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DMS-736-IR May 1996

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Abstract

The anisotropic Manev problem describes the motion of two bodies in an Euclidean space in which the gravitational force acts differently in each direction. The potential is the sum between the inverse and the inverse square of the distance, where the distance is defined such that it embodies the anisotropy of the space. Using McGehee coordinates, we blow up the collision singularity, paste a collision manifold to the phase space, study the flow on and near the collision manifold, and find a rich set of collision orbits having positive measure. In the zero-energy case we describe all possible connections between equilibria and/or cycles at collision and at infinity and find the main qualitative features of the flow.

1. Introduction

The type of anisotropic problems we tackle in this paper have been first defined by Gutzwiller in the 1970s (see [G,1970], [G,1973], [G,1977]) to find connections between classical and quantum mechanics. Gutzwiller considered the anisotropic Kepler problem, which was later extensively analyzed by Devaney [De,1978], [De,1981]. Diacu also introduced an anisotropic model for the Manev three-body problem [Di,1993]. Atela [A,1988] defined a charged isosceles three-body problem, which contains the anisotropy in a particular case. Here we add to these problems the anisotropic Manev two-body problem, which we will call for short the *anisotropic Manev problem.*

The name Manev (or Maneff in French and German spelling) is connected to a gravitational model defined by a potential of the form $\alpha/r+\beta/r^2$, where r is the distance between particles and $\alpha, \beta > 0$ are constants (see [M,1924], [M,1925], [M,1930a], [M,1930b]). In fact this potential goes back to Newton, who first tackled it in *Principia.* In Book I, Article IX, Proposition XLIV, Theorem XIV, Corollary 2, Newton claims that such a potential determines a *precesionally elliptic* orbit. In terms of a central-force problem, this means that a particle moves on an ellipse that rotates in its plane of motion. This also occurs as a problem in Goldstein's *Classical Mechanics* text. A formula for the solution is easy to obtain and was known for a long time, but its complete physical picture was only recently understood in [DD,1996], a paper that uses McGehee transformations and the qualitative theory of dynamical systems. The advantage of Manev's model over the Newtonian one is that it explains the perihelion advance of the inner planets with the same precision as relativity (see [H,1975]).

Combining Gutzwiller's anisotropy with Manev's potential, we were led to the anisotropic Manev problem. In this paper, though far from obtaining a complete picture of the global flow, we settle some local and global questions and point out the main differences between this and the anisotropic Kepler problem. We will tackle the flow near collision and in the zero-energy case. Global aspects of the flow for negative-energy and positive-energy cases will be treated somewhere else.

In Section 2 we define the problem and obtain the equations of motion. Then, in Section 3, we put into the evidence the symmetries and note that they are similar to those of the anisotropic Kepler problem. In Section 4 we blow up the singularities by using McGehee transformations and paste to the phase-space the so-called *collision manifold*, which is homeomorphic to a torus. Then we find out that the flow on the collision manifold is formed only by periodic orbits except the eight equilibria and the eight heteroclinic orbits that connect some of the equilibria (see Figure 2).

In Section 5 we study the flow near the collision manifold and obtain the first main result that shows a sensitive difference between the anisotropic Manev and Kepler problems. Using a first-return-map argument we prove that for each periodic orbit belonging to the upper/lower part of the collision manifold, there is a local two-dimensional analytic manifold of orbits ejecting/tending from/to it. The only periodic orbit for which both types of manifolds occur is the middle one, which separates the upper and lower set of periodic orbits. Physically, these manifolds correspond to solutions that eject/tend from/to a binary collision such that one particle spirals around the other infinitely many times after/before contact. Also, for each upper/lower equilibrium there is a local one-dimensional unstable/stable analytic manifold outside the collision manifold. Physically, these manifolds correspond to solutions that eject/tend from/to collision with zero angular momentum, i.e. such that their trajectory at the collision instant has a common tangent.

In Section 6 we begin the study of the flow in the zero-energy case. To determine the asymptotic behavior of the motion at infinity we define the so-called *infinity manifold*, which we paste, by suitable transformations, to the phase space. Then we see that the flows on the zero-energy manifold and on the infinity manifold are *gradient-like*, this meaning here that they are increasing with respect to one of the variables. We find out that the infinity manifold has also eight equilibria (see Figure 3) and show that there exist eight homothetic orbits connecting pairwise the lower and the upper equilibria of the collision manifold and the infinity manifold, respectively (see Figure 4). In Section 7 we determine the local nature of the equilibria of the zero-energy manifold and see that some of them are hyperbolic, whereas the others are nonhyperbolic. This information is of help in Section 8, in which we describe the flow in the zero-energy case. We find all possible connecting orbits between equilibria and/or cycles, give their physical interpretation and trace the main features of the flow in the zero-energy case.

2. Equations of motion

Consider the two-degrees-of-freedom Hamiltonian system of ordinary differential equations given by

$$\begin{cases} \dot{\mathbf{q}} = \mathbf{p} \\ \dot{\mathbf{p}} = \nabla W(\mathbf{q}), \end{cases} \tag{2.1}$$

where $\mathbf{q} = (q_1, q_2)$ and $\mathbf{p} = (p_1, p_2)$ denote the *configuration* and the *momentum* of a physical system of two particles. W is what we call the *quasihomogeneous anisotropic potential*, given by

$$W(\mathbf{q}) = \frac{1}{\sqrt{\mu q_1^2 + q_2^2}} + \frac{b}{\mu q_1^2 + q_2^2},$$

where $\mu > 0$ and $b > 0$ are parameters. In the astronomical applications of the classical Manev problem the parameter b is considered very small: of the order 10^{-10}. The equations (2.1) define the motion of two particles of unit mass in an anisotropic space, i.e. a space in which the attraction forces act differently in every direction. The above potential defines the anisotropy of the space as a function of the parameter μ. If $\mu < 1$, the attraction is the weakest in the direction of the q_1-axis and the strongest in that of the q_2-axis, the situation being reversed if $\mu > 1$. If $\mu = 1$, the space is isotropic and we are in the case of the classical Manev problem, whose global phase-space structure was completely described in [DD,1995]; therefore we will not deal with it here. Since both remaining cases have a weakest-force and a strongest-force direction, we can assume, without loss of generality, that $\mu > 1$.

The Hamiltonian function of the system (2.1) is given by

$$H\big(\mathbf{p}(t), \mathbf{q}(t)\big) = (1/2)\|\mathbf{p}(t)\|^2 - W\big(\mathbf{q}(t)\big),$$

the sum of the kinetic and potential energies, which yields the integral of energy

$$H(\mathbf{p}(t), \mathbf{q}(t)) = h. \tag{2.2}$$

However, since the force ∇W is not central, the angular momentum $L(t) = \|\mathbf{p}(t) \times \mathbf{q}(t)\|$ is *not* an integral of the system, as it is in the classical (nonanisotropic) Manev problem (see [DD,1995], [DM,1995]).

As $W: (\mathbb{R}\setminus\{\mathbf{0}\}) \to \mathbb{R}$ is real analytic, for any initial data $(\mathbf{q}(0), \mathbf{p}(0)) \in (\mathbb{R}\setminus\{\mathbf{0}\}) \times \mathbb{R}$, standard results of differential-equation theory guarantee the existence and uniqueness of an analytic solution defined on a maximal interval $[0, t^*)$, where $0 < t^* \leq \infty$. If $t^* < \infty$, the solution is said to experience a *singularity*.

A particular type of singularity, called a *collision*, occurs when $\mathbf{q}(t) \to \mathbf{0}$ as $t \to t^*$. In fact, by imitating the proof used in the classical Kepler problem (see e.g. [W,1941]), we can show that in the anisotropic Manev problem all singularities are collisions. Solutions leading to collisions as well as those coming close to collisions are of particular interest because the whole qualitative structure of the phase space depends on their behavior. We will study these solutions starting with Section 4. The next section is devoted to the study of the symmetries of our problem.

