ON A LADDER OF COMMUTING SQUARES AND THE PAIR OF FACTORS GENERATED BY THEM

by

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1. INTRODUCTION.

Let A and B be C*–algebras, then they are called an ordered pair of C*–algebras, denoted by (A,B), if B contains A. As defined in [4], there exists the map T from the set of ordered pairs of finite dimensional C*–algebras into the set of finite matrices over the integers, mapping (A,B) into T_A^B the inclusion matrix of A into B. Now let $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ be increasing sequences of finite dimensional C*–algebras, with $A_n \subset B_n$ and such that there exists the unique trace tr on $\overline{\bigcup_{n=1}^{\infty} B_n}$ and on $\overline{\bigcup_{n=1}^{\infty} A_n}$. If for each n, $E_{A_{n+1}}(B_n) \subset A_n$, then the above sequences are called a ladder of commuting squares. Furthermore, if there exists an integer h, such that for each n,

$$T_{B_n}^{B_{n+1}} = T_{B_{n+h}}^{B_{n+1+h}}, \quad T_{A_n}^{A_{n+1}} = T_{A_{n+h}}^{A_{n+1+h}}, \quad T_{A_n}^{B_n} = T_{A_{n+h}}^{B_{n+h}}$$

with $T_{A_n}^{A_{n+h}}$ and $T_{B_n}^{B_{n+h}}$ primitives, then the above sequences are called periodic. In this case using GNS construction with respect to the trace tr we will have the factors $B_\infty = \overline{\bigcup_{n=1}^{\infty} B_n}^{\,s.o.t.}$ and $A_\infty = \overline{\bigcup_{n=1}^{\infty} A_n}^{\,s.o.t.}$ and by Theorem (1.4) [4], we have

$$[B_\infty:A_\infty] = \frac{\|\vec{s}_n\|^2}{\|\vec{t}_n\|^2},$$

for n large enough, where $\vec{s}_n$ is a trace vector for A_n and $\vec{t}_n$ is a trace vector for B_n. Also, if $p \in A_\infty' \cap B_\infty$ is a projection such that $p \in \bigcup_{n=1}^{\infty} B_n$, then by Theorem (1.5) [4], $[(B_\infty)_p:(A_\infty)_p] = tr^2(p)[B_\infty:A_\infty]$. The main purpose of this article is to investigate when, in general, $H[B_\infty:A_\infty] = \ln[B_\infty:A_\infty]$, and to prove that the above equality holds in the periodic case, thus in this case by (4.5) [3], for each projection

$$p \in (A_\infty)' \cap B_\infty,\ [(B_\infty)_p:(A_\infty)_p] = tr^2(p)[B_\infty:A_\infty].$$

Finally we are going to further study the structure of $A_\infty' \cap B_\infty$. Also as in [4], we denote a periodic ladder of commuting squares by $(A_n) \subset (B_n)$, and call it a periodic sequence with dimension vectors $a_n = (a_{i,n})$, $b_n = (b_{i,n})$ and minimal central projections $e_{k,n}$, $f_{\ell,n}$.

2. THE MAIN RESULTS

Given two von Neumann algebras, $A \subset B$, with a faithful trace tr on B, then in [3], Pimsner and Popa defined

$$\lambda(B,A) = \sup\{\lambda,\ E_A(x) \geq \lambda x,\ \forall\, x \in B_+\}$$

and showed that in the case of A and B being type II_1 factors, $\lambda(B,A) = [B:A]^{-1}$. If A and B are finite C^*–algebras with dimension vectors $a = (a_i)$ and $b = (b_i)$ and the inclusion matrix $(g_{ij}) = G = T_A^B$, then by (6.3) [3] and (6.4) [3], if $g_{k\ell} \leq a_k$ for all ℓ, we have

$$H(B:A) = \sum_{k,\ell} a_k\, g_{k\ell}\, t_\ell \ln \frac{s_k b_\ell}{t_\ell a_k} \tag{1}$$

$$(\lambda(B:A))^{-1} = \max_\ell \left(\sum_k g_{k\ell}\, s_k / t_\ell\right), \tag{2}$$

and if e_k and f_ℓ represents the minimal central projections for A and B respectively, then we get

$$H(B:A) = \sum_{k,\ell} \operatorname{tr}(e_k f_\ell) \ln \frac{s_k b_\ell}{t_\ell a_k}. \tag{3}$$

