Domination Parameters of Prisms of Graphs

by

Mark Schurch
B.Sc., Simon Fraser University, 1998

A Thesis Submitted in Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

in the Department of Mathematics and Statistics

University of Victoria

Supervisor: Dr. Christina Mynhardt

Abstract

For a permutation π of the vertex set of a graph G, the graph πG is obtained from two disjoint copies G_1 and G_2 of G by joining each vertex v in G_1 to $\pi(v)$ in G_2. Hence if $\pi = \mathbf{1}$, then $\pi G = K_2 \times G$, the prism of G. For various domination parameters y we investigate lower and upper bounds for $y(\pi G)$ and $y\,(K_2 \times G)$. Specifically we look at regular domination, paired-domination, total domination, connected domination, independent domination and packings. For these domination parameters we also study graphs which obtain the upper or lower bound for $y(\pi G)$ or $y(K_2 \times G)$.

Supervisor: Dr. C. Mynhardt, (Department of Mathematics and Statistics)

Contents

List of Figures

Chapter 1

Preliminaries

In this chapter we define concepts and introduce notation which is relevant to the main body of work. In general we follow the notation and terminology of [2] which the reader may refer to for background on graph theory topics not defined here. We also discuss the motivating results for this research.

1.1 Neighbourhoods

The *open neighbourhood* of a vertex v is the set of all vertices adjacent to v in G and is denoted $N_G(v)$. Often $N_G(v)$ is written $N(v)$ if it is clear which graph is being discussed. A vertex v is said to be a *neighbour* of u if $v \in N(u)$. The *closed neighbourhood* of a vertex v, denoted by $N_G[v]$ or

$N[v]$, is the set $N(v) \cup \{v\}$. For a set $S \subseteq V(G)$ the *open neighbourhood* $N(S)$ of S is defined to be $\cup_{v \in S} N(v)$ and the closed neighbourhood of S is $N[S] = N(S) \cup S$. We will also use the set $N\{S\} = N(S) - S$. For a vertex $s \in S$ the *private neighbourhood of* s relative to S, denoted by $\text{pn}(s, S)$, is the set $N[s] - N[S - \{s\}]$, i.e. $\text{pn}(s, S) = \{v : N[v] \cap S = \{s\}\}$. We also say that a vertex v is a *private neighbour* of s (with respect to S) if $v \in \text{pn}(s, S)$.

1.2 Domination in Graphs

In this section we define various domination parameters in graphs. Refer to [6] for a general background on domination.

For $A, B \subseteq V(G)$, if $B \subseteq N[A]$, we say "*A dominates B*". We abbreviate "A dominates B" to "$A \succ B$", and "$\{a\} \succ B$" to "$a \succ B$", etc. A set $S \subseteq V(G)$ is called a *dominating set* of G if $S \succ V(G)$. The minimum cardinality of a dominating set of G is the *domination number* and is denoted $\gamma(G)$. A minimum dominating set of G is called a *γ-set* of G. A set $S \subseteq V(G)$ is called a *total dominating* set of G if every vertex of G is adjacent to a vertex in S. The minimum cardinality of a total dominating set of G is the *total domination number* $\gamma_t(G)$ and a minimum total dominating set is a *γ_t-set.*

A set $S \subseteq V(G)$ is a *connected dominating set* of G if S is a dominating set and $\langle S \rangle$ is connected, where $\langle S \rangle$ denotes the subgraph induced by S. The minimum cardinality of a connected dominating set is the *connected domination number* $\gamma_c(G)$ and a minimum connected dominating set is a γ_c*-set.* Since every connected dominating set is also a total dominating set it follows that $\gamma_t(G) \leq \gamma_c(G)$.

A *matching* of the graph G is a set of edges such that no two edges in the set have a common vertex. The largest possible matching of a graph with n vertices consists of $n/2$ edges and is called a *perfect matching.* A set $S \subseteq V(G)$ is called a *paired-dominating set* of G if S is a dominating set and $\langle S \rangle$ contains a perfect matching. The minimum cardinality of a paired-dominating set is the *paired-domination number* $\gamma_{pr}(G)$ and a minimum paired-dominating set is a γ_{pr}*-set.* Refer to [7] for more background and results on paired-domination.

In order to help visualize and understand the different types of domination we can think of each $s \in S$ as the location of a guard capable of protecting each vertex dominated by s. Then S is a dominating set if all of G is protected, and a total dominating set if, in addition, each guard is protected by another guard. For paired-domination the location of the guards must be selected such that each guard is paired with an adjacent guard to be backups for each other and a guard cannot be the backup for two guards. Both total domination and paired-domination require that there be no isolated vertices. Note that every paired-dominating set is a total dominating set and every total dominating set is a dominating set, which implies

$$\gamma(G) \leq \gamma_t(G) \leq \gamma_{pr}(G). \tag{1.1}$$

A set $S \subseteq V(G)$ is called an *independent dominating set* of G if S is both a dominating set of G and an independent set. The minimum cardinality of an independent dominating set is the *independent domination number* $i(G)$ and a minimum independent dominating set is an *i-set.* Since every independent dominating set is a dominating set it follows that $\gamma(G) \leq i(G)$ for all graphs G.

1.3 Packings

A set $S \subseteq V(G)$ is called a *packing* or *2-packing* if for each pair of vertices $u, v \in S$, $N[u] \cap N[v] = \varnothing$, i.e. $d(u, v) \geq 3$. The maximum cardinality of a packing is the *packing number* $\rho(G)$ and a maximum packing is a *ρ-set.* For any graph G, $\rho(G) \leq \gamma(G)$ and for any tree T, $\rho(T) = \gamma(T)$ [8]. The minimum cardinality of a maximal packing is the *lower packing number* $\rho_l(G)$ and a minimum maximal packing is a *ρ_l-set.* An example of a ρ-set and a ρ_l-set are given in Figures 1.1 and 1.2 respectively.

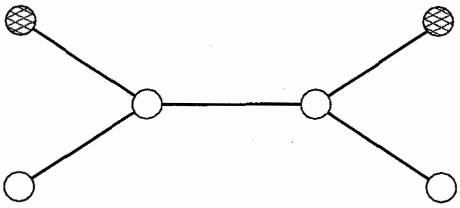

Figure 1.1: A ρ-set.

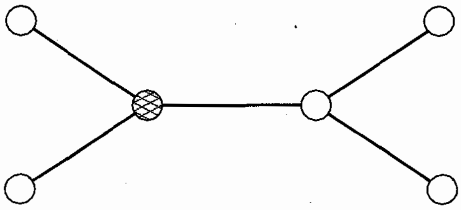

Figure 1.2: A ρ_l-set.

1.4 Graph Products

Given graphs G and H we can define a *graph product* $G \otimes H$ to be a graph with vertex set $V(G) \times V(H)$ in which $E(G \otimes H)$ depends in some way on $E(G)$ and $E(H)$. That is, in general, for all $a, b \in V(G)$ and $x, y \in V(H)$, the adjacency of (a, x) and (b, y) is determined by the adjacency, equality or non-adjacency of a and b and that of x and y. Thus for two distinct vertices of $G \otimes H$, (a, x) and (b, y) there are $3 \times 3 - 1 = 8$ possible relationships of a, b, x, and y which determine whether $(a, x)(b, y) \in E(G \otimes H)$ or $(a, x)(b, y) \notin E(G \otimes H)$. Thus using this description there are 256 different ways to define a graph product with vertex set $V(G) \times V(H)$.

We are concerned with the *Cartesian product* which will be denoted $G \times H$. The edge set of $G \times H$ is defined as

$$\begin{aligned} E(G \times H) = \{ &(a, x)(b, y) \mid a, b \in V(G), x, y \in V(H), xy \in E(H) \text{ and} \\ &a = b, \text{ or } ab \in E(G) \text{ and } x = y \}. \end{aligned} \tag{1.2}$$

There is much interest in the problem of dominating the Cartesian product of two graphs, most of which has been motivated by V. G. Vizing and his

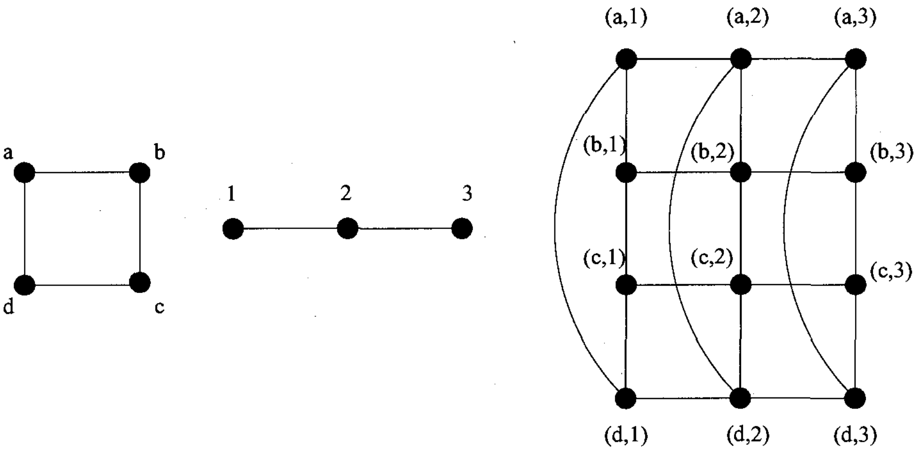

Figure 1.3: C_4, P_3 and $C_4 \times P_3$

famous conjecture. Vizing's Conjecture states that for every pair of graphs G and H, $\gamma(G \times H) \geq \gamma(G)\gamma(H)$. Vizing first discussed this problem in 1963 [10] and he formally conjectured it five years later [11]. Much work has been done on this conjecture and most of the results show that the conjecture is true for graphs with a particular property. For example, Sun [9] recently proved that Vizing's conjecture is true when $\gamma(G) = 3$. See [4] for a summary of results on Vizing's conjecture.

The graph $K_2 \times G$ is sometimes referred to as the *prism* of (or over) G. From the definition of the Cartesian product of two graphs we see that $K_2 \times G$ may be obtained from two disjoint copies G_1 and G_2 of G and joining each vertex in G_1 to its corresponding vertex in G_2.

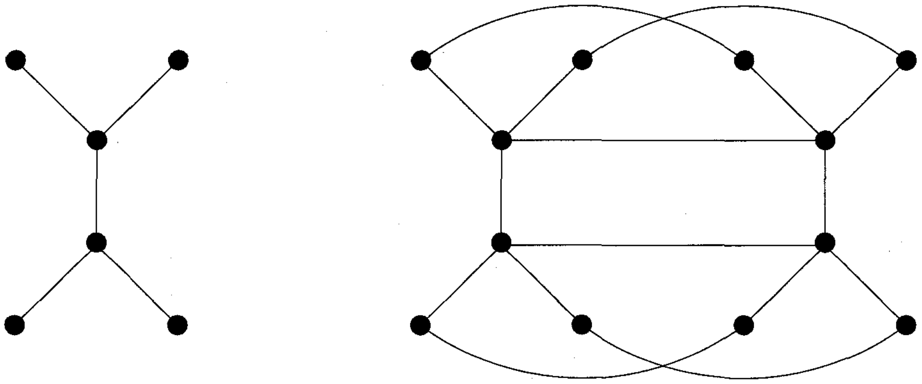

Figure 1.4: G and $K_2 \times G$

For a permutation π of the vertex set of a graph G, the graph πG is obtained from two disjoint copies G_1 and G_2 of G by joining each vertex v in G_1 to $\pi(v)$ in G_2. Hence if $\pi = \mathbf{1}$, then $\pi G = K_2 \times G$. Therefore we may also think of πG as the prism of G with respect to π.

For any vertex v of G, we denote the corresponding vertex in the subgraph G_i, $i = 1, 2$, of πG by v_i. Similarly, any set $S \subseteq V(G)$ will be denoted by S_i when considered in the subgraph G_i of πG. Also the set $S_i \subseteq V(G_i)$ (vertex $v_i \in V(G_i)$ respectively) will be denoted by S (v respectively) in G.

It is easy to see that for all graphs G, $\gamma(G) \leq \gamma(\pi G) \leq 2\gamma(G)$. Graphs for which $\gamma(K_2 \times G) = \gamma(G)$ are called *prism γ-fixers* and those for which $\gamma(K_2 \times G) = 2\gamma(G)$ are called *prism γ-doublers.* Graphs for which $\gamma(\pi G) =$

$\gamma(G)$ for all permutations π of $V(G)$, are called *universal γ-fixers* , and those for which $\gamma(\pi G) = 2\gamma(G)$ for all permutations π of $V(G)$, are called *universal γ-doublers.*

More generally, for any graph parameter $y(G)$, graphs for which $y\ (K_2 \times G) = y(G)$ are called *prism y-fixers* and those for which $y(K_2 \times G) = 2y(G)$ are called *prism y-doublers.* Graphs for which $y(\pi G) = y(G)$ for all permutations π of $V(G)$, are called *universal y-fixers,* and those for which $y(\pi G) = 2y(G)$ for all permutations π of $V(G)$, are called *universal y-doublers.*

1.5 Prism γ-Doublers and Universal γ-Doublers

In this section we state some results from [1] on prism γ-doublers and universal γ-doublers which were the motivation for this work.

Proposition 1 [1] *A graph G is a universal γ-doubler if and only if for each $X \subseteq V(G)$ with $0 < |X| < \gamma(G)$, $|V(G) - N[X]| \geq 2\gamma(G) - |X|$.*

For example, since $\gamma(C_6) = 2$ and $|N[v]| = 3$ for any vertex $v \in V(C_6)$, any set X as described in Proposition 1 satisfies the necessary requirements for C_6 to be a universal γ-doubler. Similarly, P_5, P_6 and C_5 are also universal γ-doublers.

Corollary 2 [1] *If G is a universal γ-doubler, then G has no isolated vertices and every vertex of G contained in a minimum dominating set has degree at least $\gamma(G)$.*

This result easily shows that P_n and C_n, $n \geq 7$, are not universal γ-doublers.

A set D is called an *efficient* dominating set if D is a dominating set and for each pair of vertices $u, v \in D$, $N[u] \cap N[v] = \varnothing$. Note that if D is an efficient dominating set of G then $|D| = \gamma(G)$.

Corollary 3 [1] *Let G be a universal γ-doubler of order n. If G has an efficient dominating set, then $\gamma(G) \leq \sqrt{n + 0.25} - 0.5$. Otherwise, for any nonempty packing X contained in a minimum dominating set of G we have $\gamma(G) \leq n/(|X| + 2)$.*

Corollary 4 [1] *If G is an r-regular graph which has an efficient dominating set and $r \geq \gamma(G)$, then G is a universal γ-doubler.*

For example, $G = Q_3$ [1], where $Q_1 = K_2$ and $Q_n = Q_{n-1} \times K_2$ for $n \geq 2$, satisfies the hypothesis of Corollary 4 with $r = 3$ and $\gamma(G) = 2$, hence $\gamma(\pi G) = 4$ for each permutation π of $V(G)$.

Theorem 5 [1] *A graph G is a prism γ-doubler if and only if for each pair of sets $X, Y \subseteq V(G)$ with $0 < |X| < \gamma(G)$ and $Y = V(G) - N[X]$, either*

(a) $|Y| \geq 2\gamma(G) - |X|$*, or*

(b) $|Y| = 2\gamma(G) - |X| - d$*, for some d, $1 \leq d \leq |X|$, and at least d vertices (necessarily in $N[X]$) are required to dominate $N\{X\} - N[Y]$.*

1.6 Prism γ-Fixers

In this section we include a result from [4] on prism γ-fixers.

Theorem 6 [4] *A connected graph G is a prism γ-fixer if and only if G has a γ-set W that partitions into two nonempty subsets X and D such that $G - N[X] = D$ and $G - N[D] = X$.*

An example of a graph satisfying the conditions of Theorem 6 is given in Figure 1.5. The grey vertices represent the set X and the black vertices represent the set D as described in Theorem 6.

