SPECTRUM PRESERVING LINEAR MAPS ON THE SPACE OF SELF ADJOINT OPERATORS

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§0. Introduction. Spectrum-preserving or invertibility-preserving linear maps between certain Banach algebras have been studied by several authors. See for example [10] and [11] for maps on algebras of matrices; [1, pp. 29-30] for related results; [8] for maps between commutative Banach algebras; [3] for certain positive maps between C*-algebras; and [7] for maps between algebras of operators on Banach spaces.

In §1 of this note we characterize spectrum preserving surjective linear maps between the real linear spaces of all self-adjoint operators on (real or complex) Hilbert spaces. We show that such a map ϕ takes one of the forms $\phi(A) = UAU^*$ or $\phi(A) = UA^tU^*$ for some unitary operator U, where A^t denotes the transpose of A with respect to an arbitrary, but fixed, orthonormal basis. For finite-dimensional complex Hilbert spaces, the same result was obtained in [10] under the additional assumption that ϕ preserves the multiplicity of the eigenvalues. In §2 we give a characterization of invertibility-preserving linear maps on the space of self-adjoint operators on finite dimensional real or complex Hilbert space.

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In the sequel, we use the following notation and terminology. The letters H, H_1 and H_2 denote real or complex Hilbert spaces. The algebra of all bounded linear operators on H is denoted by $B(H)$. The real linear space of all self-adjoint operators on H will be denoted by $S(H)$. By a subspace we always mean a closed linear manifold. The lattice of all closed subspaces of H will be denoted by $C(H)$. If $M \in C(H)$, then P_M denotes the orthogonal projection on M, and $M^{\perp}$ denotes the orthogonal complement of M in H. By a projection we always mean a self-adjoint idempotent. If P and Q are two projections, then we write $P \perp Q$ if $PQ = QP = 0$. If $T \in B(H)$, then $\sigma(T)$, $N(T)$ and $R(T)$ denote the spectrum, the null space and the range of T respectively. The transpose of T with respect to a fixed, but arbitrary, orthonormal basis will be denoted by T^t. A linear map ϕ from a subspace of $B(H_1)$ to a subspace of $B(H_2)$ is said to be invertibility-preserving if the image of every invertible operator is invertible. It is said to be spectrum-preserving if $\sigma(\phi(T)) = \sigma(T)$, for every T. For a non-zero $x \in H$, the rank-one operator on H defined by $u \longrightarrow (u,x)x$ will be denoted by $x \otimes x$. Note that $x \otimes x$ is a projection if and only if $\|x\| = 1$.

§1. Spectrum-preserving maps. Our main result of this section is the following.

THEOREM 1. *Let $\phi\colon S(H_1) \to S(H_2)$ be a surjective spectrum-preserving linear map. Then there exists a unitary operator $U\colon H_1 \to H_2$ such that ϕ has one of the following forms:*

(i) $\phi(A) = UAU^*$, <u>for every</u> $A \in S(H_1)$,

<u>or</u>

(ii) $\phi(A) = UA^tU^*$, <u>for every</u> $A \in S(H_1)$.

<u>Note</u>. In the real case, the two forms coincide.

Before we proceed further we should point out that the surjectivity assumption on ϕ cannot be removed as can be seen by considering the map $\phi: S(H) \to S(H \oplus H)$ given by $\phi(A) = A \oplus A$.

We begin with the following lemma.

LEMMA 1. <u>The linear map</u> ϕ <u>has the following properties</u>:

(i) ϕ <u>is injective</u>,

(ii) $\phi(I) = I$,

(iii) $\phi(P)$ <u>is a projection if and only if</u> P <u>is a projection</u>,

(iv) <u>if</u> P <u>and</u> Q <u>are two projections</u>, <u>then</u> $\phi(P) \perp \phi(Q)$ <u>if and only if</u> $P \perp Q$,

(v) $\phi(A^2) = (\phi(A))^2$ <u>for every finite rank</u> $A \in S(H_1)$.

(vi) $\phi(P)$ <u>is a rank-one projection if and only if</u> P <u>is a rank-one projection</u>.

<u>Proof</u>. (i) If $\phi(A) = 0$, then $\sigma(A) = \{0\}$. Since A is self-adjoint, it follows that $A = 0$.

