.2.2 to describe these three measures for more general Smale spaces. We also provide a new construction of the Bowen measure arising from homoclinic points.

5.1.1 Measures for SFT

For an irreducible SFT, (Σ,σ) with $N\times N$ adjacency matrix A and graph G, we can explicitly write down the measures μ_Σ, μ^s_Σ, and μ^u_Σ which we defined in section 2.2.3.

In this case the measure μ_Σ is often called the Parry measure or the Shannon-Parry measure. See for example, [12], [5], or [29] for more on the Parry measure.

From the Perron-Frobenius Theorem (eg. Thm. 4.2.3 in [16]), there exists an eigenvalue λ of A such that

1. $\lambda > 0$, and $\lambda \geq |\lambda'|$, where λ' is any other eigenvalue of A.

2. The multiplicity of λ is one, so there is a unique (up to a scalar) right (column) eigenvector associated with λ, which we will denote u_r. Furthermore $u_r > 0$ (componentwise).

3. Similarly there exists a unique left (row) eigenvector $u_l > 0$ associated with λ.

(recall from section 2.2.1 that $\log(\lambda)$ is the topological entropy). We normalize u_r and u_l such that the inner product $u_l u_r = \sum_i u_l(i)u_r(i) = 1$. The vector

$$p = (u_l(1)u_r(1), \ldots, u_l(N)u_r(N))$$

is a probability vector and the matrix

$$P_{ij} = \lambda^{-1} A_{ij} u_r(j) u_r(i)^{-1}$$

is stochastic. In other words, P_{ij} gives the probability that the $(n+1)$th vertex in a path on G is vertex j, given that the nth vertex is i. Now, since there are A_{ij} edges from vertex i to j, given an edge ξ with $i(\xi) = i$, $t(\xi) = j$ the probability that the the vertex i is followed by the edge ξ is

$$\tilde{P}_{ij} = \frac{P_{ij}}{A_{ij}} = \lambda^{-1} u_r(j) u_r(i)^{-1}.$$

For completeness we define $\tilde{P}_{ij}$ to be 0 whenever $A_{ij} = 0$.

Now for $k \leq l$ and $\xi = (\xi_k, \ldots, \xi_l)$ a path of length $(l - k + 1)$ in G, we define the cylinder set

$$U(k, l, \xi) = \{x \in X \mid x_i = \xi_i, \text{ for all } k \leq i \leq l\}.$$

We then define the measure on a cylinder set by

$$\begin{aligned}
\mu(U(k,l,\xi)) &= p(i(\xi_k))P_{i(\xi_k),t(\xi_k)}P_{i(\xi_{k+1}),t(\xi_{k+1})}\cdots P_{i(\xi_l),t(\xi_l)} \\
&= \lambda^{k-l-1}u_l(i(\xi_k))u_r(i(\xi_k))u_r(t(\xi_k))u_r(i(\xi_k))^{-1}\cdots u_r(t(\xi_l))u_r(i(\xi_l))^{-1} \\
&= \lambda^{k-l-1}u_l(i(\xi_k))u_r(t(\xi_l)).
\end{aligned}$$

In particular, for the set $V_{n,v_i,v_j}(\xi)$ we have

$$\mu(V_{n,,v_i,v_j}(\xi)) = \lambda^{-2n}u_l(i)u_r(j).$$

Recalling the local product structure on X, we have $V_{n,v_i,v_j}(\xi) \cong V^s_{n,v_i}(\xi) \times V^u_{n,v_j}(\xi)$. Where

$$\begin{aligned}
V^s_{n,v_i}(\xi) &= \{x \in X \mid x_i = \xi_i \text{ for all } i \geq -n\} \\
V^u_{n,v_j}(\xi) &= \{x \in X \mid x_i = \xi_i \text{ for all } i \leq n\}
\end{aligned}$$

We are now ready to define the measures $\mu^{s,\xi}$ and $\mu^{u,\xi}$. We have

$$\mu^{s,\xi}(V^s_{n,v_i}(\xi)) = \lambda^{-n}u_l(i)$$

and

$$\mu^{u,\xi}(V^u_{n,v_j}(\xi)) = \lambda^{-n}u_r(j).$$

Moreover, for any $x \in V_{n,,v_i,v_j}(\xi)$ we actually have

$$\mu^{s,x}(V^s_{n,v_i}(\xi)) = \lambda^{-n}u_l(i)$$

and

$$\mu^{u,x}(V^u_{n,v_j}(\xi)) = \lambda^{-n}u_r(j).$$

So, in particular, for any $x \in V_{n,,v_i,v_j}(\xi)$

$$\mu(V_{n,,v_i,v_j}(\xi)) = \mu^{s,x}(V^s_{n,v_i}(\xi))\mu^{u,x}(V^u_{n,v_j}(\xi)) = \lambda^{-2n}u_l(i)u_r(j).$$

It is straightforward to verify that these measures satisfy the conditions stated in section 2.2.3.

5.1.2 Irreducible Smale Space

We now wish to use the resolving maps to obtain the Bowen measure (and its stable and unstable components) from the above description for a SFT.

Let (x, φ) be an irreducible Smale space. As in Cor. 1.4 of [21] we can find a Smale space (Y, ψ), a SFT (Σ, σ), and factor maps $\pi_1 : \Sigma \to Y$, $\pi_2 : Y \to X$ such that

1. (Σ, σ) and (Y, ψ) are irreducible,
2. π_1 and π_2 are almost one-to-one,
3. π_1 is s-resolving and π_2 is u-resolving.

As in section 2.2.3, we can obtain the Bowen measures on Y and X by pushing forward the measure on Σ. This only requires that π_1, π_2 are almost one-to-one, not that they are resolving. Let μ_Σ, μ_Σ^s, and μ_Σ^u be the measures defined above on the SFT. Then

1. for $E \subset Y$ the Bowen measure on (Y, ψ) is $\mu_Y(E) = \mu_\Sigma(\pi_1^{-1}(E))$,
2. for $F \subset X$ the Bowen measure on (X, φ) is $\mu_X(F) = \mu_Y(\pi_2^{-1}(F)) = \mu_\Sigma((\pi_2 \circ \pi_1)^{-1}(F))$.

We now wish to define the measures on the stable and unstable foliations in (Y, ψ) and (X, φ), from μ_Σ^s, μ_Σ^u, π_1, and π_2. This is where it is important that our factor maps are resolving. We begin by stating the following 2 results which are proved in [20], [21].

Proposition 5.1. *Let (Y, ψ) and (X, φ) be irreducible Smale spaces, and $\pi : Y \to X$ be an almost one-to-one u-resolving factor map. If $x \in X$ with $\pi^{-1}(x) = \{y_1, y_2, \ldots, y_n\}$ then*

$$\pi^{-1}(V^u(x)) = \bigcup_{i=1}^{n} V^u(y_i),$$

and the union is disjoint. Moreover, for each $1 \le i \le n$

$$\pi|_{V^u(y_i)} : V^u(y_i) \to V^u(x)$$

is a homeomorphism.

Proposition 5.2. *For $y \in Y$, $V^s(y)$ is totally disconnected.*

Recall that sets of the form $V^s(y', \epsilon)$, where $y' \in V^s(y)$, $\epsilon \leq \epsilon_Y$ form a neighbourhood base for the topology on $V^s(y)$. Similarly, sets $V^u(y', \epsilon)$, $y' \in V^u(y)$ form a neighbourhood base for the topology on $V^u(y)$, and sets of the form $[V^u(y', \epsilon), V^s(y', \epsilon)]$ form a neighbourhood base for the topology on Y.

Lemma 5.3. *Let (Y, ψ) and (X, φ) be irreducible Smale spaces, and $\pi : Y \to X$ be an almost one-to-one u-resolving factor map. Fix $y \in Y$, the set $\{y' \in V^s(y) \mid \pi(y') = \pi(\tilde{y})$ for some $\tilde{y} \neq y'\}$ has μ_Y^s measure zero. In other words, $\pi|_{V^s(y)}$ is one-to-one μ_Y^s almost everywhere.*

Proof: As Y is compact, we may cover Y with a finite number of sets of the form $U_i = [V^u(z, \delta), V^s(z, \delta)]$. Fix U_i and $y \in U_i$, let $A_i = [V^u(z, \delta), y]$, $B_i = [y, V^s(z, \delta)]$, so we can write $U_i = [A_i, B_i]$.

Let $S_i = \{y' \in V^s(y, \epsilon) \cap B_i \mid \pi(y') = \pi(\tilde{y})$ for some $\tilde{y} \neq y'\}$. Since π is u-resolving, the set $U_i \cap \{y' \in Y \mid \pi(y') = \pi(z)$ for some $z \neq y'\} = [A_i, S_i]$. Now, we know that π is 1-to-1 μ_Y almost everywhere, so

$$0 = \mu_Y([A_i, S_i]) = \mu_Y^{u,y}(A_i)\mu_Y^{s,y}(S_i).$$

We also know that

$$0 \neq \mu_Y(U_i) = \mu_Y([A_i, B_i]) = \mu_Y^{u,y}(A_i)\mu_Y^{s,y}(B_i).$$

So $\mu_Y^{u,y}(A_i) \neq 0$ and thus $\mu_Y^{s,y}(S_i) = 0$. The conclusion follows. □

Let (Σ, σ), (Y, ψ), (X, φ), π_1, and π_2 be as above. Let $x \in X$ and $V^s(x_1, \delta) \subset V^s(x, \epsilon)$, $V^u(x_2, \delta) \subset V^u(x, \epsilon)$. Fix $y \in Y$ and $U(y) \subset V^u(y)$ such that $\pi_2(y) = x_2$, $\pi_2(U(y)) = V^u(x_2, \delta)$. Define measures on $V^s(x)$, $V^u(x)$ locally by

$$\begin{aligned} \mu_X^{s,x}(V^s(x_1, \delta)) &= \mu_Y^{s,y}(\pi_2^{-1}(V^s(x_1, \delta))) \\ \mu_X^{u,x}(V^u(x_2, \delta)) &= \mu_Y^{u,y}(U(y)) \end{aligned}$$

Proposition 5.4. *The measures defined above satisfy the conditions stated in section 2.2.3.*

Proof: Let $x \in X$ and let $B = V^s(x_1, \delta) \subset V^s(x, \epsilon)$, $A = V^u(x_2, \delta) \subset V^u(x, \epsilon)$. Fix $y \in Y$ and $U(y) \subset V^u(y)$ such that $\pi_2(y) = x_2$, $\pi_2(U(y)) = V^u(x_2, \delta) = A$. We need to show

1. For z close to x, $\mu_X^{u,x}(A) = \mu_X^{u,z}([A, z])$
2. $\mu_X^{u,\varphi(x)}(\varphi(A)) = \lambda \mu_X^{u,x}(A)$
3. $\mu_X([A, B]) = \mu_X^{u,x}(A) \mu_X^{s,x}(B)$
4. For z close to x, $\mu_X^{s,x}(B) = \mu_X^{s,z}([x, B])$
5. $\mu_X^{s,\varphi(x)}(\varphi(B)) = \lambda^{-1} \mu_X^{s,x}(B)$.