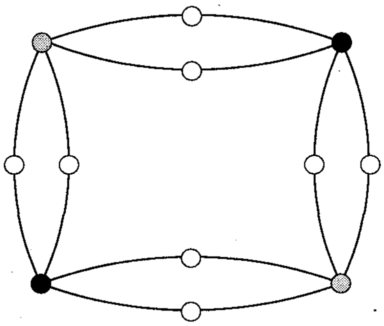

Figure 1.5: A prism γ-fixer.

1.7 An Overview

The main goal of this work is to provide similar results to those in [1] and [4] for various domination parameters. We will consider paired-domination, total domination, connected domination, independent domination and packings. For the various graph parameters our results and their proofs share common features with one another and with the results in [1] and [4]. Some results are easy observations and are included for completeness.

In Chapter 2 we investigate graphs which are prism y-doublers or universal y-doublers for various domination parameters y. In Chapter 3 we investigate graphs which are prism y-fixers with mention of universal y-fixers for some of the parameters. If possible we provide characterizations and examples of such graphs.

Chapter 2

Doublers

2.1 Paired-Domination

In this section we discuss universal γ_{pr}-doublers and prism γ_{pr}-doublers. All graphs in this section are isolate-free.

2.1.1 Universal γ_{pr}-Doublers

Observation 7 *For any isolate-free graph G and any permutation π of $V(G)$,*

$$\gamma_{pr}(\pi G) \leq 2\gamma_{pr}(G).$$

Lemma 8 *If a graph G is a prism γ_{pr}-doubler, then $\gamma_{pr}(G) = \gamma(G)$.*

Proof. Suppose G is a prism γ_{pr}-doubler and $\gamma_{pr}(G) \neq \gamma(G)$. Since $\gamma(G) \leq \gamma_{pr}(G)$ for all isolate-free graphs we have $\gamma(G) < \gamma_{pr}(G)$. Let X be any γ-set of G. Then $W = X_1 \cup X_2$ is a paired-dominating set of $K_2 \times G$ (since for each vertex $x \in X$, x_1 can be paired with x_2) and $|W| = 2\gamma(G) < 2\gamma_{pr}(G)$, a contradiction. ■

Corollary 9 *If a graph G is a universal γ_{pr}-doubler, then $\gamma_{pr}(G) = \gamma(G)$.*

Lemma 10 *If $\gamma(G) = \gamma_{pr}(G)$, then $|V(G)| \geq 2\gamma_{pr}(G)$ and there exists a paired-dominating set of cardinality $\gamma_{pr}(G) + 2i$ for $1 \leq i \leq \gamma_{pr}(G)/2$.*

Proof. Suppose $\gamma(G) = \gamma_{pr}(G)$. Then every vertex in a γ_{pr}-set X has at least one private neighbour in $V(G) - X$ or else we could form a smaller dominating set. Therefore $|V(G)| \geq 2\gamma_{pr}(G)$. To obtain paired-dominating sets of cardinality $\gamma_{pr}(G) + 2i$ for $1 \leq i \leq \gamma_{pr}(G)/2$, we observe that each pair of vertices in X can be split into two pairs since each vertex of each pair has a private neighbour in $V(G) - X$. ■

If D is a paired-dominating set of G and M a perfect matching of $\langle D \rangle$, we call M a *D-matching* of G. If M is a matching in G and v a vertex of G that does not belong to any edge of M, we call v an $\overline{M}$-vertex of G.

We now define notation that will be used throughout this section. Let

$$\left.\begin{array}{l} X \subseteq V(G) \text{ such that } 0 < |X| < \gamma_{pr}(G); \\[1em] Y = V(G) - N[X]; \\[1em] M \text{ be a maximum matching of } \langle X \rangle ; \\[1em] Z = X - V(M), \text{ i.e. } Z \text{ is the set of } \overline{M}\text{-vertices in } X; \\[1em] k = |Z|. \end{array}\right\} \tag{2.1}$$

Theorem 11 *A graph G is a universal γ_{pr}-doubler if and only if for each pair of sets $X, Y \subseteq V(G)$ as defined in* (2.1)*, $|Y| \geq 2\gamma_{pr}(G) - |X| - k - 1$.*

Proof. Suppose for some $X \subseteq V(G)$ with $0 < |X| < \gamma_{pr}(G)$, $|Y| < 2\gamma_{pr}(G) - |X| - k - 1$.

Case 1. k is even. Then $|X|$ is even. If $|Y| + k \leq \gamma_{pr}(G)$, then let D be any γ_{pr}-set of G, otherwise let D be any paired-dominating set with $|D| = |Y| + k$ if $|Y|$ is even or $|D| = |Y| + k + 1$ if $|Y|$ is odd. (A paired-dominating set of this size exists by Lemma 10.) Let π be any permutation of $V(G)$ such that $\pi(Y \cup Z) \subseteq D$ and $\langle \pi(Z) \rangle$ has a perfect matching M' which is contained in a D-matching. Then $W = X_1 \cup D_2 \succ \pi G$ and $\langle W \rangle$ contains a perfect matching in which each edge $u_2 v_2$ in M_2' is replaced by two edges $z_1 u_2$ and $z_1' v_2$, where $z, z' \in Z$. Therefore W is a paired-dominating set of πG with

$$\begin{aligned}|W| &= |X| + |D| \\ &\leq |X| + |Y| + k + 1 \\ &< |X| + (2\gamma_{pr}(G) - |X| - k - 1) + k + 1 \\ &= 2\gamma_{pr}(G),\end{aligned}$$

hence G is not a universal γ_{pr}-doubler.

Case 2. k is odd. Then $|X|$ is odd. If $|X| = \gamma_{pr}(G) - 1$, then $|Y| \leq \gamma_{pr}(G) - k - 1$. Let π be any permutation of $V(G)$ such that $\pi(Y) \subseteq X - Z$, $\pi(Z) \subseteq Z$ and $Y \subseteq \pi(X - Z)$. Then $W = X_1 \cup X_2 \succ \pi G$ and it is easy to see that $\langle W \rangle$ contains a perfect matching. Therefore W is a paired-dominating set and $|W| = 2|X| = 2\gamma_{pr}(G) - 2$, so again G is not a universal γ_{pr}-doubler. We now consider $0 < |X| < \gamma_{pr}(G) - 2$.

If $|Y| + k \leq \gamma_{pr}(G)$, then let D be any γ_{pr}-set of G, otherwise let D be a paired-dominating set of G where $|D| = |Y| + k - 1$ if $|Y|$ is even or $|D| = |Y| + k$ if $|Y|$ is odd. Let $v \in Z$ and let π be any permutation of $V(G)$ such that $\pi(Y \cup Z - v) \subseteq D$, $D - \pi(Z - v)$ contains a perfect matching M' which is contained in a D-matching and $\pi(v) = v' \in V(G) - D$. Then $W = X_1 \cup D_2 \cup \{v_2'\} \succ \pi G$ and $\langle W \rangle$ contains a perfect matching in which

v_1 is paired with v_2', each edge u_2w_2 in M_2' is replaced by two edges z_1u_2 and $z_1'w_2$, where $z, z' \in Z - v$. Therefore W is a paired-dominating set of πG and since $0 < |X| < \gamma_{pr}(G) - 2$ and $|D| \leq \gamma_{pr}(G)$,

$$\begin{aligned}|W| &= |X| + |D| + 1\\ &< 2\gamma_{pr}(G) - 1.\end{aligned}$$

Conversely, let π be a permutation of $V(G)$ such that $\gamma_{pr}(\pi G) < 2\gamma_{pr}(G) - 1$ and let W be a γ_{pr}-set of πG. Say $X_1 = W \cap V(G_1)$ and $D_2 = W \cap V(G_2)$. Without loss of generality let $|X_1| < \gamma_{pr}(G)$. Let M' be a perfect matching of $\langle W \rangle$ and $D_2' \subseteq D_2$ be the set of vertices in D_2 which are not paired with another vertex in D_2 under M'. Let $|D_2'| = k'$ and k be the number of vertices not paired in a maximum matching of $\langle X \rangle$. Note that $k \leq k'$.

If $X_1 \neq \varnothing$ then $|D_2| < 2\gamma_{pr}(G) - |X| - 1$ and each vertex of $D_2 - D_2'$ dominates at most one vertex of Y_1, while no vertex in D_2' dominates a vertex in Y_1. Therefore $|Y_1| \leq |D_2 - D_2'|$ which implies that $|Y| < 2\gamma_{pr}(G) - |X| - k' - 1 \leq 2\gamma_{pr}(G) - |X| - k - 1$.

If $X_1 = \varnothing$ then $D_2 \succ V(\pi G)$ for all permutations π, which means $D_2 = V(G_2)$ in order to dominate $V(G_1)$. Therefore $|V(G)| = |D_2| < 2\gamma_{pr}(G)$, which implies, by Lemma 10, $\gamma(G) < \gamma_{pr}(G)$. Let X' be a γ-set of G, $Y' = V(G) - N[X']$ and k' be the number of vertices not paired in a maximum matching of $\langle X' \rangle$. Then $|Y'| = 0 < 2\gamma_{pr}(G) - |X'| - k' - 1$ since $k' \leq |X'| < \gamma_{pr}(G)$. ■

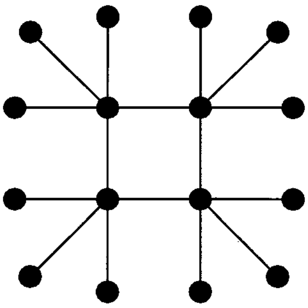

Figure 2.1: $F_4 \in \mathcal{F}$: An example of a universal γ_{pr}-doubler.

As an example of universal γ_{pr}-doublers consider the family of graphs $\mathcal{F}$. Form the graph $F_{2n} \in \mathcal{F}$ by joining each vertex of C_{2n} to $2n-1$ new vertices. Note that $\gamma_{pr}(F_{2n}) = \gamma(F_{2n}) = 2n$. Figure 2.1 shows the graph F_4. By Proposition 11, to prove that F_{2n} is a universal γ_{pr}-doubler we must show that for each pair of sets $X, Y \subseteq V(F_{2n})$ as defined in (2.1), $|Y| \geq 2\gamma_{pr}(F_{2n}) - |X| - k - 1 = 4n - |X| - k - 1$. Suppose $|X| = 2n - d$ where $1 \leq d \leq 2n-1$. Then $|Y| \geq d(2n-1)$.

If $d = 1$, then $k \geq 1$ which implies

$$\begin{aligned} 2\gamma_{pr}(F_{2n}) - |X| - k - 1 &\leq 4n - (2n-1) - 1 - 1 \\ &= 2n - 1 \\ &\leq |Y|. \end{aligned}$$

If $2 \leq d \leq 2n-1$, then $k \geq 0$ which implies

$$\begin{aligned} 2\gamma_{pr}(F_{2n}) - |X| - k - 1 &\leq 4n - (2n-d) - 1 \\ &= 2n + d - 1 \\ &\leq 2n + (2n-1) - 1 \\ &= 2(2n-1) \\ &\leq d(2n-1) \\ &\leq |Y|. \end{aligned}$$

Note that to construct a universal γ_{pr}-doubler G from C_{2n} with $\gamma_{pr}(G) = 2n$, each vertex of C_{2n} must be joined to at least $2n-1$ new vertices. If some vertex of C_{2n} is joined to more than $2n-1$ new vertices, the resulting graph is also a universal γ_{pr}-doubler.

Corollary 12 *If $\gamma(G) = \gamma_{pr}(G) = 2$, then G is a universal γ_{pr}-doubler.*

Proof. Suppose G is graph with $\gamma(G) = \gamma_{pr}(G) = 2$. Let $x \in V(G)$ and $Y = V(G) - N[x]$. Since $\gamma(G) = 2$, $|Y| \geq 1$. The result follows from Theorem 11. ■

By Corollary 12 the double star $S(k, l)$ with $k, l \geq 1$ is a universal γ_{pr}-doubler. Figure 2.2 contains the double star $S(3, 2)$.

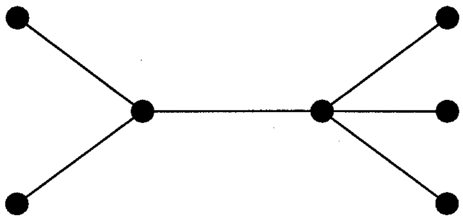

Figure 2.2: The double star $S(3, 2)$.

Corollary 13 *Let G be a universal γ_{pr}-doubler and D any γ_{pr}-set of G. For each $v \in D$, $|\mathrm{pn}(v, D)| \geq \gamma_{pr}(G) - 1$.*

Proof. Let $X = D - \{v\}$. Then $0 < |X| < \gamma_{pr}(G)$ since $\gamma_{pr}(G) \geq 2$. Therefore by Theorem 11 we have $|V(G) - N[X]| \geq 2\gamma_{pr}(G) - |X| - k - 1 = \gamma_{pr}(G) - 1$ because there is only one vertex in X which is not paired, i.e. $k = 1$. Since D is a dominating set, we have $v \succ V(G) - N[X]$. Since $X = D - \{v\}$ we have $\text{pn}(v, D) \subseteq V(G) - N[X]$, thus $|\text{pn}(v, D)| \geq \gamma_{pr}(G) - 1$. ■

The converse of Corollary 13 is shown to be false by the counterexample in Figure 2.3. In the example the black vertices represent the set D which is the only possible γ_{pr}-set of G. Clearly, for all $v \in D$ we have $|\text{pn}(v, D)| = 3 \geq \gamma_{pr}(G) - 1$. Let the circled vertices represent the set X and let $Y = V(G) - N[X]$. Then $|Y| = 2$ but by Theorem 11, in order for G to be a universal γ_{pr}-doubler, $|Y| \geq 2\gamma_{pr}(G) - |X| - k - 1 = 3$.

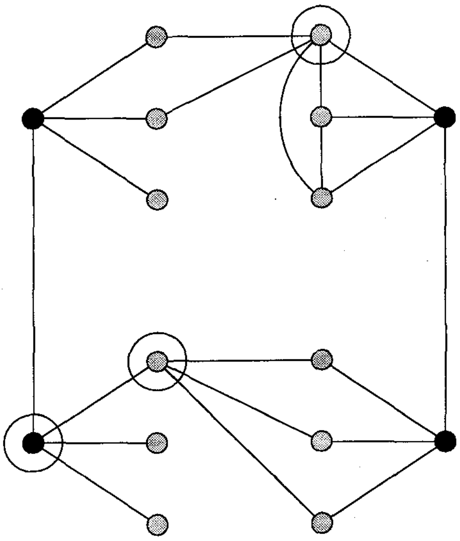

Figure 2.3: A counterexample to the converse of Corollary 13.

Corollary 14 *Let G be a universal γ_{pr}-doubler. Then every vertex of G that is contained in a minimum paired-dominating set of G has degree at least $\gamma_{pr}(G)$.*

Proof. Let $v \in V(G)$ be a vertex contained in some minimum paired-dominating set D of G. By Corollary 13, for all $v \in D$ we have $|\text{pn}(v, D)| \geq \gamma_{pr}(G) - 1$. Since D is a paired-dominating set, v has a neighbour in $D - v$, which implies $deg(v) \geq \gamma_{pr}(G)$. ■

We next present a counterexample to the converse of Corollary 14. Let $G = K_n$, $n \geq 3$. Since $\gamma_{pr}(G) = 2$ we see that every vertex contained in a minimum paired-dominating set has degree at least $\gamma_{pr}(G)$. But G is not a universal γ_{pr}-doubler since $\gamma_{pr}(\pi G) = 2$ for all permutations π of G. In fact this family of graphs is an example of universal γ_{pr}-fixers.

