# AN APPLICATION OF THE FRACTIONAL CALCULUS

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ABSTRACT. H. Silverman and E.M. Silvia introduced the class $\mathcal{R}(\alpha)$ defined by using the extremal function $S_\alpha(z) = z/(1-z)^{2(1-\alpha)}$ for the class $\mathcal{S}^*(\alpha)$ of starlike functions of order $\alpha$. The object of the present paper is to prove some distortion theorems for the fractional derivatives and fractional integrals of functions in the class $\mathcal{R}(\alpha)$.

## 1. INTRODUCTION

Let $\mathcal{S}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic and univalent in the unit disk $\mathcal{U} = \{z: |z| < 1\}$. Then a function $f(z)$ belonging to the class $\mathcal{S}$ is said to be starlike of order $\alpha$ if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad (z \in \mathcal{U}) \tag{1.2}$$

for some $\alpha$ $(0 \leqq \alpha < 1)$. We denote by $\mathcal{S}^*(\alpha)$ the class of all starlike functions of order $\alpha$. Further a function $f(z)$ belonging to the class $\mathcal{S}$ is said to be convex of order $\alpha$ if and only if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \qquad (z \in \mathcal{U}) \tag{1.3}$$

for some $\alpha$ $(0 \leqq \alpha < 1)$. Also we denote by $\mathcal{K}(\alpha)$ the class of all convex functions of order $\alpha$. We note that $f(z) \in \mathcal{K}(\alpha)$ if and only if

$zf'(z) \in \mathcal{S}^*(\alpha)$, and that $\mathcal{S}^*(\alpha) \subseteq \mathcal{S}^*(0) \equiv \mathcal{S}^*$, and $\mathcal{K}(\alpha) \subseteq \mathcal{K}(0) \equiv \mathcal{K}$ for $0 \leqq \alpha < 1$.

The classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ were first introduced by Robertson [6], and later were studied by Schild [9], MacGregor [3] and Pinchuk [5].

Now, the function

$$S_\alpha(z) = \frac{z}{(1-z)^{2(1-\alpha)}} \tag{1.4}$$

is the well-known extremal function for $\mathcal{S}^*(\alpha)$. Setting

$$C(\alpha,n) = \frac{\prod_{k=2}^{n} (k-2\alpha)}{(n-1)!} \qquad (n = 2, 3, 4, \ldots), \tag{1.5}$$

$S_\alpha(z)$ can be written in the form

$$S_\alpha(z) = z + \sum_{n=2}^{\infty} C(\alpha,n) z^n. \tag{1.6}$$

Then we observe that $C(\alpha,n)$ is decreasing in $\alpha$ and satisfies

$$\lim_{n \to \infty} C(\alpha,n) = \begin{cases} \infty & (\alpha < 1/2) \\ 1 & (\alpha = 1/2) \\ 0 & (\alpha > 1/2) \end{cases}. \tag{1.7}$$

Let $f * g(z)$ denote the convolution or Hadamard product of two functions $f(z)$ and $g(z)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \tag{1.8}$$

then

$$(1.9) \qquad f * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Then a function $f(z)$ belonging to the class $\mathcal{S}$ is said to be prestarlike of order $\alpha$ if $f(z)$ satisfies $f * S_\alpha(z) \in \mathcal{S}^*(\alpha)$ for $0 \leqq \alpha < 1$. We denote by $\mathcal{R}_\alpha$ the class of all prestarlike functions of order $\alpha$. The class $\mathcal{R}_\alpha$ was introduced by Ruscheweyh [8], who showed that a necessary and sufficient condition for $f(z)$ to be in the class $\mathcal{R}_\alpha$ is that the functional

$$(1.10) \qquad G(\alpha,z) = \frac{f * \left(S_\alpha(z)/(1-z)\right)}{f * S_\alpha(z)}$$

satisfies $\mathrm{Re}\{G(\alpha,z)\} > 1/2$ for $z \in \mathcal{U}$. Since $S_1(z) = z$, we know that $f(z)$ is prestarlike of order $1$ if and only if $\mathrm{Re}\{f(z)/z\} > 1/2$ for $z \in \mathcal{U}$. Further we note that $\mathcal{R}_0 = \mathcal{K}(0)$ and $\mathcal{R}_{1/2} = \mathcal{S}^*(1/2)$.

Let $\mathcal{T}$ be the subclass of $\mathcal{S}$ consisting of functions of the form

$$(1.11) \qquad f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geqq 0).$$

And we denote by $\mathcal{R}(\alpha)$ the class obtained by taking intersection of $\mathcal{R}_\alpha$ with $\mathcal{T}$, that is, $\mathcal{R}(\alpha) = \mathcal{R}_\alpha \cap \mathcal{T}$. The class $\mathcal{R}(\alpha)$ was introduced by Silverman and Silvia [10].

We need the followimg lemma due to Silverman and Silvia [10].

