FRACTIONAL INTEGRAL OPERATORS INVOLVING A GENERAL CLASS OF POLYNOMIALS

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In the present paper the authors derive a number of interesting expressions for the composition of certain multidimensional fractional integral operators involving a general class of polynomials with essentially arbitrary coefficients. It is shown how these fractional integral operators can be identified with elements of the algebra of functions having the multidimensional Mellin convolution as the product. Inversion formulas for the multidimensional fractional integrals are also established. The fractional integral operators studied here are fairly general in character, since (by suitably specializing the coefficients involved) the general class of polynomials can be reduced to each of the classical orthogonal polynomials, the Bessel polynomials, and numerous other classes of generalized hypergeometric polynomials studied in the literature.

1. INTRODUCTION AND DEFINITIONS

In the literature there are numerous avenues of applications of operators of fractional calculus in a wide variety of fields (see, for example, Ross [10], Srivastava and Buschman [12, p. 28 et seq.], Srivastava and Manocha [13, Chapter 5], and McBride and Roach [8]). Much of the theory of fractional calculus is based upon the familiar integral operator ${}_cD_x^{\mu}$ defined by (cf., e.g., Lavoie et al. [7]; see also Ross [10])

$$(1.1)\qquad {}_cD_x^{\mu}\, f(x) = \frac{1}{\Gamma(\mu)} \int_c^x (x-t)^{\mu-1}\, f(t)dt, \qquad \mathrm{Re}(\mu) > 0.$$

When $c = 0$, Equation (1.1) defines the classical Riemann-Liouville fractional integral of order μ. Furthermore, when $c \to \infty$, Equation (1.1) may be identified with the definition of the familiar Weyl fractional integral of order μ (see Erdélyi et al. [5, Vol. II, Chapter 13] for details).

By repeated applications of the operator ${}_cD_x^{\mu}$ to a given function of several variables, one can naturally develop the corresponding multidimensional fractional integral operator in a rather straightforward manner. The object of this paper is to present a systematic account of some interesting new generalizations of these multidimensional Riemann-Liouville and Weyl fractional integral operators.

Throughout the present paper we assume that

$$f(x_1,\ldots,x_r) \in \mathcal{A}, \tag{1.2}$$

where $\mathcal{A}$ denotes the class of functions $f(x_1,\ldots,x_r)$ for which

$$\int \cdots \int_{\Omega_r} |f(x_1,\ldots,x_r)|\,dx_1 \ldots dx_r < \infty \tag{1.3}$$

for every bounded r-dimensional region Ω_r excluding the origin, and

$$f(x_1,\ldots,x_r) = \begin{cases} O(|x_1|^{\xi_1}\cdots|x_r|^{\xi_r}), & \max\{|x_1|,\ldots,|x_r|\} \to 0, \\ O(|x_1|^{-\eta_1}\cdots|x_r|^{-\eta_r}), & \min\{|x_1|,\ldots,|x_r|\} \to \infty. \end{cases} \tag{1.4}$$

Also let $S_n^m[x]$ denote the class of polynomials introduced by Srivastava [11, p. 1, Equation (1)]:

(1.5) $$S_n^m[x] = \sum_{j=0}^{[n/m]} \frac{(-n)_{mj}}{j!} A_{n,j}\, x^j \qquad (n = 0,1,2,\ldots),$$

where $(\lambda)_j = \Gamma(\lambda+j)/\Gamma(\lambda)$, m is an arbitrary positive integer, and the coefficients $A_{n,j}$ $(n,j \geqq 0)$ are arbitrary constants, real or complex (see also Srivastava and Singh [15, p. 158, Equation (1.1)]). Then the generalized multidimensional fractional integral operators

$$\mathcal{R}^{z;m,n;\rho}_{\mu_1,\ldots,\mu_r} \quad \text{and} \quad \mathcal{W}^{z;m,n;\rho}_{\mu_1,\ldots,\mu_r}$$

are defined by

(1.6) $$\mathcal{R}^{z;m,n;\rho}_{\mu_1,\ldots,\mu_r} f(x_1,\ldots,x_r) = \frac{1}{\Gamma(\mu_1)\ \ldots\ \Gamma(\mu_r)}$$

$$\cdot \int_0^{x_1} \cdots \int_0^{x_r} (x_1-t_1)^{\mu_1-1} \ldots (x_r-t_r)^{\mu_r-1} S_n^m\left[z\left[1-\frac{t_1}{x_1}\right]^{\rho_1} \cdots \left[1-\frac{t_r}{x_r}\right]^{\rho_r}\right]$$

$$\cdot f(t_1,\ldots,t_r)dt_1 \ldots dt_r,$$

$$\mathrm{Re}(\xi_i) > -1,\ \mathrm{Re}(\mu_i+\rho_i j) > 0 \quad (i = 1,\ldots,r;\ j = 0,1,2,\ldots,[n/m])$$

