THE TWO-DIMENSIONAL THEORY OF ELECTROMAGNETIC INDUCTION IN THIN SHEETS WITH APPLICATIONS TO THE EARTH

by

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ABSTRACT

Supervisor: Professor J.T. Weaver

In this thesis, a method for studying problems of electromagnetic induction in local regions of the Earth where the principal conductivity anomalies are confined to a thin surface layer has been developed using the thin sheet approximation. A two-dimensional model is used in which the Earth is represented by a uniformly conducting half-space covered by an infinitely thin sheet of variable integrated conductivity. A quasi-static magnetic source field, in both E-polarization and B-polarization, is considered.

With the aid of the well-known boundary conditions for thin sheets, the horizontal electric field at the surface of the Earth is shown to satisfy a simple one-dimensional integral equation in both E- and B-polarizations. The method of reducing these integral equations to a linear system of algebraic equations for numerical solution is discussed in detail. The final (discrete) expressions for the fields are presented in a form suitable for programming. (A complete listing of the Fortran program is included in the appendix.)

Results obtained by this method agree well with those obtained analytically by Nicoll and Weaver (1977)

(B-polarization case). When comparing results with those obtained numerically by Fischer et al. (1978a, 1978b) (E-polarization), some differences do occur. However, the similarity of our results and those obtained from an analytical solution of Weidelt (1971) and from the finite-difference formulation of Brewitt-Taylor and Weaver (1976) indicates that our results are indeed accurate.

To illustrate the method, field values are calculated for two simple models in which the integrated conductivity changes (i) abruptly and (ii) gradually from one value to another. A more complex model depicting a cross-section through Vancouver Island is also considered.

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DEDICATION

I wish to dedicate this thesis to my mother and father for their encouragement and support during my educational career.

CHAPTER 1

INTRODUCTION

1.1 Historical review

The electromagnetic field recorded at the surface of the Earth due to sources in the ionosphere has been studied for many years with the aim of inferring information about the Earth's conductivity structure. This study may be divided into problems of a global nature and those of a local nature. Global studies are concerned with the gross features of the conductivity structure of the Earth to great depths (i.e. ~ 1000 km) whereas local studies are concerned with the finer details of the conductivity structure of the crust and upper mantle (i.e. ~ 100 km) over small regions of the Earth which may be considered flat. In recent years, there has been considerable interest in electromagnetic induction problems of a local nature in which lateral conductivity inhomogeneities occur. Problems of this last type are studied in this thesis. Problems of a global nature, which have recently been reviewed by Rikitake (1973), will not be discussed further.

A general treatment of electromagnetic induction in a flat Earth, using a conducting half-space to represent the Earth, was described in a classic paper by Price (1950).

In this paper, a complete analysis of the problem of electromagnetic induction due to an arbitrary known source field acting above a homogeneous conducting half-space, was made. His discussion included physical interpretations of his 'elementary' solutions. By a systematic application of integral transforms and the use of electric and magnetic Hertz vectors aligned normal to the surface of the conducting half-space, Weaver (1971) simplified and consolidated Price's theory. Following Weaver's procedure, Summers and Weaver (1973) generalized the analysis to include the case of an n-layered conducting half-space. Weaver (1973) has reviewed the principal features of electromagnetic induction in a multi-layered Earth.

Lateral variations in conductivity have been investigated using a variety of techniques. Several authors (D'Erceville and Kunetz, 1962; Weaver, 1963; Weaver and Thomson, 1972, and others) have used analytical approaches to the problem of a vertical discontinuity in conductivity. Analytical techniques are usually capable of treating only special problems of simple geometry but they provide valuable checks on other techniques and often give information not attainable by other methods. Numerical techniques are used to solve problems involving more complex structures. In particular, finite difference numerical methods provide one of the most general

mathematical methods for studying electromagnetic induction problems. A disadvantage of this method, though, is that considerable computer time and computer storage space is often required to solve problems involving reasonably complex structures. For some types of problems, numerical and analytical techniques may be combined to lessen this disadvantage. Analogue models, having complex conductivity structures, have also been used to study various interesting problems, such as the 'coast effect' near Japan (Roden, 1964), and the effect of channelling around Vancouver Island (Nienaber et al., 1978), which are not easily studied by mathematical techniques (see review by Dosso, 1973).

Price (1949) described another approach to the problem of conductivity anomalies confined to near the surface in which the Earth is mathematically represented by an infinitely thin sheet of variable integrated conductivity underlain by a non-conducting medium. Since then, considerable extensions to this method have been made. For example, Rikitake (1966) included a conducting half-space which was separated from the non-uniform thin sheet by a non-conducting region, in order to take into account the effect of a highly conducting mantle. In another paper by Rikitake (1968), the conducting half-space, mentioned above, was replaced by one with an undulating

surface to study the effect of upwellings of a deep mantle. In these models, the integrated conductivity varied periodically in one direction.

Many authors have used thin sheet models with discontinuities in the integrated conductivity to study the 'coast effect'. In particular, Roden (1964) studied the 'coast effect' near Japan by mathematically representing the Earth by a thin strip of width, a, and finite thickness, s, ($s \ll a$), separated from a perfectly conducting half-space by a non-conducting region. The results obtained from this model were used as a check on the validity of his analogue model measurements, mentioned earlier. A similar model (with $s \to 0$ and without the perfectly conducting half-space) was analysed by Parker (1968). Using an analytical technique, he found that the vertical magnetic field tends logarithmically to infinity in the vicinity of the edge and pointed out that previous numerical methods had not permitted this detailed investigation and thus that care should be taken in the numerical analysis to obtain accuracy despite the singular nature of the solution. Weidelt (1971), again using an analytical technique considered a model consisting of two infinitely thin half-sheets of different (finite) integrated conductivities. He also found that the vertical magnetic field tends logarithmically to

infinity near the interface between the two sheets. Furthermore, he showed that the horizontal magnetic field changed discontinuously at the interface. Both Parker and Weidelt noted the need for analytical techniques to serve as checks on numerical procedures for the modelling of arbitrary distributions of integrated conductivity.

Ashour (1971) described a more realistic model to study the 'coast effect' in which the integrated conductivity reduces quickly to zero instead of abruptly changing to zero. The field components are then continuous and finite at the coastline. An important conclusion to be drawn from this investigation is that the induced horizontal magnetic field is enhanced just off the coast and zero or very small at the coast and inland, and also that the vertical magnetic field is enhanced at the boundary even though the integrated conductivity is zero there. Ashour also mentions the importance of observing the field values, not only in the vicinity of the coastline, but also at the continental shelf.

The method of thin sheets has been applied to the study of coastal anomalies in the Southwestern United States by Schmucker (1970). In his mathematical model, a non-conducting region again separates the conducting mantle from the anomalies near the surface. An important

point discussed in his work is the limitations to Price's technique. These limitations are presented in chapter 3 of this thesis.

The thin sheet models discussed in this thesis are two-dimensional, that is, it is assumed that there is no variation in the integrated conductivity in one horizontal direction. It should be pointed out, though, that geometrically simple three-dimensional models of a circular and elliptical region of uniform integrated conductivity in an otherwise uniform thin sheet of different integrated conductivity has been studied (Ashour and Chapman, 1965). Also, many thin hemispherical cap models have been used to investigate global effects.

It is well known that two-dimensional problems may be separated into E-polarization and B-polarization cases. (The horizontal magnetic field of the source is perpendicular to the strike of the anomaly in the E-polarization case and parallel to the strike in the B-polarization case.) For the thin sheets presented so far, only the E-polarization case has been of interest since all the thin sheets have been underlain by a non-conducting region (see review by Ashour, 1973). Schmucker (1971) included, in his model, a conducting (n-layered) half-space immediately below a thin sheet having a strip of different integrated conductivity. He only investigated the E-polarization case, but with this type of model it is possible to study the B-

polarization case since electric currents may flow into the Earth. To study the B-polarization case, Bailey (1977) used a model which consisted of a uniformly conducting half-space in contact with and screened by a perfectly conducting half-plane. Nicoll and Weaver (1977) extended Bailey's problem by adding a perfectly conducting plane at a depth D, to the model, to represent the highly conducting mantle. These problems were both solved analytically using the Wiener-Hopf technique. Brewitt-Taylor (1975) used a model similar to Nicoll and Weaver but with a perfectly conducting infinitely thin strip of width, 2a, at the surface. His analytical solution was obtained by neglecting self-induction, and was therefore only approximate. He then compared his results with those obtained by applying the finite difference technique to the same model but with a more realistic strip of finite depth and conductivity (Brewitt-Taylor, 1976). Finally, Fischer, Schnegg, and Usadel (1978a, 1978b) considered the E-polarization case using Bailey's model and obtained an integral equation for the field which they solved numerically.

1.2 Summary of work in this thesis

In this thesis, the problem of electromagnetic induction in local regions of the Earth where the principal conductivity anomalies are confined to a thin surface layer, is investigated. A two-dimensional model is used in which the Earth is represented by a uniformly conducting half-space covered by an infinitely thin layer of variable integrated conductivity. This model is ideally suited to study 'coast effect' problems since the near-surface conductivity structure perpendicular to the coast changes rapidly over short distances while, parallel to the coast, the near-surface conductivity structure often does not vary significantly over great distances.

The purpose of this thesis is to present the theory (see chapter 3) for a more general thin sheet model in which both E-polarization and B-polarization cases may be discussed. Furthermore, the theory has been developed with the idea of providing boundary values for more complex three-dimensional problems.

An important feature of this model is that the variations of the conductivity with depth are eliminated from the problem so that, with the aid of the well-known boundary conditions for thin sheets, it becomes possible to express the surface horizontal electric field (in both the E-polarization and the B-polarization cases) as the solution

of a simple one-dimensional integral equation. In the E-polarization case, an improved method of accounting for the contribution of the electric field, far from the region of varying conductivity, is included in the numerical evaluation of the integral equation (see section 4.4). Also of importance is the fact that the values of the integrated conductivity at the extreme ends of the model need not be the same, which is desirable when studying realistic 'coast effect' problems. Moreover, the theory allows for a variable integrated conductivity so that the sloping continental shelf may be modelled more closely than in analytical methods. Also, a grid of variable step-size has been incorporated into the numerical method (see chapter 4) so that the edge effect due to abrupt changes in integrated conductivity may be accurately determined.

In chapter 5, some results are presented to illustrate the theory and then briefly discussed.

CHAPTER 2

EQUATIONS OF ELECTROMAGNETIC INDUCTION

2.1 Maxwell's equations

Maxwell's equations mathematically express the relationship between time-varying electric and magnetic fields. These equations can be considerably simplified, when applied to problems dealing with electromagnetic induction in the Earth, if certain assumptions are made regarding the source field and the region of interest. The appropriate differential equations, required to determine the field vectors, can then be found by using Maxwell's equations on this simplified form.

Maxwell's equations, in M.K.S. units, are

$$\overline{\nabla} \times \overline{E} = - \frac{\partial \overline{B}}{\partial t} \quad , \tag{2.1.1}$$

$$\overline{\nabla} \times \overline{H} = \overline{J} + \frac{\partial \overline{D}}{\partial t} \quad , \tag{2.1.2}$$

$$\overline{\nabla} \cdot \overline{D} = \rho \quad , \tag{2.1.3}$$

and

$$\overline{\nabla} \cdot \overline{B} = 0 \tag{2.1.4}$$

where $\overline{E}$ is the electric field intensity, $\overline{D}$ is the electric flux density, $\overline{B}$ is the magnetic flux density,

$\overline{H}$ is the magnetic field intensity, $\overline{J}$ is the volume current density, and ρ is the volume charge density. The term $\frac{\partial \overline{D}}{\partial t}$ is known as the displacement current density. ($\overline{E}$ and $\overline{B}$ will be referred to, loosely, as the electric and magnetic fields, respectively.)

As an immediate simplification, we assume that the time dependence of the electric and magnetic fields in the region of interest can be expressed as

$$\overline{P}(x,y,z,t) = \overline{P}(x,y,z)e^{i\omega t} \qquad (2.1.5)$$

where ω is the angular frequency of the source and $\overline{P}$ is any of the field vectors. In this representation, the time derivatives, $\frac{\partial \overline{P}}{\partial t}$, can be replaced by $i\omega\overline{P}$ in Maxwell's equations.

We also assume that the Earth's interior is free of primary source currents and acts as a conductor (i.e. is isotropic and obeys Ohm's law). In such a region, $\overline{B}$ and $\overline{H}$, $\overline{E}$ and $\overline{D}$, and $\overline{J}$ and $\overline{E}$ may be related as follows:

$$\overline{B} = \mu\overline{H} \quad , \qquad (2.1.6)$$

$$\overline{D} = \varepsilon\overline{E} \quad , \qquad (2.1.7)$$

and

$$\overline{J} = \sigma\overline{E} \qquad (2.1.8)$$

where μ is the permeability, ε is the permittivity

and σ is the conductivity. In air, the conductivity is approximately zero. In the Earth's crust, the conductivity ranges from approximately 10^{-6} mho/m to 10^{4} mho/m (see Table 2.1) with the mean conductivity being approximately 5×10^{-3} mho/m. The conductivity in the underlying mantle increases with depth from the mean crustal value (see Rikitake, 1973). Thus the crust is underlain by a region of higher conductivity. This point is of importance, later in this thesis, when considering the validity of representing a layer of the crust by an infinitely thin surface sheet. In air and for most materials in the Earth, the permeability and permittivity are of the order of their free-space values, μ_o and ε_o respectively ($\mu_o = 4\pi \times 10^{-7}$ H/m ; $\varepsilon_o = 8.85 \times 10^{-12}$ F/m). Exceptions to this are some ferromagnetic minerals which have permeabilities up to three times the free-space value and water which has a permittivity about 80 times the free-space value. Materials with values very much higher than these are not expected in significant quantities in the Earth. In order to simplify the computations, the permeability is assumed to be equal to its free-space value everywhere.

Furthermore, we assume that the conductivity structure in the Earth can be approximated by a number of homogeneous conducting regions. In such regions, the volume charge density can be taken as zero, since any volume charge distribution will decay quickly away to the surface of each

TABLE 2.1

Typical Values of Conductivity in the Earth's Crust

	σ (mho/m)	σ/ε (Hz)
snow	10^{-6}	10^{4}
fresh water	10^{-4}	10^{5}
dry rock	10^{-4}	10^{6}
earth (moist)	10^{-2}	10^{7}
sea water	4	10^{9}
minerals	10^{4}	10^{14}

homogeneous conducting region, independent of any applied field (see, for example, Jones, 1964). In free-space, the volume charge density is also assumed to be zero.

By substituting equations (2.1.6) through (2.1.8), with $\mu = \mu_o$, into Maxwell's equations and assuming an exponential time dependence as in equation (2.1.5), we find that, in each region of homogeneous conductivity (including $\sigma = 0$), the four Maxwell's equations reduce to two equations in two unknowns, being

$$\overline{\nabla} \times \overline{B} = (\mu_o \sigma + i\omega\mu_o \varepsilon)\, \overline{E} \quad , \qquad (2.1.9)$$

and

$$\overline{\nabla} \times \overline{E} = -i\omega \overline{B} \quad . \qquad (2.1.10)$$

The redundancy of equation (2.1.3) with $\rho = 0$ and equation (2.1.4) follows from the general vector identity

$$\overline{\nabla} \cdot (\overline{\nabla} \times \overline{P}) = 0 \quad . \qquad (2.1.11)$$

where $\overline{P}$ represents $\overline{E}$ or $\overline{B}$.

It is well-known that the displacement current term in equation (2.1.9), (i.e. $i\omega\mu_o \varepsilon$), may be neglected in geomagnetic induction studies. For this to be true in the Earth, we must have

$$\omega << \frac{\sigma}{\varepsilon} \qquad (2.1.12)$$

which is easily satisfied for values of σ in the Earth (see Table 2.1) for $\omega < 1$ kHz . To determine the frequency range in which displacement currents may be neglected in the free-space region, where $\sigma = 0$, it is convenient to express equations (2.1.9) and (2.1.10) in dimensionless form. In doing this, we define two dimensionless terms (indicated by primes) as

$$\bar{r}' = \frac{\bar{r}}{\ell} \quad \text{(i.e. } \nabla' = \ell\nabla\text{)} \tag{2.1.13}$$

and

$$\bar{B}' = \frac{\bar{B}}{B_o} \tag{2.1.14}$$

where $\bar{r}$ is the position vector, ℓ is a characteristic length in the region and B_o is a characteristic magnetic field. By substituting equations (2.1.13) and (2.1.14) into equation (2.1.10), we find that we are restricted to defining the dimensionless electric field as

$$\bar{E}' = \frac{\bar{E}}{\omega \ell B_o} \quad . \tag{2.1.15}$$

The dimensionless forms of equations (2.1.9) and (2.1.10) then become

$$\bar{\nabla}' \times \bar{B}' = i\kappa\bar{E}' \tag{2.1.16}$$

and

$$\bar{\nabla}' \times \bar{E}' = -i\bar{B}' \tag{2.1.17}$$

where $\kappa = \omega^2 \ell^2 \mu_o \varepsilon$.

Now, by substituting equation (2.1.16) into (2.1.17) or vice versa, and then using the vector identity

$$\overline{\nabla} \times (\overline{\nabla} \times \overline{P}) = \overline{\nabla}(\overline{\nabla} \cdot \overline{P}) - \nabla^2 \overline{P} \qquad (2.1.18)$$

where $\overline{P}$ may be any vector (in this case, $\overline{E}'$ or $\overline{B}'$), we obtain

$$\nabla^2 \overline{P} = -\kappa \overline{P} \quad . \qquad (2.1.19)$$

Since $\overline{E}'$ and $\overline{B}'$ are scaled so that they are of reasonable magnitudes, the right hand side of this equation will be very small if $\kappa << 1$, or in terms of frequency, if

$$\omega^2 << \frac{1}{\ell^2 \varepsilon \mu_o} \quad . \qquad (2.1.20)$$

In this case, we may approximate equation (2.1.19) by

$$\nabla^2 \overline{P} = 0 \qquad (2.1.21)$$

which is the same conclusion we would have obtained had we initially neglected the displacement current term (i.e. $i\kappa \overline{E}'$) in equation (2.1.16). Thus the frequency range over which the displacement current term may be

neglected in the free-space region (in the primed or unprimed system) is given by the inequality (2.1.20). For global studies, ℓ is of the order of the radius of the Earth (~ 6400 km) so that the inequality (2.1.20) is valid for $\omega < 10$Hz . For local studies, where the Earth is approximately flat, ℓ is considerably smaller so that the allowable values of ω are correspondingly higher. In particular, if ℓ is of the order of the height of the source (~ 100 km), then the inequality (2.1.20) is valid for $\omega < 1$ kHz. In geomagnetic induction studies, we are generally interested in $\omega < 1$ Hz so that the neglect of displacement currents is well justified.

With the neglect of displacement currents, then, the appropriate form (non-dimensionless) of Maxwell's equations becomes

$$\overline{\nabla} \times \overline{E} = -i\omega\overline{B} \qquad (2.1.22)$$

and

$$\overline{\nabla} \times \overline{B} = \mu_o \sigma \overline{E} \quad . \qquad (2.1.23)$$

By taking the curl of equation (2.1.22) and (2.1.23), and using the vector identity given by equation (2.1.18), with $\overline{P}$ representing $\overline{E}$ or $\overline{B}$, the differential equation to be satisfied by the field vectors in each homogeneous conducting region, subject to specified boundary conditions,

is found to be

$$\nabla^2\overline{P} = i\alpha^2\overline{P} \quad . \tag{2.1.24}$$

where $$\alpha^2 = \omega\mu_o\sigma \quad . \tag{2.1.25}$$

In the free-space region, equation (2.1.24) reduces to Laplace's equation,

$$\nabla^2\overline{P} = 0 \quad . \tag{2.1.26}$$

In the non-conducting region, equations (2.1.22) and (2.1.23) can be simplified to

$$\overline{B} = -\overline{\nabla}\Omega \tag{2.1.27}$$

and $$\overline{\nabla} \times \overline{E} = i\omega\overline{\nabla}\Omega \tag{2.1.28}$$

since $$\overline{\nabla} \times \overline{\nabla}a = 0 \tag{2.1.29}$$

where a is any scalar quantity. Here Ω is defined as the scalar magnetic potential. Furthermore, by taking the divergence of equation (2.1.28) and using (2.1.11), equation (2.1.26) can be replaced by

$$\nabla^2\Omega = 0 \quad . \tag{2.1.30}$$

In solving equations (2.1.24), (2.1.26) and (2.1.30), we require that the solutions and their first and second derivatives be continuous and finite in any source-free region of homogeneous conductivity and vanish when infinitely far from the source.

2.2 Two-dimensional form of Maxwell's equations

Consider the surface of the Earth to be represented by the plane $z = 0$ with the z-axis positive downward in a Cartesian coordinate system and with all sources restricted to the region $z < -h$ $(h > 0)$ above the surface. Many interesting geomagnetic induction problems may be simplified into a two-dimensional form if σ, $\overline{E}$ and $\overline{B}$ are independent of one of the coordinates (say, x). In this case, Maxwell's equations become

$$G_1(y,z) - F_2(y,z) = -i\omega X(y,z) \qquad (2.2.1)$$

$$E_2(y,z) = -i\omega Y(y,z) \qquad (2.2.2)$$

$$E_1(y,z) = i\omega Z(y,z) \quad , \qquad (2.2.3)$$

and

$$Z_1(y,z) - Y_2(y,z) = \mu_o \sigma E(y,z) \qquad (2.2.4)$$

$$X_2(y,z) = \mu_o \sigma F(y,z) \qquad (2.2.5)$$

$$X_1(y,z) = -\mu_o \sigma G(y,z) \qquad (2.2.6)$$

where $\overline{B} = (X,Y,Z)$ and $\overline{E} = (E,F,G)$. Here the Landau notation for partial derivatives has been used in which the differentiation with respect to the first and second ordered variables is indicated by the subscript $_1$ and $_2$, respectively.

In the source-free region, these equations fall into two independent groups which can be discussed separately as the E-polarization problem, governed by equations (2.2.2), (2.2.3) and (2.2.4), and the B-polarization problem, governed by equations (2.2.1), (2.2.5) and (2.2.6). Substituting equations (2.2.2) and (2.2.3) into (2.2.4) gives

$$E_{11}(y,z) + E_{22}(y,z) = i\alpha^2 E(y,z) \qquad (2.2.7)$$

in the conducting region, and

$$E_{11}(y,z) + E_{22}(y,z) = 0 \qquad (2.2.8)$$

or, in terms of magnetic potential,

$$\Omega_{11}(y,z) + \Omega_{22}(y,z) = 0 \qquad (2.2.9)$$

in the non-conducting region. Thus the E-polarization field can be written as

$$\overline{E} = (E,0,0) \quad \text{and} \quad \overline{B} = (0,Y,Z) \qquad (2.2.10)$$

with $$Y(y,z) = \frac{i}{\omega} E_2(y,z) \qquad (2.2.11)$$

and $$Z(y,z) = \frac{-i}{\omega} E_1(y,z) \qquad (2.2.12)$$

where E must satisfy (2.2.7) or (2.2.8) subject to specified boundary conditions. From equation (2.1.27), it immediately follows that

$$Y(y,z) = \frac{i}{\omega} E_2(y,z) = -\Omega_1(y,z) \qquad (2.2.13)$$

and $$Z(y,z) = \frac{-i}{\omega} E_1(y,z) = -\Omega_2(y,z) \qquad (2.2.14)$$

in $z < 0$.

Similarly the B-polarization field in the conducting region is given by

$$\overline{B} = (X,0,0) \quad \text{and} \quad \overline{E} = (0,F,G) \qquad (2.2.15)$$

with $$F(y,z) = \frac{1}{\mu_o \sigma} X_2(y,z) \qquad (2.2.16)$$

and $$G(y,z) = \frac{-1}{\mu_o \sigma} X_1(y,z) \qquad (2.2.17)$$

where $X(y,z)$ must satisfy

$$X_{11}(y,z) + X_{22}(y,z) = i\alpha^2 X(y,z) \qquad (2.2.18)$$

again subject to the given boundary conditions. Further-

more, from equation (2.1.24), we know that

$$F_{11}(y,z) + F_{22}(y,z) = i\alpha^2 F(y,z) \qquad (2.2.19)$$

and $$G_{11}(y,z) + G_{22}(y,z) = i\alpha^2 G(y,z) \quad . \qquad (2.2.20)$$

In the non-conducting region, equations (2.2.5) and (2.2.6) become

$$X_1(y,z) = 0 \quad \text{and} \quad X_2(y,z) = 0 \qquad (2.2.21)$$

which implies that,

$$X(y,z) = X_o \text{ , a constant} \quad . \qquad (2.2.22)$$

Then from equation (2.2.1), $F(y,z)$ and $G(y,z)$ must satisfy

$$G_1(y,z) - F_2(y,z) = -i\omega X_o \quad . \qquad (2.2.23)$$

CHAPTER 3

THEORY

3.1 The model

A simple two-dimensional model is used to study the resulting geomagnetic field at the surface due to currents in the ionosphere fluctuating above a small section of the Earth having near-surface inhomogeneities in conductivity. In this model, a quasi-static magnetic field located at a height h, represents the source field. The flat Earth occupies the region $z > 0$ in a right-handed Cartesian coordinate system while the region $-h < z < 0$ is assumed to be free space. The actual surface conductivity structure to be studied is contained in an integrated conductivity $\tau(y)$ given by

$$\tau(y) = \int_0^{d(y)} \bar{\sigma}(y,z)\, dz \qquad (3.1.1)$$

where $\bar{\sigma}(y,z)$ is the conductivity of the surface layer and $d(y)$ is the thickness of the surface layer. The technique, originally discussed by Price (1949), is now employed. In this technique, it is assumed that the actual surface conductivity structure can be represented by an infinitely thin layer of equivalent total integrated conductivity,

occupying the $z = 0$ plane. In this case, equation (3.1.1) becomes

$$\tau(y) = \lim_{d(y)\to 0} \int_0^{d(y)} \bar{\sigma}(y,z)\, dz \qquad (3.1.2)$$

where $\bar{\sigma}(y,z)$ mathematically becomes infinite, as $d(y)\to 0$, in such a way that $\tau(y)$ remains finite. The variations in the integrated conductivity are restricted to a limited range in y ($\ell^- < y < \ell^+$), being constant outside this region, so that

$$\tau(y) = \tau^- + \tau_a(y) \qquad (3.1.3)$$

where $\tau^- = \tau(-\infty)$ and $\tau^+ = \tau(+\infty)$

with

$$\tau_a(y) = \begin{cases} 0 & (y \leq \ell^-) \\ \tau_a(y) & (\ell^- < y < \ell^+) \\ \tau^+ - \tau^- & (\ell^+ \leq y) \end{cases}$$

The conductivity of the lower crust is represented by an infinite half-space of uniform conductivity σ, occupying the region $z > 0$ (see figure 3.1).

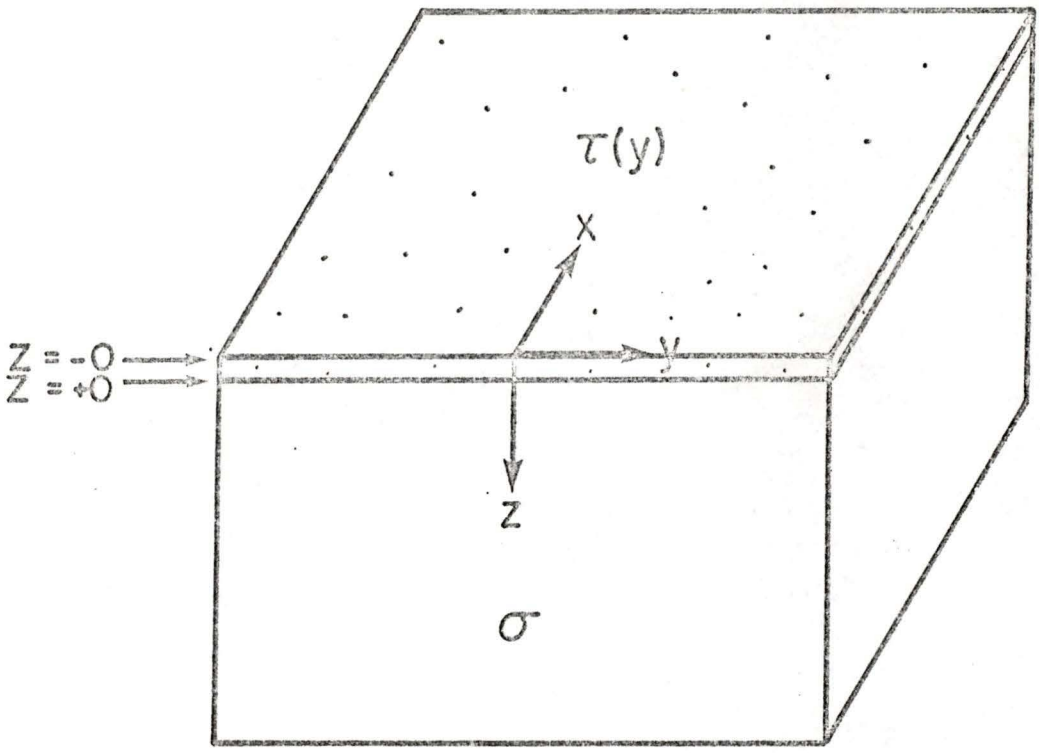

Figure 3.1 The mathematical model comprising an infinitely thin surface layer of variable integrated conductivity $\tau(y)$ covering a half-space of uniform conductivity σ.

3.2 Limitations in Price's technique

We shall see, in the next section, that the horizontal electric field must be continuous across the infinitely thin sheet if we are to use Price's technique. Thus, in the 'real' problem with a finite depth d, we require that the horizontal electric field remain approximately constant over this depth. Schmucker (1970) discussed the general restrictions on the use of Price's thin sheet approximation which this requirement imposes. He considered a three-layer model consisting of a top layer of thickness d, (terrestrial surface layers) and a poorly conducting intermediate layer of thickness h, (crust and upper mantle) overlying a highly conducting substratum. In addition, he assumed that $\delta_2 >> h$ and $\delta_3 << h$ where δ_2 and δ_3 are the skin depths in the second and third layers, respectively. Using this model, Schmucker found that the horizontal electric field was approximately constant over the depth d, if the following conditions held:

$$\text{(i)} \quad \delta_1 > 3d \quad , \quad \text{(ii)} \quad h >> d \quad ,$$

where δ_1 is the skin depth of the surface layer. This second condition implies that there should not be a good conductor at shallow depth if we are to use Price's method.