3. Symmetries

The symmetries in the anisotropic Manev problem are the same as in the anisotropic Kepler problem. The elements of the group $< S_0, S_1, S_2 >$, generated by S_0, S_1, and S_2 maps solutions of the anisotropic Manev problem into solutions. The symmetries of the generating elements are given by the formulas:

$$S_0(q_1, q_2, p_1, p_2, t) = (q_1, q_2, -p_1, -p_2, -t),$$

$$S_1(q_1, q_2, p_1, p_2, t) = (q_1, -q_2, -p_1, p_2, -t),$$

$$S_2(q_1, q_2, p_1, p_2, t) = (-q_1, q_2, p_1, -p_2, -t).$$

Notice that the symmetry S_0 implies the reversibility of the flow.

Invariant sets are those which remain invariant under the flow, i.e. if the initial condition is in an invariant set, then the whole solution is in this set. Like in [CL,1984] we can prove that the above group of symmetries defines exactly two invariant planes for the anisotropic Manev problem. These planes are

$$\Pi_1 = \{(q_1, 0, p_1, 0) | (q_1, p_1) \in (\mathbb{R} \setminus \{0\}) \times \mathbb{R}\},$$

$$\Pi_2 = \{(0, q_2, 0, p_2) | (q_2, p_2) \in (\mathbb{R} \setminus \{0\}) \times \mathbb{R}\}.$$

In each of these invariant planes the flow is given by a Hamiltonian system of degree 1. For Π_1 the Hamiltonian function is $H_1(q_1, p_1) = p_1^2/2 - 1/|q_1| - b/q_1^2$, and for Π_2 it is $H_2(q_2, p_2) = p_2^2/2 - 1/\sqrt{\mu}|q_1| - b/\mu q_1^2$. The qualitative structure of the flow in each of these invariant planes is the same, as Figure 1 shows.

Figure 1
The phase-space picture in the invariant plane Π_i.

In Figure 1, the plane is divided in two regions by the curves $p_1^2/2 - 1/|q_1| - b/q_1^2 = 0$ and $p_2^2/2 - 1/\sqrt{\mu}|q_1| - b/\mu q_1^2 = 0$, which represent the case $h = 0$ for Π_1 and Π_2, respectively, where h is the energy constant. The outside region consists of solutions with $h > 0$, whereas the inside region is filled with curves representing solutions with $h < 0$. Each curve in the positive-energy region follows an asymptote; this is either $p_i = \sqrt{2h}$, if the curve belongs to the half-plane $p_i > 0$, or $p_i = -\sqrt{2h}$, if the curve is in the half-plane $p_i < 0$. The intersections of the negative-energy curves with the q_i-axis are given by

$$Q_{1h}^- = (-1 + \sqrt{1 - 4bh})/2h, \quad Q_{1h}^+ = (-1 - \sqrt{1 - 4bh})/2h,$$

$$Q_{2h}^- = (-1 + \sqrt{1 - 4bh})/2h\sqrt{\mu}, \quad Q_{2h}^+ = (-1 - \sqrt{1 - 4bh})/2h\sqrt{\mu}.$$

4. The collision manifold

In the study of collision and near-collision solutions it is helpful to transform the system (2.1) using a method developed by McGehee [Mc,1974]. Define first the transformations of the dependent variables (space coordinates):

$$\begin{cases} r = \|\mathbf{q}\| \\ \theta = \arctan(q_2/q_1) \\ y = \dot{r} = (q_1p_1 + q_2p_2)/\|\mathbf{q}\| \\ x = r\dot{\theta} = (q_1p_2 - q_2p_1)/\|\mathbf{q}\|, \end{cases}$$

and

$$\begin{cases} v = ry \\ u = rx, \end{cases}$$

and then consider a transformation of the independent variable (time):

$$d\tau = r^{-2}dt.$$

Composing these transformations, which are analytic diffeomorphisms, the energy relation (2.2) becomes:

$$u^2 + v^2 - 2r\Delta^{-1/2} - 2b\Delta^{-1} = 2r^2h, \tag{4.1}$$

and the equations of motion (2.1) take the form:

$$\begin{cases} r' = rv \\ v' = 2r^2h + r\Delta^{-1/2} \\ \theta' = u \\ u' = (1/2)(\mu - 1)(r\Delta^{-3/2} + 2b\Delta^{-2})\sin 2\theta, \end{cases} \tag{4.2}$$

where $\Delta = \mu\cos^2\theta + \sin^2\theta$. The new variables $(r, v, \theta, u) \in (0, \infty) \times \mathbb{R} \times S^1 \times \mathbb{R}$ depend on the fictitious time τ, so the prime denotes here differentiation with respect to the new independent variable τ.

Composed with the McGehee transformations, the symmetries S_0, S_1, S_2 are changed into $\bar{S}_0, \bar{S}_1, \bar{S}_2$, where

$$\bar{S}_0(r, v, \theta, u) = (r, -v, \theta, -u),$$

$$\bar{S}_1(r, v, \theta, u) = (r, -v, -\theta, u),$$

$$\bar{S}_2(r, v, \theta, u) = (r, -v, \pi - \theta, u).$$

Notice that the sets $\{(r, v, \theta, u) \mid r = 0\}$ and $\{(r, v, \theta, u) \mid r > 0\}$ are invariant manifolds for the equations (4.2). The set

$$C = \{(r, v, \theta, u) \mid r = 0 \text{ and the energy relation (4.1) holds}\}$$

is called the *collision-ejection manifold* or simply the *collision manifold*. It replaces the set of singularities $\{(\mathbf{q}, \mathbf{p}) \mid \mathbf{q} = \mathbf{0}\}$ of the original system (2.1), with a two-dimensional manifold in the space of the new variables. This manifold is embedded in $\mathbb{R}^3 \times S^1$ and is given by the equations

$$r = 0 \quad \text{and} \quad u^2 + v^2 = 2b\Delta^{-1}. \tag{4.3}$$

This shows that C is homeomorphic to a torus (see Figure 2, in which the left and right circles are identified).

Figure 2
The flow on the collision manifold

The flow of the system (4.2) on any *constant energy surface* $\mathcal{E}_h$,

$$\mathcal{E}_h = \{(r, v, \theta, u) \mid r > 0 \text{ and the energy relation (4.1) holds}\},$$

can be extended analytically to C, where the transformed system does not have singularities. The flow on C is fictitious, in the sense that it has no physical meaning, but—due to the continuity of the solutions with respect to the initial data—its structure gives information about the behavior of the nearby flow on constant energy manifolds, that is, information about collision and near-collision solutions.

The restriction of the equations (4.2) to C yields the system

$$\begin{cases} v' = 0 \\ \theta' = u \\ u' = b(\mu - 1)\Delta^{-2}\sin 2\theta, \end{cases} \tag{4.4}$$

on the manifold C. Since $v' = 0$, the solutions of (4.4) lie on the level curves $v = \textit{constant}$ of the torus C.

There are eight *equilibria* (i.e. solutions constant in time) for the equations (4.2). In the variables (r, v, θ, u), the first four of these equilibria are: $A_0^\pm = (0, \pm\sqrt{2b/\mu}, 0, 0)$ and $A_\pi^\pm = (0, \pm\sqrt{2b/\mu}, \pi, 0)$. At these points the linearized system has the matrix

$$\begin{bmatrix} \pm\sqrt{2b/\mu} & 0 & 0 & 0 \\ 1/\sqrt{\mu} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2b(\mu-1)/\mu^2 & 0 \end{bmatrix},$$

the corresponding eigenvalues being real and taking the values: $\pm\sqrt{2b/\mu}$, 0 , $\sqrt{2b(\mu-1)}/\mu$ and $-\sqrt{2b(\mu-1)}/\mu$. The other four equilibria are $A_{\pm\pi/2}^\pm = (\pm\sqrt{2b}, \pm\pi/2, 0)$ and the linearized system at these points is given by the matrix

$$\begin{bmatrix} \pm\sqrt{2b} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2b(1-\mu) & 0 \end{bmatrix},$$

the corresponding eigenvalues being $\pm\sqrt{2b}$, 0 , $\sqrt{2b(1-\mu)}$ and $-\sqrt{2b(1-\mu)}$. Since $\mu > 1$, the last two eigenvalues are purely imaginary.