and

(1.7) $$\mathcal{W}^{z;m,n;\rho}_{\mu_1,\ldots,\mu_r} f(x_1,\ldots,x_r) = \frac{1}{\Gamma(\mu_1)\ \ldots\ \Gamma(\mu_r)}$$

$$\cdot \int_{x_1}^{\infty} \cdots \int_{x_r}^{\infty} (t_1-x_1)^{\mu_1-1} \ldots (t_r-x_r)^{\mu_r-1} S_n^m\left[z\left[1-\frac{x_1}{t_1}\right]^{\rho_1}\cdots\left[1-\frac{x_r}{t_r}\right]^{\rho_r}\right]$$

$$\cdot f(t_1,\ldots,t_r)dt_1 \ldots dt_r,$$

$$\mathrm{Re}(\eta_i) > \mathrm{Re}(\mu_i) > -\mathrm{Re}(\rho_i j) \quad (i = 1,\ldots,r;\ j = 0,1,2,\ldots,[n/m]),$$

respectively, z being a suitably bounded complex variable, and ρ denoting the r-parameter array $\rho_1, \ldots, \rho_r$.

In terms of the multidimensional fractional integral operators defined by Equations (1.6) and (1.7), we now introduce the operators given by

(1.8) $$\mathcal{J}^{z;m,n;\rho,h}_{\mu_1,\ldots,\mu_r} f(x_1,\ldots,x_r) = x_1^{-h_1-\mu_1} \ldots x_r^{-h_r-\mu_r}$$

$$\cdot \mathcal{R}^{z;m,n;\rho}_{\mu_1,\ldots,\mu_r} x_1^{h_1} \ldots x_r^{h_r} f(x_1,\ldots,x_r)$$

and

(1.9) $$\mathcal{K}^{z;m,n;\rho,h}_{\mu_1,\ldots,\mu_r} f(x_1,\ldots,x_r) = x_1^{h_1} \ldots x_r^{h_r}$$

$$\cdot \mathcal{W}^{z;m,n;\rho}_{\mu_1,\ldots,\mu_r} x_1^{-h_1-\mu_1} \ldots x_r^{-h_r-\mu_r} f(x_1,\ldots,x_r),$$

which evidently exist under the conditions derivable readily from those detailed above. Just as ρ does in Equations (1.6) and (1.7), h denotes the r-parameter array $h_1, \ldots, h_r$, with similar interpretations for σ, k, *et cetera*.

The fractional integral operators defined by Equations (1.8) and (1.9) may be looked upon as the generalizations of the repeated forms of the corresponding one-dimensional operators considered, among others, by Erdélyi [2] and Kober [6]. Moreover, for $r = 2$ and $z = \zeta = 0$, these operators would reduce immediately to the corresponding two-dimensional operators studied recently by Raina [9].

2. COMPOSITIONS OF THE MULTIDIMENSIONAL FRACTIONAL INTEGRAL OPERATORS

In view of the definition (1.8), we have

$$\mathcal{I}^{z;m,n;\rho,h}_{\lambda_1,\ldots,\lambda_r}\ \mathcal{I}^{\zeta;p,q;\sigma,k}_{\mu_1,\ldots,\mu_r}\ f(x_1,\ldots,x_r) = \frac{x_1^{-h_1-\lambda_1}\ \cdots\ x_r^{-h_r-\lambda_r}}{\Gamma(\lambda_1)\Gamma(\mu_1)\ \cdots\ \Gamma(\lambda_r)\Gamma(\mu_r)} \tag{2.1}$$

$$\cdot \int_0^{x_1}\cdots\int_0^{x_r}\prod_{i=1}^{r}\left\{s_i^{h_i-k_i-\mu_i}(x_i-s_i)^{\lambda_i-1}\right\} S_n^m\left[z\left(1-\frac{s_1}{x_1}\right)^{\rho_1}\cdots\left(1-\frac{s_r}{x_r}\right)^{\rho_r}\right]$$

$$\cdot\left[\int_0^{s_1}\cdots\int_0^{s_r}\prod_{i=1}^{r}\left\{t_i^{k_i}(x_i-t_i)^{\mu_i-1}\right\} S_q^p\left[\zeta\left(1-\frac{t_1}{s_1}\right)^{\sigma_1}\cdots\left(1-\frac{t_r}{s_r}\right)^{\sigma_r}\right]\right.$$

$$\left.\cdot\ f(t_1,\ldots,t_r)\,dt_1\ \cdots\ dt_r\right]ds_1\ \cdots\ ds_r,$$

where the integers p and q are constrained just as m and n in Definition (1.5), it being understood (for the sake of generality) that the coefficients corresponding to the polynomials $S_q^p[\varsigma]$ are $B_{q,j}$ instead of $A_{q,j}$.

