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Article

# Global Optimization and Common Best Proximity Points for Some Multivalued Contractive Pairs of Mappings

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**Abstract:** In this paper, we study a problem of global optimization using common best proximity point of a pair of multivalued mappings. First, we introduce a multivalued Banach-type contractive pair of mappings and establish criteria for the existence of their common best proximity point. Next, we put forward the concept of multivalued Kannan-type contractive pair and also the concept of weak  $\Delta$ -property to determine the existence of common best proximity point for such a pair of maps.

**Keywords:** common best proximity point; fixed point; contraction map; complete metric space; multivalued map; optimization

**JEL Classification:** 47H10; 54H25; 54E50

## 1. Preliminaries

Let  $(\mathfrak{X}, \rho)$  be a complete metric space and let  $CB(\mathfrak{X})$  denote the class of all nonempty closed and bounded subsets of the nonempty set  $\mathfrak{X}$ . For  $\mathcal{A}, \mathcal{B} \in CB(\mathfrak{X})$ , the function  $\mathcal{H} : CB(\mathfrak{X}) \times CB(\mathfrak{X}) \rightarrow [0, +\infty)$  defined by

$$\mathcal{H}(\mathcal{A}, \mathcal{B}) = \max\left\{\sup_{\xi \in \mathcal{B}} \Delta(\xi, \mathcal{A}), \sup_{\delta \in \mathcal{A}} \Delta(\delta, \mathcal{B})\right\},$$

where  $\Delta(\delta, \mathcal{B}) = \inf_{\xi \in \mathcal{B}} \rho(\delta, \xi)$ , is a metric on  $CB(\mathfrak{X})$ .

For any two non-empty subsets  $\mathcal{A}, \mathcal{B}$  of the metric space  $(\mathfrak{X}, \rho)$ , we shall use the following notations:

$$\mathcal{A}_{\mathcal{B}} = \{\theta \in \mathcal{A} : \rho(\theta, \xi) = \rho(\mathcal{A}, \mathcal{B}) \text{ for some } \xi \in \mathcal{B}\},$$

$$\mathcal{B}_{\mathcal{A}} = \{\xi \in \mathcal{B} : \rho(\theta, \xi) = \rho(\mathcal{A}, \mathcal{B}) \text{ for some } \theta \in \mathcal{A}\},$$

where  $\rho(\mathcal{A}, \mathcal{B}) = \inf\{\rho(\theta, \xi) : \theta \in \mathcal{A}, \xi \in \mathcal{B}\}$ .

For  $\mathcal{A}, \mathcal{B} \in CB(\mathfrak{X})$ , we have

$$\rho(\mathcal{A}, \mathcal{B}) \leq \mathcal{H}(\mathcal{A}, \mathcal{B}).$$

$\theta \in \mathfrak{X}$  is said to be a best proximity point (BPP, in short) of the multivalued map  $\Gamma : \mathfrak{X} \rightarrow CB(\mathfrak{X})$  if  $\Delta(\theta, \Gamma\theta) = \rho(\mathcal{A}, \mathcal{B})$ .  $v \in \mathfrak{X}$  is called a fixed point of the multivalued map  $\Gamma : \mathfrak{X} \rightarrow CB(\mathfrak{X})$  if  $v \in \Gamma v$ .

Let  $\Psi, \Omega : \mathcal{A} \rightarrow CB(\mathcal{B})$  be two multivalued maps. An element  $\theta^* \in \mathcal{A}$  is said to be a common best proximity point (CBPP, in short) of  $\Psi$  and  $\Omega$  if and only if

$$\Delta(\theta^*, \Psi\theta^*) = \rho(\mathcal{A}, \mathcal{B}) = \Delta(\theta^*, \Omega\theta^*).$$