There are eight *heteroclinic* orbits (i.e. orbits connecting two distinct equilibria) which lie in the level sets $v = \pm\sqrt{2b/\mu}$. All the other solutions are periodic (see Figure 2). Hence the structure of the flow on the collision manifold is fairly simple but it differs from that of the anisotropic Kepler problem (compare with [CL,1984]).

Our next goal will be to understand the flow and the physical behavior of the motion near the collision manifold. We will see that the flow outside the collision manifold also differs drastically from the flow of the anisotropic Kepler problem.

5. The Flow Near the Collision Manifold

For a fixed value of the constant h, the constant-energy surface $\mathcal{E}_h$ is a three-dimensional manifold, invariant under the flow of the system (4.2), embedded in the four-dimensional (r, θ, u, v)-space, and whose boundary is the two-dimensional collision manifold C. To describe the global flow, it is sufficient to describe the flow on the energy surface $\mathcal{E}_h$, for every value of the constant h. This is a difficult task. In this section we will deal only with the local behavior of the flow in the neighborhood of the collision manifold. Let us denote by P_η the periodic orbit on C having $v = \eta$, i.e.

$$P_\eta = \{(r, v, \theta, u) | r = 0, v = \eta\}.$$

Notice that for each $\eta \in (\sqrt{2b/\mu}, \sqrt{2b}) \cup (-\sqrt{2b}, -\sqrt{2b/\mu})$ there are in fact two periodic orbits (see Figure 2). However, as long as there is no confusion, we will denote each of them by the same P_η.

We can now prove the following result:

Theorem 5.1. *On the collision manifold C the equilibria $A_0^{\pm}$ and $A_\pi^{\pm}$ are saddles, whereas the equilibria $A_{\pm\pi/2}^{\pm}$ are centers. Outside the collision manifold C the equilibria $A_0^+, A_{\pm\pi/2}^+$, and A_π^+ have a one-dimensional unstable analytic manifold, whereas the equilibria $A_0^-, A_{\pm\pi/2}^-$, and A_π^- have a one-dimensional stable analytic manifold. Each periodic orbit P_η on C with $v = \eta > 0$ ($v = \eta < 0$) has a two-dimensional local unstable (stable) analytic manifold, while the periodic orbit $v = 0$ has both a two-dimensional local unstable and a two-dimensional local stable analytic manifold.*

Physical interpretation. Before proceeding with the proof, we will give the physical interpretation of the solutions described above. As we mentioned earlier, the orbits on the collision manifold have no physical meaning, so we will deal only with those existing outside the collision manifold. The solutions tending/ejecting toward/from the equilibria represent collision/ejecting orbits whose orbits have a common tangent at collision; we will call them *frontal collisions/ejections* or just *frontal collisions*, for short. The solutions tending/ejecting toward/from the cycles represent collision/ejecting orbits that spiral infinitely many times before/after collision/ejection; we will call them *spiraling collisions/ejections* or just *spiraling collisions*, for short.

Proof: The part of the theorem concerning the equilibria is obvious from the study of the eigenvalues done at the end of the previous section, so we need to deal now only with the periodic solutions P_η. For this we will distinguish between two different cases. We will first deal with the periodic orbits that go around the whole collision-manifold torus, i.e. those periodic orbits with $|\eta| < \sqrt{2b/\mu}$, and then with the ones orbits circling only the "bumps" of the collision manifold, i.e. those for which $\sqrt{2b/\mu} < |\eta| < \sqrt{2b}$. In each case we will construct the *first return map* and understand its qualitative behavior.

The first and second equations of (4.2) show that for small values of r, the variable v is increasing. Also notice that r is decreasing in the region $v < 0$ and is increasing in the region $v > 0$.

We first consider the case when $0 < \eta < \sqrt{2b/\mu}$. Let us fix such an η and an initial value for $\theta(0) = \theta_0$. Then, by the continuity of the flow, there is a neighborhood V of $(r, v) = (0, \eta)$ in $[0, \infty) \times \mathbb{R}$ such that for every solution with initial conditions $\theta(0) = \theta_0$, $(r_0, v_0) \in V$, θ strictly increases until reaching $\theta_0 + 2\pi$, and $\eta/2 < v(t) < \frac{1}{2}(\eta + \sqrt{2b/\mu})$ until θ does so. For these solutions, $\theta(t)$ can be treated as the independent variable, in which case the system (4.2) is equivalent to the *nonautonomous* system

$$\begin{cases} \dfrac{dr}{d\theta} = \dfrac{r'}{\theta'} = \dfrac{rv}{\sqrt{2r^2h + 2r\Delta^{-1/2} + 2b\Delta^{-1} - v^2}} \\[2ex] \dfrac{dv}{d\theta} = \dfrac{v'}{\theta'} = \dfrac{2r^2h + r\Delta^{-1/2}}{\sqrt{2r^2h + 2r\Delta^{-1/2} + 2b\Delta^{-1} - v^2}}, \end{cases} \tag{5.1}$$

where $u(\theta)$ is recovered using the energy relation (4.1).

The solutions to (5.1) form an analytic function

$$\Psi : V \times (\theta_0 - \epsilon, \theta_0 + 2\pi + \epsilon) \longrightarrow \mathbb{R}^2, \quad (r_0, v_0, \theta) \longmapsto (r(\theta), v(\theta)),$$

where $r(\theta)$ and $v(\theta)$ are the solutions determined by r_0, v_0 and θ_0. By analyticity, this function can be expanded in a power series in θ,

$$\Psi(r_0, v_0, \theta) = \begin{bmatrix} r_0 + \frac{dr}{d\theta}\mid_{(r_0,v_0)} (\theta - \theta_0) + \frac{1}{2}\frac{d^2r}{d\theta^2}\mid_{(r_0,v_0)} (\theta - \theta_0)^2 + \dots \\ v_0 + \frac{dv}{d\theta}\mid_{(r_0,v_0)} (\theta - \theta_0) + \frac{1}{2}\frac{d^2v}{d\theta^2}\mid_{(r_0,v_0)} (\theta - \theta_0)^2 + \dots \end{bmatrix}.$$

The first return map ψ is given by $\psi(r_0, v_0) = \Psi(r_0, v_0, \theta_0 + 2\pi)$. Explicitly,

$$\psi(r_0, v_0) = \begin{bmatrix} \psi_1(r_0, v_0) \\ \psi_2(r_0, v_0) \end{bmatrix} = \begin{bmatrix} r_0 + 2\pi\frac{dr}{d\theta}\mid_{(r_0,v_0)} + 2\pi^2\frac{d^2r}{d\theta^2}\mid_{(r_0,v_0)} + \dots \\ v_0 + 2\pi\frac{dv}{d\theta}\mid_{(r_0,v_0)} + 2\pi^2\frac{d^2v}{d\theta^2}\mid_{(r_0,v_0)} + \dots \end{bmatrix}. \quad (5.2)$$

Of interest are the eigenvalues of $D\psi|_{(0,\eta)}$, the total derivative of ψ at the fixed point $(0, \eta)$. By the chain rule, since $\frac{dr}{d\theta}$ and $\frac{dv}{d\theta}$ are multiples of r, all their higher derivatives will be as well. This property ensures that the derivative has the form

$$D\psi(0, \eta) = \begin{bmatrix} \frac{\partial\psi_1}{\partial r}(0, \eta) & 0 \\ \frac{\partial\psi_2}{\partial r}(0, \eta) & 1 \end{bmatrix}, \quad (5.3)$$

where $\frac{\partial\psi_1}{\partial r}(0, \eta)$ and $\frac{\partial\psi_2}{\partial r}(0, \eta)$ remain to be determined. The zero entry ensures that the eigenvalues of $D\psi|_{(0,\eta)}$ are 1 and $\frac{\partial\psi_1}{\partial r}(0, \eta)$. To draw the desired conclusion, we need an estimate on $\frac{\partial\psi_1}{\partial r}(0, \eta)$.

