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Abstract: This article presents the study of certain analytic functions defined by bounded radius rotations associated with conic domain. Many geometric properties like coefficient estimate, radii problems, arc length, integral representation, inclusion results and growth rate of coefficients of Taylor's series representation are investigated. By varying the parameters in results, several well-known results in literature are obtained as special cases.

Keywords: functions of bounded boundary and bounded radius rotations; subordination; functions with positive real part; uniformly starlike and convex functions

1. Introduction

Let $\mathcal{A}$ denote the family of complex valued functions f which are holomorphic (analytic) in $\mathfrak{E} = \{z \in \mathbb{C} : |z| < 1\}$ and are normalized through the conditions $f(0) = 0$ and $f'(0) = 1$. That is, for $f \in \mathcal{A}$, one may have its series form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in \mathfrak{E}. \tag{1}$$

The class $\mathcal{UCV}$ is comprised those univalent functions $f(z)$ by which every circular arc $\mathfrak{C} \subset \mathfrak{E}$, with center at $\mathfrak{E}$, is mapped onto the convex arc and such functions are known as uniformly convex functions. This class was first introduced by Goodman [1]. The interesting analytic condition of class $\mathcal{UCV}$ was given in [2] and is stated as follows:

$$\mathcal{UCV} = \left\{ f \in \mathcal{A} : \Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \left|\frac{zf''(z)}{f'(z)}\right|, \ z \in \mathfrak{E} \right\}.$$

Kanas et al. [3] further generalized the class $\mathcal{UCV}$ by introducing the class of k-uniformly convex functions, named as k-$\mathcal{UCV}$, $k \geq 0$ and the class k-$\mathcal{ST}$ of corresponding k-starlike functions. The class k-$\mathcal{UCV}$ is defined as follows:

$$k\text{-}\mathcal{UCV} = \left\{ f \in \mathcal{A} : \Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > k\left|\frac{zf''(z)}{f'(z)}\right|, \ z \in \mathfrak{E} \right\}.$$

They, in addition, discussed these classes geometrically and established connections with the conic domains

$$\mathcal{G}_k = \left\{ u + iv \,; u^2 > k^2 \left((u-1)^2 + v^2 \right) \right\}. \tag{2}$$

It is important to mention that the class k-$\mathcal{UCV}$ was studied much earlier with some extra conditions but without geometrical interpretation. The class k-$\mathcal{UCV}$ is defined geometrically in a way that the common region of $\mathfrak{E}$ and the disk $|\mathfrak{D}| \leq k$ is mapped onto a convex domain by these univalent functions. Thus, the notion of convexity got the generalized version of k-uniform convexity. If $k = 0$, Then, the center $\mathfrak{D}$ shifts to origin and thus k-$\mathcal{UCV}$ takes the form of $\mathcal{C}$, the family of convex univalent functions.

The domain $\mathcal{G}_k$ represents conic regions for certain values of parameter k, that is, it gives an elliptic region for $k > 1$, the hyperbolic region (right branch) for $0 < k < 1$ and the parabolic region when $k = 1$. For more details, see [3–6]. The domain $\mathcal{G}_{k,\beta}$, which is generalization of $\mathcal{G}_k$ is given as:

$$\mathcal{G}_{k,\beta} = (1-\beta)\,\mathcal{G}_k + \beta,$$

where

$$\beta = \begin{cases} [0,1)\,, & \text{if} \quad k \in [0,1]\,, \\ \left[0, 1 - \frac{\sqrt{k^2-1}}{k}\right), & \text{if } k > 1. \end{cases} \tag{3}$$

For details, see [7]. The function which gives the boundary curves of these conical regions is denoted by $\varphi_{k,\beta}(z)$ which is holomorphic in $\mathfrak{E}$ and maps $\mathfrak{E}$ onto $\mathcal{G}_{k,\beta}$ such that $\varphi_{k,\beta}(z) = 1$ and $\varphi'_{k,\beta}(0) > 1$ and is defined as:

$$\varphi_{k,\beta}(z) = \begin{cases} \frac{1+(1-2\beta)z}{1-z}, & k = 0, \\ 1 + \frac{2(1-\beta)}{\pi^2}\left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & k = 1, \\ 1 + \frac{2(1-\beta)}{1-k^2}\sinh^2\left[\left(\frac{2}{\pi}\cos^{-1} k\right)\tan^{-1} \mathrm{h}\sqrt{z}\right], & 0 < k < 1, \\ 1 + \frac{(1-\beta)}{k^2-1}\sin\left[\frac{\pi}{2R(t)}\int_0^{\frac{u(z)}{\sqrt{t}}} \frac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}dx\right] + \frac{1}{k^2-1}, & k > 1. \end{cases} \tag{4}$$

For the detailed study of this function, we refer the readers to see [3,6].

Let k-$\mathcal{P}(\beta)$ denote the family of holomorphic functions $q(z)$ with $q(0) = 1$ and $q(z) \prec \varphi_{k,\beta}(z)$ for $z \in \mathfrak{E}$, where the notion "$\prec$" denotes the familiar subordinations. It is pertinent to have

$$k\text{-}\mathcal{P}(\beta) \subset \mathcal{P}\left(\frac{k+\beta}{1+k}\right) \subset \mathcal{P},$$

where $\mathcal{P}$ is the family of functions with a positive real part. In addition, for $q \in k$-$\mathcal{P}(0)$, we have

$$|\arg q(z)| \leq \frac{\lambda\pi}{2},$$

where

$$\lambda = \frac{2}{\pi}\tan^{-1}(1/k). \tag{5}$$

Therefore, one may write

$$q(z) = h^{\lambda}(z)\,, \quad h(z) \in \mathcal{P}.$$

Definition 1. *Let the function $q(z)$ be holomorphic in $\mathfrak{E}$ with $q(0)=1$. Then, $q \in k\text{-}\mathcal{P}_m(\beta)$, if for $m \geq 2$, $k \geq 0$, $z \in \mathfrak{E}$ and β is given by Equation (3), we have*

$$q(z) = \left(\frac{m}{4}+\frac{1}{2}\right) q_1(z) - \left(\frac{m}{4}-\frac{1}{2}\right) q_2(z),$$

where $q_1(z), q_2(z) \in k\text{-}\mathcal{P}(\beta)$ [8].