1. We can find $y' \in \pi^{-1}(z)$ such that y' is 'close' to y. Then,

$$\mu_X^{u,x}(A) = \mu_Y^{u,y}(U(y)) = \mu_Y^{u,y'}([U(y), y']) = \mu_X^{u,z}(\pi([U(y), y']))$$

but $\pi([U(y), y']) = [\pi(U(y)), \pi(y')]$ and $\pi(U(y)) = A$, $\pi(y') = z$ so we have

$$\mu_X^{u,x}(A) = \mu_X^{u,z}(\pi([U(y), y'])) = \mu_X^{u,z}([\pi(U(y)), \pi(y')]) = \mu_X^{u,z}([A, z]).$$

2. $$\mu_X^{u,\varphi(x)}(\varphi(A)) = \mu_Y^{u,\psi(y)}(\psi(U(y))) = \lambda \mu_Y^{u,y}(U(y)) = \lambda \mu_X^{u,x}(A).$$

3. Recall that for any $w \in \Sigma$, $V^s(w)$ has a neighbourhood base which consists of sets which are compact and open. Furthermore, any open set can be written as a countable union of such basic sets which are pairwise disjoint and with diameters arbitrarily small. As π_1 is s-resolving, $\pi_1|_{V^s(w)} : V^s(w) \to V^s(\pi_1(w))$ is a homeomorphism, and thus $V^s(\pi_1(w))$ has a neighbourhood base with the same properties. Now B is open, so $\pi_2^{-1}(B)$ is open, hence we can write

$$\pi_2^{-1}(B) = \bigcup_i \tilde{B}_i$$

where the $\tilde{B}_i$ are pairwise disjoint and have diameter $< \epsilon_Y/2$. Choose $y_i \in \tilde{B}_i$ for each i and $x_i = \pi_2(y_i)$, $B_i = \pi_2(\tilde{B}_i)$. Let $A_i = [A, x_i]$ and U_i such that $y_i \in U_i$, $\pi_2 : U_i \to A_i$ is a homeomorphism. Then we can write

$$\pi_2^{-1}([A, B]) = \bigcup_i [U_i, \tilde{B}_i]$$

moreover, the union is disjoint. Now

$$\mu_X([A,B]) = \mu_Y(\pi_2^{-1}([A,B])) = \mu_Y(\bigcup_i [U_i, \tilde{B}_i])$$
$$= \sum_i \mu_Y[U_i, \tilde{B}_i] = \sum_i \mu_Y^{u,y_i}(U_i)\mu_Y^{s,y_i}(\tilde{B}_i).$$

Now $\mu_Y^{u,y_i}(U_i) = \mu_X^{u,x_i}(A_i) = \mu_X^{u,x}(A)$ for all i (by part 1 of this prop.) So we have

$$\mu_X([A,B]) = \mu_X^{u,x}(A)\sum_i \mu_Y^{s,y_i}(\tilde{B}_i) = \mu_X^{u,x}(A)\mu_Y^{s,y}(\pi_2^{-1}(B)) = \mu_X^{u,x}(A)\mu_X^{s,x}(B).$$

4. We can find $y' \in \pi_2^{-1}(z)$ such that y' is 'close' to y. Let x_i, y_i, B_i, $\tilde{B}_i$ be as in part 3. Let $z_i = [z, x_i]$, $B(z)_i = [z, B_i]$, $y'_i = [y', y_i]$ and $\tilde{B(z)}_i = [y', \tilde{B}_i]$. Then $z_i = \pi_2(y'_i)$, $B(z)_i = \pi_2(\tilde{B}(z)_i)$, and $\cup B(z)_i = [z, B]$ so

$$\mu_X^{s,x}(B) = \mu_Y^{s,y}(\pi_2^{-1}(B)) = \sum_i \mu_Y^{s,y_i}(\tilde{B}_i)$$
$$= \sum_i \mu_Y^{s,y'_i}(\tilde{B}(z)_i) = \sum_i \mu_X^{s,z_i}(B(z)_i) = \mu_X^{s,z}([z,B])$$

5.

$$\mu_X^{s,\varphi(x)}(\varphi(B)) = \mu_Y^{s,\psi(y)}(\pi_2^{-1}(\varphi(B))) = \mu_Y^{s,\psi(y)}(\psi(\pi^{-1}(B)))$$
$$= \lambda^{-1}\mu_Y^{s,y}(\pi_2^{-1}(B)) = \lambda^{-1}\mu_X^{s,x}(B)$$

□

There are completely analogous results relating the measures μ_Y, μ_Y^s, μ_Y^u to μ_Σ, μ_Σ^s, μ_Σ^u via the s-resolving almost one-to-one factor map π_1. We state without proof: for $V^s(y_1,\delta) \subset V^s(y,\epsilon)$, $V^u(y_2,\delta) \subset V^u(y,\epsilon)$, fix $e \in \pi_1^{-1}(y)$ and $U(e) \subset V^s(e)$ such that $\pi_1(U(e)) = V^s(y_1,\delta)$ then

$$\begin{aligned}
\mu_Y([V^u(y_2,\delta), V^s(y_1,\delta)]) &= \mu_\Sigma(\pi_1^{-1}([V^u(y_2,\delta), V^s(y_1,\delta)])) \\
\mu_Y^{s,y}(V^s(y_1,\delta)) &= \mu_\Sigma^{s,e}(U(e)) \\
\mu_Y^{u,y}(V^u(y_2,\delta)) &= \mu_\Sigma^{u,e}(\pi_1^{-1}(V^u(y_2,\delta)))
\end{aligned}$$

5.1.3 Bowen measure from Homoclinic points

In this section we describe a construction of the Bowen measure for an irreducible Smale space from homoclinic points. The construction is reminiscent of Bowen's construction from periodic points in [3], but we were unable to find this construction in the literature. In [3] the unique entropy maximizing φ-invariant probability measure is constructed as the weak-$*$ limit of the sequence μ_n, where μ_n is defined as follows. Let $S_n = \cup_{k=1}^{n} Per_k(X, \varphi)$ then

$$\mu_n = \frac{1}{\#S_n} \sum_{z \in S_n} \delta_z.$$

In our construction we use points which are homoclinic to a given point (or more generally, heteroclinic to a pair of given points) instead of periodic points. It is worth noting that in Bowen's construction each μ_n is a φ-invariant probability measure. It our case, the measures constructed are not φ-invariant, but in the limit we recover φ-invariance.

We first prove the result for a mixing Smale Space, then use the spectral decomposition result (Prop. 2.14) to prove it for irreducible Smale space.

Let (X, φ) be a mixing Smale space. We start with the mixing case, since in the case of a mixing SFT (Σ, σ) the adjacency matrix A is primitive. This allows us to use the result of Thm. 4.5.12 in [16], which says $\lim_{n\to\infty} \lambda^{-n} A^n = u_r u_l$. This result is critical in the proof of Prop. 5.8.

Definition 5.5. *For x, $y \in X$, $B \subset V^u(x)$ and $C \subset V^s(y)$ open with compact closure. We define*

$$h^k_{B,C} = \varphi^k(B) \cap \varphi^{-k}(C)$$

and the measure

$$\mu^k_{B,C} = \frac{1}{\#h^k_{B,C}} \sum_{z \in h^k_{B,C}} \delta_z.$$

Remark 5.6.

- *Points $z \in h^k_{B,C}$ are unstably equivalent to x and stably equivalent to y, or heteroclinic. In addition, if $x = y$ then $h^k_{B,C}$ consists of points which are homoclinic to x (hence the use of h).*

- *As $V^u(x)$ and $V^s(y)$ are transverse foliations and $\varphi^k(B)$ and $\varphi^{-k}(C)$ have compact closure for each k, $\#h^k_{B,C}$ is finite for each k.*

- *$h^k_{B,C}$ may be empty, and hence $\mu^k_{B,C}$ may not be well defined for some positive integers k. However, for given B, C there exists a K such that for all $k > K$, $\mu^k_{B,C}$ is well defined. Since we will be interested in the (weak-$*$) limit of these measures as $k \to \infty$ we will not be concerned with the finite number of k's for which our definition is not valid.*

We now state the main result of this section for *mixing* Smale space.

Theorem 5.7. *For each function $f \in C(X)$ we have*

$$\lim_{k\to\infty} \int_X f d\mu^k_{B,C} = \int_X f d\mu_X,$$

where μ_X is the Bowen measure. In other words $\mu^k_{B,C} \to \mu_X$ in the weak-$*$ topology (independent of B, C).

To prove Prop. 5.7 we first establish the result for a mixing SFT and use the machinery of resolving maps to obtain the more general result.

Let (Σ, σ) be a mixing SFT. Fix $x, y \in \Sigma$, $n, m \in \mathbb{Z}$ and define

$$\begin{aligned} B &= \{z \in \Sigma \mid z_i = x_i \ \forall i \leq n\} \subset V^u(x, \epsilon_\Sigma) \\ C &= \{z \in \Sigma \mid z_i = y_i \ \forall i \geq -m+1\} \subset V^s(y, \epsilon_\Sigma) \end{aligned}$$

Proposition 5.8. *For each function $f \in C(\Sigma)$ we have*

$$\lim_{k\to\infty} \int_\Sigma f d\mu^k_{B,C} = \int_\Sigma f d\mu_\Sigma,$$

In other words, $\mu^k_{B,C} \to \mu_\Sigma$ in the weak-$*$ topology (independent of B, C).

Proof: It suffices to prove the result for a function of the form $e_l(\xi) = \chi_{E_l(\xi)}$. Where $E_l(\xi) = V_{l,v_{i'},v_{j'}}(\xi)$. Ie $i(\xi_{-l}) = v_{i'}$, and $t(\xi_l) = v_{j'}$. Let $t(x_n) = v_i$, and $i(y_{-m}) = v_j$. Now for $k \geq max\{n+l, m+l\}$

$$\int_\Sigma e_l(\xi) d\mu^k_{B,C} = \mu^k_{B,C}(E_l(\xi)) = \frac{\#E_l(\xi) \cap h^k_{B,C}}{\#h^k_{B,C}}.$$

The number of points in $E_l(\xi) \cap h^k_{B,C}$ is equal to the number of paths of length $k-(n+l)$ from $t(\sigma^k(x)_{-k+n}) = t(x_n) = v_i$ to $i(\xi_{-l+1}) = v_{i'}$, which equals $A^{k-(n+l)}_{ii'}$, times the number of paths of length $k-(n+l)$ from $t(\xi_l) = v_{j'}$ to $i(\sigma^{-k}(y)_{k-m+1}) =$

$i(y_{-m+1}) = v_j$, or $A_{j'j}^{k-(m+l)}$. The number of points in $h_{B,C}^k$ is the number of paths from $t(\sigma^k(x)_{-k+n}) = t(x_n) = v_i$ to $i(\sigma^{-k}(y)_{k-m+1}) = i(y_{-m+1}) = v_j$, or $A_{ij}^{2k-(n+m)}$. We therefore have

$$\int_\Sigma e_l(\xi) d\mu_{B,C}^k = \frac{A_{ii'}^{k-(n+l)} A_{j'j}^{k-(m+l)}}{A_{ij}^{2k-(n+m)}},$$

and

$$\begin{aligned}
\lim_{k\to\infty} \int_\Sigma & e_l(\xi) d\mu_{B,C}^k \\
&= \lim_{k\to\infty} \frac{A_{ii'}^{k-(n+l)} A_{j'j}^{k-(m+l)}}{A_{ij}^{2k-(n+m)}} \\
&= \lim_{k\to\infty} \frac{e_i A^{k-(n+l)} e_{i'} e_{j'} A^{k-(m+l)} e_j}{e_i A^{2k-(n+m)} e_j} \\
&= \lambda^{-2l} \frac{e_i \lim_k (\lambda^{-k+(n+l)} A^{k-(n+l)}) e_{i'} e_{j'} \lim_k (\lambda^{-k+(m+l)} A^{k-(m+l)}) e_j}{e_i \lim_k (\lambda^{-2k+(n+m)} A^{2k-(n+m)}) e_j} \\
&= \lambda^{-2l} \frac{e_i (u_r u_l) e_{i'} e_{j'} (u_r u_l) e_j}{e_i (u_r u_l) e_j} \quad \text{(by Thm. 4.5.12 in [16])} \\
&= \lambda^{-2l} \frac{u_r(i) u_l(i') u_r(j') (u_l(j)}{u_r(i) u_l(j)} \\
&= \lambda^{-2l} u_l(i') u_r(j') \\
&= \mu_\Sigma(V_{l, v_{i'}, v_{j'}}(\xi)) \\
&= \int_\Sigma e_l(\xi) d\mu_\Sigma.
\end{aligned}$$

□

We now wish to prove this result for more general open sets (with compact closure) $B' \subset V^u(x)$, $C' \subset V^s(x)$. To do this we will first need the following lemmas.