**Remark 1.**

1. In the metric space  $(CB(\mathfrak{S}), \mathcal{H})$ ,  $\theta \in \mathfrak{S}$  is a fixed point of  $\Gamma$  if and only if  $\Delta(\theta, \Gamma\theta) = 0$ . In general,  $\theta \in \Gamma\xi$  if and only if  $\Delta(\theta, \Gamma\xi) = 0$  for any  $\theta, \xi \in \mathfrak{S}$ .
2. For two closed sets  $\mathcal{A}, \mathcal{B}$ , when  $\mathcal{A} \cap \mathcal{B} \neq \emptyset$ , we have  $\rho(\mathcal{A}, \mathcal{B}) = 0$ . In that case, a fixed point and a BPP are identical.
3. The function  $\Delta$  is continuous in the sense that if  $\theta_n \rightarrow \theta$  as  $n \rightarrow +\infty$ , then  $\Delta(\theta_n, \mathcal{A}) \rightarrow \Delta(\theta, \mathcal{A})$  as  $n \rightarrow +\infty$  for any  $\mathcal{A} \subseteq \mathfrak{S}$ .
4. A CBPP is an element at which the functions  $\theta \rightarrow \Delta(\theta, \Psi\theta)$  and  $\theta \rightarrow \Delta(\theta, \Omega\theta)$  achieve a global minimum, for  $\Delta(\theta, \Psi\theta) \geq \rho(\mathcal{A}, \mathcal{B})$  and  $\Delta(\theta, \Omega\theta) \geq \rho(\mathcal{A}, \mathcal{B})$  for all  $\theta \in \mathcal{A}$ .

The following lemmas are significant in the present context.

**Lemma 1 ([1,2]).** Let  $(\mathfrak{S}, \rho)$  be a metric space and  $\mathcal{A}, \mathcal{B} \in CB(\mathfrak{S})$ . Then

1.  $\Delta(\theta, \mathcal{B}) \leq \rho(\theta, \gamma)$  for any  $\gamma \in \mathcal{B}$  and  $\theta \in \mathfrak{S}$ ;
2.  $\Delta(\theta, \mathcal{B}) \leq \mathcal{H}(\mathcal{A}, \mathcal{B})$  for any  $\theta \in \mathcal{A}$ .

**Lemma 2 ([3]).** Let  $\mathcal{A}, \mathcal{B} \in CB(\mathfrak{S})$  and let  $\theta \in \mathcal{A}$ . If  $p > 0$ , then there exists  $\xi \in \mathcal{B}$  such that

$$\rho(\theta, \xi) \leq \mathcal{H}(\mathcal{A}, \mathcal{B}) + p.$$

In general, we may not obtain a point  $\xi \in \mathcal{B}$  such that

$$\rho(\theta, \xi) \leq \mathcal{H}(\mathcal{A}, \mathcal{B}).$$

But when  $\mathcal{B}$  is compact, then such a point  $\xi$  exists, i.e.,  $\rho(\theta, \xi) \leq \mathcal{H}(\mathcal{A}, \mathcal{B})$ .

The notion of  $P$ -property was introduced by Sankar Raj [4]. Further, the idea of weak  $P$  property was put forward by Zhang et al. [5] to improve the results of Caballero et al. [6] on Geraghty-contractions.

**Definition 1 ([4]).** Let  $(\mathfrak{S}, \rho)$  be a metric space and  $\mathcal{A}, \mathcal{B}$  be two non-empty subsets of  $\mathfrak{S}$  such that  $\mathcal{A} \cap \mathcal{B} \neq \emptyset$ . The pair  $(\mathcal{A}, \mathcal{B})$  satisfies the  $P$ -property if and only if  $\rho(\theta_1, \xi_1) = \rho(\mathcal{A}, \mathcal{B}) = \rho(\theta_2, \xi_2)$  implies  $\rho(\theta_1, \theta_2) = \rho(\xi_1, \xi_2)$ , where  $\theta_1, \theta_2 \in \mathcal{A}$  and  $\xi_1, \xi_2 \in \mathcal{B}$ .

