

***CLOSE-TO-CONVEXITY, STARLIKENESS
AND CONVEXITY OF CERTAIN
ANALYTIC FUNCTIONS***

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Abstract

The main object of the present paper is to derive several sufficient conditions for close-to-convexity, starlikeness and convexity of certain (normalized) analytic functions. Relevant connections of some of the results is obtained in this paper with those in earlier works are also provided.

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1. INTRODUCTION AND DEFINITIONS

Let \mathcal{A} denote the class of functions f normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

which are *analytic* in the *open* unit disk

$$\mathcal{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Also let $\mathcal{S}^*(\alpha)$, $\mathcal{K}(\alpha)$, and $\mathcal{C}(\alpha)$ denote the subclasses of \mathcal{A} consisting of functions which are, respectively, starlike, convex, and *close-to-convex* of order α ($0 \leq \alpha < 1$). Thus we have (see, for details, Duren [1] and Goodman [2]; see also Srivastava and Owa [6])

$$\mathcal{S}^*(\alpha) := \left\{ f : f \in \mathcal{A} \text{ and } \Re \left(\frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in \mathcal{U}; 0 \leq \alpha < 1) \right\}, \quad (1.2)$$

$$\mathcal{K}(\alpha) := \left\{ f : f \in \mathcal{A} \text{ and } \Re \left(1 + \frac{zf''(z)}{f'(z)} \right) < \alpha \quad (z \in \mathcal{U}; 0 \leq \alpha < 1) \right\}, \quad (1.3)$$

and

$$\mathcal{C}(\alpha) := \left\{ f : f \in \mathcal{A} \text{ and } \Re \left(\frac{f'(z)}{g(z)} \right) > \alpha \quad (z \in \mathcal{U}; 0 \leq \alpha < 1; g \in \mathcal{K}) \right\}, \quad (1.4)$$

where, for convenience,

$$\mathcal{S}^* := \mathcal{S}^*(0), \quad \mathcal{K} := \mathcal{K}(0), \quad \text{and} \quad \mathcal{C} := \mathcal{C}(0). \quad (1.5)$$

Next, with a view to recalling the principle of subordination between analytic functions, let the functions f and g be analytic in \mathcal{U} . Then we say that the function f is *subordinate* to g if there exists a function h analytic in \mathcal{U} with

$$h(0) = 0 \quad \text{and} \quad |h(z)| < 1 \quad (z \in \mathcal{U}), \quad (1.6)$$

such that

$$f(z) = g(h(z)) \quad (z \in \mathcal{U}). \quad (1.7)$$

We denote this subordination by

$$f(z) \prec g(z). \quad (1.8)$$

In particular, if the function g is *univalent* in \mathcal{U} , the subordination (1.8) is equivalent to (*cf.* [1, p. 190])

$$f(0) = g(0) \quad \text{and} \quad f(\mathcal{U}) \subset g(\mathcal{U}). \quad (1.9)$$

Recently, R. Singh and S. Singh [5] proved several interesting results involving univalence and starlikeness of functions $f \in \mathcal{A}$. In our attempt here to generalize these results of Singh and Singh [5], we are let naturally to several sufficient conditions for close-to-convexity, starlikeness, and convexity of functions $f \in \mathcal{A}$.

The following lemma (popularly known as *Jack's lemma*) will be required in our present investigation.

Lemma 1 (cf. Jack [3]; see also Miller and Mocanu [4]). *Let the (non-constant) function $w(z)$ be analytic in \mathcal{U} with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in \mathcal{U}$, then*

$$z_0 w'(z_0) = c w(z_0),$$

where c is a real number and $c \geq 1$.

2. SUFFICIENT CONDITIONS FOR CLOSE-TO-CONVEXITY

Our first result (Theorem 1 below) provides a sufficient condition for close-to-convexity of functions $f \in \mathcal{A}$.

