# *MAJORIZATION BY STARLIKE FUNCTIONS OF COMPLEX ORDER*

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**Abstract**

The authors investigate several majorization problems involving starlike and convex functions of complex order as well as functions belonging to a certain class $\mathcal{R}(\lambda, \gamma)$ which they introduce here. Relevant connections of the main results obtained in this paper with those given by earlier workers on the subject are also pointed out.

## 1. Introduction and Definitions

Let the functions $f(z)$ and $g(z)$ be analytic in the *open* unit disk

$$\mathcal{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Following MacGregor [5], we say that $f(z)$ is majorized by $g(z)$ in $\mathcal{U}$ and write

$$f(z) \ll g(z) \quad (z \in \mathcal{U}) \tag{1.1}$$

if there exists a function $\phi(z)$, analytic in $\mathcal{U}$, such that

$$|\phi(z)| \leqq 1 \quad \text{and} \quad f(z) = \phi(z)g(z) \quad (z \in \mathcal{U}). \tag{1.2}$$

The majorization (1.1) is closely related to the concept of quasi-subordination between analytic functions, which was considered recently by (for example) Altintaş and Owa [1].

The *main* object of this paper is to investigate the problem of majorization of certain analytic functions by starlike and convex functions of *complex* order.

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A function $f(z)$, analytic in $\mathcal{U}$, is said to be *starlike of complex order* $\gamma$ $(\gamma \in \mathbb{C} \setminus \{0\})$ in $\mathcal{U}$, that is, $f \in \mathcal{S}(\gamma)$, if and only if

$$\frac{f(z)}{z} \neq 0 \quad \text{and} \quad \operatorname{Re}\left\{1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > 0 \tag{1.3}$$

$$(z \in \mathcal{U}; \quad \gamma \in \mathbb{C} \setminus \{0\}).$$

Furthermore, a function $f(z)$, analytic in $\mathcal{U}$, is said to be *convex of complex order* $\gamma$ $(\gamma \in \mathbb{C} \setminus \{0\})$ in $\mathcal{U}$, that is, $f \in \mathcal{C}(\gamma)$, if and only if

$$f'(z) \neq 0 \quad \text{and} \quad \operatorname{Re}\left\{1 + \frac{1}{\gamma}\frac{zf''(z)}{f'(z)}\right\} > 0 \tag{1.4}$$

$$(z \in \mathcal{U}; \quad \gamma \in \mathbb{C} \setminus \{0\}).$$

Clearly, we have the following relationships:

$$\mathcal{S}(1-\alpha) = \mathcal{S}^*(\alpha) \quad \text{and} \quad \mathcal{C}(1-\alpha) = \mathcal{K}(\alpha) \quad (0 \leqq \alpha < 1), \tag{1.5}$$

where $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ denote, respectively, the familiar classes of (*normalized*) starlike and convex functions of order $\alpha$ in $\mathcal{U}$, which were introduced by Robertson [9] (see also Srivastava and Owa [12]).

The classes $\mathcal{S}(\gamma)$ and $\mathcal{C}(\gamma)$ were considered by Nasr and Aouf [7] and Wiatrowski [13], respectively. We also note that

$$\mathcal{S}\left((1-\beta)e^{-i\alpha}\cos\alpha\right) = \mathcal{L}_{\alpha,\beta} \quad \left(-\frac{\pi}{2} < \alpha < \frac{\pi}{2};\ 0 \leqq \beta < 1\right) \tag{1.6}$$

and

$$\mathcal{S}\left(e^{-i\alpha}\cos\alpha\right) = \mathcal{L}_{\alpha,0} = \mathcal{S}_\alpha \quad \left(-\frac{\pi}{2} < \alpha < \frac{\pi}{2}\right), \tag{1.7}$$

where $\mathcal{L}_{\alpha,\beta}$ denotes the class of $\alpha$-spiral functions of order $\beta$, investigated by Libera [4], and $\mathcal{S}_\alpha$ denotes the class of spiral-like functions introduced by Špaček in 1933 (*cf.* [10] and [11]; see also Duren [2, p. 52 *et seq.*]).