In view of the hypothesis (1.2), the inversion of the order of integration is guaranteed by a known multidimensional extension of Fubini's theorem, provided further that

$$(2.2)\qquad \begin{cases} \mathrm{Re}(\lambda_i+\rho_i j) > 0,\ \mathrm{Re}(\mu_i+\sigma_i \ell) > 0,\ \mathrm{Re}(k_i+\xi_i) > -1 \\ (i = 1,\ldots,r;\ j = 0,1,2,\ldots,[n/m];\ \ell = 0,1,2,\ldots,[q/p]). \end{cases}$$

Under these conditions, we thus find from (2.1) that

$$(2.3)\qquad \mathcal{J}^{z;m,n;\rho,h}_{\lambda_1,\ldots,\lambda_r}\ \mathcal{J}^{\varsigma;p,q;\sigma,k}_{\mu_1,\ldots,\mu_r}\ f(x_1,\ldots,x_r) = \frac{x_1^{-h_1-\lambda_1}\cdots x_r^{-h_r-\lambda_r}}{\Gamma(\lambda_1)\Gamma(\mu_1)\cdots\Gamma(\lambda_r)\Gamma(\mu_r)}$$

$$\cdot \int_0^{x_1}\cdots\int_0^{x_r} t_1^{k_1}\cdots t_r^{k_r}\, f(t_1,\ldots,t_r)\,\Lambda(t_1,\ldots,t_r)\,dt_1\cdots dt_r,$$

where, for convenience,

$$(2.4)\qquad \Lambda(t_1,\ldots,t_r) = \int_{t_1}^{x_1}\cdots\int_{t_r}^{x_r} \prod_{i=1}^{r}\left\{s_i^{h_i-k_i-\mu_i}(x_i-s_i)^{\lambda_i-1}(s_i-t_i)^{\mu_i-1}\right\}$$

$$\cdot\, S_n^m\left[z\left[1-\frac{s_1}{x_1}\right]^{\rho_1}\cdots\left[1-\frac{s_r}{x_r}\right]^{\rho_r}\right] S_q^p\left[\varsigma\left[1-\frac{t_1}{s_1}\right]^{\sigma_1}\cdots\left[1-\frac{t_r}{s_r}\right]^{\sigma_r}\right] ds_1\cdots ds_r.$$

The multiple integral in (2.4) can be evaluated fairly easily by setting

$$u_i = \frac{x_i - s_i}{x_i - t_i} \qquad (i = 1,\ldots,r) \tag{2.5}$$

and appealing to the definition (1.5) and an elementary integral [5, Vol. I, p. 310, Equation (24)]. Upon substituting the resulting expression for $\Lambda(t_1,\ldots,t_r)$ in (2.3), we finally obtain

$$\mathcal{I}^{z;m,n;\rho,h}_{\lambda_1,\ldots,\lambda_r}\ \mathcal{I}^{\varsigma;p,q;\sigma,k}_{\mu_1,\ldots,\mu_r}\ f(x_1,\ldots,x_r) = \frac{x_1^{-k_1-\lambda_1-\mu_1}\cdots x_r^{-k_r-\lambda_r-\mu_r}}{\Gamma(\lambda_1+\mu_1)\cdots\Gamma(\lambda_r+\mu_r)} \tag{2.6}$$

$$\cdot \int_0^{x_1}\cdots\int_0^{x_r}\prod_{i=1}^{r}\left\{t_i^{k_i}(x_i-t_i)^{\lambda_i+\mu_i-1}\right\}\mathcal{F}^{m,n;\rho,h}_{p,q;\sigma,k}(t_1,\ldots,t_r)$$

$$\cdot\, f(t_1,\ldots,t_r)dt_1\cdots dt_r,$$

where

$$\mathcal{F}^{m,n;\rho,h}_{p,q;\sigma,k}(t_1,\ldots,t_r) = \sum_{j=0}^{[n/m]}\sum_{\ell=0}^{[q/p]}(-n)_{mj}\,(-q)_{p\ell}\,A_{n,j}\,B_{q,\ell}\,\frac{z^j}{j!}\,\frac{\varsigma^\ell}{\ell!} \tag{2.7}$$

$$\cdot\prod_{i=1}^{r}\left\{\frac{(\lambda_i)_{\rho_i j}\,(\mu_i)_{\sigma_i\ell}}{(\lambda_i+\mu_i)_{\rho_i j+\sigma_i\ell}}\left[1-\frac{t_i}{x_i}\right]^{\rho_i j+\sigma_i\ell}\right.$$

$$\cdot \; {}_2F_1\left[\begin{matrix}\lambda_i+\rho_i j,\ \mu_i+\sigma_i \ell-h_i+k_i; \\ \\ \lambda_i+\mu_i+\rho_i j+\sigma_i \ell;\end{matrix}\ 1-\frac{t_i}{x_i}\right]\Bigg\}$$

in terms of the Gaussian hypergeometric function ${}_2F_1$, where (for nonnegative integers p and q, $p \leqq q+1$)

$$(2.8) \qquad {}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p; \\ \\ \beta_1,\ldots,\beta_q;\end{matrix}\ z\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!}$$

$$(|z| < \infty \ \text{ if } \ p \leqq q;\ |z| < 1 \ \text{ if } \ p = q+1).$$

Upon introducing some obvious parametric interchanges, and making use of Euler's transformation [4, p. 64, Equation (23)], it is not difficult to verify the commutativity of the fractional integral operators involved in (2.6).

