LIMITING PROBABILITIES FOR REPEATED ROLLS OF A DIE

(Mathematics Magazine Problem 1217)

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1217. A die is rolled repeatedly. Let p_n be the probability that the accumulated score is at some time equal to n. Find $\lim_{n\to\infty} p_n$. [David Callan, Lafayette College.]

<u>Solution by Bruce R. Johnson, University of Victoria</u>.

More generally, we will solve the problem using a fair r-sided die with equally likely possible outcomes $1, 2, \ldots, r$. We will give three different solutions; the first relies on probabilistic intuition, the second uses Markov Chain Theory, and the third uses Renewal Theory.

<u>Solution 1</u> (Intuitive). For positive integer n and $j \in \{1,2,\ldots,r\}$, we let $A_{n,j}$ denote the event that j occurs on the roll that causes the accumulated sum to become $\geq n$ for the first time. Since in the long run the outcomes $1, 2, \ldots, r$ will occur in equal proportions and since the accumulated sum will advance j units each time j is rolled, it follows that, for large n, $P(A_{n,j})$ will be approximately proportional to j. That is,

$$\lim_{n\to\infty} P(A_{n,j}) = j/(1+2+\ldots+r), \quad j = 1, 2, \ldots, r.$$

Therefore,

$$\lim_{n\to\infty} P(\text{accumulated sum ever equals } n)$$

$$= \lim_{n\to\infty} \sum_{j=1}^{r} P(A_{n,j})\, P(\text{accumulated sum ever equals } n \mid A_{n,j})$$

$$= \lim_{n\to\infty} \sum_{j=1}^{r} P(A_{n,j}) \cdot (1/j)$$

$$= \sum_{j=1}^{r} \big(j/(1+2+\ldots+r)\big)(1/j)$$

$$= r/(1+2+\ldots+r) = 2/(1+r).$$

Hence, for a standard die the answer is $\lim_{n\to\infty} p_n = 2/7$.

Solution 2 (Markov Chain Theory). For every nonnegative integer n we define

$$X_n = n - (\text{largest accumulated sum that is } \le n).$$

Then $\{X_n\}_{n=0}^{\infty}$ is an ergodic Markov chain with state space $= \{0,1,\ldots,r-1\}$ and transition matrix $\mathcal{P}$ given by

$$\mathcal{P} = \begin{bmatrix} \frac{1}{r} & \frac{r-1}{r} & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{r-1} & 0 & \frac{r-2}{r-1} & 0 & 0 & \cdots & 0 \\ \frac{1}{r-2} & 0 & 0 & \frac{r-3}{r-2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & & & \\ \frac{1}{2} & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2} \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

To find the stationary distribution $(\pi_0,\pi_1,\ldots,\pi_{r-1})$ for this Markov chain we solve the linear system

$$\sum_{j=0}^{r} \pi_j = 1, \quad (\pi_0,\pi_1,\ldots,\pi_{r-1})\,\mathcal{P} = (\pi_0,\pi_1,\ldots,\pi_{r-1}),$$

and obtain

$$\pi_j = 2(r-j)/r(1+r), \quad j = 0,\ 1,\ \ldots,\ r-1.$$

Therefore,

$$\lim_{n\to\infty} P(\text{accumulated sum ever equals } n)$$

$$= \lim_{n\to\infty} P(X_n = 0) = \pi_0 = 2/(1+r).$$

Solution 3 (Renewal Theory). For each positive integer j we let T_j denote the outcome of the jth roll. Then $T_1, T_2, \ldots$ are independent random variables with common distribution F, where F denotes the discrete uniform distribution on $1, 2, \ldots, r$. We let $\{N(t): t \ge 0\}$ be the renewal process determined by the interarrival times $T_1, T_2, \ldots$. That is,

$$N(t) = \text{number of indices } n \text{ for which } T_1 + T_2 + \ldots + T_n \le t.$$

For $t \ge 0$ we consider the "current life" random variable

$$\delta_t = t - (T_1 + T_2 + \ldots + T_{N(t)}). \quad \text{(Of course, } \delta_t = t \text{ if } N(t) = 0.)$$

For fixed $y \ge 0$ we let $A_y(t) = P(\delta_t \le y)$. Since

$$P(\delta_t \le y \mid T_1 = x) = \begin{cases} A_y(t-x) & \text{if } x \le t \\ 1 & \text{if } x > t \text{ and } y \ge t \\ 0 & \text{if } x > t \text{ and } y < t, \end{cases}$$

it follows that $A_y(t)$ satisfies the renewal equation

$$A_y(t) = \int_{[0,\infty)} P(\delta_t \le y \mid T_1 = x)\, dF(x)$$

$$= a_y(t) + \int_{[0,t]} A_y(t-x)\, dF(x), \quad \text{where}$$

$$a_y(t) = \begin{cases} 1 - F(t) & \text{if } 0 \le t \le y \\ 0 & \text{if } t > y. \end{cases}$$

Therefore, from the Basic Renewal Theorem (see [1], page 191), we obtain

$$\lim_{n\to\infty} A_y(n) = \frac{2}{1+r} \sum_{k=0}^{\infty} a_y(k)$$

$$= \frac{2}{1+r} \sum_{k=0}^{[y]} \left(1-F(k)\right).$$

Hence,

$$\lim_{n\to\infty} P(\text{accumulated sum ever equals } n)$$

$$= \lim_{n\to\infty} P(\delta_n = 0) = \lim_{n\to\infty} A_0(n) = 2/(1+r).$$

REFERENCES

[1] S. Karlin and H.M. Taylor, A First Course in Stochastic Processes, Academic Press, New York, 1975.

[2] S.M. Ross, Introduction to Probability Models, 2nd ed., Academic Press, New York, 1980.