Lemma 5.9. *Let (Σ, σ) be a mixing SFT. Fix $x, y \in \Sigma$, $n, m \in \mathbb{Z}$ and define*

$$\begin{aligned}
B &= \{z \in \Sigma \mid z_i = x_i \ \forall i \leq n\} \subset V^u(x, \epsilon_\Sigma) \\
C &= \{z \in \Sigma \mid z_i = y_i \ \forall i \geq -m+1\} \subset V^s(y, \epsilon_\Sigma).
\end{aligned}$$

Then

$$\lim_{k\to\infty} \lambda^{-2k} \# h_{B,C}^k = \mu^u(B) \mu^s(C).$$

Proof: Let $t(x_n^1) = v_i$ and $i(y_{-m+1}) = v_j$, We then have

$$\begin{aligned}
\lim_{k\to\infty} \lambda^{-2k} h_{B,C}^k &= \lim_{k\to\infty} \lambda^{-2k} A_{ij}^{2k-(n+m)} \\
&= \lambda^{-(n+m)} \lim_{k\to\infty} \lambda^{-2k+n+m} e_i A^{2k-(n+m)} e_j \\
&= \lambda^{-(n+m)} e_i u_r u_l e_j \\
&= \lambda^{-n} u_r(i) \lambda^{-m} u_l(j) \\
&= \mu^u(B)\mu^s(C).
\end{aligned}$$

□

Lemma 5.10. *Let $B \subset V^u(x)$, $C \subset V^s(y)$ be open and compact. Then*

$$\lim_{k\to\infty} \lambda^{-2k} \# h_{B,C}^k = \mu_\Sigma^u(B)\mu_\Sigma^s(C).$$

Proof: If B, and C are clopen, then each is a finite disjoint union of cylinder sets of the form in above. Let

$$B = \sum_{i=1}^{n} B_i, \quad C = \sum_{i=1}^{m} C_i,$$

then for fixed k the h_{B_i,C_j}^k are pairwise disjoint and $\cup_{i,j} h_{B_i,C_j}^k = h_{B,C}^k$. Using Lemma 5.9 we can now write

$$\begin{aligned}
\lim_{k\to\infty} \lambda^{-2k} \# h_{B,C}^k &= \lim_{k\to\infty} \sum_{i,j} \lambda^{-2k} \# h_{B_i,C_j}^k \\
&= \sum_{i,j} \lim_{k\to\infty} \lambda^{-2k} \# h_{B_i,C_j}^k \\
&= \sum_{i,j} \mu^u(B_i)\mu^s(C_j) \\
&= \mu^u(B)\mu^s(C).
\end{aligned}$$

□

Lemma 5.11. *Let $B \subset V^u(x)$, $C \subset V^s(y)$ be open with compact closure. Then*

$$\lim_{k\to\infty} \lambda^{-2k} \# h_{B,C}^k = \mu_\Sigma^u(B)\mu_\Sigma^s(C).$$

Proof: Fix $\epsilon > 0$ We can find sets $B_1 \subseteq B \subseteq B_2 \subset V^u(x)$ and $C_1 \subseteq C \subseteq C_2 \subset V^s(x)$ such that $B_1,\ B_2,\ C_1$ and C_2 are compact and open and

$$\mu^u(B_2)\mu^s(C_2) - \epsilon < \mu^u(B)\mu^s(C) < \mu^u(B_1)\mu^s(C_1) + \epsilon.$$

Notice that $\#h^k_{B_1,C_1} \leq \#h^k B, C \leq \#h^k B_2, C_2$, so

$$\begin{aligned}
\mu^u(B)\mu^s(C) - \epsilon &< \mu^u(B_1)\mu^s(C_1) \\
&= \lim_{k\to\infty} \lambda^{-2k}\#h^k_{B_1,C_1} \\
&\leq \liminf_{k\to\infty} \lambda^{-2k}\#h^k_{B,C}
\end{aligned}$$

and

$$\begin{aligned}
\mu^u(B)\mu^s(C) + \epsilon &> \mu^u(B_2)\mu^s(C_2) \\
&= \lim_{k\to\infty} \lambda^{-2k}\#h^k_{B_2,C_2} \\
&\geq \limsup_{k\to\infty} \lambda^{-2k}\#h^k_{B,C}.
\end{aligned}$$

As this hold for all $\epsilon > 0$ we have

$$\limsup_{k\to\infty} \lambda^{-2k}\#h^k_{B,C} \leq \mu^u(B)\mu^s(C) \leq \liminf_{k\to\infty} \lambda^{-2k}\#h^k_{B,C}$$

and hence

$$\lim_{k\to\infty} \lambda^{-2k}\#h^k_{B,C} = \mu^u_\Sigma(B)\mu^s_\Sigma(C).$$

$\square$

We are now ready to prove the more general version of Prop. 5.8.

Proposition 5.12. *The result of Prop. 5.8 holds with $B \subset V^u(x)$, $C \subset V^s(x)$ open with compact closure.*

Proof: As in the proof of Prop. 5.4, we can write

$$B = \bigcup_i B_i, \quad C = \bigcup_j C_i$$

where each $V^u_{n_i}(x_i)$, $V^s_{m_j}(y_j)$ is of the form considered in Prop. 5.8, and the unions are disjoint. Abusing notation slightly we write

$$h^k = h^k_{B,C}, \quad \mu^k = \mu^k_{B,C}$$

and

$$h^k_{ij} = h^k_{B_i,C_j}, \ \mu^k_{ij} = \mu^k_{B_i,C_j}$$

Notice that for fixed k the h^k_{ij}'s are pairwise disjoint and $\cup_{i,j} h^k_{ij} = h^k$. We can write

$$\lim_{k\to\infty} \int_\Sigma f d\mu^k = \lim_{k\to\infty} \sum_{i,j} \frac{\#h^k_{ij}}{\#h^k} \int_\Sigma f d\mu^k_{ij}.$$

Now let $M = sup_{z\in\Sigma}|f(z)|$, which is finite as f is continuous and Σ is compact. For each k, $\sum_{i,j} \frac{\#h^k_{ij}}{\#h^k} = 1$ so for any $I \in \mathbb{N}$ we can write

$$1 = \lim_{k\to\infty} \sum_{i,j} \frac{\#h^k_{ij}}{\#h^k} = \lim_{k\to\infty} \sum_{i,j}^{I} \frac{\#h^k_{ij}}{\#h^k} + \lim_{k\to\infty} \sum_{i>I,j>I} \frac{\#h^k_{ij}}{\#h^k}.$$

We also know that

$$1 = \sum_{i,j} \frac{\mu^u(B_i)\mu^s(C_j)}{\mu^u(B)\mu^s(C)}$$

and we may choose I large enough so that

$$\lim_{k\to\infty} \sum_{i>I,j>I} \frac{\#h^k_{ij}}{\#h^k} < \frac{\epsilon}{2M},$$

and

$$|\sum_{i,j}^{I} \frac{\mu^u(B_i)\mu^s(C_j)}{\mu^u(B)\mu^s(C)} - 1| < \frac{\epsilon}{2M}.$$

Using Lemma 5.11 and Prop. 5.8 we now have

$$\begin{aligned}
\left|\lim_{k\to\infty}\int_\Sigma f d\mu^k - \int_\Sigma f d\mu\right| &= \left|\lim_{k\to\infty}\sum_{i,j}\frac{\#h_{ij}^k}{\#h^k}\int_\Sigma f d\mu_{ij}^k - \int_\Sigma f d\mu\right| \\
&= \left|\lim_{k\to\infty}\sum_{i,j}^{I}\frac{\#h_{ij}^k}{\#h^k}\int_\Sigma f d\mu_{ij}^k + \lim_{k\to\infty}\sum_{i,j>I}\frac{\#h_{ij}^k}{\#h^k} - \int_\Sigma f d\mu\right| \\
&= \left|\sum_{i,j}^{I}\lim_{k\to\infty}\frac{\lambda^{-2k}\#h_{ij}^k}{\lambda^{-2k}\#h^k}\int_\Sigma f d\mu_{ij}^k + \lim_{k\to\infty}\sum_{i,j>I}\frac{\#h_{ij}^k}{\#h^k} - \int_\Sigma f d\mu\right| \\
&= \left|\sum_{i,j}^{I}\frac{\mu^u(B_i)\mu^s(C_j)}{\mu^u(B)\mu^s(C)}\int_\Sigma f d\mu + \lim_{k\to\infty}\sum_{i,j>I}\frac{\#h_{ij}^k}{\#h^k} - \int_\Sigma f d\mu\right| \\
&\leq \left|\int_\Sigma f d\mu\left(\sum_{i,j}^{I}\frac{\mu^u(B_i)\mu^s(C_j)}{\mu^u(B)\mu^s(C)} - 1\right)\right| + \left|\lim_{k\to\infty}\sum_{i,j>I}\frac{\#h_{ij}^k}{\#h^k}\right| \\
&\leq \left|M\left(\sum_{i,j}^{I}\frac{\mu^u(B_i)\mu^s(C_j)}{\mu^u(B)\mu^s(C)} - 1\right)\right| + \left|M\lim_{k\to\infty}\sum_{i,j>I}\right| \\
&< M\frac{\epsilon}{2M} + M\frac{\epsilon}{2M} \\
&= \epsilon.
\end{aligned}$$

This holds for all $\epsilon > 0$ so

$$\lim_{k\to\infty}\int_\Sigma f d\mu^k = \int_\Sigma f d\mu$$

□

We now wish to extend this result to the mixing Smale space case. We first state the following Lemma.

Lemma 5.13. *Let $\pi : (X, \varphi) \to (Y, \psi)$ be an almost 1-to-1 s-resolving factor map. There exists a constant M such that if $y \in Y$, $B \subset V^u(y)$ open with compact closure, and $B' = \pi^{-1}(B)$, then $B' = \cup_1^m B'_i$ where the union is disjoint, $m \leq M$ and $B'_i \subset V^u(x_i)$ for some $x_i \in X$.*

Proof: Let ϵ_π be as in Lemma 3.2 of [20], and cover X with ϵ_π-balls. Let M be the minimum number of sets in such a cover, and label them $\{U_i\}_1^M$. Now choose $x_i \in B' \cap U_i$ for each i such that this set is non-empty, and let $\tilde{B}_i = B' \cap U_i$. We show that $\tilde{B}_i \subset V^u(x_i)$. Suppose $x \in \tilde{B}_i$, then $[x, x_i] \in V^u(x_i) \cap U_i$. Now $d(x, x_i) < \epsilon_\pi$, so from Lemma 3.2 of [20], $\pi([x, x_i] = [\pi(x), \pi(x_i)]$. $\pi(x), \pi(x_i)$ are both in $B \subset V^u(y)$ so they are unstably equivalent. Therefore $[\pi(x), \pi(x_i)] = \pi(x)$. We have that $\pi([x, x_i]) = \pi(x)$, so by Lemma 3.3 of [20] x and $[x, x_i]$ are unstably equivalent. $[x, x_i]$ is also unstably equivalent to x_i, so x is unstably equivalent to x_i znd we have shown that $\tilde{B}_i \subset V^u(x_i)$.