**Definition 2 ([5]).** Let  $(\mathfrak{S}, \rho)$  be a metric space and  $\mathcal{A}, \mathcal{B}$  be two non-empty subsets of  $\mathfrak{S}$  such that  $\mathcal{A} \cap \mathcal{B} \neq \emptyset$ . The pair  $(\mathcal{A}, \mathcal{B})$  satisfies the weak  $P$ -property if and only if  $\rho(\theta_1, \xi_1) = \rho(\mathcal{A}, \mathcal{B}) = \rho(\theta_2, \xi_2)$  implies  $\rho(\theta_1, \theta_2) \leq \rho(\xi_1, \xi_2)$ , where  $\theta_1, \theta_2 \in \mathcal{A}$  and  $\xi_1, \xi_2 \in \mathcal{B}$ .

The following well known lemma will be used in the sequel.

**Lemma 3.** If  $\{\theta_n\}$  is a sequence in a complete metric space  $(\mathfrak{S}, \rho)$  such that  $\rho(\theta_{n+1}, \theta_n) \leq \lambda\rho(\theta_n, \theta_{n-1})$  for all  $n \in \mathbb{N}$ , where  $\lambda \in (0, 1)$ , then  $\{\theta_n\}$  is a Cauchy sequence.

BPPs under different types of contractive conditions have been studied in [7–15]. Moreover, BPPs for different kinds of multivalued mappings have been studied in [16–19]. Some more relevant works may be found in [20–24].

In this paper, we put forward the idea of multivalued Banach-type contractive pair (MVBCP, in short) and with the help of weak  $P$  property, establish conditions under which such a pair admits a CBPP. Next, we define the notion of weak  $\Delta$ -property and a multivalued Kannan-type contractive pair (MVKCP, in short) and prove an existence of CBPP result for that pair.

## 2. Common Best Proximity Point for MVBCP

In this section, first we define a MVBCP. The corresponding CBPP result follows.

**Definition 3.** Let  $(\mathfrak{S}, \rho)$  be a metric space and  $\mathcal{A}, \mathcal{B}$  be two non-empty subsets of  $\mathfrak{S}$ . The pair of mappings  $\Psi, \Omega : \mathcal{A} \rightarrow CB(\mathcal{B})$  is said to be a MVBCP if there exists  $\tau \in [0, 1)$  such that

$$\mathcal{H}(\Omega\theta, \Psi\zeta) \leq \tau\rho(\theta, \zeta)$$

for all  $\theta, \zeta \in \mathfrak{S}$ .

**Theorem 1.** Let  $(\mathfrak{S}, \rho)$  be a complete metric space and  $\mathcal{A}, \mathcal{B}$  be two non-empty closed subsets of  $\mathfrak{S}$  such that  $\mathcal{A}_B \neq \emptyset$  and that the pair  $(\mathcal{A}, \mathcal{B})$  satisfies the weak  $P$ -property. Let the pair of mappings  $\Psi, \Omega : \mathcal{A} \rightarrow CB(\mathcal{B})$  be a MVBCP such that  $\Psi\theta$  and  $\Omega\theta$  are compact for each  $\theta \in \mathcal{A}$ , and further  $\Psi\theta \subseteq \mathcal{B}_A$  and  $\Omega\theta \subseteq \mathcal{B}_A$  for all  $\theta \in \mathcal{A}_B$ . Then  $\Psi$  and  $\Omega$  have a CBPP.