The expression for the composition of the form (2.6), but involving the fractional integral operator defined by (1.9), can easily be derived in a similar manner, and we have

$$(2.9) \qquad \mathcal{K}^{z;m,n;\rho,h}_{\lambda_1,\ldots,\lambda_r}\ \mathcal{K}^{\varsigma;p,q;\sigma,k}_{\mu_1,\ldots,\mu_r}\ f(x_1,\ldots,x_r) = \frac{x_1^{h_1} \cdots x_r^{h_r}}{\Gamma(\lambda_1+\mu_1) \cdots \Gamma(\lambda_r+\mu_r)}$$

$$\cdot \int_{x_1}^{\infty} \cdots \int_{x_r}^{\infty} \prod_{i=1}^{r} \left\{ t_i^{-h_i-\lambda_i-\mu_i} (t_i-x_i)^{\lambda_i+\mu_i-1} \right\} \mathcal{G}^{m,n;\rho,h}_{p,q;\sigma,k}(t_1,\ldots,t_r)$$

$$\cdot\ f(t_1,\ldots,t_r)\,dt_1 \cdots dt_r,$$

where

$$(2.10)\qquad \mathcal{G}^{m,n;\rho,h}_{p,q;\sigma,k}(t_1,\ldots,t_r) = \sum_{j=0}^{[n/m]} \sum_{\ell=0}^{[q/p]} (-n)_{mj}\,(-q)_{p\ell}\,A_{n,j}\,B_{q,\ell}\,\frac{z^j}{j!}\,\frac{\varsigma^\ell}{\ell!}$$

$$\cdot \prod_{i=1}^{r} \left\{ \frac{(\lambda_i)_{\rho_i j}\,(\mu_i)_{\sigma_i \ell}}{(\lambda_i+\mu_i)_{\rho_i j+\sigma_i \ell}} \left[1 - \frac{x_i}{t_i}\right]^{\rho_i j+\sigma_i \ell} \right.$$

$$\left. \cdot\ {}_2F_1\left[\begin{matrix}\mu_i+\sigma_i\ell,\ \lambda_i+\rho_i j+h_i-k_i; \\ \lambda_i+\mu_i+\rho_i j+\sigma_i\ell;\end{matrix}\ 1-\frac{x_i}{t_i}\right]\right\},$$

provided that the hypothesis (1.2) holds true, and [cf. Equation (2.2)]

$$(2.11)\qquad \begin{cases}\mathrm{Re}(\lambda_i+\rho_i j) > 0,\ \mathrm{Re}(\mu_i+\sigma_i\ell) > 0,\ \mathrm{Re}(h_i+\eta_i) > 0 \\ (i = 1,\ldots,r;\ j = 0,1,2,\ldots,[n/m];\ \ell = 0,1,2,\ldots,[q/p]).\end{cases}$$

Next we consider a composition of the fractional integral operators defined by (1.8) and (1.9). Indeed, for one such composition of a mixed type, it follows from the definitions (1.8) and (1.9) that

$$(2.12)\qquad \mathcal{K}^{\varsigma;p,q;\sigma,k}_{\mu_1,\ldots,\mu_r}\ \mathcal{J}^{z;m,n;\rho,h}_{\lambda_1,\ldots,\lambda_r}\ f(x_1,\ldots,x_r) = \frac{x_1^{k_1}\ \cdots\ x_r^{k_r}}{\Gamma(\lambda_1)\Gamma(\mu_1)\ \cdots\ \Gamma(\lambda_r)\Gamma(\mu_r)}$$

$$\cdot \int_{x_1}^{\infty} \cdots \int_{x_r}^{\infty} \prod_{i=1}^{r} \left\{ s_i^{-h_i-k_i-\lambda_i-\mu_i} (s_i-x_i)^{\mu_i-1} \right\} S_q^p\left[\varsigma\left[1-\frac{x_1}{s_1}\right]^{\sigma_1} \cdots \left[1-\frac{x_r}{s_r}\right]^{\sigma_r}\right]$$

$$\cdot \left[\int_0^{s_1} \cdots \int_0^{s_r} \prod_{i=1}^{r} \left\{ t_i^{h_i} (s_i - t_i)^{\lambda_i - 1} \right\} S_n^m \left[z \left[1 - \frac{t_1}{s_1} \right]^{\rho_1} \cdots \left[1 - \frac{t_r}{s_r} \right]^{\rho_r} \right] \right.$$