At this point the sets $\tilde{B}_i$ may not be pairwise disjoint. Suppose $x \in \tilde{B}_i \cap \tilde{B}_j$. Then from above $x \in V^u(x_i) \cap V^u(x_j)$ and thus $V^u(x_i) = V^u(x_j)$. We then replace these two sets with the union $\tilde{B}_i \cup \tilde{B}_j$. Similarly, if x is in n of the $\tilde{B}_i$ sets, then all n of these sets are contained in one unstable set and we union them all together. In this manner we arrive at a collection of sets B'^m_{i1} such that $m \leq M$, the B'_i's are pairwise disjoint, $\cup_1^m B'_i = B'$, and $B'_i \subset V^u(x_i)$ for some x_i. $\square$

The following proposition allows us to extend our result from the mixing SFT case to the mixing Smale space case.

Proposition 5.14. *Let (X, φ), and (Y, ψ) be mixing Smale spaces, $\pi : X \to Y$ an almost 1-to-1 (s or u) resolving factor map, and suppose the conclusion of Prop. 5.7 holds for (X, φ). Then the conclusion of Prop. 5.7 holds for (Y, ψ).*

Proof: Suppose π is s-resolving (the u-resolving case is completely analogous). Let y_1, $y_2 \in Y$ and $B \subset V^u(y_1)$, $C \subset V^s(y_2)$, and let

$$h_Y^k = h_{B,C}^k, \quad \mu_Y^k = \mu_{B,C}^k.$$

Now, set $B' = \pi_1^{-1}(B)$. By Lemma 5.13 $B' = \cup_1^m B'_i$, where the union is disjoint and $B_i \subset V^u(x_1^i)$ for some $x_1^i \in X$. Also, fix $x_2 \in \pi_1^{-1}(y_2)$, and set C' such that $\pi_1 : C' \to C$ is a homeomorphism, so $C' \in V^s(x_2)$. Now set

$$h_X^k = h_{B',C'}^k = \bigcup_1^M h_{B'_i,C'}^k, \quad \mu_X^k = \mu_{B',C}^k = \sum_1^M \frac{\# h_{B'_i,C'}^k}{\# h_{B',C'}^k} \mu_{B'_i,C'}^k.$$

Notice that since $h_X^k \subset V^s(\varphi^{-k}(x_2))$ and π is s-resolving, π is one-to-one (and hence bijective) on h_X^k. In other words $h_Y^k = \pi(h_X^k)$, and therefore $\mu_X^k = (\mu_Y^k \circ \pi)$. Also recall from Lemma 5.11 that

$$\lim_{k \to \infty} \frac{\# h_{B'_i,C'}^k}{\# h_{B',C'}^k} = \frac{\mu^u(B'_i)}{\mu^u(B')}.$$

Now, for $f \in C(Y)$

$$
\begin{aligned}
\int_Y f d\mu_Y &= \int_X (f \circ \pi) d\mu_X, \quad \text{as } \pi \text{ is 1-1 } \mu_X \text{ a.e.} \\
&= \lim_{k\to\infty} \int_X (f \circ \pi) d\mu^k_{B'_i,C'} \quad \text{for any } i\text{, by hypothesis} \\
&= (\lim_{k\to\infty} \int_X (f \circ \pi) d\mu^k_{B'_i,C'}) \sum_1^m \frac{\mu^u(B'_i)}{\mu^u(B')} \\
&= (\lim_{k\to\infty} \sum_1^m \frac{\# h^k_{B'_i,C'}}{\# h^k_{B',C'}} \int_X (f \circ \pi) d\mu^k_{B'_i,C'}) \\
&= \lim_{k\to\infty} \int_X (f \circ \pi) d\mu^k_X \\
&= \lim_{k\to\infty} \int_X (f \circ \pi) d(\mu^k_Y \circ \pi) \\
&= \lim_{k\to\infty} \int_Y f d\mu^k_Y
\end{aligned}
$$

□

Proof of Theorem 5.7: Recall that for the mixing Smale space (X, φ) we can find mixing Smale spaces (Y, ψ) and (Σ, σ) (a SFT). As well as almost 1-to-1 factor maps $\pi_1 : \Sigma \to Y$, $\pi_2 : Y \to X$ such that π_1 is s-resolving and π_2 is u-resolving. The conclusion then follows from Cor. 5.12 and 2 applications of Prop. 5.14. □

Now suppose (X, φ) is an irreducible Smale space (not necessarily mixing). As in Prop. 2.14 we can find a mixing Smale space (Y, ψ) and $N \in \mathbb{N}$ such that $X \cong Y \times \{1, 2, \dots, N\}$ and for $y \in Y$

$$
\varphi(y, i) = \begin{cases} (y, i+1) & \text{if } i < N \\ (\psi(y), 1) & \text{if } i = N \end{cases}
$$

Definition 5.15. *For (X, φ), (Y, ψ) as above, $x,\ y \in Y$, U^u_x, U^s_y as in Defn. 5.5, $i,\ j, l \in \{1, 2, \dots, N\}$ such that $i + l = j - l$ modulo N, $k \in \mathbb{N}$ let*

$$
U^u_{(x,i)} = U^u_x \times \{i\} \subset (X, \varphi),\ U^s_{(y,j)} = U^s_y \times \{j\} \subset (X, \varphi),
$$

$$
h^{k,l}_{U^u_{(x,i)}, U^s_{(y,j)}} = \varphi^{Nk+l}(U^u_{(x,i)}) \cap \varphi^{-Nk-l}(U^s_{(y,j)}),
$$

and

$$\mu_X^{k,l}(U^u_{(x,i)}, U^s_{(y,j)}) = \frac{1}{N \cdot \#h^{k,l}} \sum_{z \in h^{k,l}} \delta_z.$$

Remark 5.16. *In Defn. 5.15*

- *$i + l = j - l$ modulo N implies $h^{k,l}_{U^u_{(x,i)}, U^s_{(y,j)}}$ is non-empty.*
- *The factor of $\frac{1}{N}$ in $\mu_X^{k,l}(U^u_{(x,i)}, U^s_{(y,j)})$ arises from the fact that the measure is supported on $Y \times \{i+l\}$ and hence we want $\mu_X^{k,l}(U^u_{(x,i)}, U^s_{(y,j)})(X) = \frac{1}{N}$.*
- *$\mu_X^{k,l}(U^u_{(x,i)}, U^s_{(y,j)}) \to \frac{1}{N}\mu_Y$, as in Prop. 5.7.*

We now have the following analogue of Prop. 5.7.

Proposition 5.17. *Let (X, φ) be an irreducible Smale space with (Y, ψ) as above. Fix x_i, $y_i \in Y \times \{i\}$ and let $U^u_{(x_i,i)}$, $U^s_{(y_i,i)}$, be as in Defn. 5.15 for each $1 \le i \le N$. For $n = Nk + l \in \mathbb{N}$ (with $0 \le l \le N-1$). If*

$$\mu_X^n = \sum_{m=1}^{N} \mu_X^{k,l}(U^u_{(x_{m-l},m-l)}, U^s_{(y_{m+l},m+l)}) \text{ where } m-l,\ m+l \text{ are mod } N$$

then for $f \in C(X)$

$$\lim_{n\to\infty} \int_X f d\mu_X^n = \int_X f d\mu_X.$$

In other words $\mu_X^n \to \mu_X$ in the weak-$*$ topology.

Proof: Notice that $\mu_X|_{Y\times\{i\}} = \frac{1}{N}\mu_Y$. Let $f_i = f|_{Y\times\{i\}}$ then

Now fix $l \in \{0, 1, \ldots, N-1\}$ and consider the subsequence

$$\begin{aligned} \lim_{n\to\infty} \int_X f d\mu_X^n &= \lim_{k\to\infty} \int_X f d\mu_X^{Nk+l} \\ &= \sum_{m=1}^{N} \lim_{k\to\infty} \int_Y f_m d\mu_X^{k,l}(U^u_{(x_{m-l},m-l)}, U^s_{(y_{m+l},m+l)} \\ &= \sum_{m=1}^{N} \frac{1}{N} \int_Y f_i d\mu_Y \\ &= \sum_{m=1}^{N} \int_X f_i d\mu_X|_{Y\times\{i\}} \\ &= \int_X f d\mu_X. \end{aligned}$$

Now, since this limit exists for each $l \in \{0, 1, \ldots, N-1\}$ and is independent of l, we have

$$\lim_{n\to\infty} \int_X f d\mu_X^n = \int_X f d\mu_X.$$

$\square$

5.2 Traces

In this section we define traces on several of the algebras discussed above and show that these traces are ring/module homomorphisms where this makes sense. In the case of a mixing Smale space, the fact that the trace is a ring homomorphism is due to Putnam ([19]). We extend this result to the irreducible case, and use a similar argument to prove the module homomorphisms. We then show that the traces we have defined arise as an asymptotic of the usual trace on $\mathfrak{B}(l^2(V^h(P)))$.

Let (X, φ) be an irreducible Smale space. Then as in [19], the formula

$$\tau^H(f) = \int_X f(x,x) d\mu(x)$$

where μ is the Bowen measure, defines a trace on $C_c(G^h)$. This extends to a bounded trace on $H(X, \varphi)$. Furthermore, the formulae

$$\begin{aligned} \tau^S(f) &= \int_{V^u(P)} f(x,x) d\mu_u^x(x) \text{ and} \\ \tau^U(f) &= \int_{V^s(P)} f(x,x) d\mu_s^x(x) \end{aligned}$$

define traces on the algebras $C_c(G^s)$ and $C_c(G^u)$ respectively, however these traces are not bounded when extended to $S(X, \varphi, P)$, $U(X, \varphi, P)$. Recall that $G^s(X, \varphi, P)$ is stable equivalence restricted to $V^u(P)$, the unstable set of P. This is why we must integrate with respect to μ_u^x to obtain a trace on $C_c(G^s)$. We also define a trace on $C(H, \alpha)$ by

$$\tau^{CH}(f) = \int_0^1 \tau^H(f(t)) dt.$$

For $f \in M_n(A)$ where A is $C(H, \alpha)$, $H(X, \varphi)$, $S(X, \varphi, P)$, or $U(X, \varphi, P)$, we define the trace in the obvious way:

$$\tau^*(f) = \sum_{i=1}^{n} \tau^*(f_{ii}).$$

The proof that these are indeed traces is in [19]. We include a proof of the trace property only in the case of the $S(X, \varphi, P)$ algebra, as our definition of $S(X, \varphi, P)$ differs from the definition in [19] (we use the definition in [20], [22]).

Proposition 5.18. *Let $a,\ b \in C_c(G^s)$, then*

$$\tau^S(ab) = \tau^S(ba).$$

Proof: It suffices to consider a with support $V(h^u_{y_a}(y_a), y_a, h^u y_a, \delta_a)$, b with support $V(h^u_{y_b}(y_b), y_b, h^u y_b, \delta_b)$. Then

$$\begin{aligned}
\tau^S(ab) &= \int_{V^u(P)} ab(x,x)d\mu^x_u(x) \\
&= \int_{V^u(P)} \sum_{(x,y)\in G^s} a(x,y)b(y,x)d\mu^x_u(x)
\end{aligned}$$

Each summand if non-zero only if $x = h^u_{y_a}(y)$ and $y = h^u_{y_b}(x)$. So the sum is non-zero only if $x = h^u_{y_a} h^u_{y_b}(x)$ (equivalently $y = h^u_{y_b} h^u_{y_a}(y)$, in which case we have.

$$\begin{aligned}
\tau^S(ab) &= \int_{V^u(P)} a(h^u_{y_a}(y), y)b(y, h^u_{y_a}(y))d\mu_u^{h^u_{y_a}(y)}(h^u_{y_a}(y)) \\
&= \int_{V^u(P)} b(y, h^u_{y_a}(y))a(h^u_{y_a}(y), y)d\mu^y_u(y) \\
&= \tau^S(ba)
\end{aligned}$$

□

Remark 5.19. *In [19] the trace properties are proved for a* mixing *Smale space. The irreducilbe case follows from the decomposition results of sections 2.5, and 3.3, and remark 4.11. However, in section 5.2.3 when we consider asymptotics, we are forced to restrict ourselves to the mixing case.*

5.2.1 Trace Homomorphisms for Irreducible SFT

We first consider the case of a SFT.