**Proof.** Fix  $\theta_0 \in \mathcal{A}_B$  and choose  $\zeta_0 \in \Omega\theta_0 \subseteq \mathcal{B}_A$ . By the definition of  $\mathcal{B}_A$ , we choose  $\theta_1 \in \mathcal{A}_B$  such that

$$\rho(\theta_1, \zeta_0) = \rho(\mathcal{A}, \mathcal{B}). \tag{1}$$

If  $\zeta_0 \in \Omega\theta_1 \cap \Psi\theta_1$ , then we have

$$\rho(\mathcal{A}, \mathcal{B}) \leq \Delta(\theta_1, \Psi\theta_1) \leq \rho(\theta_1, \zeta_0) = \rho(\mathcal{A}, \mathcal{B}), \text{ since } \zeta_0 \in \Psi\theta_1,$$

and

$$\rho(\mathcal{A}, \mathcal{B}) \leq \Delta(\theta_1, \Omega\theta_1) \leq \rho(\theta_1, \zeta_0) = \rho(\mathcal{A}, \mathcal{B}), \text{ since } \zeta_0 \in \Omega\theta_1.$$

Thus  $\rho(\mathcal{A}, \mathcal{B}) = \Delta(\theta_1, \Psi\theta_1) = \Delta(\theta_1, \Omega\theta_1)$ , i.e.,  $\theta_1$  is a CBPP of  $\Psi$  and  $\Omega$ . Therefore, assume that  $\zeta_0 \notin \Omega\theta_1 \cap \Psi\theta_1$ . Consider the case  $\zeta_0 \notin \Psi\theta_1$ .

Since  $\Psi\theta_1$  is compact, by Lemma 2 and the definition of MVBCP, there exist  $\zeta_1 \in \Psi\theta_1 \subseteq \mathcal{B}_A$  and  $\tau \in [0, 1)$  such that

$$0 < \Delta(\zeta_0, \Psi\theta_1) < \rho(\zeta_0, \zeta_1) \leq \mathcal{H}(\Omega\theta_0, \Psi\theta_1) \leq \tau\rho(\theta_0, \theta_1). \tag{2}$$

Since  $\zeta_1 \in \mathcal{B}_A$ , there exists  $\theta_2 \in \mathcal{A}_B$  such that

$$\rho(\theta_2, \zeta_1) = \rho(\mathcal{A}, \mathcal{B}). \tag{3}$$

From (1), (3) and weak  $P$ -property, we have that

$$\rho(\theta_1, \theta_2) \leq \rho(\zeta_0, \zeta_1). \tag{4}$$

From (2) and (4), we have that

$$\rho(\theta_1, \theta_2) \leq \rho(\zeta_0, \zeta_1) \leq \tau\rho(\theta_0, \theta_1). \tag{5}$$

If  $\zeta_1 \in \Omega\theta_2 \cap \Psi\theta_2$ , then like earlier we can show that  $\theta_2$  is a CBPP of  $\Omega$  and  $\Psi$ . Thus assume that  $\zeta_1 \notin \Omega\theta_2 \cap \Psi\theta_2$ . Consider the case  $\zeta_1 \notin \Omega\theta_2$ . Since  $\Omega\theta_2$  is compact, there exists  $\zeta_2 \in \Omega\theta_2$  such that

$$0 < \Delta(\xi_1, \Omega\theta_2) < \rho(\xi_1, \xi_2) \leq \mathcal{H}(\Omega\theta_2, \Psi\theta_1) \leq \tau\rho(\theta_1, \theta_2). \tag{6}$$

Since  $\xi_2 \in \Omega\theta_2 \subseteq \mathcal{B}_A$ , there exists  $\theta_3 \in \mathcal{A}_B$  such that

$$\rho(\theta_3, \xi_2) = \rho(\mathcal{A}, \mathcal{B}). \tag{7}$$

From (3), (7) and weak P-property, we have that

$$\rho(\theta_2, \theta_3) \leq \rho(\xi_1, \xi_2). \tag{8}$$

Also, from (5) and (6),

$$\rho(\xi_1, \xi_2) \leq \tau\rho(\xi_0, \xi_1). \tag{9}$$

Continuing in this way, we obtain two sequences  $\{\theta_n\}$  and  $\{\xi_n\}$  in  $\mathcal{A}_B$  and  $\mathcal{B}_A$  respectively, satisfying