Now let $(A_n) \subset (B_n)$ be a ladder of commuting squares and let us denote

$$T_{A_n}^{B_n} = G_n = (g_{ijn}).$$

Then by (2.6) [3] and (3.4) [3],

$$H(B_\infty : A_\infty) = \lim H(B_n : A_n) \text{ increasingly} \tag{4}$$

$$\lambda^{-1}(B_\infty, A_\infty) = \lim \lambda^{-1}(B_n, A_n) \text{ increasingly.} \tag{5}$$

Also note that if $(A_n) \subset (B_n)$ is periodic then we have the following rigidity properties:

Lemma. *Let* $(A_n) \subset (B_n)$ *be a periodic sequence. Then for* n *large enough and each* ℓ,

$$\lambda^{-1}(B_n,A_n) = [B_\infty:A_\infty] = \sum_k g_{k\ell n} \frac{s_{k,n}}{t_{\ell,n}}.$$

Proof.

$$\sum_k g_{k\ell n} \frac{s_{k,n}}{t_{\ell,n}} = \sum_{k,m} g_{k\ell n}\, g_{kmn} \frac{t_{m,n}}{t_{\ell,n}} = \frac{(G_n^t\, G_n\, \vec{t}_n)_\ell}{t_{\ell,n}}$$

and by Theorem (1.4) [4], for n large enough $G_n^t\, G_n\, \vec{t}_n = [B_\infty:A_\infty]\vec{t}_n$, thus

$$\sum_k g_{k\ell n} \frac{s_{k,n}}{t_{\ell,n}} = [B_\infty:A_\infty] = \lambda^{-1}(B_n,A_n).$$

Q.E.D.

THEOREM 1: *Let* $(A_n) \subset (B_n)$ *be a periodic sequence, then*

$$H(B_\infty:A_\infty) = \ell n([B_\infty:A_\infty]).$$

Proof. First note that

$$H(B_n:A_n) = \sum_{k,\ell} tr(e_{k,n} f_{\ell,n})\ \ell n \left[\frac{s_{k,n} b_{\ell,n}}{t_{\ell,n} a_{k,n}}\right]$$

$$\le \ell n \sum_{k,\ell} tr(e_{k,n} f_{\ell,n}) \frac{s_{k,n} b_{\ell,n}}{t_{\ell,n} a_{k,n}},$$

and since ℓn is strictly concave, the equality holds only when all the terms $\frac{s_{k,n} b_{\ell,n}}{t_{\ell,n} a_{k,n}}$ are equal and furthermore

$$\sum_{k,\ell} tr(e_{k,n} f_{\ell,n}) \frac{s_{k,n} b_{\ell,n}}{t_{\ell,n} a_{k,n}} = \sum_{k,\ell} \frac{g_{k,\ell} s_{k,n}}{t_{\ell,n}} tr(f_{\ell,n}) = [B_\infty:A_\infty].$$

Now by (1.6) [4], $\vec{s}_n$ is a multiple of the Perron vector $\vec{v}$ with Perron eigenvalue μ for $T_{A_n}^{A_{n+k}}$ and $\left[\frac{1}{\mu}\right]^{\ell}\vec{a}_{n+\ell k}$ converges to a positive multiple of $\vec{v}$, which implies that for n large enough $\frac{a_{k_1,n}}{a_{k_2,n}}$ is close enough to $\frac{s_{k_1,n}}{s_{k_2,n}}$. Similarly for n large enough $\frac{b_{k_1,n}}{b_{k_2,n}}$ is close enough to $\frac{t_{k_1,n}}{t_{k_2,n}}$, thus for n large enough the terms $\frac{s_{k,n}b_{\ell,n}}{t_{\ell,n}a_{k,n}}$ are close enough to each other, hence,

$$H(B_\infty:A_\infty) = \lim H(B_n:A_n) = \ell n([B_\infty:A_\infty]).$$

Q.E.D.