Corollary 15 *If G is a universal γ_{pr}-doubler of order n, then $\gamma_{pr}(G) \leq \sqrt{n}$.*

Proof. By Corollary 13, if D is a γ_{pr}-set of G then $|\text{pn}(v, D)| \geq \gamma_{pr}(G) - 1$ for each $v \in D$. Hence $n \geq \gamma_{pr}(G) + \gamma_{pr}(G)(\gamma_{pr}(G) - 1) = \gamma_{pr}(G)^2$, so that $\gamma_{pr}(G) \leq \sqrt{n}$. ■

Define an *efficient* paired-dominating set D as a γ_{pr}-set such that for any two vertices $u, v \in D$, $N(u) \cap N(v) = \varnothing$.

Observation 16 *If G is regular and has an efficient paired-dominating set D, then $|D| = \gamma_{pr}(G)$.*

Proof. Suppose to the contrary that G is an r-regular graph with an efficient paired-dominating set D but $|D| > \gamma_{pr}(G)$. Since D is an efficient paired-dominating set of G, $|V(G)| = r|D|$. Let X be a γ_{pr}-set of G. Then $|V(G)| \leq r|X|$, which is a contradiction since $|X| < |D|$. ■

Corollary 17 *If G is an r-regular graph and has an efficient paired-dominating set with $r \geq \gamma_{pr}(G)$, then G is a universal γ_{pr}-doubler.*

Proof. Let G be a graph which satisies the hypothesis. Let $X \subseteq V(G)$, with $0 < |X| < \gamma_{pr}(G)$ and k be the number of vertices not paired in maximum matching of $\langle X \rangle$; then $|N[X]| \leq r|X| + k$. Since G contains an efficient paired-dominating set, $|V(G)| = r\gamma_{pr}(G)$. Let $Y = V(G) - N[X]$. Then $|Y| \geq r\gamma_{pr}(G) - r|X| - k$ and since $r \geq \gamma_{pr}(G)$ we have $|Y| \geq \gamma_{pr}(G)(\gamma_{pr}(G) - |X|) - k$.

Suppose $|X| = \gamma_{pr}(G) - 1$. Then $|Y| \geq \gamma_{pr}(G) - k = 2\gamma_{pr}(G) - |X| - k - 1$ so it follows by Theorem 11 that G is a universal γ_{pr}-doubler.

Suppose $|X| < \gamma_{pr}(G) - 1$. Then $|Y| \geq 2\gamma_{pr}(G)$, so we again have satisfied the necessary requirements of Theorem 11. ■

Corollary 17 allows us to construct a family of regular graphs which are universal γ_{pr}-doublers. Construct the family of graphs $\mathcal{H}$ as follows. Begin by taking a cycle of even length, C_{2n} and removing every second edge. For each removed edge uv, take a copy of the complete bipartite graph $K_{r-1,r-1}$, $r \geq 2n$ with partite sets U, V, and join u to each of the vertices in U and v to each of the vertices in V. See Figure 2.4 for an example of the graph in $\mathcal{H}$ constructed from C_4. Let $H \in \mathcal{H}$. It is easy to see that H is r-regular so we need only show that H has an efficient paired-dominating set of cardinality less than or equal to r. First we note that dominating a graph $H \in \mathcal{H}$ which was formed by starting with a cycle of length $2n$ is equivalent to dominating C_{4n} and it is easy to see that $\gamma_{pr}(C_{4n}) = 2n \leq r$. Moreover, the $2n$ vertices which originally made up the graph of C_{2n} form an efficient paired-dominating set of H. Hence by Corollary 17, H is a universal γ_{pr}-doubler.

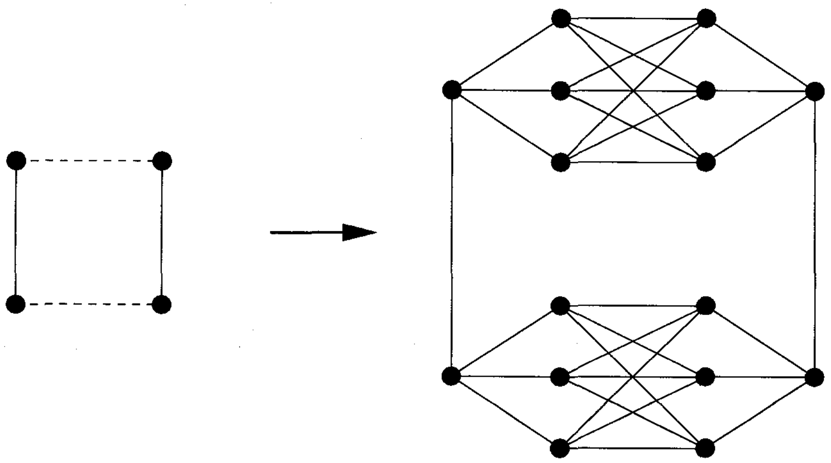

Figure 2.4: The graph in $\mathcal{H}$ constructed from C_4.

2.1.2 Prism γ_{pr}-Doublers

Observation 18 *For any isolate-free graph G, $\gamma_{pr}(K_2 \times G) \leq 2\gamma(G)$.*

Theorem 19 *A graph G is a prism γ_{pr}-doubler if and only if for each pair of sets X, Y as defined in* (2.1), *either*

(*a*) $|Y| \geq 2\gamma_{pr}(G) - |X| - k - 1$, *or*

(*b*) $|Y| = 2\gamma_{pr}(G) - |X| - k - d - 1$ *where* $d \geq 1$, *and if* $A \subseteq N[X] - Z$ *dominates* $N\{X\} - N[Y] - N[Z]$ *and* $\langle A \cup Y \rangle$ *has a perfect matching, then* $|A| \geq d + 1$.

Proof. Suppose $\gamma_{pr}(K_2 \times G) = 2\gamma_{pr}(G)$ and consider any pair of sets X, Y as defined in (2.1) and a maximum matching M of $\langle X \rangle$. If $|Y| \geq 2\gamma_{pr}(G) - |X| - k - 1$ then we are done, so assume $|Y| = 2\gamma_{pr}(G) - |X| - k - d - 1$ for some $d \geq 1$. Suppose to the contrary that there exists a set $A \subseteq N[X] - Z$ such that $A \succ N\{X\} - N[Y] - N[Z]$ and $\langle A \cup Y \rangle$ has a perfect matching M^*, but $|A| \leq d$. If $|A| = d$, then $|A \cup Y| = 2\gamma_{pr}(G) - |X| - k - 1$ is an odd number since $|X|$ and k have the same parity. But $\langle A \cup Y \rangle$ has a perfect matching, a contradiction. Hence $|A| \leq d - 1$. Consider the set $W \subseteq V(K_2 \times G)$ where $W = X_1 \cup Y_2 \cup A_2 \cup Z_2$. By our choice of X and Y it follows that $X_1 \cup Y_2 \succ V(G_1)$. Further, since $A_2 \succ N\{X_2\} - N[Y_2] - N[Z_2]$, $X_1 \cup Y_2 \cup A_2 \cup Z_2 \succ V(G_2)$. Thus $W \succ K_2 \times G$. Moreover, $M \cup M^* \cup \{z_1 z_2 : z \in Z\}$ is a perfect matching of $\langle W \rangle$, and it follows that W is a paired-dominating set of $K_2 \times G$. But

$$\begin{aligned} |W| &= |X| + |Y| + |Z| + |A| \\ &\leq |X| + (2\gamma_{pr}(G) - |X| - k - 1 - d) + k + (d - 1) \\ &= 2\gamma_{pr}(G) - 2, \end{aligned}$$

a contradiction, hence (b) holds.

Conversely, suppose $\gamma_{pr}(K_2 \times G) < 2\gamma_{pr}(G) - 1$ and let $W = X_1 \cup D_2$ be a minimum paired-dominating set of $K_2 \times G$ and M be a perfect matching of $\langle W \rangle$. Let Z_1 be the set of vertices in $\langle X_1 \rangle$ which are not paired under M and $k^* = |Z|$. Without loss of generality we may assume $|X| < \gamma_{pr}(G)$ and $|D| < 2\gamma_{pr}(G) - |X| - 1$.

Suppose $X = \varnothing$. Then $D_2 \succ K_2 \times G$ which means $D_2 = V(G_2)$, i.e., $D = V(G)$. Therefore $|W| = |D| = |V(G)| \leq 2\gamma_{pr}(G) - 2$. By Lemma 10, this implies that $\gamma(G) < \gamma_{pr}(G)$. Let X' be a γ-set of G, M' be a maximum matching of $\langle X' \rangle$, Z' the set of $\overline{M'}$-vertices in X' and $k' = |Z'|$. Note that $k' > 0$ since X' is not a paired-dominating set of G. Since $X' \succ G$, we have $Y' = V(G) - N[X'] = \varnothing$. Since $0 < |X'| < \gamma_{pr}(G)$, X', Y' are equivalent to the sets X, Y as defined in (2.1). But $2\gamma_{pr}(G) - |X'| - k' - 1 \geq 2\gamma_{pr}(G) - 2|X'| - 1 > 0 = |Y'|$, hence Y' does not satisfy (a).

If Y' does not satisfy (b) then we are done, so suppose that Y' satisfies (b). Then $|Y'| = 0 = 2\gamma_{pr}(G) - |X'| - k' - 1 - d$ which means $d = 2\gamma_{pr}(G) - |X'| - k' - 1$. Let $A' = X' - Z'$. Then $A' \subseteq N[X'] - Z'$, $A' \succ N\{X'\} - N[Y'] - N[Z']$

and since $Y' = \varnothing$, M' is a perfect matching of $\langle A' \cup Y' \rangle$. But

$$\begin{aligned} |A'| &= |X'| - k' \\ &= 2|X'| - |X'| - k' \\ &< 2\gamma_{pr}(G) - |X'| - k' - 1 \\ &= d, \end{aligned}$$

contradicting (b).

Now suppose $X \neq \varnothing$. Let $Y = V(G) - N[X]$ and let $A = D - Z - Y$. Then $A \subseteq N[X] - Z$, $A \cup Y$ contains a perfect matching and $A \succ N\{X\} - N[Y] - N[Z]$. Let M be a maximum matching of $\langle X \rangle$ and let $k = |X - V(M)|$. Note that $k \leq k^*$. Since $Y = D - Z - A$, $|Y| < 2\gamma_{pr}(G) - |X| - k^* - |A| - 1 \leq 2\gamma_{pr}(G) - |X| - k - |A| - 1$. Thus we have $|Y| = 2\gamma_{pr}(G) - |X| - k - d - 1$ for some value of d with $d > |A|$, hence (b) does not hold. ■

Observation 20 [7] *If a vertex $u \in V(G)$ is adjacent to a leaf, then u is in every paired-dominating set of G.*

Corollary 21 *If every vertex which is contained in a minimum paired-dominating set of G is adjacent to at least one leaf, then G is a prism γ_{pr}-doubler.*

Proof. Let G be a graph which satisfies the hypothesis. Then by Observation 20, $\gamma_{pr}(G) = k$ where k is the number of vertices in G which are adjacent to a leaf. Let $u \in V(G)$ be a vertex which is adjacent to the leaves $v^1, v^2, ...v^r$ where $r \geq 1$. Then for all such vertices $u \in V(G)$, every paired-dominating set of $K_2 \times G$ contains at least two vertices from the set $\{u_1, u_2, v_1^1, v_2^1, v_1^2, v_2^2, ..., v_1^r, v_2^r\}$. Thus by Observation 18, $\gamma_{pr}(K_2 \times G) = 2k$.
■

Now we introduce a method of producing graphs which are prism γ_{pr}-doublers but not universal γ_{pr}-doublers. Suppose a graph G satisfies the hypothesis of Corollary 21, hence is a prism γ_{pr}-doubler. If there exists a vertex v in a γ_{pr}-set of G such that $deg(v) < \gamma_{pr}(G)$, then by Corollary 14, G is not a universal γ_{pr}-doubler. Therefore, if we take any graph H with $n \geq 4$ vertices such that H contains a perfect matching and $\delta(H) \leq n - 2$, adding a leaf to each vertex of H will produce a graph G which is a prism γ_{pr}-doubler but not a universal γ_{pr}-doubler. For example, add a leaf to each vertex of P_{2n} or C_{2n} for $n \geq 2$.

2.2 Total Domination

In this section we discuss universal γ_t-doublers and prism γ_t-doublers. All graphs in this section are isolate-free.

2.2.1 Universal γ_t-Doublers

Observation 22 *For any isolate-free graph G and any permutation π of $V(G)$, $\gamma_t(\pi G) \leq 2\gamma_t(G)$.*

Lemma 23 *If G is isolate-free and a prism γ_t-doubler, then $\gamma(G) = \gamma_t(G)$.*

Proof. Suppose G is a graph without isolated vertices and $\gamma(G) \neq \gamma_t(G)$. This implies $\gamma(G) < \gamma_t(G)$ since $\gamma(G) \leq \gamma_t(G)$ for all graphs without isolated vertices. Let X be a γ-set of G. Then $W = X_1 \cup X_2$ is a total dominating set for $K_2 \times G$ and $|W| = 2|X| < 2\gamma_t(G)$. ■

Corollary 24 *If G is isolate-free and a universal γ_t-doubler, then $\gamma(G) = \gamma_t(G)$.*

Lemma 25 *If for any graph G, $\gamma(G) = \gamma_t(G)$, then $|V(G)| \geq 2\gamma_t(G)$.*

Proof. If $\gamma_t(G)$ is defined, then G is isolate-free and the result follows. ■

Theorem 26 *A graph G is a universal γ_t-doubler if and only if for each pair of sets $X, Y \subseteq V(G)$ with $0 < |X| < \gamma_t(G)$ and $Y = V(G) - N[X]$, $|Y| \geq 2\gamma_t(G) - |X| - k$, where k is the number of isolated vertices in $\langle X \rangle$.*

Proof. Consider any set $X \subseteq V(G)$ with $0 < |X| < \gamma_t(G)$, and let $Y = V(G) - N[X]$ and Z be the set of isolated vertices in $\langle X \rangle$. Suppose $|Y| < 2\gamma_t(G) - |X| - k$ where $k = |Z|$. If $|Y| + k \leq \gamma_t(G)$, let D be any γ_t-set of G, if not let D be any total dominating set with $|D| = |Y| + k$. Let π be any permutation of $V(G)$ such that $\pi(Y \cup Z) \subseteq D$. Then $W = X_1 \cup D_2 \succ \pi G$, so W is a total dominating set of πG with $|W| \leq 2\gamma_t(G) - 1$. Hence G is not a universal γ_t-doubler.

Conversely, let π be a permutation of $V(G)$ such that $\gamma_t(\pi G) < 2\gamma_t(G)$ and $W = X_1 \cup D_2$ be a γ_t-set of πG. Without loss of generality let $|X| < \gamma_t(G)$, which means $|D| < 2\gamma_t(G) - |X|$.

Suppose $X = \varnothing$. Then $D = V(G)$, hence $|V(G)| < 2\gamma_t(G)$. By Lemma 25 we have $\gamma(G) < \gamma_t(G)$. Let X' be any γ-set of G, k' be the number of isolated vertices in $\langle X' \rangle$ and $Y' = V(G) - N[X']$. Then $|Y'| = 0$, $0 < |X'| < \gamma_t(G)$ and $2\gamma_t(G) - |X'| - k' \geq 2 > |Y'|$. Hence Y' does not satisfy the hypothesis.