Schmucker's assumptions (i.e. $\delta_2 >> h$ and $\delta_3 << h$) imply that an extreme conductivity contrast exists between the conductivities of the crust (σ_2) and mantle (σ_3). For example, when $h = 100$ km and $3d < \delta_1 < \infty$ in his model, we find that σ_2 must be much less than 10^{-10} mho/m and σ_3 must be infinite. By using a three-layer model similar to Schmucker's (1970), the attenuation of the tangential electric field over a surface layer of depth d, can be determined for more realistic conductivities of the crust and mantle. From the theory for n-layered structures described in a review paper by Weaver (1973), we find that

$$\frac{E(d)}{E(0)} = \frac{U_1}{V_1 + V_2 - \dfrac{U_2^2}{V_2 + \dfrac{\sqrt{2i}}{\delta_3}}} \qquad (3.2.1)$$

where $$U_n = \frac{\operatorname{csch}\ (\sqrt{2i}\, d_n/\delta_n)}{\delta_n}$$

and $$V_n = \frac{\coth\ (\sqrt{2i}\, d_n/\delta_n)}{\delta_n} \qquad (n = 1, 2)$$

with $d = d_1$ and $h = d_2$.

Using this equation, we can determine the order of attenuation to be expected over a surface layer in coastal regions ($\sigma_1 \approx 4$ mho/m) where the depth of the

ocean is not likely to be more than 5 km. The results, illustrated in figure 3.2, show the limitations of Price's method. We see that, for a mantle at shallow depth (~ 50 km), the attenuation is particularly sensitive to the conductivity of the mantle (10^{-1} mho/m $< \sigma_2 < \infty$) whereas for a mantle at greater depth (500 km), the attenuation is affected very little by the conductivity of the mantle, being somewhat more sensitive to the conductivity of the crustal region ($0 < \sigma_2 < 10^{-2}$ mho/m). Furthermore, as the depth of the surface layer nears or becomes greater than its skin depth, the attenuation becomes large. In particular, for coastal anomalies, we see that the attenuation tends to increase rapidly for frequencies greater than about 0.25 mHz (~ 1 cycle per hour) which is in rough agreement with Schmucker (1970, 1971).

As a further example, we consider a typical model, in prospecting geophysics, consisting of a layer of overburden covering more resistive bedrock. From figure 3.3, we find that Price's approximation is valid for frequencies up to about 1 kHz, for a 50 m layer of overburden of conductivity 10^{-2} mho/m lying on a basement of conductivity 10^{-4} mho/m.

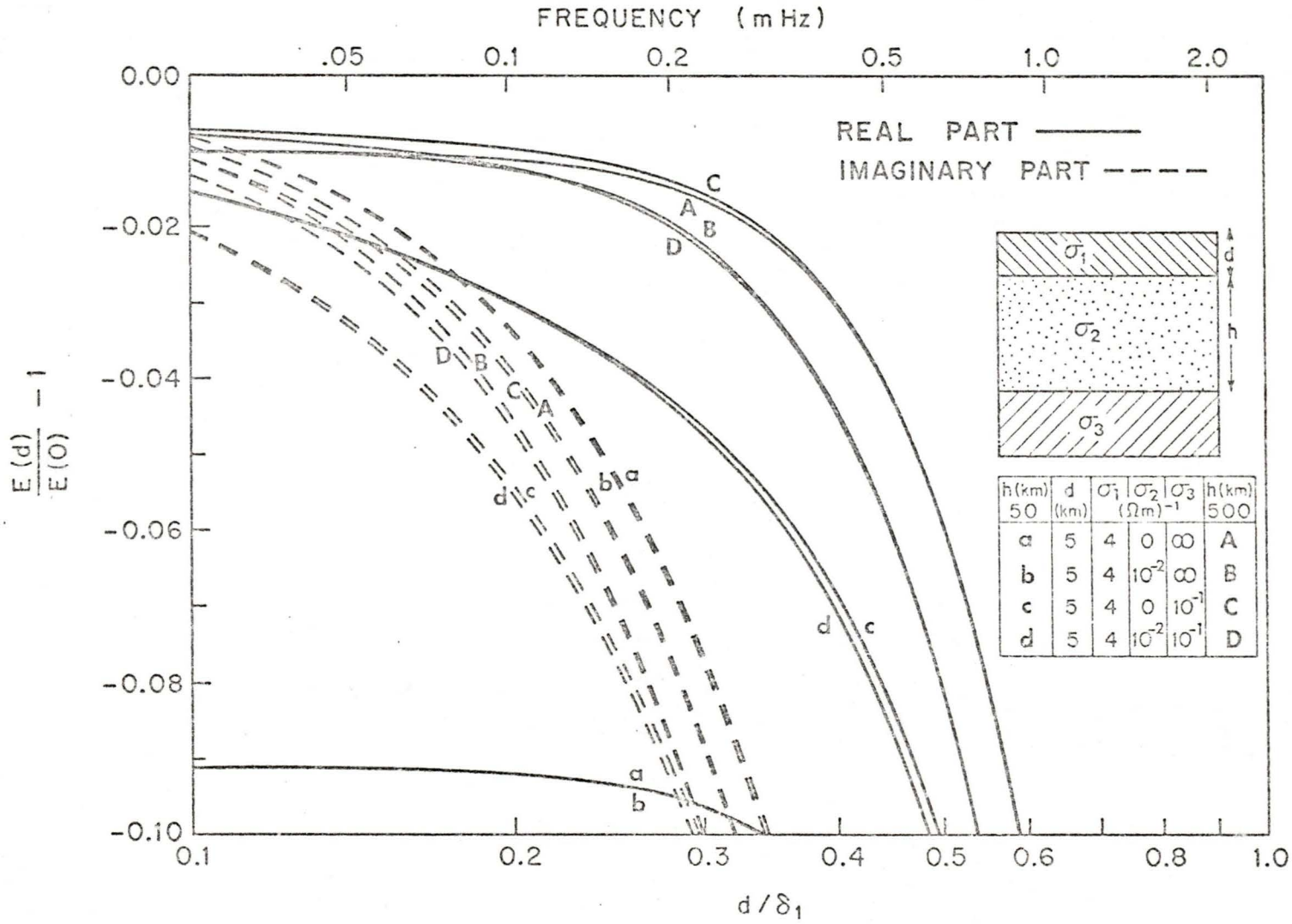

Figure 3.2 Attenuation of the horizontal electric field in the top layer of a three-layer model (i.e. ocean, crust, mantle) as a function of the depth (in skin depths) of the top layer.

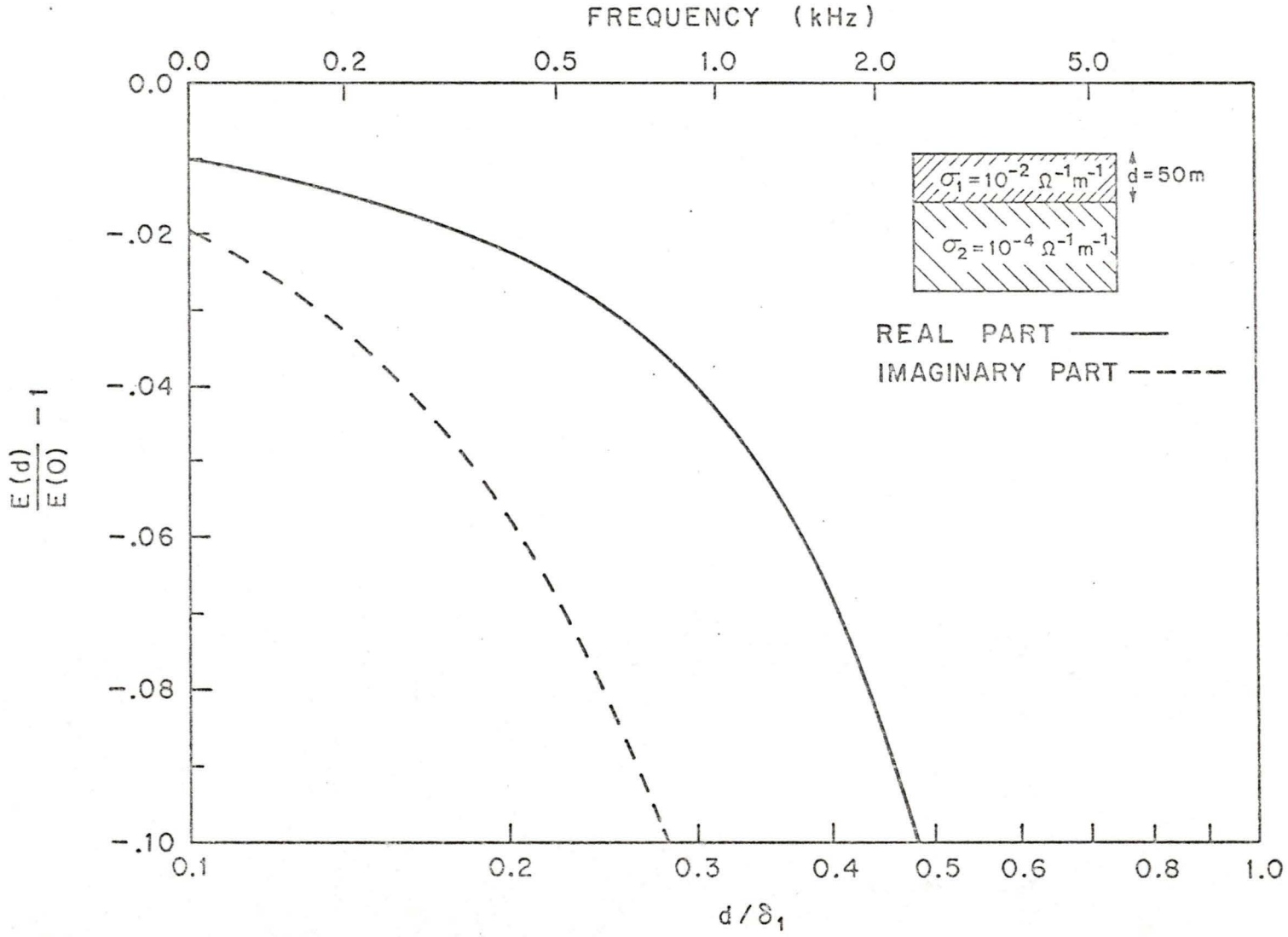

Figure 3.3 Attenuation of the horizontal electric field in the top layer of a two-layer model (overburden, basement rock) as a function of the depth (in skin depths) of the top layer.

3.3 Normal field

It is convenient to begin the study of this problem by considering the whole surface layer to be of uniform integrated conductivity τ_n, thus reducing the problem to one variable z. The resulting field, defined as the normal field and denoted by the subscript n, can be used to form the boundary values for the more complex problem of a surface layer of variable integrated conductivity. The problem is further simplified by first considering the E-polarization field for one variable,

$$\bar{E}_n(z) = (E_n(z),0,0) \quad \text{and} \quad \bar{B}_n = (0,Y_n(z),0) \tag{3.3.1}$$

with $$Y_n(z) = \frac{i}{\omega} E_n'(z) \tag{3.3.2}$$

and $$Y_n'(z) = -\mu_o \sigma E_n(z) \tag{3.3.3}$$

where $E_n(z)$ must now satisfy

$$E_n''(z) = i\alpha^2 E_n(z) \qquad \text{in} \quad z > 0 \tag{3.3.4}$$

and $$E_n''(z) = 0 \qquad \text{in} \quad z < 0 \quad . \tag{3.3.5}$$

Here the primes (') indicate differentiation.

The relationship between the electric and magnetic field just above and just below the infinitely thin sheet,

which is required to solve equations (3.3.4) and (3.3.5), can easily be determined by considering a uniform cover of finite depth d, and then letting $d \to 0$. The integral form of equation (2.1.23),

$$\oint_C \overline{E} \cdot \overline{d\ell} = -i\omega \int_S \overline{B} \cdot \overline{dA} \qquad (3.3.6)$$

obtained by using Stoke's theorem, can be employed to show that the horizontal electric field must be continuous across the infinitely thin sheet. Let c be a rectangular circuit enclosing the surface S, having two sides parallel and arbitrarily close to the upper and lower boundaries of the surface cover, as in figure 3.4a. As the uniform surface layer becomes infinitely thin (see figure 3.4b), the right-hand side of equation (3.3.6) goes to zero, since

$$\lim_{d\to 0} \left| \int_S \overline{B} \cdot \overline{dA} \right| = \lim_{d\to 0} \left| \int_{-0}^{d+0} \int_0^{\ell} Y_n(z)\, dx\, dz \right| \leq \lim_{d\to 0} |M\ell d| = 0 \qquad (3.3.7)$$

where M is the maximum value of $|Y_n(z)|$ in $-0 < z < d+0$ which must be finite for any real problem. For this one-dimensional problem, the horizontal electric field is constant and in the x-direction so equation (3.3.6) becomes

$$E_n(-0)\ell - E_n(+0)\ell = 0 \qquad (3.3.8)$$

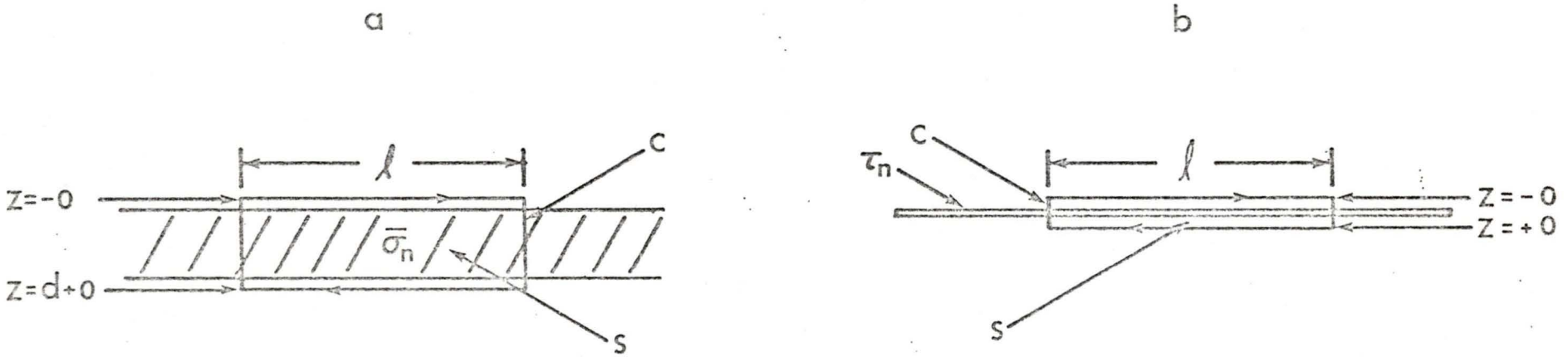

Figure 3.4 A circuit C enclosing a surface S, as d → 0 .

which implies that

$$E_n(-0) = E_n(+0) = E_n \quad . \tag{3.3.9}$$

Similarly the horizontal magnetic field can be shown to be discontinuous across the infinitely thin sheet by considering the integration of equation (3.3.3) from just above the upper boundary ($z = -0$) to just below the lower boundary ($z = d+0$) of the surface, giving

$$-\int_{-0}^{d+0} Y_n'(z)\,dz = \mu_o \int_{-0}^{d+0} E_n(z)\,\overline{\sigma_n}(z)\,dz \quad . \tag{3.3.10}$$

Using the mean-value theorem, $E_n(z)$ can be expressed as

$$E_n(z) = E_n(-0) + zE_n'(a) \tag{3.3.11}$$

where $(-0 \leq a \leq z)$.

If this expression for $E_n(z)$ is substituted into equation (3.3.10) and the surface layer made infinitely thin, then the right-hand side of equation (3.3.10) becomes

$$\lim_{d\to 0} \mu_o \int_{-0}^{d+0} E_n(z)\overline{\sigma_n}(z)dz = \mu_o E_n(-0) \lim_{d\to 0} \int_{-0}^{d+0} \overline{\sigma_n}(z)dz + \mu_o E_n'(-0) \lim_{d\to 0} \int_{-0}^{d+0} z\overline{\sigma_n}(z)dz$$

$$= \mu_o \tau_n E_n \qquad (3.3.12)$$

since

$$\lim_{d\to 0} \left| \int_{-0}^{d+0} z\overline{\sigma_n}(z)dz \right| \leq \tau_n \lim_{d\to 0} |d| = 0 \qquad (3.3.13)$$

while the left-hand side is just

$$-\lim_{d\to 0} \int_{-0}^{d+0} Y_n'(z)dz = -\lim_{d\to 0} \left[Y_n(d+0) - Y_n(-0) \right]$$

$$= Y_n(-0) - Y_n(+0) \qquad (3.3.14)$$

which implies that

$$Y_n(-0) - Y_n(+0) = \mu_o \tau_n E_n \quad . \qquad (3.3.15)$$

Equations (3.3.9) and (3.3.15) give the surface boundary conditions for the normal field. The argument used here to determine these conditions has only relied

on the requirement that the fields be finite (i.e. an M may be found) and continuous (i.e. an ℓ may be found) and that an integrated conductivity may be defined in the manner described. Thus, under these conditions, the surface boundary conditions we have derived apply also to an infinitely thin layer with varying integrated conductivity.

The continuity conditions can be used to express the surface electric field in terms of the surface magnetic field. The solution of equation (3.3.4), which vanishes as $z \to +\infty$, is

$$E_n(z) = E_n(+0)e^{-\sqrt{i}\alpha z} = E_n e^{-\sqrt{i}\alpha z} \qquad (3.3.16)$$

so that equation (3.3.2) then yields

$$Y_n(z) = \frac{-i}{\omega}\sqrt{i}\alpha E_n(z) \quad . \qquad (3.3.17)$$

Thus, the horizontal magnetic field just below the surface layer is

$$Y_n(+0) = \frac{-i}{\omega}\sqrt{i}\alpha E_n \quad . \qquad (3.3.18)$$

It follows from equation (3.3.3) that the horizontal magnetic field must be constant in the non-conducting

region, so that in $z < 0$,

$$Y_n(z) = Y_n(-0) = Y_o \text{ , a constant.} \qquad (3.3.19)$$

Hence, by integrating equation (3.3.2), we obtain the electric field in the region $z < 0$ as

$$E_n(z) = E_n - i\omega z Y_o \quad . \qquad (3.3.20)$$

Finally, we obtain

$$E_n = \frac{i\omega}{\sqrt{i}\alpha} \left(\frac{Y_o}{1 + \frac{\sqrt{i}\alpha\tau_n}{\sigma}} \right) \qquad (3.3.21)$$

by substituting equation (3.3.18) into (3.3.15), and

$$Y_n(+0) = \frac{Y_o}{1 + \frac{\sqrt{i}\alpha\tau_n}{\sigma}} \qquad (3.3.22)$$

by substituting equation (3.3.21) back into (3.3.18).

At this point, it is convenient to define the term 'skin depth' which is used often in this thesis. By taking the modulus of equation (3.3.16),

$$\left|E_n(z)\right| = \left|E_n e^{-(1+i)\alpha z/\sqrt{2}}\right| = |E_n|\ e^{-\alpha z/\sqrt{2}} \qquad (3.3.23)$$

it can easily be seen that the amplitude of the field is attenuated with depth by the factor $e^{-\alpha z/\sqrt{2}}$. The depth at which the field is attenuated to e^{-1} of its surface value is defined as the skin depth δ , and is given by

$$\delta = \sqrt{\frac{2}{\omega\mu_o\sigma}} = \frac{\sqrt{2}}{\alpha} \quad . \qquad (3.3.24)$$

The B-polarization fields, in one dimension, are

$$\overline{B}_n(z) = (X_n(z),0,0) \quad \text{and} \quad \overline{E}_n(z) = (0,F_n(z),0) \qquad (3.3.25)$$

with $$F_n'(z) = i\omega X_n(z) \qquad (3.3.26)$$

and $$X_n'(z) = \mu_o\sigma F_n(z) \qquad (3.3.27)$$

where $X_n(z)$ must satisfy

$$X_n''(z) = i\alpha^2 X_n(z) \qquad (z > 0) \qquad (3.3.28)$$

and $$X_n''(z) = 0 \qquad (z < 0) \quad . \qquad (3.3.29)$$

For a surface layer of uniform integrated conductivity, the B-polarization problem is identical to the E-polarization problem, except for a rotation of the source field through 90 degrees in the horizontal plane. Thus, by replacing $E_n(z)$ by $F_n(z)$, and $Y_n(z)$ by $-X_n(z)$ in any of the E-polarization equations, we obtain the corresponding B-polarization equations. In particular, the boundary conditions across the infinitely thin sheet are

$$F_n(-0) = F_n(+0) = F_n \qquad (3.3.30)$$

and $$X_n(-0) - X_n(+0) = -\mu_o \tau_n F_n \quad . \qquad (3.3.31)$$

Furthermore, at the surface, the B-polarization fields are given by

$$X_n(-0) = X_o \quad , \quad \text{a constant} \quad , \qquad (3.3.32)$$

$$X_n(+0) = \frac{X_o}{1 + \frac{\sqrt{i}\alpha\tau_n}{\sigma}} \quad , \qquad (3.3.33)$$

and $$F_n = \frac{-i\omega}{\sqrt{i}\alpha} \left(\frac{X_o}{1 + \frac{\sqrt{i}\alpha\tau_n}{\sigma}} \right) \quad . \qquad (3.3.34)$$

3.4 Anomalous fields in E-polarization

We now consider the more complex E-polarization problem of an integrated conductivity which varies as in equation (3.1.3). In this case, the total electric field is found by solving the differential equation

$$\frac{\partial^2}{\partial y^2} E_t(y,z) + \frac{\partial^2}{\partial z^2} E_t(y,z) = i\alpha^2 E_t(y,z) \qquad (3.4.1)$$

$$(\alpha = 0 \quad \text{in} \quad z < 0)$$

with the corresponding magnetic fields being

$$Y_t(y,z) = \frac{i}{\omega} \frac{\partial}{\partial z} E_t(y,z) \quad , \qquad (3.4.2)$$

and $$Z_t(y,z) = \frac{-i}{\omega} \frac{\partial}{\partial y} E_t(y,z) \qquad (3.4.3)$$

where the subscript t denotes the total fields. The boundary conditions for the total fields across the infinitely thin surface layer immediately follow from the normal fields by simply replacing the one-dimensional normal fields by the two-dimensional total fields and

noting that

$$\lim_{d\to 0} \left| \int_{-0}^{d+0} \frac{\partial}{\partial y} Z_t(y,z)\, dz \right| \leq M \lim_{d\to 0} |d| = 0 \qquad (3.4.4)$$

where M is the maximum value of $\left|\frac{\partial}{\partial y} Z_t(y,z)\right|$ between $z = -0$ and $z = d+0$. Of course, the normal integrated conductivity, τ_n, must be replaced by the varying integrated conductivity $\tau(y)$, as given in equation (3.1.3). The surface boundary conditions for the total fields, then, are

$$E_t(y,+0) = E_t(y,-0) = E_t(y,0) \quad , \qquad (3.4.5)$$

$$Y_t(y,-0) - Y_t(y,+0) = \mu_o \tau(y)\, E_t(y,0) \quad , \qquad (3.4.6)$$

and $$Z_t(y,+0) = Z_t(y,-0) = Z_t(y,0) \qquad (3.4.7)$$

where the continuity of $Z_t(y,z)$ follows from equations (3.4.3) and (3.4.5).

The boundary conditions for the total fields at $y = \pm\infty$ and $z = \pm 0$ are also required. Far from the region of varying integrated conductivity in the negative

and positive y directions, we assume that the fields behave like the normal fields discussed earlier. Furthermore, we know, from Jones and Price (1970), that the total horizontal magnetic field at the surface due to a uniform source field is the same at the extreme values of y, even if the conductivity structures in $z > 0$ at $y = -\infty$ and $y = +\infty$ are different. Thus we have that

$$Y_t(-\infty,-0) = Y_t(+\infty,-0) = Y_o \qquad (3.4.8)$$

The boundary values for the total electric field at the surface and the total horizontal magnetic field just below the surface are then obtained by simply replacing τ_n in equations (3.3.21) and (3.3.22) by τ^- and τ^+ respectively. The resulting boundary values are denoted by

$$E_t(-\infty,0) = E_n^- \quad , \quad E_t(+\infty,0) = E_n^+ \quad , \qquad (3.4.9)$$

and

$$Y_t(-\infty,+0) = Y_n^-(+0) \quad , \quad Y_t(+\infty,+0) = Y_n^+(+0). \qquad (3.4.10)$$

In addition, we have that

$$Z_t(\pm\infty,0) = 0 \qquad (3.4.11)$$

since $Z_t(u,0) = \frac{\partial}{\partial y} E_t(u,0)$ and $E_t(u,0)$ is assumed constant, far from the region of varying integrated conductivity.

The fields, obtained by subtracting the normal fields at $y = -\infty$ from the total fields, are defined as the anomalous fields and are given by

$$E(y,z) = E_t(y,z) - E_n(z) \quad , \qquad (3.4.12)$$

$$Y(y,z) = Y_t(y,z) - Y_n(z) = \frac{i}{\omega} E_2(y,z) \quad , \qquad (3.4.13)$$

and $$Z(y,z) = Z_t(y,z) = \frac{-i}{\omega} E_1(y,z) \qquad (3.4.14)$$

where $E(y,z)$ must satisfy

$$E_{11}(y,z) + E_{22}(y,z) = i\alpha^2 E(y,z) \qquad (3.4.15)$$

$$\text{in} \quad z > 0 \ ,$$

and $E_{11}(y,z) + E_{22}(y,z) = 0 \quad \text{in} \quad z < 0 \ . \qquad (3.4.16)$

From the surface boundary conditions for the total and normal fields and the definition of the anomalous fields, it follows that

$$E(y,-0) = E(y,+0) = E(y,0) \quad , \tag{3.4.17}$$

$$Z(y,-0) = Z(y,+0) = Z(y,0) \quad , \tag{3.4.18}$$

and

$$Y(y,-0) - Y(y,+0) = \mu_o \tau(y) E(y,0) + \mu_o (\tau(y) - \tau^-) E_n^- . \tag{3.4.19}$$

In particular, equation (3.4.19) is found by subtracting equation (3.3.15) from (3.4.6) and then using equation (3.4.13). The boundary values at $y = \pm\infty$ are found to be

$$E(-\infty,0) = 0 \quad , \quad E(+\infty,0) = E_n^+ - E_n^- \quad , \tag{3.4.20}$$

$$Y(-\infty,+0) = 0 \quad , \quad Y(+\infty,+0) = Y_n^+(+0) - Y_n^-(+0) \quad , \tag{3.4.21}$$

and $$Y(\pm\infty,-0) = 0 \quad , \quad Z(\pm\infty,0) = 0 \quad . \tag{3.4.22}$$

Clearly these anomalous fields are caused by currents induced in the region of varying integrated conductivity and thus are entirely of internal origin. This property

of the anomalous fields is significant and is used in determining an expression for $Y(y,-0)$.

Equation (3.4.19), which links the anomalous horizontal magnetic field just below and just above the surface, is of importance here, as it is this equation which is used to determine the anomalous fields along the surface. We are about to show that $Y(y,-0)$ and $Y(y,+0)$ can be expressed as integrals over the surface values of the electric field. By substituting these integral equations back into equation (3.4.19), we then have an equation written only in terms of the surface electric field. This equation may be solved numerically using just a one-dimensional grid in the y-direction, rather than a two-dimensional grid in the y-z plane which is normally required to solve two-dimensional induction problems numerically. This is an important feature of the work presented in this thesis since it means that both the computer time and the storage space required for the numerical solution of geomagnetic induction problems with surface anomalies are considerably reduced. Once the anomalous electric field is determined, the horizontal and vertical magnetic fields just above and below the surface can easily be obtained by using the forthcoming integral equations for $Y(y,-0)$ and $Y(y,+0)$, and equation (3.4.14) for $Z(y,0)$.