Now, the denominators in (5.1) are bounded on $V \times (\theta_0 - \epsilon, \theta_0 + 2\pi + \epsilon)$, so there exists an $M > 0$ such that

$$\frac{dr}{d\theta} \geq Mr(\theta)v(\theta) \geq \frac{M\eta}{2}r(\theta), \quad (5.4)$$

for $\theta \in [0, 2\pi]$ and for all solutions $\big(r(\theta), v(\theta)\big)$ starting in V. Integration with respect to θ from 0 to 2π yields the inequality

$$\psi_1(r_0, v_0) \geq r_0 e^{M\pi v_0},$$

for all $(r_0, v_0) \in V$.

We will need the following Tauberian lemma: *If $f(x) > Kx$, where K is a constant, and $f(0) = 0$, then $f'(0) \geq K$.* This is obvious since $f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h} \geq K$.

Fix now v_0. Then, since $\Psi_1(0, v_0) = 0 = 0 \cdot e^{M\pi v_0}$, by the above Tauberian lemma the slope of the curve $r_0 \longmapsto \psi_1(r_0, v_0)$ at $r_0 = 0$ is greater than that of $r_0 \longmapsto e^{M\pi v_0} \cdot r_0$. This is equivalent to

$$\frac{\partial}{\partial r}\psi_1|_{(0,v_0)} \geq e^{M\pi v_0},$$

which is greater than 1 for all $(r, v) \in V$. Thus, the periodic orbit P_η has a two-dimensional unstable manifold of orbits ejecting from it outside C. A similar argument shows that for $-\sqrt{2b/\mu} < \eta < 0$, we have

$$\frac{\partial\psi_1}{\partial r}(0, \eta) \leq e^{M\eta\pi} < 1, \quad (5.5)$$

and so the corresponding periodic orbit P_η has a *stable manifold* of orbits approaching it. These orbits represent solutions in which a collision (or ejection) occurs as the particles spin around each other, in contrast to the classical Newtonian case in which collisions are frontal: the particles asymptotically approach each other radially, i.e. following ultimately a definite direction.

We will now see what happens in the case $\eta = 0$. For this we will apply an extension of a theorem due to Casasayas, Fontich, and Nunes (see [CF,1992]):

Let $F = (F_1, F_2)$ *be an analytic function from a neighborhood of* $(0,0)$ *in* $\mathbb{R}^2$ *to* $\mathbb{R}^2$, *such that*

(i) $F(0,y) = (0,y)$,

(ii) $DF(0,0) = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ *with* $c > 0$,

(iii) $\alpha = D_x D_y F_1(0,0) > 0$.

Then there exist stable and unstable manifolds of $(0,0)$ *which are, locally, graphs of analytic functions, that is,* $W^s_{loc}(\delta) = \{(x, \varphi^s(x)) | x \in (-\delta, 0]\}$ *and* $W^u_{loc}(\delta) = \{(x, \varphi^u(x)) | x \in [0, \delta)\}$, *where* $\varphi^{s,u} \sim (\alpha/2c)x^2$. (By W^s_{loc} and W^u_{loc} we have denoted the *local stable manifold* and the *local unstable manifold*, respectively.)

The proof of this result follows identically the one in [CF,1992], the only difference being that, in that paper, the value of the constant c equals 1. We will prove now that the conditions of this theorem are fulfilled by our first return map ψ.

Now, we the solutions starting at $(r, v) = (0, v_0)$ are periodic, so *(i)* is clearly satisfied. From (5.2), by direct calculation we see that *(ii)* is fulfilled with $c = 1 + \pi\sqrt{2\mu/b}$. For *(iii)*, we have

$$\alpha = D_x D_y \psi_1(0,0) = \lim_{v_0 \to 0} \frac{D_x \psi_1(0, v_0) - D_x \psi_1(0, -v_0)}{2v_0}$$
$$\geq \lim_{v_0 \to 0} \frac{e^{M\pi v_0} - e^{-M\pi v_0}}{2v_0} = M\pi > 0.$$

This means that each of the two periodic orbits P_η with $\eta = 0$ have both a stable manifold of approaching orbits and an unstable manifold of ejecting orbits. This completes the first part of the proof.

Let us now describe the flow near a periodic solution P_η with $\sqrt{2b/\mu} < \eta < \sqrt{2b}$ and $\theta > 0$. (The case with $\theta < 0$ or $-\sqrt{2b} < \eta < -\sqrt{2b/\mu}$ is similar.) We start by shifting the origin of the frame from 0 to $\pi/2$ and change the variable θ to a variable ϕ. When moving from 0 to 2π, the new angular varable ϕ rotates around the v-axis, which now goes through $A^+_{\pi/2}$ instead of A_0 (see Figure 2), such that it allows us to describe only the periodic orbits P_η with $v > \sqrt{2b/\mu}$ and $\theta > 0$. This is done by defining the transformation

$$\begin{cases} u = w \sin\phi \\ \theta = \dfrac{\pi}{2} + w\cos\phi. \end{cases}$$

With this transformation the system (4.2) changes to

$$\begin{cases} r' = rv \\ v' = 2r^2h + r\tilde{\Delta}^{-1/2} \\ \phi' = -\sin^2\phi + (1/2)(\mu - 1)w^{-1}(r\tilde{\Delta}^{-3/2} + 2b\tilde{\Delta}^{-2})\cos\phi\sin(2w\cos\phi) \\ w' = -(1/2)(\mu - 1)(r\tilde{\Delta}^{-3/2} + 2b\tilde{\Delta}^{-2})\sin\phi\sin(2w\cos\phi) + w\sin\phi\cos\phi, \end{cases} \tag{5.6}$$

and the energy relation becomes

$$w^2\sin^2\phi + v^2 - 2r\tilde{\Delta}^{-1/2} - 2b\tilde{\Delta}^{-1} = 2r^2h, \tag{5.7}$$

where $\tilde{\Delta} = \mu\sin^2(w\cos\phi) + \cos^2(w\cos\phi)$.

Now, proceed as in the other case. In an appropriate neighborhood of the periodic orbit, we have $\phi' \neq 0$, so the nonautonomous system

$$\begin{cases} \dfrac{dr}{d\phi} = \dfrac{r'}{\phi'} \\ \dfrac{dw}{d\phi} = \dfrac{v'}{\phi'} \end{cases}$$

is analytic in that neighborhood, where the energy relation (5.7) is used to recover w (the positive root). Using the same methods as before, the matrix

$$D\Phi(0,\eta) = \begin{bmatrix} \frac{\partial\Phi_1}{\partial r}(0,\eta) & 0 \\ \frac{\partial\Phi_2}{\partial r}(0,\eta) & 1 \end{bmatrix} \tag{5.8}$$

is the derivative of the Poincaré map $\Phi(r,v)$ on the section $\phi = 0$. The right column is the same as in (5.3) because again r' and v' are multiples of r. Since ϕ' is bounded, the same inequalities as the ones used for $\frac{dr}{d\theta}$ show that $\frac{\partial\Phi_1}{\partial r}(0,\eta) > 1$ for $\sqrt{2b} < \eta < \sqrt{2b/\mu}$. So each periodic solution with $\sqrt{2b} < \eta < \sqrt{2b/\mu}$ has an unstable manifold of ejecting solutions. Similarly each solution with $-\sqrt{2b/\mu} < \eta < -\sqrt{2b}$ has a stable manifold of approaching solutions, in which the angular momentum oscillates without bound, and the particles stay within an acute angle from the weak axis of the force. This completes the proof of the theorem.