Suppose $X \neq \varnothing$. Let Z be the set of isolated vertices in $\langle X \rangle$, $k = |Z|$ and $Y = V(G) - N[X]$. Note that each vertex in Z_1 is adjacent to a vertex in

D_2. Let $D_2' \subseteq D_2$ be the set of vertices in D_2 which are adjacent, in πG, to a vertex in Z_1; hence $|D_2'| = |Z| = k$. Then $D_2 - D_2'$ dominates Y_1 and each vertex in $D_2 - D_2'$ dominates at most one vertex in Y_1, while no vertex in D_2' dominates a vertex in Y_1. Hence $|Y| \leq |D_2 - D_2'| < 2\gamma_t(G) - |X| - k$. ■

Corollary 27 *Let G be a universal γ_t-doubler. Then every vertex of G that is contained in a minimum total dominating set has degree at least $\gamma_t(G) + 1$.*

Proof. Suppose a graph G is a universal γ_t-doubler. Let $v \in V(G)$ be a vertex contained in a minimum total dominating set D of G. Let $X = D - v$ and note that $X \neq \varnothing$ because a total dominating set has at least two vertices. By Theorem 26, we have $|V(G) - N[X]| \geq 2\gamma_t(G) - |X| - k = \gamma_t(G) + 1 - k$, where k is the number of isolated vertices in $\langle X \rangle$. Since D is a dominating set, we have $v \succ V(G) - N[X]$, so v is adjacent to all $\gamma_t(G) + 1 - k$ vertices in $V(G) - N[X]$. Also, since D is a total dominating set, v is adjacent to all of the k isolated vertices in $\langle X \rangle$ which implies $deg(v) \geq \gamma_t(G) + 1$. ■

Figure 2.5 is an example of a graph G which is a universal γ_t-doubler with $\gamma_t(G) = 2$. By Corollary 27, every vertex contained in a minimum total dominating set of G has degree three or more. The only minimum total dominating set of G is indicated by the black vertices which indeed have the required degree.

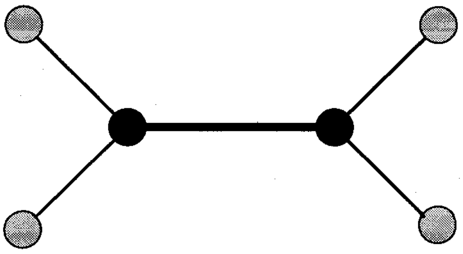

Figure 2.5: An example of a universal γ_t-doubler.

We can see that the converse of Corollary 27 does not hold by the same counterexample ($G = K_n$, $n \geq 3$) and similar argument used for Corollary 14.

If a graph G is a universal γ_{pr}-doubler it is not necessarily a universal γ_t-doubler. For example, $G = P_4$, by Corollary 12, is a universal γ_{pr}-doubler but, by Corollary 27, is not a universal γ_t-doubler. Also, if a graph G is a universal γ_t-doubler it is not necessarily a universal γ_{pr}-doubler. Consider, for example, the star $K_{1,r}$ where $r \geq 2$. It is easy to verify that $K_{1,r}$ is a universal γ_t-doubler but, by Corollary 9, $K_{1,r}$ is not a universal γ_{pr}-doubler.

2.2.2 Prism γ_t-Doublers

We now define notation that will be used throughout this section. Let

$$\left.\begin{aligned}&X \subseteq V(G) \text{ such that } 0 < |X| < \gamma_t(G);\\ &Y = V(G) - N[X];\\ &Z \text{ be the set of isolated vertices in } \langle X\rangle;\\ &k = |Z|.\end{aligned}\right\} \tag{2.2}$$

Theorem 28 *A graph G is a prism γ_t-doubler if and only if for each pair of sets X, Y as defined in* (2.2)*, either*

(a) $|Y| \geq 2\gamma_t(G) - |X| - k$, *or*

(b) $|Y| = 2\gamma_t(G) - |X| - k - d$ *where* $d \geq 1$, *and if* $A \subseteq N[X] - Z$ *dominates* $N\{X\} - N[Y] - N[Z]$ *and the only isolated vertices in* $\langle Z \cup A \cup Y\rangle$ *are in* Z, *then* $|A| \geq d$.

Proof. Suppose $\gamma_t(K_2 \times G) = 2\gamma_t(G)$ and consider any sets X, Y and Z as defined in (2.2). If $|Y| \geq 2\gamma_t(G) - |X| - k$ then we are done, so assume $|Y| = 2\gamma_t(G) - |X| - k - d$ for some $d \geq 1$. Suppose to the contrary that there exists a set $A \subseteq N[X] - Z$ such that $A \succ N\{X\} - N[Y] - N[Z]$ and the only isolated vertices in $\langle Z \cup A \cup Y\rangle$ are in Z, but $|A| < d$. Let

$W = X_1 \cup Z_2 \cup A_2 \cup Y_2$. Then $W \succ V(G)$ and $\langle W \rangle$ is isolate-free. Therefore W is a total dominating set of $K_2 \times G$ but

$$\begin{aligned}|W| &= |X| + |Z| + |Y| + |A| \\ &< |X| + k + (2\gamma_t(G) - |X| - k - d) + d \\ &= 2\gamma_t(G),\end{aligned}$$

a contradiction.

Conversely, suppose $\gamma_t(K_2 \times G) < 2\gamma_t(G)$ and let $W = X_1 \cup D_2$ be a minimum total dominating set of $K_2 \times G$. Let Z be the set of isolated vertices in $\langle X \rangle$ and $k = |Z|$. Without loss of generality we may assume $|X| < \gamma_t(G)$ and $|D| < 2\gamma_t(G) - |X|$.

Suppose $X = \varnothing$. Then $D_2 \succ K_2 \times G$ which means $D_2 = V(G_2)$, i.e. $D = V(G)$. Therefore $|W| = |D| = |V(G)| \leq 2\gamma_t(G) - 1$. By Lemma 25 this implies that $\gamma(G) < \gamma_t(G)$. Let X' be a γ-set of G, Z' the set of isolated vertices in $\langle X' \rangle$ and $k' = |Z'|$. Note that $k' > 0$ because X' is not a total dominating set of G. Since $X' \succ G$, we have $Y' = V(G) - N[X'] = \varnothing$. Since $0 < |X'| < \gamma_t(G)$, X', Y' are equivalent to the sets X, Y as defined in (2.2). But $2\gamma_t(G) - |X'| - k' \geq 2\gamma_t(G) - 2|X'| > 0 = |Y'|$, hence Y' does not satisfy (a).

If Y' does not satisfy (b), then we are done, so suppose that Y' satisfies (b). Then $|Y'| = 0 = 2\gamma_t(G) - |X'| - k' - d$ which means $d = 2\gamma_t(G) - |X'| - k'$. Let $A' = X' - Z'$. Then $A' \subseteq N[X'] - Z'$, $A' \succ N\{X'\} - N[Y'] - N[Z']$ and since $Y' = \varnothing$, the only isolated vertices in $\langle Z' \cup A' \cup Y' \rangle$ are in Z'. But

$$\begin{aligned} |A'| &= |X'| - k' \\ &= 2|X'| - |X'| - k' \\ &< 2\gamma_t(G) - |X'| - k' \\ &= d, \end{aligned}$$

contradicting (b).

Now suppose $X \neq \varnothing$. Let $Y = V(G) - N[X]$ and $A = D - Z - Y$. Then $A \subseteq N[X] - Z$, the only isolated vertices in $\langle Z \cup A \cup Y \rangle$ are in Z and $A \succ N\{X\} - N[Y] - N[Z]$. Since $Y = D - Z - A$, $|Y| < 2\gamma_t(G) - |X| - k - |A|$. Thus we have $|Y| = 2\gamma_t(G) - |X| - k - d$ for some value of d with $d > |A|$, hence (b) does not hold. ■

2.3 Connected Domination

Unlike in the cases of regular domination, paired-domination and total domination, for connected domination $\gamma_c(\pi G)$ is not bounded by $2\gamma_c(G)$ but instead by $2\gamma_c(G)+1$ as shown in Proposition 30.

A graph G is called a *universal γ_c-maximizer* if $2\gamma_c(G) \leq \gamma_c(\pi G) \leq 2\gamma_c(G)+1$ for all permutations π of $V(G)$.

In this section we discuss universal γ_c-maximizers and prism γ_c-doublers. All graphs in this section are connected.

2.3.1 Universal γ_c-Maximizers

Observation 29 *For any connected graph G, $\gamma_c(K_2 \times G) \leq 2\gamma_c(G)$.*

Proposition 30 *For any connected graph G and any permutation π of $V(G)$, $\gamma_c(\pi G) \leq 2\gamma_c(G)+1$.*

Proof. Let G be a graph and π be a permutation of $V(G)$ such that $\gamma_c(\pi G) > 2\gamma_c(G)+1$. Let D be a γ_c-set of G and $W = D_1 \cup D_2$. Then $W \succ V(\pi G)$ but $\langle W \rangle$ is not connected or else we have a contradiction. Let $X = \pi(D)$. Necessarily, $X \cap D = \varnothing$. Since D is a dominating set, $D \succ X$. Then for any vertex $v \in X$, v_1 is adjacent to some vertex in D_1 and to some vertex in D_2

because $\pi(D) = X$. But then $W' = W \cup \{v_1\}$ is a connected dominating set of πG, a contradiction. ■

The double star $G = S(k, l)$ with $k, l \geq 2$ is an example of a graph for which there exists a permutation π of $V(G)$ such that $\gamma_c(\pi G) = 2\gamma_c(G) + 1$. This equality holds as long as in πG the support vertices of $\langle G_1 \rangle$ are not adjacent to the support vertices of $\langle G_2 \rangle$. (Figure 2.6.)

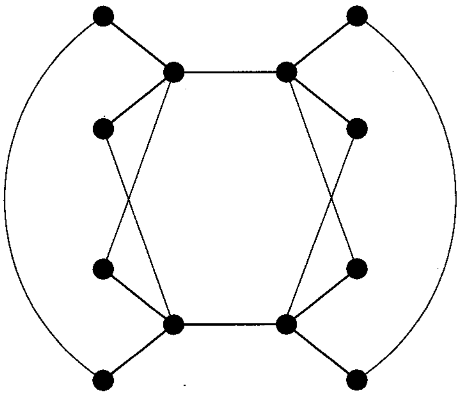

Figure 2.6: $G = S(2, 2)$ and $\gamma_c(\pi G) = 2\gamma_c(G) + 1 = 5$

Now we state a property of the graphs for which there exists a permutation of the vertex set such that equality holds for the upper bound provided in Proposition 30.

Proposition 31 *For a graph G, if there exists a permutation π of $V(G)$ such $\gamma_c(\pi G) = 2\gamma_c(G) + 1$, then at least $V(G)/2$ vertices are contained in no γ_c-set of G.*

Proof. Let G be a graph and π a permutation of $V(G)$ such that $\gamma_c(\pi G) = 2\gamma_c(G) + 1$. Let Y be the set of vertices which are contained no γ_c-set of G and let $X = V(G) - Y$. Suppose that $|X| > V(G)/2$. Then $X' = X \cap \pi(X) \neq \varnothing$. Let $v \in X'$ and D be a γ_c-set of G which contains v and U be γ_c-set which contains $\pi(v)$. Then $D_1 \cup U_2$ is a connected dominating set of πG, a contradiction. ■

To show that the converse of Proposition 31 is not true consider the graph $G = P_4$. It is easy to see that at most $V(G)/2$ vertices are contained in a γ_c-set of G. Label the vertices of G as a, b, c and d where a and d are the leaves of G and b is the support vertex for a. Let $X = \{b, c\}$. If $\pi(b) \in X$ (or $\pi(c) \in X$), then $X_1 \cup X_2$ is a connected dominating set of πG, hence $\gamma_c(\pi G) \leq 2\gamma_c(G)$. Suppose then that $\pi(b) \in \{a, d\}$. Without loss of generality we may assume $\pi(b) = a$ and $\pi(c) = d$. Then $W = \{a_1, b_1, b_2, c_2\}$ is a connected dominating set of πG. Hence, for all permutations π of $V(G)$, $\gamma_c(\pi G) \leq 2\gamma_c(G)$.

Observation 32 *If G is a universal γ_t-doubler and $\gamma_t(G) = \gamma_c(G)$, then G*

is a universal γ_c-maximizer.

Proof. This follows from the fact that for any connected graph G, $\gamma_t(G) \leq \gamma_c(G)$. ■

Observation 33 *If a graph G is a universal γ_c-maximizer, then $\gamma(G) = \gamma_c(G)$.*

Proof. Suppose that $\gamma(G) < \gamma_c(G)$. Let D be a γ-set of G and C be a γ_c-set of G. Let π be a permutation of $V(G)$ such that $\pi(C) \subseteq D$. Then $W = C_1 \cup D_2$ is a connected dominating set of πG and $|W| < 2\gamma_c(G)$, thus G is not a universal γ_c-maximizer. ■

Proposition 34 *A graph G is a universal γ_c-maximizer if and only if for each pair of sets $X, Y \subseteq V(G)$ with $0 < |X| < \gamma_c(G)$ and $Y = V(G) - N[X]$, $|Y| \geq 2\gamma_c(G) - |X| - k$, where k is the number of components of $\langle X \rangle$.*

Proof. Consider any set $X \subseteq V(G)$ with $0 < |X| < \gamma_c(G)$. Suppose $|Y| < 2\gamma_c(G) - |X| - k$ where k is the number of components of $\langle X \rangle$. Let X' be a subset of X which contains exactly one vertex from each component of $\langle X \rangle$.

If $|Y|+k \leq \gamma_c(G)$, let D be any γ_c-set of G, if not let D be any connected dominating set of G with $|D| = |Y| + k$. Let π be any permutation of $V(G)$ such that $\pi(Y \cup X') \subseteq D$. Then for $W = X_1 \cup D_2$, $\langle W \rangle$ is connected and $W \succ \pi G$. Hence W is a connected dominating set of πG with $|W| = |X| + |D| < 2\gamma_c(G)$.

Conversely, let π be a permutation of $V(G)$ such that $\gamma_c(\pi G) < 2\gamma_c(G)$ and let $W = X_1 \cup D_2$ be a γ_c-set of πG. Without loss of generality let $|X_1| < \gamma_c(G)$, which means $|D_2| < 2\gamma_c(G) - |X_1|$.

Suppose $X_1 = \varnothing$. Then $D_2 \succ V(G_1)$ which means $D_2 = V(G_2)$ and $|V(G)| < 2\gamma_c(G)$. Let $v \in D$. Then $D_2 - \{v_2\}$ dominates all of πG except for the vertex x_1 where $x = \pi^{-1}(v)$. Let $u \in N(\pi^{-1}(v))$. Then $W' = \{u_1\} \cup (D_2 - \{v_2\}) \succ V(\pi G)$ and $|W'| = |W| < 2\gamma_c(G)$. Let $X' = \{u\}$ and let $Y' = V(G) - N[X']$. Then $|Y'| \leq |V(G)| - 2 < 2\gamma_c(G) - 2 = 2\gamma_c(G) - |X'| - k'$ where k' is the number of components of $\langle X' \rangle$. Hence Y' does not satisfy the hypothesis.