The integral equations for $Y(y,-0)$ and $Y(y,+0)$ just mentioned, can be obtained by using a Fourier transform in the y variable, which is defined, as in Sneddon (1951), to be

$$\hat{f}(\eta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{i\eta y} \, dy \quad , \tag{3.4.23}$$

with the inverse transform being

$$f(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\eta) e^{-i\eta y} \, d\eta \quad . \tag{3.4.24}$$

An important relationship between Fourier transformed functions is provided by the Faltung theorem which states that

$$\int_{-\infty}^{\infty} \hat{f}(\eta) \, \hat{g}(\eta) \, e^{-i\eta y} d\eta = \int_{-\infty}^{\infty} f(u) \, g(y-u) \, du \quad . \tag{3.4.25}$$

In order to determine $Y(y,-0)$ in integral form, we consider the non-conducting region ($z < 0$) in which the anomalous magnetic field components can be expressed in terms of an anomalous magnetic scalar potential $\Omega(y,z)$, by the equations

$$Y(y,z) = -\Omega_1(y,z) \qquad (3.4.26)$$

and $$Z(y,z) = -\Omega_2(y,z) \qquad (3.4.27)$$

where $\Omega(y,z)$ satisfies

$$\Omega_{11}(y,z) + \Omega_{22}(y,z) = 0 \quad . \qquad (3.4.28)$$

Following the work of Weaver (1964), we take the Fourier transform of equation (3.4.28) to obtain

$$\hat{\Omega}_{22}(\eta,z) = \eta^2 \hat{\Omega}(\eta,z) \quad . \qquad (3.4.29)$$

We require the solution to this equation which vanishes as $z \to -\infty$, since the anomalous potential must be of internal origin. It is

$$\hat{\Omega}(\eta,z) = \hat{\Omega}(\eta,0)\ e^{|\eta|z} \qquad (3.4.30)$$

with the inverse Fourier transform giving,

$$\Omega(y,z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\Omega}(\eta,0) e^{|\eta| z} e^{-i\eta z} d\eta \ . \qquad (3.4.31)$$

If we differentiate this equation with respect to z and let $z \to 0$, it follows from equation (3.4.27) that

$$Z(y,0) = -\Omega_2(y,-0) = \frac{-1}{\sqrt{2\pi}} \int_{\infty}^{\infty} |\eta| \hat{\Omega}(\eta,-0) e^{-i\eta y} d\eta \qquad (3.4.32)$$

which from equation (3.4.24) implies

$$\hat{\Omega}(\eta,-0) = \frac{\hat{Z}(\eta,0)}{|\eta|} \ . \qquad (3.4.33)$$

Substituting this into equation (3.4.31) and differentiating with respect to y, we have

$$Y(y,-0) = -\Omega_1(y,-0) = \frac{-i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{Z}(\eta,0) \ \mathrm{sgn}(\eta) e^{-i\eta y} d\eta. \qquad (3.4.34)$$

After applying the Faltung theorem, this becomes

$$Y(Y,-0) = \frac{-1}{\pi} \int_{-\infty}^{\infty} \frac{Z(u,0)}{y-u} \, du \qquad (3.4.35)$$

since the inverse Fourier transform of $\mathrm{sgn}(\eta) = \frac{\eta}{|\eta|}$ is known to be $\frac{-i}{y}\sqrt{\frac{2}{\pi}}$ (Lighthill, 1958). Here, the

bar across the integral sign signifies that the Cauchy principal value (C.P.V.) of the integral must be taken at $u = y$ for the improper integral to exist. From equation (3.4.14) we know that

$$Z(y,0) = \frac{-i}{\omega} E_1(y,0) \qquad (3.4.36)$$

so that the desired integral expression for the anomalous horizontal magnetic field at the surface is

$$Y(y,-0) = \frac{i}{\omega\pi} \int_{-\infty}^{\infty} \frac{E_1(u,0)}{y-u}\, du \quad . \qquad (3.4.37)$$

Following the general procedure used to determine equation (3.4.37), an integral expression for $Y(y,+0)$ can be found. We begin by determining an expression for the anomalous electric field in the conducting region by taking a Fourier transform of equation (3.4.15) to give

$$\hat{E}_{22}(\eta,z) = (\eta^2 + i\alpha^2)\ \hat{E}(\eta,z) \qquad (3.4.38)$$

with the solution, which vanishes as $z \to \infty$, being

$$\hat{E}(\eta,z) = \hat{E}(\eta,0)\ e^{-\gamma z} \qquad (3.4.39)$$

where $\gamma = \eta^2 + i\alpha^2$. (3.4.40)

The exponential term in equation (3.4.39) can be considered as a Fourier transform $\hat{g}(\eta)$, of some function $g(y)$ which is

$$\begin{aligned} g(y) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\gamma z} e^{-i\eta y} \, d\eta \\ &= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-\gamma z} \cos \eta y \, d\eta \\ &= \sqrt{\frac{2}{\pi}} \; \frac{Z\sqrt{i}\alpha \; K_1(\sqrt{i}\alpha\sqrt{y^2+z^2})}{\sqrt{y^2+z^2}} \end{aligned} \tag{3.4.41}$$

where K_1 is a modified Bessel function of the second kind of order 1. The last step in equation (3.4.41) is a tabulated result (Erdélyi et al., 1954, eq. 1.4.26). In order to simplify the notation, we define a function

$$P(y,z) = \frac{K_1(\sqrt{i}\alpha\sqrt{y^2+z^2})}{\sqrt{y^2+z^2}} \quad . \tag{3.4.42}$$

Now by taking the inverse Fourier transform of equation (3.4.39) and using the Faltung theorem, an expression for the anomalous electric field in the conducting region is obtained, giving

$$E(y,z) = \frac{z\sqrt{i}\alpha}{\pi} \int_{-\infty}^{\infty} E(u,0)\ P(y-u,z)\ du \quad . \qquad (3.4.43)$$

In arriving at equation (3.4.43), we have tacitly assumed that $E(y,0) = 0$ at $y = \pm\infty$ (i.e. $E_n^+ = E_n^-$) as this is a requirement of taking the Fourier transform of the second derivative with respect to y in equation (3.4.15). Now from a tabulated result (Gradshteyn and Ryzhik, 1965, eq. 6.596#3), we know that

$$\int_0^{\infty} P(y,z)\ dy = \sqrt{\frac{\pi}{2\sqrt{i}\alpha z}}\ K_{1/2}(\sqrt{i}\alpha z) \qquad (\alpha > 0) \qquad (3.4.44)$$

which, together with the tabulated result (Abramowitz and Stegun, 1964, eq. 10.2.17)

$$\sqrt{\frac{\pi}{2w}}\ K_{1/2}(w) = \left(\frac{\pi}{2w}\right) e^{-w}, \quad (w, \text{ a complex number})$$

yields

$$\frac{z\sqrt{i}\alpha}{\pi} \int_{-\infty}^{\infty} P(y,z)\ dy = e^{-\sqrt{i}\alpha z} \quad . \qquad (3.4.45)$$

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Using this last equation, equation (3.4.42) may be re-written as

$$E(y,z) = z\sqrt{i}\alpha\, J(y,z) + E(y,0)\, e^{-\sqrt{i}\alpha z} \qquad (3.4.46)$$

where

$$J(y,z) = \frac{1}{\pi} \int_{-\infty}^{\infty} [E(u,0) - E(y,0)]\, P(y-u,z)\, du$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} [E(y+s,0) - E(y,0)]\, P(s,z)\, ds \quad . \qquad (3.4.47)$$

In this latter form, we see that $J(y,z) \to 0$ as $y \to \pm\infty$. Thus the solution for $E(y,z)$ is actually valid for more general boundary conditions since $E(y,z)$ approaches the expected one-dimensional solution at $y = +\infty$, being

$$E(+\infty,z) = E_t(+\infty,z) - E_n^-(z) = (E_n^+ - E_n^-)\, e^{-\sqrt{i}\alpha z}$$

$$(z \geq 0)$$

even when $E_n^+ \neq E_n^-$. By the Uniqueness theorem, then,

equation (3.4.46) is the one and only solution to equation (3.4.15) which satisfies the required boundary conditions.

We now differentiate equation (3.4.46) with respect to z to obtain

$$E_2(y,z) = \sqrt{i}\alpha\left[J(y,z) + zJ_2(y,z) - e^{-\sqrt{i}\alpha z}E(y,0)\right] \quad (3.4.48)$$

The second term on the right above vanishes at $z = +0$, since it can be shown (see appendix B) that $J_2(y,z)$ exists at $z = +0$. Hence, the desired integral expression for the anomalous horizontal magnetic field just below the surface follows from equation (3.4.13), being

$$Y(y,+0) = \frac{i}{\omega} E_2(y,+0) = \frac{i\sqrt{i}\alpha}{\omega}\left[J(y,+0) - E(y,0)\right] \quad (3.4.49)$$

where $$J(y,+0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left[E(u,0) - E(y,0)\right] P(y-u,+0)\, du \quad (3.4.50)$$

and $$P(y-u,+0) = \frac{K_1(\sqrt{i}\alpha|y-u|)}{|y-u|} \quad . \quad (3.4.51)$$

Here, the C.P.V. of the integral is taken at $u = y$ for the improper integral to exist.

By substituting equation (3.3.21) into (3.4.19) and rearranging, we finally obtain the desired expression for the anomalous electric field at the surface

$$E(y,0) = \frac{\omega\sigma}{\alpha^2\tau(y)}\left[Y(y,-0) - Y(y,+0)\right]$$

$$- \frac{i\omega}{\sqrt{i}\alpha}\left(\frac{Y_o}{1 + \frac{\sqrt{i}\alpha\tau^-}{\sigma}}\right)\left(\frac{\tau(y) - \tau^-}{\tau(y)}\right) \qquad (3.4.52)$$

where $Y(y,-0)$ and $Y(y,+0)$ are now known in terms of $E(u,0)$ by equations (3.4.37) and (3.4.49), respectively. (In the above equation, we have also used $\alpha^2 = \omega\mu_o\sigma$ as defined in equation (2.1.25).)

In the next chapter we will determine the anomalous electric field numerically using this equation. After $E(u,0)$ is determined, $Y(y,-0)$, $Y(y,+0)$ and $Z(y,0)$ may easily be calculated. The total fields can then be found by simply adding the normal field values (at $y = -\infty$) to the anomalous field results.

3.5 Anomalous fields in B-polarization

The anomalous B-polarization equations can be determined by following the same procedure used to find the anomalous E-polarization equations. Again the anomalous field components are found by subtracting the normal field components (at $y = -\infty$) from the corresponding total field components. The whole procedure is considerably simplified, though, since from section 2.2, we already know that the total horizontal magnetic field above the surface must be constant. As this must agree with the normal field at $y = -\infty$, it follows that in $z < 0$

$$X_t(y,z) = X_o \qquad (3.5.1)$$

which implies that

$$X(y,-0) = X_t(y,-0) - X_o = 0 \quad . \qquad (3.5.2)$$

The anomalous horizontal magnetic field in the conducting region may be found by solving the differential equation

$$X_{11}(y,z) + X_{22}(y,z) = i\alpha^2 X(y,z) \quad . \qquad (3.5.3)$$

The solution to this equation has already been determined in section 3.4, as

$$X(y,z) = \frac{\sqrt{i}\alpha z}{\pi} \int_{-\infty}^{\infty} [X(u,+0) - X(y,+0)] P(y-u,z)\, du$$

$$+ X(y,+0)\ e^{-\sqrt{i}\alpha z} \qquad (3.5.4)$$

with the derivative with respect to z, at $z = +0$, being

$$X_2(y,+0) = \frac{\sqrt{i}\alpha}{\pi} \int_{-\infty}^{\infty} [X(u,+0) - X(y,+0)] P(y-u,+0)\, du$$

$$- \sqrt{i}\alpha X(y,+0) \quad . \qquad (3.5.5)$$

From equations (2.2.16), (2.2.17), (3.3.25) and (3.3.23) and the definition of the anomalous fields, the anomalous electric fields, in $z > 0$, are given by

$$F(y,z) = \frac{1}{\mu_o \sigma} X_2(y,z) \qquad (3.5.6)$$

and

$$G(y,z) = \frac{-1}{\mu_o \sigma} X_1(y,z) \quad . \qquad (3.5.7)$$

The boundary conditions for the anomalous fields (i) across the surface, and (ii) at $y = \pm\infty$ ($z = \pm 0$), easily follow from the argument used to determine equations (3.4.17) through (3.4.22) and are

$$\text{(i)} \quad F(y,+0) = F(y,-0) = F(y,0) \tag{3.5.8}$$

$$X(y,+0) - X(y,-0) = X(y,+0)$$

$$= \mu_o \tau(y)\, F(y,0) + \mu_o (\tau(y) - \tau^-) F_n^- \quad , \tag{3.5.9}$$

and

$$\text{(ii)} \quad F(-\infty,0) = 0 \ , \quad F(+\infty,0) = F_n^+ - F_n^- \quad , \tag{3.5.10}$$

$$X(-\infty,+0) = 0 \ , \quad X(+\infty,+0) = X_n^+(+0) - X_n^-(+0) \ , \tag{3.5.11}$$

$$X(\pm\infty,-0) = 0 \ , \ G(\pm\infty,+0) = 0 \tag{3.5.12}$$

where F_n^- and F_n^+, and $X_n^-(+0)$ and $X_n^+(+0)$ are obtained by replacing τ_n in equations (3.3.32) and (3.3.31) by τ^- and τ^+, respectively.

As in E-polarization, the equation describing the surface boundary condition for the anomalous horizontal magnetic field is important, since it is this equation

which can be used to determine $X(y,+0)$ which, in turn, is used to determine $F(y,0)$ and $G(y,+0)$. By substituting equation (3.3.32) into (3.5.9) and remembering that $\alpha^2 = \omega\mu_o\sigma$ and $X(y,-0) = 0$, we obtain

$$X(y,+0) = \frac{\alpha^2\tau(y)}{\omega\sigma} F(y,0) - \frac{\sqrt{i}\alpha(\tau(y) - \tau^-)X_o}{\sigma\ (1 + \frac{\sqrt{i}\alpha\tau^-}{\sigma})} \qquad (3.5.13)$$

where from equations (3.5.5) and (3.5.6), $F(y,0)$ is known in terms of $X(u,+0)$ as

$$F(y,0) = \frac{\omega\sqrt{i}}{\alpha} \left\{ \frac{1}{\pi} \int_{-\infty}^{\infty} [X(u,+0) - X(y,+0)]\ P(y-u,+0)\ du \right.$$

$$\left. - X(y,+0) \right\} \qquad (3.5.14)$$

From equation (3.5.7), the vertical electric field just below the surface layer is given by

$$G(y,+0) = \frac{-\omega}{\alpha^2} X_1(y,+0) \ . \qquad (3.5.15)$$

These equations could be evaluated numerically in a similar way to the corresponding E-polarization equations.

In this thesis, we prefer to determine the horizontal electric field directly since it is more convenient to have $E(y,0)$ and $F(y,0)$, rather than $E(y,0)$ and $X(y,+0)$, immediately available for boundary values in three-dimensional problems. We will now show that an expression entirely in terms of the surface values of the horizontal electric field may be found. From equations (2.2.19), (2.2.20) and (2.2.1) and the definition of anomalous fields, it follows, in $z > 0$, that

$$F_{11}(y,z) + F_{22}(y,z) = i\alpha^2 F(y,z) \quad , \tag{3.5.16}$$

$$G_{11}(y,z) + G_{22}(y,z) = i\alpha^2 G(y,z) \quad , \tag{3.5.17}$$

and $$F_2(y,z) - G_1(y,z) = i\omega X(y,z) \quad . \tag{3.5.18}$$

From equation (3.4.49) in the previous section, we know that solving equation (3.5.16) leads to the expression

$$F_2(y,+0) = \sqrt{i}\alpha \left\{ \frac{1}{\pi} \int_{-\infty}^{\infty} [F(u,0) - F(y,0)] P(y-u,+0)\, du - F(y,0) \right\} \tag{3.5.19}$$

An expression for $G_1(y,+0)$ can be obtained by applying a Fourier transform in the y-variable to equation (3.5.17). The solution, which vanishes as $z \to \infty$, is

$$\hat{G}(\eta,z) = \hat{G}(\eta,+0)\ e^{-\gamma z} \qquad (3.5.20)$$

where $\gamma^2 = \eta^2 + i\alpha^2$.

By differentiating with respect to z at $z = +0$, and rearranging, we find that

$$\hat{G}(\eta,z) = \frac{-\hat{G}_2(\eta,+0)\ e^{-\gamma z}}{\gamma} \quad . \qquad (3.5.21)$$

Now, by taking the inverse Fourier transform of the above equation, we obtain

$$G(y,z) = \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-\gamma z}}{\gamma}\ \hat{G}_2(\eta,+0)\ e^{-i\eta y}\ d\eta \qquad (3.5.22)$$

which, after differentiating with respect to y, becomes

$$G_1(y,z) = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{\eta\ e^{-\gamma z}}{\gamma}\right) \hat{G}_2(\eta,+0)\ e^{-i\eta y}\ d\eta \quad . \qquad (3.5.23)$$

From tabulated results (Erdélyi et al., 1954, eq. 1.4.27 and eq. 2.4.36), we know that

$$\frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-\gamma z}}{\gamma} e^{-i\eta y} \, d\eta = -\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \frac{e^{-\gamma z}}{\gamma} \cos \eta y \, d\eta$$

$$= -\sqrt{\frac{2}{\pi}} K_0 (\alpha \sqrt{i} \sqrt{y^2+z^2}) \qquad (3.5.24)$$

and

$$\frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\eta e^{-\gamma z}}{\gamma} e^{-i\eta y} \, d\eta = -\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \frac{\eta e^{-\gamma z}}{\gamma} \sin \eta y \, d\eta$$

$$= -\sqrt{i}\alpha y \, P(y,z) \qquad (3.5.25)$$

and since the divergence of the electric field is zero, we also know that, in $z > 0$

$$G_2(y,z) = -F_1(y,z) \quad . \qquad (3.5.26)$$

Thus, by applying the Faltung theorem to equations (3.5.22) and (3.5.23) and using the above information, we finally obtain expressions for the vertical electric field and its

derivative with respect to y, just below the surface,

$$G(y,+0) = \frac{1}{\pi} \int_{-\infty}^{\infty} F_1(u,0)\ K_0(\sqrt{i}\alpha|y-u|)\ du$$

$$= \frac{-\sqrt{i}\alpha}{\pi} \int_{-\infty}^{\infty} F(u,0)\,(y-u)\ P(y-u,+0)\ du \qquad (3.5.27)$$

and

$$G_1(y,+0) = \frac{-\sqrt{i}\alpha}{\pi} \int_{-\infty}^{\infty} F_1(u,0)\,(y-u)\ P(y-u,+0)\ du\ . \qquad (3.5.28)$$

The second step in equation (3.5.27) is obtained by integration by parts. The desired expression for the horizontal electric field at the surface is found by substituting equation (3.5.9) into (3.5.18), giving

$$\frac{i\alpha^2\tau(y)}{\sigma} F(y,0) = F_2(y,+0) - G_1(y,+0) - i\alpha^2\left(\frac{\tau(y)-\tau^-}{\sigma}\right) F_n^- \qquad (3.5.29)$$

where $F_2(y,+0)$ and $G_1(y,+0)$ are known in terms of $F(y,0)$ by equations (3.5.19) and (3.5.28), respectively.

In the next chapter, we will determine the horizontal electric field at the surface numerically using the above equation. The horizontal magnetic field and the vertical electric field just below the surface can then be calculated using equations (3.5.13) and (3.5.15), respectively.

CHAPTER 4

NUMERICAL METHOD

4.1 Introduction

We begin this chapter by expressing the field equations of chapter 3 in dimensionless form so that the results will be as general as possible. We do this by scaling the length terms, u and y, with respect to the skin depth, $\delta = \frac{\sqrt{2}}{\alpha}$, of the lower half-space, i.e.

$$u' = \frac{u}{\delta} = \frac{\alpha u}{\sqrt{2}} \quad \text{and} \quad y' = \frac{y}{\delta} = \frac{\alpha y}{\sqrt{2}} \tag{4.1.1}$$

and by defining new dimensionless fields to be

$$\bar{E}' = \frac{\bar{E}}{\omega \delta B_o} \tag{4.1.2}$$

and

$$\bar{B}' = \frac{\bar{B}}{B_o} \tag{4.1.3}$$

where $B_o = Y_o$ in E-polarization and $B_o = X_o$ in B-polarization. Furthermore, we denote the dimensionless form of the integrated conductivity and of the function $P(y-u,+0)$ by

$$\tau'(y') = \frac{\tau(y)}{\sigma\delta} \tag{4.1.4}$$

and

$$P'(y'-u',+0) = \delta P(y-u,+0) \quad . \tag{4.1.5}$$

Here all dimensionless terms are denoted by primes (').
We may now write the E-polarization equations ((3.4.52), (3.4.37), (3.4.49) and (3.4.36)) in dimensionless form as

$$E'(y',0) = \left(\frac{Y'(y',-0) - Y'(y',+0)}{2\tau'(y')}\right) + \left(\frac{\sqrt{2i}}{1+\sqrt{2i}\tau'^{-}}\right)\left(\frac{\tau'^{-} - \tau'(y')}{2\tau'(y')}\right), \qquad (4.1.6)$$

$$Y'(y',-0) = \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{E_1'(u',0)}{y'-u'}\, du' \quad , \qquad (4.1.7)$$

$$Y'(y',+0) = i\sqrt{2i} \left\{ \frac{1}{\pi} \int_{-\infty}^{\infty} [E'(u',0) - E'(y',0)]\, P'(y'-u',+0)\, du' - E'(y',0) \right\} \quad , \qquad (4.1.8)$$

and $$Z'(y',0) = -iE_1'(y',0) \qquad (4.1.9)$$

and the B-polarization equations ((3.5.26), (3.5.13), (3.5.15), (3.5.19) and (3.5.25)) as

$$F'(y',0) = -i\left(\frac{F_2'(y',+0) - G_1'(y',+0)}{2\tau'(y')}\right) + \left(\frac{\sqrt{2i}}{1+\sqrt{2i}\tau'^{-}}\right)\left(\frac{\tau'^{-} - \tau'(y')}{2\tau'(y')}\right), \quad (4.1.10)$$

$$X'(y',+0) = 2\tau'(y')F'(y',0) + \left(\frac{\sqrt{2i}}{1+\sqrt{2i}\tau'^{-}}\right)(\tau'^{-} - \tau'(y')), \quad (4.1.11)$$

and $$G'(y',+0) = \frac{-1}{2} X_1'(y',+0) \quad (4.1.12)$$

where

$$F_2'(y',+0) = \sqrt{2i}\left\{\frac{1}{\pi}\int_{-\infty}^{\infty}[F'(u',0) - F'(y',0)]P'(y'-u',+0)\,du' - F'(y',0)\right\} \quad (4.1.13)$$

and

$$G_1'(y',+0) = \frac{-\sqrt{2i}}{\pi}\int_{-\infty}^{\infty}(y'-u')F_1'(u',0)P'(y'-u',+0)\,du' . \quad (4.1.14)$$

To simplify the notation, we will drop the primes from the above equations, in this chapter, since we will only be concerned with dimensionless terms.

The above equations can not be solved analytically so we must use numerical techniques. In order to express these equations in a form suitable for numerical evaluation, we first assume that the fields reach their boundary values at some finite points, y_1, and y_M (see figure 4.1) and remain at these values out to $y = \pm\infty$. (Note that, as an alternative in the E-polarization case, we can use the improved end condition described in section 4.4.) Within these finite boundaries, we choose a set of grid points y_j of variable step-size (see figure 4.1) at which we will determine the unknown field values. As a general rule, we choose the grid points in such a way that there are more points per given distance in the region of varying integrated conductivity. The accuracy of the resulting field values in a given region is generally increased by increasing the number of grid points and/or by extending the boundary points. One further point to mention is that the integrated conductivity is initially defined in the region between consecutive grid points. As shown by Brewitt-Taylor and Weaver (1976), the value at the grid point is found by taking a weighted average, given by

$$\tau(y_j) = \frac{k_{j-1}\tau(y_{j-\frac{1}{2}}) + k_j\tau(y_{j+\frac{1}{2}})}{k_{j-1} + k_j}$$

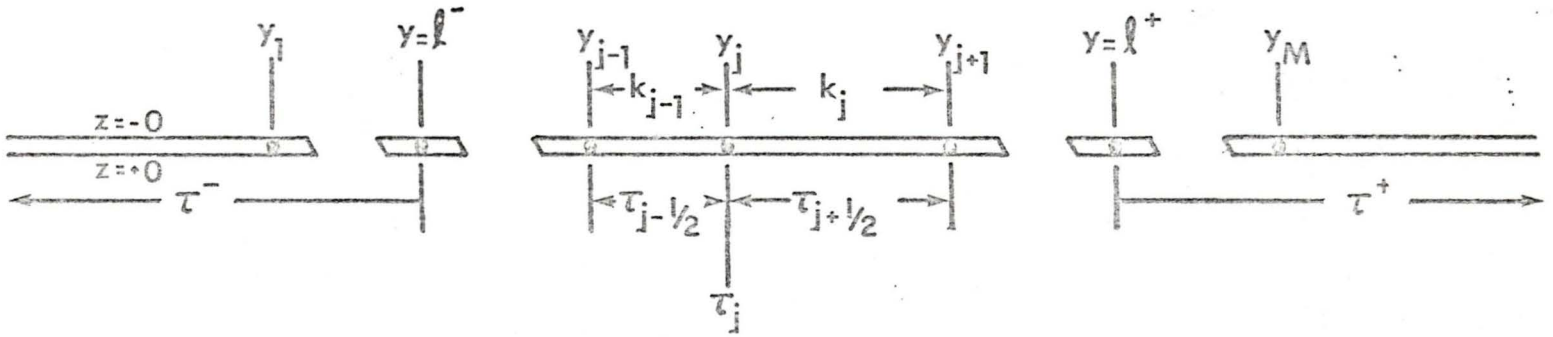

Figure 4.1 The set of grid points and the limited region of varying integrated conductivity ($\ell^- < y < \ell^+$).

where $\tau(y_{j-\frac{1}{2}})$ denotes the constant value of the integrated conductivity in the interval between y_{j-1} and y_j, and k_j denotes the distance between any consecutive grid points, y_j and y_{j+1} (i.e. $k_j = y_{j+1} - y_j$).

In the next section, we will see that at each of the M-2 grid points, y_j (j = 2,3,..., M-1), equations (4.1.6) and (4.1.10) can be written in terms of M-2 unknowns. We thus have a system of equations which may easily be expressed in matrix form. The unknown fields, $E(y_j,0)$ and $F(y_j,0)$, at each grid point can then be determined by using Gaussian elimination. Once $E(y_j,0)$ and $F(y_j,0)$ are determined, the other field components may easily be found.

4.2 Numerical form of E-polarization equations

It is convenient to show first the method used to evaluate numerically the integral expression for $Y(u,-0)$, assuming $E(u,0)$ is known. The same method also applies to the integral expression for $Y(u,+0)$. The required discrete form of equation (4.1.6) is then easily found.

We begin by writing the integral in equation (4.1.7) as a sum of integrals

$$-i\pi Y(y_m,-0) = \int_{-\infty}^{\infty} \frac{E_1(u,0)}{y_m-u}\,du$$

$$= \int_{-\infty}^{y_1} \frac{E_1(u,0)}{y_m-u}\,du + \int_{y_{m-1}}^{y_{m+1}} \frac{E_1(u,0)}{y_m-u}\,du$$

$$+ \int_{y_M}^{\infty} \frac{E_1(u,0)}{y_m-u}\,du$$

$$+ \sum_{\substack{j=2 \\ (j\neq m,m+1)}}^{M-1} \int_{y_{j-1}}^{y_j} \frac{E_1(u,0)}{y_m-u}\,du \qquad (4.2.1)$$

Here the first and third terms on the right vanish, since over these intervals, $E(u,0)$ is assumed constant (see section 4.1), and thus the derivative is zero.

Hartmann (1963) has pointed out that the effect of putting $E_1(u,0) = 0$ for $-\infty < u < y_1$ and $y_M < u < \infty$ causes results which are inaccurate, because the influence of $\frac{1}{y_m - u}$ in the infinite integral is cut off at y_1 and y_M. For a given region, though, this effect can be decreased to a desired accuracy by extending the boundary points. (A much better method of overcoming this effect is described in section 4.4.) Each of the remaining terms in equation (4.2.1) are evaluated by a variation of a method suggested by Hartmann (1963).

We now analyse the term involving the singularity. We begin by expanding $E(u,0)$ about y_m in a Taylor's series to second order, to give

$$E(u,0) = E(y_m,0) + (u-y_m)E_1(y_m,0) + \frac{(u-y_m)^2}{2} E_{11}(y_m,0) \quad (4.2.2)$$

which, on differentiation, becomes

$$E_1(u,0) = E_1(y_m,0) + (u-y_m)E_{11}(y_m,0) \quad (4.2.3)$$

$$(y_{m-1} \leq u \leq y_{m+1}) .$$

Three point finite-difference (F.D.) forms are used to evaluate $E_1(y_m,0)$ and $E_{11}(y_m,0)$. These forms can be determined, for variable step-size grids, by again using

equation (4.2.2). In this case, we write

$$E(y_{m+1},0) = E(y_m,0) + k_m E_1(y_m,0) + \frac{k_m^2}{2} E_{11}(y_m,0) \qquad (4.2.4)$$

and

$$E(y_{m-1},0) = E(y_m,0) - k_{m-1} E_1(y_m,0) + \frac{k_{m-1}^2}{2} E_{11}(y_m,0) \qquad (4.2.5)$$

where $$k_j = y_{j+1} - y_j \quad . \qquad (4.2.6)$$

The required finite difference expression for $E_1(u,0)$ is found by multiplying equation (4.2.4) by k_{m-1}^2 and equation (4.2.5) by k_m^2 and subtracting. The resulting equation can then be rearranged to give

$$E_1(y_m,0) = a_m E(y_{m-1},0) + b_m E(y_m,0) + c_m E(y_{m+1},0) \qquad (4.2.7)$$

where

$$a_m = \frac{-k_m^-}{k_{m-1} k_m^+} \quad , \quad b_m = \frac{k_m^-}{k_m k_{m-1}} \quad , \quad c_m = \frac{k_{m-1}}{k_m k_m^+} \qquad (4.2.8)$$

with $$k_m^+ = k_m + k_{m-1} \quad , \quad k_m^- = k_m - k_{m-1} \quad . \qquad (4.2.9)$$

Similarly, by multiplying equation (4.2.4) by k_{m-1} and equation (4.2.5) by k_m and adding, we obtain

$$E_{11}(y_m,0) = p_m E(y_{m-1},0) + q_m E(y_m,0) + r_m E(y_{m+1},0) \qquad (4.1.10)$$

where

$$p_m = \frac{2}{k_{m-1}k_m + } \ , \ q_m = \frac{-2}{k_m k_{m-1}} \ , \ r_m = \frac{2}{k_m k_m + } \ . \qquad (4.1.11)$$

To simplify notation, we will denote the fields at the general grid point y_j, by the grid subscript alone (i.e. $E_j = E(y_j,0)$). Using equation (4.2.3) and the F.D. forms for $E_1(y_m,0)$ and $E_{11}(y_m,0)$, we may now write

$$\int_{y_{m-1}}^{y_{m+1}} \frac{E_1(u,0)}{y_m - u}\,du = E_1(y_m,0) \int_{y_{m-1}}^{y_{m+1}} \frac{du}{y_m - u} + E_{11}(y_m,0) \int_{y_{m-1}}^{y_{m+1}} \frac{u-y_m}{y_m - u}\,du$$

$$= -(a_m E_{m-1} + b_m E_m + c_m E_{m+1}) \ell n \left(\frac{k_m}{k_{m-1}} \right)$$

$$- (p_m E_{m-1} + q_m E_m + r_m E_{m+1})\ k_m + \qquad (4.1.12)$$

where the first integral on the right-hand side has been evaluated by taking the C.P.V. of the integral as follows,

$$-\int_{y_{m-1}}^{y_{m+1}} \frac{du}{y_m - u} = \int_{-k_{m-1}}^{k_m} \frac{dv}{v} = \lim_{\varepsilon \to 0} \left\{ \int_{-k_{m-1}}^{-\varepsilon} + \int_{\varepsilon}^{k_m} \right\} \frac{dv}{v}$$

$$= \lim_{\varepsilon \to 0} \left\{ \int_{\varepsilon}^{k_m} - \int_{\varepsilon}^{k_{m-1}} \right\} \frac{dv}{v}$$

$$= \lim_{\varepsilon \to 0} \left\{ \ell n\, k_m - \ell n\, \varepsilon - \ell n\, k_{m-1} + \ell n\, \varepsilon \right\}$$

$$= \ell n \left(\frac{k_m}{k_{m-1}} \right) \quad . \tag{4.2.13}$$

Clearly the representation of $E_1(u,0)$ by equation (4.2.3) is equivalent to fitting a quadratic through the values of $E(u,0)$ at y_{m-1}, y_m and y_{m+1} and then differentiating with respect to u.