Finally, let $\mathcal{R}(\lambda, \gamma)$ denote the class of functions $h(z)$ of the form:

$$h(z) = 1 - \sum_{n=1}^{\infty} c_n z^n \quad (c_n \geqq 0), \tag{1.8}$$

which are analytic in $\mathcal{U}$ and satisfy the inequality:

$$\left|h(z) + \lambda z h'(z) - 1\right| < |\gamma| \tag{1.9}$$

$$(z \in \mathcal{U}; \quad \operatorname{Re}(\lambda) \geqq 0; \quad \gamma \in \mathbb{C} \setminus \{0\}).$$

The class $\mathcal{R}(\lambda, \beta)$ $(\operatorname{Re}(\lambda) \geqq 0;\ 0 \leqq \beta < 1)$ was considered recently by Altintaş and Owa [1].

## 2. Majorization Problems for the Classes $\mathcal{S}(\gamma)$ and $\mathcal{C}(\gamma)$

We begin by proving

**Theorem 1.** *Let the function $f(z)$ be analytic in $\mathcal{U}$ and suppose that $g \in \mathcal{S}(\gamma)$. If $f(z)$ is majorized by $g(z)$ in $\mathcal{U}$, then*

$$|f'(z)| \leqq |g'(z)| \quad (|z| \leqq r_1), \tag{2.1}$$

*where*

$$r_1 = r_1(\gamma) := \frac{3 + |2\gamma - 1| - \sqrt{9 + 2\,|2\gamma - 1| + |2\gamma - 1|^2}}{2\,|2\gamma - 1|}. \tag{2.2}$$

*Proof.* Since $g \in \mathcal{S}(\gamma)$, we readily find from (1.3) that, if

$$p(z) := 1 + \frac{1}{\gamma}\left(\frac{zg'(z)}{g(z)} - 1\right) \quad (\gamma \in \mathbb{C} \setminus \{0\}), \tag{2.3}$$

then

$$\operatorname{Re}\{p(z)\} > 0 \quad (z \in \mathcal{U}) \tag{2.4}$$

and

$$p(z) = \frac{1 + w(z)}{1 - w(z)} \quad (w \in \Omega), \tag{2.5}$$

where $\Omega$ denotes the well-known class of bounded analytic functions $w(z)$ in $\mathcal{U}$, which satisfy the conditions (*cf.*, *e.g.*, Goodman [3, p. 58]):

$$w(0) = 0 \quad \text{and} \quad |w(z)| \leqq |z| \quad (z \in \mathcal{U}). \tag{2.6}$$

From (2.3) and (2.5), we obtain

$$\frac{zg'(z)}{g(z)} = \frac{1 + (2\gamma - 1)w(z)}{1 - w(z)}, \tag{2.7}$$

which, in view of (2.6), immediately yields the inequality:

$$|g(z)| \leqq \frac{(1 + |z|)\,|z|}{1 - |2\gamma - 1| \cdot |z|}\,|g'(z)| \quad (z \in \mathcal{U}). \tag{2.8}$$

Next, since $f(z)$ is majorized by $g(z)$ in $\mathcal{U}$, from (1.2) we have

$$f'(z) = \phi'(z)g(z) + \phi(z)g'(z) \quad (z \in \mathcal{U}). \tag{2.9}$$

Thus, noting that $\phi \in \Omega$ satisfies the inequality (*cf.* Nehari [8, p. 168]):

$$|\phi'(z)| \leqq \frac{1 - |\phi(z)|^2}{1 - |z|^2} \quad (z \in \mathcal{U}), \tag{2.10}$$

and making use of (2.8) and (2.10) in (2.9), we get

$$|f'(z)| \leqq \frac{(1 - |z|)(1 - |2\gamma - 1| \cdot |z|)|\phi(z)| + \left(1 - |\phi(z)|^2\right)|z|}{(1 - |z|)(1 - |2\gamma - 1| \cdot |z|)}\,|g'(z)| \tag{2.11}$$

$$(z \in \mathcal{U}),$$

which, upon setting

$$|z| = r \quad \text{and} \quad |\phi(z)| = \rho \quad (0 \leqq \rho \leqq 1), \tag{2.12}$$

yields

$$|f'(z)| \leqq \frac{\Theta(\rho)}{(1-r)(1-|2\gamma-1|\,r)}\,|g'(z)| \quad (z \in \mathcal{U}), \tag{2.13}$$