Taking $k=0$ and $\beta=0$, the class $\mathcal{P}_m$ introduced by Pinchuk [9] is obtained. In addition, $k\text{-}\mathcal{P}_2(\beta) = k\text{-}\mathcal{P}(\beta)$, $0\text{-}\mathcal{P}_m(\beta) = \mathcal{P}_m(\beta)$ and $0\text{-}\mathcal{P}_2(\beta) = \mathcal{P}(\beta)$, where $\mathcal{P}_m(\beta)$ and $\mathcal{P}(\beta)$ were introduced in [9].

It is noted that $k\text{-}\mathcal{P}_m(\beta)$ is a convex set. Noor [8] introduced the classes $k\text{-}\mathcal{UV}^m(\beta)$ and $k\text{-}\mathcal{UR}^m(\beta)$ of k-uniformly bounded boundary and radius rotation of order β corresponding to the class $k\text{-}\mathcal{P}_m(\beta)$.

Now, we consider the following new subclasses of holomorphic functions.

Definition 2. *A function $f \in \mathcal{A}$ is known to be in $k\text{-}\mathcal{UR}_s^m(\beta)$, $k \geq 0$, $m \geq 2$ and β is given by Equation (3), if*

$$\frac{2zf'(z)}{f(z)-f(-z)} \in k\text{-}\mathcal{P}_m(\beta), \quad (z \in \mathfrak{E}).$$

Definition 3. *A function $f \in \mathcal{A}$ is known to be in the class $k\text{-}\mathcal{B}_s^m(\alpha,\beta)$, $\alpha > 0$, $k \geq 0$, $m \geq 2$ and β is given by Equation (3), if there exists $g \in k\text{-}\mathcal{UR}_s^m(\beta)$ such that*

$$\Re\left\{\frac{zf'(z)}{f(z)}\left(\frac{2f(z)}{g(z)-g(-z)}\right)^{\alpha}\right\} > k\left|\frac{zf'(z)}{f(z)}\left(\frac{2f(z)}{g(z)-g(-z)}\right)^{\alpha}-1\right|,$$

or equivalently

$$\frac{zf'(z)}{f(z)}\left(\frac{2f(z)}{g(z)-g(-z)}\right)^{\alpha} \in k\text{-}\mathcal{P}(0).$$

It is pertinent to note that, by assigning specific values to parameters α, β, m and k in $k\text{-}\mathcal{UR}_s^m(\beta)$ and $k\text{-}\mathcal{B}_s^m(\alpha,\beta)$, several well-known subclasses of holomorphic and univalent functions are obtained, from which some are listed below:

1. $0\text{-}\mathcal{UR}_s^m(\beta) = \mathcal{R}_s^m(\beta)$, introduced by Bhargava et al. [10].
2. For $m=2$ and $\alpha=0$, we obtain the class $k\text{-}\mathcal{ST}_s(\beta)$, and $k\text{-}\mathcal{UK}_s(\beta)$, for details, we refer to [8].
3. $0\text{-}\mathcal{UR}_s^2(0) = \mathcal{S}_s^*$, for details, see [11].

Throughout the article, we shall consider, unless otherwise stated, that $m \geq 2$, $\alpha > 0$, $k \geq 0$ and β is given by Equation (3).

2. Preliminary Lemmas

Lemma 1. *[12] Let $k \in [0,\infty)$ and $\varphi_{k,\beta}(z)$ be defined by Equation (4). If*

$$\varphi_{k,\beta}(z) = 1 + Q_1 z + Q_2 z^2 + \cdots,$$

Then,

$$Q_1 = \begin{cases} \frac{2\beta A^2}{1-k^2} & 0 \leq k < 1, \\ \frac{8\beta}{\pi^2} & k = 1, \\ \frac{\pi^2\beta}{4\sqrt{t}(k^2-1)R^2(t)(1+t)} & k > 1, \end{cases} \tag{6}$$

and

$$Q_2 = \begin{cases} \frac{(A^2+2)}{3} Q_1 & 0 \leq k < 1, \\ \frac{2}{3} Q_1 & k = 1, \\ \frac{4R^2(t)\left(t^2+6t+1\right)-\pi^2}{24\sqrt{t}R^2(t)(1+t)} Q_1 & k > 1, \end{cases} \tag{7}$$

where

$$A = \frac{2\cos^{-1} k}{\pi},$$

and $t \in (0,1)$ *is taken such that* $k = \cosh\left(\frac{\pi R'(t)}{R(t)}\right)$, $R(t)$ *is the Legendre's complete elliptic integral of the first kind.*

To proceed our main results, the following Lemmas proved by Pommerenke [13] and Golusin [14] are needed.

Lemma 2. *Let the holomorphic function* $p \in \mathcal{P}$. *Then [13]*

$$\frac{1}{2\pi} \int_0^{2\pi} |p(z)|^2 \, d\theta \leq \frac{1+3r^2}{1-r^2}.$$

Lemma 3. *Let the function* $s_1(z)$ *be starlike in* $\mathfrak{E}$. *Then [14],*
(i) : *there exists* ξ *with* $|\xi| = r$ *such that for all* z, $|z| = r$

$$|z-\xi| \, |s_1(z)| \leq \frac{2r^2}{1-r^2}$$

(ii)

$$\frac{r}{(1+r)^2} \leq s_1(z) \leq \frac{r}{(1-r)^2}.$$

3. Main Results

Theorem 1. *Let* $f \in k\text{-}\mathcal{UR}_s^m(\beta)$. *Then, the odd function*

$$\phi(z) = \frac{f(z) - f(-z)}{2}$$

belongs to $k\text{-}\mathcal{UR}^m(\beta)$.