Away from the singularity, it is assumed that $E(u,0)$ varies approximately linearly between consecutive grid points so that $E_1(u,0)$ may be expressed by a simple two point F.D. form, given by

$$E_1(u,0) = \frac{E_j - E_{j-1}}{y_j - y_{j-1}} \qquad \begin{pmatrix} y_1 \le y_{j-1} \le u \le y_j \le y_{m-1} \\ y_{m+1} \le y_{j-1} \le u \le y_j \le y_M \end{pmatrix} \tag{4.2.14}$$

so that

$$\sum_{\substack{j=2 \\ (j\neq m,m+1)}}^{M} \int_{Y_{j-1}}^{Y_j} \frac{E_1(u,0)}{Y_m - u}\, du = \sum_{\substack{j=2 \\ (j\neq m,m+1)}}^{M} \left(\frac{E_j - E_{j-1}}{Y_j - Y_{j-1}}\right) \int_{Y_{j-1}}^{Y_j} \frac{du}{Y_m - u}$$

$$= - \sum_{\substack{j=2 \\ (j\neq m,m+1)}}^{M} \left\{ \left(\frac{E_j}{Y_j - Y_{j-1}}\right) \ell n \left(\frac{Y_j - Y_m}{Y_{j-1} - Y_m}\right) - \left(\frac{E_{j-1}}{Y_j - Y_{j-1}}\right) \ell n \left(\frac{Y_j - Y_m}{Y_{j-1} - Y_m}\right) \right\}$$

$$= - \sum_{\substack{j=2 \\ (j\neq m-1,m,m+1)}}^{M-1} E_j \left\{ \frac{1}{k_{j-1}} \ell n \left(\frac{Y_j - Y_m}{Y_{j-1} - Y_m}\right) - \frac{1}{k_j} \ell n \left(\frac{Y_{j+1} - Y_m}{Y_j - Y_m}\right) \right\}$$

$$+ \frac{E_1}{k_1} \ell n \left(\frac{Y_2 - Y_m}{Y_1 - Y_m}\right) - \frac{E_{m-1}}{k_{m-2}} \ell n \left(\frac{k_{m-1}}{k^{+}_{m-1}}\right) + \frac{E_{m+1}}{k_{m+1}} \ell n \left(\frac{k^{+}_{m+1}}{k_m}\right)$$

$$- \frac{E_M}{k_{M-1}} \ell n \left(\frac{Y_M - Y_m}{Y_{M-1} - Y_m}\right) \qquad (4.2.15)$$

where E_1 stands for $E(Y_1,0)$ and is not to be confused with the partial derivative $E_1(u,0)$. We may now write equation (4.2.1) in a form suitable for programming on a computer, as

$$-i\pi\, Y(Y_m,-0) \cong \sum_{j=1}^{M} Q_{mj} E_j \qquad (4.2.16)$$

where

$$Q_{mj} = \begin{cases} \dfrac{1}{k_j} \ell n \left(\dfrac{y_m - y_{j+1}}{y_m - y_j} \right) - \dfrac{1}{k_{j-1}} \ell n \left(\dfrac{y_m - y_j}{y_m - y_{j-1}} \right) & \begin{pmatrix} 2 \leq j \leq m-2 \\ m+2 \leq j \leq M-1 \end{pmatrix} \\ \dfrac{1}{k_1} \ell n \left(\dfrac{y_m - y_2}{y_m - y_1} \right) & (j = 1) \\ a_m \ell n \left(\dfrac{k_{m-1}}{k_m} \right) - p_m k_m^+ - \dfrac{1}{k_{m-2}} \ell n \left(\dfrac{k_{m-1}}{k_{m-1}^+} \right) & (j = m-1) \\ b_m \ell n \left(\dfrac{k_{m-1}}{k_m} \right) - q_m k_m^+ & (j = m) \\ c_m \ell n \left(\dfrac{k_{m-1}}{k_m} \right) - r_m k_m^+ - \dfrac{1}{k_{m+1}} \ell n \left(\dfrac{k_m}{k_{m-1}^+} \right) & (j = m+1) \\ \dfrac{1}{k_{M-1}} \ell n \left(\dfrac{y_m - y_{M-1}}{y_m - y_M} \right) & (j = M) \end{cases} \qquad (4.2.17)$$

with special cases when $m = 2$ and $m = M-1$, being

$$Q_{21} = a_2 \ell n\left(\frac{k_1}{k_2}\right) - p_2 k_2^{+}$$

(4.2.18)

and $$Q_{M-1,M} - c_{M-1} \ell n\left(\frac{k_{M-1}}{k_m}\right) - r_{M-1} k_{M-1}^{+} \quad .$$

Here, m varies from 2 to M-1 . (Unless the improved boundary condition of section 4.4 is used, the term Q_{m1} is not required since E_1 is zero by the boundary conditions. Here and later, terms associated with E_1 will be included for the sake of completeness.)

Following the procedure used to evaluate $Y(y_m,-0)$ the integral expression for the horizontal magnetic field just below the surface, given by equation (4.1.8), may be written as

$$-\sqrt{\frac{i}{2}}\,\pi Y(y_m,+0) = \int_{-\infty}^{\infty} [E(u,0) - E(y_m,0)]\, P(y_m-u,+0)\, du - \pi E(y_m,0)$$

$$= \left\{ \int_{-\infty}^{Y_1} + \int_{Y_M}^{\infty} \right\} E(u,0) P(y_m-u,+0)\, du + \sum_{\substack{j=2 \\ j\neq m,m+1}}^{M} \left\{ \int_{Y_{j-1}}^{Y_j} E(u,0) P(y_m-u,+0)\, du \right\}$$

$$-\left[\pi + \left\{ \int_{-\infty}^{y_{m-1}} + \int_{y_{m+1}}^{\infty} \right\} P(y_m-u,+0)\ du\right]E_m$$

$$+ \int_{y_{m-1}}^{y_{m+1}} \left[E(u,0) - E(y_m,0)\right] P(y_m-u,+0)\ du$$

$$= E_1 \int_{-\infty}^{y_1} P(y_m-u,+0)\ du + E_M \int_{y_M}^{\infty} P(y_m-u,+0)\ du$$

$$+ \sum_{j=2}^{m-1} \left\{ E_j \int_{y_{j-1}}^{y_j} P(y_m-u,+0)\ du + \left(\frac{E_j - E_{j-1}}{y_j - y_{j-1}}\right) \int_{y_{j-1}}^{y_j} (u-y_j) P(y_m-u,+0)\ du \right\}$$

$$+ \sum_{j=m+1}^{M-1} \left\{ E_j \int_{y_j}^{y_{j+1}} P(y_m-u,+0)\ du + \left(\frac{E_{j+1} - E_j}{y_{j+1} - y_j}\right) \int_{y_j}^{y_{j+1}} (u-y_j) P(y_m-u,+0)\ du \right\}$$

$$- E_m\left[\pi + \left\{ \int_{-\infty}^{y_{m-1}} + \int_{y_{m+1}}^{\infty} \right\} P(y_m-u,+0)\ du\right]$$

$$+ E_1(y_m,0) \int_{y_{m-1}}^{y_{m+1}} (u-y_m) P(y_m-u,+0)\ du$$

$$+ \frac{E_{11}(y_m,0)}{2} \int_{y_{m-1}}^{y_{m+1}} (u-y_m)^2 P(y_m-u,+0)\ du \qquad (4.2.19)$$

Here we have represented $E(u,0)$ near the singularity $(y_{m-1} < u < y_{m+1})$ by equation (4.2.2) and between consecutive grid points away from the singularity by

$$E(u,0) = E_j + (u-y_j)\left(\frac{E_j - E_{j-1}}{y_j - y_{j-1}}\right) \qquad (4.2.20)$$

$$\left(\begin{array}{c} y_1 \leq y_{j-1} \leq u \leq y_j \leq y_{m-1} \\ \\ y_{m+1} \leq y_{j-1} \leq u \leq y_j \leq y_M \end{array} \right)$$

We now denote the integral coefficients in equation (4.2.19) by the symbols U_{mj}, V_{mj} and W_m as follows:

$$U_{mj} = \begin{cases} -\sqrt{2i}\int_{-\infty}^{y_1} P(y_m-u,+0)\ du & (j=1) \\ -\sqrt{2i}\int_{y_{j-1}}^{y_j} P(y_m-u,+0)\ du & (2 \le j \le m-1) \\ \sqrt{2i}\,\pi + \sqrt{2i}\left\{\int_{-\infty}^{y_{m-1}} + \int_{y_{m+1}}^{\infty}\right\} P(y_m-u,+0)\ du & (j=m) \\ -\sqrt{2i}\int_{y_j}^{y_{j+1}} P(y_m-u,+0)\ du & (m+1 \le j \le M-1) \\ -\sqrt{2i}\int_{y_M}^{\infty} P(y_m-u,+0)\ du & (j=M) \end{cases} \tag{4.2.21}$$

$$V_{mj} = \begin{cases} \sqrt{2i}\int_{y_{j-1}}^{y_j} (u-y_j)P(y_m-u,+0)\ du & (2 \le j \le m-1) \\ -\sqrt{2i}\int_{y_{m-1}}^{y_{m+1}} (u-y_m)P(y_m-u,+0)\ du & (j=m) \\ -\sqrt{2i}\int_{y_j}^{y_{j+1}} (u-y_j)P(y_m-u,+0)\ du & (m+1 \le j \le M-1) \end{cases} \tag{4.2.22}$$

$$W_m = \frac{-\sqrt{2i}}{2}\int_{y_{m-1}}^{y_{m+1}} (u-y_m)^2\ P(y_m-u,+0)\ du \tag{4.2.23}$$

Each of these terms is written in a form suitable for programming in appendix A. In doing this, the handbook by Abramowitz and Stegun (1964) has been used extensively.

Using the above defined coefficients we may write equation (4.2.19) as

$$i\pi Y(y_m,+0) = \sum_{j=1}^{M} R_{mj} E_j \qquad (4.2.25)$$

where

$$R_{mj} = \begin{cases} U_{m1} - \dfrac{V_{m2}}{k_1} & (j = 1) \\ U_{mj} + \dfrac{V_{mj}}{k_{j-1}} - \dfrac{V_{mj+1}}{k_j} & (2 \leq j \leq m-2) \\ U_{mm-1} + \dfrac{V_{mm-1}}{k_{m-2}} + a_m V_{mm} + p_m W_m & (j = m-1) \\ U_{mm} + b_m V_{mm} + q_m W_m & (j = m) \\ U_{mm+1} - \dfrac{V_{mm+1}}{k_{m+1}} + c_m V_{mm} + r_m W_m & (j = m+1) \\ U_{mj} - \dfrac{V_{mj}}{k_j} + \dfrac{V_{mj-1}}{k_{j-1}} & (m+2 \leq j \leq M-1) \\ U_{mM} + \dfrac{V_{mM-1}}{k_{M-1}} & (j = M) \end{cases} \qquad (4.2.26)$$

with special cases when $m = 2$ and $m = M-1$, being

$$R_{21} = U_{m1} + a_2 V_{22} + p_2 W_2$$

and $$R_{M-1\,M} = U_{M-1\,M} + c_{M-1} V_{M-1\,M-1} + r_{M-1} W_{M-1} \quad .$$

Again, m varies from 2 to $M-1$.

Finally, equation (4.1.6) can be written as

$$E_m = \frac{i}{2\pi\tau_m} \left\{ \sum_{j=1}^{M} (Q_{mj} + R_{mj}) E_j + i\sqrt{2i}\pi \frac{(\tau_m - \tau^-)}{(1 + \sqrt{2i}\tau^-)} \right\} \qquad (4.2.27)$$

As mentioned in section 4.1, there are $M-2$ equations of this form in the $M-2$ unknowns, E_j $(j = 2,\ldots,M-1)$. (E_1 and E_M are the known boundary values.) The E_j's may be determined by expressing the system of equations in matrix form as

$$[A][E] = [B] \qquad (4.2.28)$$

where

$$A_{mj} = Q_{mj} + R_{mj} + 2\pi i \delta_{mj} \tau_m$$

and (4.2.29)

$$B_m = \frac{(1-i)\pi(\tau_m - \tau^-)}{(1 + \sqrt{2i}\tau^-)} - (R_{m1} + Q_{m1}) E_1 - (R_{mM} + Q_{mM}) E_M \ ,$$

and then using Gaussian elimination. Here, δ_{mj} is the Kronecker delta.

With the horizontal electric field known at the grid points, the horizontal magnetic field just above the surface can easily be determined from equation (4.2.16). The horizontal magnetic field just below the surface could be determined from equation (4.2.25) but, with $Y(y_m,-0)$ known, we can find $Y(y_m,+0)$ by simply rearranging equation (4.1.6). The vertical magnetic field $Z(u,0)$ can be determined at the midpoint between consecutive grid points by

$$Z(y_{m-\frac{1}{2}},0) = -i\left(\frac{E_m - E_{m-1}}{y_m - y_{m-1}}\right) \qquad (4.2.30)$$

where $$y_{m-\frac{1}{2}} = \frac{y_m + y_{m-1}}{2} \quad . \qquad (4.2.31)$$

The total fields can then be obtained by adding the normal fields at $y = -\infty$ to the corresponding anomalous fields. In dimensionless form, the normal fields are

$$E_n^-(0) = \sqrt{\frac{i}{2}}\left(\frac{1}{1 + \sqrt{2i}\tau^-}\right) \quad , \qquad (4.2.32)$$

$$Y_n^-(+0) = \left(\frac{1}{1 + \sqrt{2i\tau^-}} \right) \quad , \tag{4.2.33}$$

$$Y_n^-(-0) = 1 \quad , \tag{4.2.34}$$

and $$Z_n^-(0) = 0 \quad . \tag{4.2.35}$$

Here the subscript n follows the notation used in section 3.3 for normal fields and the postscripted - sign indicates that these are the fields determined at $y = -\infty$.

4.3 Numerical form of B-polarization equations

Having described the procedure required to express the E-polarization equations in numerical form, we can now easily express the B-polarization equations in the appropriate form. By comparing equation (4.1.13) with (4.1.8), the integral expression for $F_2(y_m,+0)$ may be written immediately as

$$F_2(y_m,+0) = \frac{-i}{\pi} \sum_{j=1}^{M} R_{mj} F_j \qquad (4.3.1)$$

where the R_{mj}'s are the same coefficients as given by equation (4.2.26). Furthermore, following the method of presentation used in the previous section and remembering that we have assumed $F(u,0)$ is constant (i.e. $F_1(u,0) = 0$) in the regions $-\infty < u < y_1$ and $y_M < u < \infty$, we write

$$G_1(y_m,+0) = -\frac{\sqrt{2i}}{\pi} \int_{-\infty}^{\infty} (y_m-u) F_1(u,0) P(y_m-u,+0)\, du$$

$$= \frac{-\sqrt{2i}}{\pi} \left[\left\{ \int_{-\infty}^{y_1} + \int_{y_M}^{\infty} \right\} (y_m - u) F_1(u,0) P(y_m - u, +0)\ du \right.$$

$$+ \sum_{\substack{j=2 \\ (j \neq m, m+1)}}^{M} \left\{ \int_{y_{j-1}}^{y_j} (y_m - u) F_1(y,0) P(y_m - u, +0)\ du \right\}$$

$$\left. + \int_{y_{m-1}}^{y_{m+1}} (y_m - u) F_1(u,0) P(y_m - u, +0)\ du \right]$$

$$= \frac{1}{\pi} \left[\sum_{j=2}^{m-1} \left(\frac{F_j - F_{j-1}}{y_j - y_{j-1}} \right) \tilde{V}_{mj} + F_1(y_m, 0)\ \tilde{V}_{mm} + F_{11}(y_m, 0)\ \tilde{W}_m \right.$$

$$\left. + \sum_{j=m+1}^{M-1} \left(\frac{F_{j+1} - F_j}{y_{j+1} - y_j} \right) \tilde{V}_{mj} \right]$$

$$= \frac{1}{\pi} \sum_{j=1}^{M} S_{mj} F_j \qquad (4.3.2)$$

where

$$\tilde{V}_{mj} = \begin{cases} \sqrt{2i} \displaystyle\int_{y_{j-1}}^{y_j} (u - y_m) P(y_m - u, +0)\ du & (2 \leq j \leq m-1) \\ \sqrt{2i} \displaystyle\int_{y_{m-1}}^{y_{m+1}} (u - y_m) P(y_m - u, +0)\ du & (j = m) \\ \sqrt{2i} \displaystyle\int_{y_j}^{y_{j+1}} (u - y_m) P(y_m - u, +0)\ du & (m+1 \leq j \leq M-1) \end{cases} \qquad (4.3.3)$$

$$\tilde{W}_m = \sqrt{2i} \int_{y_{m-1}}^{y_{m+1}} (y_m - u)^2 P(y_m - u, +0)\ du \qquad (4.3.4)$$

and

$$S_{mj} = \begin{cases} \dfrac{-\tilde{V}_{m2}}{k_1} & (j=1) \\ \dfrac{\tilde{V}_{mj}}{k_{j-1}} - \dfrac{\tilde{V}_{mj+1}}{k_j} & (2 \le j \le m-2) \\ a_m \tilde{V}_{mm} + p_m \tilde{W}_m + \dfrac{\tilde{V}_{mm-1}}{k_{m-2}} & (j=m-1) \\ b_m \tilde{V}_{mm} + q_m \tilde{W}_m & (j=m) \\ c_m \tilde{V}_{mm} + r_m \tilde{W}_m - \dfrac{\tilde{V}_{mm+1}}{k_{m+1}} & (j=m+1) \\ \dfrac{\tilde{V}_{mj-1}}{k_{j-1}} - \dfrac{\tilde{V}_{mj}}{k_j} & (m+2 \le j \le M-1) \\ \dfrac{\tilde{V}_{mM-1}}{k_{M-1}} & (j=M) \end{cases} \qquad (4.3.5)$$

with special cases when $m = 2$ and $m = M-1$, being

$$S_{21} = a_2 \tilde{V}_{22} + p_2 \tilde{W}_2 \tag{4.3.6}$$

and $$S_{M-1\,M} = c_{M-1}\, \tilde{V}_{M-1\,M-1} + r_{M-1} \tilde{W}_{M-1} \quad .$$

Both $\tilde{V}_{mj}$ and $\tilde{W}_m$ are written in a form suitable for programming in appendix A. Now, by substituting equations (4.3.1) and (4.3.2) into (4.1.10), we finally obtain $M-2$ equations of the form

$$F_m = \frac{-1}{2\pi i \tau_m} \left[\sum_{j=1}^{M} (R_{mj} + S_{mj}) F_j + i\sqrt{2i}\pi \, \frac{\tau^- - \tau_m}{1 + \sqrt{2i}\tau^-} \right] \tag{4.3.7}$$

which can be written in matrix form as

$$[A][F] = [B] \tag{4.3.8}$$

where

$$A_{mj} = R_{mj} + S_{mj} + 2\pi i \delta_{mj} \tau_m \tag{4.3.9}$$

and $$B_m = (1-i)\pi \left(\frac{\tau^- - \tau_m}{1 + \sqrt{2i}\tau^-} \right) - (R_{m1} + S_{m1}) F_1 - (R_{mM} + S_{mM}) F_M \quad .$$

The F_j's can then be determined using Gaussian elimination. With the F_j's known, it is a simple matter to calculate X_m using equation (4.1.11). The vertical electric field just below the surface can be calculated at the mid-point between consecutive grid points by writing equation (4.1.12) as

$$G(y_{m-\frac{1}{2}},+0) = -\frac{1}{2}\left(\frac{X_m - X_{m-1}}{y_m - y_{m-1}}\right) \quad . \tag{4.3.10}$$

The total fields are again determined by adding the normal fields at $y = -\infty$ to the corresponding anomalous fields. Here, the normal fields, in dimensionless units, are given by

$$F_n^-(0) = -\sqrt{\frac{i}{2}}\left(\frac{1}{1+\sqrt{2i}\tau^-}\right) \tag{4.3.11}$$

$$X_n^-(+0) = \frac{1}{1+\sqrt{2i}\tau^-} \tag{4.3.12}$$

and $$G_n^-(+0) = 0 \quad . \tag{4.3.13}$$

It is interesting to note that if we had evaluated the horizontal magnetic field directly by substituting equation (3.5.14) into (3.5.13), the resulting expression would lead to an ill-conditioned matrix when high conductivity contrasts are studied.

4.4 Improved boundary conditions for E-polarization

It has been shown (Weaver and Brewitt-Taylor, 1978) that

$$E_1(u,0) \rightarrow \frac{D^{\pm}}{u^2} \qquad (4.4.1)$$

and

$$E(u,0) \rightarrow E_n^{\pm} - \frac{D^{\pm}}{u} \qquad (4.4.2)$$

as $u \rightarrow \pm\infty$ (D^-, D^+, constants). In section 4.2, it was assumed that $E(u,0)$ is constant (and, thus, $E_1(u,0) = 0$) in the regions $-\infty < u < y_1$ and $y_M < u < \infty$. This assumption can be greatly improved (in the above intervals) by including the information given in relations (4.4.1) and (4.4.2). In the next chapter, it will be shown that the inclusion of these modifications in our numerical technique gives much better results than our earlier assumption and considerably reduces the computer time and storage space required to determine the values of the field components in the region of interest (i.e. varying conductivity).

We begin by assuming that

$$E_1(u,0) = \begin{cases} \dfrac{D^-}{u^2} & (-\infty < u \leq y_1) \\ \dfrac{D^+}{u^2} & (y_M \leq u < \infty) \end{cases} \qquad (4.4.3)$$

and

$$E(u,0) = \begin{cases} E_n^- - \dfrac{D^-}{u} & (-\infty < u \leq y_1) \\ E_n^+ - \dfrac{D^+}{u} & (Y_M \leq u < \infty) \end{cases} \qquad (4.4.4)$$

for y_1 and y_M sufficiently far from the region of interest. By evaluating equation (4.4.4) at $u = y_1$ and $u = y_M$, we immediately see that

$$D^- = y_1 (E_n^- - E_1) \qquad (4.4.5)$$

and

$$D^+ = y_M (E_n^+ - E_M) \quad . \qquad (4.4.6)$$

(Note that E_1 and E_M are no longer the values at $u = \pm\infty$.) Furthermore, we can use two-point F.D. forms to represent numerically the derivatives in equation (4.4.3) at the end grid points, giving

$$D^- = y_1^2 \left(\frac{E_2 - E_1}{y_2 - y_1} \right) \qquad (4.4.7)$$

and

$$D^+ = y_M^2 \left(\frac{E_M - E_{M-1}}{y_M - y_{M-1}} \right) \quad . \qquad (4.4.8)$$

Now by substituting D^- and D^+ back into equations (4.4.5) and (4.4.6) and solving for E_1 and E_M, we obtain

$$E_1 = s^- E_n^- + t^- E_2 \qquad (4.4.9)$$

and $$E_M = s^+ E_n^+ + t^+ E_{M-1} \qquad (4.4.10)$$

where $$s^- = \frac{y_2 - y_1}{y_2 - 2y_1} \quad , \quad t^- = \frac{y_1}{2y_1 - y_2} \quad ,$$

$$s^+ = \frac{y_M - y_{M-1}}{2y_M - y_{M-1}} \quad , \quad t^+ = \frac{y_M}{2y_M - y_{M-1}} \quad .$$

Finally we can eliminate E_1 and E_M from equations (4.4.5) and (4.4.6), to give

$$D^- = \left(\frac{y_1^2}{2y_1 - y_2} \right) (E_n^- - E_2) \qquad (4.4.11)$$

and $$D^+ = \left(\frac{y_M^2}{2y_M - y_{M-1}} \right) (E_n^+ - E_{M-1}) \quad . \qquad (4.4.12)$$

Modifications can now be made to the expressions for the horizontal magnetic field just above and below the

surface (given by equations (4.2.16) and (4.2.25)) to include the new boundary conditions. Clearly, we can write

$$-i\pi Y(y_m,-0) = \sum_{j=2}^{M-1} Q_{mj}E_j + Q_{m1}E_1 + Q_{mM}E_M + D^-H_m^- + D^+H_m^+ \qquad (4.4.13)$$

and

$$-i\pi Y(y_m,+0) = \sum_{j=2}^{M-1} R_{mj}E_j + (R_{m1}-U_{m1})E_1 + U_{m1}E_n^- - D^-I_m^- + (R_{mM}-U_{mM})E_M + U_{mM}E_n^+ - D^+I_m^+ \qquad (4.4.14)$$

where $H_m^- = \int_{-\infty}^{y_1} \frac{du}{u^2(y_m-u)}$, $H_m^+ = \int_{y_M}^{\infty} \frac{du}{u^2(y_m-u)}$

and $I_m^- = -\sqrt{2i}\int_{-\infty}^{y_1} \frac{P(y_m-u,+0)}{u}\,du$,

$$I_m^+ = -\sqrt{2i}\int_{y_M}^{\infty} \frac{P(y_m-u,+0)}{u}\,du \quad .$$

Both H_m^- and H_m^+ are easily found to be

$$H_m^- = \begin{cases} -\frac{1}{y_m}\left[\frac{1}{y_1} + \frac{1}{y_m}\ln\left(1 - \frac{y_m}{y_1}\right)\right] & (y_m \neq 0) \\ \\ \frac{1}{2y_1^2} & (y_m = 0) \end{cases} \qquad (4.4.15)$$

and

$$H_m^+ = \begin{cases} \frac{1}{y_m}\left[\frac{1}{y_M} + \frac{1}{y_m}\ln\left(1 - \frac{y_m}{y_M}\right)\right] & (y_m \neq 0) \\ \\ -\frac{1}{2y_M^2} & (y_m = 0) \end{cases} \qquad (4.4.16)$$

whereas I_m^- and I_m^+ are evaluated numerically and the procedure is described in detail at the end of appendix A. Now by substituting the expressions for D^-, D^+, E_1 and E_M in equations (4.4.9) through (4.4.12) into equations (4.4.13) and (4.4.14), we can define new coefficients $\tilde{Q}_{mj}$ and $\tilde{R}_{mj}$ such that

$$-i\pi Y(y_m,-0) = \sum_{j=2}^{M-1} \tilde{Q}_{mj}E_j + \tilde{Q}_{m1}E_n^- + Q_{mM}E_n^+ \qquad (4.4.17)$$

where

$$\tilde{Q}_{mj} = \begin{cases} s^-Q_{m1} + y_1 t^- H_m^- & (j=1) \\ Q_{m2} + t^-(Q_{m1} - y_1 H_m^-) & (j=2) \\ Q_{mj} & (3 \le j \le M-2) \\ Q_{mM-1} + t^+(Q_{mM} - y_M H_m^+) & (j=M-1) \\ s^+Q_{mM} + y_M t^+ H_m^+ & (j=M) \end{cases}$$

and

$$i\pi Y(y_m,+0) = \sum_{j=2}^{M-1} \tilde{R}_{mj}E_j + \tilde{R}_{m1}E_n^- + \tilde{R}_{mM}E_n^+ \qquad (4.4.18)$$

where

$$\tilde{R}_{mj} = \begin{cases} U_{m1} + s^-(R_{m1} - U_{m1}) - y_1 t^- I_m^- & (j=1) \\ R_{m2} + t^-(R_{m1} - U_{m1} + y_1 I_m^-) & (j=2) \\ R_{mj} & (3 \le j \le M-2) \\ R_m + t^+(R_{mM} - U_{mM} + y_M I_m^+) & (j=M-1) \\ U_{mM} + s^+(R_{mM} - U_{mM}) - y_M t^+ I_m^+ & (j=M) \end{cases}$$

(Note that we still have M-2 equations in M-2 unknowns as in section 4.2.)

CHAPTER 5

RESULTS AND DISCUSSION

5.1 Introductory remarks

Following the notation of the previous chapter, we define the dimensionless form of the total fields (indicated by primes) as

$$\bar{E}_t' = \frac{\bar{E}_t}{\omega\delta B_o}$$

and

$$\bar{B}_t' = \frac{\bar{B}_t}{B_o}$$

where $B_o = Y_o$ in E-polarization and $B_o = X_o$ in B-polarization. Furthermore, in this chapter, we denote the total fields just below and above the surface by superscripts as

$$A_t^{\pm} \equiv A(y, \pm 0)$$

where A_t represents X_t, Y_t, or G_t. We also recall, from section 4.1, that the dimensionless grid points and integrated conductivity are given by

$$y' = \frac{y}{\delta}$$

and $$\tau' = \frac{\tau}{\sigma\delta} = \sqrt{\frac{\pi\mu_o f}{\sigma}}\ \tau \quad . \qquad (5.1.1)$$

By taking a quick glance at equation (4.2.17), (4.2.26), (4.3.5) and the appropriate terms in appendix A, we find that the integral coefficients Q_{mj}, R_{mj} and S_{mj} are dependent only on the set of dimensionless grid points. Hence, these coefficients need be calculated only once for each grid and stored for future use. From equations (4.2.29) and (4.3.9), we also know that the matrix of coefficients A_{mj} and B_m, and consequently, the resulting field values, are known entirely in terms of the above integral coefficients, the dimensionless values of the integrated conductivity and the given boundary values. Thus we immediately see that a given set of dimensionless field results may be scaled to represent the field values for various conductivity structures and source frequencies since, for a given τ', the model parameters f, σ, and τ may be varied in accordance with equation (5.1.1). We must remember, though, that the limitations imposed by the thin sheet approximation somewhat restrict the choice of values for the model parameters.

5.2 Demonstration of the effectiveness of the improved boundary condition for E-polarization problems

The effect of the improved E-polarization boundary condition, discussed in section 4.4, can be seen by setting up a 75 point grid (-8.5 → 8.5) and then comparing results obtained from the full grid with results obtained from a 45 point subset of this grid (-2.5 to 2.5). To illustrate this effect, a model is chosen in which the integrated conductivity is

$$\tau' = \tau/\sigma\delta = \begin{cases} 10^4 & (y' < 0) \\ 0 & (y' > 0) \end{cases}$$

and the grid (symmetrical about $y' = 0$) is

$$y' = y/\delta = \begin{cases} 0,.05,.1,.15,.2,.25,.3,.35,.4,.45,.5,.6,.7, \\ .8,.9,1,1.2,1.4,1.6,1.8,2,2.25,2.5,2.75,3, \\ 3.25,3.5,3.75,4,4.5,5,5.5,6,6.5,7,7.5,8,8.5 \end{cases}.$$

The values for the field components at some discrete points are given in Table 5.1. As expected, the field values, determined with or without the correction, are almost identical when the full grid is used. With the correction term applied to the partial grid, the agreement is still very good whereas, without the correction, the values are considerably different. The largest relative differences

TABLE 5.1

A comparison of field values, showing the effect of the improved boundary conditions for E-polarization problems.