Suppose now that $X_1 \neq \varnothing$. Note that each component of $\langle X_1 \rangle$ has a vertex that is adjacent to a vertex in D_2 (and vice versa). Let k be the number of components of $\langle X_1 \rangle$ and $X_1' \subseteq X_1$ a set of cardinality k containing one vertex from each component of $\langle X_1 \rangle$ such that each vertex in X_1' is adjacent to a vertex in D_2. Let $D_2' \subseteq D_2$ be the set of vertices in D_2 which are adjacent to a vertex in X_1'; hence $|D_2'| = k$. Then $D_2 - D_2'$ dominates Y_1 and each vertex in $D_2 - D_2'$ dominates at most one vertex in Y_1, while no vertex in D_2' dominates a vertex in Y_1. Therefore $|Y_1| \leq |D_2 - D_2'| = |D_2| - k$, so $|Y| < 2\gamma_c(G) - |X| - k$. Hence Y does not satisfy the hypothesis. ■

Corollary 35 *If G is a universal γ_c-maximizer with $n \geq 3$, then every vertex of G that is contained in a minimum connected dominating set has degree at least $\gamma_c(G) + 1$.*

Proof. Suppose G is a universal γ_c-maximizer. Let $v \in V(G)$ be a vertex contained in some minimum connected dominating set D of G. If $\gamma_c(G) = 1$, $deg(v) = n - 1 \geq 2$. If $\gamma_c(G) > 1$, then let $X = D - v$. Since we have $0 < |X| < \gamma_c(G)$, by Proposition 34, $|V(G) - N[X]| \geq 2\gamma_c(G) - |X| - k = \gamma_c(G) + 1 - k$, where k is the number of components in $\langle X \rangle$. Since D is a connected dominating set, $v \in N[X]$; hence v is adjacent to each vertex in $V(G) - N[X]$. Also, since $\langle D \rangle$ is connected, v is adjacent to each of the k components of $\langle X \rangle$, which implies $deg(v) \geq \gamma_c(G) + 1$. ■

2.3.2 Prism γ_c-Doublers

Observation 36 *If* $\gamma(G) < \gamma_c(G)$ *and there exists a* γ*-set* D *and a* γ_c*-set* C *of* G *such that* $D \subseteq C$*, then* G *is not a prism* γ_c*-doubler.*

Proof. Let G be a graph which satisfies the hypothesis. Let D be a γ-set of G and C a γ_c-set of G such that $D \subseteq C$. Then $D_1 \cup C_2$ is a connected dominating set of $K_2 \times G$ where $|D_1 \cup C_2| < 2\gamma_c(G)$. ■

Proposition 37 *If a graph* G *is prism* γ_c*-doubler, then for every vertex* v *contained in a* γ_c*-set* D *of* G*,* $\text{pn}(v, D) \neq \varnothing$*.*

Proof. Suppose, to the contrary, that $\gamma_c(K_2 \times G) = \gamma_c(G)$ and there exists a vertex v which is contained in a γ_c-set D of G such that $\text{pn}(v, D) = \varnothing$. Then $D - \{v\} \succ V(G)$. Let $W = D_1 \cup (D_2 - \{v_2\})$. Then $W \succ V(K_2 \times G)$ and $\langle W \rangle$ is connected, but $|W| = 2\gamma_c(G) - 1$, a contradiction. ■

By Proposition 37, P_n with $n \geq 4$ and C_n with $n \geq 5$ are not prism γ_c-doublers. To see that the converse of Proposition 37 is not true a counterexample is provided in Figure 2.7. (Note that the black vertices represent a γ_c-set for each graph and for G this γ_c-set is unique).

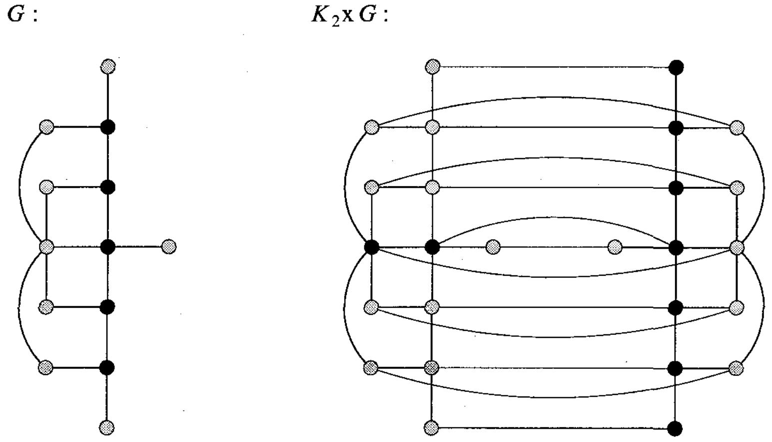

Figure 2.7: A counterexample for the converse of Proposition 37.

2.4 Independent Domination

In this section we discuss the upper bound for $i(\pi G)$ and prism i-doublers.

2.4.1 An Upper Bound for $i(\pi G)$

Unlike the cases of other domination parameters discussed in previous sections, $i(\pi G)$ is not bounded above by $2i(G)$ or $2i(G) + 1$. Consider, for example, the star $K_{1,n}$, $n \geq 2$, with central vertex v and the set U of leaves. Since $v \succ K_{1,n}$, $i(K_{1,n}) = 1$. Let π be any permutation of $V(K_{1,n})$ that fixes v. Then no i-set of πG contains both v_1 and v_2. Furthermore, it is easy to see that any i-set of πG is of the form $X_1 \cup D_2$, where $X_1, D_2 \neq \varnothing$, $X \cap D = \varnothing$ and $X \cup D = U$, so that $i(\pi G) = n = |V(G)| - 1$.

We show that in all cases, $i(\pi G) \leq |V(G)|$, and characterize the graphs for which equality holds. We need two more definitions. For any $A, B \subseteq V(G)$, we denote the set of edges joining vertices of A to vertices of B by $E(A, B)$. The *corona* $\mathrm{cor}(H)$ of a graph H is the graph obtained by adding a pendant edge to each vertex of H.

Theorem 38 *For any graph G of order n and any permutation π of $V(G)$, $i(G) \leq i(\pi G) \leq n$. Moreover if G is connected, then $i(\pi G) = n$ for some permutation π of $V(G)$ if and only if G is K_1 or K_2.*

Proof. Suppose that the lower bound does not hold. Let G be a graph and π a permutation of $V(G)$ such that $i(\pi G) < i(G)$. Let $W = X_1 \cup D_2$ be an i-set of πG. Then $X \succ V(G) - \pi^{-1}(D)$ and $X \cup \pi^{-1}(D) \succ V(G)$. Moreover, for each $d \in D$, $\mathrm{pn}(\pi^{-1}(d), X \cup \pi^{-1}(D)) \subseteq \{\pi^{-1}(d)\}$. If $X \cup \pi^{-1}(D)$ is independent we have a contradiction so we may assume that $X \cup \pi^{-1}(D)$ is not independent. Let D' be the set of all vertices in $\pi^{-1}(D)$ which are adjacent to a vertex in X and let $D'' = \pi^{-1}(D) - D'$. Since $X \succ V(G) - D''$, $X \cup D'' \succ V(G)$. If $X \cup D''$ is independent we have a contradiction so we now assume $X \cup D''$ is not independent. Let D^* be an i-set of $\langle D'' \rangle$; then $X \cup D^*$

is independent. Since $X \succ V(G) - D''$ and $D^* \succ D''$, $X \cup D^* \succ V(G)$, a contradiction.

For the upper bound we only need to observe that for any two vertices $v_1, u_2 \in V(\pi G)$, where $u = \pi(v)$, at most one of v_1 and u_2 is in an i-set of πG.

For $G = K_1$ or $G = K_2$ and any permutation π of $V(G)$, it is easily verified that $i(\pi G) = |V(G)|$. Suppose then that G is connected, $n \geq 3$ and $i(\pi G) = n$ for some permutation π of $V(G)$. Let $W = X_1 \cup D_2$ be an i-set of πG, where $|X| = k \leq |D|$, and $D' = \pi^{-1}(D)$. Necessarily, $X \cap D' = \varnothing$. Since $|X_1 \cup D_2| = n$, it follows that $V(G) = X \cup D'$. Similarly, if $\pi(X) = X'$, then $X' \cap D = \varnothing$ and $V(G) = X' \cup D$. Moreover, since X and D are independent and G is connected with at least three vertices, $X, D \neq \varnothing$ and each vertex in X (respectively D) is adjacent to a vertex in D' (respectively X').

In πG, consider any vertex $a_1 \in X_1$ and any vertex $b_1 \in D'_1$ adjacent to a_1. Let $\pi(b) = c$; then b_1 is adjacent to $c_2 \in D_2$. Moreover, c_2 is adjacent to some vertex $d_2 \in X'_2$. Let $\pi^{-1}(d) = e$; then d_2 is adjacent to $e_1 \in X_1$.

Suppose $e_1 \neq a_1$. Let $W' = (W - \{e_1, c_2\}) \cup \{d_2\}$. Note that $\text{pn}(e_1, W - \{c_2\}) \subseteq \{e_1, d_2\}$ because $a_1 \succ b_1$, and $\text{pn}(c_2, W - \{e_1\}) \subseteq \{c_2, d_2\}$. Since $c_2 \succ \{e_1, d_2, c_2\}$, W' is a dominating set of πG. Moreover, d_2 is not adjacent to any vertex in $X_1 - \{e_1\}$. Suppose d_2 is adjacent to $x_2 \in D_2 - \{c_2\}$. Let $\pi^{-1}(x) = y$. Then $\mathrm{pn}(x_2, W') \subseteq \{y_1\}$. If $\mathrm{pn}(x_2, W') = \{y_1\}$, let $W'' = (W' - x_2) \cup \{y_1\}$ and note that $\langle W'' \rangle$ has one edge less than $\langle W' \rangle$. If $\mathrm{pn}(x_2, W') = \varnothing$, let $W'' = W' - x_2$. Repeat this procedure to remove all neighbours of d_2 in D_2. The resulting set W^* is an independent dominating set of πG with $|W^*| < n$, a contradiction.

Therefore we may assume that $e_1 = a_1$, so that

$$\langle a_1, b_1, c_2, d_2 \rangle \text{ is a 4-cycle in } \pi G. \tag{2.3}$$

Now if b_1 is adjacent to some vertex $e_1 \in X_1 - \{a_1\}$, we may proceed as above with the roles of a_1 and e_1 interchanged, hence we also assume that

$$N_G(b) \cap X = \{a\}. \tag{2.4}$$

Suppose a_1 is adjacent to $b_1' \neq b_1 \in D_1'$. Let $\pi(b') = c' \neq c$ and note that b_1' is adjacent, in πG, to c_2', and c_2' is adjacent to $d_2' \in X_2'$. If $d_2' = d_2$, let $U' = (W - \{a_1, c_2'\}) \cup \{b_1'\}$. As above, $U' \succ \pi G$ and $\{b_1'\} \cup (D_2 - \{c_2'\})$ is independent. If U' is independent, let $U^* = U'$. Otherwise, b_1' is adjacent to $x_1' \in X_1 - \{a_1\}$, and we proceed similar to the case for W' to obtain an independent dominating set U^* of πG with $|U^*| < n$, a contradiction. On the other hand, if $d_2' \neq d_2$, let $e' = \pi^{-1}(d')$; then $e_1' \neq a_1$ and we may define

$V' = (W - \{e'_1, d'_2\}) \cup \{c'_2\}$ and proceed as in the case of W' above. Thus we may also assume that

$$N_G(a) = \{b\}. \tag{2.5}$$

Since $G \neq K_2$, it follows that $k \geq 2$. Repeating this argument for each vertex in X, we deduce from (2.4) and (2.5) that $E(X, N(X))$ consists of k independent edges, and the only other edges in $\langle E(X, N(X))\rangle$ are edges between vertices in $N(X) \subseteq D'$. Similarly, $E(D, N(D))$ consists of $|D|$ independent edges, and the only other edges in $\langle E(D, N(D))\rangle$ are edges between vertices in $N(D) \subseteq X'$. Moreover, by (2.3), for each edge $ab \in E(X, N(X))$ there is a unique edge $cd \in E(D, N(D))$ such that $\langle a_1, b_1, d_2, c_2\rangle$ is a 4-cycle in πG, i.e. such that $\pi(a) = c$ and $\pi(b) = d$. Since $|X| = |X'| = k$ and $|D| = |D'|$, it follows from the above statement that

(a) $|D| = k = |X|$, i.e. πG consists of k disjoint copies, say $a_{i1}, b_{i1}, c_{i2}, d_{i2}, a_{i1}$, $i = 1, ..., k$, of C_4 and additional edges in $\langle\{b_{i1} : i = 1, ..., k\}\rangle$ and $\langle\{d_{i1} : i = 1, ..., k\}\rangle$, therefore

(b) $X = D$ and $X' = D'$, and

(c) $\langle D'\rangle$ is a connected graph of order k and $G = \text{cor}(\langle D'\rangle)$.

Since $\langle D'\rangle$ is connected and $n \geq 3$, in πG there exist two disjoint copies of

C_4, say a_1, b_1, c_2, d_2, a_1 and $a'_1, b'_1, c'_2, d'_2, a'_1$, such that $b' \in N_G(b)$. Suppose $d \in N_G(d')$. Let $W' = (W - \{a_1, a'_1, c_2, c'_2\}) \cup \{b_1, d'_2\}$. Since $b_1 \succ \{a_1, b_1, b'_1, c_2\}$ and $d'_2 \succ \{a'_1, c'_2, d_2, d'_2\}$, by the above statement (a), W' is a dominating set of πG. Moreover, since $G = \mathrm{cor}(\langle D' \rangle)$ and $D' = X'$, W' is independent. Thus W' is an independent dominating set of πG with $|W'| < n$, a contradiction. Suppose then that $d \notin N_G(d')$. Let $W' = (W - \{a_1, a'_1, c_2, c'_2\}) \cup \{b_1, d_2, d'_2\}$. By similar arguments we again have that W' is an independent dominating set of πG with $|W'| < n$, a contradiction. ■

2.4.2 Prism i-Doublers

Lemma 39 *For a graph* G, $i(K_2 \times G) = \min\{|D| + |D'|\}$, *where the minimum is taken over all disjoint independent sets* D, D' *such that* $D \succ V(G) - D'$ *and* $D' \succ V(G) - D$.

Proof. Let $W = X_1 \cup Y_2$ be an i-set of $K_2 \times G$ and $k = \min\{|D| + |D'|\}$, where the minimum is taken over all disjoint independent sets D, D' such that $D \succ V(G) - D'$ and $D' \succ V(G) - D$. Necessarily, X and Y are disjoint independent sets of G and $X \succ V(G) - Y$ and $Y \succ V(G) - X$. Thus $|W| = i(K_2 \times G) \geq k$. For any two disjoint independent sets D and D' of G

such that $D \succ V(G) - D'$ and $D' \succ V(G) - D$, $D_1 \cup D_2'$ is an independent dominating set of $K_2 \times G$. Thus $i(K_2 \times G) = k$. ■

Theorem 40 *The following statements are equivalent for any graph G.*

(*i*) *G is a prism i-doubler.*

(*ii*) *There exist two disjoint independent sets, D and D', such that $D \succ V(G) - D'$, $D' \succ V(G) - D$, where $|D| + |D'| = 2i(G)$, and for any two independent sets $X, Y \subseteq V(G)$ with $0 \leq |X| < i(G)$ and $Y = V(G) - N[X]$, either*

(*a*) *$|Y| \geq 2i(G) - |X|$, or*

(*b*) *$|Y| = 2i(G) - |X| - d$, where $d \geq 1$, and if an independent set $A \subseteq N\{X\} - N[Y]$ dominates $N\{X\} - N[Y]$, then $|A| \geq d$.*

(*iii*) *$2i(G) = \min\{|D| + |D'|\}$, where the minimum is taken over all disjoint independent sets D, D' such that $D \succ V(G) - D'$ and $D' \succ V(G) - D$.*

Proof. By Lemma 39, (*i*) and (*iii*) are equivalent statements, therefore we need only show that (*i*) implies (*ii*) and vice versa.

Suppose $i(K_2 \times G) = 2i(G)$. Let $W = D_1 \cup D'_2$ be an i-set of $K_2 \times G$. Since W is an independent set of $K_2 \times G$, D and D' are each independent sets of G and $D \cap D' = \varnothing$. Moreover, $D \succ V(G) - D'$ and $D' \succ V(G) - D$ or else $D_1 \cup D'_2 \not\succ V(K_2 \times G)$.