Remark 5.20. *In contrast to the results on measures in section 5.1, the trace results for irreducible Smale space in section 5.2.2 are not extensions of results for the SFT case (this section). In other words, the results of this section follow as a particular case of the results in section 5.2.2. We include these results here both as a motivating example, and because the computations can all be made explicitly in the SFT case.*

We are interested in the induced trace on the K_0 groups of the algebras. We begin with the trace on $H(\Sigma, \sigma)$. The formula

$$\tau^H(f) = \int_X f(x,x) d\mu(x)$$

defined on $C_c(G^h(\Sigma, \sigma))$ extends to a bounded trace on $H(\Sigma, \sigma)$. Now, as $H(\Sigma, \sigma)$ is AF, $K_0(H(\Sigma, \sigma))$ is generated by the rank one projections in $H(\Sigma, \sigma)$. Each rank one projection is homotopic to some function of the form $e_{N,v_i,v_j}(\xi, \xi)$, so we compute (letting the left/right PF eigenvectors be u_l/u_r)

$$\begin{aligned} \tau^H(e_{N,v_i,v_j}(\xi,\xi)) &= \int_X e_{N,v_i,v_j}(\xi,\xi)(x,x) d\mu(x) \\ &= \mu(V_{N,v_i,v_j}(\xi)) \\ &= \lambda^{-2N} u_l(i) u_r(j) \\ &= \lambda^{-2N} u_l e_{ij} u_r \end{aligned}$$

Extending this to all of $K_0(H(\Sigma, \sigma))$ we have the following. For $[X, N] \in K_0(H(\Sigma, \sigma))$,

$$\tau^H[X, N] = \lambda^{-2N} u_l X u_r.$$

Now for $f \in M_n(C(H, \alpha))$ we have $\tau^{CH}(f) = \int_{[0,1]} \tau^H(f(t))dt$, but in the case that f is a projection, $\tau^H(f(t))$ is constant, so we can write $\tau^{CH}(f) = \tau^H(f(0))$ (f gives a homotopy from $f(t)$ to $f(0)$, hence $f(t)$ is unitarily equivalent to $f(0)$). Recalling that $K_0(C(H, \alpha)) \cong ker(id - \alpha_*) \subset K_0(H(\Sigma, \sigma))$ with the embedding given by evaluation at 0, we can immediately write down the trace on $K_0(C(H, \alpha))$ by restricting the trace on $K_0(H(\Sigma, \sigma))$ to $ker(id - \alpha_*)$. For $[X, N] \in K_0(C(H, \alpha))$,

$$\tau^{CH}[X, N] = \lambda^{-2N} u_l X u_r.$$

We now consider the trace on $K_0(S(\Sigma,\sigma,P))$ and $K_0(U(\Sigma,\sigma,P))$. Here we must be a little bit more careful, as the formula

$$\tau^S(f) = \int_{V^u(P)} f(x,x)d\mu_u^x(x)$$

defines a trace on $C_c(G^s(\Sigma,\sigma,P))$, but does not extend to a bounded trace on $S(\Sigma,\sigma,P)$. However, as $K_0(S(\Sigma,\sigma,P))$ is generated by the rank one projections, and each of these is homotopic to a function of the form $e_{N,v_j}(\xi,\xi)$ we can simply compute in a straightforward manner.

$$\begin{aligned}
\tau^S(e_{N,v_j}(\xi,\xi)) &= \int_X e_{N,v_j}(\xi,\xi)(x,x)d\mu_u^x \\
&= \mu_u^x(V^u_{N,v_j}) \\
&= \lambda^{-N}u_r(j) \\
&= \lambda^{-N}e_j u_r.
\end{aligned}$$

Where e_j is the jth canonical basis (row) vector. Extending to all of $K_0(S(\Sigma,\sigma,P))$ we have the following. For $[v,N] \in K_0(S(\Sigma,\sigma,P))$

$$\tau^S[v,N] = \lambda^{-N} v u_r.$$

Similarly, for $[w,N] \in K_0(U(\Sigma,\sigma,P))$ (so w is now thought of as a column vector) we have

$$\tau^U[w,N] = \lambda^{-N} u_l w.$$

The trace yields group homomorphisms

$$\begin{aligned}
\tau^{CH} : K_0(C(H,\alpha)) &\to \mathbb{R}, \\
\tau^S : K_0(S(\Sigma,\sigma,P)) &\to \mathbb{R}, \text{ and} \\
\tau^U : K_0(U(\Sigma,\sigma,P)) &\to \mathbb{R}.
\end{aligned}$$

We now show that the trace is in fact a ring/module homomorphism. In the case of a SFT, this is a straightforward computation. We start with the following lemma, which is a well known fact from linear algebra.

Lemma 5.21. *Let $X \in C(A)$, then u_l (resp. u_r) is a left (right) eigenvector for X with eigenvalue c_X.*

Proof:

$$u_l XA = u_l AX = \lambda u_l X.$$

So $u_l X$ is a left eigenvector for A with eigenvalue λ. As the left Perron eigenvector is unique, we must have that $u_l X = c_X u_l$ for some constant c_X. The proof that u_r is a right eigenvector with eigenvalue $\tilde{c}_X$ is completely analogous. To see that $c_X = \tilde{c}_X$ consider

$$c_X = c_X u_l u_r = u_l X u_r = \tilde{c}_X u_l u_r = \tilde{c}_X.$$

□

Proposition 5.22. *$\tau^{CH} : K_0(C(H,\alpha)) \to \mathbb{R}$ is a ring homomorphism. Moreover, it respects the module structures of $K_0(S(\Sigma,\sigma,P))$, $K_0(U(\Sigma,\sigma,P))$.*

Proof: Let $[X,N]$, $[Y,M] \in K_0(C(H,\alpha))$, $[v,K] \in K_0(S)$, $[w,L] \in K_0(U)$ then

$$\begin{aligned}
\tau^{CH}([X,N] * [Y,M]) &= \tau^{CH}([XY, N+M]) \\
&= \lambda^{-2N-2M} u_l XY u_r \\
&= \lambda^{-2N-2M} c_X c_Y u_l u_r \\
&= \lambda^{-2N} \lambda^{-2M} c_X c_Y u_l u_r u_l u_r \quad (u_l u_r = 1) \\
&= \lambda^{-2N} u_l X u_r \lambda^{-2M} u_l Y u_r \\
&= \tau^{CH}[X,N] \tau^{CH}[Y,M].
\end{aligned}$$

Similarly

$$\begin{aligned}
\tau^{U}([X,N] * [w,L]) &= \tau^{U}([Xw, 2N+L]) \\
&= \lambda^{-2N-L} u_l X w \\
&= \lambda^{-2N-L} c_X u_l w \\
&= \lambda^{-2N} \lambda^{-L} u_l X u_r u_l w \\
&= \lambda^{-2N} u_l X u_r \lambda^{-L} u_l w \\
&= \tau^{CH}[X,N] \tau^{U}[w,L],
\end{aligned}$$

and

$$\begin{aligned}
\tau^S([v,K]*[X,N]) &= \tau^S([vX, 2N+K]) \\
&= \lambda^{-2N-K} vXu_r \\
&= \lambda^{-2N-K} c_X v u_r \\
&= \lambda^{-2N}\lambda^{-K} u_l X u_r v u_r \\
&= \lambda^{-K} v u_r \lambda^{-2N} u_l X u_r \\
&= \tau^S[v,K]\tau^{CH}[X,N].
\end{aligned}$$

□

5.2.2 Irreducible Smale Space

We first consider the case of a mixing Smale space, the irreducible case will then follow from the decomposition of an irreducible Smale space into a union of mixing Smale spaces. Let (X, φ) be a mixing Smale space. We prove that the Trace is a ring/module homomorphism. The proof for the ring structure appears in [19]. The proof for the module structure is similar. We include both here for completeness. We begin with a technical Lemma.

Lemma 5.23. *Let $f \in C_c(X)$, $g \in C_c(V^s(x_g))$, then*

$$\lim_{n\to+\infty} \int_{V^s(x_g)} f(\varphi^{-n}(x))g(x)d\mu^s(x) = \int_X f(x)d\mu(x) \int_{V^s(x_g)} g(x)d\mu^s(x).$$

Proof: It suffices to prove the result for f supported on a set of the form $V_f = [V^u(x_f, \delta_f), V^s(x_f, \delta_f)]$, g supported on $V^s(x_g, \delta_g)$. Define the function $\tilde{g}$ with support $V_{\tilde{g}} = [V^u(x_g, \epsilon_X/2), V^s(x_g, \delta_g)]$ by $\tilde{g}(x) = g([x_g, x])$. Let $K^u = \mu^u(V^u(x_g, \epsilon_X/2))$, $K^s = \mu^s(V^s(x_g, \delta_g))$. We first show

$$\lim_{n\to+\infty} \left| K^u \int_{V^s(x_g)} f(\varphi^{-n}(x))g(x)d\mu^s(x) - \int_X f(\varphi^{-n}(x))\tilde{g}(x)d\mu(x) \right| = 0.$$

Let $M_g = sup|g(x)|$ and fix $\epsilon > 0$. By the uniform continuity of f we may find $\delta > 0$ such that $d(x, y) < \delta$ implies $|f(x) - f(y)| < \epsilon/(K^s K^u M_g)$. There exists $N \in \mathbb{N}$ such that for all $n > N$, and for all $x,\ y \in V_f$ such that $x \sim_s y$ $d(\varphi^{-n}(x), \varphi^{-n}(y)) < \epsilon_X$

implies $d(x,y) < \delta$. Now for $n > N$, $x \in V^s(x_g, \delta_g) \cap \varphi^n(V_f)$ we have

$$\begin{aligned}
&\left| K^u f(\varphi^{-n}(x))g(x) - \int_{V^u(x_g,\epsilon_X/2)} f(\varphi^{-n}([y,x]))\tilde{g}([y,x])d\mu^u(y) \right| \\
&\quad = \left| \int_{V^u(x_g,\epsilon_X/2)} f(\varphi^{-n}(x))g(x)d\mu^u(y) - \int_{V^u(x_g,\epsilon_X/2)} f(\varphi^{-n}([y,x]))g(x)d\mu^u(y) \right| \\
&\quad \le \int_{V^u(x_g,\epsilon_X/2)} \left| f(\varphi^{-n}(x)) - f(\varphi^{-n}([y,x])) \right| |g(x)| d\mu^u(y) \\
&\quad \le \int_{V^u(x_g,\epsilon_X/2)} \frac{\epsilon}{M} |g(x)| d\mu^u(y) \\
&\quad = \frac{\epsilon}{K^u K^s M_g} |g(x)| K^u \\
&\quad \le \frac{\epsilon}{K^s}
\end{aligned}$$