$$\left. \cdot f(t_1, \ldots, t_r) dt_1 \cdots dt_r \right] ds_1 \cdots ds_r .$$

Upon interchanging the order of integration, again by appealing to the aforementioned multidimensional extension of Fubini's theorem, the inner multiple integral becomes

$$\int_{\max(x_1,t_1)}^{\infty} \cdots \int_{\max(x_r,t_r)}^{\infty} \prod_{i=1}^{r} \left\{ s_i^{-h_i - k_i - \lambda_i - \mu_i} (s_i - t_i)^{\lambda_i - 1} (s_i - x_i)^{\mu_i - 1} \right\}$$

$$\cdot S_n^m \left[z \left[1 - \frac{t_1}{s_1} \right]^{\rho_1} \cdots \left[1 - \frac{t_r}{s_r} \right]^{\rho_r} \right] S_q^p \left[\varsigma \left[1 - \frac{x_1}{s_1} \right]^{\sigma_1} \cdots \left[1 - \frac{x_r}{s_r} \right]^{\sigma_r} \right] ds_1 \cdots ds_r ,$$

which can be evaluated by setting

(i) $s_i = \dfrac{x_i}{u_i}$ if $x_i > t_i$ $\qquad (i = 1, \ldots, r)$

and

(ii) $s_i = \dfrac{t_i}{u_i}$ if $x_i < t_i$ $\qquad (i = 1, \ldots, r)$.

The other composition of a mixed type can be handled similarly, and we finally obtain

(2.13) $$\mathcal{K}^{\zeta;p,q;\sigma,k}_{\mu_1,\ldots,\mu_r}\, \mathcal{I}^{z;m,n;\rho,h}_{\lambda_1,\ldots,\lambda_r}\, f(x_1,\ldots,x_r)$$

$$= \mathcal{I}^{z;m,n;\rho,h}_{\lambda_1,\ldots,\lambda_r}\, \mathcal{K}^{\zeta;p,q;\sigma,k}_{\mu_1,\ldots,\mu_r}\, f(x_1,\ldots,x_r) = \frac{x_1^{-h_1-\lambda_1-\mu_1} \cdots x_r^{-h_r-\lambda_r-\mu_r}}{\Gamma(\lambda_1) \cdots \Gamma(\lambda_r)}$$

$$\cdot \int_0^{x_1} \cdots \int_0^{x_r} \prod_{i=1}^{r} \left\{ t_i^{h_i} (x_i-t_i)^{\lambda_i+\mu_i-1} \right\} \mathcal{U}^{m,n;\rho,h}_{p,q;\sigma,k}(t_1,\ldots,t_r) f(t_1,\ldots,t_r)\, dt_1 \cdots dt_r$$

$$+ \frac{x_1^{k_1} \cdots x_r^{k_r}}{\Gamma(\mu_1) \cdots \Gamma(\mu_r)} \int_{x_1}^{\infty} \cdots \int_{x_r}^{\infty} \prod_{i=1}^{r} \left\{ t_i^{-k_i-\lambda_i-\mu_i} (t_i-x_i)^{\lambda_i+\mu_i-1} \right\}$$

$$\cdot\, \mathcal{V}^{m,n;\rho,h}_{p,q;\sigma,k}(t_1,\ldots,t_r) f(t_1,\ldots,t_r)\, dt_1 \cdots dt_r,$$

where

(2.14) $$\mathcal{U}^{m,n;\rho,h}_{p,q;\sigma,k}(t_1,\ldots,t_r) = \sum_{j=0}^{[n/m]} \sum_{\ell=0}^{[q/p]} (-n)_{mj}\, (-q)_{p\ell}\, A_{n,j}\, B_{q,\ell}\, \frac{z^j}{j!} \frac{\zeta^\ell}{\ell!}$$

$$\cdot \prod_{i=1}^{r} \left\{ \frac{(\mu_i)_{\sigma_i \ell}}{(h_i+k_i+1)_{\mu_i+\sigma_i \ell}} \left[1 - \frac{t_i}{x_i}\right]^{\rho_i j+\sigma_i \ell} \right.$$

$$\left. \cdot\, {}_2F_1\left[\begin{matrix} \mu_i+\sigma_i\ell,\ h_i+k_i+\lambda_i+\mu_i+\rho_i j+\sigma_i\ell; \\ h_i+k_i+\mu_i+\sigma_i\ell+1; \end{matrix}\ \frac{t_i}{x_i}\right] \right\}$$

and

$$(2.15)\qquad \mathscr{V}^{m,n;\rho,h}_{p,q;\sigma,k}(t_1,\ldots,t_r) = \sum_{j=0}^{[n/m]} \sum_{\ell=0}^{[q/p]} (-n)_{mj}\,(-q)_{p\ell}\,A_{n,j}\,B_{q,\ell}\,\frac{z^j}{j!}\,\frac{\zeta^\ell}{\ell!}$$