**(B1)**  $\xi_{2n} \in \Omega\theta_{2n} \subseteq \mathcal{B}_A$  and  $\xi_{2n+1} \in \Psi\theta_{2n+1} \subseteq \mathcal{B}_A$ ,

**(B2)**  $\rho(\theta_{n+1}, \xi_n) = \rho(\mathcal{A}, \mathcal{B})$ ,

**(B3)**  $\rho(\theta_n, \theta_{n+1}) \leq \tau\rho(\theta_{n-1}, \theta_n)$  and  $\rho(\xi_n, \xi_{n+1}) \leq \tau\rho(\xi_{n-1}, \xi_n)$ ,

for each  $n = 0, 1, 2, \dots$

From **(B3)** and Lemma 3, we observe that  $\{\theta_n\}$  and  $\{\xi_n\}$  both are Cauchy sequences. Since  $\mathcal{A}$  and  $\mathcal{B}$  are closed subsets of a complete metric space, we conclude that  $\mathcal{A}$  and  $\mathcal{B}$  both are complete subspaces.

Hence, there exists  $\theta \in \mathcal{A}$  and  $\xi \in \mathcal{B}$  such that  $\theta_n \rightarrow \theta$  and  $\xi_n \rightarrow \xi$  as  $n \rightarrow +\infty$ .

We claim that  $\Omega\theta_n$  converges to  $\Omega\theta$ . Indeed, if  $m > n$ , then

$$\begin{aligned} \mathcal{H}(\Omega\theta_n, \Omega\theta) &\leq \mathcal{H}(\Omega\theta_n, \Psi\theta_m) + \mathcal{H}(\Psi\theta_m, \Omega\theta) \\ &\leq \tau[\rho(\theta_n, \theta_m) + \rho(\theta_m, \theta)] \\ &\rightarrow 0 \text{ as } n \rightarrow +\infty. \end{aligned}$$

Similarly, we can show that  $\Psi\theta_n$  converges to  $\Psi\theta$ .

From **(B2)** we have that

$$\rho(\theta_{n+1}, \xi_n) = \rho(\mathcal{A}, \mathcal{B})$$

for each  $n = 0, 1, 2, \dots$

This implies

$$\lim_{n \rightarrow +\infty} \rho(\theta_{n+1}, \xi_n) = \rho(\theta, \xi) = \rho(\mathcal{A}, \mathcal{B}). \tag{10}$$

Again, we claim that  $\xi \in \Omega\theta \cap \Psi\theta$ . Since  $\xi_{2n} \in \Omega\theta_{2n}$ , we have

$$\begin{aligned} \lim_{n \rightarrow +\infty} \Delta(\xi_{2n}, \Omega\theta) &\leq \lim_{n \rightarrow +\infty} \mathcal{H}(\Omega\theta_{2n}, \Omega\theta) = 0, \text{ (since } \Omega\theta_n \text{ converges to } \Omega\theta) \\ \implies \Delta(\xi, \Omega\theta) &= 0. \end{aligned}$$

Hence  $\xi \in \Omega\theta$ .

Also since  $\xi_{2n+1} \in \Psi\theta_{2n+1}$ , we have

$$\begin{aligned} \lim_{n \rightarrow +\infty} \Delta(\xi_{2n+1}, \Psi\theta) &\leq \lim_{n \rightarrow +\infty} \mathcal{H}(\Psi\theta_{2n+1}, \Psi\theta) = 0, \text{ (since } \Psi\theta_n \text{ converges to } \Psi\theta) \\ \implies \Delta(\xi, \Psi\theta) &= 0. \end{aligned}$$