We now have

$$\begin{aligned}
&\left| K^u \int_{V^s(x_g)} f(\varphi^{-n}(x))g(x)d\mu^s(x) - \int_X f(\varphi^{-n}([y,x]))\tilde{g}([y,x])d\mu([y,x]) \right| \\
&\qquad = \left| K^u \int f(\varphi^{-n}(x))g(x)d\mu^s(x) - \int f(\varphi^{-n}([y,x]))\tilde{g}([y,x])d(\mu^u \times \mu^s) \right| \\
&\qquad = \left| K^u \int f(\varphi^{-n}(x))g(x)d\mu^s(x) - \iint f(\varphi^{-n}([y,x]))\tilde{g}([y,x])d\mu^u(y)d\mu^s(x) \right| \\
&\qquad = \left| \int (K^u f(\varphi^{-n}(x))g(x) - \int f(\varphi^{-n}([y,x]))\tilde{g}([y,x])d\mu^u(y))d\mu^s(x) \right| \\
&\qquad \le \int \left| K^u f(\varphi^{-n}(x))g(x) - \int f(\varphi^{-n}([y,x]))\tilde{g}([y,x])d\mu^u(y) \right| d\mu^s(x) \\
&\qquad \le \int \frac{\epsilon}{K^s} d\mu^s \\
&\qquad = \frac{\epsilon}{K^s} K^s \\
&\qquad = \epsilon.
\end{aligned}$$

Hence we have

$$\lim_{n \to +\infty} |K^u \int_{V^s(x_g)} f(\varphi^{-n}(x))g(x)d\mu^s(x) - \int_X f(\varphi^{-n}(x))\tilde{g}(x)d\mu(x)| = 0.$$

or

$$\lim_{n \to +\infty} K^u \int_{V^s(x_g)} f(\varphi^{-n}(x))g(x)d\mu^s(x) = \lim_{n \to +\infty} \int_X f(\varphi^{-n}(x))\tilde{g}(x)d\mu(x).$$

Now, as φ is strong mixing with respect to μ, as in [29] we have

$$\lim_{n\to+\infty}\int_X f(\varphi^{-n}(x))\tilde{g}(x)d\mu(x) = \int_X f(\varphi^{-n}(x))d\mu(x)\int_X \tilde{g}(x)d\mu(x).$$

Finally, from the definition of $\tilde{g}$ it is clear that

$$\int_X \tilde{g}(x)d\mu(x) = K^u\int_{V^s(x_g)} g(x)d\mu^s(x).$$

Putting this all together we have

$$\lim_{n\to+\infty}\int_{V^s(x_g)} f(\varphi^{-n}(x))g(x)d\mu^s(x) = \int_{V^s(x_g)} g(x)d\mu^s(x).$$

□

Lemma 5.24. *For $f,\ g\in C_c(G^h(X,\varphi))$, $a\in C_c(G^u(X,\varphi,P))$, $b\in C_c(G^s(X,\varphi,P))$ we have*

1. $\lim_{n\to+\infty}\tau^H(\alpha^n(f)\alpha^{-n}(g)) = \tau^H(f)\tau^H(g)$,
2. $\lim_{n\to+\infty}\tau^U(\alpha^n(f)a) = \tau^H(f)\tau^U(a)$,
3. $\lim_{n\to+\infty}\tau^S(b\alpha^{-n}(f)) = \tau^S(b)\tau^H(f)$.

Proof: It suffices to consider functions f, g supported on sets of the form $V_f = V(x_f, y_f, h_f, \delta_f)$, $V_g = V(x_g, y_g, h_g, \delta_g)$, a supported on some $V_a = V(x_a, y_a, h_a^s, \delta_a)$ and b supported on some $V_b = V(x_b, y_b, h_b^u, \delta_b)$.

1. First suppose $h_f \neq id$, then

$$\tau^H(f) = \int_X f(x,x)d\mu(x) = 0,$$

and

$$\begin{aligned}\tau^H(\alpha^n(f)\alpha^{-n}(g)) &= \int_X \sum_{y\in V^h(x)} \alpha^n(f)(x,y)\alpha^{-n}(g)(y,x)d\mu(x)\\ &= \int\sum f(\varphi^{-n}(x),\varphi^{-n}(y))g(\varphi^n(y),\varphi^n(x))d\mu(x).\end{aligned}$$

The sum reduces to a single term where $\varphi^n(x) = h_g\varphi^n(y)$ or $x = \varphi^{-n}h_g\varphi^n(y)$, so we have

$$\tau^H(\alpha^n(f)\alpha^{-n}(g)) = \int_X f(\varphi^{-2n}h_g\varphi^n(y), \varphi^{-n}(y))g(\varphi^n(y), h_g\varphi^n(y))d\mu(x).$$

The integrand is equal to 0 unless $h_f\varphi^{-2n}h_g\varphi^n(y) = \varphi^{-n}(y)$. We show that for sufficiently large n, the set

$$\left\{y \in X \mid \varphi^n(y) \in V_g,\ \varphi^{-2n}h_g\varphi^n(y) \in V_f,\ h_f\varphi^{-2n}h_g\varphi^n(y) = \varphi^{-n}(y)\right\}$$

is empty. If, for some $z \in V_f$, $h_f(z) = z$, then $h_f = id$, so we can can find $\delta > 0$ such that $d(z, h_f(z)) > \delta$ for all $z \in V_f$. Also, for n large enough we have

$$d(\varphi^{-2n}(x), \varphi^{-2n}h_g(x)) < \delta$$

for all $x \in V_g$. Set $x = \varphi^n(y)$, then we have

$$d(\varphi^{-n}(y), \varphi^{-2n}h_g\varphi^n(y)) < \delta$$

but

$$d(\varphi^{-2n}h_g\varphi^n(y), h_z\varphi^{-2n}h_g\varphi^n(y)) > \delta$$

so $h_f\varphi^{-2n}h_g\varphi^n(y) \neq \varphi^{-n}(y)$. Hence

$$\tau^H(\alpha^n(f)\alpha^{-n}(g)) = 0 = \tau^H(f)\tau^H(g).$$

Similarly, if $h_g \neq id$ then

$$\tau^H(\alpha^n(f)\alpha^{-n}(g)) = 0 = \tau^H(f)\tau^H(g).$$

Finally, if $h_f = id$ and $h_g = id$ then

$$\tau^H(\alpha^n(f)\alpha^{-n}(g)) = \int_X f(\varphi^{-n}(x), \varphi^{-n}(x))g(\varphi^n(x), \varphi^n(x))d\mu(x)$$

and as n tends to infinity, this tends to

$$\int_X f(x,x)d\mu(x) \int_X g(x,x)d\mu(x) = \tau^H(f)\tau^H(g)$$

as φ is strong mixing with respect to μ (see for example [29]).

2. We again begin by considering the case that $h_f \neq id$, in this case $\tau^H(f) = 0$ and

$$\tau^S(\alpha^n(f)a) = \int_{V^s(P)} f(\varphi^{-n}(x), \varphi^{-n}(y))a(y, x)d\mu^s(x)$$

where $h_a^s(y) = x$, so we can write

$$\tau^S(\alpha^n(f)a) = \int V^s(P) f(\varphi^{-n} h_a^s(y), \varphi^{-n}(y))a(y, h_a^s(y))d\mu^s(x)$$

As in part 1, the integrand is zero except on the set

$$\left\{y \in X \mid y \in V_a,\ \varphi^{-n} h_a^s(y) \in V_f,\ \varphi^n h_f \varphi^{-n} h_a^s(y) = y\right\},$$

which we show is empty for n sufficiently large. As before we can find $\delta > 0$ such that

$$d(z, h_f(z)) > \delta \text{ for all } z \in V_f.$$

Also, for n large enough

$$d(\varphi^{-n}(x), \varphi^{-n} h_a^s(x)) < \delta$$

for all $x \in V_a$. so we have

$$d(\varphi^{-n}(y), \varphi^{-n} h_a^s(y)) < \delta$$

and

$$d(\varphi^{-n} h_a^s(y), h_f \varphi^{-n} h_a^s(y)) > \delta$$

so $h_f \varphi^{-n} h_a^s(y) \neq \varphi^{-n}(y)$, or $\varphi^n h_f \varphi^{-n} h_a^s(y) \neq y$. So the set is empty and

$$\lim_{n \to +\infty} \tau^S(\alpha^n(f)a) = 0 = \tau^H(f)\tau^S(a).$$

Similarly, the result holds if $h_a^s \neq id$, so once again we are left to consider the case $h_a^s = id$, and $h_f = id$. In this case we have

$$\tau^S(\alpha^n(f)a) = \int_{V^s(P)} f(\varphi^{-n}(x), \varphi^{-n}(x))a(x, x)d\mu^s(x).$$

Now, using Lemma 5.23 we have

$$\lim_{n\to+\infty} \tau^S(\alpha^n(f)a) = \int_X f(x,x)d\mu(x) \int_{V^s(P)} a(x,x)d\mu^s(x) = \tau^H(f)\tau^S(a)$$

(more detail here?)

3. This is completely analogous to part 2.

□

Theorem 5.25. *Let (X,φ) be a mixing Smale space and $C(H,\alpha)$ the mapping cylinder of the corresponding homoclinic algebra. With $\tau^{CH}(\cdot)$ as defined above, $\tau^{CH}_* : K_0(C(H,\alpha)) \to \mathbb{R}$ is a ring homomorphism. Moreover, it respects the module structure of $K_0(S(\Sigma,\sigma,P))$ and $K_0(U(\Sigma,\sigma,P))$.*

Proof: As we know the trace is a group homomorphism, it suffices to show that it preserves the multiplicative structure. The first statement is as in [19], but we include it here for completeness. Consider $p \in P_n(C(H,\alpha))$, $q \in P_m(C(H,\alpha))$ and recall the notation

$$((p\times q)_t)_{(ij)(kl)} = \frac{\alpha_t(p_{(ik)})\alpha_{-t}a_{(jl)} + \alpha_{-t}a_{(jl)}\alpha_t(p_{(ik)})}{2}$$

so

$$[p]_0[q]_0 = \lim_{t\to+\infty} [\chi_{(1/2,\infty)}((p\times q)_t)]_0.$$

Also

$$\lim_{t\to\infty} ||\chi_{(1/2,\infty)}((p\times q)_t) - (p\times q)_t|| = 0,$$

and, for sufficiently large t

$$\tau^{CH}(\chi_{(1/2,\infty)}((p\times q)_t)) = \tau^H(\chi_{(1/2,\infty)}((p\times q)_t)(0)) = \tau^H(\chi_{(1/2,\infty)}((p\times q)_t(0))).$$

Putting this all together with Lemma 5.24 we have

$$\begin{aligned}
\tau_*^{CH}([p]_0[q]_0) &= \lim_{t\to\infty} \tau^{CH}(\chi_{(1/2,\infty)}((p\times q)_t)) \\
&= \lim_{t\to\infty} \tau^{H}(\chi_{(1/2,\infty)}((p\times q)_t(0))) \\
&= \lim_{t\to\infty} \tau^{H}((p\times q)_t(0)) \\
&= \lim_{n\to\infty} \tau^{H}((p\times q)_n(0)) \\
&= \lim_{n\to\infty} \sum_i \sum_j \tau^{H}((((p\times q)_n(0))_{(ij)(ij)}) \\
&= \lim_{n\to\infty} \sum_{i,j} \tau^{H}(\frac{\alpha^n(p(0))_{(ii)}\alpha^{-n}(q(0))_{(jj)} + \alpha^{-n}q(0)_{(jj)}\alpha^n(p(0))_{(ii)}))}{2}) \\
&= \lim_{n\to\infty} \sum_{i,j} \tau^{H}(\alpha^n(p(0))_{(ii)}\alpha^{-n}(q(0))_{(jj)}) \\
&= \sum_{i,j} \tau^{H}(p(0)_{(ii)})\tau^{H}(q(0)_{(jj)}) \\
&= \tau^{H}(p(0))\tau^{H}(q(0)) \\
&= \tau_*^{CH}([p]_0)\tau_*^{CH}([q]_0)
\end{aligned}$$