The following consequence of Theorem 5.1 is now obvious:

Corollary 5.2. *The set of initial data leading to collisions has positive measure. More precisely, the set of initial data leading to frontal collisions has zero measure, whereas the set of initial data leading to spiraling collisions has positive measure.*

Remark. Observe that the anisotropic Kepler problem lacks the spiraling orbits we have found in the anisotropic Manev problem. This *black-hole effect* appears because of the inverse-square term added to the Newtonian potential in the Manev case. The occurence of this phenomenon is independent on the anisotropy of the space, for such orbits exist in the classical Manev two- and three-body problem too (see [DD,1996], [Di,1996]).

We will further consider the zero-energy manifold and in what remains of this paper will give a complete qualitative description of the flow in this particular case.

6. The Zero-Energy Manifold

In this section we will study the flow on $\mathcal{E}_0$, i.e. the case of the zero energy level $h = 0$, and compare the flow on the collision manifold with the one on the so-called *infinity manifold*, which we will define below. In order to understand the global flow we will analyze the invariant submanifolds associated to the equilibrium points and to the periodic orbits on the collision manifold, as well as their possible intersections.

Let us start by giving a characteristic property of the flow on $\mathcal{E}_0$. For this we need the following definition (for details see [Mc,1974] or [Di,1992]):

Definition 6.1. A flow having only isolated equilibria is called *gradient-like* with respect to one of the coordinates, if every nonequilibrium solution increases on that coordinate.

With this definition we can now prove the following:

Lemma 6.2. *The flow on $\mathcal{E}_0$ is gradient-like with respect to the v-coordinate.*

Proof: For $h = 0$, the second equation in (4.2) shows that $v' > 0$ for all values of θ.

To study the asymptotic behavior at infinity, we will apply a suitable blow-up transformation. Since the potential is a *quasihomogeneous function* (i.e. the sum of homogeneous functions of different degrees—as defined in [Di,1996]), this transformation is slightly different from the one used in the case of the collision.

Taking $h = 0$ and $\rho = 1/r$, the equations (4.2) become

$$\begin{cases} \rho' = -\rho v, \\ v' = \rho^{-1}\Delta^{-1/2}, \\ \theta' = u, \\ u' = [(\mu - 1)/2](\rho^{-1}\Delta^{-3/2} + 2b\Delta^{-2}) \sin 2\theta, \end{cases} \tag{6.1}$$

and the energy relation takes the form

$$\rho(u^2 + v^2) - 2\Delta^{-1/2} - 2b\rho\Delta^{-1} = 0. \tag{6.2}$$

Rescaling the velocities by using the transformations $\bar{v} = \rho^{\frac{1}{2}} v, \bar{u} = \rho^{\frac{1}{2}} u$, and rescaling the (independent) time variable by defining the transformation $d\tau = \rho^{\frac{1}{2}} ds$, the equations (6.1) take the form

$$\begin{cases} \dot{\rho} = -\rho\bar{v}, \\ \dot{\bar{v}} = -(1/2)\bar{v}^2 + \Delta^{-1/2}, \\ \dot{\theta} = \bar{u}, \\ \dot{\bar{u}} = -(1/2)\bar{v}\bar{u} + [(\mu - 1)/2](\Delta^{-3/2} + 2b\rho\Delta^{-2}) \sin 2\theta, \end{cases} \tag{6.3}$$

where the dot denotes differentiation with respect to the new (fictitious) time variable s. In the new coordinates the energy relation becomes

$$\bar{u}^2 + \bar{v}^2 - 2\Delta^{-1/2} - 2b\rho\Delta^{-1} = 0. \tag{6.4}$$

Analogously to the collision manifold, we define the *infinity manifold* I,

$$I = \{(\rho, \bar{v}, \theta, \bar{u}) \mid \rho = 0 \text{ and } \bar{u}^2 + \bar{v}^2 = 2\Delta^{-1}\},$$

which is also homeomorphic to a torus (see Figure 3).

The flow on I is given by

$$\begin{cases} \dot{\bar{v}} = (1/2)\bar{u}^2, \\ \dot{\theta} = \bar{u}, \\ \dot{\bar{u}} = -(1/2)\bar{v}\bar{u} + [(\mu - 1)/2]\Delta^{-3/2}\sin 2\theta. \end{cases} \tag{6.5}$$

As in the case of the collision manifold, the flow on the infinity manifold is fictitious in the sense that it has no physical meaning. But once again, using the continuity of the solutions with respect to the initial data, the structure of the flow on the infinity manifold will allow us to draw conclusions about the behavior of the flow near infinity. Therefore let us now study the flow on I.

Figure 3
The flow on the infinity manifold

There are eight equilibrium points on the infinity manifold I, which, in the new $(\bar{v}, \theta, \bar{u})$-coordinates, have the form $B_0^{\pm} = (\pm\sqrt{2\mu^{-1/2}}, 0, 0)$, $B_\pi^{\pm} = (\pm\sqrt{2\mu^{-1/2}}, \pi, 0)$, and $B_{\pm\pi/2}^{\pm} = (\pm\sqrt{2}, \pm\pi/2, 0)$. We can now show that the flow on I is fairly simple:

Lemma 6.3. *The flow on I is gradient like with respect to the $\bar{v}$-coordinate.*

Proof: From the first equation of system (6.5) we see that $\dot{\bar{v}} \geq 0$. If $\dot{\bar{v}} = 0$, then $\bar{u} = 0$, so $\ddot{\bar{v}} = \bar{u}\dot{\bar{u}} = 0$. But $\dddot{\bar{v}} = \dot{\bar{u}}^2 + \bar{u}\ddot{\bar{u}} = \dot{\bar{u}}^2$, which is 0 only at the equilibria and is positive otherwise. This completes the proof.

We now define the notion of a *central configuration*, a concept that plays an important role both in the Newtonian case as well as in the one considered here.

Definition 6.4. The configuration q_0 is called central if $\nabla W(q_0)$ is parallel to q_0.

In the classical Newtonian case (or in general for quasihomogeneous potentials in an isotropic space), finding central configurations means finding *homothetic* orbits, for if the particles are released with zero velocities from a central configuration, they move homothetically toward a total collision. But this might fail to be true in our case. However, the following proposition shows that the property remains true, i.e. homothetic orbits do exist in the anisotropic Manev problem.

Proposition 6.5. *There exist eight orbits connecting the respective equilibrium points on the collision manifold C and on the infinity manifold I.*

Proof: Recall that the equilibrium points are defined in each of the blow-up coordinate systems: at collision and at infinity. This defines two different charts, carrying in each case the corresponding time scales for the differential equations.

In the chart containing the infinity manifold I, the homothetic orbits are given by the equations:

$$\dot{\rho} = -\rho\bar{v}, \quad \dot{\bar{v}} = -\frac{1}{2}\bar{v}^2 + \mu^{-\frac{1}{2}}, \qquad \text{if } \theta = 0 \text{ or } \theta = \pi,$$

$$\dot{\rho} = -\rho\bar{v}, \quad \dot{\bar{v}} = -\frac{1}{2}\bar{v}^2 + 1, \qquad \text{if } \theta = \pm\pi/2.$$

In the chart containing the collision manifold C, the homothetic orbits are given by the equations:

$$r' = rv, \quad v' = v^2 - \mu^{-\frac{1}{2}}r - b\mu^{-1}, \qquad \text{if } \theta = 0, \text{ or } \theta = \pi,$$

$$r' = rv, \quad v' = v^2 - r - b, \qquad \text{if } \theta = \pm\pi/2.$$

Using the energy relation and correspondingly changing the time scales, by straightforward computation we see that the set of eight orbits described in the infinity-manifold chart is identical to the eight-element set described in the collision-manifold chart. This completes the proof.

Remark. Notice that the α- and ω-limits of the four homothetic orbits described above are equilibrium points on C and I.