Proof. Let $f \in k\text{-}\mathcal{UR}_s^m(\beta)$ and consider

$$\phi(z) = \frac{f(z) - f(-z)}{2}.$$

Logarithmic differentiation of the above relation yields

$$\frac{\phi'(z)}{\phi(z)} = \frac{f'(z) + f'(-z)}{f(z) - f(-z)},$$

or, equivalently,

$$\frac{z\phi'(z)}{\phi(z)} = \frac{1}{2}\left[q(z) + q(-z)\right],$$

where

$$q(z)=\frac{2zf'(z)}{f(z)-f(-z)} \quad \text{and} \quad q(-z)=\frac{2(-z)f'(-z)}{f(-z)-f(z)}.$$

Because $f(z)\in k\text{-}\mathcal{UR}_s^m(\beta)$, then, there exist $p_1(z),\ p_2(z)\in k\text{-}\mathcal{P}(\beta)$ such that

$$q(z)=\frac{2zf'(z)}{f(z)-f(-z)}=\left(\frac{m}{4}+\frac{1}{2}\right)p_1(z)-\left(\frac{m}{4}-\frac{1}{2}\right)p_2(z).$$

Therefore, we have

$$\frac{z\phi'(z)}{\phi(z)}=\left(\frac{m}{4}+\frac{1}{2}\right)\frac{p_1(z)+p_1(-z)}{2}-\left(\frac{m}{4}-\frac{1}{2}\right)\frac{p_2(z)+p_2(-z)}{2}.$$

Since $k\text{-}\mathcal{P}(\beta)$ is a convex set, we have

$$\frac{p_i(z)+p_i(-z)}{2}\in k\text{-}\mathcal{P}(\beta),\ \ i=1,2.$$

Thus, we have that

$$\frac{z\phi'(z)}{\phi(z)}\in k\text{-}\mathcal{P}_m(\beta),\ \ (z\in\mathfrak{E}),$$

and hence $\phi(z)\in k\text{-}\mathcal{UR}^m(\beta)$. □

When we take $m=2$, the following result, proved by Noor [8], is obtained.

Corollary 1. *Let $f\in k\text{-}\mathcal{ST}_s(\beta)$. Then,*

$$\phi(z)=\frac{1}{2}\left[f(z)-f(-z)\right]$$

belongs to $k\text{-}\mathcal{ST}(\beta)$.

Corollary 2. *Let $f\in\mathcal{R}_s^m(\beta)$. Then,*

$$\phi(z)=\frac{1}{2}\left[f(z)-f(-z)\right]$$

belongs to $\mathcal{R}^m(\beta)$.

Theorem 2. *If $f\in k\text{-}\mathcal{UR}_s^m(\beta)$, then*

$$f'(z)=\frac{p(z)}{2}\ \exp\left\{\int_0^z\frac{1}{2\xi}\left(p(\xi)+p(-\xi)-2\right)d\xi\right\} \tag{8}$$

for some $p(z)\in k\text{-}\mathcal{P}_m(\beta)$.

Proof. Let $f\in k\text{-}\mathcal{UR}_s^m(\beta)$. Then, by definition, one may have

$$\frac{2zf'(z)}{f(z)-f(-z)}=p(z),\ p(z)\in k\text{-}\mathcal{P}_m(\beta). \tag{9}$$

Simple computation leads us to

$$\frac{f(z)-f(-z)}{z}=\exp\left\{\int_0^z\frac{1}{2\xi}\left(p(\xi)+p(-\xi)-2\right)d\xi\right\}. \tag{10}$$

Using (9) in (10), we can easily obtain (8). □

When we take $m = 2$, the above result takes the following form, proved by Noor [8].

Corollary 3. *If $f \in k\text{-}\mathcal{ST}_s(\beta)$, then*

$$f'(z) = \frac{p(z)}{2} \exp\left\{ \int_0^z \frac{1}{2\xi} \left(p(\xi) + p(-\xi) - 2\right) d\xi \right\}$$

for $p(z) \in k\text{-}\mathcal{P}(\beta)$.

When $m = 2$, $k = 0$ and $\beta = 0$. Then, we have the following result, proved in [11].

Theorem 3. *Let $f \in k\text{-}\mathcal{UR}_s^m(\beta)$ be of the form (1). Then,*

$$|a_2| \leq \frac{m}{8} |Q_1|, \tag{11}$$

where Q_1 is given by (6).

Proof. Let $f \in k\text{-}\mathcal{UR}_s^m(\beta)$ and let it be of the form (9). Then,

$$f''(z) = \frac{p'(z)}{2} \exp\left\{ \int_0^z \frac{p(\xi) + p(-\xi) - 2}{2\xi} d\xi \right\} + \frac{p(z)}{2} \left\{ \frac{f(z) - f(-z)}{z} \right\}'. \tag{12}$$

From (12), we have $f''(0) = \frac{p'(0)}{2}$. It is well known that $|p'(0)|$ in the class $k\text{-}\mathcal{P}_m(\beta)$ is $|p'(0)| \leq \frac{m}{2} |Q_1|$, where Q_1 is given by (6). Thus, we get (11). □

Corollary 4. *The following disk is contained in the range of every function from $k\text{-}\mathcal{UR}_s^m(\beta)$.*

$$|w| < \frac{8}{16 + m|Q_1|},$$

where Q_1 is given by (6).

Proof. According to the Koebe theorem, each omitted value w satisfies

$$|w| > \frac{1}{2 + |a_2|}. \tag{13}$$

Using (13) and Theorem 3, we get the required result. □

By using the similar technique as used in [11], we have the following result.

Theorem 4. *Let $f \in k\text{-}\mathcal{UR}_s^m(\beta)$. Then, for $z = re^{i\theta}$ and $0 \leq \theta_1 < \theta_2 \leq 2\pi$,*

$$\int_{\theta_1}^{\theta_2} \Re\left(\frac{zf'(z)}{f(z)}\right) d\theta > -(1 - \beta_1)\left(\frac{m}{2} - 1\right)\pi,$$

for

$$\beta_1 = \frac{\beta + k}{1 + k}. \tag{14}$$

Theorem 5. *Let $f(z) \in k\text{-}\mathcal{B}_s^m(\alpha,\beta)$. Then, for $z = re^{i\theta}$,*

$$\int_{\theta_1}^{\theta_2} \Re J(\alpha, f(z))\, d\theta > -\left(\alpha(1-\beta_1)\left(\frac{m}{2}-1\right)+\sigma\right)\pi, \tag{15}$$

where $0 \le \theta_1 < \theta_2 \le 2\pi$, β_1 is defined by (14) and

$$J(\alpha, f(z)) = \left(1+\frac{zf''(z)}{f'(z)}\right) + (\alpha-1)\frac{zf'(z)}{f(z)}. \tag{16}$$