Corollary. *Let* p *be a projection in* $A'_\infty \cap B_\infty$, *then,*

$$[(B_\infty)_p:(A_\infty)_p] = tr^2(p)[B_\infty:A_\infty] .$$

Proof. Use (4.5) [3].

Q.E.D.

Theorem 2. *Let* $(A_n) \subset (B_n)$ *be a ladder of commuting squares with* $[B_\infty:A_\infty] < \infty$ *and such that* $\lim\limits_{n\to\infty} \sup g_{k\ell n} < \infty$, *then* $H(B_\infty:A_\infty) = \ell n([B_\infty:A_\infty])$ *if and only if*

$$\lim_{n\to\infty} \frac{s_{k,n}b_{\ell,n}}{t_{\ell,n}a_{k,n}} = [B_\infty:A_\infty] = \lim_{n\to\infty} g_{k\ell n}^2 \frac{tr(e_{k,n})tr(f_{\ell,n})}{tr^2(e_{k,n}f_{\ell,n})}$$

for all k,ℓ *for which* $\lim\sup tr(e_{k,n}f_{\ell,n}) \neq 0$, *and* $\lim\limits_{n\to\infty} \frac{s_{k,n}b_{\ell,n}}{t_{\ell,n}a_{k,n}} = 0$ *otherwise.*

Proof. By (6.3) [3], we have

$$\lambda^{-1}(B_n, A_n) = \max_{\ell} \left[\sum_k tr(e_{k,n}) \frac{g^2_{k\ell n}}{tr(e_{k,n} f_{\ell,n})} \right] \tag{6}$$

$$H(B_n : A_n) = \sum_{k,\ell} tr(e_{k,n} f_{\ell,n}) \, \ell n \left[\frac{g^2_{k\ell n} \, tr(e_{k,n}) \, tr(f_{\ell,n})}{tr^2(e_{k,n} f_{\ell,n})} \right], \tag{7}$$

where the sums are taken over those indices for which $e_{k,n} f_{\ell,n} \neq 0$, thus

$$H(B_\infty : A_\infty) = \lim H(B_n : A_n) \leq \lim \ell n \left[\sum_{k,\ell} \frac{g^2_{k\ell n} \, tr(e_{k,n}) \, tr(f_{\ell,n})}{tr(e_{k,n} f_{\ell,n})} \right] = \ell n [B_\infty : A_\infty]$$

where the sum is taken over k,ℓ, for which $e_{k,n} f_{\ell,n} \neq 0$ and $\lim_{n \to \infty} \sup tr(e_{k,n} f_{\ell,n}) \neq 0$. But by the above there exists $d < \infty$, such that

$$\frac{g^2_{k\ell n} \, tr(e_{k,n}) \, tr(f_{\ell,n})}{tr(e_{k,n} f_{\ell,n})} < d$$

for all n, hence by the strict concavity of the function ℓn we are done.

Q.E.D.

Now let p be a projection in $A'_\infty \cap B_\infty$, then by the above Corollary

$$[(B_\infty)_p : (A_\infty)_p] = tr^2(p) [B_\infty : A_\infty],$$

so in order to compute $[(B_\infty)_p : (A_\infty)_p]$, we have to find the value of $tr(p)$. In particular if p is a minimal projection we will get a value of the index for two subfactors with a trivial relative commutant. To find the value of $tr(p)$, first note that since $A'_\infty \cap B_\infty$ is a finite dimensional C*–algebra, for given $\epsilon > 0$, there exists an integer n_0, such that for all

$x \in A_\infty' \cap B_\infty$, with $x \geq 0$, $\|x\| \leq 1$, and for all $n \geq n_0$, $\|E_{B_n}(x) - x\|_2 < \epsilon$. This shows that the projections in $A_\infty' \cap B_\infty$ can be approximated by the projections in $A_n' \cap B_n$ for n large enough.