where the function $\Theta(\rho)$ defined by

$$\Theta(\rho) := -r\rho^2 + (1-r)(1-|2\gamma-1|\,r)\rho + r \quad (0 \leqq \rho \leqq 1) \tag{2.14}$$

takes on its maximum value at $\rho = 1$ with $r = r_1(\gamma)$ given by (2.2). Furthermore, if $0 \leqq \sigma \leqq r_1(\gamma)$, where $r_1(\gamma)$ is given by (2.2), then the function $\Lambda(\rho)$ defined by

$$\Lambda(\rho) := -\sigma\rho^2 + (1-\sigma)\,(1-|2\gamma-1|\,\sigma)\,\rho + \sigma \tag{2.15}$$

is seen to be an increasing function on the interval $0 \leqq \rho \leqq 1$, so that

$$\Lambda(\rho) \leqq \Lambda(1) = (1-\sigma)\,(1-|2\gamma-1|\,\sigma) \tag{2.16}$$

$$(0 \leqq \rho \leqq 1; \quad 0 \leqq \sigma \leqq r_1(\gamma)).$$

Hence, upon setting $\rho = 1$ in (2.13), we conclude that the inequality in (2.1) holds true for $|z| \leqq r_1(\gamma)$, where $r_1(\gamma)$ is given by (2.2).

In view of the first relationship in (1.5), if we set $b = 1$ in Theorem 1, we immediately obtain

**Corollary 1** (*cf.* MacGregor [5, p. 96, Theorem 1B]). *Let the function $f(z)$ be analytic in $\mathcal{U}$ and suppose that*

$$g \in \mathcal{S}^* := \mathcal{S}^*(0) \subset \mathcal{S}, \tag{2.17}$$

*where $\mathcal{S}$ denotes the class of (normalized) analytic and univalent functions in $\mathcal{U}$. If $f(z)$ is majorized by $g(z)$ in $\mathcal{U}$, then*

$$|f'(z)| \leqq |g'(z)| \quad \left(|z| \leqq 2 - \sqrt{3}\right). \tag{2.18}$$

Since (*cf.*, *e.g.*, MacGregor [6, p. 71])

$$f \in \mathcal{K} := \mathcal{K}(0) \subset \mathcal{S} \Rightarrow f \in \mathcal{S}^*\left(\frac{1}{2}\right), \tag{2.19}$$

by letting $\gamma \to \frac{1}{2}$ in Theorem 1, we have

**Corollary 2** (*cf.* MacGregor [5, p. 96, Theorem 1C]). *Let the function $f(z)$ be analytic in $\mathcal{U}$ and suppose that $g \in \mathcal{K}$. If $f(z)$ is majorized by $g(z)$ in $\mathcal{U}$, then*

$$|f'(z)| \leqq |g'(z)| \quad \left(|z| \leqq \frac{1}{3}\right). \tag{2.20}$$

The proof of our next result (Theorem 2 below) is based essentially upon the following lemma.

**Lemma 1.** *If $f \in \mathcal{C}(\gamma)$ $(\gamma \in \mathbb{C} \setminus \{0\})$, then $f \in \mathcal{S}\left(\frac{1}{2}\gamma\right)$, that is,*

$$\mathcal{C}(\gamma) \subset \mathcal{S}\left(\frac{1}{2}\gamma\right) \quad (\gamma \in \mathbb{C} \setminus \{0\}). \tag{2.21}$$

*Proof.* We first rewrite the known assertion (2.19) in the form:

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0 \Rightarrow \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\frac{1}{2} \quad (z\in\mathcal{U}), \tag{2.22}$$

which readily yields

$$1+\frac{zf''(z)}{f'(z)}=\frac{1-w(z)}{1+w(z)} \Rightarrow \frac{zf'(z)}{f(z)}=\frac{1}{1+w(z)} \quad (w\in\Omega). \tag{2.23}$$

Now, making use of (2.23), it is easily seen that

$$\begin{aligned} 1+\frac{1}{\gamma}\frac{zf''(z)}{f(z)} &= \frac{\gamma+(\gamma-2)w(z)}{\gamma\{1+w(z)\}} \\ \Rightarrow 1+\frac{2}{\gamma}\left(\frac{zf'(z)}{f(z)}-1\right) &= \frac{\gamma+(\gamma-2)w(z)}{\gamma\{1+w(z)\}} \quad (w\in\Omega), \end{aligned} \tag{2.24}$$

and the assertion (2.21) of Lemma 1 follows immediately from (2.24).