(ii) Since $\phi(I)$ is self-adjoint with spectrum $\{1\}$, we get $\phi(I) = I$.

(iii) This follows from the fact that a self-adjoint operator is a projection if and only if its spectrum is a subset of $\{0,1\}$.

(iv) For two projections P and Q, the operator $P + Q$ is a projection if and only if $P \perp Q$. Since $\phi(P+Q) = \phi(P) + \phi(Q)$, (iv) follows from (iii).

(v) If $A \in S(H_1)$ is of finite rank, then $A = \sum_{i=1}^{k} c_i E_i$, where the E_i's are projections with $E_i \perp E_j$, $i \neq j$. Now using (iv) we have,

$$\phi(A^2) = \phi\left(\sum_{i=1}^{k} c_i^2 E_i\right) = \sum_{i=1}^{k} c_i^2 \phi(E_i) = \left(\sum_{i=1}^{k} c_i \phi(E_i)\right)^2 = (\phi(A))^2.$$

(vi) A projection has rank one if and only if it cannot be written as the sum of two nonzero mutually perpendicular projections. This together with (i), (iii) and (iv) proves (vi). ■

We also need the following version of Lemma 4 of [7]. For completeness, we give the proof.

LEMMA 2. <u>Let</u> $T \in B(H)$, $x \in H$ <u>and</u> $\lambda \notin \sigma(T)$. <u>Then</u> $\lambda \in \sigma(T + x \otimes x)$ <u>if and only if</u> $((\lambda-T)^{-1}x, x) = 1$.

<u>Proof</u>. If $((\lambda-T)^{-1}x, x) = 1$, then

$$(T + x \otimes x)(\lambda-T)^{-1}x = T(\lambda-T)^{-1}x + x = \lambda(\lambda-T)^{-1}x,$$

and hence λ is an eigenvalue of $T + x \otimes x$. Conversely, if $\lambda \in \sigma(T + x \otimes x)$, then by a variant of the Fredholm alternative, λ is an eigenvalue of $T + x \otimes x$ and so there exists a nonzero vector $y \in H$ such that $(T + x \otimes x)y = \lambda y$. Thus $y = (y,x)(\lambda-T)^{-1}x$. This implies that $((\lambda-T)^{-1}x, x) = 1$. ■

<u>Proof of Theorem 1</u>. We treat the real and complex cases separately.

Case I. Complex Hilbert spaces. We first consider finite dimensional spaces. (It is obvious that H_1 and H_2 have the same dimension.) We extend ϕ to a map from $\mathcal{B}(H_1)$ onto $\mathcal{B}(H_2)$ by the formula $\phi(A+iB) = \phi(A) + i\phi(B)$ for $A, B \in S(H_1)$. It follows from Lemma 1(v), that ϕ is a Jordan isomorphism from $\mathcal{B}(H_1)$ to $\mathcal{B}(H_2)$. By a result from ring theory (see [6], p. 50) ϕ is either an algebra isomorphism or an algebra anti-isomorphism, and hence there exists a bijective linear map $S: H_1 \to H_2$ such that ϕ takes one of the forms $\phi(A) = SAS^{-1}$ or $\phi(A) = SA^tS^{-1}$. Since $\phi(A^*) = \phi(A)^*$, we have $S = \alpha U$ for a scalar α and a unitary U. It follows that $\phi(A) = UAU^*$ for every $A \in S(H_1)$ or $\phi(A) = UA^tU^*$ for every $A \in S(H_1)$.

We now consider the infinite-dimensional case. Define $L: C(H_1) \to C(H_2)$ by $L(M) = \phi(P_M)H_2$. Using Lemma 1(iv), it is easy to see that L is an order-isomorphism between $C(H_1)$ and $C(H_2)$, i.e. $L(M) \subseteq L(N)$ if and only if $M \subseteq N$. It follows easily that L is a lattice-isomorphism. By a result of Fillmore and Longstaff [5, Theorem 1], there exists a bicontinuous linear or conjugate linear bijection $S: H_1 \to H_2$ such that $L(M) = SM$, for every $M \in C(H_1)$. This fact, together with Lemma 1(iii, iv), imply that $\phi(P) = SPS^{-1}$ for every projection $P \in S(H_1)$. But every self-adjoint operator is a real-linear combination of a finite number of projections (see [4] and [12]), therefore $\phi(A) = SAS^{-1}$ for every $A \in S(H_1)$.