Proof. Let

$$\frac{zf'(z)}{f(z)}\left(\frac{2f(z)}{g(z)-g(-z)}\right)^{\alpha} = h^{\sigma}(z),$$

where $h(z) \in \mathcal{P}$,

$$\frac{(zf'(z))'}{f'(z)} + (\alpha-1)\frac{zf'(z)}{f(z)} = \frac{\sigma z h'(z)}{h(z)} + \frac{\alpha z\phi'(z)}{\phi(z)},$$

$$\int_{\theta_1}^{\theta_2}\left[\frac{(zf'(z))'}{f'(z)} + (\alpha-1)\frac{zf'(z)}{f(z)}\right] d\theta = \sigma\int_{\theta_1}^{\theta_2}\frac{zh'(z)}{h(z)}d\theta + \alpha\int_{\theta_1}^{\theta_2}\frac{z\phi'(z)}{\phi(z)}d\theta,$$

where $\phi(z)$ is an odd function of the form

$$\phi(z) = \frac{1}{2}\left[g(z) - g(-z)\right].$$

Since $g(z) \in k\text{-}\mathcal{UR}_s^m(\beta)$ and by Theorem 1 $\phi(z) \in k\text{-}\mathcal{UR}^m(\beta) \subset \mathcal{R}^m(\beta_1)$, therefore, by using Theorem 4, we have

$$\int_{\theta_1}^{\theta_2}\Re\left(\frac{z\phi'(z)}{\phi(z)}\right) d\theta > -(1-\beta_1)\left(\frac{m}{2}-1\right)\pi. \tag{17}$$

In addition, we observe that, for $h(z) \in \mathcal{P}$,

$$\begin{aligned}\frac{\partial}{\partial\theta}\arg h\left(re^{i\theta}\right) &= \frac{\partial}{\partial\theta}\Re\left\{-i\ln h\left(re^{i\theta}\right)\right\}\\ &= \Re\left\{\frac{re^{i\theta}h'\left(re^{i\theta}\right)}{h\left(re^{i\theta}\right)}\right\}.\end{aligned}$$

Therefore,

$$\int_{\theta_1}^{\theta_2}\Re\left\{\frac{re^{i\theta}h'\left(re^{i\theta}\right)}{h\left(re^{i\theta}\right)}\right\} d\theta = \arg h\left(re^{i\theta_2}\right) - \arg h\left(re^{i\theta_1}\right),$$

which takes the form

$$\left|\int_{\theta_1}^{\theta_2}\Re\left\{\frac{re^{i\theta}h'\left(re^{i\theta}\right)}{h\left(re^{i\theta}\right)}\right\} d\theta\right| = \left|\arg h\left(re^{i\theta_2}\right) - \arg h\left(re^{i\theta_1}\right)\right|.$$

This implies that

$$\max_{h\in\mathcal{P}(\beta)}\left|\int_{\theta_1}^{\theta_2}\Re\left(\frac{re^{i\theta}h'\left(re^{i\theta}\right)}{h\left(re^{i\theta}\right)}\right) d\theta\right| = \max_{h\in\mathcal{P}(\beta)}\left|\arg h\left(re^{i\theta_2}\right) - \arg h\left(re^{i\theta_1}\right)\right|. \tag{18}$$

Since $h(z) \in \mathcal{P}$, thus

$$\left| h(z) - \frac{1+r^2}{1-r^2} \right| \leq \frac{2r}{1-r^2}.$$

Thus, the values $h(z)$ are contained in the circle of Apollonius with diameter end points $\frac{1-r}{1-r}$ and $\frac{1+r}{1-r}$ and radius $\frac{2r}{1-r^2}$. Thus, the maximum of $|\arg h(z)|$ is attained at points where tangent ray from origin to the circle can be drawn, that is, when

$$\arg h(z) = \pm \sin^{-1}\left(\frac{2r}{1-r^2}\right).$$

Now,

$$\max_{h \in \mathcal{P}(\beta)} \left| \int_{\theta_1}^{\theta_2} \Re\left(\frac{re^{i\theta} h'\left(re^{i\theta}\right)}{h\left(re^{i\theta}\right)}\right) d\theta \right| \leq 2\sin^{-1}\left(\frac{2r}{1-r^2}\right).$$

This implies that

$$\max_{h \in \mathcal{P}(\beta)} \left| \int_{\theta_1}^{\theta_2} \Re\left(\frac{re^{i\theta} h'\left(re^{i\theta}\right)}{h\left(re^{i\theta}\right)}\right) d\theta \right| \leq \pi - 2\cos^{-1}\left(\frac{2r}{1-r^2}\right). \tag{19}$$

Thus,

$$\int_{\theta_1}^{\theta_2} \Re J(\alpha, f(z))\, \mathrm{d}\theta > -\left(\alpha(1-\beta_1)\left(\frac{m}{2}-1\right)+\sigma\right)\pi + 2\sigma\cos^{-1}\left(\frac{2r}{1-r^2}\right),$$

which gives

$$\int_{\theta_1}^{\theta_2} \Re J(\alpha, f(z))\, \mathrm{d}\theta > -\left(\alpha(1-\beta_1)\left(\frac{m}{2}-1\right)+\sigma\right)\pi, \qquad (r \to 1).$$

This completes the proof. □

For talking $k = 0$, we obtain the integral representation for the class $\mathcal{T}_s^m(\beta)$.

Corollary 5. *Let $f \in \mathcal{T}_s^m(\beta)$. Then, for $z = re^{i\theta}$,*

$$\int_{\theta_1}^{\theta_2} \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) \mathrm{d}\theta > -\sigma\pi,$$

where $0 \leq \theta_1 < \theta_2 \leq 2\pi$.

Theorem 6. *Let $f \in k\text{-}\mathcal{B}_s^m(\alpha, \beta)$. Then, for $\frac{\alpha}{2-\sigma}(m+2)(1-\beta_1) > 1$,*

$$L_r f(z) \leq \begin{cases} C(\alpha, \sigma, m, \beta_1)\, \mathfrak{M}^{1-\alpha}(r) \left(\frac{1}{1-r}\right)^{\alpha\left(\frac{m}{2}+1\right)(1-\beta_1)-1+\sigma}, & 0 < \alpha \leq 1, \\ \\ C(\alpha, \sigma, m, \beta_1)\, \mathfrak{m}^{1-\alpha}(r) \left(\frac{1}{1-r}\right)^{\alpha\left(\frac{m}{2}+1\right)(1-\beta_1)-1+\sigma}, & \alpha > 1, \end{cases}$$

where $\mathfrak{M}(r) = \max_{|z|=r} |f(z)|$, $\mathfrak{m}(r) = \min_{|z|=r} |f(z)|$ and $C(\alpha, \sigma, m, \beta_1)$ is a constant depending upon α, σ, m and β_1 only.