$$\cdot \prod_{i=1}^{r} \left\{ \frac{(\lambda_i)_{\rho_i j}}{(h_i+k_i+1)_{\lambda_i+\rho_i j}} \left[1-\frac{x_i}{t_i}\right]^{\rho_i j+\sigma_i \ell} \right.$$

$$\left. \cdot\, {}_2F_1\left[\begin{matrix}\lambda_i+\rho_i j,\ h_i+k_i+\lambda_i+\mu_i+\rho_i j+\sigma_i \ell; \\ h_i+k_i+\lambda_i+\rho_i j+1;\end{matrix}\ \frac{x_i}{t_i}\right]\right\},$$

provided that the hypothesis (1.2) holds true, and [cf. Equations (2.2) and (2.11)]

$$(2.16)\qquad \begin{cases} \mathrm{Re}(\lambda_i+\rho_i j) > 0,\ \mathrm{Re}(\mu_i+\sigma_i \ell) > 0,\ \mathrm{Re}(h_i+k_i) > -1,\ \mathrm{Re}(h_i+\xi_i) > -1, \\ \mathrm{Re}(k_i+\eta_i) > 0 \quad (i = 1,\ldots,r;\ j = 0,1,2,\ldots,[n/m];\ \ell = 0,1,2,\ldots,[q/p]). \end{cases}$$

3. MULTIDIMENSIONAL MELLIN TRANSFORMS AND CONVOLUTIONS

For a multivariable function $f(x_1,\ldots,x_r) \in \mathcal{A}$, the multidimensional Mellin transform is defined, as usual, by (cf. [14, Part I, p. 125, Equation (3.5)])

$$(3.1)\quad \mathcal{M}\{f(x_1,\ldots,x_r):\ s_1,\ldots,s_r\} = \int_0^\infty \cdots \int_0^\infty x_1^{s_1-1} \cdots x_r^{s_r-1}\, f(x_1,\ldots,x_r)\,dx_1 \ldots dx_r,$$

provided that

$$-\mathrm{Re}(\xi_i) < \mathrm{Re}(s_i) < \mathrm{Re}(\eta_i), \quad \forall i \in \{1,\ldots,r\}.$$

We also define the multidimensional Mellin convolution of two functions $f(x_1,\ldots,x_r)$ and $g(x_1,\ldots,x_r)$ by

$$(3.2) \qquad (f * g)(x_1,\ldots,x_r) = (g * f)(x_1,\ldots,x_r)$$

$$= \int_0^\infty \cdots \int_0^\infty u_1^{-1} \cdots u_r^{-1}\, f\left(\frac{x_1}{u_1},\ldots,\frac{x_r}{u_r}\right) g(u_1,\ldots,u_r)\,du_1 \cdots du_r,$$

provided that the multiple integral exists. Then the fractional integral operators defined by (1.8) and (1.9) can readily be expressed as multidimensional Mellin convolutions in the following forms:

$$(3.3) \qquad \mathcal{I}^{z;m,n;\rho,h}_{\mu_1,\ldots,\mu_r} f(x_1,\ldots,x_r) = \left[I_{\mu,\rho,h;z,m,n} * f\right](x_1,\ldots,x_r),$$

where

$$(3.4) \qquad I_{\mu,\rho,h;z,m,n}(x_1,\ldots,x_r) = \prod_{i=1}^{r} \left\{ \frac{x_i^{-h_i-\mu_i}(x_i-1)^{\mu_i-1}}{\Gamma(\mu_i)} H(x_i-1) \right\}$$

$$\cdot\, S_n^m\left[z\left(\frac{x_1-1}{x_1}\right)^{\rho_1} \cdots \left(\frac{x_r-1}{x_r}\right)^{\rho_r}\right];$$

$$\mathcal{K}^{z;m,n;\rho,h}_{\mu_1,\ldots,\mu_r} f(x_1,\ldots,x_r) = \left[K_{\mu,\rho,h;z,m,n} * f\right](x_1,\ldots,x_r), \tag{3.5}$$

where

$$K_{\mu,\rho,h;z,m,n}(x_1,\ldots,x_r) = \prod_{i=1}^{r} \left\{ \frac{x_i^{h_i}(1-x_i)^{\mu_i - 1}}{\Gamma(\mu_i)} H(1-x_i) \right\} \cdot S_n^m\left[z(1-x_1)^{\rho_1}\ldots(1-x_r)^{\rho_r}\right], \tag{3.6}$$

$H(x)$ being the Heaviside unit function.