LEMMA. Let the function $f(z)$ be defined by (1.1). Then $f(z)$ is in the class $\mathcal{R}(\alpha)$ if and only if

$$(1.12) \qquad \sum_{n=2}^{\infty} (n-\alpha)C(\alpha,n)a_n \leqq 1 - \alpha.$$

The object of the present paper is to prove distortion theorems for the fractional derivatives and fractional integrals of $f(z)$ belonging to the class $\mathcal{R}(\alpha)$.

## 2. FRACTIONAL CALCULUS

Many essentially equivalent definitions of the fractional calculus, that is, the fractional derivatives and the fractional integrals, have been given in the literature (*cf.*, *e.g.*, [1], [2, Chapter 13], [7], [11, p. 28 *et seq.*], and [13]). We find it to be convenient to recall here the following definitions which were used recently by Owa [4] (and by Srivastava and Owa [12]).

DEFINITION 1. The fractional integral of order $\lambda$ is defined, for a function $f(z)$, by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} \, d\zeta, \tag{2.1}$$

where $\lambda > 0$, $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

DEFINITION 2. The fractional derivative of order $\lambda$ is defined, for a function $f(z)$, by

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} \, d\zeta, \tag{2.2}$$

where $0 \leqq \lambda < 1$, $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed as in Definition 1 above.

DEFINITION 3. Under the hypotheses of Definition 2, the fractional derivative of order $(n+\lambda)$ is defined by

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^{\lambda} f(z), \tag{2.3}$$

where $0 \leqq \lambda < 1$, and $n \in \mathcal{N}_0 = \{0,1,2, \ldots\}$.

THEOREM 1. Let the function $f(z)$ defined by (1.11) be in the class $\mathcal{R}(\alpha)$ with $0 \leqq \alpha \leqq 1/2$. Then

$$\left|D_z^{-\lambda} f(z)\right| \geqq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)} \left\{1 - \frac{1}{(2+\lambda)(2-\alpha)} |z|\right\} \tag{2.4}$$

and

$$\left|D_z^{-\lambda} f(z)\right| \leqq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)} \left\{1 + \frac{1}{(2+\lambda)(2-\alpha)} |z|\right\} \tag{2.5}$$

for $\lambda > 0$ and $z \in \mathcal{U}$. The bounds in (2.4) and (2.5) are sharp.

PROOF. Let

$$\begin{aligned} F(z) &= \Gamma(2+\lambda) z^{-\lambda} D_z^{-\lambda} f(z) \\ &= z - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2+\lambda)}{\Gamma(n+1+\lambda)} a_n z^n \end{aligned} \tag{2.6}$$

for $\lambda > 0$.

Then, since $[\Gamma(n+1)\Gamma(2+\lambda)]/\Gamma(n+1+\lambda)$ is a decreasing function of $n$ $(n \geqq 2)$, we have

$$\frac{\Gamma(n+1)\Gamma(2+\lambda)}{\Gamma(n+1+\lambda)} \leqq \frac{2}{2+\lambda} \tag{2.7}$$

for $\lambda > 0$ and $n \geqq 2$.

Further, we note that $C(\alpha,n+1) \geqq C(\alpha,n)$ for $0 \leqq \alpha \leqq 1/2$ and $n \geqq 2$ by means of (1.5). Consequently, by using the lemma, we have

$$(2-\alpha)C(\alpha,2)\sum_{n=2}^{\infty} a_n \leqq \sum_{n=2}^{\infty} (n-\alpha)C(\alpha,n)a_n \leqq 1-\alpha \tag{2.8}$$

which implies that

$$\sum_{n=2}^{\infty} a_n \leqq \frac{1}{2(2-\alpha)} . \tag{2.9}$$

Hence we see that

$$\begin{aligned} |F(z)| &\geqq |z| - |z|^2 \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2+\lambda)}{\Gamma(n+1+\lambda)} a_n \\ &\geqq |z| - \left(\frac{2}{2+\lambda}\right)|z|^2 \sum_{n=2}^{\infty} a_n \\ &\leqq |z| - \frac{1}{(2+\lambda)(2-\alpha)} |z|^2 \end{aligned} \tag{2.10}$$

which yields (2.4), and

$$\begin{aligned} |F(z)| &\leqq |z| + |z|^2 \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2+\lambda)}{\Gamma(n+1+\lambda)} a_n \\ &\leqq |z| + \left(\frac{2}{2+\lambda}\right)|z|^2 \sum_{n=2}^{\infty} a_n \end{aligned} \tag{2.11}$$

$$\leq |z| + \frac{1}{(2+\lambda)(2-\alpha)} |z|^2$$

which gives (2.5). Finally, by taking the function $f(z)$ defined by

$$D_z^{-\lambda} f(z) = \frac{z^{1+\lambda}}{\Gamma(2+\lambda)} \left\{1 - \frac{1}{(2+\lambda)(2-\alpha)} z\right\}, \tag{2.12}$$

we can show that the bounds of the theorem are sharp.