Upon replacing $\gamma$ in Theorem 1 by $\frac{1}{2}\gamma$, if we apply Lemma 1, we have

**Theorem 2.** *Let the function $f(z)$ be analytic in $\mathcal{U}$ and suppose that $g\in\mathcal{C}(\gamma)$. If $f(z)$ is majorized by $g(z)$ in $\mathcal{U}$, then*

$$|f'(z)| \leqq |g'(z)| \quad (|z|\leqq r_2), \tag{2.25}$$

*where*

$$r_2=r_2(\gamma):=\frac{3+|\gamma-1|-\sqrt{9+2|\gamma-1|+|\gamma-1|^2}}{2|\gamma-1|}. \tag{2.26}$$

## 3. A Majorization Problem for the Class $\mathcal{R}(\lambda,\gamma)$

We begin by proving Lemma 2 and Lemma 3 below, which will be required in our investigation of the majorization problem for the class $\mathcal{R}(\lambda,\gamma)$.

**Lemma 2.** *If the function $h(z)$ defined by (1.8) is in the class $\mathcal{R}(\lambda,\gamma)$, then*

$$\sum_{n=1}^{\infty} c_n \leqq \frac{|\gamma|}{1+\operatorname{Re}(\lambda)}. \tag{3.1}$$

**Proof.** Assume that $h(z)$ is defined by (1.8) and is in the class $\mathcal{R}(\lambda,\gamma)$, so that the condition (1.9) readily yields

$$\left|-\sum_{n=1}^{\infty}(1+\lambda n)\,c_n z^n\right| < |\gamma| \quad (z\in\mathcal{U}). \tag{3.2}$$

Since $|\operatorname{Re}(z)|\leqq|z|$ for any $z\in\mathbb{C}$, we find from (3.2) that

$$\operatorname{Re}\left\{\sum_{n=1}^{\infty}(1+\lambda n)\,c_n z^n\right\} < |\gamma| \quad (z\in\mathcal{U}). \tag{3.3}$$

Now choose values of $z$ on the real axis and let $z \to 1-$ through real values. Then the inequality (3.3) yields

$$\sum_{n=1}^{\infty} \{1 + n\,\mathrm{Re}(\lambda)\}\, c_n \leqq |\gamma|\,, \tag{3.4}$$

which implies the assertion (3.1) of Lemma 1, since (by definition) $\mathrm{Re}(\lambda) \geqq 0$ and $n \in \mathbb{N}$.

**Lemma 3.** *If the function $h(z)$ defined by (1.8) is in the class $\mathcal{R}(\lambda, \gamma)$, then*

$$1 - \frac{|\gamma|}{1+\mathrm{Re}(\lambda)}\,|z| \leqq |h(z)| \leqq 1 + \frac{|\gamma|}{1+\mathrm{Re}(\lambda)}\,|z| \quad (z \in \mathcal{U})\,. \tag{3.5}$$

**Proof.** Since $h(z)$ is defined by (1.8), we have

$$1 - |z| \sum_{n=1}^{\infty} c_n \leqq |h(z)| \leqq 1 + |z| \sum_{n=1}^{\infty} c_n \quad (z \in \mathcal{U})\,, \tag{3.6}$$

which, in view of Lemma 2, immediately yields the assertion (3.5) of Lemma 3.

Finally, we prove

**Theorem 3.** *Let the functions $f(z)$ and $g(z)$ be analytic in $\mathcal{U}$ and suppose that the function $g(z)$ is so normalized that it also satisfies the following inclusion property:*

$$\frac{zg'(z)}{g(z)} \in \mathcal{R}(\lambda, \gamma). \tag{3.7}$$

*If $f(z)$ is majorized by $g(z)$ in $\mathcal{U}$, then*

$$\left|f'(z)\right| \leqq \left|g'(z)\right| \quad (|z| \leqq r_3)\,, \tag{3.8}$$