Proof. We know that

$$L_r f(z) = \int_0^{2\pi} \left| z f'(z) \right| d\theta, \ \ z = re^{i\theta}, \ 0 < r < 1.$$

Since $f \in k\text{-}\mathcal{B}_s^m(\alpha, \beta)$, thus

$$\frac{z f'(z)}{f^{1-\alpha}(z)} \left(\frac{2}{g(z) - g(-z)} \right)^{\alpha} = p^{\sigma}(z), \ \ p(z) \in \mathcal{P}.$$

By Theorem 1, we have for $g \in k\text{-}\mathcal{UR}_s^m(\beta)$, the function

$$\phi(z) = \frac{1}{2}\left[g(z) - g(-z) \right] \in k\text{-}\mathcal{UR}^m(\beta),$$

which yields

$$z f'(z) = (f(z))^{1-\alpha} (\phi(z))^{\alpha} p^{\sigma}(z).$$

Therefore, we have

$$\begin{aligned} L_r f(z) &\leq \int_0^{2\pi} |f(z)|^{1-\alpha} |\phi(z)|^{\alpha} |p(z)|^{\sigma} d\theta \\ &\leq \mathfrak{M}^{1-\alpha}(r) \int_0^{2\pi} |\phi(z)|^{\alpha} |p(z)|^{\sigma} d\theta. \end{aligned}$$

Since $\phi(z) \in k\text{-}\mathcal{UR}^m(\beta) \subset \mathcal{R}^m(\beta_1)$, we have

$$\phi(z) = \frac{(s_1(z))^{\left(\frac{m}{4}+\frac{1}{2}\right)}}{(s_2(z))^{\left(\frac{m}{4}-\frac{1}{2}\right)}}, \ \ s_1, s_2 \in k\text{-}\mathcal{UR}^2(\beta).$$

Since $k\text{-}\mathcal{UR}^2(\beta) \subset \mathcal{S}^*(\beta_1)$, so we can write

$$s_i(z) = z \left(\frac{\phi_i(z)}{z} \right)^{1-\beta_1}, \ \ \text{for } i = 1, 2 \text{ and } \phi_i(z) \in \mathcal{S}^*.$$

Thus, for odd functions $s_1(z), s_2(z) \in \mathcal{S}^*(\beta_1)$, we have

$$\begin{aligned} L_r(f(z)) &\leq \mathfrak{M}^{1-\alpha}(r) \int_0^{2\pi} |z|^{\beta_1} \left| \frac{(\phi_1(z))^{(1-\beta_1)\left(\frac{m}{4}+\frac{1}{2}\right)}}{(\phi_2(z))^{(1-\beta_1)\left(\frac{m}{4}-\frac{1}{2}\right)}} \right|^{\alpha} |p(z)|^{\sigma} d\theta \\ &\leq \mathfrak{M}^{1-\alpha}(r) \int_0^{2\pi} \frac{|(\phi_1(z))|^{\alpha\left(\frac{m}{4}+\frac{1}{2}\right)(1-\beta_1)}}{|(\phi_2(z))|^{\alpha\left(\frac{m}{4}-\frac{1}{2}\right)(1-\beta_1)}} |p(z)|^{\sigma} d\theta \\ &\leq \mathfrak{M}^{1-\alpha}(r) \int_0^{2\pi} \frac{2^{\alpha\left(\frac{m}{2}-1\right)(1-\beta_1)}}{r^{\alpha\left(\frac{m}{4}-\frac{1}{2}\right)(1-\beta_1)}} |(\phi_1(z))|^{\alpha\left(\frac{m}{4}+\frac{1}{2}\right)(1-\beta_1)} |p(z)|^{\sigma} d\theta \\ &= \frac{\mathfrak{M}^{1-\alpha}(r)\, 2^{\alpha\left(\frac{m}{2}-1\right)(1-\beta_1)}}{r^{\alpha\left(\frac{m}{4}-\frac{1}{2}\right)(1-\beta_1)}} \int_0^{2\pi} |(\phi_1(z))|^{\alpha\left(\frac{m}{4}+\frac{1}{2}\right)(1-\beta_1)} |p(z)|^{\sigma} d\theta. \end{aligned}$$

Now, by making use of Holder's inequality, with $m_1 = 2/2-\sigma$ and $m_2 = 2/\sigma$ such that $(1/m_1) + (1/m_2) = 1$, we have

$$L_r(f(z)) \leq \frac{\mathfrak{M}^{1-\alpha}(r)\ \pi\, 2^{\alpha\left(\frac{m}{2}-1\right)(1-\beta_1)+1}}{r^{\alpha\left(\frac{m}{4}-\frac{1}{2}\right)(1-\beta_1)}} \left(\frac{1}{2\pi}\int_0^{2\pi} |p(z)|^2\, d\theta\right)^{\frac{\sigma}{2}}$$

$$\times \left(\frac{1}{2\pi}\int_0^{2\pi} |\phi_1(z)|^{\frac{\alpha}{2-\sigma}\left(\frac{m}{2}+1\right)(1-\beta_1)}\, d\theta\right)^{\frac{2-\sigma}{2}}.$$

By using Lemma 2 and distortion results, we obtain

$$L_r(f(z)) \leq \frac{\mathfrak{M}^{1-\alpha}(r)\ \pi\, 2^{\alpha\left(\frac{m}{2}-1\right)(1-\beta_1)+1}}{r^{\alpha\left(\frac{m}{4}-\frac{1}{2}\right)(1-\beta_1)}} \left(\frac{1+3r^2}{1-r^2}\right)^{\frac{\sigma}{2}} \left(\frac{1}{2\pi}\int_0^{2\pi} \frac{r^{\frac{\alpha}{2-\sigma}\left(\frac{m}{2}+1\right)(1-\beta_1)}}{|1-re^{i\theta}|^{\frac{2\alpha}{2-\sigma}\left(\frac{m}{2}+1\right)(1-\beta_1)}}\, d\theta\right)^{\frac{2-\sigma}{2}}$$

$$= \frac{\mathfrak{M}^{1-\alpha}(r)\ \pi\, 2^{\alpha\left(\frac{m}{2}-1\right)(1-\beta_1)+1}}{r^{\alpha\left(\frac{m}{4}-\frac{1}{2}\right)(1-\beta_1)}} r^{\frac{\alpha}{2-\sigma}\left(\frac{m}{2}+1\right)(1-\beta_1)} \left(\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{|1-re^{i\theta}|^{\frac{2\alpha}{2-\sigma}\left(\frac{m}{2}+1\right)(1-\beta_1)}}\, d\theta\right)^{\frac{2-\sigma}{2}} \left(\frac{1+3r^2}{1-r^2}\right)^{\frac{\sigma}{2}}.$$