If S is linear, the fact that SAS^{-1} is a self-adjoint operator for every $A \in S(H_1)$ implies that $S = \alpha U$ for a scalar α and a unitary U. Therefore $\phi(A) = UAU^*$ for every $A \in S(H_1)$.

If S is conjugate linear, then we can write $S = RJ$ where R is linear and J is the conjugation operator with respect to an orthonormal basis $\mathcal{B}$ for H_1; i.e. $(Jx,e) = (e,x)$ for every $e \in \mathcal{B}$. It follows that $\phi(A) = RA^tR^{-1}$

where A^t is the transpose of A with respect to $\mathcal{B}$. As before, R is a scalar multiple of a unitary operator V and $\phi(A) = VA^tV^*$ for every $A \in S(H_1)$. This ends the proof for the complex case.

Case II. Real Hilbert spaces. We need to consider two-dimensional spaces separately. In this case, let $\mathcal{B}_1 = \{x_1, x_2\}$ be an orthonormal basis for H_1. Then $\phi(x_i \otimes x_i)$, $i = 1, 2$, is a rank-one projection by Lemma 1(vi). Since $(x_1 \otimes x_1) \perp (x_2 \otimes x_2)$, we have $\phi(x_1 \otimes x_1) \perp \phi(x_2 \otimes x_2)$. Thus we can choose an orthonormal basis $\mathcal{B}_2 = \{y_1, y_2\}$ for H_2 such that $\phi(x_i \otimes x_i) = y_i \otimes y_i$, $i = 1, 2$. Replacing operators by their matrices relative to the bases $\mathcal{B}_1$ and $\mathcal{B}_2$ we have

$$\phi\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \phi\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

By part (v) of Lemma 1, we have that, $\phi\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$ is a symmetric matrix of the form $\begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix}$ with $\alpha^2 + \beta^2 = 1$. For every $c \in \mathbb{R}$, we have

$$\phi\left(c\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} c+\alpha & \beta \\ \beta & -\alpha \end{bmatrix},$$

and so the matrices $\begin{bmatrix} c & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} c+\alpha & \beta \\ \beta & -\alpha \end{bmatrix}$ have the same spectrum. This implies that $\alpha = 0$ and $\beta = \pm 1$. Replacing y_1 by $-y_1$ if necessary, we may assume that $\phi\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Since $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}$ is a basis for $S(H_1)$ we have $\phi(A) = UAU^*$ for every $A \in S(H_1)$, where

U is the unitary operator mapping $\mathcal{B}_1$ to $\mathcal{B}_2$.

Now assume that $\dim \mathcal{H}_1$ (and hence $\dim \mathcal{H}_2$) ≥ 3. As in case I, the equation $L(M) = \phi(P_M)\mathcal{H}_2$ defines a lattice-isomorphism L between $C(\mathcal{H}_1)$ and $C(\mathcal{H}_2)$. A result of Mackey, implicit in [9], asserts that, except in dimension 2, every such lattice-isomorphism L is induced by an invertible linear operator $S: \mathcal{H}_1 \to \mathcal{H}_2$ via the equation $L(M) = SM$. (In finite-dimensional spaces, Mackey's result is simply the Fundamental Theorem of Projective Geometry [2, p. 44]). As in case I, we get that S is a scalar multiple of a unitary operator U and that $\phi(P) = UPU^*$ for every projection P in $S(\mathcal{H}_1)$.

At this point, the proof diverges from that of Case I; we don't know whether every self-adjoint operator on a real Hilbert space is a linear combination of projections. Instead we use a technique which we used in [7].