Hence  $\xi \in \Psi\theta$ . Therefore,

$$\xi \in \Omega\theta \cap \Psi\theta. \tag{11}$$

Finally, using (10) and (11) we have that

$$\begin{aligned} \rho(\mathcal{A}, \mathcal{B}) &\leq \Delta(\theta, \Psi\theta) \leq \rho(\theta, \xi) = \rho(\mathcal{A}, \mathcal{B}) \\ \implies \Delta(\theta, \Psi\theta) &= \rho(\mathcal{A}, \mathcal{B}), \end{aligned}$$

and

$$\begin{aligned} \rho(\mathcal{A}, \mathcal{B}) &\leq \Delta(\theta, \Omega\theta) \leq \rho(\theta, \xi) = \rho(\mathcal{A}, \mathcal{B}) \\ \implies \Delta(\theta, \Omega\theta) &= \rho(\mathcal{A}, \mathcal{B}), \end{aligned}$$

Hence  $\theta$  is a CBPP of  $\Omega$  and  $\Psi$ .  $\square$

Next, we present an example in which the pair  $(\mathcal{A}, \mathcal{B})$  satisfies only the weak  $P$ -property but not the  $P$ -property.

**Example 1.** Consider  $\mathfrak{S} = \mathbb{R}^2$  with the Euclidean metric  $\rho$ . Let  $\mathcal{A} = \{(-5, 0), (0, 1), (5, 0)\}$  and  $\mathcal{B} = \{(\theta, \xi) : \xi = 2 + \sqrt{2 - \theta^2}, \theta \in [-\sqrt{2}, \sqrt{2}]\}$ . Then  $\rho(\mathcal{A}, \mathcal{B}) = \sqrt{3}$  and  $\mathcal{A}_{\mathcal{B}} = \{(0, 1)\}$ ,  $\mathcal{B}_{\mathcal{A}} = \{(\sqrt{2}, 2), (-\sqrt{2}, 2)\}$ .

Define a pair of multivalued maps  $\Omega, \Psi : \mathcal{A} \rightarrow CB(\mathcal{B})$  in the following manner:

$$\Omega(-5, 0) = \{(0, 2 + \sqrt{2})\}, \Omega(0, 1) = \{(-\sqrt{2}, 2), (0, 2 + \sqrt{2})\}, \Omega(5, 0) = \{(-1, 3), (1, 3)\},$$

and

$$\Psi(-5, 0) = \{(-\sqrt{2}, 2), (-1, 3)\}, \Psi(0, 1) = \{(\sqrt{2}, 2)\}, \Psi(5, 0) = \{(\sqrt{2}, 2), (1, 3)\}.$$

By routine calculations, it is easy to check that the condition

$$\mathcal{H}(\Omega\theta, \Psi\xi) \leq \tau\rho(\theta, \xi)$$

is satisfied for all  $\theta, \xi \in \mathfrak{S}$  and for  $\tau = \frac{19}{20} \in [0, 1)$ .

Thus the pair  $\Psi, \Omega$  is a MVBCP.

Finally, we observe that

$$\rho((0, 1), (\sqrt{2}, 2)) = \rho((0, 1), (-\sqrt{2}, 2)) = \sqrt{3} = \rho(\mathcal{A}, \mathcal{B}),$$

but

$$\rho((0, 1), (0, 1)) = 0 < \rho((\sqrt{2}, 2), (-\sqrt{2}, 2)) = 2\sqrt{2}.$$

Thus,  $(\mathcal{A}, \mathcal{B})$  satisfies weak  $P$ -property, but not the  $P$ -property. Therefore, all conditions of Theorem 1 are satisfied and since  $\Delta((0, 1), \Psi(0, 1)) = \Delta((0, 1), \Omega(0, 1)) = \sqrt{3} = \rho(\mathcal{A}, \mathcal{B})$ , we conclude that  $(0, 1)$  is a CBPP of  $\Psi$  and  $\Omega$ .

### 3. Common Best Proximity Point for MVKCP

In this section, we define the concepts of weak  $\Delta$ -property and a MVKCP. Combining these two concepts, we establish a CBPP result.