Consider any pair of independent sets X, Y with $0 \leq |X| < i(G)$ and $Y = V(G) - N[X]$. If $|Y| \geq 2i(G) - |X|$ then we are done. So we assume $|Y| = 2i(G) - |X| - d$ where $d \geq 1$. Suppose there is an independent set $A \subseteq N\{X\} - N[Y]$ such that $|A| \leq d - 1$ and $A \succ N\{X\} - N[Y]$. Since $A \subseteq N\{X\} - N[Y]$ it follows that $A \cup Y$ is an independent set and $W = X_1 \cup Y_2 \cup A_2$ is an independent dominating set of $K_2 \times G$. But

$$\begin{aligned} |W| &= |X| + |Y| + |A| \\ &\leq |X| + 2i(G) - |X| - d + d - 1 \\ &= 2i(G) - 1, \end{aligned}$$

a contradiction, hence (b) holds.

Conversely, suppose that $i(K_2 \times G) \neq 2i(G)$. If $i(K_2 \times G) > 2i(G)$ then there do not exist sets D and D' as described in the hypothesis, otherwise $D_1 \cup D'_2 \succ K_2 \times G$, a contradiction. We may then assume that $i(K_2 \times G) < 2i(G)$. Let $W = X_1 \cup D_2$ be an i-set of $K_2 \times G$ and note that $X \cap D = \varnothing$.

Let $Y = V(G) - N[X]$ and note that $Y \subseteq D$ since $D_2 \succ Y_1$. Hence X and Y are both independent sets. Without loss of generality we may assume that $|X| < i(G)$. Since $Y \subseteq D$, $|Y| < 2i(G) - |X|$. Hence (a) does not hold and $|Y| = 2i(G) - |X| - d$ where $d \geq 1$. Let $A = D - Y$. Since D is independent and $X \cup D = \varnothing$, $A \subseteq N\{X\} - N[Y]$ and $Y \cup A = D \succ V(G) - X$. It follows that $A \succ N\{X\} - N[Y]$. But

$$\begin{aligned} |A| &= |D| - |Y| \\ &< (2i(G) - |X|) - (2i(G) - |X| - d) \\ &= d, \end{aligned}$$

a contradiction, hence (b) does not hold. ■

To illustrate the use of Theorem 40 we show that P_6 is a prism i-doubler. We label the vertices of P_6 from left to right by $v_1, v_2, v_3, v_4, v_5, v_6$. Let $D = \{v_1, v_4\}$ and $D' = \{v_3, v_6\}$. Then $D \succ V(G) - D'$, $D' \succ V(G) - D$, $|D| + |D'| = 2i(G)$ and D, D' are clearly disjoint independent sets. Now we observe any pair of sets X, Y such that $0 \leq |X| < i(P_6) = 2$ and $Y = V(G) - N[X]$. If $0 \leq |X| < 2$ then Y is not an independent set. Thus by Theorem 40, P_6 is a prism i-doubler.

2.5 Packings

2.5.1 Prism ρ-Doublers

Observation 41 *For any graph G and any permutation π of $V(G)$, $\rho(G) \leq \rho(\pi G) \leq 2\rho(G)$.*

Proposition 42 *A graph G is a prism ρ-doubler if and only if G contains two maximum packings D and X such that $D \cup X$ is independent.*

Proof. Suppose $\rho(K_2 \times G) = 2\rho(G)$. Let $W = X_1 \cup D_2$ be a ρ-set of $K_2 \times G$. It follows that $|D| = |X| = \rho(G)$ since it is not possible for either $|X|$ or $|D|$ to be greater than $\rho(G)$. Moreover, for any pair of vertices $u_1, v_2 \in W$, $d_{K_2 \times G}(u_1, v_2) \geq 3$ and $d_G(u, v) \geq 2$. Thus D and X are each ρ-sets of G and $D \cup X$ is independent.

Conversely, suppose that G contains two ρ-sets, D and X, such that $D \cup X$ is independent. Then $W = X_1 \cup D_2$ is a packing of $K_2 \times G$ of order $2\rho(G)$ and by Observation 41 this is a ρ-set for $K_2 \times G$. ■

An example of prism ρ-doublers are the graphs formed by adding at least two leaves to each vertex of P_n where $n \geq 2$ (See Figure 2.8).

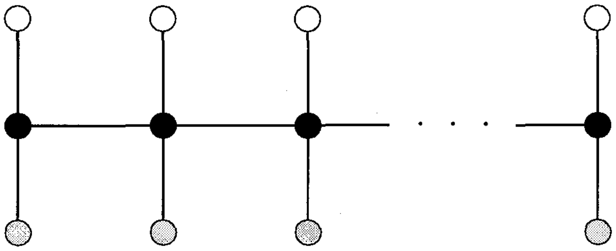

Figure 2.8: A prism ρ-doubler.

2.5.2 An Upper Bound for $\rho_l(\pi G)$?

Unlike the case of $\rho(\pi G)$, $\rho_l(\pi G)$ is not bounded above by $2\rho_l(G)$. Consider, for example, the spider $S(2, 2, ..., 2)$ shown in Figure 2.9. For $G = S(2_{(1)}, 2_{(2)}, ..., 2_{(k)})$, $\rho_l(G) = 1$ while $\rho_l(K_2 \times G) = k$.

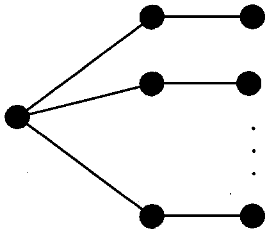

Figure 2.9: S(2,2,...,2)

We have not determined a general upper bound for $\rho_l(\pi G)$.

Chapter 3

Fixers

It is easily verified for $G = \overline{K}_n$ that $\gamma(\pi G) = \gamma(G)$ for all permutations π of $V(G)$ but it is not known whether $\overline{K}_n$ is the only graph for which this is true. In this chapter, for various domination parameters y, we only consider prism y-fixers except when dealing with connected domination and packings.

3.1 Regular Domination

3.1.1 Prism γ-Fixers

A set D is a 2-*dominating set* of G if every vertex in $V(G) - D$ is adjacent to at least two vertices in D. The 2-*domination number* $\gamma_{\times 2}(G)$ is the minimum cardinality of a 2-dominating set.

The following theorem is similar to Theorem 6 [4].

Theorem 43 *A graph G is a prism γ-fixer if and only if there exists an independent 2-dominating set W of G with $|W| = \gamma(G)$ which can be partitioned into two sets, X and D, such that $X \succ V(G) - D$ and $D \succ V(G) - X$.*

Proof. Suppose that G is a prism γ-fixer. Let $X_1 \cup D_2$ be a dominating set of $K_2 \times G$ such that $|X_1| + |D_2| = \gamma(G)$. If $X \cap D \neq \varnothing$ then $|X \cup D| < \gamma(G)$ and $X \cup D \succ V(G)$, a contradiction. Let $W = X \cup D$. Since $X \cap D = \varnothing$ and $X_1 \cup D_2 \succ V(K_2 \times G)$ we have that $X \succ V(G) - D$ and $D \succ V(G) - X$, which also implies that $W = X \cup D$ is a 2-dominating set of G. We now need only show that W is an independent set. Suppose not. Then there exist adjacent vertices $u, v \in X \cup D$. Since $X \cup D$ is a 2-dominating set and $u \succ v$, we have that $((X \cup D) - \{v\}) \succ V(G)$, a contradiction.

Conversely, suppose G satisfies the hypothesis. Then $X_1 \cup D_2 \succ V(K_2 \times G)$ and since $|X_1 \cup D_2| = |W| = \gamma(G)$, we are done. ■

Figure 3.1 is an example of a prism γ-fixer while Figure 3.2 illustrates the necessity of the ability to partition the independent 2-dominating set as outlined in Theorem 43. Since the independent dominating set shown in

Figure 3.2 is the only one which is also a γ-set and it cannot be partitioned as outlined in Theorem 43, this graph is not a prism γ-fixer. In each figure the circled vertices represent the independent 2-dominating set.

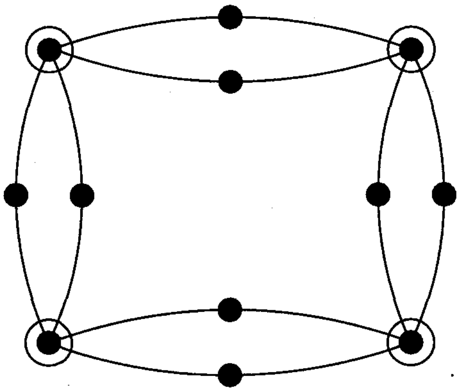

Figure 3.1: An example of a prism γ-fixer

Corollary 44 *If a graph G is a prism γ-fixer, then $\gamma(G) = i(G)$.*

Proof. For all graphs G, $\gamma(G) \leq i(G)$. Suppose G is a prism γ-fixer. Then by Theorem 43, G contains an independent set which is also a γ-set. Hence $\gamma(G) = i(G)$. ■

Corollary 45 *If an isolate-free graph G is a prism γ-fixer, then $\delta(G) \geq 2$.*

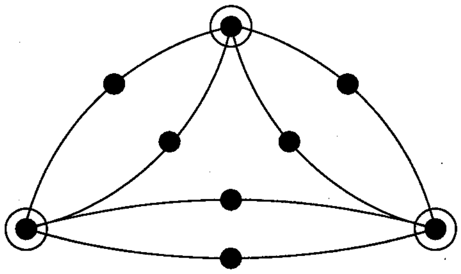

Figure 3.2: A graph with an independent $\gamma_{\times 2}(G)$-set and $\gamma(K_2 \times G) > \gamma(G)$.

Proof. Let G be an isolate-free graph and a prism γ-fixer. Suppose $\delta(G) = 1$. By Theorem 43, G has an independent γ-set D which is also a 2-dominating set. Let $v \in V(G)$ be a vertex of degree one and u be adjacent to v. If $u \in D$, then $v \notin D$ since D is independent. Hence v is not adjacent to two vertices in D, a contradiction. Therefore $u \notin D$ and $v \in D$. Since $u \in V(G) - D$ it has another neighbour in D besides v; let x be such a vertex. Then $D' = (D - \{v, x\}) \cup \{u\} \succ V(G)$ since each neighbour of x in $V(G) - D$ is adjacent to some vertex in $D - \{v, x\}$. But $|D'| < |D| = \gamma(G)$, a contradiction. Thus we have that $\delta(G) \neq 1$ and since G is isolate-free, $\delta(G) \geq 2$. ■

3.2 Independent Domination

3.2.1 Prism i-Fixers

Corollary 46 *If G is a prism γ-fixer, then G is also a prism i-fixer.*

Proof. Suppose G is a prism γ-fixer. Then by Theorem 43 and the fact that for any graph, $\gamma(G) \leq i(G)$, we know that W as defined in the proof of Theorem 43 is an i-set of G and $X_1 \cup D_2$ is an i-set of the same cardinality for $K_2 \times G$. ■

Theorem 47 *A graph G is a prism i-fixer if and only if there exists an independent 2-dominating set W of G with $|W| = i(G)$ which can be partitioned into two sets, X and D, such that $X \succ V(G) - D$ and $D \succ V(G) - X$.*

Proof. The proof is very similar to that of Theorem 43. Suppose G is a prism i-fixer. Let $X_1 \cup D_2$ be an i-set of $K_2 \times G$; then $|X_1| + |D_2| = i(G)$. Necessarily, $X \cap D = \varnothing$ or else $X_1 \cup D_2$ is not an independent set. Then $X \succ V(G) - D$ and $D \succ V(G) - X$. Let $W = X \cup D$. It follows that W is a 2-dominating set of G. Suppose that $X \cup D$ is not independent. Let X' be the set of all vertices in X which are adjacent to a vertex in D. Then

$W' = (X - X') \cup D$ is independent. Since $D \succ V(G) - X$ and $D \succ X'$, we have that $W' \succ V(G)$, a contradiction.

Conversely, suppose G satisfies the hypothesis. Then $W = X_1 \cup D_2 \succ V(K_2 \times G)$. Since $|W| = i(G)$, we are done. ■

A family of graphs $\mathcal{F}$ which are prism i-fixers is given by the complete bipartite graphs $K_{r,s}$ with $r, s \geq 2$. Suppose $G \in \mathcal{F}$ and let $V_1, V_2 \in V(G)$ represent the partite sets of G such that $|V_1| \leq |V_2|$. Then V_1 is an i-set of G. Also, each vertex in V_2 is dominated at least twice so V_1 is a 2-dominating set. Furthermore, for any partition of V_1 into sets X and D, $X \succ V(G) - D$ and $D \succ V(G) - X$. Therefore by Theorem 47, G is a prism i-fixer.

Corollary 48 *A graph G with $i(G) = 2$ is a prism i-fixer if and only if there exists an i-set which is also a 2-dominating set.*

Proof. Let G be an isolate-free graph and a prism i-fixer with $i(G) = 2$. Then by Theorem 47 there exists an i-set which is also a 2-dominating set.

Conversely, suppose that $i(G) = 2$ and G contains an i-set $W = \{u, v\}$ which is also a 2-dominating set. Since W is a 2-dominating set each vertex in $V(G) - W$ is dominated by both u and v. Therefore $u \succ V(G) - v$ and

$v \succ V(G) - u$. So we now have satisfied the necessary conditions of Theorem 47 for G to be a prism i-fixer. ■

Figure 3.3 shows an example of a graph G which is a prism i-fixer but not a prism γ-fixer. For the graph G in Figure 3.3, $\gamma(G) = 3$ while $i(G) = 4$, hence by Corollary 44, G is not a prism γ-fixer. The black vertices represent an i-set which satisfies the necessary conditions of Theorem 47 for a graph to be a prism i-fixer, while the circled vertices represent represent a γ-set.

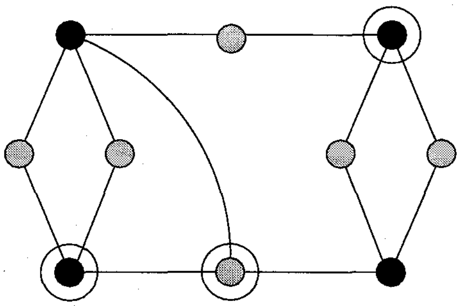

Figure 3.3: An example if a prism i-fixer which is not a prism γ-fixer.

3.3 Paired-Domination

3.3.1 Prism γ_{pr}-fixers

Lemma 49 *For any isolate-free graph G,*

$$\gamma_{pr}(G) \leq \gamma_{pr}(K_2 \times G) \leq 2\gamma(G).$$

Proof. To prove the upper bound $\gamma_{pr}(K_2 \times G) \leq 2\gamma(G)$ we note that for any γ-set D of G, $D_1 \cup D_2$ is a paired-dominating set of $K_2 \times G$. For the lower bound, let us assume to the contrary that there exists a graph G such that $\gamma_{pr}(K_2 \times G) < \gamma_{pr}(G)$. Let $W = X_1 \cup D_2$ be a γ_{pr}-set of $K_2 \times G$ and let $W' = X \cup D$. Then $W' \succ V(G)$ and $|W'| = |X| + |D| - |X \cap D|$. If $X \cap D = \varnothing$, then $\langle W' \rangle$ contains a perfect matching, hence W' is a paired-dominating set of G, a contradiction. So we assume that $X \cap D \neq \varnothing$. Let M be a maximum matching of $\langle W' \rangle$ and Z the set of $\overline{M}$-vertices. Then $|Z| \leq |X \cap D|$. We form a new dominating set W^* by adding the minimum number of vertices to W' such that each vertex in Z is paired. If this is not possible for a vertex in Z we simply remove it since in this case it is already dominated and has no private neighbours. Since $\langle W^* \rangle$ contains a perfect matching, W^* is a paired-dominating set of G. But $|W^*| \leq |W|$, a contradiction. ■

Corollary 50 *If* $\gamma_{pr}(G) = 2\gamma(G)$*, then* G *is a prism* γ_{pr}*-fixer.*

By Corollary 50 and the fact that $\gamma_{pr}(P_5) = \gamma_{pr}(C_5) = 4$, the graphs P_5 and C_5 are prism γ_{pr}-fixers.