* × 10^{-7}

		FULL GRID WITH CORRECTION		FULL GRID WITHOUT CORRECTION		PARTIAL GRID WITH CORRECTION		PARTIAL GRID WITHOUT CORRECTION	
	y/δ	Real	Imag	Real	Imag	Real	Imag	Real	Imag
$\frac{E_t}{\omega\epsilon y_o}$	-2.0	.533 *	- .028 *	.533 *	- .028 *	.533 *	- .027 *	.532 *	- .029 *
	-1.0	.559 *	- .058 *	.559 *	- .058 *	.559 *	- .058 *	.558 *	- .061 *
	-0.5	.622 *	- .116 *	.622 *	- .116 *	.622 *	- .115 *	.619 *	- .119 *
	-0.1	1.017 *	- .347 *	1.016 *	- .347 *	1.018 *	- .345 *	1.000 *	- .355 *
	-0.05	1.316 *	- .491 *	1.316 *	- .492 *	1.317 *	- .489 *	1.303 *	- .502 *
	0.0	4.985 *	-1.996 *	4.983 *	-1.998 *	4.990 *	-1.986 *	4.931 *	-2.044 *
	0.05	.0515	.1210	.0516	.1210	.0513	.1212	.0527	.1197
	0.1	.0777	.1781	.0778	.1780	.0774	.1783	.0796	.1760
	0.5	.1962	.3728	.1963	.3726	.1953	.3734	.2010	.3672
	1.0	.2867	.4595	.2869	.4592	.2852	.4604	.2960	.4502
	2.0	.3880	.5054	.3883	.5050	.3837	.5063	.4204	.4902
$\frac{y_t}{y_o}$	-2.0	1.070	- .0546	1.070	- .0550	1.070	- .0539	1.069	- .0579
	-1.0	1.126	- .0909	1.126	- .0914	1.127	- .0899	1.125	- .0964
	-0.5	1.219	- .1443	1.219	- .1449	1.219	- .1429	1.216	- .1521
	-0.1	1.651	- .3583	1.651	- .3591	1.652	- .3557	1.642	- .3726
	-0.05	1.957	- .4988	1.957	- .4997	1.958	- .4955	1.944	- .5166
	0.0	3.140	1.000	3.139	-1.002	3.142	- .9946	3.113	-1.031
	0.05	.656	.0061	.656	.0057	.656	.0072	.656	- .0005
	0.1	.664	.0133	.664	.0130	.664	.0144	.664	.0064
	0.5	.726	.0578	.726	.0574	.726	.0592	.728	.0481
	1.0	.794	.0841	.794	.0836	.793	.0860	.798	.0707
	2.0	.887	.0859	.887	.0853	.884	.0886	.902	.0651
$\frac{z_t}{y_o}$	-1.9	-.015 *	- .014 *	-.015 *	- .014 *	-.015 *	- .014 *	-.016 *	- .013 *
	-1.1	-.053 *	- .047 *	-.053 *	- .047 *	-.053 *	- .047 *	-.054 *	- .046 *
	-0.55	-.177 *	- .215 *	-.177 *	- .215 *	-.176 *	- .216 *	-.180 *	- .211 *
	-0.025	-.003	- .007	-.003	- .007	-.003	- .007	-.003	- .007
	0.025	2.425	-1.021	2.424	-1.022	2.426	-1.016	2.398	-1.045
	0.55	.247	- .213	.246	- .213	.247	- .212	.239	- .220
	1.1	.087	- .136	.087	- .136	.088	- .134	.080	- .149
	1.9	.017	- .072	.016	- .072	.016	- .068	.014	- .110

are noted in the imaginary components.

In order not to waste computer time and storage space, the improved boundary condition is used in all of the following E-polarization calculations.

5.3 Comparison with other work

In this section, results obtained by the method described in this thesis are compared with those calculated by other authors for the particular model of an infinitely conducting half-sheet, i.e.

$$\tau' = \begin{cases} \infty & (y' < 0) \\ 0 & (y' > 0) \end{cases}$$

covering a conducting half-space. In determining our results for both polarizations, we have approximated $\tau' = \infty$ by $\tau' = 10^{12}$ and used the following 75 point grid (symmetrical about $y' = 0$)

$$y' = \begin{cases} .0,.01,.02,.04,.06,.08,.1,.125,.15,.175,.2,.25,.3, \\ .35,.4,.45,.5,.6,.7,.8,.9,1,1.2,1.4,1.6,1.8,2.0, \\ 2.2,2.4,2.6,2.8,3.0,3.2,3.4,3.6,3.8,4,4.2 \quad . \end{cases}$$

Fischer et al. (1978a,1978b) have obtained the solution for this particular model in the E-polarization case, by incorporating directly into the theory the fact that the horizontal electric field must be zero at the surface of the infinitely conducting half-sheet. They solved their integral equation (which can be shown to be a special case of our equation (3.4.52)) numerically by assuming the electric field is constant between each grid point and by using an evenly-spaced grid. Initially, they used

90 grid points between 0 and 9. Then, to improve the resolution in the vicinity of the origin, the field values were recalculated using 80 grid points between 0 and 2. (The advantages of incorporating uneven grid spacing and the improved boundary conditions into the theory as we have done can be seen immediately.) On comparing our results with those of Fischer et al., we see from figures 5.1 and 5.3 that the real and imaginary parts of the horizontal electric field and the vertical magnetic field agree well. The real and imaginary parts of the horizontal magnetic field also agree well for $y' < 0$ but quite different results are obtained for $y' > 0$ (see figure 5.2). At the interface ($y' = 0$), we find that the real and imaginary parts of the horizontal magnetic field are discontinuous, diverging as $y' \to 0$ and smoothly approaching the finite values, .65 and 0, as $y' \to +0$. Fischer et al. suggest a different value of -1.14 for the real part of the magnetic field at $y' = +0$. To help resolve this discrepancy, we note that Weidelt (1971) obtained an analytical solution to a similar problem by using the Weiner-Hopf technique. (He considered a model having an infinitely conducting half-sheet separated from an infinitely conducting sheet by a non-conducting region.) In his paper (Weidelt, 1971, fig. 6),

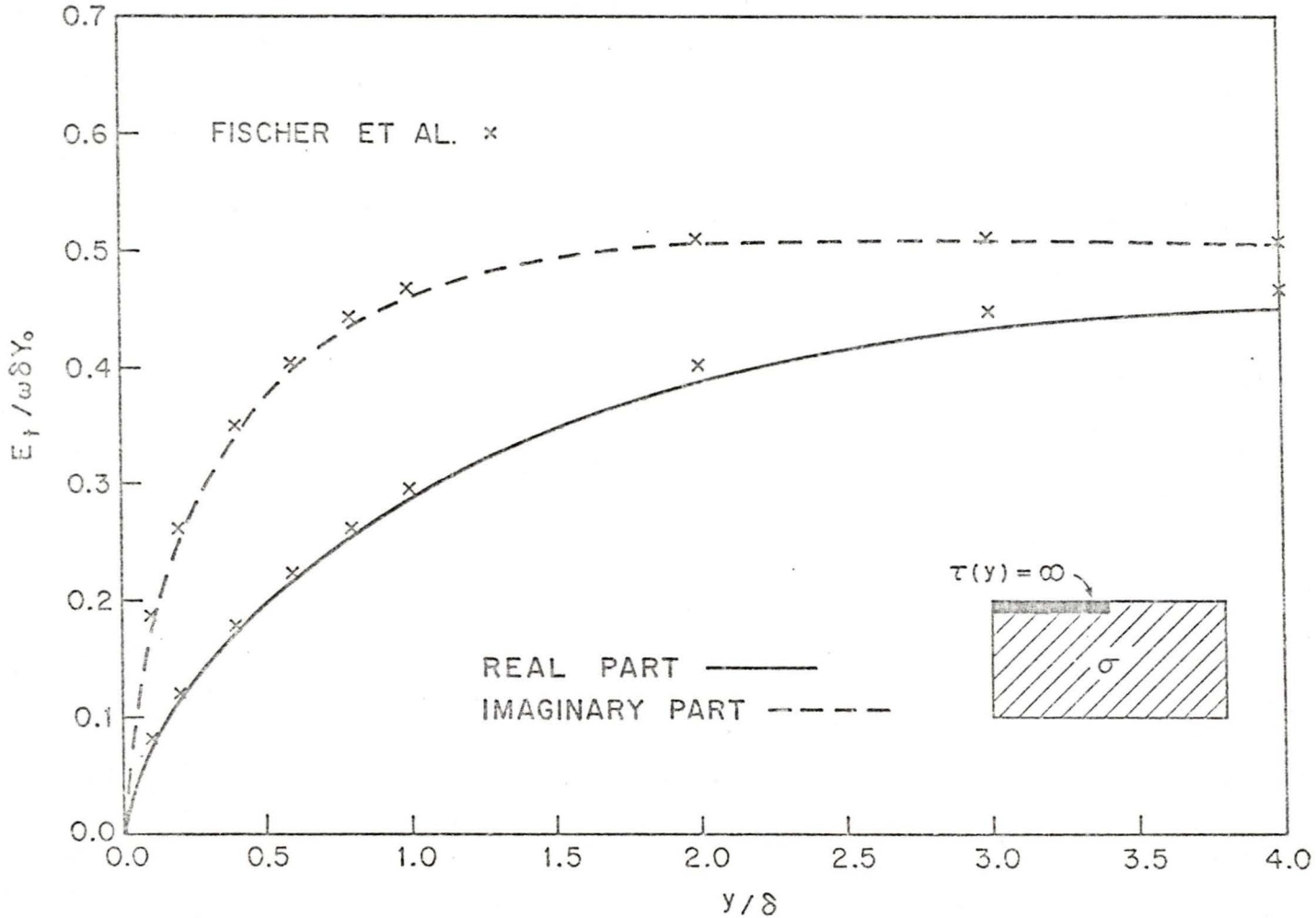

Figure 5.1 The real and imaginary parts of the horizontal electric field (z = 0) for a perfectly conducting thin sheet (z = 0, y < 0) covering a conducting half-space shown in the diagram (E-polarization). (Results are compared with those of Fischer et al. (1978b).)

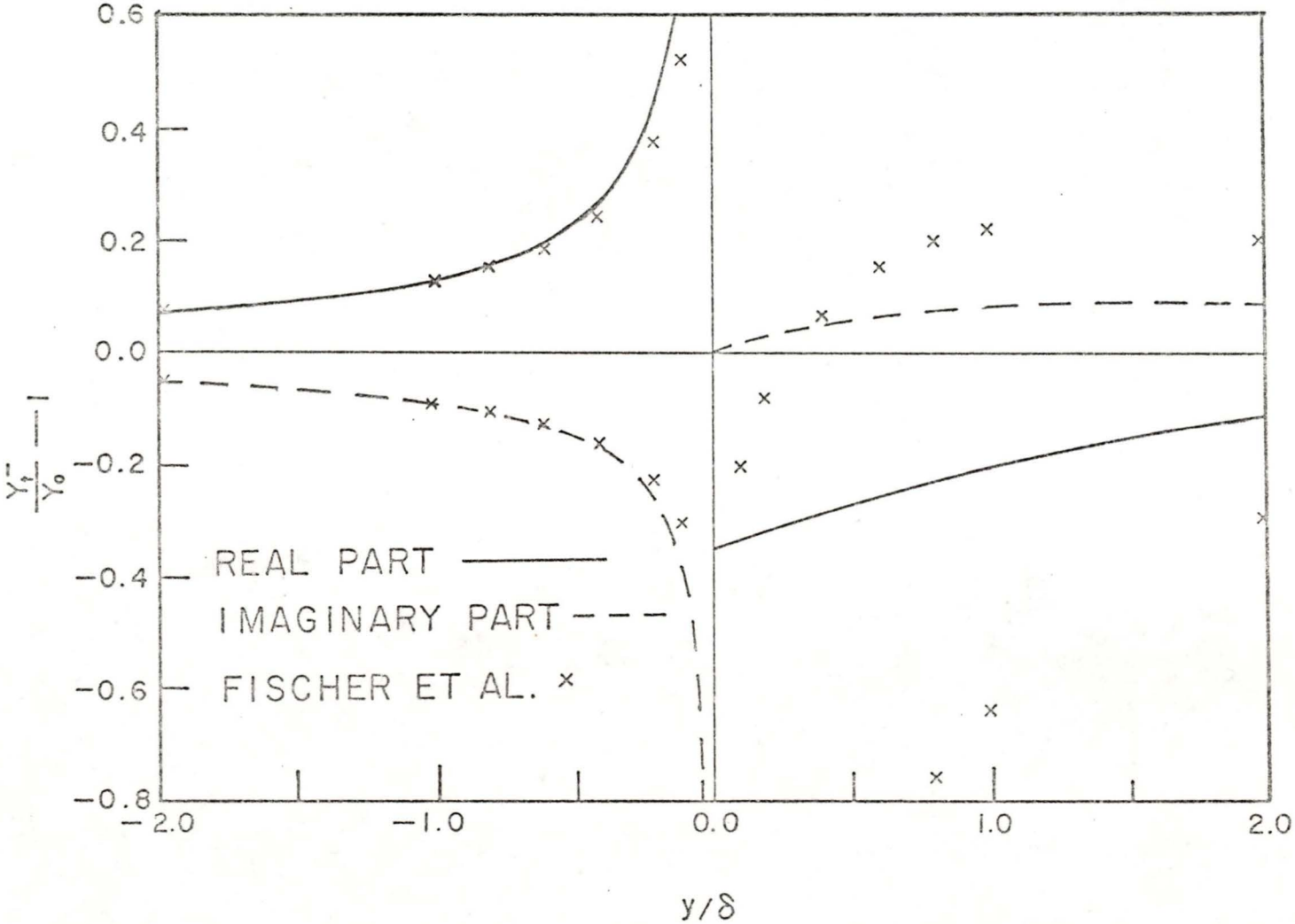

Figure 5.2 The real and imaginary parts of the horizontal magnetic field (z = -0) for the model shown in Figure 5.1 (E-polarization). (Results are compared with those of Fischer et al. (1978b).)

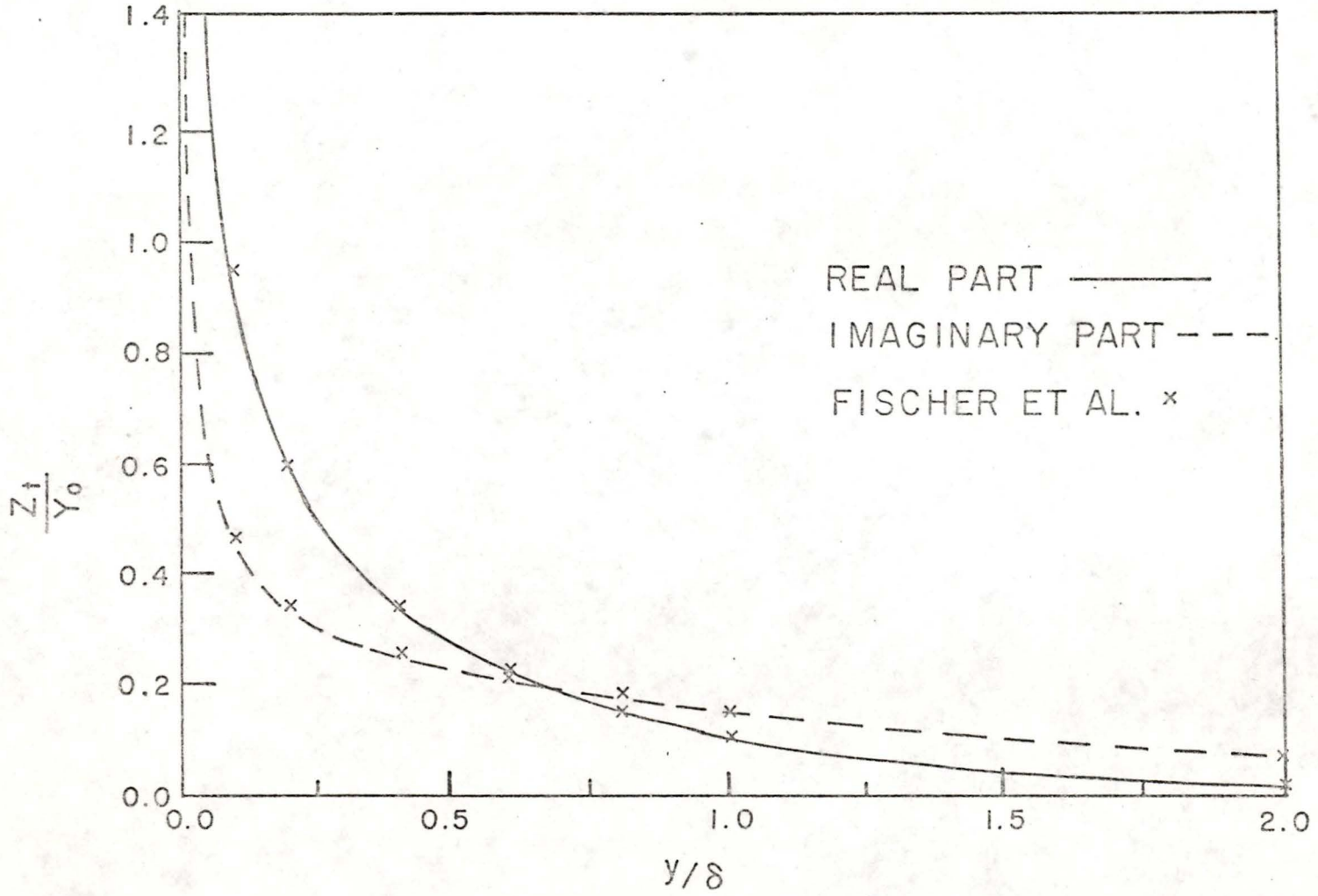

Figure 5.3 The real and imaginary parts of the vertical magnetic field (z = 0) for the model shown in Figure 5.1 (E-polarization). (Results are compared with those of Fischer et al. (1978b).)

we find, at $y' = +0$, that

$$\frac{H_{xi} - H_o}{H_o} \approx -.67 \qquad (5.3.1)$$

where H_o denotes the uniform external magnetic field and H_{xi} denotes the internal horizontal magnetic field and also that total horizontal field H_x , is given by

$$H_x = H_{xi} + H_o \quad .$$

Hence, we obtain $H_x(\pm\infty,0) = 2H_o$ since $H_{xi}(\pm\infty,0) = H_o$. Clearly, equation (5.3.1) becomes

$$\frac{H_x(+0,0)}{2H_o} \approx .67$$

which compares directly with our result of

$$\frac{Y_t(+0,-0)}{Y_o} \approx .65 \quad .$$

This leads us to believe that our results for the horizontal magnetic field for $y' > 0$ are indeed reasonable. As a further check on our numerical procedure, work is continuing at the University of Victoria to solve the problem of E-polarization and B-polarization induction in two half-sheets of finite

integrated conductivity covering a conducting half-space.

From figures 5.4, 5.5, and 5.6, all field component values, in the B-polarization case, agree well with field values obtained by Nicoll and Weaver (1977) whose analytical solution was determined by applying the Weiner-Hopf technique to the problem.

Later, in section 5.5, we also compare our results with those determined by a finite difference method and obtain excellent agreement. This gives additional verification of the validity of our numerical scheme.

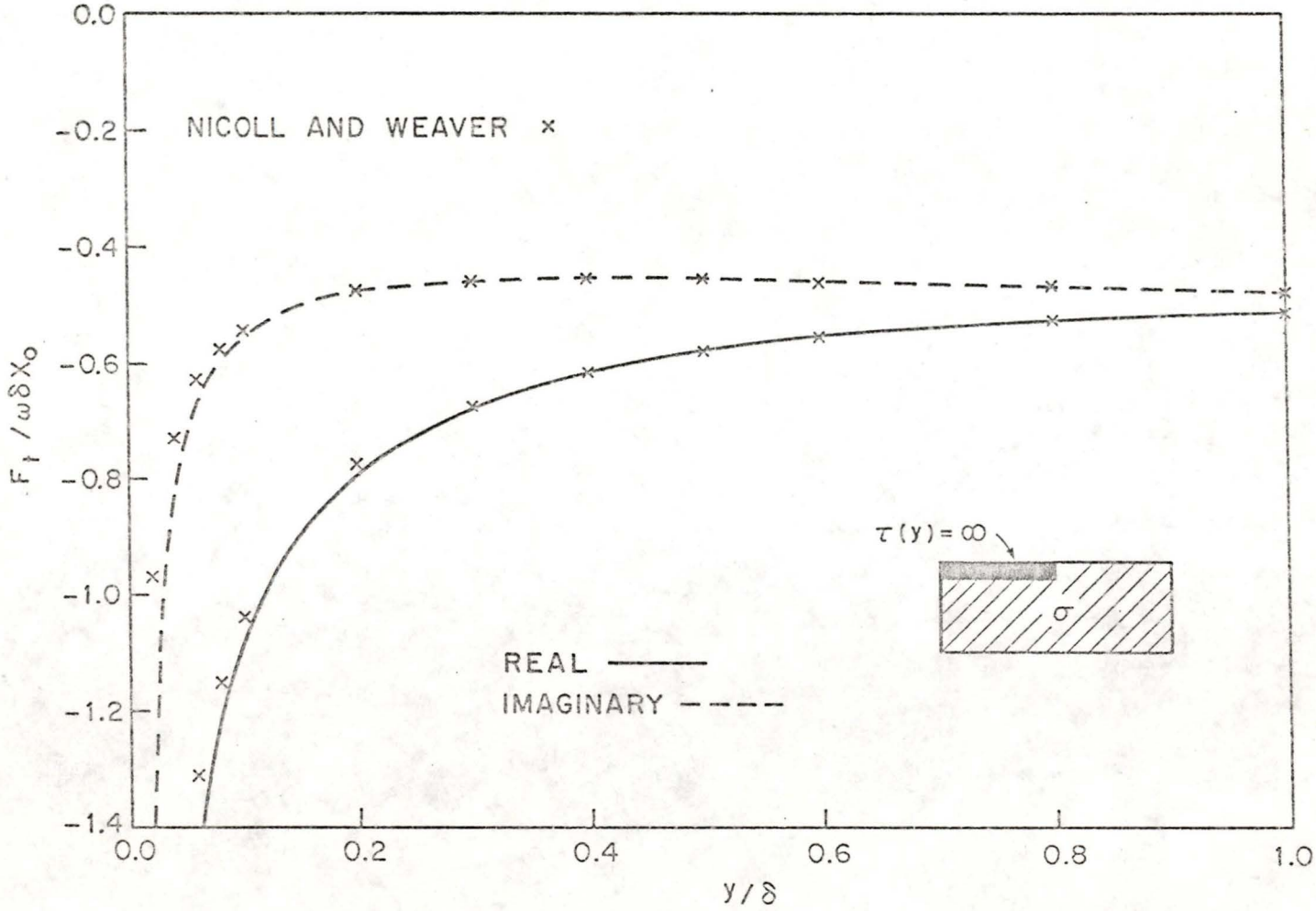

Figure 5.4 The real and imaginary parts of the horizontal electric field (z = 0) for a perfectly conducting thin sheet (z = 0, y < 0) covering a conducting half-space shown in the diagram (B-polarization). (Results are compared with those of Nicoll and Weaver (1977).)

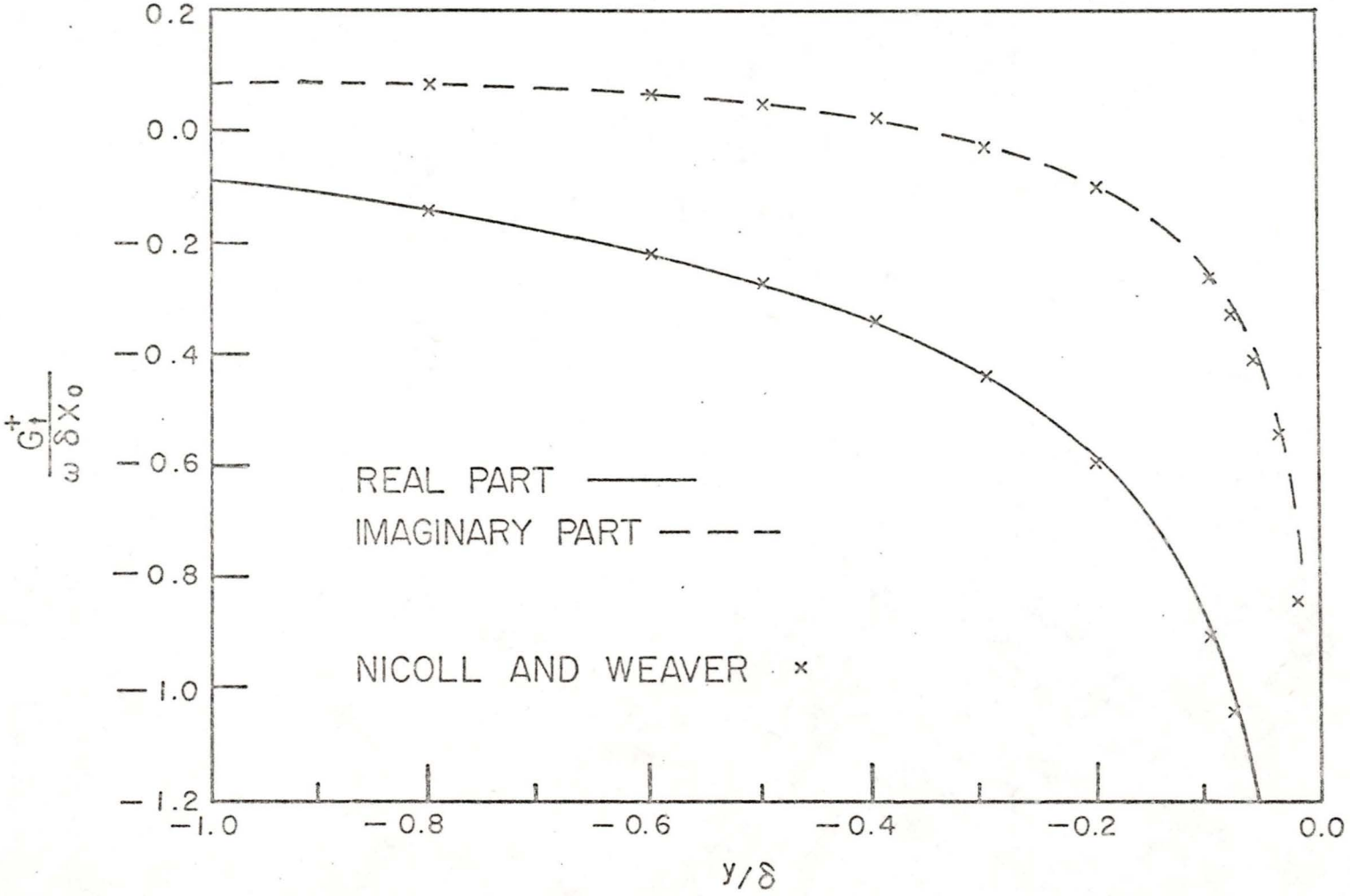

Figure 5.5 The real and imaginary parts of the vertical electric field (z = +0) for the model shown in Figure 5.4 (B-polarization). (Results are compared with those of Nicoll and Weaver (1977).)

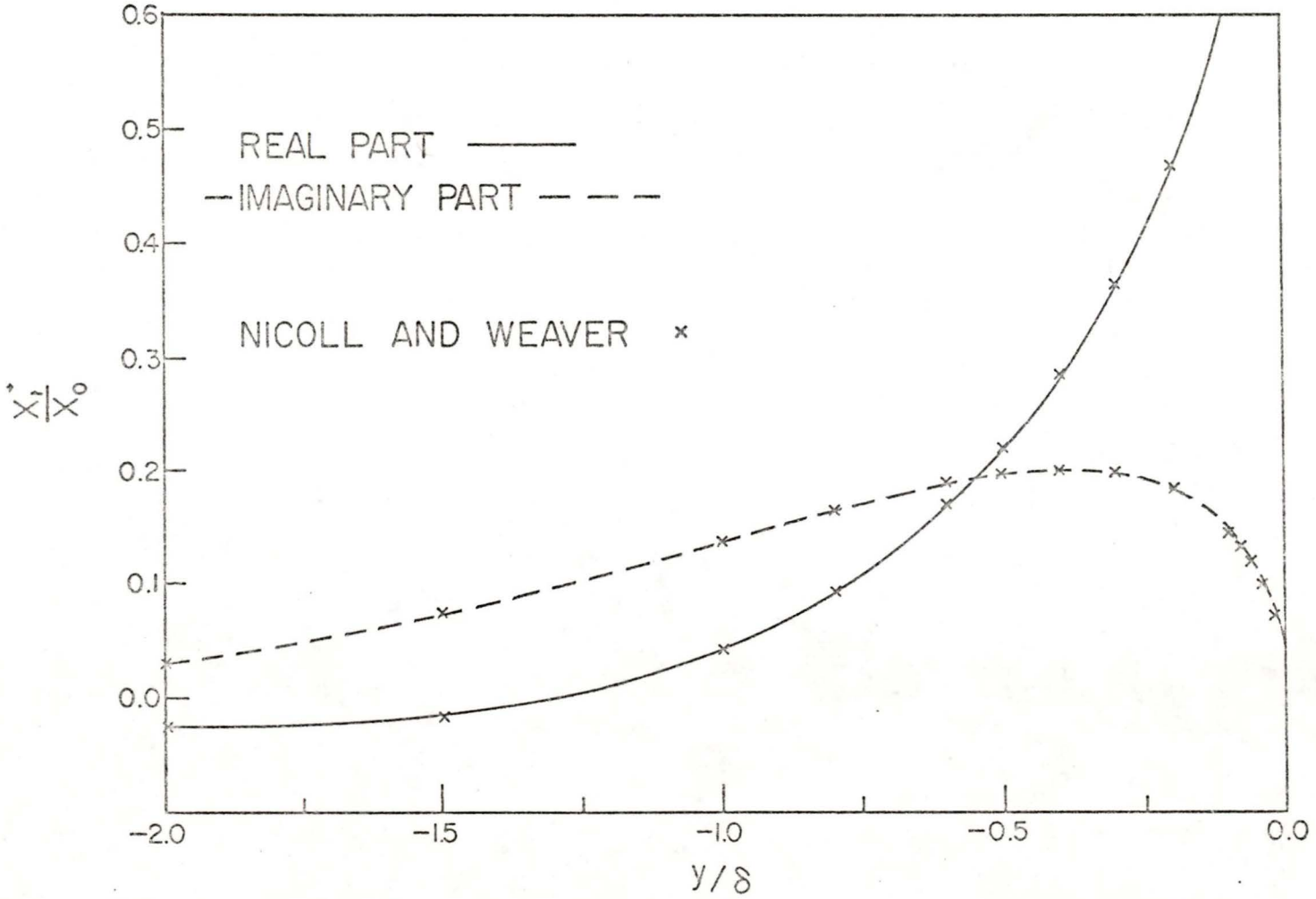

Figure 5.6 The real and imaginary parts of the horizontal magnetic field (z = +0) for the model shown in Figure 5.4 (B-polarization). (Results are compared with those of Nicoll and Weaver (1977).)

5.4 Comparison of abrupt and gradual changes in integrated conductivity

In this section, two similar models are considered in which τ' changes from 2π to π. In the first model, τ' changes abruptly at $y' = 0$ while, in the second model, τ' changes smoothly over one-half skin-depth (see figures 5.7 through 5.13).

For the E-polarization case, with an abrupt change in τ', the real component of the vertical magnetic field diverges as the interface is approached while the real part of the horizontal magnetic field jumps discontinuously at the interface. The remaining E-polarization components are continuous in this region. As in section 5.3, this is consistent with results obtained by Weidelt (1971, figure 2) for a similar model of two half-sheets having integrated conductivities $\tau_1 = 1$ and $\tau_2 = 4$, respectively, overlying a non-conducting region.