**Definition 4.** Consider the metric space  $(CB(\mathfrak{S}), \mathcal{H})$  and let  $\mathcal{A}, \mathcal{B}$  be two non-empty subsets in  $CB(\mathfrak{S})$  such that  $\mathcal{A}_B \neq \emptyset$ . The pair  $(\mathcal{A}, \mathcal{B})$  is said to have the weak  $\Delta$ -property if and only if  $\Delta(\theta, \mathcal{U}) = \rho(\mathcal{A}, \mathcal{B}) = \Delta(\xi, \mathcal{V})$  implies  $\rho(\theta, \xi) \leq \mathcal{H}(\mathcal{U}, \mathcal{V})$ , for all  $\theta, \xi \in \mathcal{A}_B$  and  $\mathcal{U}, \mathcal{V} \subseteq \mathcal{B}_A$ .

**Definition 5.** Let  $(\mathfrak{S}, \rho)$  be a metric space and  $\mathcal{A}, \mathcal{B}$  be two non-empty subsets of  $\mathfrak{S}$ . The pair of mappings  $\Psi, \Omega : \mathcal{A} \rightarrow CB(\mathcal{B})$  ( $\Psi$  and  $\Omega$  may be identical) is said to be a multivalued Kannan-type contractive pair (MVKCP, in short) if there exists  $\lambda \in [0, 1)$  such that

$$\mathcal{H}(\Omega\theta, \Psi\xi) \leq \frac{\lambda}{2} [\Delta(\theta, \Omega\theta) + \Delta(\xi, \Psi\xi) - 2\rho(\mathcal{A}, \mathcal{B})] \tag{12}$$

for all  $\theta, \xi \in \mathfrak{S}$ .

**Remark 2.** If  $\Psi, \Omega$  is an MVKCP, the condition (12) is satisfied when  $\Psi = \Omega$  as well.

**Definition 6 ([25]).** Let  $(\mathfrak{S}, \rho)$  be a metric space and  $R$  be a self-map on  $\mathfrak{S}$ .  $R$  is said to be a Kannan mapping if there exists  $0 \leq \lambda < \frac{1}{2}$  such that

$$\rho(R\theta, R\xi) \leq \lambda \{ \rho(\theta, R\theta) + \rho(\xi, R\xi) \},$$

for all  $\theta, \xi \in \mathfrak{S}$ .

**Remark 3.** If  $(\mathfrak{S}, \rho)$  is a complete metric space, then a Kannan mapping on  $\mathfrak{S}$  possesses a unique fixed point.

Now we present the main result of this section.

**Theorem 2.** Let  $(\mathfrak{S}, \rho)$  be a complete metric space and  $\mathcal{A}, \mathcal{B}$  be two non-empty closed subsets of  $\mathfrak{S}$  such that  $\mathcal{A}_B \neq \emptyset$  and that the pair  $(\mathcal{A}, \mathcal{B})$  satisfies the weak  $\Delta$ -property. Let the pair of mappings  $\Psi, \Omega : \mathcal{A} \rightarrow CB(\mathcal{B})$  be a MVKCP such that  $\Psi\theta \subseteq \mathcal{B}_A$  and  $\Omega\theta \subseteq \mathcal{B}_A$  for all  $\theta \in \overline{\mathcal{A}}_B$ . Then  $\Psi$  and  $\Omega$  have a CBPP.

**Proof.** Define the map  $\Gamma : \Omega(\overline{\mathcal{A}}_B) \rightarrow \mathcal{A}_B$  by

$$\Gamma(S) = \{ \theta \in \mathcal{A}_B : \Delta(\theta, S) = \rho(\mathcal{A}, \mathcal{B}) \}, \tag{13}$$

for all  $S \in \overline{\mathcal{A}}_B$ . The map  $\Gamma$  is well defined, for if  $\Gamma(S) = \theta_1$  and  $\Gamma(S) = \theta_2$ , then  $\Delta(\theta_1, S) = \rho(\mathcal{A}, \mathcal{B})$  and  $\Delta(\theta_2, S) = \rho(\mathcal{A}, \mathcal{B})$ . By weak  $\Delta$ -property, we have  $\rho(\theta_1, \theta_2) \leq \mathcal{H}(S, S) = 0$ , i.e.,  $\theta_1 = \theta_2$ .