For a unit vector $h \in \mathcal{H}_1$ we have $\phi(h \otimes h) = U(h \otimes h)U^* = Uh \otimes Uh$, and hence $\phi(x \otimes x) = Ux \otimes Ux$, for every $x \in \mathcal{H}_1$. Let $A \in S(\mathcal{H}_1)$ and let $\lambda > \|A\|$. Since ϕ preserves the spectrum, we have $\lambda \in \sigma(A + x \otimes x)$ if and only if $\lambda \in \sigma(\phi(A) + Ux \otimes Ux)$ and so by Lemma 2, we have that $((\lambda - A)^{-1}x, x) = 1$ if and only if $((\lambda - \phi(A))^{-1}\, Ux, Ux) = 1$. Since both $(\lambda - A)^{-1}$ and $(\lambda - \phi(A))^{-1}$ are positive operators, it follows that for every $x \in \mathcal{H}_1$ we have

$$((\lambda - A)^{-1}x, x) = ((\lambda - \phi(A))^{-1}\, Ux, Ux).$$

Replacing λ by $\frac{1}{t}$, we get

$$((1 - tA)^{-1}x, x) = ((1 - t\phi(A))^{-1}\, Ux, Ux)$$

for $0 < t < \|A\|^{-1}$. The same equation holds for $t = 0$ since U is unitary. Taking the derivative from the right at $t = 0$ we get $(Ax, x) = (\phi(A)Ux, Ux)$ for every $x \in \mathcal{H}_1$. By polarization, we get $\phi(A) = UAU^*$. ∎

We have the following immediate corollary.

COROLLARY 1. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two complex Hilbert spaces and $\phi\colon \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$ a surjective, adjoint-preserving linear map. If ϕ preserves the spectrum for self-adjoint operators, then ϕ takes one of the forms given in Theorem 1. In particular ϕ preserves the spectrum of every operator.

§2. Invertibility-preserving maps. Now we consider invertibility-preserving linear maps $\phi\colon \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$. For finite-dimensional spaces, we give a characterization of such maps similar to the result of Theorem 1. We note, in passing, that under the assumption that $\phi(I) = I$, the map ϕ is invertibility-preserving if and only if $\sigma(\phi(T)) \subseteq \sigma(T)$ for every $T \in \mathcal{S}(\mathcal{H})$. Our result, below, shows that in finite-dimensional spaces, the above conditions imply that ϕ preserves the spectrum.

THEOREM 2. Let $\mathcal{H}$ be a real or complex finite-dimensional Hilbert space and let $\phi\colon \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$ be an invertibility preserving linear map. Then there exists an invertible operator S on $\mathcal{H}$ such that ϕ has one of the following forms.

(1) $\phi(A) = \pm SAS^*$, for every $A \in \mathcal{S}(\mathcal{H})$,

or

(2) $\phi(A) = \pm SA^tS^*$, for every $A \in \mathcal{S}(\mathcal{H})$.

Before we start the proof, we require the following lemma.

LEMMA 3. Let $A \in S(H)$, H finite-dimensional. Then $\sigma(A+B) \cap \sigma(B)$ is nonempty for every $B \in S(H)$ if and only if $A = 0$.

Proof. The result is obvious if $A = 0$. If $A \neq 0$ then by the spectral theorem $A = A_1 \oplus A_2$ where A_1 is of rank-one and A_2 is invertible. (The second direct summand may be absent). We will construct a self-adjoint operator B of the form $B_1 \oplus kI$ such that $\sigma(A+B) \cap \sigma(B)$ is empty. We first construct B_1 such that $\sigma(A_1+B_1) \cap \sigma(B_1) = \emptyset$. We have $A_1 = \pm x \otimes x$ for a nonzero vector x. Choose a self-adjoint operator B_1 such that none of its eigenvectors is orthogonal to x. (This is equivalent to choosing an orthonormal basis such that every entry of the matrix of A_1 is nonzero and then choosing B_1 to be an operator whose matrix is diagonal with distinct eigenvalues). If $\lambda \in \sigma(A_1+B_1) \cap \sigma(B_1)$, then there exists a nonzero vector y such that $(A_1+B_1)y = \lambda y$, i.e. $(y,x)x = \pm(\lambda-B_1)y$. If $(y,x) = 0$, then y is an eigenvector of B_1 orthogonal to x, a contradiction. If $(y,x) \neq 0$, then $x \in R(\lambda-B_1)$ and so x is orthogonal to $N(\lambda-B_1)$, again contradicting the choice of B_1. We conclude that $\sigma(A_1+B_1) \cap \sigma(B_1)$ is empty. Now, by taking k large enough, it is easy to see that $\sigma(A+B) \cap \sigma(B)$ is empty. ∎

Proof of Theorem 2. The proof is divided into several steps. We will write S for $S(H)$.

