VARIANT OF BUFFON'S NEEDLE PROBLEM

(Amer. Math. Monthly Advanced Problem 6644)

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6644. *Proposed by Peter Rogerson, SUNY at Buffalo.*

A needle with length L between 1 and $\sqrt{2}$ is tossed at random upon the Cartesian plane. Find the probability that it comes to rest not crossing any line of the form $x = m$ or $y = n$, where m and n are integers. (The case $L < 1$ is treated on pp. 255–256 of J.V. Uspensky's *Introduction to Mathematical Probability*, McGraw Hill, 1937. Uspensky attributes that case of the problem to Laplace.)

Solution by Bruce R. Johnson, University of Victoria, Victoria, B.C., Canada. We will show that the desired probability is

$$\frac{2}{\pi}\left[\sin^{-1}\frac{1}{L} - \cos^{-1}\frac{1}{L} + 2\sqrt{L^2-1} - 1 - \frac{L^2}{2}\right].$$

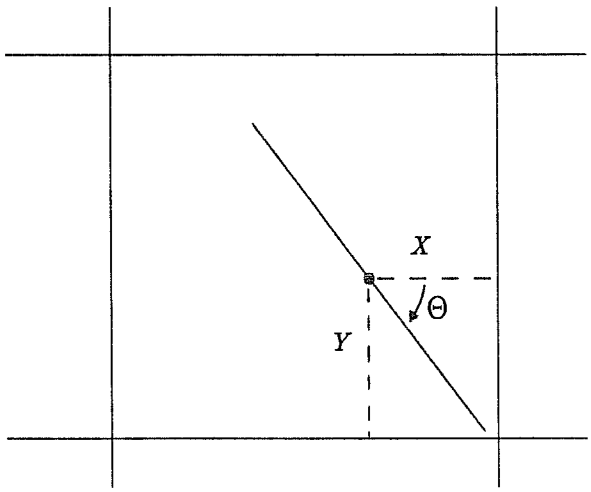
Numerical values of this probability for a few selected values of L are given in the following table.

Needle Length L	1.0	1.1	1.2	1.3	$\sqrt{2}$
P(Needle not crossing)	.0451	.0146	.00387	.00051	0

Let X = distance from center of needle to nearest line of form $x = m$, where m is an integer;

Y = distance from center of needle to nearest line of form $y = n$, where n is an integer;

Θ = angle having vertex at center of needle, initial ray parallel to x-axis and directed toward nearest line of form $x = m$ (integer), and terminal ray the extension of the needle's half that intersects nearest line of form $y = n$ (integer).



With probability one X, Y, Θ are well–defined random variables that are independent such that X and Y are uniformly distributed over the interval $(0,\frac{1}{2})$ and Θ is uniformly distributed over the interval $(0,\pi)$. By first expressing the desired probability in terms of X, Y, Θ, and then conditioning on Θ, we obtain

P(Needle not crossing any lines)

$$= P\left[\frac{L}{2}\cos\Theta < X,\ \frac{L}{2}\sin\Theta < Y,\quad 0 < \Theta \le \frac{\pi}{2}\right] + P\left[\frac{L}{2}\cos(\pi\text{-}\Theta) < X,\ \ \frac{L}{2}\sin(\pi\text{-}\Theta) < Y,\ \ \frac{\pi}{2} < \Theta < \pi\right]$$

$$= 2P\left[\frac{L}{2}\cos\Theta < X,\ \frac{L}{2}\sin\Theta < Y,\quad 0 < \Theta \le \frac{\pi}{2}\right]$$

$$= 2\int_0^{\pi} P\left[\frac{L}{2}\cos\Theta < X,\ \frac{L}{2}\sin\Theta < Y,\ 0 < \Theta \le \frac{\pi}{2}\ \middle|\ \Theta = \theta\right] f_\Theta(\theta)\ d\theta$$

$$= 2\int_0^{\pi/2} P\left[\frac{L}{2}\cos\theta < X\right] P\left[\frac{L}{2}\sin\theta < Y\right] \frac{1}{\pi}\ d\theta, \quad \text{by independence of} \quad X,\ Y,\ \Theta$$

$$= 2\int_{\cos^{-1}(1/L)}^{\sin^{-1}(1/L)} (1 - L\cos\theta)(1 - L\sin\theta)\ \frac{1}{\pi}\ d\theta$$

$$= \frac{2}{\pi}\left[\sin^{-1}\frac{1}{L} - \cos^{-1}\frac{1}{L} + 2\sqrt{L^2-1} - 1 - \frac{L^2}{2}\right].$$