To give a total description of the global flow in the zero-energy case, notice that, on $\mathcal{E}_0$ and outside the collision manifold C, the flow is gradient-like with respect to the v coordinate, but it is not gradient-like with respect to $\bar{v}$, on $\mathcal{E}_0$ and outside the infinity manifold I. Moreover, the so-called *zero velocity curve* is empty. This is easy to see, for the zero velocity curve is defined as the set of phase-space points for which the momentum coordinate p is zero in the energy relation (2.2). In McGehee coordinates this corresponds to taking $v = 0$ and $u = 0$ in the energy relation (4.1).

Recall from Theorem 5.1 that for each η, the periodic orbit P_η has, at least locally, a two-dimensional stable (unstable) manifold if $\eta < 0$ ($\eta > 0$); if $\eta = 0$, then P_0 has both a stable and an unstable two-dimensional submanifold. Also recall that, without loss of generality, we can take $\mu > 1$.

Due to the gradient-like structure of the global flow on $\mathcal{E}_0$, the invariant manifolds corresponding to periodic orbits P_η with $\eta \leq 0$ cannot intersect invariant manifolds corresponding to periodic orbits P_η with $\eta \geq 0$.

7. The Local Structure

Before going deeper into the global structure of the flow on $\mathcal{E}_0$, we have to analyze the hyperbolic character of the equilibrium points. The computations are set in McGehee coordinates for the equilibrium points belonging to the total collision manifold C and in infinity-blow-up coordinates for those belonging to the infinity manifold I.

We begin with McGehee coordinates. From the energy relation (4.1), for $h = 0$ we obtain

$$r = (1/2)(u^2 + v^2)\Delta^{1/2} - b\Delta^{-1/2}.$$

Substituting r in the equations of motion (4.2) in McGehee coordinates, we obtain the system

$$\begin{cases} v' = (1/2)(u^2 + v^2) - b\Delta^{-1} \\ \theta' = u, \\ u' = [(\mu - 1)/4][(u^2 + v^2)\Delta^{-1} + 2b\Delta^{-2}) \sin 2\theta. \end{cases} \tag{7.1}$$

The matrix of the attached linear system of variables (v, θ, u), at the equilibrium points $A_0^{\pm} = (\pm\sqrt{2b/\mu}, 0, 0)$ and $A_\pi^{\pm} = (\pm\sqrt{2b/\mu}, \pi, 0)$, is

$$\begin{bmatrix} \pm\sqrt{2b/\mu} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2b(\mu - 1)/\mu^2 & 0 \end{bmatrix},$$

having the eigenvalues $\lambda_1 = \pm\sqrt{2b/\mu}, \lambda_2 = \sqrt{2}b(\mu - 1)/\mu, \lambda_3 = -\sqrt{2}b(\mu - 1)/\mu$, which, since $\mu > 1$, shows that these equilibrium points are hyperbolic. The equilibria A_0^- and A_π^- have a two-dimensional stable manifold and a one-dimensional unstable manifold, whereas the equilibria A_0^+ and A_π^+ have a one-dimensional stable manifold and a two-dimensional unstable manifold.

The matrix of the attached linear system of variables (v, θ, u), at the equilibrium points $A_{\pm\pi/2}^{\pm} = (\pm\sqrt{2b}, \pm\pi/2, 0)$ is

$$\begin{bmatrix} \pm\sqrt{2b/\mu} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -2b(\mu - 1) & 0 \end{bmatrix},$$

with eigenvalues $\lambda_1 = \pm\sqrt{2b}$, $\lambda_2 = i\sqrt{2b(\mu - 1)}$ and $\lambda_3 = -i\sqrt{2b(\mu - 1)}$. This is in agreement with the fact that these equilibria are α- or ω-limits of the corresponding homothetic orbits and that the structure of the flow restricted to the collision manifold proves them to be centers. So these equilibrium points are *not* hyperbolic.

We pass now to the infinity-blow-up coordinates. For $h = 0$, the corresponding energy relation (6.4) is equivalent to

$$2\rho b = (\bar{u}^2 + \bar{v}^2)\Delta - 2\Delta^{1/2}.$$

Substitution of ρ in the equations (6.3) gives

$$\begin{cases} \dot{\bar{v}} = -(1/2)\bar{v}^2 + \Delta^{-1/2}, \\ \dot{\theta} = \bar{u}, \\ \dot{\bar{u}} = -(1/2)\bar{v}\bar{u} + [(\mu - 1)/2][(\bar{v}^2 + \bar{u}^2)\Delta^{-1} - \Delta^{-3/2}] \sin 2\theta. \end{cases} \tag{7.2}$$

The matrix of the attached linear system of variables $(\bar{v}, \theta, \bar{u})$ at the equilibrium points $B_0^{\pm} = (\pm\sqrt{2\mu^{-1/2}}, 0, 0)$ and $B_\pi^{\pm} = (\pm\sqrt{2\mu^{-1/2}}, \pi, 0)$ is

$$\begin{bmatrix} \mp\sqrt{2\mu^{-1/2}} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & (\mu-1)/(2\mu^{3/2}) & \mp\sqrt{2\mu^{-1/2}} \end{bmatrix}.$$

The eigenvalues of this matrix are

$$\lambda_1 = \mp\sqrt{2\mu^{-1/2}},$$

$$\lambda_2 = \mp(2\mu)^{-1/4}/2 + (1/2)\sqrt{(2\mu)^{-1/2} + 2(\mu-1)\mu^{3/2}},$$

$$\lambda_3 = \mp(2\mu)^{-1/4}/2 - (1/2)\sqrt{(2\mu)^{-1/2} + 2(\mu-1)\mu^{3/2}},$$

so the equilibria are hyperbolic. The equilibria B_0^- and B_π^- have a one-dimensional stable manifold and a two-dimensional unstable manifold, whereas the equilibria B_0^+ and B_π^+ have a two-dimensional stable manifold and a one-dimensional unstable manifold.

Finally, the corresponding matrix of the linear system in variables $(\bar{v}, \theta, \bar{u})$, at the equilibria $B_{\pm\pi/2}^{\pm} = (\pm\sqrt{2}, \pm\pi/2, 0)$, is

$$\begin{bmatrix} \mp\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -(\mu-1)(2\mu^{1/2}-1)/2\mu^{3/2} & \mp\sqrt{2}/2 \end{bmatrix},$$

with eigenvalues

$$\lambda_1 = \mp\sqrt{2},$$

$$\lambda_2 = \mp\sqrt{2}/4 + (1/2)\sqrt{1/2 - 2(\mu-1)(2\mu^{1/2}-1)\mu^{-3/2}},$$

$$\lambda_3 = \mp\sqrt{2}/4 - (1/2)\sqrt{1/2 - 2(\mu-1)(2\mu^{1/2}-1)\mu^{-3/2}},$$

the sign being chosen with respect to the upper $\pm$ sign of B. This shows that the equilibria are also hyperbolic. In fact $B_{\pm\pi/2}^+$ are sinks, whereas $B_{\pm\pi/2}^-$ are sources. Like in the previous case, the eigenvalue λ_1 corresponds to the homothetic orbit, while the others correspond to the flow restricted to the infinity manifold.

Thus, the behavior of our flow for $\rho = 0$ is in agreement with what we know about the flow on the total collision manifold for the anisotropic Kepler problem (see [CL,1984]).

Now that the local structure of the flow on $\mathcal{E}_0$ at the equilibria is understood, we can go further towards filling in the global picture of the zero-energy case.

8. More Connecting Orbits

To complete the global description of the flow on the zero-energy manifold $\mathcal{E}_0$, we must analyze the stable and unstable submanifolds of the saddle points. Recall that the flow near the saddle points on the collision manifold has two separatrices given by the heteroclinic orbits joining these equilibria. We will now prove the following result, which describes the connecting orbits on $\mathcal{E}_0$, offering the qualitative features of the flow on the zero-energy manifold.