Similarly in B-polarization, for an abrupt change in τ', the real part of the vertical electric field diverges as $y' \to 0$ and the real part of the horizontal electric field jumps discontinuously at $y' = 0$, while the remaining field components are continuous in the region.

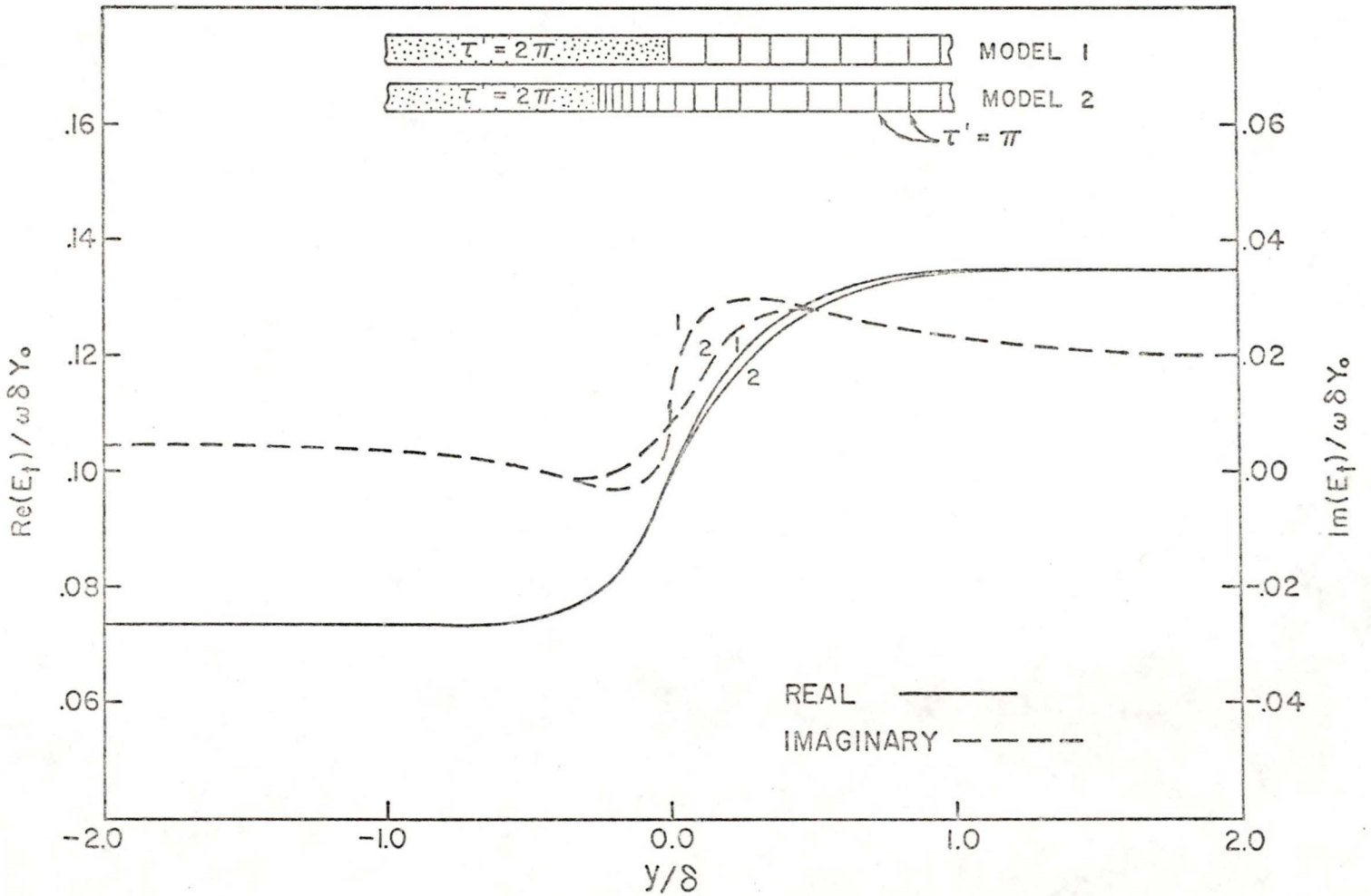

Figure 5.7 The effect of abrupt and gradual changes in integrated conductivity on the horizontal electric field at the surface (E-polarization).

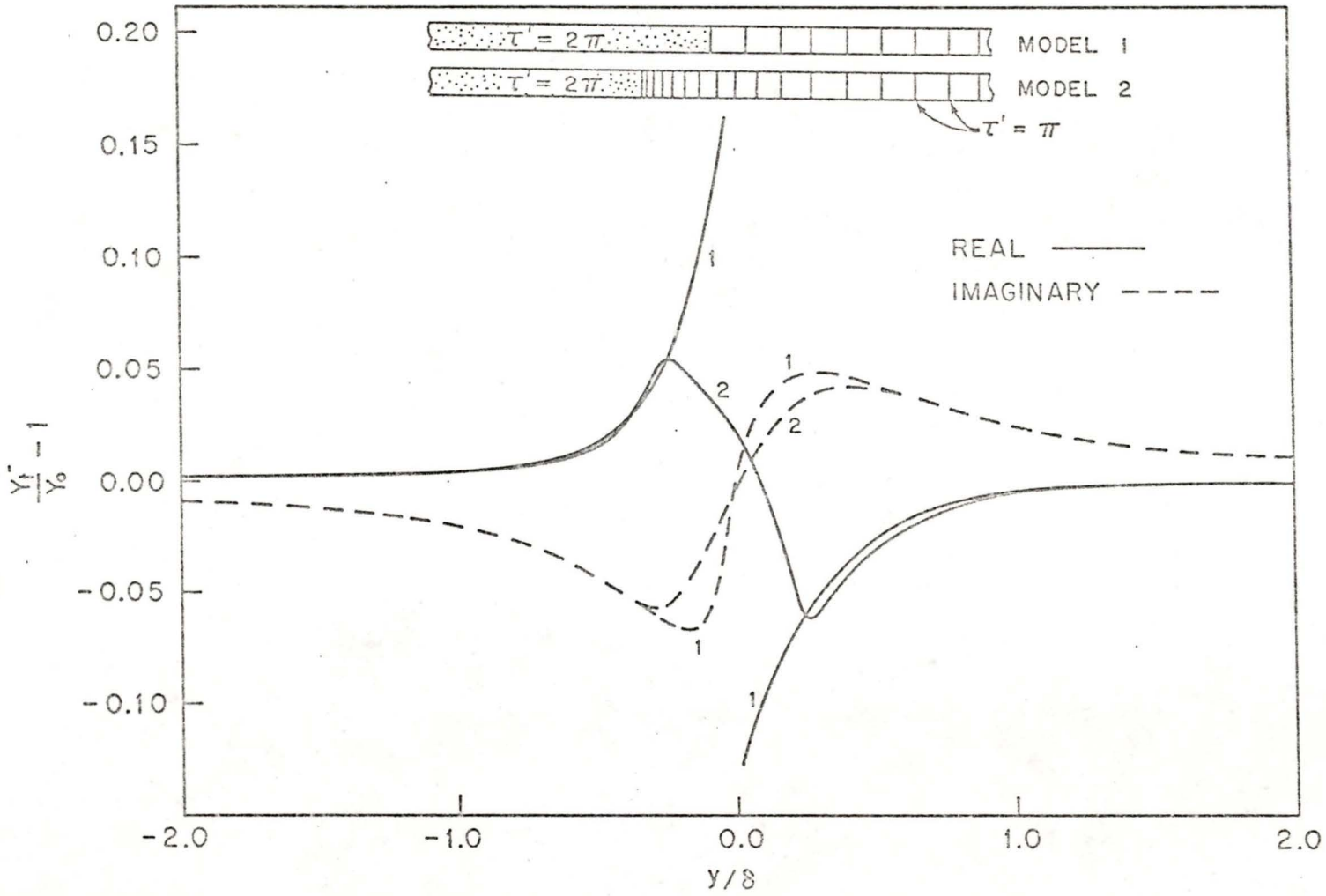

Figure 5.8 The effect of abrupt and gradual changes in integrated conductivity on the horizontal magnetic field at the surface (E-polarization).

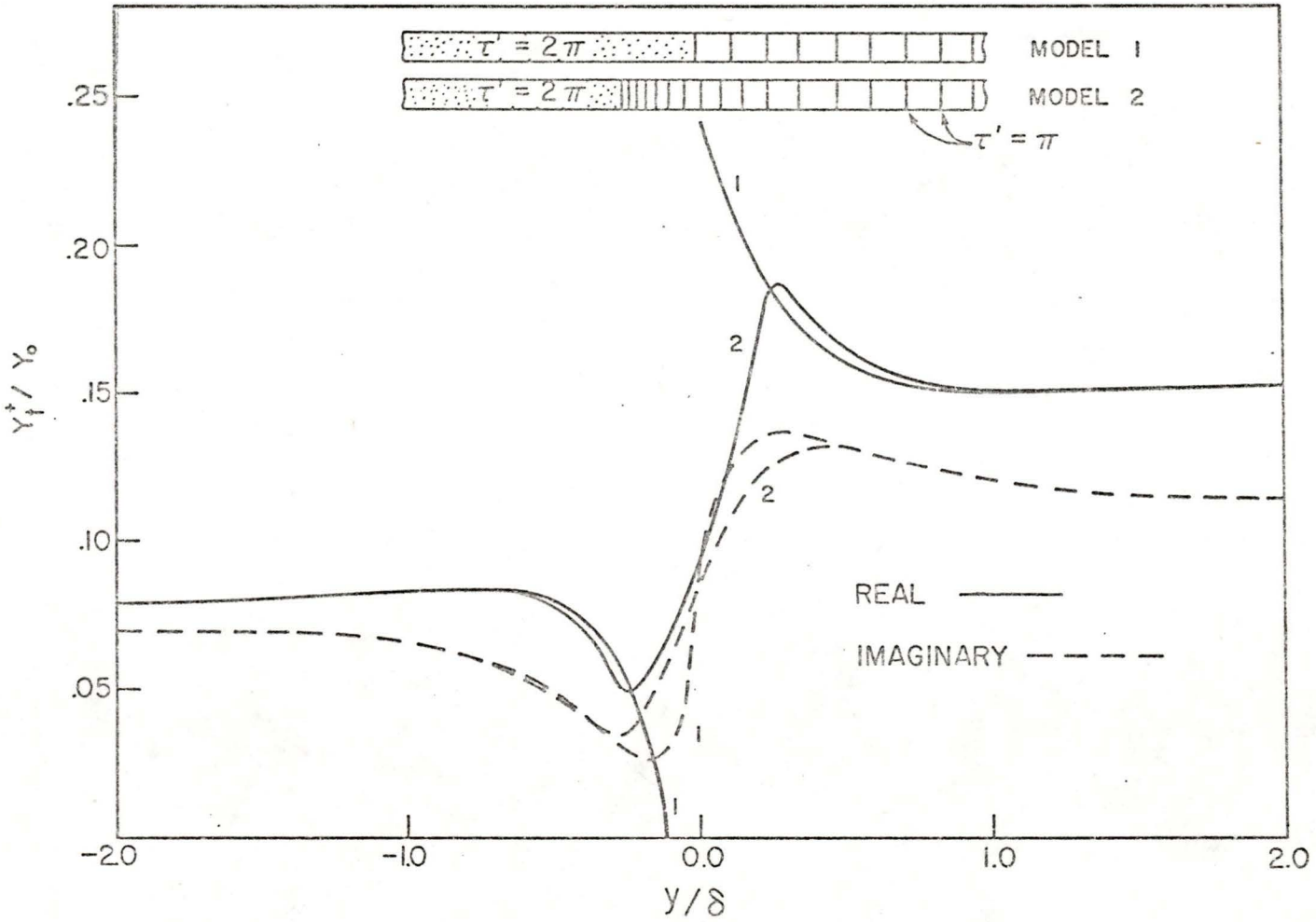

Figure 5.9 The effect of abrupt and gradual changes in integrated conductivity on the horizontal magnetic field just below the infinitely thin sheet (E-polarization).

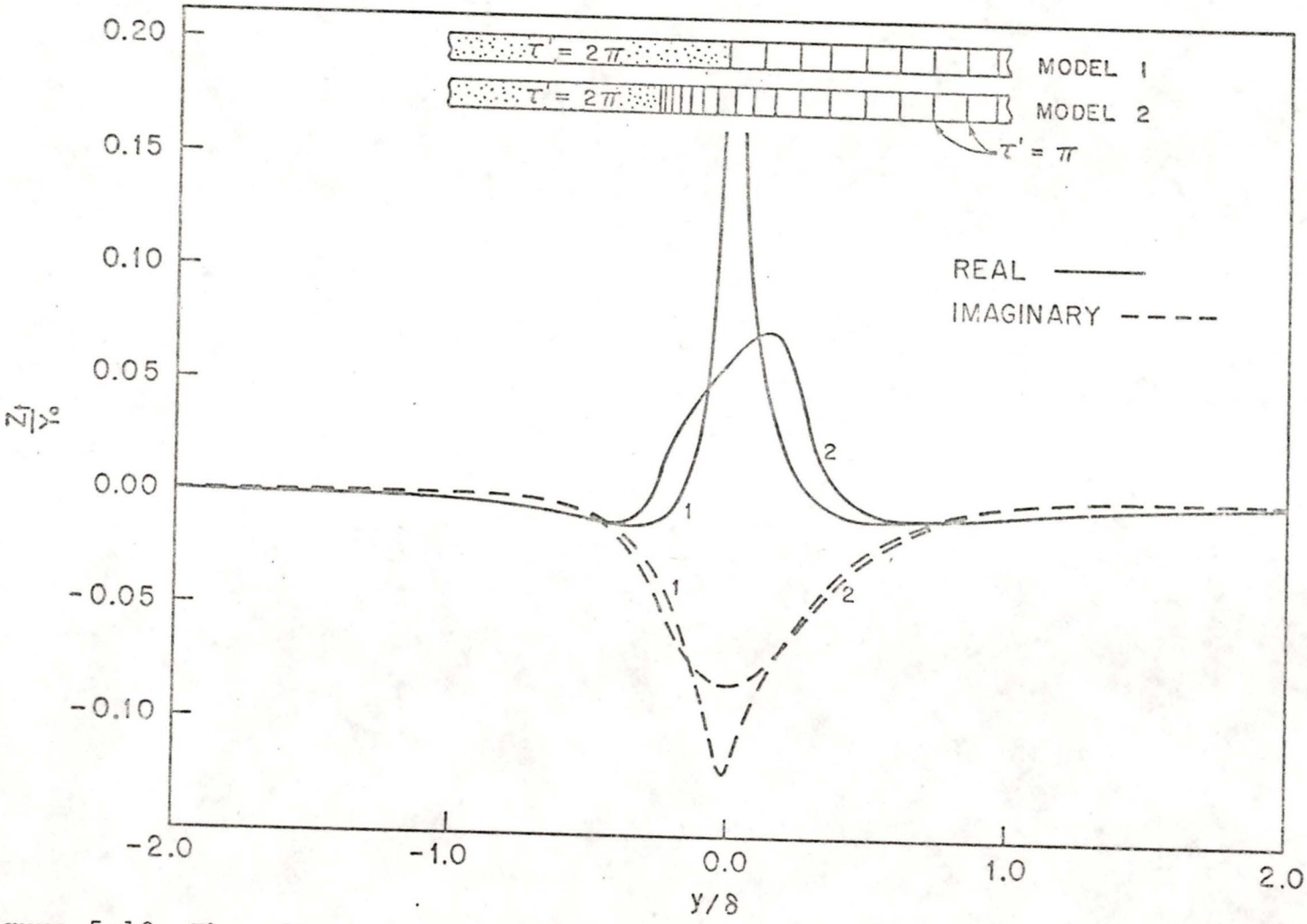

Figure 5.10 The effect of abrupt and gradual changes in integrated conductivity on the vertical magnetic field at the surface (E-polarization).

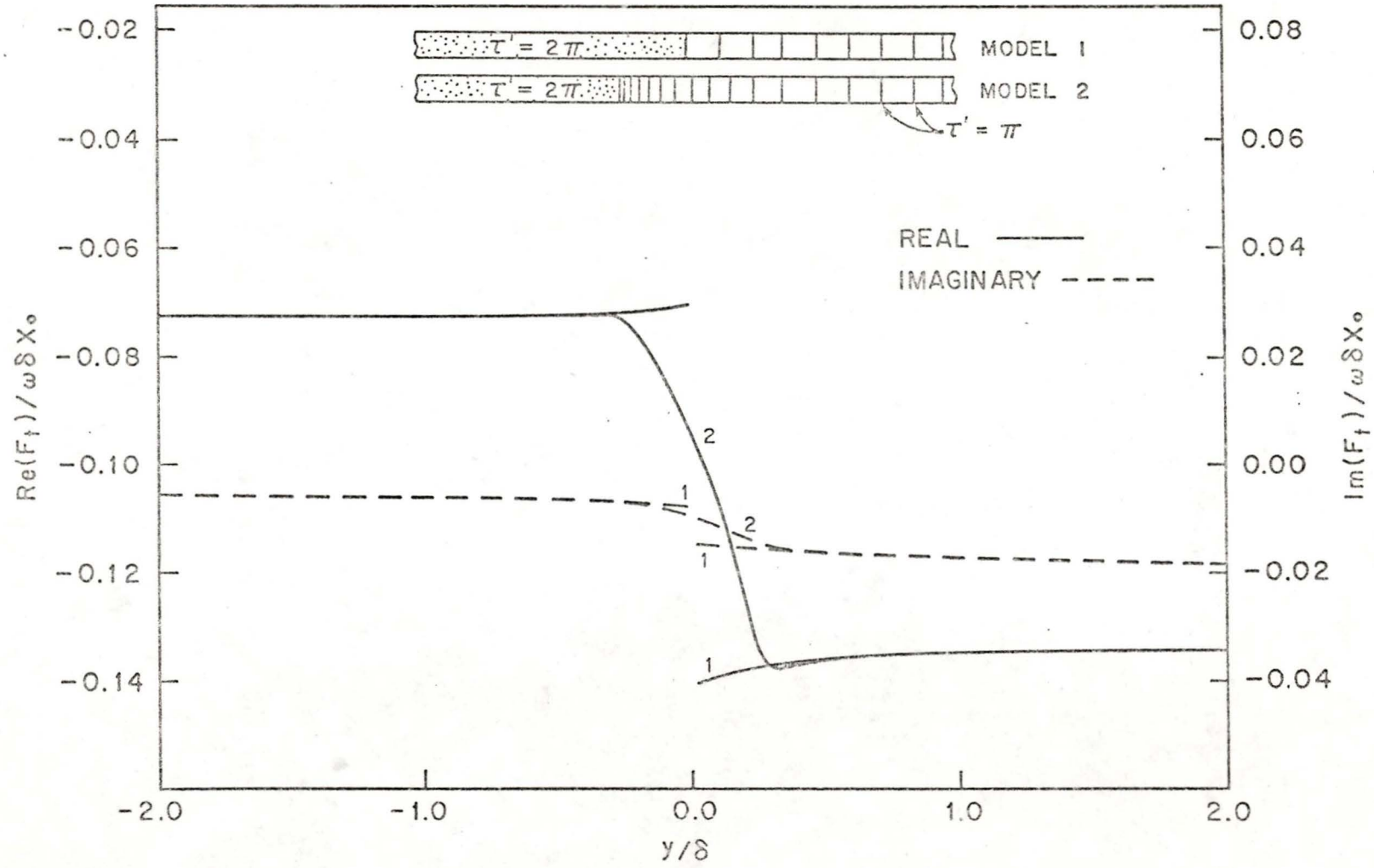

Figure 5.11 The effect of abrupt and gradual changes in integrated conductivity on the horizontal electric field at the surface (B-polarization).

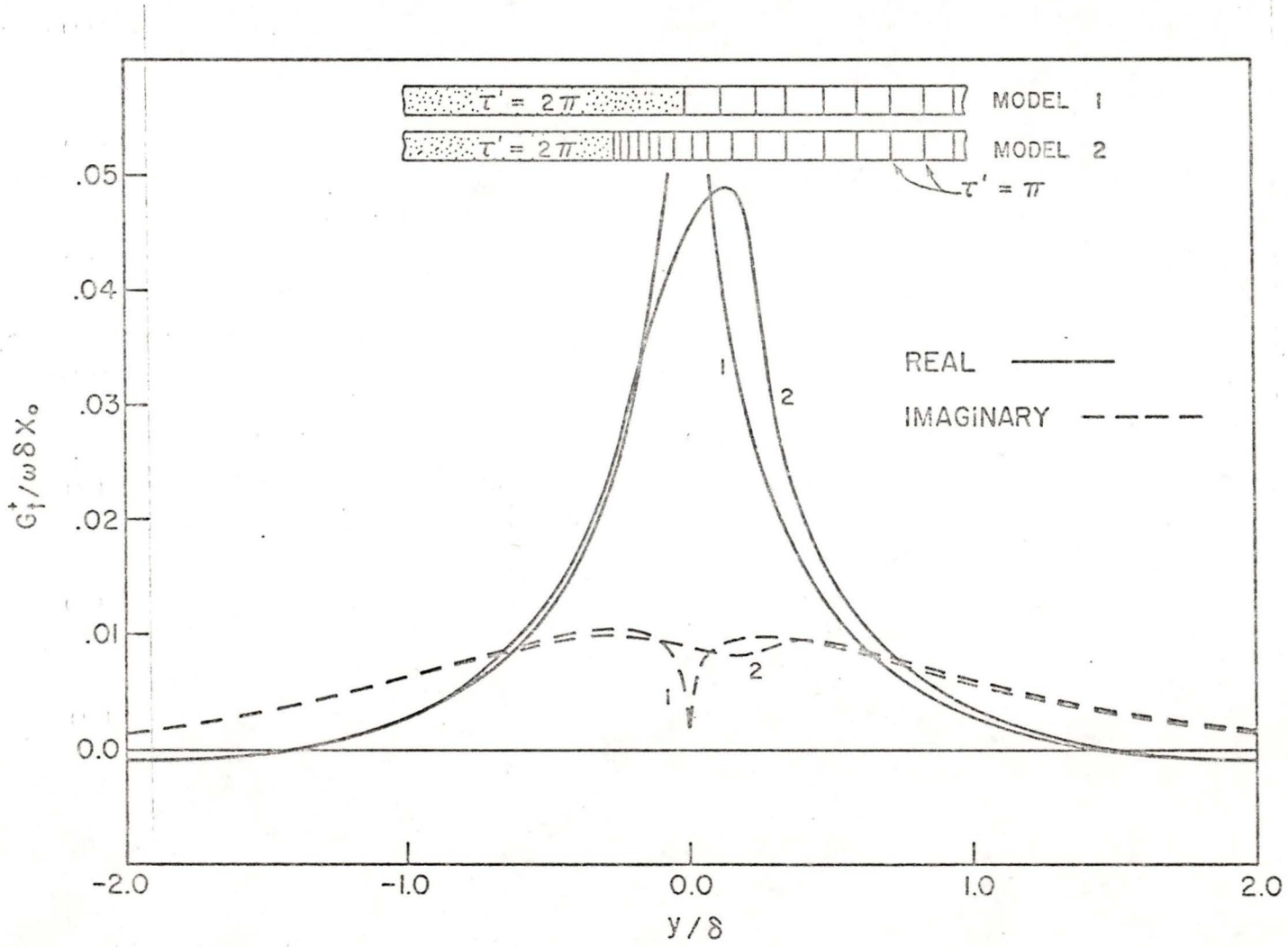

Figure 5.12 The effect of abrupt and gradual changes in integrated conductivity on the vertical electric field just below the infinitely thin sheet (B-polarization).

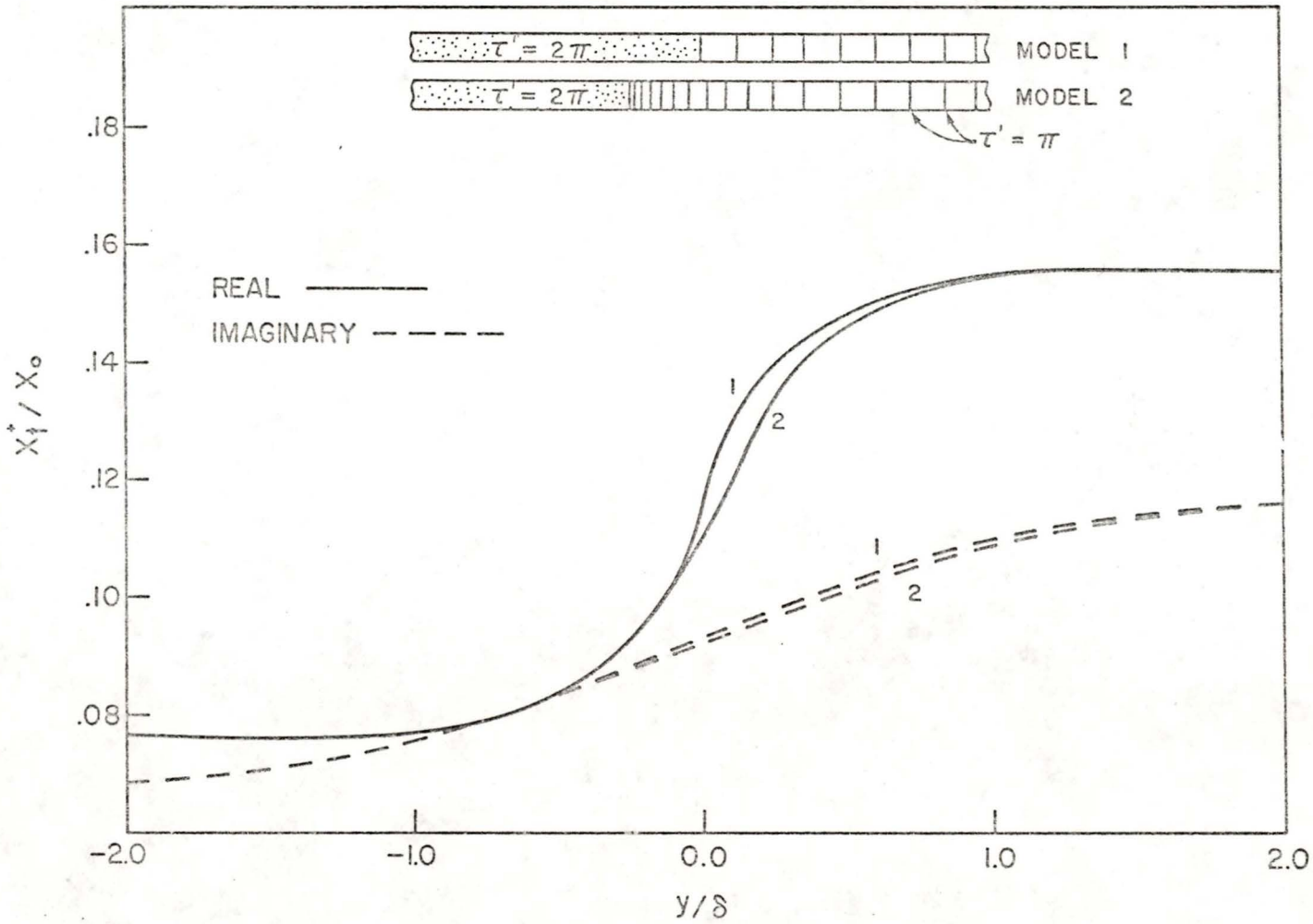

Figure 5.13 The effect of abrupt and gradual changes in integrated conductivity on the horizontal magnetic field just below the infinitely thin sheet (B-polarization).

When τ' changes smoothly from one value to the other, all field components remain continuous, in both polarizations, while crossing the zone of varying integrated conductivity (see Ashour, 1971). It is important to notice, with this model, that the peaks in the field components have shifted away from $y' = 0$. This indicates that for realistic results the sloping continental shelf should be modelled more closely than by an abrupt change in conductivity. In the next section, a coast model, which includes a continental slope, is considered.

5.5 Vancouver Island Model

As a final illustration of the method developed in this thesis, we consider a two-dimensional E-polarization model roughly representing a cross-section through Vancouver Island which includes the continental slope, the continental shelf and the inner strait between the island and the mainland.

We have compared our results for the amplitude of the horizontal and vertical magnetic fields at the surface with results obtained using the finite-difference formulation of Brewitt-Taylor and Weaver (1976). As seen in figure 5.14, the agreement is excellent. It is interesting to note that the computer time required to determine the fields at the surface using their numerical procedure (with a 59 point surface grid by 40 point vertical grid) is approximately 15 minutes compared to about 1 minute using our method for the same 59 point surface grid. Also, extra disk space must be used in their method.