Theorem 1. *Let the function $f \in \mathcal{A}$ satisfy the inequality:*

$$\Re \left(1 + \frac{z f''(z)}{f'(z)} \right) > \frac{1 + 3\alpha}{2(1 + 3\alpha)} \quad (z \in \mathcal{U}; 0 \leq \alpha < 1). \quad (2.1)$$

Then

$$\Re \{f'(z)\} > \frac{1 - \alpha}{2} \quad (z \in \mathcal{U}; 0 \leq \alpha < 1) \quad (2.2)$$

or, equivalently,

$$f \in \mathcal{C} \left(\frac{1 - \alpha}{2} \right) \quad (0 \leq \alpha < 1). \quad (2.3)$$

Proof. We begin by defining a function w by

$$f'(w) = \frac{1 + \alpha w(z)}{1 + w(z)} \quad (w(z) \neq -1; z \in \mathcal{U}; 0 \leq \alpha < 1). \quad (2.4)$$

Then, clearly, w is analytic in \mathcal{U} with $w(0) = 0$. We also find from (2.4) that

$$1 + \frac{z f''(z)}{f'(z)} = 1 + \frac{\alpha z w'(z)}{1 + \alpha w(z)} - \frac{z w'(z)}{1 + w(z)} \quad (z \in \mathcal{U}). \quad (2.5)$$

Suppose now that there exists a point $z_0 \in \mathcal{U}$ such that

$$|w(z_0)| = 1 \quad \text{and} \quad |w(z)| < 1 \quad \text{when} \quad |z| < |z_0|. \quad (2.6)$$

Then, by applying Lemma 1, we have

$$z_0 w'(z_0) = c w(z_0) \quad \left(c \geq 1; w(z_0) = e^{i\theta}; \theta \in \mathbb{R} \right). \quad (2.7)$$

Thus we find from (2.5) and (2.7) that

$$\begin{aligned} \Re \left(1 + \frac{z_0 f''(z_0)}{f'(z_0)} \right) &= 1 + \Re \left(\frac{c\alpha e^{i\theta}}{1 + \alpha e^{i\theta}} \right) - \Re \left(\frac{ce^{i\theta}}{1 + e^{i\theta}} \right) \\ &= 1 + \frac{c\alpha (\alpha + \cos \theta)}{1 + \alpha^2 + 2\alpha \cos \theta} - \frac{c}{2} \\ &\leq \frac{1 + 3\alpha}{2(1 + \alpha)} \quad (z_0 \in \mathcal{U}; 0 \leq \alpha < 1), \end{aligned}$$

which obviously contradicts our hypothesis (2.1). It follows that

$$|w(z)| < 1 \quad (z \in \mathcal{U}),$$

that is, that

$$\left| \frac{1 - f'(z)}{f'(z) + \alpha} \right| < 1 \quad (z_0 \in \mathcal{U}; 0 \leq \alpha < 1). \quad (2.8)$$

This evidently completes the proof of Theorem 1.

Theorem 2. *If the function $f \in \mathcal{A}$ satisfies the inequality:*

$$\Re \left(1 + \frac{zf''(z)}{f'(z)} \right) < \frac{3 + 2\alpha}{2 + \alpha} \quad (z_0 \in \mathcal{U}; 0 \leq \alpha < 1), \quad (2.9)$$

then

$$|f'(z) - 1| < 1 + \alpha \quad (z_0 \in \mathcal{U}; 0 \leq \alpha < 1). \quad (2.10)$$

Proof. Our proof of Theorem 2, also based upon Lemma 1, is much akin to that of Theorem 1. Indeed, in place of the definition (2.4), here we let the function w be given by

$$f'(z) = (1 + \alpha)w(z) + 1 \quad (z_0 \in \mathcal{U}; 0 \leq \alpha < 1). \quad (2.11)$$

The details may be omitted.

Remark 1. Since the inequality (2.10) implies that

$$\Re \{f'(z)\} > -\alpha \quad (z_0 \in \mathcal{U}; 0 \leq \alpha < 1), \quad (2.12)$$

by setting $\alpha = 0$ in Theorem 2, we readily obtain

Corollary 1 (Singh and Singh [5, p. 311, Corollary 2]). *If the function $f \in \mathcal{A}$ satisfies the inequality:*

$$\Re \left(1 + \frac{zf''(z)}{f'(z)} \right) < \frac{3}{2} \quad (z \in \mathcal{U}), \quad (2.13)$$

then

$$|f'(z) - 1| < 1 \quad (z \in \mathcal{U}), \quad (2.14)$$

that is, $f \in \mathcal{C}$.