We now extend this result to determine a necessary and sufficient condition for a graph to be a prism γ_{pr}-fixer. For the following result we will define a weak partition (i.e., the sets of the partition may be empty) of the pairs of a γ_{pr}-set S into three sets D, Y and Z such that

$$\left.\begin{array}{l} D \succ V(G) - S - X, \\ Z \succ V(G) - S - X \text{ and} \\ Y = S - D - Z \\ \text{where } X = V(G) - (S \cup (N[Z] \cap N[D])), \text{ i.e., } X \text{ consists of} \\ \text{the vertices of } V(G) - S \text{ that are not dominated by vertices} \\ \text{from both } D \text{ and } Z. \end{array}\right\} \tag{3.1}$$

Figure 3.4 illustrates examples of such a partition where circles represent D, squares represent Z, triangles represent Y and the non-filled vertices represent X.

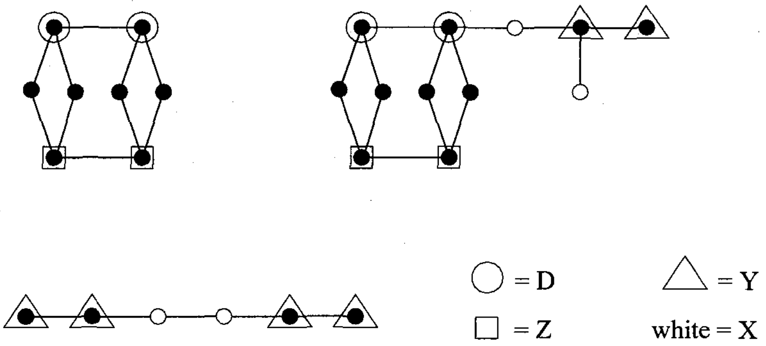

Figure 3.4: Examples of the partition defined in (3.1).

Theorem 51 *A graph G is a prism γ_{pr}-fixer if and only if G has a γ_{pr}-set S in which the pairs can be weakly partitioned into three sets, D, Z and Y, such that $D \succ V(G) - S - X$, $Z \succ V(G) - S - X$, where $X = V(G) - (S \cup (N[D] \cap N[Z]))$, and there exists an independent set Y' of order $|Y|/2$ which dominates $Y \cup X$.*

Proof. Suppose that G is a prism γ_{pr}-fixer. Let $S = D_1' \cup Z_2'$ be a γ_{pr}-set of $K_2 \times G$ and M a perfect matching of $\langle S \rangle$ in which as many vertices as possible are matched with their own image. Let Y' be the set of vertices of $V(G)$ which are paired with themselves in $\langle S \rangle$ under M. Let $D = D' - Y'$, $Z = Z' - Y'$ and $S' = D \cup Z \cup Y'$. Necessarily, $S' \succ V(G)$ and $|S'| = |D| + |Z| - |D \cap Z| + |Y'|$. We now claim that $D \cap Z = \varnothing$. Suppose not. Then we have two possible cases of intersection.

Case 1. There exists a pair of vertices $u, v \in V(G)$ which are paired under M in either $\langle D_1 \rangle$ or $\langle Z_2 \rangle$ (not both by our choice of M) and $u, v \in D \cap Z$. Without loss of generality we will assume that u_2 and v_2 are paired in $\langle Z_2 \rangle$. Suppose in $\langle S' \rangle$, we preserve the pairings of those vertices paired in $\langle D_1 \rangle$ and pair as many as possible in $\langle Z - D \rangle$. Using this pairing the number of vertices not paired in $\langle S' \rangle$ is strictly less than $|Y'| + |D \cap Z|$. We now form a new set S^* by adding the minimum number of vertices to S' such that $\langle S^* \rangle$ contains a perfect matching. (In the forming of S^*, if any vertex in $\langle S' \rangle$ cannot be paired then it is removed since in this case it is already dominated and has no private neighbours.) Then S^* is a paired-dominating set, but $|S^*| < |D| + |Z| + 2|Y'| = |S|$, a contradiction.

Case 2. There exist vertices $u_1, v_1, v_2, w_2 \in S$ such that $u_1v_1, v_2w_2 \in M$ and $u_2, w_1 \notin S$. If $\mathrm{pn}(w_2, Z_2) = \varnothing$ then we have a smaller paired-dominating set of G by taking $S' = D \cup (Z - w) \cup Y'$ and forming the set S^* as outlined previously. Suppose then that $\mathrm{pn}(w_2, Z_2) \neq \varnothing$. It follows that for each $x_2 \in \mathrm{pn}(w_2, Z_2)$, $D_1 \succ \{x_1\}$. Therefore we again have a smaller paired-dominating set of G by taking $S' = D \cup (Z - w) \cup Y'$ and forming the set S^* from S' as outlined previously.

Hence $D \cap Z = \varnothing$. Suppose that Y' is not independent. Then in a maximum matching of $\langle S' \rangle$, at most $|Y'|-2$ vertices are not paired. We now form the set S^* by adding the minimum number of vertices to S' such that $\langle S^* \rangle$ contains a perfect matching. Then $|S^*| \leq |Z| + |D| + 2|Y'| - 2 < |S|$, a contradiction. Let Y be a set formed by pairing each vertex $v \in Y'$ with a vertex $u \in V(G) - Z - D - Y'$. This formation of the set Y is possible or else we could form a smaller paired-dominating set, a contradiction. It follows that $D \cup Z \cup Y$ is a paired-dominating set of G and D, Z and Y are all disjoint. Let $X = V(G) - (S \cup (N[D] \cap N[Z]))$. Then $D \succ V(G) - S - X$, $Z \succ V(G) - S - X$ and Y' is an independent set of order $|Y|/2$ which dominates $Y \cup X$.

Conversely, suppose that G has a γ_{pr}-set in which the pairs can be partitioned into three sets, D, Z and Y as defined in (3.1), and there exists an independent set Y' of order $|Y|/2$ which dominates $Y \cup X$ (X is also as defined in (3.1)). Then $S = Z_1 \cup D_2 \cup Y_1' \cup Y_2'$ is a paired-dominating set for $K_2 \times G$ and $|S| = \gamma_{pr}(G)$. Hence by Lemma 49, S is a γ_{pr}-set of $K_2 \times G$. ■

The graph shown in Figure 3.5 is an example of a prism γ_{pr}-fixer and we use it to illustrate Theorem 51. Note that $\gamma_{pr}(G) = 6$ and the set $S =$

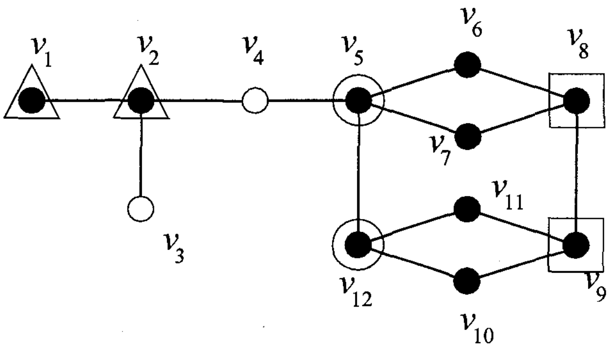

Figure 3.5: An example of a prism γ_{pr}-fixer.

$\{v_1, v_2, v_5, v_{12}, v_8, v_9\}$ is a γ_{pr}-set of G. Let $D = \{v_5, v_{12}\}$, $Z = \{v_8, v_9\}$ and $Y = \{v_1, v_2\}$. Let $X = V(G) - (S \cup (N[D] \cap N[Z]))$, i.e. $X = \{v_3, v_4\}$. Then $D \succ V(G) - S - X$ and $Z \succ V(G) - S - X$. Let $Y' = \{v_2\}$. Then Y' is an independent set of order $|Y|/2$ which dominates $Y \cup X$. Thus the graph in Figure 3.5 satisfies the necessary requirements of Theorem 51 for G to be a prism γ_{pr}-fixer.

Some other examples of prism γ_{pr}-fixers include K_n for $n \geq 2$, P_3, P_5, P_6, P_9, C_5, C_6 and C_9. Furthermore, this list contains all the paths and cycles which are prism γ_{pr}-fixers.

3.4 Total Domination

3.4.1 Prism γ_t-fixers

Lemma 52 *For any graph G,* $\gamma_t(K_2 \times G) \leq 2\gamma(G)$ *and for any permutation* π *of* $V(G)$, $\gamma_t(G) \leq \gamma_t(\pi G)$.

Proof. For any γ-set D of G, $D_1 \cup D_2$ is a total dominating set of $K_2 \times G$, hence $\gamma_t(K_2 \times G) \leq 2\gamma(G)$. Let G be a graph and π a permutation of $V(G)$ such that $\gamma_t(\pi G) < \gamma_t(G)$. Let $W = X_1 \cup D_2$ be a γ_t-set of πG, $D' = \pi^{-1}(D)$ and $k = |X \cap D'|$. It follows that for $W' = X \cup D'$, $\langle W' \rangle$ has at most k isolated vertices and $|W'| = \gamma_t(\pi G) - k < \gamma_t(G) - k$. Therefore at most k more vertices are required to obtain a total dominating set of G with cardinality $|W'| + k < \gamma_t(G)$, a contradiction. ■

Corollary 53 *If* $\gamma_t(G) = 2\gamma(G)$, *then G is a prism* γ_t*-fixer.*

Proposition 54 *A graph G is a prism* γ_t*-fixer if and only if G is a prism* γ_{pr}*-fixer and* $\gamma_t(G) = \gamma_{pr}(G)$.

Proof. Suppose that G is a prism γ_t-fixer and $\gamma_t(G)$ is odd. Let $W = X_1 \cup D_2$ be a γ_t-set of $K_2 \times G$ and $k = |X \cap D|$. Since $\gamma_t(G)$ is odd there exists a set

$U \subseteq W$ such that $\langle U \rangle$ is connected and $|U| \geq 3$. Let $U = Z_1 \cup Y_2$. Suppose that either $Z_1 = \varnothing$ or $Y_2 = \varnothing$. Without loss of generality say $Y_2 = \varnothing$. Let $z_1 \in Z_1$ such that z_1 is not a support vertex of $\langle Z_1 \rangle$. Then $D \succ \mathrm{pn}(z, X)$. Let $W' = (X \cup D) - \{z\}$. Then $W' \succ V(G)$ and $|W'| = |W| - k - 1$. Note that W' is not necessarily a total dominating set since $\langle W' \rangle$ may contain isolated vertices. Since there are at most k isolated vertices, at most k more vertices are required to make W' into a total dominating set. Hence $\gamma_t(G) < \gamma_t(K_2 \times G)$, a contradiction.

Suppose now that $Z_1 \neq \varnothing$ and $Y_2 \neq \varnothing$. Let $W' = X \cup D$. Then $|W'| = |W| - k$. But since $Z_1 \neq \varnothing$ and $Y_2 \neq \varnothing$, there are at most $k-1$ isolated vertices in $\langle W' \rangle$ and again we have $\gamma_t(G) < \gamma_t(K_2 \times G)$, a contradiction.

Suppose that $\gamma_t(G)$ is even. Let $W = X_1 \cup D_2$ be a γ_t-set of $K_2 \times G$. By the previous arguments there does not exist a connected set $U \subseteq W$ with $|U| \geq 3$. Therefore W is a γ_{pr}-set of $K_2 \times G$ since for any graph $\gamma_t(G) \leq \gamma_{pr}(G)$. By Lemma 49, $\gamma_{pr}(G) = \gamma_{pr}(K_2 \times G)$, implying that G is a prism γ_{pr}-fixer.

Conversely, suppose that G is a prism γ_{pr}-fixer and $\gamma_t(G) = \gamma_{pr}(G)$. Since $\gamma_t(G) \leq \gamma_{pr}(G)$ and any paired-dominating set is also a total dominating set, it follows from Lemma 52 that G is also a prism γ_t-fixer. ■

P_5 and P_6 are examples of prism γ_{pr}-fixers but only P_6 is a prism γ_t-fixer. Note that $\gamma_{pr}(P_5) = 4$ and $\gamma_t(P_5) = 3$ while $\gamma_{pr}(P_6) = \gamma_t(P_6) = 4$.

3.5 Connected Domination

3.5.1 Prism γ_c-Fixers

Proposition 55 *A graph G is a prism γ_c-fixer if and only if $G = K_1$.*

Proof. If $G = K_1$, then clearly G is a prism γ_c-fixer.

Suppose $\gamma_c(K_2 \times G) = \gamma_c(G)$. Let $W = X_1 \cup D_2$ be a γ_c-set of $K_2 \times G$. Without loss of generality we may assume that $|X| \leq \gamma_c(G)/2$. If $X = \varnothing$, then $D = V(G)$. Hence $|V(G)| = \gamma_c(G)$. Also $G \neq K_2$, since $\gamma_c(K_2) = 1 < |V(G)|$. Hedeteimi and Laskar in [5] show that for any connected graph with $n \geq 3$, $\gamma_c(G) \leq n - 2$. Therefore if $X = \varnothing$, then $G = K_1$.

Suppose $0 < |X| \leq \gamma_c(G)/2$. Since $\langle W \rangle$ is connected it follows that $X \cap D \neq \varnothing$ which implies $X \cup D < \gamma_c(G)$. But this is a contradiction since $\langle X \cup D \rangle$ is connected and $X \cup D \succ V(G)$. ■

Corollary 56 *A graph G is a universal γ_c-fixer if and only if $G = K_1$.*

3.6 Packings

3.6.1 Universal ρ-Fixers

Theorem 57 *A graph G is a universal ρ-fixer if and only if for each pair (not necessarily disjoint) of packings X, Y of G with $|X| + |Y| = \rho(G) + 1$, either $|V(G) - N[X]| < \rho(G) - |X| + 1$ or $|V(G) - N[Y]| < \rho(G) - |Y| + 1$.*

Proof. Let G be a graph and π a permutation of $V(G)$ such that $\rho(G) < \rho(\pi G)$. Let $W = X_1 \cup Y_2$ be a packing of πG such that $|W| = \rho(G) + 1$. Since W is a packing of πG, $\pi^{-1}(Y) \cap N[X] = \varnothing$ and $\pi(X) \cap N[Y] = \varnothing$. Hence $|V(G) - N[X]| \geq |Y| = \rho(G) - |X| + 1$ and $|V(G) - N[Y]| \geq |X| = \rho(G) - |Y| + 1$.

Conversely, suppose to the contrary that there exists a pair of packings X, Y of G with $|X|+|Y| = \rho(G)+1$, such that $|V(G)-N[X]| \geq \rho(G)-|X|+1$ and $|V(G) - N[Y]| \geq \rho(G) - |Y| + 1$. Let π be a permutation of $V(G)$ such that $\pi(X) \subseteq V(G) - N[Y]$ and $\pi^{-1}(Y) \subseteq V(G) - N[X]$. Then $W = X_1 \cup Y_2$ is a packing of πG. Hence $\rho(\pi G) \geq \rho(G) + 1$ and G is not a universal ρ-fixer.