For the second part, consider $p \in P_n(C(H,\alpha))$, and $a \in P_m(U)$ with $Tr(a) < \infty$. Recall the notation

$$((p\times a)_t)_{(ij)(kl)} = \frac{\alpha_t(p(0)_{(ik)})a_{(jl)} + a_{(jl)}\alpha_t(p(0)_{(ik)})}{2},$$

so

$$[p]_0[a]_0 = \lim_{t\to\infty}[\chi_{(1/2,\infty)}((p\times a)_t)]_0.$$

Also

$$\lim_{t\to\infty} ||\chi_{(1/2,\infty)}((p\times a)_t) - (p\times a)_t|| = 0.$$

Combining this with Lemma 5.24 we have

$$
\begin{aligned}
\tau_*^U([p]_0[a]_0) &= \lim_{t\to\infty} \tau^U(\chi_{(1/2,\infty)}((p\times a)_t)) \\
&= \lim_{t\to\infty} \tau^U((p\times a)_t) \\
&= \lim_{n\to\infty} \tau^U((p\times a)_n) \\
&= \lim_{n\to\infty} \sum_i \sum_j \tau^U(((p\times a)_n)_{(ij)(ij)}) \\
&= \lim_{n\to\infty} \sum_{i,j} \tau^U(\frac{\alpha^n(p(0))_{(ii)}a_{(jj)} + a_{(jj)}\alpha^n(p(0)_{(ii)}))}{2}) \\
&= \lim_{n\to\infty} \sum_{i,j} \tau^U(\alpha^n(p(0))_{(ii)}a_{(jj)}) \\
&= \sum_{i,j} \tau^H(p(0)_{(ii)})\tau^U(a_{(jj)}) \\
&= \tau^H(p(0))\tau^U(a) \\
&= \tau_*^{CH}([p]_0)\tau_*^U([a]_0)
\end{aligned}
$$

Now suppose $a \in P_m(U(\Sigma,\sigma,P))$ is such that $\tau^U(a) = \infty$. Let $\{a^k\}$ be a sequence in $M_m(C_c(G^u))$ that converges in norm to a, and such that $k < Tr(a_k) < \infty$. By Lemma 5.24 and the above calculation we know that

$$\lim_{t\to\infty} \tau^U((p\times a^k)_t) = \tau^H(p(0))\tau^U(a^k) > k\tau^H(p(0))$$

so

$$\tau_*^U([p]_0[a]_0) = \lim_{k\to\infty} \tau^H(p(0))\tau^U(a^k) = \infty = \tau_*^{CH}([p]_0)\tau_*^U([a]_0)$$

The corresponding result for the right-module $K_0(S(\Sigma,\sigma,P))$ is completely analogous.

□

Now let (X,φ) be an irreducible Smale space with (Y,ψ), N as in Prop. 2.14. We wish to prove Theorem 5.25 in this case. We first note that the trace τ_X^s on $S(X,\varphi,P)$,

$$\tau_X^{\cdot}(a) = \frac{1}{N}\sum_1^N \tau_Y^{\cdot}(a_i),$$

where a_i is in the i^{th} summand of $S(X,\varphi,P) \cong \bigoplus_1^N S(Y,\psi,\tilde{P})$. Similar formulas hold for $U(X,\varphi,P)$ and $H(X,\varphi)$. In the case of $C(H_X,\alpha_\varphi)$ we have

$$\tau_X^{CH}(a) = \tau_Y^{CH}\tilde{a}$$

where $a \mapsto \tilde{a}$ is the isomorphism from $C(H_X,\alpha_\varphi)$ to $C(H_Y,\alpha_\psi)$ (Prop. 3.33). The results of Theorem 5.25 now follow almost immediately in the irreducible case. For example, let $[a]_0 \in K_0(S(X,\varphi,P))$, $[b]_0 \in K_0(C(H_X,\alpha_\varphi))$.

$$\begin{aligned}
\tau_X^s([a]_0[b]_0) &= \tau_X^s\big(([a_1]_0[\tilde{b}]_0,\ldots,[a_N]_0[\tilde{b}])\big) \\
&= \frac{1}{N}\sum_1^N tr_Y^s([a_i]_0[\tilde{b}]_0) \\
&= \frac{1}{N}\sum_1^N tr_Y^s([a_i]_0)\tau_y^{CH}([\tilde{b}]_0) \\
&= \tau_X^s([a]_0)\tau_X^{CH}([b]_0).
\end{aligned}$$

The other results of Theorem 5.25 follow similarly.

5.2.3 Asymptotics

In this section we restrict to the case that (X,φ) is a *mixing* Smale space and show how the above traces on $S(X,\varphi,P)$ and $U(X,\varphi,P)$ are related to asymptotics of the usual trace on $\mathfrak{B}(l^2(V^h(P)))$. As discussed in section 2.3 we consider $S(X,\varphi,P)$ and $U(X,\varphi,P)$ to be subalgebras of $\mathfrak{B}(l^2(V^h(P)))$. For an element of $S(X,\varphi,P)$ or $U(X,\varphi,P)$ we can thus take the trace as an operator on this Hilbert space. Ie fix the countable basis $\{\delta_x \mid x \in V^h(P)\}$, then for $A \in \mathfrak{B}(l^2(V^h(P)))$,

$$Tr(A) = \sum_{x\in V^h(P)} < A\delta_x, \delta_x >$$

In general $S(X,\varphi,P)$ and $U(X,\varphi,P)$ do not consist of trace class operators w.r.t. this trace. However, recall from Prop. 2.21 that for $a \in S(X,\varphi,P)$ and $b \in U(X,\varphi,P)$, $ab \in \mathcal{K}$. Furthermore, from the proof of Prop. 2.21, if $a \in C_c(G^s(X,\varphi,P))$, $b \in C_c(G^u(X,\varphi,P))$, then ab is a finite rank operator. Recall also, Prop. 2.22, that

$$\lim_{k\to+\infty} ||\alpha^k(a)\alpha^{-k}(b) - \alpha^{-k}(b)\alpha^k(a)|| = 0.$$

The result we wish to prove is

Theorem 5.26. *Let (X, φ) be a mixing Smale space with topological entropy* $\log(\lambda)$ *and corresponding algebras $S(X, \varphi, P)$, $U(X, \varphi, P)$. If $a \in S(X, \varphi, P)$, $b \in U(X, \varphi, P)$ then*

$$\lim_{k \to +\infty} \lambda^{-2k} Tr\left(\alpha^k(a)\alpha^{-k}(b)\right) = \tau^S(a)\tau^U(b).$$

As we have done before, we will first prove the result for a mixing SFT, then use resolving maps to obtain the general result. We start with the following lemma which is true for general mixing Smale space.

Lemma 5.27. *Let (X, φ) be a mixing Smale space. Let $a \in C_c(G^s(X, \varphi, P))$ be supported on a set of the form $V(x_a, y_a, h^u_{y_a}, \delta_a)$, and $b \in C_c(G^u(X, \varphi, P))$ supported on $V(x_b, y_b, h^s_{y_b}, \delta_b)$. Let $V_a = r(V(x_a, y_a, h^u_{y_a}, \delta_a))$, $V_b = s(x_b, y_b, h^s_{y_b}, \delta_b))$. If $h^u_{y_a} = h^s_{y_b} = id$ then for each $k \in \mathbb{N}$*

$$Tr(\alpha^k(a)\alpha^{-k}(b)) = \sum_{w \in \varphi^k(V_a) \cap \varphi^{-k}(V_b)} a(\varphi^{-k}(w), \varphi^{-k}(w))b(\varphi^k(w), \varphi^k(w))$$

otherwise

$$\lim_{k \to +\infty} Tr(\alpha^k(a)\alpha^{-k}(b)) = 0$$

Proof: If $x \neq h^u_{y_b}(y)$, $a(x, y) = 0$. Abusing notation slightly, let $a(y) = a(h^u_{y_a}(y), y)$. Similarly, let $b(y) = b(h^s_{y_b}(y), y)$. Now

$$\begin{aligned} Tr(\alpha^k(a)\alpha^{-k}(b)) &= \sum_{w \in V^h(P)} < \alpha^k(a)\alpha^{-k}(b)\delta_w, \delta_w > \\ &= \sum_{w \in V^h(P)} a(\varphi^{-2k} h^s_{y_b} \varphi^k(w)) b(\varphi^k(w)) < \delta_{\varphi^k h^u_{y_a} \varphi^{-2k} h^s_{y_b} \varphi^k(w)}, \delta_w > . \end{aligned}$$

Suppose $h^u_{y_a} = h^s_{y_b} = id$, then we have

$$Tr(\alpha^k(a)\alpha^{-k}(b)) = \sum_{w \in V^h(P)} a(\varphi^{-k}(w)) b(\varphi^k(w)) < \delta_w, \delta_w > .$$

$a(\varphi^{-k}(w)) \neq 0$ only if $w \in \varphi^k(V_a)$, and $b(\varphi^k(w)) \neq 0$ only if $w \in \varphi^{-k}(V_b)$, so

$$Tr(\alpha^k(a)\alpha^{-k}(b)) = \sum_{w \in \varphi^k(V_a) \cap \varphi^{-k}(V_b)} a(\varphi^{-k}(w)) b(\varphi^k(w)).$$

Now suppose $h^u_{y_a} \neq id$.

$$Tr(\alpha^k(a)\alpha^{-k}(b)) = \sum_{w \in E_k} a(\varphi^{-2k} h^s_{y_b} \varphi^k(w)) b(\varphi^k(w)) < \delta_{\varphi^k h^u_{y_a} \varphi^{-2k} h^s_{y_b} \varphi^k(w)}, \delta_w >$$

Where

$$E_k = \{w \in V^h(P) |\ \varphi^k(w) \in V_b,\ \varphi^{-2k} h^s_b \varphi^k(w) \in V_a,\ \varphi^k h^u_{y_a} \varphi^{-2k} h^s_b \varphi^k(w) = w\}.$$

We show that for large enough k, the set E_k is empty (this is essentially the same argument used in the proof of Prop. 5.24). We can $\delta > 0$ such that

$$d(z, h^u_{y_a}(z)) > \delta$$

for all $z \in V_a$. Now, we can find K sufficiently large so that for all $k > K$ we have

$$d(\varphi^{-2k}(y), \varphi^{-2k} h^s_{y_b}(y)) < \delta$$

for all $y \in V_b$. So for $w \in E_k$, $\varphi^k(w) \in V_b$ and thus

$$d(\varphi^{-k}(w), \varphi^{-2k} h^s_{y_b}(\varphi^k(w))) < \delta$$

but $\varphi^{-2k} h^s_{y_b}(\varphi^k(w)) \in V_a$ so

$$d(\varphi^{-2k} h^s_{y_b}(\varphi^k(w)), h^u_{y_a} \varphi^{-2k} h^s_{y_b}(\varphi^k(w)) > \delta$$

so $\varphi^{-k}(w) \neq h^u_{y_a} \varphi^{-2k} h^s_{y_b}(\varphi^k(w))$, contradicting $w \in E_k$. Hence for large k, E_k is empty and

$$\lim_{k \to \infty} Tr(\alpha^k(a)\alpha^{-k}(b)) = 0.$$

A similar argument gives the result in the case $h^s_{y_b} \neq 0$. □

We now prove Theorem 5.26 in the case of a mixing SFT.