THEOREM 3. *Let* p *be a projection in* $A_\infty' \cap B_\infty$, *then*

$$[(B_\infty)_p : (A_\infty)_p] \geq \limsup_{k,\ell,n} \frac{\operatorname{tr}(f_{\ell,n})}{\operatorname{tr}(e_{k,n})},$$

where the limit is taken over k,ℓ,n, *for which* $e_{k,n} \cdot f_{\ell,n} \neq 0$.

Proof. As in [4], the inclusion of $A_n \subset B_n$ is specified by the matrix $T_{A_n}^{B_n}$ or by the Bratteli diagram of the inclusion, as defined in [1]. Now using the Bratteli diagram of the inclusion of $A_n \subset B_n$, we get

$$A_n' \cap B_n = \sum_{k,\ell} Q_{k,\ell.n}\, e_{k,n}\, f_{k,n},$$

where $Q_{k,\ell,n}$ is a type $I_{g_{k,\ell,n}}$ factor. But for each k, $A_\infty' \cap B_\infty$ is isomorphic to $(A_\infty' \cap B_\infty)e_{k,n}$, and the minimal central projections in $(A_n' \cap B_n)_{e_{k,n}}$ are of the form $e_{k,n}\, f_{\ell,n}$. Hence if $q_{k,\ell,n}$ is a minimal projection in $(A_n' \cap B_n)e_{k,n}\, f_{\ell,n}$, then

$$\operatorname{tr}(q_{k,\ell,n}) = a_{k,n}\, t_{\ell,n} = \frac{\operatorname{tr}(e_{k,n} f_{\ell,n})}{g_{k\ell n}},$$

and this implies that for any projection p in $A_\infty' \cap B_\infty$, we have

$$\mathrm{tr}(p) \geq \limsup_{k,\ell,n} \frac{\mathrm{tr}(e_{k,n}, f_{\ell,n})}{g_{k\ell n}\ \mathrm{tr}(e_{k,n})},$$

when the limit is taken over all k,ℓ,n for which $g_{k\ell n} \neq 0$, and now we are done.

Q.E.D.

Next we are going to show that actually $A'_\infty \cap B_\infty$ is isomorphic to a subalgebra of $(A'_n \cap B_n)_{e_{k,n}}$ for each k. This fact is also coherent with Theorem (1.7) [4], but in order to prove this we need to modify one of the Christensen's results.

Let L be a Type II_1 von Neumann algebra with a faithful normal trace tr and let M and N be sub von Neumann algebras of L. Then we write $M \overset{\delta}{\subset} N$ if, for all x in M with $\|x\| \leq 1$, $\|x - E_N(x)\|_2 \leq \delta$. Furthermore, suppose $e_1, \cdots, e_\ell$ are the minimal central projections of N and let e_N be the orthogonal projection onto the subspace $L^2(N,\mathrm{tr})$ of $L^2(L,\mathrm{tr})$.

Lemma. *Suppose* N *is finite dimensional, then if* δ *is small enough there exists a homomorphism* φ *of* M *into* N, *with* $\varphi(I) = I$, *and such that* $\|\varphi(x) - E_N(x)\|_2$ *is small enough.*

Proof. Note that $e_1, \cdots, e_\ell$ are also the minimal central projections of a semifinite von Neumann algebra $\langle L, e_N\rangle = \{L, e_N\}''$, thus there exists a unique faithful normal semifinite trace tr on $\langle L, e_N\rangle$, such that

$$\mathrm{tr}(e_k e_N) = \mathrm{tr}(e_k), \qquad k = 1,2,\cdots,e.$$

For all $x \in L$,

$$\mathrm{tr}(x e_N) = \mathrm{tr}(E_N(x) e_N) = \mathrm{tr}\left[\sum_k E_N(x) e_k e_N\right] = \mathrm{tr}(x).$$