Next we prove

Theorem 3. *If the function $f \in \mathcal{A}$ satisfies the inequality:*

$$|f'(z) - 1|^\beta |zf''(z)|^\gamma < \frac{(1 - \alpha)^{\beta + \gamma}}{2^{\beta + 2\gamma}} \quad (z \in \mathcal{U}; 0 \leq \alpha < 1; \beta, \gamma \geq 0), \quad (2.15)$$

then

$$\Re \{f'(z)\} > \frac{1 + \alpha}{2} \quad (z \in \mathcal{U}; 0 \leq \alpha < 1). \quad (2.16)$$

Proof. We define the function w by

$$f'(z) = \frac{1 + \alpha w(z)}{1 + w(z)} \quad (w(z) \neq -1; z \in \mathcal{U}; 0 \leq \alpha < 1). \quad (2.17)$$

Then, clearly, w is analytic in \mathcal{U} with $w(0) = 0$. We also find from (2.17) that

$$|f'(z) - 1|^\beta |zf''(z)|^\gamma = \frac{(1 - \alpha)^{\beta + \gamma} |w(z)|^\beta |zw'(z)|^\gamma}{|1 + w(z)|^{\beta + 2\gamma}} \quad (z \in \mathcal{U}). \quad (2.18)$$

Supposing now that there exists a point $z_0 \in \mathcal{U}$ such that

$$|w(z_0)| = 1 \quad \text{and} \quad |w(z)| < 1 \quad \text{when} \quad |z| < |z_0|,$$

if we apply Lemma 1 just as we did in the proof of Theorem 1, we shall obtain

$$\begin{aligned} |f'(z_0) - 1|^\beta |z_0 f''(z_0)|^\gamma &= \frac{(1 - \alpha)^{\beta + \gamma} c^\gamma}{|1 + e^{i\theta}|^{\beta + 2\gamma}} \\ &\geq \frac{(1 - \alpha)^{\beta + \gamma}}{2^{\beta + 2\gamma}} \quad (z_0 \in \mathcal{U}; 0 \leq \alpha < 1), \end{aligned}$$

which obviously contradicts our hypothesis (2.15). Thus we have

$$|w(z)| < 1 \quad (z \in \mathcal{U}),$$

which implies that

$$\left| \frac{f'(z) - 1}{f'(z) - \alpha} \right| < 1 \quad (z \in \mathcal{U}; 0 \leq \alpha < 1), \quad (2.19)$$

that is, that (2.16) holds true.

By letting

$$\beta = \gamma - 1 = 0$$

in Theorem 2, we arrive at

Corollary 2. *If the function $f \in \mathcal{A}$ satisfies the inequality:*

$$|zf''(z)| < \frac{1 - \alpha}{4} \quad (z \in \mathcal{U}; 0 \leq \alpha < 1), \quad (2.20)$$

then

$$\Re \{f'(z)\} > \frac{1 + \alpha}{2} \quad (z \in \mathcal{U}; 0 \leq \alpha < 1). \quad (2.21)$$

Remark 2. An analogous result (which apparently is *not* contained in Corollary 2) was proven earlier by Singh and Singh [5, p. 310, Corollary 1], which asserted that, if the function $f \in \mathcal{A}$ satisfies the inequality:

$$|zf''(z)| < 1 \quad (z \in \mathcal{U}),$$

then $f \in \mathcal{C}$.

3. STARLIKENESS AND CONVEXITY

In this section, we first prove the following result (Theorem 4 below), which involve the already introduced principle of subordination between analytic functions (see Section 1).

Theorem 4. *If the function $f \in \mathcal{A}$ satisfies the inequality:*

$$\Re \left(1 + \frac{zf''(z)}{f'(z)} \right) < \begin{cases} \frac{5\lambda - 1}{2(\lambda + 1)} & (z \in \mathcal{U}; 1 < \lambda \leq 2) \\ \frac{\lambda + 1}{2(\lambda - 1)} & (z \in \mathcal{U}; 2 < \lambda < 3) \end{cases} \quad (3.1)$$

for some $\lambda (1 < \lambda < 3)$, then

$$\frac{zf'(z)}{f(z)} \prec \frac{\lambda(1-z)}{\lambda-z}. \quad (3.2)$$

The result is sharp for the function f given by

$$f(z) = z \left(1 - \frac{z}{\lambda} \right)^{\lambda-1}. \quad (3.3)$$

Proof. Let us define the function w by

$$\frac{zf'(z)}{f(z)} = \frac{z[1-w(z)]}{\lambda-w(z)} \quad (w(z) \neq \lambda; z \in \mathcal{U}; 1 < \lambda < 3). \quad (3.4)$$