From our results, for the frequency of 1 cph (~ .3 mHz), there are minima in the vertical field at the island coastline facing the continent and just off the island coastline facing the ocean. There are maxima at the continental coastline and the island coastline facing the deep ocean with the main maximum occurring well out over the continental slope. Slight

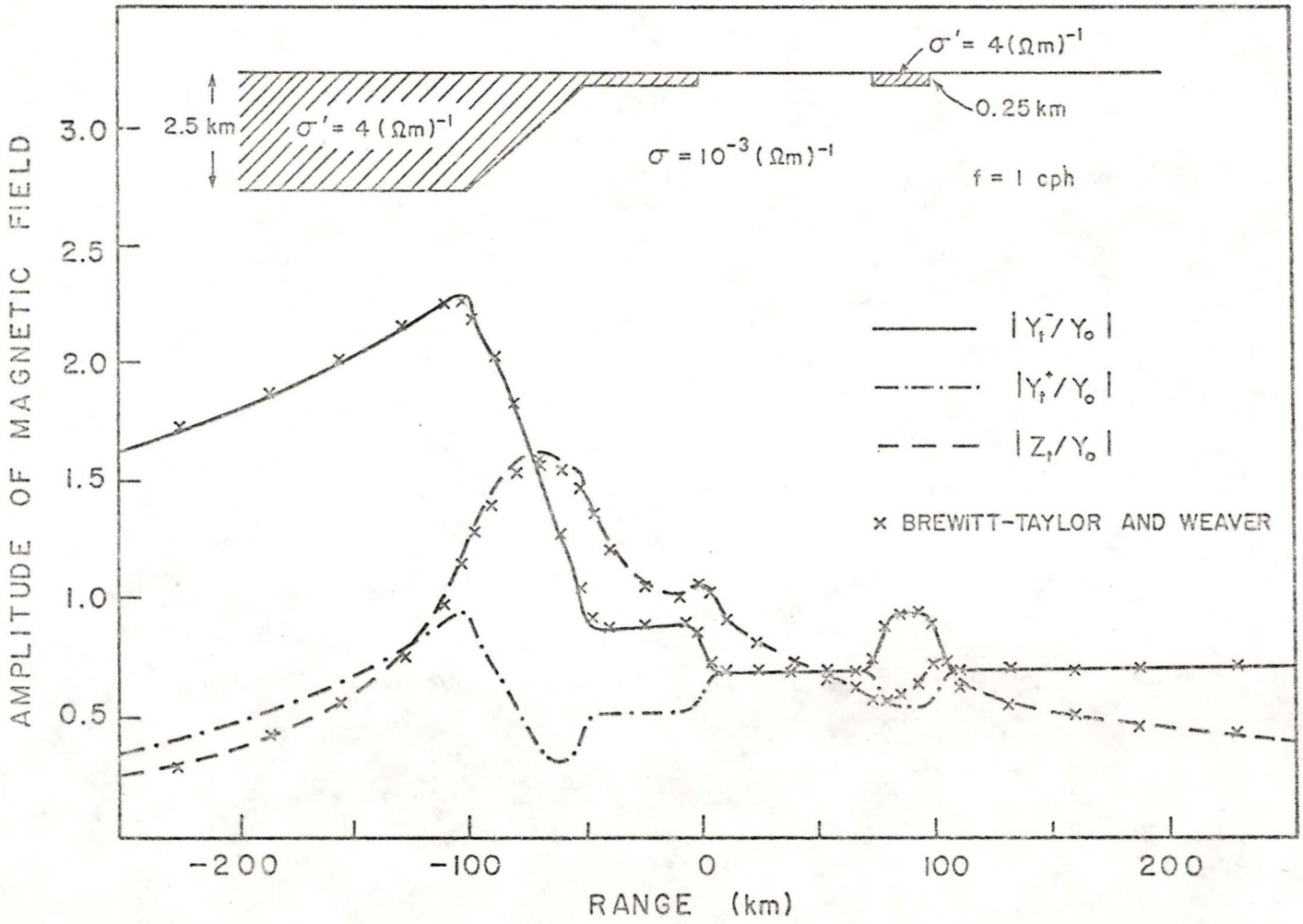

Figure 5.14 Amplitudes of the horizontal magnetic field at the surface and below the ocean and the vertical magnetic field at the surface for the model of Vancouver Island (E-polarization).

enhancements of the field also occur at the beginning and end of the continental slope. The amplitude of the surface horizontal field decreases considerably from a maximum obtained where the deep ocean first meets the continental slope and then clearly distinguishes the continental shelf by a levelling off of the field followed by an abrupt change at the coastline. The island and strait are similarly distinguished by a levelling off of the field and abrupt changes at the edges. The horizontal magnetic field at the ocean bottom has a maximum at the point where the deep ocean meets the continental slope and has a minimum just before reaching the continental shelf. Along the continental shelf and at the bottom of the strait, the field roughly behaves like a mirror image of the surface field.

In the region of the continental slope and shelf, we see that the fields are quite perturbed at this frequency. Thus, when trying to infer information about the conductivity structure of the Earth in this region, we conclude that field measurements should be taken either at the surface or bottom of the ocean as well as over the land.

5.6 Summary and future work

In this thesis, a method for studying problems of electromagnetic induction in two-dimensional conductivity anomalies confined to near the Earth's surface has been developed using the thin sheet approximation. Both E-polarization and B-polarization cases have been considered.

A number of important features have been included in the numerical evaluation of the integral equations used to determine the field values. In particular, the electric field is approximated by a quadratic function near the singularity and by a linear function between consecutive grid points elsewhere. Also, even or uneven grid-spacing may be used. Furthermore, the integrated conductivity may vary over a limited range along the y-axis and may have different values at the extreme ends of the model. Its value at a grid point is determined by a weighted average of the conductivity in the two adjoining regions. Additionally, the fact that the horizontal electric field along the surface, $E(u,0)$, far from the region of varying conductivity, dies off as $1/u$ has been incorporated into the procedure.

In section 5.3, it has been shown that the accuracy of our results is excellent when compared to the analytical solution of Nicoll and Weaver (1977) (B-polarization

case) but that some differences do occur when comparing our results with those of Fischer et al. (1978a,1978b) (E-polarization case). However, the similarity between our results and those obtained by Weidelt (1971) analytically leads us to conclude that our calculations are indeed accurate. For the complex model of Vancouver Island (E-polarization case), our results have also been compared with those obtained by the finite-difference formulation of Brewitt-Taylor and Weaver (1976) which requires much more computer time and storage space than our method. The excellent agreement between the results obtained by these two different approaches is further verification of the validity of our procedure. Moreover, the considerable saving in computer time and storage space provides one of the main reasons for developing the theory using the thin sheet approximation.

It has been mentioned previously that work is in progress to determine analytical solutions to some geometrically simple thin sheet models (using the Weiner-Hopf technique) in order to provide further checks on our numerical procedure. Work is also being done to extend the two-dimensional theory to include a layered half-space in the present model. As a final comment, it should be pointed out that a two-dimensional model

can not be used to study the additional effect of channelling of electric currents around an island. The extension of this theory to analyse problems in three-dimensions would give a mathematical technique for studying this 'channelling effect' which would be more practical, in terms of computer time and storage space, than a finite-difference formulation. The extension to three-dimensions is presently being developed.

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APPENDIX A

EVALUATION OF INTEGRAL COEFFICIENTS CONTAINING BESSEL FUNCTIONS

In order to determine the coefficients U_{mj} , V_{mj} , and W_m (defined in section 4.2), $\tilde{V}_{mj}$ and $\tilde{W}_m$ (defined in section 4.3) and I_m^- and I_m^+ (defined in section 4.4) in a form suitable for programming on a computer, some expressions for the integrals of Modified Bessel functions of the second kind are required. From the handbook by Abramowitz and Stegun (1964), appropriate series representations of $K_0(w)$ and $K_1(w)$ are available almost directly. Also, from tabulated results (Abramowitz and Stegun, 1964, eq. 11.4.23 and eq. 11.1.9), we know that

$$\int_0^\infty K_0(w)\, dw = \frac{\pi}{2} \qquad (A.1)$$

and

$$\int_0^z K_0(w) = z \sum_{k=0}^{\infty} a_k \left(\frac{z}{2}\right)^{2k} \qquad (A.2)$$

where $z = \sqrt{2i}\, x$, (x , real and positive)

$a_k = p_k + iq_k$,

$$p_o = 1 - \gamma + \frac{\ell n\ 2}{2} - \ell n\ x \ ,$$

$$q_o = \frac{-\pi}{4} \quad ,$$

$$p_k = \frac{p_o + r_k + 1/(2k+1)}{(k!)^2(2k+1)} \quad ,$$

$$q_k = \frac{q_o}{(k!)^2(2k+1)} \quad ,$$

$$r_k = \sum_{j=2}^{k} \frac{1}{j} \quad ,$$

γ = .57721566490153 , Euler's constant.

After some rearrangements, equation (A.2) may be written as

$$A(z) = \int_0^z K_0(w)\, dw = x\left[(R(x) - S(x)) + i(R(x) + S(x))\right] \qquad (A.3)$$

where

$$R(x) = \sum_{k=0}^{\infty} (-1)^k b_k u^{2k} \quad , \quad S(x) = \sum_{k=0}^{\infty} (-1)^k c_k u^{2k} \quad ,$$

$$b_k = 32^k(p_{2k} - 32 \cdot u \cdot q_{2k+1}) \quad , \quad c_k = 32^k(q_{2k} + 32 \cdot u \cdot p_{2k+1}) \quad ,$$

$$u = \left(\frac{x}{8}\right)^2 \quad .$$

Both $R(x)$ and $S(x)$ are convergent alternating series of real terms. If these series are terminated when $k = n$, it is well known that the remainder is less than the first term neglected for $n > N$, provided that the terms of the remainder are monotonically decreasing. For instance, for $x \leq 8$, $N = 2$ (see Table A.1). For convenience in programming, the above series representation is used for $x < 8$ with $n = 10$. In this case, the maximum remainder for $R(x)$ is then $|b_{11}| = .33 \times 10^{-10}$ and, for $S(x)$, is $|c_{11}| = .16 \times 10^{-10}$. Note that the coefficients b_k and c_k are multiplied by $\left(\frac{x}{8}\right)^{4k}$ so that for $x < 8$, the terms in $R(x)$ and $S(x)$ die off extremely quickly.

Now by using equation (A.1), we can define the following two functions:

$$\tilde{H}(z) = \int_0^z K_0(w)\, dw = \begin{cases} A(z) & (x < 8) \\ \frac{\pi}{2} - B(z) & (x \geq 8) \end{cases} \tag{A.4}$$

and

$$\tilde{I}(z) = \int_z^\infty K_0(w)\, dw = \begin{cases} \frac{\pi}{2} - A(z) & (x < 8) \\ B(z) & (x \geq 8) \end{cases} \tag{A.5}$$

A series representation of $B(z) = \int_z^\infty K_0(w)\, dw$, for $x \geq 8$, is given in the handbook by Abramowitz and Stegun

TABLE A.1

Coefficients used to determine A(z) to the desired accuracy

k	$\lvert b_k \rvert$			$\lvert c_k \rvert$		
	x = .001	x = 1.0	x = 8.0	x = .001	x = 1.0	x = 8.0
1	$.43 \times 10^{3}$	$.77 \times 10^{2}$	$.71 \times 10^{2}$	$.40 \times 10^{2}$	$.37 \times 10^{2}$	$.84 \times 10^{2}$
2	$.18 \times 10^{4}$	$.40 \times 10^{3}$	$.14 \times 10^{3}$	$.15 \times 10^{3}$	$.15 \times 10^{3}$	$.15 \times 10^{3}$
3	$.15 \times 10^{4}$	$.37 \times 10^{3}$	$.11 \times 10^{3}$	$.13 \times 10^{3}$	$.12 \times 10^{3}$	$.94 \times 10^{2}$
4	$.38 \times 10^{3}$	$.10 \times 10^{3}$	$.30 \times 10^{2}$	$.32 \times 10^{2}$	$.31 \times 10^{2}$	$.23 \times 10^{2}$
5	$.39 \times 10^{2}$	$.11 \times 10^{2}$	$.35 \times 10^{1}$	$.32 \times 10^{1}$	$.32 \times 10^{1}$	$.25 \times 10^{1}$
6	$.20 \times 10^{1}$	$.58 \times 10^{0}$	$.20 \times 10^{0}$	$.16 \times 10^{0}$	$.16 \times 10^{0}$	$.13 \times 10^{0}$
7	$.53 \times 10^{-1}$	$.16 \times 10^{-1}$	$.58 \times 10^{-2}$	$.42 \times 10^{-2}$	$.42 \times 10^{-2}$	$.35 \times 10^{-2}$
8	$.84 \times 10^{-3}$	$.27 \times 10^{-3}$	$.99 \times 10^{-4}$	$.66 \times 10^{-4}$	$.65 \times 10^{-4}$	$.56 \times 10^{-4}$
9	$.83 \times 10^{-5}$	$.27 \times 10^{-5}$	$.10 \times 10^{-5}$	$.64 \times 10^{-6}$	$.64 \times 10^{-6}$	$.55 \times 10^{-6}$
10	$.54 \times 10^{-7}$	$.18 \times 10^{-7}$	$.71 \times 10^{-8}$	$.41 \times 10^{-8}$	$.41 \times 10^{-8}$	$.36 \times 10^{-8}$
11	$.23 \times 10^{-9}$	$.80 \times 10^{-10}$	$.33 \times 10^{-10}$	$.18 \times 10^{-10}$	$.18 \times 10^{-10}$	$.16 \times 10^{-10}$
12	$.74 \times 10^{-12}$	$.25 \times 10^{-12}$	$.11 \times 10^{-12}$	$.55 \times 10^{-13}$	$.55 \times 10^{-13}$	$.50 \times 10^{-13}$

(1964, eq. 11.1.18) (see computer listing - subroutine BESSEL - for further details).

From the well-known recurrence relations for modified Bessel functions of the second kind (Abramowitz and Stegun, 1964, eq. 9.6.26 and eq. 9.6.27), we know that

$$K_1(w) = -K_0'(w)$$

and

$$\frac{K_1(w)}{w} = -K_0(w) - K_1'(w)$$

Furthermore, both $K_0(w)$ and $K_1(w)$ vanish at infinity. With these relations, we find that

$$\int_{z_1}^{z_2} \frac{K_1(w)}{w}\, dw = -\int_{z_1}^{z_2} [K_0(w) + K_1'(w)]\, dw = M(z_2) - M(z_1) \quad , \qquad (A.6)$$

$$\int_{z_1}^{\infty} \frac{K_1(w)}{w}\, dw = -\int_{z_1}^{\infty} [K_0(w) + K_1'(w)]\, dw = L(z_1) \quad , \qquad (A.7)$$

$$\int_{z_1}^{z_2} K_1(w)\, dw = -\int_{z_1}^{z_2} K_0'(w)\, dw = K_0(z_1) - K_0(z_2) \quad , \qquad (A.8)$$

$$\int_{0}^{z_1} wK_1(w)\, dw = -wK_0(w) \Big|_0^{z_1} + \int_0^{z_1} K_0(w)\, dw = N(z_1) \quad , \qquad (A.9)$$

where $L(z) = -\tilde{I}(z) + K_1(z)$,

$M(z) = -\tilde{H}(z) - K_1(z)$,

and $N(z) = -zK_0(z) + \tilde{H}(z)$.

Here, $z_1 = \sqrt{2i}\, x_1$ and $z_2 = \sqrt{2i}\, x_2$ with $0 < x_1 < x_2$.
Now, by using the above results, we can express the coefficients U_{mj} , V_{mj} , W_m , $\tilde{V}_{mj}$ and $\tilde{W}_m$, given by equations (4.2.21), (4.2.22), (4.2.23), (4.3.3) and (4.3.4) respectively, as

$$U_{mj} = \begin{cases} -\sqrt{2i}\, L(g_{m1}) & (j = 1) \\ \sqrt{2i}\,[M(g_{mj}) - M(g_{mj-1})] & (2 \leq j \leq m-1) \\ -\sqrt{2i}\,[M(g_{mm-1}) + M(g_{mm+1})] & (j = m) \\ \sqrt{2i}\,[M(g_{mj}) - M(g_{mj+1})] & (m+1 \leq j \leq M-1) \\ -\sqrt{2i}\, L(g_{mM}) & (j = M) \end{cases} \qquad (A.10)$$

$$V_{mj} = \begin{cases} K_0(g_{mj}) - K_0(g_{mj-1}) + (y_m - y_j)U_{mj} & (2 \leq j \leq m-1) \\ K_0(g_{mm+1}) - K_0(g_{mm-1}) & (j = m) \\ K_0(g_{mj+1}) - K_0(g_{mj}) + (y_m - y_j)U_{mj} & (m+1 \leq j \leq M-1) \end{cases} \quad (A.11)$$

$$W_m = \frac{-1}{2}\left[N(g_{mm+1}) + N(g_{mm-1})\right] \quad (A.12)$$

$$\tilde{V}_{mj} = \begin{cases} K_0(g_{mj-1}) - K_0(g_{mj}) & (2 \leq j \leq m-1) \\ K_0(g_{mm-1}) - K_0(g_{mm+1}) & (j = m) \\ K_0(g_{mj}) - K_0(g_{mj+1}) & (m+1 \leq j \leq M-1) \end{cases} \quad (A.13)$$

and $\tilde{W}_m = N(g_{mm+1}) + N(g_{mm-1})$ (A.14)

where $g_{mj} = \sqrt{2i}\,|y_m - y_j|$. In the above, we have taken the Cauchy Principal Value of the integral denoted by V_{mm} and noted that $(u-y_j) = (u-y_m) + (y_m-y_j)$ in V_{mj} $(m \neq j)$.

Finally, consider the integrals I_m^- and I_m^+, defined as

$$I_m^- = -\sqrt{2i} \int_{-\infty}^{y_1} \frac{P(y_m-u,+0)}{u}\, du$$

and

$$I_m^+ = -\sqrt{2i} \int_{y_M}^{\infty} \frac{P(y_m-u,+0)}{u}\, du \quad ,$$

and recall that $P(y_m-u,+0) = \dfrac{K_1(\sqrt{2i}\,|y_m-u|)}{|y_m-u|}$. For $y_m \neq 0$, we make a change of variable $(w = \sqrt{2i}\,|y_m-u|)$ and note that

$$\frac{1}{(w \pm \sqrt{2i}\ y_m)} = \pm \frac{1}{\sqrt{2i}\ y_m} \left(\frac{1}{w} - \frac{1}{w \pm \sqrt{2i}\ y_m} \right).$$

Next, we integrate the term involving $\left(\dfrac{1}{w \pm \sqrt{2i}\ y_m} \right)$ by parts. For $y_m = 0$, we make the same change in variable and integrate by parts directly. The result is

$$I_m^{\pm} = \begin{cases} \dfrac{1}{y_m} \left(-\sqrt{2i}\ L(\sqrt{2i}\ h_m^{\pm}) + \dfrac{K_0(\sqrt{2i}\ h_m^{\pm})}{h_0^{\pm}} + \tilde{M}^{\pm}(\sqrt{2i}\ h_m^{\pm}) \right) & (y_m \neq 0) \\ \pm \left(\dfrac{K_0(\sqrt{2i}\ h_0^{\pm})}{h_0^{\pm}} + \tilde{N}(\sqrt{2i}\ h_0^{\pm}) \right) & (y_m = 0) \end{cases} \qquad (A.15)$$

where

$$\tilde{M}^{\pm}(z) = -\sqrt{2i}\int_{z}^{\infty} \frac{K_0(w)\ dw}{(w \pm \sqrt{2i}\ y_m)^2} \qquad (A.16)$$

and

$$\tilde{N}(z) = -2i\int_{z}^{\infty} \frac{K_0(w)\,dw}{w^3} \qquad (A.17)$$

with $h_m^- = y_m - y_1$, $h_m^+ = y_M - y_m$,

$h_0^- = -y_1$, $h_0^{\pm} = y_M$.

To evaluate $\tilde{M}^{\pm}(\sqrt{2i}\ h_m^{\pm})$, we discretize the integral as

$$-\sqrt{2i}\int_{\sqrt{2i}\,h_m^{\pm}}^{\infty} \frac{K_0(w)\ dw}{(w \pm \sqrt{2i}\ y_m)^2} = -2i\left\{\int_{\sqrt{2i}\,h_m^{\pm}}^{\sqrt{2i}\,s_n} + \sum_{j=n}^{159}\int_{\sqrt{2i}\,s_j}^{\sqrt{2i}\,s_{j+1}} + \int_{\sqrt{2i}\,s_N}^{\infty}\right\} \frac{K_0(w)\ dw}{(w \pm \sqrt{2i}\ y_m)^2}$$

where $s_N = 8$, $s_j = .02\ j$ $(1 \le j \le 400)$, and $s_{n-1} < h_m^{\pm} \le s_n$, and then replace $1/(w \pm \sqrt{2i}\ y_m)^2$ by

the constant value

$$\frac{1}{4i}\left[\frac{1}{(h_m^{\pm} \pm y_m)^2} + \frac{1}{(s_n \pm y_m)^2}\right]$$

in the interval $h_m^{\pm} < \frac{w}{\sqrt{2i}} < s_n$, and similarly, by

$$\frac{1}{4i}\left[\frac{1}{(s_j \pm y_m)^2} + \frac{1}{(s_{j+1} \pm y_m)^2}\right]$$

in each of the remaining intervals, $\sqrt{2i}\ s_j < w \leq \sqrt{2i}\ s_{j+1}$. The resulting expression is then

$$\tilde{M}^{\pm}(\sqrt{2i}\ h_m^{\pm}) \approx \left(\frac{i-1}{4}\right)\left\{\left[\frac{1}{(h_m^{\pm} \pm y_m)^2} + \frac{1}{(s_n \pm y_m)^2}\right]\left[\tilde{H}(\sqrt{2i}\ s_n) - \tilde{H}(\sqrt{2i}\ h_m^{\pm})\right]\right.$$

$$\left. + \sum_{j=n}^{399}\left[\frac{1}{(s_j \pm y_m)^2} + \frac{1}{(s_{j+1} \pm y_m)^2}\right]\left[\tilde{H}(\sqrt{2i}\ s_{j+1}) - \tilde{H}(\sqrt{2i}\ s_j)\right]\right\} \tag{A.18}$$

Similarly, we find that

$$\tilde{N}(\sqrt{2i}\ h_0^{\pm}) \approx \frac{(i-1)}{2} \left\{ \left[\frac{1}{h_0^{\pm 3}} + \frac{1}{s_n^{3}} \right] \left[\tilde{H}(\sqrt{2i}\ s_n) - \tilde{H}(\sqrt{2i}\ h_0^{\pm}) \right] \right.$$

$$\left. + \sum_{j=n}^{399} \left[\frac{1}{s_j^{3}} + \frac{1}{s_{j+1}^{3}} \right] \left[\tilde{H}(\sqrt{2i}\ s_{j+1}) - \tilde{H}(\sqrt{2i}\ s_j) \right] \right\} \qquad \text{(A.19)}$$

In the above, we have assumed that

$$\int_{\sqrt{2i}\ s_N}^{\infty} \frac{K_0(w)\ dw}{(w \pm \sqrt{2i}\ y_m)^2} \quad \text{and} \quad \int_{\sqrt{2i}\ s_N}^{\infty} \frac{K_0(w)\ dw}{w^3}$$

are negligible.

APPENDIX B

VERIFICATION THAT $J_2(+0)$ EXISTS

To show that $J_2(z)$ exists at $z = +0$, we need to be familiar with two theorems concerning uniform convergence (u.c.) and differentiation of certain integrals (Kaplan, 1973, pp. 447-449). The first theorem may be stated as follows:

If $|g(s,z)| < h(s)$ on $0 \leq z \leq 1$ and if $\int_1^\infty h(s)\,ds$ exists, then $\int_1^\infty g(s,z)\,ds$ is u.c. on $0 \leq z \leq 1$.

The second theorem may be stated as:

Given $\int_1^\infty g(s,z)\,ds$ where $g_2(s,z)$ exists and $\int_1^\infty g_2(s,z)\,ds$ is u.c. on $0 \leq z \leq 1$, then

$$\frac{\partial}{\partial z}\left\{\int_1^\infty g(s,z)\,ds\right\} = \int_1^\infty g_2(s,z)\,ds \quad \text{on} \quad 0 \leq z \leq 1 .$$

After scaling all lengths in equation (3.4.47) by $1/\alpha$ and rearranging, we have that

$$J(z) = \frac{1}{\pi}\int_0^\infty f(y,s)\,P(s,z)\,ds \qquad (B.1)$$

where $f(y,s) = E(y+s,0) + E(y-s,0) - 2E(y,0)$ (B.2)

and $P(s,z) = \dfrac{K_1(\sqrt{i}\,\sqrt{s^2 + z^2})}{\sqrt{s^2 + z^2}}$. (B.3)

For convenience in this verification, we further subdivide $J(z)$ as

$$J(z) = I(z) + H(z) \tag{B.4}$$

where $I(z) = \int_0^1 f(y,s)\, P(s,z)\, ds$ (B.5)

and $H(z) = \int_1^\infty f(y,s)\, P(s,z)\, ds$. (B.6)

We now wish to show that $H'(+0)$ exists. We first use the Mean Value Theorem (M.V.T.) to immediately state that

$$|f(y,s)| < M\, s^2 \tag{B.7}$$

where $|E_{11}(\bar{s},0)| < M$ for some $\bar{s}$, $0 < \bar{s} < 1$. Then, from a tabulated result (Abramowitz and Stegun, 1964, eq. 9.6.28)

$$P_2(s,z) = \frac{-\sqrt{i}\, z\, K_2(\sqrt{i}\,\sqrt{s^2 + z^2})}{(s^2 + z^2)} \quad ,$$

we obtain

$$|P_2(s,z)| \leq \frac{z}{s^2+z^2}|K_2(\sqrt{i}\sqrt{s^2+z^2})| \leq \frac{1}{s^2}|K_2(\sqrt{i}\sqrt{s^2+z^2})| \quad (B.8)$$

on $0 \leq z \leq 1$. From another tabulated result (Abramowitz and Stegun, 1964, eq. 9.6.23)

$$K_2(\omega) = \frac{\omega^2}{3} \int_1^\infty e^{-\omega t}(t^2-1)^{3/2} \, dt \quad ,$$

we find that

$$|K_2(\nu+i\nu)| < \nu^2 \int_1^\infty e^{-\nu t} t^3 \, dt = e^{-\nu}(\nu + 3 + \frac{6}{\nu} + \frac{6}{\nu^2}) \quad (B.9)$$

since

$$|(t^2-1)^{3/2}| < |t^3| \quad ,$$

$$\left|\frac{(1+i)^2}{3}\right| = \frac{2}{3} < 1 \quad ,$$

and $|e^{-i\omega t}| \leq 1$.

Now, since $\nu = \sqrt{\frac{s^2 + z^2}{2}}$ and $s \geq 1$ on $0 \leq z \leq 1$, we see that

$$\frac{s}{\sqrt{2}} \leq \nu \leq \frac{\sqrt{s^2+1}}{\sqrt{2}} \leq s$$

which implies

$$|K_2(\sqrt{i}\sqrt{s^2+z^2})| < e^{-s/\sqrt{2}}\left(s + 3 + \frac{6\sqrt{2}}{s} + \frac{12}{s^2}\right) < Dse^{-s/\sqrt{2}} \qquad (B.10)$$

where $D = (16 + 6\sqrt{2})$. After substituting this into (B.8), we obtain

$$|P_2(s,z)| < \frac{De^{-s/\sqrt{2}}}{s} \quad . \qquad (B.11)$$

Finally, from the inequalities given by (B.7) and (B.11), we can determine that

$$|f(y,s)|\,|P_2(s,z)| < Ms^2\,\frac{De^{-s/\sqrt{2}}}{s} = Nse^{-s/\sqrt{2}} \qquad (B.12)$$

on $0 \leq z \leq 1$.

and that

$$\int_1^\infty Nse^{-s/\sqrt{2}}\,ds = (2 + \sqrt{2})\,Ne^{-1/\sqrt{2}} \quad , \qquad (B.13)$$

a finite value .

By the first theorem, $\int_1^\infty f(y,s)\,P_2(s,z)\,ds$ is u.c. on $0 \leq z \leq 1$, and by the second theorem

$$H'(z) = \int_1^\infty f(y,s)\,P_2(s,z)\,ds \quad \text{on } 0 \leq z \leq 1 \ .$$

In particular, $H'(+0)$ exists.

Now consider $I(z)$. By the M.V.T., we can write that

$$|P(s,z) - P(s,0)| = z|P_2(s,c)| = z\left|\frac{cK_2(\sqrt{i}\sqrt{s^2+c^2})}{s^2 + c^2}\right| \qquad (B.14)$$

for some c between 0 and 1. From (B.9), we immediately see that

$$|K_2(\nu + i\nu)| \leq e^{-\nu}\left(\nu + 3 + \frac{6}{\nu} + \frac{6}{\nu^2}\right) < \frac{16}{\nu^2} \qquad (B.15)$$

since, in this case,

$$\nu = \sqrt{\frac{s^2 + c^2}{2}}$$

and thus $0 < \nu < 1$. After substituting (B.15) into (B.14), we obtain

$$|P(s,z) - P(s,0)| < \frac{16zc}{s^2\nu^2} = \frac{32zc}{s^2(s^2 + c^2)} \quad . \qquad (B.16)$$

From (B.5), (B.7) and (B.16), we find that

$$\left|\frac{I(z) - I(0)}{z}\right| \leq \frac{1}{z}\int_0^1 |f(y,s)||P(s,z) - P(s,0)|\,ds$$

$$< \frac{1}{z}\int_0^1 \left(Ms^2 \cdot \frac{32cz}{s^2(s^2 + c^2)}\right) ds$$

$$= 32\,M\tan^{-1}\frac{1}{c} \quad . \qquad (B.17)$$

Now, as $z \to +0$ in the above, we obtain the desired result that $|I'(+0)| < 16\,M\pi$. Since both $I'(+0)$ and $H'(+0)$ exist, we may finally state that

$$\frac{\partial}{\partial z}\left\{\frac{z\sqrt{i}}{\pi}\,J(z)\right\}_{z=+0} = \frac{\sqrt{i}}{\pi}\,J(+0) \quad . \tag{B.18}$$

APPENDIX C

LISTING OF THE FORTRAN PROGRAM

This program calculates the electromagnetic field components in both E- and B-polarization for the model of a thin sheet of variable integrated conductivity covering a conducting half-space. As near as possible, the program follows the notation used in chapter 4.