*where $r_3 = r_3(\lambda, \gamma)$ is the root of the cubic equation:*

$$|\gamma|\, r^3 - \left\{1 + \mathrm{Re}(\lambda) r^2\right\} - \{2 + |\gamma| + 2\,\mathrm{Re}(\lambda)\}\, r + 1 + \mathrm{Re}(\lambda) = 0, \tag{3.9}$$

*which lies in open interval $(0, 1)$.*

**Proof.** For an appropriately normalized analytic function $g(z)$ satisfying the inclusion property (3.7), we find from the assertion (3.5) of Lemma 3 that

$$\left|\frac{zg'(z)}{g(z)}\right| \geqq 1 - \frac{|\gamma|}{1+\mathrm{Re}(\lambda)} r \quad (|z| = r;\ 0 < r < 1) \tag{3.10}$$

or, equivalently, that

$$\left|g'(z)\right| \leqq \frac{\{1 + \mathrm{Re}(\lambda)\}\, r}{1 + \mathrm{Re}(\lambda) - |\gamma|\, r} \cdot \left|g'(z)\right| \quad (|z| = r;\ 0 < r < 1)\,. \tag{3.11}$$

Since $f(z) \ll g(z)$ in $\mathcal{U}$, there exists an analytic function $w \in \Omega$ such that

$$f(z) = w(z) g(z) \quad \text{and} \quad |w(z)| < 1 \quad (z \in \mathcal{U})\,. \tag{3.12}$$

Thus, in view of (3.11) and just as in the proof of Theorem 1, we have [*cf.* Equation (2.10)]

$$|w(z)| \leqq \frac{1 - |w(z)|^2}{1 - |z|^2} \quad (z \in \mathcal{U}) \tag{3.13}$$

and

$$|f'(z)| \leqq \left(|w(z)| + \frac{1-|w(z)|^2}{1-r^2} \cdot \frac{\{1+\mathrm{Re}(\lambda)\}\, r}{1+\mathrm{Re}(\lambda) - |\gamma|\, r}\right) |g'(z)|$$

$$= \frac{\Psi(\rho)}{(1-r^2)\{1+\mathrm{Re}(\lambda) - |\gamma|\, r\}} |g'(z)| \quad (z \in \mathcal{U};\ 0 < r < 1), \tag{3.14}$$

where we have set $|w(z)| = \rho$ and the function $\Psi(\rho)$ defined by

$$\Psi(\rho) := \{1+\mathrm{Re}(\lambda)\}\, r + \left(1-r^2\right)\{1+\mathrm{Re}(\lambda) - |\gamma|\, r\}\, \rho$$
$$- \{1+\mathrm{Re}(\lambda)\}\, r\rho^2 \qquad (0 \leqq \rho \leqq 1) \tag{3.15}$$

takes on its maximum value at $\rho = 1$ with $r = r_3(\lambda,\gamma)$ given by (3.9). Moreover, if

$$0 \leqq \sigma \leqq r_3(\lambda,\gamma),$$

where $r_3(\lambda,\gamma)$ is the root of the cubic equation (3.9) such that

$$0 < r_3(\lambda,\gamma) < 1, \tag{3.16}$$

then the function $\Xi(\rho)$ defined by

$$\Xi(\rho) := \{1+\mathrm{Re}(\lambda)\}\, \sigma + \left(1-\sigma^2\right)\{1+\mathrm{Re}(\lambda) - |\gamma|\, \sigma\}\, \rho$$
$$- \{1+\mathrm{Re}(\lambda)\}\, \sigma\rho^2 \tag{3.17}$$

is seen to be an increasing function on the interval $0 \leqq \rho \leqq 1$, so that

$$\Xi(\rho) \leqq \Xi(1) = \left(1-\sigma^2\right)\{1+\mathrm{Re}(\lambda) - |\gamma|\, \sigma\} \tag{3.18}$$

$$(0 \leqq \rho \leqq 1; \quad 0 \leqq \sigma \leqq r_3(\lambda,\gamma)).$$

Consequently, upon setting $\rho = 1$ in (3.14), we complete the proof of Theorem 3.

For $|\gamma| = 1 - \beta$ $(0 \leqq \beta < 1)$, Theorem 3 would immediately yield a result which is essentially equivalent to the one given earlier by Altintaş and Owa [1, p. 182, Theorem 1].

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