This implies that

$$\begin{aligned} L_r(f(z)) &\leq \mathfrak{M}^{1-\alpha}(r)\, \pi^{\frac{\sigma}{2}}\, 2^{\alpha\left(\frac{m}{2}-1\right)(1-\beta_1)+1+\sigma} \left(\frac{1}{(1-r)^{\frac{2\alpha}{2-\sigma}\left(\frac{m}{2}+1\right)(1-\beta_1)-1}}\right)^{\frac{2-\sigma}{2}} \left(\frac{1}{1-r}\right)^{\frac{\sigma}{2}} \\ &= \mathfrak{M}^{1-\alpha}(r)\, \pi^{\frac{\sigma}{2}}\, 2^{\alpha\left(\frac{m}{2}-1\right)(1-\beta_1)+1+\sigma} \left(\frac{1}{1-r}\right)^{\left(\frac{2-\sigma}{2}\right)\left(\frac{2\alpha}{2-\sigma}\left(\frac{m}{2}+1\right)(1-\beta_1)-1\right)} \left(\frac{1}{1-r}\right)^{\frac{\sigma}{2}} \\ &= C(\alpha,\sigma,m,\beta_1)\, \mathfrak{M}^{1-\alpha}(r) \left(\frac{1}{1-r}\right)^{\alpha\left(\frac{m}{2}+1\right)(1-\beta_1)-1+\sigma}, \end{aligned}$$

where

$$C(\alpha,\sigma,m,\beta_1) = \pi^{\frac{\sigma}{2}}\, 2^{\alpha\left(\frac{m}{2}-1\right)(1-\beta_1)+1+\sigma}$$

is a constant depending upon α, σ, m and β_1 only. Similarly, for $\alpha > 1$, we have

$$L_r(f(z)) \leq C(\alpha,\sigma,m,\beta_1)\ \mathfrak{m}^{\alpha-1}(r) \left(\frac{1}{1-r}\right)^{\alpha\left(\frac{m}{2}+1\right)(1-\beta_1)-1+\sigma}.$$

□

Theorem 7. *Let $f \in k\text{-}\mathcal{B}_s^m(\alpha,\beta)$. Then, for $n \geq 2$ and $\frac{\alpha}{2-\sigma}(m+2)(1-\beta_1) > 1$,*

$$|a_n| \leq \begin{cases} C_1(\alpha,\sigma,m,\beta_1)\ \mathfrak{M}^{1-\alpha}(n)\,(n)^{\alpha\left(\frac{m}{2}+1\right)(1-\beta_1)-2+\sigma}, & 0 < \alpha \leq 1, \\ C_1(\alpha,\sigma,m,\beta_1)\ \mathfrak{m}^{\alpha-1}(n)\,(n)^{\alpha\left(\frac{m}{2}+1\right)(1-\beta_1)-2+\sigma}, & \alpha > 1, \end{cases}$$

where β_1 is given by (14) and $\mathfrak{m}$, $\mathfrak{M}$ are the same as in Theorem 6 and $C_1(\alpha,\sigma,m,\beta_1)$ is a constant.

Proof. Since $z = re^{i\theta}$, Cauchy theorem gives

$$na_n = \frac{1}{2\pi r^n}\int_0^{2\pi} zf'(z)\, e^{-in\theta}\, d\theta,$$

which reduces to

$$\begin{aligned} |na_n| &= \left| \frac{1}{2\pi r^n} \int_0^{2\pi} zf'(z)\, e^{-in\theta} d\theta \right| \\ &\leq \frac{1}{2\pi r^n} \int_0^{2\pi} \left| zf'(z)\, e^{-in\theta} \right| d\theta. \end{aligned}$$

Therefore,

$$n\,|a_n| \leq \frac{1}{2\pi r^n} L_r f(z).$$

Now, using Theorem 6 for $0 < \alpha \leq 1$, we have

$$n\,|a_n| \leq \frac{1}{2\pi r^n} C(\alpha, \sigma, m, \beta_1)\ \mathfrak{M}^{1-\alpha}(r) \left(\frac{1}{1-r} \right)^{\alpha\left(\frac{m}{2}+1\right)(1-\beta_1)-1+\sigma}.$$

Putting $r = 1 - \frac{1}{n}$, we have

$$|a_n| \leq C_1(\alpha, \sigma, m, \beta_1)\ \mathfrak{M}^{1-\alpha}(r)\,(n)^{\alpha\left(\frac{m}{2}+1\right)(1-\beta_1)-2+\sigma}.$$

Similarly, we obtain the required result for $\alpha > 1$. □

Theorem 8. *Let $f \in k\text{-}\mathcal{B}_s^m(\alpha, \beta)$. Then, for $\frac{\alpha}{2-\sigma}(m+2)(1-\beta_1) > 1$,*

$$||a_{n+1}| - |a_n|| \leq \begin{cases} \mathfrak{M}^{1-\alpha}(r)\ C_2(\alpha, \sigma, m, \beta_1)\,(n)^{\alpha\left(\frac{m}{2}+1\right)(1-\beta_1)+\sigma-3}, & 0 < \alpha \leq 1, \\ \\ \mathfrak{m}^{1-\alpha}(r)\ C_2(\alpha, \sigma, m, \beta_1)\,(n)^{\alpha\left(\frac{m}{2}+1\right)(1-\beta_1)+\sigma-3}, & \alpha > 1, \end{cases}$$

where $\mathfrak{m}(r) = \min_{|z|=r} |f(z)|$, $\mathfrak{M}(r) = \max_{|z|=r} |f(z)|$ and $C_2(\alpha, \sigma, m, \beta_1)$ is a constant depending upon α, σ, m and β_1 only.