Then, clearly, w is analytic in \mathcal{U} with $w(0) = 0$. By logarithmic differentiation of both sides of (3.4), we also find that

$$1 + \frac{zf''(z)}{f'(z)} = \frac{\lambda[1-w(z)]}{\lambda-w(z)} - \frac{zw'(z)}{1-w(z)} + \frac{zw'(z)}{\lambda-w(z)} \quad (z \in \mathcal{U}). \quad (3.5)$$

Assuming now that there exists a point $z_0 \in \mathcal{U}$ such that

$$|w(z_0)| = 1 \quad \text{and} \quad |w(z)| < 1 \quad \text{when} \quad |z| < |z_0|,$$

if we apply Lemma 1 just as we did in the proof of Theorem 1, we shall obtain

$$\begin{aligned}
& \Re \left(1 + \frac{z_0 f'(z_0)}{f'(z_0)} \right) \\
&= \Re \left(\frac{\lambda(1 - e^{i\theta})}{\lambda - e^{i\theta}} \right) - \Re \left(\frac{ce^{i\theta}}{1 - e^{i\theta}} \right) + \Re \left(\frac{ce^{i\theta}}{\lambda - e^{i\theta}} \right) \\
&= \frac{\lambda(\lambda + 1)(1 - \cos \theta)}{1 + \lambda^2 - 2\lambda \cos \theta} + \frac{c}{2} + \frac{c(\lambda \cos - 1)}{1 + \lambda^2 - 2\lambda \cos \theta} \\
&= \frac{\lambda + 1}{2} + \frac{(\lambda^2 - 1)(c + 1 - \lambda)}{2(1 + \lambda^2 - 2\lambda \cos \theta)} \\
&\geq \frac{\lambda + 1}{2} + \frac{(\lambda^2 - 1)(2 - \lambda)}{2(1 + \lambda^2 - 2\lambda \cos \theta)} \quad (z_0 \in \mathcal{U}; 1 < \lambda < 3),
\end{aligned}$$

which yields the inequality

$$\Re \left(1 + \frac{z_0 f''(z_0)}{f'(z_0)} \right) \geq \begin{cases} \frac{5\lambda - 1}{2(\lambda + 1)} & (z_0 \in \mathcal{U}; 1 < \lambda \leq 2) \\ \frac{\lambda + 1}{2(\lambda - 1)} & (z_0 \in \mathcal{U}; 2 < \lambda < 3). \end{cases} \quad (3.6)$$

Since (3.6) obviously contradicts our hypothesis (3.1), we conclude that

$$|w(z)| < 1 \quad (z \in \mathcal{U}),$$

that is, that

$$\left| \frac{zf'(z)}{f(z)} - \frac{\lambda}{\lambda + 1} \right| < \frac{\lambda}{\lambda + 1} \quad (z \in \mathcal{U}; 1 < \lambda < 3), \quad (3.7)$$

which implies the subordination (3.2) asserted by Theorem 4.

Finally, for the function f given by (3.3), we have

$$\frac{zf'}{f(z)} = \frac{\lambda(1 - z)}{\lambda - z}, \quad (3.8)$$

which evidently completes our proof of Theorem 4.

Remark 3. A special case of Theorem 4 when $\lambda = 2$ was given earlier by Singh and Singh [5, p. 313, Theorem 6].

Lastly, since

$$f(z) \in \mathcal{K}(\alpha) \iff zf'(z) \in \mathcal{S}^*(\alpha) \quad (0 \leq \alpha < 1), \quad (3.9)$$

whose special case, when $\alpha = 0$, is the familiar Alexander Theorem (*cf.*, *e.g.*, Duren [1, p. 43, Theorem 2.12]), Theorem 4 can be applied in order to deduce

Corollary 3. *If the function $f \in \mathcal{A}$ satisfies the inequality:*

$$\Re \left(\frac{2zf''(z) + z^2f'''(z)}{f'(z) + zf''(z)} \right) < \begin{cases} \frac{3(\lambda - 1)}{2(\lambda + 1)} & (z \in \mathcal{U}; 1 < \lambda \leq 2) \\ \frac{3 - \lambda}{2(\lambda - 1)} & (z \in \mathcal{U}; 2 < \lambda < 3) \end{cases} \quad (3.10)$$

for some λ ($1 < \lambda < 3$), then

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{\lambda(1 - z)}{\lambda - z}. \quad (3.11)$$

The result is sharp for the function f given by

$$f'(z) = \left(1 - \frac{z}{\lambda}\right)^{\lambda-1}. \quad (3.12)$$

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