Theorem 8.1. *For every lower cycle of the collision manifold (including the median one), there exist orbits ejecting from a lower equilibrium of the infinity manifold and tending to this cycle; there exist orbits ejecting from every upper cycle of the collision manifold (including the median one) and tending to an upper equilibrium of the infinity manifold. There do not exist orbits connecting cycles, or cycles and equilibria, of the collision manifold. There exist noncollision orbits connecting lower and upper equilibria of the infinity manifold.*

Physical interpretation. The zero-energy solutions ejecting from a lower equilibrium and tending to a lower cycle are escape-collision orbits unbounded at time $-\infty$ and ending in a spiraling collision at a finite time. Symmetrically, solutions ejecting form an upper cycle and tending to an upper equilibrium of the infinity manifold are collision-escape orbits starting from a spiraling ejection at a finite time and becoming unbounded at time $+\infty$. The next statement tells that there do not exist solutions starting from a collision at a finite time and ending in a collision at a later finite time, independently on the qualitative nature of the collisions, i.e. spiraling or frontal. The last sentence affirms the existence of orbits unbounded at time $-\infty$, which also become unbounded at time $+\infty$ without experiencing any kind of collision.

Proof: We will first prove the existence of connecting orbits between each cycle P_η of the collision manifold and equilibria of the infinity manifold. For this let $P_\eta, \eta \leq 0$, be a cycle and take an orbit belonging to the two-dimensional local manifold of orbits tending to P_η. Since the flow is gradient-like with respect to the variable v on $\mathcal{E}_0$ outside the collision maniold, the chosen orbit must come from one of the lower equilibria of the infinity manifold. Indeed, this happens because there are no other equilibria or cycles below P_η having an unstable manifold of positive dimension. Similarly we can prove the existence of orbits connecting every $P_\eta, \eta \geq 0$, with upper equilibria of the infinity manifold.

The nonexistence of orbits connecting cycles of the collision manifold and of orbits connecting cycles with equilibria of the collision manifold follows again from the gradient-like property of the flow and the nonexistence of unstable manifolds for the lower equilibria and the cycles P_η with $\eta < 0$ and the nonexistence of stable manifolds for the upper equilibria and the cycles P_η with $\eta > 0$. Also, because of the gradient-like property there are no homoclinic connections for the cycles or the equilibria.

To prove the existence of orbits connecting lower and upper equilibria of the infinity manifold take an initial condition in the plane $v = 0$ of $\mathcal{E}_0$, close to but outside the collision manifold. Due to the gradient-like property and the fact that the equilibria and the cycles of the collision manifold have no stable manifolds, this orbit will have to end at one of the upper equilibria of the infinity manifold. This completes the proof of the theorem.

We now prove two properties that will give us a better image of the flow on the zero-energy manifold. Using the symmetries proved in Section 2, these properties can be obviously extended to other equilibria. First denote by W^s_X the stable manifold and by W^u_X the unstable manifold of the equilibrium X and by $\overline{W^s_X}$ the topological closure of W^s_X.

Proposition 8.2. $\overline{W^s_{A^-_\pi}} \bigcap W^s_{A^-_0} \neq \emptyset$.

Proof: We will use the Hartman-Grobman theorem and analyze the linear part of the equations (7.1) in a neighborhood of the equilibrium point A^-_0. The linear part of (7.1) has the form

$$\begin{cases} v' = -v_0 v, \\ \theta' = u, \\ u' = K\theta, \end{cases} \tag{8.1}$$

where $K = b(\mu^2 - 1)/\mu^2$ and $v_0 = \sqrt{2b/\mu}$. Using standard results of the theory of linear differential equations, after a suitable transformation the equations (8.1) take the form

$$\begin{cases} x' = -v_0 x, \\ y' = \sqrt{K} y, \\ z' = -\sqrt{K} z, \end{cases} \tag{8.2}$$

where x, y, and z are the new variables obtained after performing the transformation. For any initial condition (x_0, y_0, z_0), the equations (8.2) have the solution

$$\begin{cases} x(t) = x_0 e^{-v_0 t}, \\ y(t) = y_0 e^{\sqrt{K} t}, \\ z(t) = z_0 e^{-\sqrt{K} t}. \end{cases} \tag{8.3}$$

Assuming that $x_0, y_0 \neq 0$. Raising now the first equation in (8.3) to the power $\sqrt{K}/v_0$ we obtain the relation $x^{\sqrt{K}/v_0} = x_0^{\sqrt{K}/v_0} e^{-\sqrt{K}/v_0}$, which if multiplied with the second equation in (8.3) yields $x^{\sqrt{K}/v_0} y = x_0^{\sqrt{K}/v_0} y_0 \neq 0$. By the Hartman-Grobman Theorem a similar relation exists between the variables v and θ of the equations (7.1) in a small neighborhood of the equilibrium point A^-_0. Then let us take a small rectangle contained in the region $\theta > 0$ having one side on $W^s_{A^-_\pi}$ and shift it, backwards in time, along the flow. Having now a relation between the sides of the rectangle as the one existing between the variables v and θ, if we start close enough to A^-_0, the rectangle will not degenerate to a point. Thus, the local hyperbolic structure near A^-_0 implies the desired result. This completes the proof.

Proposition 8.3. *$W^u_{B^-_0}$ and $W^s_{A^-_0}$ intersect transversally along the homothetic orbit connecting A^-_0 and B^-_0.*

Proof: Consider a small rectangle having one side on the branch of $W^u_{B^-_0}$ that lies on the infinity manifold I and the other side on the homothetic orbit leaving B^-_0. Shifting the rectangle forwards in time along the homothetic orbit connecting B^-_0 with A^-_0, we see that the points of the rectangle close to the homothetic orbit pass close to A^-_0. Using the differentiability of the flow in a neighborhood of the homothetic orbit, we see that, when reaching A_0^-, the rectangle has a limiting horizontal angle in the θ-u plane. Because of the two-dimensional hyperbolic structure of the flow on C at the point A_0^-, this limiting angle must correspond to one of the two separatrices. If the limiting separatrix is $C \bigcap W^s_{A_0^-}$, then $W^s_{A_0^-} \equiv W^u_{B_0^-}$, which is absurd, because this invariant manifold would intersect the stable manifolds of the periodic orbits P_η, giving rise to points whose ω-limit is at the same time A_0^- and some periodic orbit. Remember that we have obtained a three-dimensional flow eleminating the coordinate r from the energy relation, and that the intersection between the stable manifolds of the periodic orbits P_η and $W^s_{A_0^-}$ takes place in this three-dimensional space. The above contradiction implies that $W^u_{A_0^-} \subset \overline{W^u_{B_0^-}}$. The transversality of the above separatrices on C at A_0^- shows that the two dimensional invariant manifolds $W^u_{B_0^-}$ and $W^s_{A_0^-}$ intersect transversally along the homothetic orbit. This proves the proposition.

We have thus described the main features of the flow on the zero-energy manifold.

Acknowledgement. Florin Diacu was supported in part by the NSERC Grant OGP0122045. Ernesto Lacomba and Ernesto Perez received partial support from the Conacyt Grant 1772-E9210.