Proof. We know that, for $\xi \in \mathfrak{E}$ and $n \geq 1$,

$$|(n+1)\,\xi a_{n+1} - na_n| \leq \frac{1}{2\pi r^{n+1}} \int_0^{2\pi} |z - \xi|\,|zf'(z)|\, d\theta, \quad z = re^{i\theta},\ 0 < r < 1,\ 0 \leq \theta \leq 2\pi.$$

As $f \in k\text{-}\mathcal{B}_s^m(\alpha, \beta)$, thus

$$\frac{zf'(z)}{f(z)} \left(\frac{2f(z)}{g(z) - g(-z)} \right)^{\alpha} = p^{\sigma}(z), \quad p \in \mathcal{P}.$$

From Theorem 4, we have

$$\phi(z) = \frac{1}{2}\,[g(z) - g(-z)] \in k - \mathcal{UR}^m(\beta) \qquad \text{for } g \in k - \mathcal{UR}_s^m(\beta).$$

This leads us to

$$zf'(z) = (f(z))^{1-\alpha}\,(\phi(z))^{\alpha}\,p^{\sigma}(z).$$

Thus, for $\xi \in \mathfrak{E}$ and $n \geq 1$, we have

$$\left|(n+1)\,\xi a_{n+1} - n a_n\right| \leq \frac{M^{1-\alpha}(r)}{2\pi r^{n+1}} \int_0^{2\pi} |z-\xi|\,|\phi(z)|^{\alpha}\,|p(z)|^{\sigma}\,d\theta.$$

Since $\phi(z) \in k\text{-}\mathcal{UR}^m(\beta) \subset \mathcal{R}^m(\beta_1)$, therefore, for $\phi_1(z), \phi_2(z) \in \mathcal{S}^*$, we have

$$\begin{aligned} \left|\xi a_{n+1}(n+1) - n a_n\right| &\leq \frac{M^{1-\alpha}(r)}{2\pi r^{n+1}} \int_0^{2\pi} |z|^{\alpha\beta_1}\,|z-\xi| \left| \frac{(\phi_1(z))^{(1-\beta_1)\left(\frac{m}{4}+\frac{1}{2}\right)}}{(\phi_2(z))^{(1-\beta_1)\left(\frac{m}{4}-\frac{1}{2}\right)}} \right|^{\alpha} |p(z)|^{\sigma}\,d\theta \\ &\leq \frac{M^{1-\alpha}(r)}{2\pi r^{n+1}} \int_0^{2\pi} |z-\xi|\, \frac{|(\phi_1(z))|^{\alpha(1-\beta_1)\left(\frac{m}{4}+\frac{1}{2}\right)}}{|(\phi_2(z))|^{\alpha(1-\beta_1)\left(\frac{m}{4}-\frac{1}{2}\right)}}\,|p(z)|^{\sigma}\,d\theta. \end{aligned}$$

Using Lemma 3(i), we have

$$\begin{aligned} &\left|(n+1)\,\xi a_{n+1} - n a_n\right| \leq \frac{2^{\alpha\left(\frac{m}{2}-1\right)(1-\beta_1)} M^{1-\alpha}(r)}{2\pi r^{n+1+\alpha\left(\frac{m}{4}-\frac{1}{2}\right)(1-\beta_1)}} \\ &\times \int_0^{2\pi} |z-\xi|\,|(\phi_1(z))|\,|(\phi_1(z))|^{\alpha(1-\beta_1)\left(\frac{m}{4}+\frac{1}{2}\right)-1}\,|p(z)|^{\sigma}\mathrm{d}\theta. \end{aligned}$$

Now, using Lemma 3(ii), we have

$$\left|(n+1)\,\xi a_{n+1} - n a_n\right| \leq \frac{2^{\alpha\left(\frac{m}{2}-1\right)(1-\beta_1)} M^{1-\alpha}(r)}{2\pi r^{n-1+\alpha\left(\frac{m}{4}-\frac{1}{2}\right)(1-\beta_1)}(1-r)} \int_0^{2\pi} |(\phi_1(z))|^{\alpha(1-\beta_1)\left(\frac{m}{4}+\frac{1}{2}\right)-1}\,|p(z)|^{\sigma}\mathrm{d}\theta.$$

Now, using Cauchy–Schwarz inequality, we have

$$\begin{aligned} \left|(n+1)\,\xi a_{n+1} - n a_n\right| &\leq \frac{2^{\alpha\left(\frac{m}{2}-1\right)(1-\beta_1)} M^{1-\alpha}(r)}{r^{n-1+\alpha\left(\frac{m}{4}-\frac{1}{2}\right)(1-\beta_1)}(1-r)} \left(\frac{1}{2\pi}\int_0^{2\pi} |p(z)|^2\,d\theta\right)^{\frac{\sigma}{2}} \\ &\times \left(\frac{1}{2\pi}\int_0^{2\pi} |(\phi_1(z))|^{\frac{\alpha(1-\beta_1)\left(\frac{m}{2}+1\right)-2}{2-\sigma}}\,d\theta\right)^{\frac{2-\sigma}{2}}. \end{aligned}$$

By using Lemma 2 and distortion results, we obtain

$$\begin{aligned} &\left|(n+1)\,\xi a_{n+1} - n a_n\right| \leq \frac{2^{\alpha\left(\frac{m}{2}-1\right)(1-\beta_1)} M^{1-\alpha}(r)}{r^{n-1+\alpha\left(\frac{m}{4}-\frac{1}{2}\right)(1-\beta_1)}(1-r)} \left(\frac{1+3r^2}{1-r^2}\right)^{\frac{\sigma}{2}} \\ &\times \left(\frac{1}{2\pi}\int_0^{2\pi} \frac{r^{\frac{1}{2-\sigma}\left\{\alpha\left(\frac{m}{2}+1\right)(1-\beta_1)-2\right\}}}{|1-re^{i\theta}|^{\frac{2}{2-\sigma}\left\{\left(\alpha\frac{m}{2}+1\right)(1-\beta_1)-2\right\}}}\,d\theta\right)^{\frac{2-\sigma}{2}} \\ &\leq \frac{C_2(\alpha,\sigma,m,\beta_1) M^{1-\alpha}(r) r^{\alpha(1-\beta_1)-n}}{(1-r)^{1+\frac{\sigma}{2}}} \left(\frac{1}{(1-re^{i\theta})^{\frac{2}{2-\sigma}\left\{\alpha\left(\frac{m}{2}+1\right)(1-\beta_1)-2\right\}-1}}\right)^{\frac{2-\sigma}{2}} \\ &\leq \frac{C_2(\alpha,\sigma,m,\beta_1) M^{1-\alpha}(r)}{r^{n-1}(1-r)^{1+\frac{\sigma}{2}}} \left(\frac{1}{(1-r)^{\alpha\left(\frac{m}{2}+1\right)(1-\beta_1)+\frac{\sigma}{2}-3}}\right) \\ &\leq \frac{C_2(\alpha,\sigma,m,\beta_1) M^{1-\alpha}(r)}{r^{n-1}(1-r)^{\alpha\left(\frac{m}{2}+1\right)(1-\beta_1)+\sigma-2}}, \end{aligned}$$