By appealing to certain known results [5, Vol. I, p. 311, Equations (31) and (32)], it is not difficult to derive the multidimensional Mellin transforms of the functions defined by (3.4) and (3.6). We thus have

$$\mathcal{M}\{I_{\mu,\rho,h;z,m,n}(x_1,\ldots,x_r): s_1,\ldots,s_r\} = \theta(s_1,\ldots,s_r) = \sum_{j=0}^{[n/m]} (-n)_{mj} A_{n,j} \frac{z^j}{j!} \prod_{i=1}^{r} \left\{ \frac{(\mu_i)_{\rho_i j}}{(h_i - s_i + 1)_{\mu_i + \rho_i j}} \right\}, \tag{3.7}$$

$$\text{Re}(\mu_i + \rho_i j) > 0, \ \text{Re}(h_i - s_i) > -1 \quad (i = 1,\ldots,r;\ j = 0,1,2,\ldots,[n/m]);$$

$$\mathcal{M}\{K_{\mu,\rho,h;z,m,n}(x_1,\ldots,x_r): s_1,\ldots,s_r\} = \phi(s_1,\ldots,s_r) = \sum_{j=0}^{[n/m]} (-n)_{mj} A_{n,j} \frac{z^j}{j!} \prod_{i=1}^{r} \left\{ \frac{(\mu_i)_{\rho_i j}}{(h_i + s_i)_{\mu_i + \rho_i j}} \right\}, \tag{3.8}$$

$$\text{Re}(\mu_i + \rho_i j) > 0, \ \text{Re}(h_i + s_i) > 0 \quad (i = 1,\ldots,r;\ j = 0,1,2,\ldots,[n/m]).$$

These last results (3.7) and (3.8) in conjunction with (3.3) and (3.4) immediately yield

$$(3.9)\qquad \mathcal{M}\left\{\mathcal{I}^{z;m,n;\rho,h}_{\mu_1,\ldots,\mu_r} f(x_1,\ldots,x_r):\ s_1,\ldots,s_r\right\}$$

$$= \mathcal{M}\{f(x_1,\ldots,x_r):\ s_1,\ldots,s_r\}\theta(s_1,\ldots,s_r)$$

and

$$(3.10)\qquad \mathcal{M}\left\{\mathcal{K}^{z;m,n;\rho,h}_{\mu_1,\ldots,\mu_r} f(x_1,\ldots,x_r):\ s_1,\ldots,s_r\right\}$$

$$= \mathcal{M}\{f(x_1,\ldots,x_r):\ s_1,\ldots,s_r\}\phi(s_1,\ldots,s_r),$$

which are valid under the conditions corresponding appropriately to those stated with (3.1), (3.7), and (3.8).

In view of the relationships (3.3) and (3.5), we can conclude that the properties of commutativity and associativity hold true for the fractional integral operators involved, by merely identifying these operators with elements of the algebra of functions having the multidimensional Mellin convolution (3.2) as the product.

4. INVERSION FORMULAS

Making use of an inversion theorem for the multidimensional Mellin transform (3.1), given recently by Srivastava and Panda [14, Part I, p. 125, Lemma 2], we can readily deduce from (3.9) and (3.10) the following inversion formulas for the fractional integral operations defined by (1.8) and (1.9):

(4.1)
$$f(x_1,\ldots,x_r) = \frac{1}{(2\pi\omega)^r}\int_{c_1-\omega\infty}^{c_1+\omega\infty}\cdots\int_{c_r-\omega\infty}^{c_r+\omega\infty}\frac{x_1^{-s_1}\cdots x_r^{-s_r}}{\theta(s_1,\ldots,s_r)}$$

$$\cdot\, \mathcal{M}\left\{\mathcal{I}^{z;m,n;\rho,h}_{\mu_1,\ldots,\mu_r} f(x_1,\ldots,x_r):\ s_1,\ldots,s_r\right\}ds_1\ \ldots\ ds_r,$$

where $\omega = \sqrt{-1}$, and $\theta(s_1,\ldots,s_r)$ is given by (3.7);

(4.2)
$$f(x_1,\ldots,x_r) = \frac{1}{(2\pi\omega)^r}\int_{c_1-\omega\infty}^{c_1+\omega\infty}\cdots\int_{c_r-\omega\infty}^{c_r+\omega\infty}\frac{x_1^{-s_1}\cdots x_r^{-s_r}}{\phi(s_1,\ldots,s_r)}$$

$$\mathcal{M}\left\{\mathcal{K}^{z;m,n;\rho,h}_{\mu_1,\ldots,\mu_r} f(x_1,\ldots,x_r):\ s_1,\ldots,s_r\right\}ds_1\ \ldots\ ds_r,$$

where $\phi(s_1,\ldots,s_r)$ is given by (3.8).

The precise conditions under which the inversion formulas (4.1) and (4.2) are valid are easily obtainable from those required and stated above for the existence of the various fractional integral operators and of their multidimensional Mellin transforms involved (see also [14, Part I,p. 125, Lemma 2]).