Step 1. ϕ is injective (and hence bijective). To prove this assume that $\phi(A) = 0$. For any $T \in S$, if $\lambda I - T$ is invertible, then $\phi(I)\left(\lambda - \phi(I)^{-1}\phi(T)\right)$ is also invertible. It follows that $\sigma\left(\phi(I)^{-1}\phi(T)\right) \subseteq \sigma(T)$. Using this for $T = A + B$ and $T = B$, we get

$$\sigma\left(\phi(I)^{-1}\phi(B)\right) \subseteq \sigma(A+B) \cap \sigma(B)$$

for every $B \in \mathcal{S}$. By Lemma 3, we get that $A = 0$ and ϕ is injective.

Step 2. $\phi(I) = \pm D^2$ for an invertible positive operator D, i.e. $\phi(I)$ is either strictly positive or strictly negative. The invertibility of $\phi(I)$ is obvious. Let $C = \phi^{-1}(I)$, so $\phi(I-\lambda C) = \phi(I) - \lambda I$. It follows that if $\lambda \in \sigma(\phi(I))$, then $\lambda \neq 0$ and $\lambda^{-1} \in \sigma(C)$. Therefore it suffices to show that $C \geq 0$ or $C \leq 0$. Assume to the contrary that C has two eigenvalues λ_1 and λ_2 with $\lambda_1\lambda_2 < 0$. Identifying operators with matrices, we may write C in the form $\begin{bmatrix} \lambda_2 & 0 \\ 0 & \lambda_2 \end{bmatrix} \oplus C_1 \oplus 0$ where C_1 is invertible. (The second or the third direct summands may be absent). Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus 0 \oplus I$. We have that, for for every nonzero real λ, the operator $C + \lambda A$ is invertible. It follows that $I + \lambda\phi(A)$ is invertible for every real λ. This implies that $\phi(A) = 0$, contradicting the fact that ϕ is injective.

Step 3. Define $\psi\colon \mathcal{S} \to \mathcal{S}$ by $\psi(A) = \pm D^{-1}\phi(A)D^{-1}$. The map ψ is an invertibility-preserving bijective linear map and $\psi(I) = I$. It follows easily that $\sigma(\psi(A)) \subseteq \sigma(A)$ for every $A \in \mathcal{S}$.

Step 4. ψ preserves the spectrum. In view of Step 3, it suffices to show that ψ^{-1} preserves invertibility. To prove this, assume that $A \in \mathcal{S}$ and $\psi(A)$ is invertible. Let $A = \sum_{i=1}^{n} c_i E_i$ be the spectral representation of A. By examining the proof of Lemma 1(iii), (iv), we see that the "if parts" are valid under the weaker assumptions that $\sigma(\phi(A)) \subseteq \sigma(A)$ for every $A \in \mathcal{S}$.

We conclude that $\psi(E_i)$, $i = 1,2,\ldots,n$ are mutually orthogonal nonzero projections. We have $\psi(A) = \sum_{i=1}^{n} c_i \psi(E_i)$, and since $\psi(A)$ is invertible, we have $c_i \neq 0$, $1 \leq i \leq n$, and hence A is invertible.

Step 5. Now we apply Theorem 1 to ψ to get a unitary operator $U: \mathcal{H} \to \mathcal{H}$ such that $\psi(A) = UAU^*$ for every $A \in \mathcal{S}$ or $\psi(A) = UA^tU^*$ for every $A \in \mathcal{S}$. If $S = DU$, then $\phi(A) = \pm SAS^*$ or $\phi(A) = \pm SA^tS^*$. ∎

COROLLARY 2. Let $\mathcal{H}$ be a complex finite dimensional Hilbert space and $\phi: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ an adjoint preserving linear map. If ϕ preserves invertibility for self-adjoint operators, then ϕ takes one of the forms given by Theorem 2. In particular ϕ preserves invertibility on $\mathcal{B}(\mathcal{H})$.

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