where $C_2(\alpha,\sigma,m,\beta_1)$ is a constant. Now, putting $|\xi| = \frac{n}{n+1}$, we obtain

$$n\,\big||a_{n+1}| - |a_n|\big| \leq \frac{C_2(\alpha,\sigma,m,\beta_1)\,M^{1-\alpha}(r)}{r^{n-1}(1-r)^{\alpha\left(\frac{m}{2}+1\right)(1-\beta_1)+\sigma-2}}.$$

Now, taking $r = 1 - \frac{1}{n}$ $(n \to \infty)$,we have

$$C_2(\alpha, \sigma, m, \beta_1)\, \mathfrak{M}^{1-\alpha}(r)\,(n)^{\alpha\left(\frac{m}{2}+1\right)(1-\beta_1)+\sigma-3}, \quad 0 < \alpha \leq 1.$$

Similarly for $\alpha > 1$, we have

$$||a_{n+1}| - |a_n|| \leq C_2(\alpha, \sigma, m, \beta_1)\ m^{\alpha-1}(r)\,(n)^{\alpha\left(\frac{m}{2}+1\right)(1-\beta_1)+\sigma-3}.$$

Thus, the result follows. □

Theorem 9. *Let $f \in k\text{-}\mathcal{B}_s^m(\alpha, \beta)$ for $\alpha > 0$. Then, $f(z)$ is $\frac{1}{\alpha}$−convex for $|z| < r_m^*$,*

$$r_m^* = \frac{2\alpha}{(\alpha m + 2\sigma - \alpha\beta_1 m) + \sqrt{(\alpha m + 2\sigma - \alpha\beta_1 m)^2 - 4\alpha^2(1 - 2\beta_1)}}, \quad \alpha > 0.$$

Proof. Let

$$zf'(z) = (f(z))^{1-\alpha}(\phi(z))^{\alpha} h^{\sigma}(z),$$

where $\frac{g(z)-g(-z)}{2} = \phi(z) \in k\text{-}\mathcal{UR}^m(\beta) \subset \mathcal{R}^m(\beta_1)$ and $h(z) \in \mathcal{P}$. Differentiating logarithmically, we obtain

$$\frac{1}{\alpha}\left(\frac{(zf'(z))'}{f'(z)}\right) + \left(1 - \frac{1}{\alpha}\right)\frac{zf'(z)}{f(z)} = \frac{z\phi'(z)}{\phi(z)} + \frac{\sigma}{\alpha}\frac{zh'(z)}{h(z)}.$$

We can write

$$\begin{aligned} \Re\left\{\frac{1}{\alpha}\left(\frac{(zf'(z))'}{f'(z)}\right) + \left(1 - \frac{1}{\alpha}\right)\frac{zf'(z)}{f(z)}\right\} &= \Re\left(\frac{z\phi'(z)}{\phi(z)}\right) + \frac{\sigma}{\alpha}\Re\left(\frac{zh'(z)}{h(z)}\right) \\ &> \Re\left(\frac{z\phi'(z)}{\phi(z)}\right) - \frac{\sigma}{\alpha}\left|\frac{zh'(z)}{h(z)}\right|. \end{aligned}$$

Now, using the distortion results for the classes $\mathcal{R}^m(\beta_1)$ and $\mathcal{P}$, we have

$$\begin{aligned} \Re\left\{\frac{1}{\alpha}\left(\frac{(zf'(z))'}{f'(z)}\right) + \left(1 - \frac{1}{\alpha}\right)\frac{zf'(z)}{f(z)}\right\} &\geq \beta_1 + \frac{(1-\beta_1)\left(1 - mr + r^2\right)}{1 - r^2} - \frac{2\sigma r}{\alpha(1 - r^2)} \\ &= \frac{\alpha\beta_1(1 - r^2) + \alpha(1 - \beta_1)\left(1 - mr + r^2\right) - 2\sigma r}{\alpha(1 - r^2)} \\ &\geq \frac{\alpha(1 - 2\beta_1) r^2 - (\alpha m + 2\sigma - \alpha\beta_1 m) r + \alpha}{\alpha(1 - r^2)}, \end{aligned}$$

taking

$$\alpha(1 - 2\beta_1) r^2 - (\alpha m + 2\sigma - \alpha\beta_1 m) r + \alpha = 0,$$

$$r_m^* = \frac{(\alpha m + 2\sigma - \alpha\beta_1 m) \pm \sqrt{(\alpha m + 2\sigma - \alpha\beta_1 m)^2 - 4\alpha^2(1 - 2\beta_1)}}{2\alpha(1 - 2\beta_1)}.$$

Since $0 \leq r < 1$,

$$\begin{aligned} r_m^* &= \frac{(\alpha m + 2\sigma - \alpha\beta_1 m) - \sqrt{(\alpha m + 2\sigma - \alpha\beta_1 m)^2 - 4\alpha^2(1 - 2\beta_1)}}{2\alpha(1 - 2\beta_1)} \\ &= \frac{2\alpha}{(\alpha m + 2\sigma - \alpha\beta_1 m) + \sqrt{(\alpha m + 2\sigma - \alpha\beta_1 m)^2 - 4\alpha^2(1 - 2\beta_1)}}, \quad \alpha > 0. \end{aligned}$$

This completes the proof. □

4. Conclusions

In this article, we have presented certain analytic functions defined by bounded radius rotations associated with conic domain. We have investigated many geometric properties like coefficient estimate, radii problems, arc length, integral representation, inclusion results and growth rate of coefficients of Taylor's series representation. By varying the parameters in results, several well-known results in literature have been shown as special cases.

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