5. APPLICATIONS INVOLVING CLASSICAL ORTHOGONAL POLYNOMIALS

By assigning suitable special values to the coefficients $A_{n,j}$ in (1.5), the polynomials $S_n^m[x]$ can be reduced to each of the various classical orthogonal polynomials such as the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, the Hermite polynomials $H_n(x)$, and the Laguerre polynomials $L_n^{(\alpha)}(x)$, and indeed also the Gegenbauer (or ultraspherical) polynomials $C_n^{\nu}(x)$, the Legendre polynomials $P_n(x)$, and the Tchebycheff polynomials $T_n(x)$ and $U_n(x)$ of the first and second kinds. Since (see, e.g., [13, p. 74])

$$(5.1) \qquad L_n^{(\alpha)}(x) = \lim_{\beta \to \infty} \left\{ P_n^{(\alpha,\beta)}\left[1 - \frac{2x}{\beta}\right] \right\}$$

and

$$(5.2) \qquad H_n(x) = (-1)^n\, 2^{n/2}\, n! \lim_{\alpha \to \infty} \left\{ \alpha^{-n/2}\, L_n^{(\alpha)}(\alpha + x\sqrt{2\alpha}) \right\},$$

and since each of the other classical orthogonal polynomials listed above are contained in $P_n^{(\alpha,\beta)}(x)$ for special values of α and β, it would suffice to indicate how our results can be applied to deduce the corresponding properties of fractional integral operators involving the classical Jacobi polynomials:

$$(5.3) \qquad P_n^{(\alpha,\beta)}(x) = \sum_{j=0}^{n} \binom{n+\alpha}{n-j} \binom{n+\beta}{j} \left[\frac{x-1}{2}\right]^j \left[\frac{x+1}{2}\right]^{n-j}.$$

Upon comparing the definitions (1.5) and (5.3), it is easily observed that each of our results in the preceding sections can be reduced in terms of the classical Jacobi polynomials by merely setting

$$m = 1 \quad \text{and} \quad A_{n,j} = \binom{n+\alpha}{n} \frac{(\alpha+\beta+n+1)_j}{(\alpha+1)_j},$$

and then replacing $S_n^1[x]$ by $P_n^{(\alpha,\beta)}(1-2x)$. The details may be omitted.

6. FURTHER APPLICATIONS AND SPECIAL CASES

Some further interesting applications of our results include those involving such generalized hypergeometric polynomials as the Bessel polynomials

(6.1) $$y_n(x,\alpha,\beta) = \sum_{j=0}^{n} \binom{n}{j} \binom{n+\alpha+j-2}{j} j! \left(\frac{x}{\beta}\right)^j$$

in which case

(6.2) $$m = 1,\ A_{n,j} = (\alpha+n-1)_j,\ S_n^1[x] \to y_n(-\beta x,\alpha,\beta),$$

the Gould-Hopper polynomials

(6.3) $$g_n^m(x,h) = \sum_{j=0}^{[n/m]} \frac{n!}{j!(n-mj)!} h^j x^{n-mj}$$

for which

(6.4) $$A_{n,j} = 1,\ S_n^m[x] \to (-1)^n \left(\frac{x}{h}\right)^{n/m} g_n^m\left[-\left(\frac{h}{x}\right)^{1/m}, h\right],$$

and the (substantially more general) Brafman polynomials [cf. Equation (2.8)]

(6.5) $$B_n^m[\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q:x]$$

$$= {}_{m+p}F_q\left[\begin{matrix}\Delta(m;-n),\ \alpha_1,\ldots,\alpha_p; \\ \beta_1,\ldots,\beta_q;\end{matrix}\ x\right]$$

for which

$$(6.6)\qquad A_{n,j} = \frac{(\alpha_1)_j \cdots (\alpha_p)_j}{(\beta_1)_j \cdots (\beta_q)_j}, \quad S_n^m[x] \to B_n^m[\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q:xm^m],$$

where $\Delta(m;\lambda)$ abbreviates the m-parameter array

$$\frac{\lambda}{m}, \frac{\lambda+1}{m}, \ldots, \frac{\lambda+m-1}{m},$$

m being a positive integer (cf., e.g., [13, pp. 75-76]; see also [15, pp. 160-161]).

Indeed, each of our results involving the polynomials $S_n^m[x]$ will apply also to such other hypergeometric polynomials as the extended Jacobi polynomials and their various generalizations studied in the literature (see, for details, [15, pp. 161-162]).

In their special cases when $r = 2$ and $z = \zeta = 0$, many of our assertions yield the corresponding results given recently by Raina [9]. As a matter of fact, the special cases of our composition formulas (2.6), (2.9), and (2.13) when $z = \zeta = 0$ would provide multidimensional extensions of several interesting results due to Buschman [1, p. 100, Equation (2.4)] and Erdélyi [3, pp. 166-167, Equations (6.1), (6.2), and (6.3)].

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