From (13), we have  $\Delta(\Gamma(\Omega\theta), \Omega\theta) = \rho(\mathcal{A}, \mathcal{B})$  and  $\Delta(\Gamma(\Omega\xi), \Omega\xi) = \rho(\mathcal{A}, \mathcal{B})$  for any  $\theta, \xi \in \overline{\mathcal{A}}_B$ .

Again, using the weak  $\Delta$ -property, we have

$$\begin{aligned} \rho(\Gamma(\Omega\theta), \Gamma(\Omega\xi)) &\leq \mathcal{H}(\Omega\theta, \Omega\xi) \\ &\leq \frac{\lambda}{2} [\Delta(\theta, \Omega\theta) + \Delta(\xi, \Omega\xi) - 2\rho(\mathcal{A}, \mathcal{B})] \\ &\leq \frac{\lambda}{2} [\rho(\theta, \Gamma(\Omega\theta)) + \Delta(\Gamma(\Omega\theta), \Omega\theta) + \rho(\xi, \Gamma(\Omega\xi)) + \Delta(\Gamma(\Omega\xi), \Omega\xi) - 2\rho(\mathcal{A}, \mathcal{B})] \\ &= \frac{\lambda}{2} [\rho(\theta, \Gamma(\Omega\theta)) + \rho(\xi, \Gamma(\Omega\xi)) - 2\rho(\mathcal{A}, \mathcal{B})], \end{aligned}$$

for any  $\theta, \xi \in \overline{\mathcal{A}}_B$  and  $\lambda \in [0, 1)$ .

It means that the composition map  $\Gamma \circ \Omega : \overline{\mathcal{A}}_B \rightarrow \overline{\mathcal{A}}_B$  is a Kannan map from  $\overline{\mathcal{A}}_B$  to itself, which is a complete metric space.

Thus,  $\Gamma \circ \Omega$  has a unique fixed point  $\theta_1$ , i.e.,  $\Gamma \circ \Omega(\theta_1) = \theta_1 \in \mathcal{A}_B$ , which implies that  $\Delta(\theta_1, \Omega(\theta_1)) = \rho(\mathcal{A}, \mathcal{B})$ .

Similarly, we can define  $\Pi : \Psi(\overline{\mathcal{A}}_B) \rightarrow \mathcal{A}_B$  and obtain a unique fixed point  $\theta_2$  of  $\Pi \circ \Psi$  and consequently  $\Delta(\theta_2, \Psi(\theta_2)) = \rho(\mathcal{A}, \mathcal{B})$ .

Using the weak  $\Delta$ -property, we have that

$$\begin{aligned} \rho(\theta_1, \theta_2) &\leq \mathcal{H}(\Omega\theta_1, \Psi\theta_2) \\ &\leq \frac{\lambda}{2} [\Delta(\theta_1, \Omega\theta_1) + \Delta(\theta_2, \Psi\theta_2) - 2\rho(\mathcal{A}, \mathcal{B})] \\ &= 0, \end{aligned}$$

which implies that  $\theta_1 = \theta_2 = \theta$  (say).

Therefore,  $\Delta(\theta, \Omega(\theta)) = \Delta(\theta, \Psi(\theta)) = \rho(\mathcal{A}, \mathcal{B})$ . Thus  $\theta$  is a CBPP of  $\Omega$  and  $\Psi$ .  $\square$

#### 4. Conclusions

The concepts of MVBCP, MVKCP and weak  $\Delta$ -property have been introduced in this paper. Using weak  $P$ -property, a CBPP result has been proved for a MVBCP and using the weak  $\Delta$ -property, a similar result has been established for a MVKCP. The current study is interesting because the proof of our main theorem in Section 2 provides us with a scheme on how to find a CBPP for two multivalued maps. An application of the same has also been discussed in Example 1.

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