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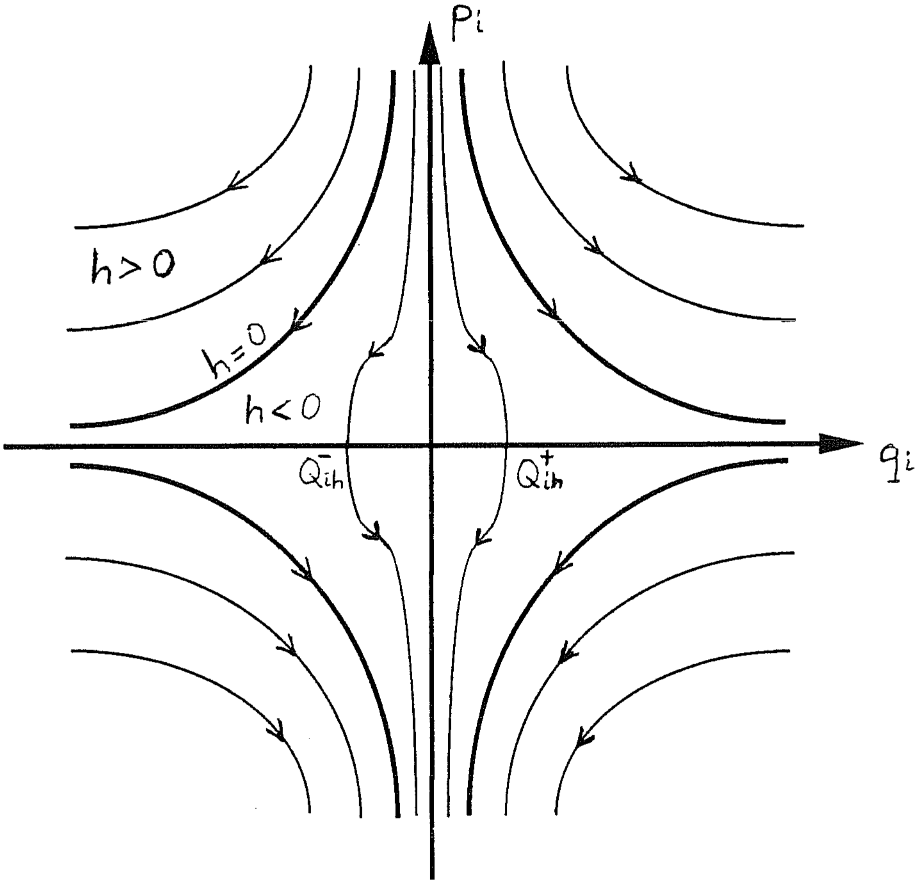

Figure 1

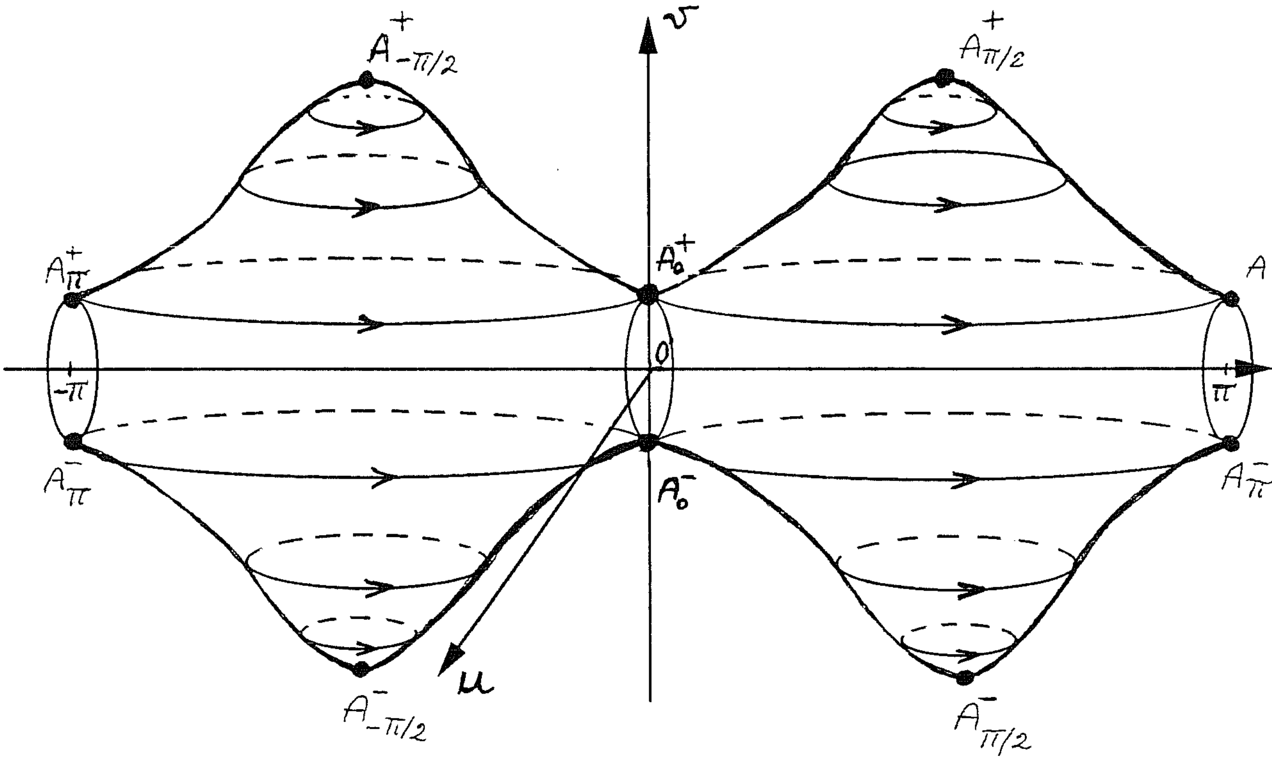

Figure 2

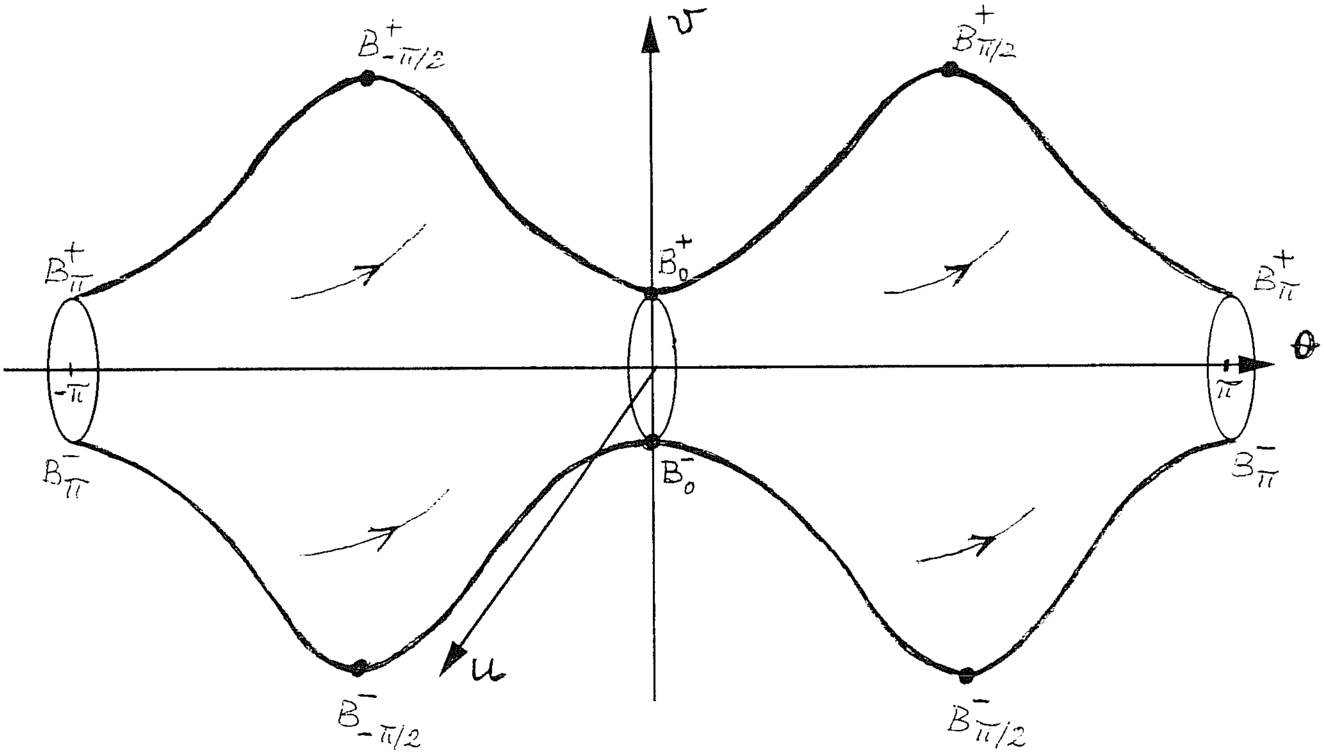

Figure 3