Now using the same arguments as in (1.10) [2], we get that if $\delta < 2^{-\frac{1}{2}}$, then there exists a projection $g \in M' \cap \langle M, e_N \rangle$, such that

$$\|g - e_N\|_2^{tr} = \left[tr\left[(g-e_N)(g-e_N)\right]\right]^{\frac{1}{2}} \le 2^{\frac{1}{4}}\, \delta^{\frac{1}{2}} (1 - 2^{\frac{1}{4}}\, \delta^{\frac{1}{2}})^{-1},$$

and

$$|1 - tr(g)| < 2^{\frac{1}{2}}\, \delta (1 - 2^{\frac{1}{4}}\, \delta^{\frac{1}{2}})^{-2}.$$

Now suppose there are integers $i_1, \cdots, i_\ell$, such that Ne_k is a type I_{i_k} factor. Hence if f_k is a minimal projection in Ne_k, then $e_N f_k$ is a minimal projection in $\langle L, e_N \rangle$. Following the same arguments as in Corollary (4.3) [2], if δ is small enough then $tr(ge_k) = tr(e_N e_k)$ for all k. Now we can apply the arguments of Lemma (4.1) [2], to get the desired homomorphism φ.

Q.E.D.

THEOREM 4. *For* n *large enough* $A'_\infty \cap B_\infty$ *is isomorphic to a subalgebra of* $(A'_n \cap B_n)_{e_{k,n}}$ *for each* k.

Proof. By (1.8) [4], $\liminf tr(e_{k,n}) > 0$. Thus for n large enough,

$$(A'_\infty \cap B_\infty)_{e_{k,n}} \overset{\delta}{\subset} (A'_n \cap B_n)_{e_{k,n}}$$

with δ small enough. Now we can use the above Lemma and we are done.

Q.E.D.

Finally, we are going to prove a theorem that will help us to identify $A'_\infty \cap B_\infty$, also in order to facilitate the calculation, we assume that $A_n \subset B_n$ is a periodic sequence with

$h = 1$, and note that by the above for n large enough there exists an isomorphism φ_n of $A'_\infty \cap B_\infty$ into $A'_n \cap B_n$, such that if x is in $A'_\infty \cap B_\infty$, with $\|x\| \leq 1$, then $\|x + \varphi_n(x)\|_2$ is small enough.

Theorem 5. *For* n *large enough,* $T^{A'_n \cap B_{n+1}}_{\varphi_n(A'_\infty \cap B_\infty)} = T^{A'_n \cap B_{n+1}}_{\varphi_{n+1}(A'_\infty \cap B_\infty)}$.

Proof. The minimal central projections of $A'_n \cap B_{n+1}$ are of the form $e_{k,n} f_{\ell,n+1}$ and using the same arguments as before, there exists $d > 0$, such that for n large enough $tr(e_{k,n} f_{\ell,n+1}) > d$, if $e_{k,n} f_{\ell,n+1} \neq 0$, hence there exists $d_1 > 0$ such that for each minimal projection g_n in $A'_n \cap B_{n+1}$ $tr(g_n) > d_1$. Since d_1 is independent from n, we get that for n large enough, and for any minimal central projection q of $A'_\infty \cap B_\infty$, $tr(\varphi_n(q)) = tr(\varphi_{n+1}(q))$, so now we can apply Corollary (3.2) [3] and the proof is complete.

Q.E.D.

Now by the above theorem, we get

$$T^{A'_n \cap B_{n+1}}_{A'_n \cap B_n} T^{A'_n \cap B_n}_{\varphi_n(A'_\infty \cap B_\infty)} = T^{A'_n \cap B_{n+1}}_{A'_{n+1} \cap B_{n+1}} T^{A'_{n+1} \cap B_{n+1}}_{\varphi_{n+1}(A'_\infty \cap B_\infty)}.$$

Since all the above matrices have simple structure, and since the set

$$\left\{ T^{A'_n \cap B_n}_{\varphi_n(A'_\infty \cap B_\infty)} \right\}_{n=n_0}^{\infty}$$

is a finite set, we can possibly compute the structure of $A'_\infty \cap B_\infty$ using the above formula.

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