Proposition 5.28. *Theorem 5.26 is true for* (Σ, σ) *a SFT.*

Proof: It suffices to consider the case $a = e_{N,v_j}(\xi_1, \xi_2) \in C_c(G^s(\Sigma, \sigma, P))$, and $b = e_{M,v_i}(\eta_1, \eta_2) \in C_c(G^u(\Sigma, \sigma, P))$. Suppose first that $\xi_1 \neq \xi_2$, then both sides of the equation are equal to zero by Lemma 5.27 (similarly if $\eta_1 \neq \eta_2$). We now suppose $a = e_{N,v_j}(\xi, \xi)$, $b = e_{M,v_i}(\eta, \eta)$. As in Lemma 5.27, The rank of the operator $\alpha^k(a)\alpha^{-k}(b)$

is equal to the number of points in the intersection of $\sigma^k(V^u_{N,v_j}(\xi))$ and $\sigma^{-k}(V^s_{M,v_i}(\eta))$. For $2k \geq N+M$, the number of points of intersection is equal to the number of paths in G of length $2k-(N+M)$ from v_j to v_i, or $A^{2k-(N+M)}_{ji}$. So for $2k \geq N+M$ we have

$$Tr(\alpha^k(a)\alpha^{-k}(b)) = A^{2k-(N+M)}_{ji}.$$

So

$$\begin{aligned}
\lim_{k\to\infty} \lambda^{-2k} Tr(\alpha^k(a)\alpha^{-k}(b)) &= \lim_{k\to\infty} \lambda^{-2k} A^{2k-(N+M)}_{ji} \\
&= \lambda^{-(N+M)} \lim_{k\to\infty} \lambda^{-2k+(N+M)} A^{2k-(N+M)}_{ji} \\
&= \lambda^{-(N+M)} \lim_{n\to\infty} \lambda^{-n} A^n_{ji} \\
&= \lambda^{-(N+M)} \lim_{n\to\infty} \lambda^{-n} e_j A^n e_i \\
&= \lambda^{-(N+M)} e_j (\lim_{n\to\infty} \lambda^{-n} A^n) e_i \\
&= \lambda^{-(N+M)} e_j (u_r u_l) e_i \\
&= \lambda^{-(N+M)} u_r(j) u_l(i) \\
&= \lambda^{-N} u_r(j) \lambda^{-M} u_l(i) \\
&= \tau^S(a)\tau^U(b)
\end{aligned}$$

□

The following Lemma allows us to use Prop. 5.28 to prove Theorem 5.26.

Lemma 5.29. *Let (X,φ) and (Y,ψ) be mixing Smale spaces and $\pi : X \to Y$ an almost 1-to-1 resolving (u or s) factor map. If (X,φ) satisfies the conclusion of Prop. 5.26, then so does (Y,ψ).*

Proof: Suppose π is s-resolving (the u-resolving case is analogous). It suffices to consider $a \in C_c(G^s(Y,\psi,\pi(P)))$ supported on a set of the form $V(h^u_a(y_a), y_a, h^u_a, \delta_a)$, $b \in C_c(G^u(Y,\psi,\pi(P)))$ supported on a set of the form $V(h^s_b(y_b), y_b, h^s_b, \delta_b)$. Moreover, if $h^u_a \neq id$, or $h^s_b \neq id$ both sides of the equation are equal to zero (Lemma 5.27). In the case $h^u_a = h^s_b = id$ let $V^u_a = r(V(h^u_a(y_a), y_a, h^u_a, \delta_a)) = s(V(h^u_a(y_a), y_a, h^u_a, \delta_a))$ and $V^s_b = r(V(h^s_b(y_b), y_b, h^s_b, \delta_b)) = s(V(h^s_b(y_b), y_b, h^s_b, \delta_b))$. We then have

$$Tr(\alpha^k(a)\alpha^{-k}(b)) = \sum_{z\in h^k} \alpha^k(a)(z,z)\alpha^{-k}(z,z) = \sum_{z\in h^k} a(\psi^{-k}(z),\psi^{-k}(z))b(\psi^k(z),\psi^k(z)),$$

where $h_k = \psi^k(V_a^u) \cap \psi^{-k}(V_b^s)$. Now let $U_a^u = \pi^{-1}(V_a^u))$, and fix $x_b \in \pi^{-1}(y_b)$ and $U_b^s \subset V^s(e_b, \epsilon_X)$ such that $pi|_{U_b^s}$ is a homeomorphism onto V_b^s. Define the function $\tilde{a} \in C_c(G^s(X, \varphi, P))$ with support $U_a^u \times U_a^u$ such that $\tilde{a} = a \circ (\pi \times \pi)$. Similarly define $\tilde{b}$ with support $U_b^s \times U_b^s$ such that $\tilde{b} = b \circ (\pi \times \pi)$. Let $\tilde{h}_k = \varphi^k(U_a^u) \cap \varphi^{-k}(U_b^s)$. Then π restricted to $\tilde{h}_k$ is $1-1$ and $\pi(\tilde{h}_k) = h_k$. Now we have

$$\begin{aligned} Tr(\alpha^k(a)\alpha^{-k}(b)) &= \sum_{z \in h^k} a(\psi_{-k}(z), \psi_{-k}(z)) b(\psi^k(z), \psi^k(z)) \\ &= \sum_{x \in \tilde{h}^k} \tilde{a}(\varphi^{-k}(x), \varphi^{-k}(x)) b(\varphi^k(x), \varphi^k(x)) \\ &= Tr(\alpha^k(\tilde{a})\alpha^{-k}(\tilde{b})). \end{aligned}$$

Also

$$\begin{aligned} Tr(\tilde{a}) &= \int_{V^u(P)} \tilde{a}(z,z) d\mu_X^u(z) \\ &= \int_{U_a^u} (a \circ (\pi \times \pi))(z,z) d(\mu_Y^u \circ \pi)(z) \\ &= \int_{V_a^u} a(y,y) d\mu_Y^u(y) \\ &= Tr(a). \end{aligned}$$

Similarly,

$$\begin{aligned} Tr(\tilde{b}) &= \int_{V^s(P)} \tilde{b}(z,z) d\mu_X^s(z) \\ &= \int_{U_b^s} (b \circ (\pi \times \pi))(z,z) d(\mu_Y^s \circ \pi)(z) \\ &= \int_{V_b^s} b(y,y) d\mu_Y^s(y) \\ &= Tr(b). \end{aligned}$$

Putting these together we have

$$\begin{aligned} \lim_{k\to\infty} \lambda^{-2k} Tr(\alpha^k(a)\alpha^{-k}(b)) &= \lim_{k\to\infty} \lambda^{-2k} Tr(\alpha^k(\tilde{a})\alpha^{-k}(\tilde{b})) \\ &= Tr(\tilde{a})Tr(\tilde{b}) \quad \text{by Prop. 5.28} \\ &= Tr(a)Tr(b). \end{aligned}$$

□

The proof of Theorem 5.26 is now straightforward.

Proof of Theorem 5.26: For (X, φ) mixing, let (Σ, σ), (Y, ψ), π_1, and π_2 be as in section 5.1.2. The result then follows from Prop. 5.28 and two applications of Lemma 5.29. □

Bibliography

[1] B. Blackadar. *K-theory for Operator Algebras.* Springer-Verlag, Berlin-Heidelberg-New York, 1986.

[2] R. Bowen. Markov partitions for axiom a diffeomorphisms. *Amer. J. Math.*, 92:725–747, 1970.

[3] R. Bowen. Periodic points and measures for axiom a diffeomorphisms. *Trans. Amer. Math. Soc.*, 154:377–397, 1971.

[4] M. Boyle, B. Marcus, and P. Trow. Resolving maps and the dimension group for shifts of finite type. *Memoirs A.M.S.*, 70(377), 1987.

[5] M. Brin and G. Stuck. *Introduction to Dynamical Systems.* Cambridge University Press, Cambridge-New York-Melbourne-Madrid-Cape Town, 2002.

[6] A. Connes. An analogue of the Thom isomorphism for crossed products of a C*-algebra by an action of $\mathbb{R}$. *Advances in Math.*, 39:31–55, 1981.

[7] A. Connes and N. Higson. Déformations, morphisms asymptotiques et K-théorie bivariant. *C.R. Acad. Sci. Series I*, 311:101–106, 1990.

[8] J. Cuntz and W. Krieger. A class of C*-algebras and topological Markov chains. *Invent. Math.*, 56:251–268, 1980.

[9] D. Fried. Finitely presented dynamical systems. *Ergodic Theory and Dynamical Systems*, 7:489–507, 1987.

[10] N. Jacobson. *Lectures in Abstract Algebra*, volume II. Van Nostrand, 1953.

[11] J. Kaminker, I. Putnam, and J. Spielberg. Operator algebras and hyperbolic dynamics. In S. Doplicher, R. Longo, J.E. Roberts, and L. Zsido, editors, *Operator Algebras and Quantum Field Theory.* International Press, 1997.

[12] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*, volume 54 of *Encyclopedia of Math. and its Appl.* Cambridge University Press, Cambridge-New York-Melbourne, 1997.

[13] K.H. Kim and F.W. Roush. Williams' conjecture is false for reducible subshifts. *J. Amer. Math. Soc.*, 5:213–215, 1992.

[14] K.H. Kim and F.W. Roush. The williams conjecture is false for irreducible subshifts. *Annals of Math.*, 149:545–558, 1999.

[15] W. Krieger. On dimension functions and topological Markov chains. *Invent. Math.*, 56:239–250, 1980.

[16] D. Lind and B. Marcus. *An Introduction to Symbolic Dynamics and Coding.* Cambridge University Press, Cambridge-New York-Melbourne, 1995.

[17] P.S. Muhly, J.N. Renault, and D.P. Williams. Equivalence and isomorphism for groupoid C^*-algebas. *J. Operator Th.*, 17:3–22, 1987.

[18] J.R. Munkres. *Topology.* Prentice Hall, 2000.

[19] I. F. Putnam. C*-algebras from Smale spaces. *Canad. J. Math.*, 48:175–195, 1996.

[20] I. F. Putnam. The functoriality of the C*-algebras associated with hyperbolic dynamical systems. *J. London Math. Soc.*, 62(2):873–884, 2000.

[21] I. F. Putnam. Lifting factor maps to resolving maps. *Israel J. Math.*, 146:253–280, 2005.

[22] I. F. Putnam and J. Spielberg. The structure of C^*-algebras associated with hyperbolic dynamical systems. *J. Func. Anal.*, 163:279–299, 1999.

[23] M. Rørdam, F. Larsen, and J. Laustsen. *An Introduction to K-Theory for C*-Algebras*, volume 49 of *London Mathematical Society Student Texts.* Cambridge University Press, Cambridge-New York-Melbourne, 2000.

[24] J.N. Renault. *A groupoid approach to C^*-algebras.* Lecture Notes in Math. Springer-Verlag, 1980.

[25] D. Ruelle. *Thermodynamic Formalism*, volume 5 of *Encyclopedia of Math. and its Appl.* Addison-Wesley, Reading, 1978.

[26] D. Ruelle. Noncommutative algebras for hyperbolic diffeomorphisms. *Invent. Math.*, 93:1–13, 1988.

[27] D. Ruelle and D. Sullivan. Currents, flows and diffeomorphisms. *Topology*, 14:319–327, 1975.

[28] S.Smale. Differentiable dynamical systems. *Bull. Amer. Math. Soc.*, 73:747–817, 1967.

[29] P. Walters. *An Introduction to Ergodic Theory.* Graduate Texts in Mathematics. Springer-Verlag, Berlin-Heidelberg-New York, 1982.

[30] R.F. Williams. Classification of subshifts of finite type. *Annals of Math.*, 98:120–153, 1973.