COROLLARY 1. Let the function $f(z)$ defined by (1.11) be in the class $\mathcal{R}(\alpha)$ with $0 \leqq \alpha \leqq 1/2$. Then $D_z^{-\lambda} f(z)$ is included in a disk with its center at the origin and radius $r_0$ given by

$$r_0 = \frac{1}{\Gamma(2+\lambda)} \left\{1 + \frac{1}{(2+\lambda)(2-\alpha)}\right\}. \tag{2.13}$$

THEOREM 2. Let the function $f(z)$ defined by (1.11) be in the class $\mathcal{R}(\alpha)$ with $0 \leqq \alpha \leqq 1/2$. Then

$$\left|D_z^{\lambda} f(z)\right| \geqq \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)} \left\{1 - \frac{1}{2-\alpha} |z|\right\} \tag{2.14}$$

and

$$\left|D_z^{\lambda} f(z)\right| \leqq \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)} \left\{1 + \frac{1}{2-\alpha} |z|\right\} \tag{2.15}$$

for $0 \leqq \lambda < 1$ and $z \in \mathcal{U}$. The bounds in (2.14) and (2.15) are sharp.

PROOF. We define the function

$$(2.16) \qquad P(z) = \Gamma(2-\lambda) z^{\lambda} D_z^{\lambda} f(z)$$

$$= z - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n z^n$$

for $0 \leqq \lambda < 1$. In view of the lemma, we have

$$(2.17) \qquad \left(\frac{2-\alpha}{2}\right) C(\alpha,2) \sum_{n=2}^{\infty} n\, a_n \leqq \sum_{n=2}^{\infty} (n-\alpha) C(\alpha,n) a_n \leqq 1 - \alpha$$

which gives

$$(2.18) \qquad \sum_{n=2}^{\infty} n\, a_n \leqq \frac{1}{2-\alpha}\,.$$

Note that

$$(2.19) \qquad 1 \leqq \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} < n$$

for $0 \leqq \lambda < 1$ and $n \geqq 2$, and $C(\alpha,n+1) \geqq C(\alpha,n)$ for $0 \leqq \alpha \leqq 1/2$ and $n \geqq 2$. Hence we get

$$(2.20) \qquad |P(z)| \geqq |z| - |z|^2 \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n$$

$$\geqq |z| - |z|^2 \sum_{n=2}^{\infty} n\, a_n$$

$$\geqq |z| - \frac{1}{2-\alpha} |z|^2$$

and

$$(2.21)\qquad |P(z)| \leqq |z| + |z|^2 \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n$$

$$\leqq |z| + |z|^2 \sum_{n=2}^{\infty} n\, a_n$$

$$\leqq |z| + \frac{1}{2-\alpha} |z|^2 .$$

Thus the inequalities (2.14) and (2.15) follow from (2.20) and (2.21), respectively. Further, we can see that the bounds of the theorem are sharp for the function $f(z)$ defined by

$$(2.22)\qquad D_z^{\lambda} f(z) = \frac{z^{1-\lambda}}{\Gamma(2-\lambda)} \left\{1 - \frac{1}{2-\alpha} z\right\}.$$

This completes the proof of the theorem.

COROLLARY 2. Let the function $f(z)$ defined by (1.11) be in the class $\mathcal{R}(\alpha)$ with $0 \leqq \alpha \leqq 1/2$. Then $D_z^{\lambda} f(z)$ is included in a disk with its center at the origin and radius $r_1$ given by

$$(2.23)\qquad r_1 = \frac{3-\alpha}{(2-\alpha)\Gamma(2-\lambda)} .$$

THEOREM 3. Let the function $f(z)$ defined by (1.11) be in the class $\mathcal{R}(\alpha)$ with $0 \leqq \alpha \leqq 1/2$. Then

$$(2.24)\qquad \left|D_z^{1-\lambda} f(z)\right| \leqq \frac{|z|^{\lambda}}{\Gamma(2+\lambda)} \left\{(1+\lambda) + \left(\frac{1}{2-\alpha}\right)|z|\right\}$$

for $\lambda > 0$ and $z \in \mathcal{U}$. The result (2.24) is sharp.

PROOF. Let $F(z)$ be defined by (2.6). Then, with the aid of (2.18), we can see that

$$(2.25)\qquad |F'(z)| \leqq 1 + |z| \sum_{n=2}^{\infty} \frac{n\,\Gamma(n+1)\Gamma(2+\lambda)}{\Gamma(n+1+\lambda)}\, a_n$$

$$\leqq 1 + \left(\frac{2}{2+\lambda}\right)|z| \sum_{n=2}^{\infty} n\, a_n$$

$$\leqq 1 + \frac{2}{(2+\lambda)(2-\alpha)}\,|z|.$$

By using (2.5) of Theorem 1, we know that the inequality (2.25) gives the estimate (2.24) we require. Further, the result is sharp for the function $f(z)$ defined by (2.12).

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