■

Corollary 58 *If a graph G is a universal ρ-fixer, then $\rho(G) \leq 2$.*

Proof. Let G be a connected graph with $\rho(G) \geq 3$ and X be a ρ-set of G. Since G is a connected graph with $\rho(G) \geq 3$ and by the definition of a packing, for each vertex $v \in X$, $|\text{pn}(v, X)| \geq 2$. Let $x \in X$ and $X' = X - \{x\}$. Let $Y \subseteq X$ with $|Y| = 2$. Then $|X'| + |Y| = \rho(G) + 1$. Furthermore, $|V(G) - N[X']| \geq 2 = \rho(G) - |X'| + 1$ and $|V(G) - N[Y]| \geq 2(\rho(G) - 2) \geq \rho(G) - 1 = \rho(G) - |Y| + 1$. Hence by Theorem 57, G is not a universal ρ-fixer. ■

Corollary 59 *If $\rho(G) \leq 2$ and every ρ-set of G is a dominating set of G, then G is universal ρ-fixer.*

Proof. Let G be a graph where every ρ-set of G is a dominating set and $\rho(G) \leq 2$. Let X,Y be a pair of packings of G with $|X| + |Y| = \rho(G) + 1 \leq 3$. Since $\rho(G) \leq 2$ it follows that either $|X| = \rho(G)$ or $|Y| = \rho(G)$. Without loss of generality we assume $|X| = \rho(G)$. Then X is a ρ-set of G and by our choice of G, X is also a dominating set of G. Furthermore, since $\rho(G) - |X| + 1 = 1$ and $|V(G) - N[X]| = 0$ we have $|V(G) - N[X]| < \rho(G) - |X| + 1$. Hence by Theorem 57, G is a universal ρ-fixer. ■

By Corollary 59 it is easy to verify that C_6 and the family of graphs formed by adding an edge between a vertex $v \in K_n$ and a vertex $u \in K_m$, where $n, m \geq 1$, are examples of universal ρ-fixers.

3.6.2 Prism ρ-Fixers

Proposition 60 *A graph G is a prism ρ-fixer if and only if for each pair of packings $X, Y \subseteq V(G)$ such that $X \cup Y$ is independent, $|X| + |Y| \leq \rho(G)$.*

Proof. Suppose $\rho(K_2 \times G) = \rho(G)$ but for some pair of packings $X, Y \subseteq V(G)$ where $X \cup Y$ is independent, $|Y| + |X| > \rho(G)$. Then for each pair of vertices x, y, where $x \in X$ and $y \in Y$, we have $d_G(x, y) \geq 2$ which means $d_{K_2 \times G}(x_1, y_2) \geq 3$. Then $X_1 \cup Y_2$ is a packing set of $K_2 \times G$ of cardinality $|X| + |Y| > \rho(G)$, a contradiction.

Conversely, suppose that G is not a prism ρ-fixer. Then $\rho(K_2 \times G) > \rho(G)$. Let $W = X_1 \cup Y_2$ be a ρ-set of $K_2 \times G$; necessarily we have that $|X| + |Y| > \rho(G)$. Since W is a packing for $K_2 \times G$, $X \cup Y$ is independent. ■

We can form a family of prism ρ-fixers $\mathcal{F}$ by taking P_4 and joining any number of copies of P_2 to either support vertex of P_4. See Figure 3.6 for an example of a graph in $\mathcal{F}$. We now use Proposition 60 to show that the graphs in $\mathcal{F}$ are prism ρ-fixers. Let $G \in \mathcal{F}$. Then $\rho(G) = k$ where k is the number of leaves on the graph G. Since the largest independent set of G also has cardinality k it follows that for each pair of packings $X, Y \subseteq V(G)$ such that $X \cup Y$ is independent, $|X| + |Y| \leq \rho(G)$. Thus by Proposition 60, G is a prism ρ-fixer. Note that for any graph $F \in \mathcal{F}$, where $F \neq P_4$, $\rho(F) > 2$. Hence by Corollary 59, F is not a universal ρ-fixer.

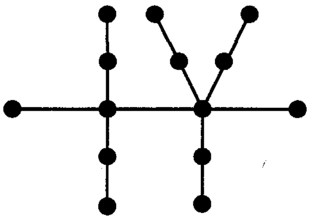

Figure 3.6: An example of a graph in $\mathcal{F}$.

Proposition 61 *If a graph G is a prism ρ-fixer, then every ρ-set of G is also a dominating set of G.*

Proof. Suppose that G is a prism ρ-fixer but G contains a ρ-set Q which is not a dominating set. Let $v \in V(G) - N[Q]$. Then $W = Q_1 \cup \{v_2\}$ is a packing of $K_2 \times G$, which implies $\rho(K_2 \times G) > \rho(G)$, a contradiction. ■

We can see that the converse of Proposition 61 does not hold by observing the graph P_7. Note that $\rho(P_7) = 3$ and the only ρ-set of P_7 is also a dominating set, but $\rho(K_2 \times P_7) = 4$. (See Figure 3.7, where the circled vertices represent maximum packings for each graph.)

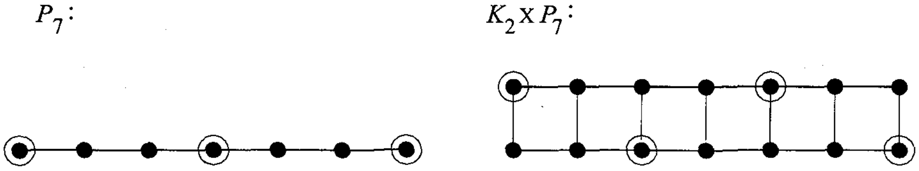

Figure 3.7: Maximum packings in P_7 and $K_2 \times P_7$.

3.6.3 Universal ρ_l-Fixers

Proposition 62 *For any graph G and any permutation π of $V(G)$, $\rho_l(G) \leq \rho_l(\pi G)$.*

Proof. Let G be a graph and π a permutation of $V(G)$ such that $\rho_l(G) > \rho_l(\pi G)$. Let $W = X_1 \cup D_2$ be a ρ_l-set of πG. Then $X_1 \neq \varnothing$ and $D_2 \neq \varnothing$, otherwise W is also a maximal packing of G, a contradiction. Since $\rho_l(G) > |W|$, there exists a set $Y \subseteq V(G) - X$ with $|Y| > |D|$ such that $X \cup Y$ is a packing of G. Therefore $d_G(x, y) \geq 3$ for each $x \in X$ and $y \in Y$. Since

W is a maximal packing of πG, for each $y \in Y$ there exists $u \in D$ such that $d_G(\pi^{-1}(u), y) \leq 1$. By the pigeonhole principle there exists $u \in D$ and $y, y' \in Y$ such that $d_G(\pi^{-1}(u), y) \leq 1$ and $d_G(\pi^{-1}(u), y') \leq 1$. But then $d_G(y, y') \leq 2$, contradicting the fact that $X \cup Y$ is a packing of G. ■

Proposition 63 *If a graph G contains a ρ_l-set which is also a dominating set, then G is a universal ρ_l-fixer.*

Proof. Suppose $P \subseteq V(G)$ is a minimum maximal packing set of G which is also a dominating set. Then P_1 is a maximal packing set of πG for any permutation π since for any vertex $v_2 \in V(G_2)$, $N[v_2] \cap N[P_1] = N[v_2] \cap V(G_1) \neq \varnothing$. ■

An example of a universal ρ_l-fixer is P_6. The two support vertices are a ρ_l-set and a dominating set of P_6. (See Figure 3.8).

Figure 3.8: An example of a universal ρ_l-fixer.

3.6.4 Prism ρ_l-Fixers

We first mention a few notes on packings of G and $K_2 \times G$.

(i) If $W = X_1 \cup D_2$ is a maximal packing of $K_2 \times G$, then $X \cup D$ is not necessarily a packing of G, but if it is, then the maximality of W in $K_2 \times G$ ensures the maximality of $X \cup D$ in G.

(ii) If $W = X_1 \cup D_2$ is a maximal packing of $K_2 \times G$ and $X \cup D$ is not a packing of G, then there exists a set $D' \subseteq D$ such that for each vertex $v \in D'$ there exists a vertex $u \in X$ such that $d_G(u, v) = 2$.

(iii) If $X \cup D$ is a maximal packing in G, then $X_1 \cup D_2$ is a packing, but not necessarily a maximal packing, in $K_2 \times G$.

Definition: $N_k[S]$ is the set of all vertices, including S, within a distance k from S.

Proposition 64 *A graph G is a prism ρ_l-fixer if and only if there exists an independent set D, with $|D| = \rho_l(G)$, which can be partitioned into packings $X, Y \subseteq V(G)$ such that $V(G) - N[Y] \subseteq N_2[X]$ and $V(G) - N[X] \subseteq N_2[Y]$.*

Proof. Suppose that G is a prism ρ_l-fixer. Let $W = X_1 \cup Y_2$ be a ρ_l-set of $K_2 \times G$. Then $X \cup Y$ is an independent set of G. Suppose that

$V(G) - N[Y] \not\subseteq N_2[X]$ or $V(G) - N[X] \not\subseteq N_2[Y]$. Without loss of generality we may assume the former. Let $v \in V(G) - N[Y] - N_2[X]$. Then for each $u_1 \in X_1$, $d_{K_2 \times G}(u_1, v_1) \geq 3$ and for each vertex $w_2 \in Y_2$, $d_{K_2 \times G}(w_2, v_1) \geq 3$, a contradiction since $X_1 \cup Y_2$ is a ρ_l-set of $K_2 \times G$.

Conversely, suppose that G contains an independent set D, with $|D| = \rho_l(G)$, which can be partitioned into packings $X, Y \subseteq V(G)$ such that $V(G) - N[Y] \subseteq N_2[X]$ and $V(G) - N[X] \subseteq N_2[Y]$. Then $W = X_1 \cup Y_2$ is a maximal packing of $K_2 \times G$ and by Proposition 62, W is also a ρ_l-set of $K_2 \times G$. ■

Some examples of prism ρ_l-fixers are P_6, P_7, C_6, C_7 and C_8 which are shown in Figure 3.9. In the figure the circled vertex in each graph represents the packing X and the vertex with a square around it represents the packing Y as defined in Proposition 64. It is easy to verify that these choices for X and Y satisfy the necessary requirements for these graphs to be prism ρ_l-fixers. The spider $S(3, 3, ..., 3)$ shown in Figure 3.10 is also a prism ρ_l-fixer. Any partition of the support vertices (vertices adjacent to leaves) of $S(3, 3, ..., 3)$ gives us our packings X and Y as defined in Proposition 64.

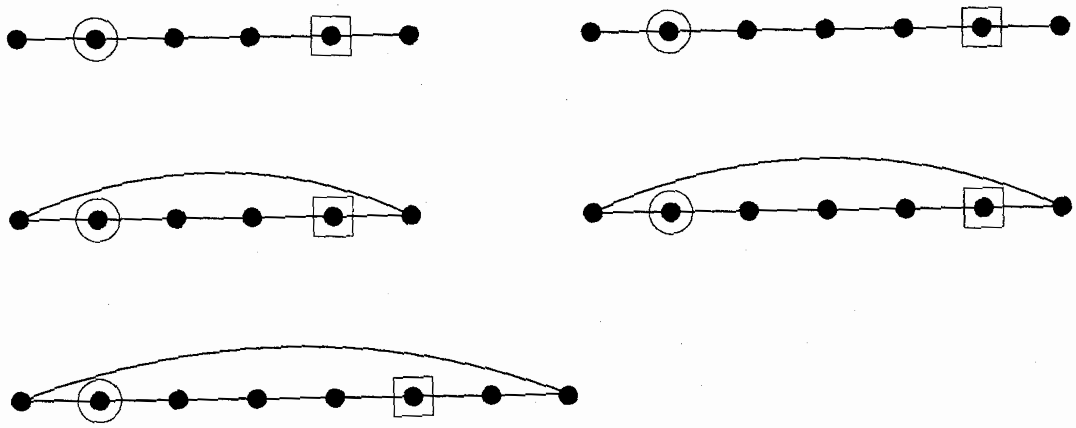

Figure 3.9: Examples of prism ρ_l-fixers.

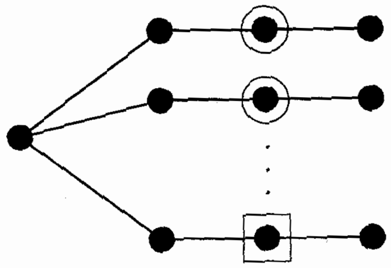

Figure 3.10: $S(3,3,...,3)$

Conclusion

In this thesis we investigated various domination parameters for the graphs πG and $K_2 \times G$. In particular, we looked at $\gamma(G)$, $\gamma_{pr}(G)$, $\gamma_t(G)$, $\gamma_c(G)$, $i(G)$, $\rho(G)$ and $\rho_l(G)$. The parameters y which satisfy $y(G) \leq y(\pi G) \leq 2y(G)$ for all permutations π of $V(G)$ are $y \in \{\gamma, \gamma_{pr}, \gamma_t, \rho\}$. For the parameter i we have $i(G) \leq i(\pi G) \leq |V(G)|$, for γ_c, $\gamma_c(G) \leq \gamma_c(\pi G) \leq 2\gamma_c(G) + 1$ and for ρ_l, $\rho_l(G) \leq \rho_l(\pi G)$, for all permutations π of $V(G)$.

Universal y-doublers are graphs for which $y(\pi G) = 2y(G)$ for all permutations π of $V(G)$. We characterized universal y-doublers for $y \in \{\gamma_{pr}, \gamma_t\}$. We also characterized universal γ_c-maximizers, the graphs for which $2\gamma_c(G) \leq \gamma_c(\pi G) \leq 2\gamma_c + 1$ for all permutations π of $V(G)$. Furthermore, K_1 and K_2 are the only graphs for which $i(\pi G) = |V(G)|$ for any permutations π of $V(G)$.

Prism y-doublers are graphs for which $y(K_2 \times G) = 2y(G)$. We characterized prism y-doublers for $y \in \{\gamma_{pr}, \gamma_t, i, \rho\}$.

Prism y-fixers are graphs for which $y(K_2 \times G) = y(G)$. We characterized prism y-fixers for $y \in \{\gamma, \gamma_{pr}, \gamma_t, \gamma_c, i, \rho, \rho_l\}$. In particular, K_1 is the only graph for which $\gamma_c(K_2 \times G) = \gamma_c(G)$ and thus, is the only graph which is a universal γ_c-fixer. Also included in this work is a characterization of universal ρ-fixers.

Some open problems include a characterization of universal y-fixers for $y \in \{\gamma, \gamma_{pr}, \gamma_t, i, \rho_l\}$. Also unknown is a characterization for universal ρ-doublers, prism γ_c-doublers and prism ρ_l-doublers..

Bibliography

[1] A. P. Burger, C. M. Mynhardt and W. D. Weakley, On the domination number of prisms of graphs, *Discussiones Math. Graph Theory* **24** (2004), 303–318.

[2] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, Third Edition, Chapman & Hall, London (1996).

[3] B. L. Hartnell and D. F. Rall, On dominating the Cartesian product of a graph and K_2, *Discussiones Math. Graph Theory* **24** (2004), 389–402.

[4] B. L. Hartnell and D. F. Rall, Domination in Cartesian products: Vizing's conjecture, in: T. Haynes, S. Hedetniemi and P. Slater, *Domination in Graphs—Advanced Topics*, Marcel Dekker, New York, (1998) 163–189.

[5] S. Hedetniemi and R. Laskar, Connected domination in graphs, *Graph theory and combinatorics(Cambridge 1983)*, Academic Press, London, (1984), 209–218

[6] T. Haynes, S. Hedetniemi and P. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York (1998).

[7] T. Haynes and P. Slater, Paired-domination in graphs, *Networks* **32** (1998), 199–206.

[8] A. Meir and J. W. Moon, Relations between packing and covering numbers of a tree, *Pacific J. Math.* **61** (1975), 225–233.

[9] L. Sun, A result on Vizing's conjecture, *Discrete Math.* **275** (2004), 363–366.

[10] V.G. Vizing, The Cartesian product of graphs, *Vyčisl. Sistemy* **9** (1963), 30–43.

[11] V. G. Vizing, Some unsolved problems in graph theory, *Uspehi Mat. Nauk* **23** (1968), 117–134.