It should be noted that the listing of the subroutine LINSYS which is used to calculate the horizontal electric fields by Gaussian elimination is not included here. Information on this subroutine is available from the University of Victoria Computer Centre.

```
C --- THIS PROGRAM CALCULATES 2-DIMENSIONAL ELECTROMAGNETIC INDUCTION   GRN00100
C             FIELDS FOR THZ MODEL OF A THIN SHEET OF VARIABLE          GRN00200
C             INTEGRATED CONDUCTIVITY COVERING A CONDUCTING HALF-SPACE. GRN00300
C                                                                       GRN00400
C------------------------------- MAIN ------------------------------GRN00500
C --- READS DATA, SETS PROGRAM CONSTANTS AND CALLS REQUIRED ROUTINES.   GRN00600
C                                                                       GRN00700
      IMPLICIT COMPLEX*16 (E,F,G,I,X,Y,Z),REAL*8(A-D,K,O-W)             GRN00800
      REAL*8 Y                                                          GRN00900
      COMMON/A0/ IA,IB,IC,ROOT2,PID2,PID4,PID8,D2,D8,SJ,NP,NN,NDIMA,NRHSGRN01000
      COMMON/A1/ Y(75),DIC(75),KKK(75),XX(75),YY(75),YP(75),ZZ(75)      GRN01100
      COMMON/A2/ DEL,SIG,DFREQ,OMEG,I,IPW,O,W,T,PI,M1,MM,M2,ML,M3,MK,JERGRN01200
      COMMON/A3/ ZH(75,75),ZS(75,75),QQ(75,75)                          GRN01300
      COMMON/A5/ A(75),B(75),C(75),P(75),Q(75),R(75),AA(75),KK(75)      GRN01400
      COMMON/A6/ ZUL(75),ZUR(75),ZKOL(75),ZKOR(75)                      GRN01500
C                                                                       GRN01600
    5 FORMAT(10D7.0)                                                    GRN01700
    6 FORMAT(2D15.5,11I2)                                               GRN01800
C                                                                       GRN01900
C --- PROGRAM CONSTANTS                                                 GRN02000
      O = 0.0D0                                                         GRN02100
      W = 1.0D0                                                         GRN02200
      T = 2.0D0                                                         GRN02300
      I = (0.0D0,1.0D0)                                                 GRN02400
      IPW = (1.0D0,1.0D0)                                               GRN02500
      PI = DARCOS(-1.0D0)                                               GRN02600
      PID2 = .5D0*PI                                                    GRN02700
      PID4 = .25D0*PI                                                   GRN02800
      PID8 = .125D0*PI                                                  GRN02900
      ROOT2 = DSQRT(T)                                                  GRN03000
      IA = W/IPW                                                        GRN03100
      IB = IPW/ROOT2                                                    GRN03200
      IC = (I-W)/ROOT2                                                  GRN03300
      D2 = W/8.D0**2                                                    GRN03400
      D8 = W/8.D0**8                                                    GRN03500
      UO = PI*4.0D-7                                                    GRN03600
      SJ = .02D0                                                        GRN03700
      NP = 400                                                          GRN03800
      NDIMA = 73                                                        GRN03900
      NRHS = 1                                                          GRN04000
C                                                                       GRN04100
C-----------------------------------------------------------------------GRN04200
C     PROGRAM OPTIONS                                                   GRN04300
C     TO INPUT GRID AND INTEGRATED CONDUCTIVITY IN KILOMETERS AND MHOS, GRN04400
C           SET JDIM = 0.                                               GRN04500
C     TO INPUT GRID AND INTEGRATED CONDUCTIVITY IN DIMENSIONLESS UNITS, GRN04600
C           SET JDIM = 1.                                               GRN04700
C     TO HAVE RESULTS PUNCHED ONTO CARDS, SET JC = 1. JN IS THEN SOME   GRN04800
C           NUMBER WHICH IS PUNCHED ONTO THE CARDS TO INDICATE THE      GRN04900
C           PARTICULAR SET OF RESULTS.                                  GRN05000
C     TO CALCULATE MATRIX COEFFICIENTS AND FIELDS, SET JCF = 0.         GRN05100
C     TO CALCULATE MATRIX COEFFICIENTS AND STORE ON UNIT 11 (DISC),     GRN05200
C           SET JCF = 1.  (FIELDS ARE ALSO DETERMINED.)                 GRN05300
C     TO USE COEFFICIENTS STORED ON UNIT 11 (DISC) FROM A PREVIOUS      GRN05400
C           RUN, SET JCF = 2.  (FIELDS ARE ALSO DETERMINED.)            GRN05500
C     TO CALCULATE MATRIX COEFFICIENTS ONLY AND STORE ON UNIT 11 (DISC),GRN05600
C           SET JCF = 3.  (FIELDS ARE NOT DETERMINED.)                  GRN05700
C     TO CALCULATE B-POLARIZATION FIELDS, SET JBP = 1.                  GRN05800
C     TO CALCULATE E-POLARIZATION FIELDS, SET JEP = 1.                  GRN05900
C     TO DETERMINE MATRIX COEFFICIENTS FOR IMPROVED BOUNDARY CONDITIONS GRN06000
C           FOR E-POLARIZATION PROBLEMS, SET JCOR = 1.                  GRN06100
C     TO RESTART PROGRAM, SET JR = 1.                                   GRN06200
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C     IF DIMENSIONLESS GRID POINTS ARE THE SAME FOR THE NEXT RUN, THE   GRN06300
C           MATRIX COEFFICIENTS NEED NOT BE RECALCULATED.  SEE SECTIONGRN06400
C           5.1 IN THESIS.  (IE. SET JCF = 1 IN THE FIRST RUN, THEN   GRN06500
C           SET JCF = 2 IN FUTURE RUNS.)                              GRN06600
C ------------------------------------------------------------------------GRN06700
C                                                                     GRN06800
C --- READ INPUT DATA (FREQUENCY,CONDUCTIVITY,END GRID POINTS,OPTIONS) GRN06900
  11  READ(5,6) DFREQ,SIG,M1,MM,JBP,JEP,JC,JCF,JT,JN,JDIM,JM,JCOR     GRN07000
C                                                                     GRN07100
C --- MORE PROGRAM CONSTANTS.                                         GRN07200
      M2 = M1 + 1                                                     GRN07300
      M3 = M1 + 2                                                     GRN07400
      MK = MM - 2                                                     GRN07500
      ML = MM - 1                                                     GRN07600
      NN = ML - M1                                                    GRN07700
      OMEG  = T*PI*DFREQ                                              GRN07800
      DEL   = DSQRT(T/(OMEG*U0*SIG))                                  GRN07900
C                                                                     GRN08000
C --- READ INTEGRATED CONDUCTIVITY                                    GRN08100
      READ(5,5)  (DIC(J),J=M1,ML)                                     GRN08200
C                                                                     GRN08300
C --- READ COEFFICIENTS FROM STORAGE.                                 GRN08400
      IF(JCF.EQ.2)  READ(11)                                          GRN08500
     1 ((ZR(M,J),J=M1,MM),M=M1,MM),((ZS(M,J),J=M1,MM),M=M1,MM),(ZKOL(J),GRN08600
     1 ZKOR(J),ZUL(J),ZUR(J),J=M1,MM),((QQ(M,J),J=M1,MM),M=M1,MM),    GRN08700
     1 (Y(J),KK(J),AA(J),A(J),B(J),C(J),P(J),Q(J),R(J),J=M1,MM)       GRN08800
C                                                                     GRN08900
      IF(JCF.EQ.2) GO TO 10                                           GRN09000
C                                                                     GRN09100
C --- READ GRID DATA.                                                 GRN09200
      READ(5,5)  (Y(J),J=M1,MM)                                       GRN09300
C                                                                     GRN09400
C --- IF JDIM = 0, CONVERT KILOMETERS TO DIMENSIONLESS UNITS.         GRN09500
      IF(JDIM.EQ.1) GOTO 9                                            GRN09600
      DSCALE = 1.D3/DEL                                               GRN09700
      DO 1 J = M1,MM                                                  GRN09800
   1  Y(J) = Y(J) * DSCALE                                            GRN09900
C                                                                     GRN10000
C --- CALCULATION OF GRID STEP-SIZES.                                 GRN10100
   9  DO 2 J = M1,ML                                                  GRN10200
      M      = J + 1                                                  GRN10300
      KK(J)  = Y(J+1) - Y(J)                                          GRN10400
   2  AA(M)  = W/KK(J)                                                GRN10500
C                                                                     GRN10600
C --- CALCULATION OF FINITE DIFFERENCE COEFFICIENTS.                  GRN10700
      DO 3 J = M2,ML                                                  GRN10800
      K  = KK(J)                                                      GRN10900
      K1 = KK(J-1)                                                    GRN11000
      KP = K + K1                                                     GRN11100
      KM = K - K1                                                     GRN11200
      A(J) = -K/(KP*K1)                                               GRN11300
      B(J) = KM/(K*K1)                                                GRN11400
      C(J) = K1/(KP*K)                                                GRN11500
      P(J) = T/(KP*K1)                                                GRN11600
      Q(J) = -T/(K*K1)                                                GRN11700
   3  R(J) = T/(KP*K)                                                 GRN11800
C                                                                     GRN11900
C --- CALCULATION OF DIMENSIONLESS AVERAGED INTEGRATED CONDUCTIVITY.  GRN12000
C --- IF JDIM = 0, CONVERT MHOS TO DIMENSIONLESS UNITS.               GRN12100
  10  SD = W                                                          GRN12200
      IF(JDIM.EQ.0) SD = W/(SIG*DEL)                                  GRN12300
      DO 4 J = M2,ML                                                  GRN12400
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    4  KKK(J) = SD*(KK(J)*DIC(J)*KK(J-1)*DIC(J-1))/(KK(J)+KK(J-1))          GRN12500
       KKK(M1) = SD*DIC(M2)                                                 GRN12600
       KKK(MM) = SD*DIC(ML)                                                 GRN12700
C                                                                           GRN12800
       IF(JCF.NE.2) CALL COEFF                                              GRN12900
C                                                                           GRN13000
C --- STORE COEFFICIENTS FOR FUTURE USE.                                    GRN13100
       IF(JCF.EQ.1.OR.JCF.EQ.3) WRITE(11)                                   GRN13200
      1 ((ZR(M,J),J=M1,MM),M=M1,MM),((ZS(M,J),J=M1,MM),M=M1,MM),(ZKOL(J),   GRN13300
      1 ZKOR(J),ZUL(J),ZUR(J),J=M1,MM),((QQ(M,J),J=M1,MM),M=M1,MM),         GRN13400
      1 (Y(J),KK(J),AA(J),A(J),B(J),C(J),P(J),Q(J),R(J),J=M1,MM)            GRN13500
       IF(JCF.EQ.3) STOP                                                    GRN13600
C                                                                           GRN13700
C --- CALCULATION OF B-POLARIZATION FIELDS.                                 GRN13800
       IF(JBP.NE.1) GOTO 7                                                  GRN13900
       CALL BPOL                                                            GRN14000
       CALL RESULT(1,JBP,JEP,JC,JCF,JT,JN,JDIM,JM,JCOR)                    GRN14100
C                                                                           GRN14200
C --- CALCULATION OF E-POLARIZATION FIELDS.                                 GRN14300
    7  IF(JEP.NE.1) GOTO 8                                                  GRN14400
       IF(JCOR.EQ.1) CALL EPOLC                                             GRN14500
       CALL EPOL(JCOR)                                                      GRN14600
       CALL RESULT(2,JBP,JEP,JC,JCF,JT,JN,JDIM,JM,JCOR)                    GRN14700
    8  CONTINUE                                                             GRN14800
C                                                                           GRN14900
       REWIND 11                                                            GRN15000
       IF(JM.EQ.1) GOTO 11                                                  GRN15100
C                                                                           GRN15200
       STOP                                                                 GRN15300
       END                                                                  GRN15400
C                                                                           GRN15500
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C---------------------------------- COEFF ------------------------------GRN15600
C --- DETERMINATION OF MATRIX COEFFICIENTS FOR E-POL AND B-POL PROBLEMS GRN15700
C              (SEE CHAPTER 4)                                          GRN15800
C                                                                       GRN15900
      SUBROUTINE COEFF                                                  GRN16000
      IMPLICIT COMPLEX*16(E,F,G,I,X,Y,Z),REAL*8(A-D,K,O-W)              GRN16100
      REAL*8 Y                                                          GRN16200
      COMMON/A1/ Y(75),DIC(75),KKK(75),XX(75),YY(75),YP(75),ZZ(75)      GRN16300
      COMMON/A2/ DEL,SIG,DFREQ,OMEG,I,IPW,O,W,T,PI,M1,MM,M2,ML,M3,MK,JERGRN16400
      COMMON/A3/ ZR(75,75),ZS(75,75),QQ(75,75)                          GRN16500
      COMMON/A5/ A(75),B(75),C(75),P(75),Q(75),R(75),AA(75),KK(75)      GRN16600
      COMMON/A6/ ZUL(75),ZUR(75),ZKOL(75),ZKOR(75)                      GRN16700
      COMMON/WS/ ZKO(75),ZM(75),ZN(75),ZU(75),ZV(75),ZW(75),            GRN16800
     1           ZZV(75),ZZW(75),DL(75)                                 GRN16900
C                                                                       GRN17000
C --- CALCULATION OF COEFFICIENTS ZUL AND ZUR.                          GRN17100
      CALL BESSEL(M1)                                                   GRN17200
      CALL BESSEL(MM)                                                   GRN17300
C                                                                       GRN17400
      DO 10 M = M2,ML                                                   GRN17500
      MN = M - 1                                                        GRN17600
      MP = M + 1                                                        GRN17700
      CALL BESSEL(M)                                                    GRN17800
      KP = Y(MP) - Y(MN)                                                GRN17900
C                                                                       GRN18000
      DL(M) = DLOG((Y(MP)-Y(M))/(Y(M)-Y(MN)))                           GRN18100
      DO 30 J = M2,MM                                                   GRN18200
   30 IF(J.LT.M.OR.J.GT.MP) DL(J) = DLOG((Y(M)-Y(J))/(Y(M)-Y(J-1)))     GRN18300
C                                                                       GRN18400
C --- CALCULATION OF U, V, W, V(TILDA) AND W(TILDA).  (SEE APPENDIX A)  GRN18500
      DO 60 J = M2,ML                                                   GRN18600
      DD = Y(M)-Y(J)                                                    GRN18700
      IF(J.GE.M) GOTO 61                                                GRN18800
      ZU(J) = IPW*(ZM(J)-ZN(J-1))                                       GRN18900
      ZV(J) = ZKO(J)-ZKO(J-1)+DD*ZU(J)                                  GRN19000
      ZZV(J) = ZKO(J-1)-ZKO(J)                                          GRN19100
      ZZW(J) = ZN(J-1)-ZN(J)+DD*ZZV(J)                                  GRN19200
      GOTO 60                                                           GRN19300
   61 IF(J.LE.M) GOTO 60                                                GRN19400
      ZU(J) = IPW*(ZM(J)-ZM(J+1))                                       GRN19500
      ZV(J) = ZKO(J+1)-ZKO(J)+DD*ZU(J)                                  GRN19600
      ZZV(J) = ZKO(J)-ZKO(J+1)                                          GRN19700
      ZZW(J) = ZN(J+1)-ZN(J)+DD*ZZV(J)                                  GRN19800
   60 CONTINUE                                                          GRN19900
      ZU(M) = -IPW*(ZM(MP)+ZM(MN))                                      GRN20000
      ZV(M) = ZKO(MP)-ZKO(MN)                                           GRN20100
      ZW(M) = -.5D0*(ZN(MP)+ZN(MN))                                     GRN20200
      ZZV(M) = ZKO(MN) - ZKO(MP)                                        GRN20300
      ZZW(M) = ZN(MP) + ZN(MN)                                          GRN20400
C                                                                       GRN20500
C --- CALCULATE COEFFICIENTS FOR Y(U,-0) - SEE EQUATION 4.2.16 - AND    GRN20600
C          FOR Y(U,+0) AND G1(U,+0) - SEE EQUATIONS 4.2.25 AND 4.3.2.  GRN20700
      DO 20 J = M2,ML                                                   GRN20800
      IF(J.GE.MN) GOTO 21                                               GRN20900
      QQ(M,J) = AA(J+1)*DL(J+1)-AA(J)*DL(J)                             GRN21000
      ZS(M,J) = AA(J)*ZZV(J)-AA(J+1)*ZZV(J+1)                           GRN21100
      ZR(M,J) = ZU(J)+AA(J)*ZV(J)-AA(J+1)*ZV(J+1)                       GRN21200
      GOTO 20                                                           GRN21300
   21 IF(J.LE.MP) GOTO 20                                               GRN21400
      QQ(M,J) = AA(J+1)*DL(J+1)-AA(J)*DL(J)                             GRN21500
      ZR(M,J) = ZU(J)+AA(J)*ZV(J-1)-AA(J+1)*ZV(J)                       GRN21600
      ZS(M,J) = AA(J)*ZZV(J-1)-AA(J+1)*ZZV(J)                           GRN21700
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   20 CONTINUE                                                          GRN21800
      QQ(M,M) = -B(M)*DL(M) - Q(M)*KP                                   GRN21900
      ZR(M,M) = ZU(M) + B(M)*ZV(M) + Q(M)*ZW(M)                         GRN22000
      ZS(M,M)   = B(M)*ZZV(M) + Q(M)*ZZW(M)                             GRN22100
      IF(M.EQ.M2) GOTO 1                                                GRN22200
      QQ(M,MN) = -(AA(MN)*DL(MN) + A(M)*DL(M) + P(M)*KP)                GRN22300
      QQ(M,M1) = AA(M2)*DL(M2)                                          GRN22400
      ZR(M,MN) = ZU(MN) + AA(MN)*ZV(MN) + A(M)*ZV(M) + P(M)*ZW(M)       GRN22500
      ZR(M,M1) = ZUL(M) - AA(M2)*ZV(M2)                                 GRN22600
      ZS(M,MN) = A(M)*ZZV(M) + P(M)*ZZW(M) + AA(MN)*ZZV(MN)             GRN22700
      ZS(M,M1) = -AA(M2)*ZZV(M2)                                        GRN22800
      GOTO 2                                                            GRN22900
    1 QQ(M,M1) = -A(M)*DL(M) - P(M)*KP                                  GRN23000
      ZR(M,M1) = ZUL(M) + A(M)*ZV(M) + P(M)*ZW(M)                       GRN23100
      ZS(M,M1) = A(M)*ZZV(M) + P(M)*ZZW(M)                              GRN23200
    2 IF(M.EQ.ML) GOTO 3                                                GRN23300
      QQ(M,MP) = AA(M+2)*DL(M+2) - C(M)*DL(M) - R(M)*KP                 GRN23400
      QQ(M,MM) = -AA(MM)*DL(MM)                                         GRN23500
      ZR(M,MP) = ZU(MP) - AA(M+2)*ZV(MP) + C(M)*ZV(M) + R(M)*ZW(M)      GRN23600
      ZR(M,MM) = ZUR(M) + AA(MM)*ZV(ML)                                 GRN23700
      ZS(M,MP) = C(M)*ZZV(M) + R(M)*ZZW(M) - AA(M+2)*ZZV(MP)            GRN23800
      ZS(M,MM) =  AA(MM)*ZZV(ML)                                        GRN23900
      GOTO 10                                                           GRN24000
    3 QQ(M,MM) = -C(M)*DL(M) - R(M)*KP                                  GRN24100
      ZR(M,MM) = ZUR(M) + C(M)*ZV(M) + R(M)*ZW(M)                       GRN24200
      ZS(M,MM) = C(M)*ZZV(M) + R(M)*ZZW(M)                              GRN24300
   10 CONTINUE                                                          GRN24400
      RETURN                                                            GRN24500
      END                                                               GRN24600
C                                                                       GRN24700
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C---------------------------------- BESSEL ------------------------------GRN24800
C --- CALCULATION OF BESSEL FUNCTIONS.  (SEE A+S AND APPENDIX A)          GRN24900
C --- A+S - DENOTES REFERENCE TEXT:  HANDBOOK OF MATHEMATICAL FUNCTIONS   GRN25000
C           BY ABRAMOWITZ AND STEGUN, 1964.                               GRN25100
C                                                                         GRN25200
      SUBROUTINE BESSEL(M)                                                GRN25300
      IMPLICIT COMPLEX*16(E,F,G,I,X,Y,Z),REAL*8(A-D,K,O-W)                GRN25400
      REAL*8 Y,X,X8,X8DX                                                  GRN25500
      COMPLEX*16 CDEXP                                                    GRN25600
      COMMON/A0/ IA,IB,IC,ROOT2,PID2,PID4,PID8,D2,D8,SJ,NP,NN,NDIMA,NRHSGRN25700
      COMMON/A1/ Y(75),DIC(75),KKK(75),XX(75),YY(75),YP(75),ZZ(75)        GRN25800
      COMMON/A2/ DEL,SIG,DFREQ,OMEG,I,IPW,O,W,J,PI,M1,MM,M2,ML,M3,MK,JERGRN25900
      COMMON/A6/ ZUL(75),ZUR(75),ZKUL(75),ZKUR(75)                        GRN26000
      COMMON/WS/ ZK0(75),ZM(75),ZN(75)                                    GRN26100
C                                                                         GRN26200
C --- A+S - 9.11.11                                                       GRN26300
      ZTHETA(U)= -.3926991D0*I+U*(.0110486D0-.0110485D0*I+U*(             GRN26400
     1-.0009765D0*I+U*(-.0000906D0-.0000901D0*I+U*(-.0000252D0+U*(        GRN26500
     2-.0000034D0+.0000051D0*I+U*(.0000006D0+.0000019D0*I))))))          GRN26600
C                                                                         GRN26700
C --- A+S - 9.11.14                                                       GRN26800
      ZPHI(U)= .7071068D0*IPW+U*(-.0625001D0-.0000001D0*I+U*(-.0013813D0GRN26900
     1+.0013811D0*I+U*(.0000005D0+.0002452D0*I+U*(.0000346D0+.0000338D0 GRN27000
     2*I+U*(.0000117D0-.0000024D0*I+U*(.0000016D0-.0000032D0*I))))))     GRN27100
C                                                                         GRN27200
C --- A+S - 9.11.9                                                        GRN27300
      FF(U,V) = DSQRT(PID2/U)*CDEXP(-U*IB+ZTHETA(-V))                     GRN27400
C                                                                         GRN27500
C --- A+S - 9.11.1                                                        GRN27600
      BER(U,V,SV) =W+V*(113.77777774D0+V*(2.64191397D0+V*(.00122552D0)))GRN27700
     1-SV*(64.D0+V*(32.36345652D0+V*(.08349609D0+V*.00000901D0)))         GRN27800
C                                                                         GRN27900
C --- A+S - 9.11.2                                                        GRN28000
      BEI(U,V,SV) = (16.D0+V*(72.81777742D0+V*(.52185615D0+V*.00011346D0GRN28100
     1))-SV*(113.77777774D0+V*(10.56765779D0+V*.01103667D0)))*U*U*D2      GRN28200
C                                                                         GRN28300
C --- A+S - 9.11.5                                                        GRN28400
      BERP(U,V,SV) = (SV*(14.22222222D0+V*(.66047849D0+V*.00045957D0))    GRN28500
     1-(4.D0+V*(6.0687481D0+V*(.02609253D0+V*.00000394D0))))*U*U*U*D2     GRN28600
C                                                                         GRN28700
C --- A+S - 9.11.6                                                        GRN28800
      BEIP(U,V,SV)=(.5D0+V*(11.37777772D0+V*(.14677204D0+V*.00004609D0))GRN28900
     1-SV*(10.66666666D0+V*(2.31167514D0+V*.00379386D0)))*U              GRN29000
C                                                                         GRN29100
      DO 30 J = M1,MM                                                     GRN29200
      IF(J.EQ.M) GOTO 30                                                  GRN29300
      D = DABS(Y(M)-Y(J))                                                 GRN29400
      X = D * ROOT2                                                       GRN29500
      X8 = D8*X**8                                                        GRN29600
      SQX8 = DSQRT(X8)                                                    GRN29700
      KFACT = -DLOG(.5D0 * X)                                             GRN29800
      Z = D * IPW                                                         GRN29900
      Z2 = D2*Z*Z                                                         GRN30000
      IF(X.GT.8.D0) GOTO 40                                               GRN30100
      BERX  =  BER(X,X8,SQX8)                                             GRN30200
      BEIX  =  BEI(X,X8,SQX8)                                             GRN30300
      BERPX = BERP(X,X8,SQX8)                                             GRN30400
      BEIPX = BEIP(X,X8,SQX8)                                             GRN30500
C                                                                         GRN30600
C --- A+S - 9.11.3                                                        GRN30700
      KER = KFACT*BERX+PID4*BEIX-.57721566D0-SQX8*(59.05819744D0+X8*      GRN30800
     1(60.60977451D0+X8*(.19636347D0+X8*.00026458D0)))+X8*               GRN30900
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```
     2(171.36272133D0+X8*(5.65539121D0+X8*.0030969900))                    GRN31000
C                                                                          GRN31100
C --- A+S - 9.11.4                                                        GRN31200
      KEI = KFACT*BEIX-PID4*BERX+(X*X*D2)*(6.76454936D0+X8*(               GRN31300
     1124.2356965D0+X8*(1.17509064D0+X8*.00029532D0)))-(X*X*D2)*           GRN31400
     2SQX8*(142.91827687D0+X8*(21.30060904D0+X8*.02695875D0))              GRN31500
C                                                                          GRN31600
C --- A+S - 9.11.7                                                        GRN31700
      KERP = KFACT*BERPX-(BERX/X)+PID4*BEIPX+(X*X*X*D2)*(-3.6911373400     GRN31800
     1-X8*(11.36433272D0+X8*(.06136358D0+X8*.00001075D0)))+(X*X*X*D2)      GRN31900
     2*(SQX8*(21.42034017D0+X8*(1.4138478D0+X8*.00116137D0)))              GRN32000
C                                                                          GRN32100
C --- A+S - 9.11.8                                                        GRN32200
      KEIP = KFACT*BEIPX-(BEIX/X)-PID4*BERPX+X*.21139217D0-X*SQX8*(        GRN32300
     113.39858846D0+X8*(4.65950823D0+X8*.00926707D0))+X*X8*                GRN32400
     2(19.41182758D0+X8*(.33049424D0+X8*.00011997D0))                      GRN32500
C                                                                          GRN32600
C --- A+S - 9.9.2                                                         GRN32700
      ZKO(J) = KER + I*KEI                                                 GRN32800
C                                                                          GRN32900
C --- A+S - 9.9.2 , 9.9.17                                                GRN33000
      ZK1 = IC * (KERP + I*KEIP)                                           GRN33100
      GO TO 50                                                             GRN33200
  40  X8DX = 8.D0/X                                                        GRN33300
C                                                                          GRN33400
C --- A+S - 9.11.9                                                        GRN33500
      ZKO(J) = FF(X,X8DX)                                                  GRN33600
C                                                                          GRN33700
C --- A+S - 9.9.2, 9.9.17, 9.11.12                                        GRN33800
      ZK1 = (N-I)*ZKO(J)*ZPH1(-X8DX)                                       GRN33900
C                                                                          GRN34000
  50  CALL INZKO(D,Z,ZH,ZI)                                                GRN34100
C                                                                          GRN34200
C --- A+S - 11.4.23                                                       GRN34300
      IF(M.EQ.M1) ZOL(J) = IPW*(ZI-ZK1)                                    GRN34400
      IF(M.EQ.MM) ZOR(J) = IPW*(ZI-ZK1)                                    GRN34500
      ZH(J) = -ZH-ZK1                                                      GRN34600
      ZH(J) = IA*ZH - D*ZKO(J)                                             GRN34700
  30  CONTINUE                                                             GRN34800
      ZKOL(M) = ZKO(M1)                                                    GRN34900
      ZKOR(M) = ZKO(MM)                                                    GRN35000
C                                                                          GRN35100
      RETURN                                                               GRN35200
      END                                                                  GRN35300
C                                                                          GRN35400
```

```
C------------------------------------ INTK0 ------------------------------GRN35500
C --- CALCULATION OF INTEGRAL OVER K0. (SEE APPENDIX A)                    GRN35600
C                                                                          GRN35700
      SUBROUTINE INTK0(D,Z,ZH,ZI)                                          GRN35800
      IMPLICIT COMPLEX*16(E,P,G,I,X,Y,Z),REAL*8(A-D,K,O-W)                 GRN35900
      REAL*8 Y,X                                                           GRN36000
      COMPLEX*16 CDEXP                                                     GRN36100
      COMMON/A0/ IA,IB,IC,ROOT2,PID2,PID4,PID8,D2,D8,SJ,NP,NN,NDIMA,NRHSGRN36200
      COMMON/A2/ DEL,SIG,DFREQ,OMEG,I,IPW,O,W,P,PI,M1,MM,M2,ML,M3,MK,JERGRN36300
      COMMON/SV/ AA(23),PP(23),Q(23),P(23),B(11),C(11),Q0,P00,IR4,NQ     GRN36400
C                                                                          GRN36500
C --- A+S - 11.1.18                                                        GRN36600
      ZF(G) = 1.25331414D0+G*(.11190289D0+G*(.02576646D0+G*(.00933994D0+GRN36700
     1     G*(.00417454D0+G*(.00163271D0+G*.00033934D0)))))                GRN36800
C                                                                          GRN36900
      IF(NQ.NE.0) GO TO 1                                                  GRN37000
      NQ = 1                                                               GRN37100
      IR4 = DCOS(PID8)+I*DSIN(PID8)                                        GRN37200
      P00 = W - .5772156649015300 + .5D0*DLOG(T)                           GRN37300
      R = 0                                                                GRN37400
      Q0 = -PID4                                                           GRN37500
      AA(1) = 32.D0/3.D0                                                   GRN37600
      Q(1) = AA(1)*Q0                                                      GRN37700
      PP(1) = AA(1)*(P00+W/3.D0)                                           GRN37800
      DO 2 N = 2,23                                                        GRN37900
      A = W*N                                                              GRN38000
      AA(N) = 32.D0*(A+A-W)/((A+A+W)*A*A)*AA(N-1)                          GRN38100
      Q(N) = AA(N)*Q0                                                      GRN38200
      R = R + W/A                                                          GRN38300
    2 PP(N) = AA(N)*(P00+R+W/(W+A+A))                                      GRN38400
C                                                                          GRN38500
    1 IF(D.GE.8.D0) GOTO 5                                                 GRN38600
C                                                                          GRN38700
C --- A+S - 11.1.9 (SEE APPENDIX A)                                        GRN38800
      DL = DLOG(D)                                                         GRN38900
      DO 3 N = 1,23                                                        GRN39000
    3 P(N) = PP(N) - AA(N)*DL                                              GRN39100
      DD = D*D*D2                                                          GRN39200
      B0 = P00 - DL - DD*Q(1)                                              GRN39300
      C0 = Q0 + DD*P(1)                                                    GRN39400
      DO 4 N = 1,11                                                        GRN39500
      B(N) = P(N+N) - DD*Q(N+N+1)                                          GRN39600
    4 C(N) = Q(N+N) + DD*P(N+N+1)                                          GRN39700
      D4 = -DD*DD                                                          GRN39800
      R = B0+D4*(B(1)+D4*(B(2)+D4*(B(3)+D4*(B(4)+D4*(B(5)+D4*(B(6)+D4*     GRN39900
     1     (B(7)+D4*(B(8)+D4*(B(9)+D4*(B(10)+D4*B(11))))))))))            GRN40000
      S = C0+D4*(C(1)+D4*(C(2)+D4*(C(3)+D4*(C(4)+D4*(C(5)+D4*(C(6)+D4*     GRN40100
     1     (C(7)+D4*(C(8)+D4*(C(9)+D4*(C(10)+D4*C(11))))))))))            GRN40200
      ZH = D*((R-S) + I*(R+S))                                             GRN40300
      ZI = PID2 - ZH                                                       GRN40400
C                                                                          GRN40500
      GOTO 6                                                               GRN40600
C                                                                          GRN40700
    5 GG = -7.D0/Z                                                         GRN40800
      X = ROOT2*D                                                          GRN40900
      ZI = ZF(GG)/(IR4*DSQRT(X)*CDEXP(X))                                  GRN41000
      ZH = PID2 - ZI                                                       GRN41100
    6 CONTINUE                                                             GRN41200
C                                                                          GRN41300
      RETURN                                                               GRN41400
      END                                                                  GRN41500
C                                                                          GRN41600
```

```
C---------------------------------- EPOLC ------------------------------GRN41700
C --- CALCULATION OF COEFFICIENTS FOR IMPROVED BOUNDARY CONDITION       GRN41800
C          FOR E-POLARIZATION PROBLEMS (SEE SECTION 4.4 AND APPENDIX A) GRN41900
C                                                                        GRN42000
      SUBROUTINE EPOLC                                                   GRN42100
      IMPLICIT COMPLEX*16(E,F,G,I,X,Y,Z),REAL*8(A-D,K,O-W)               GRN42200
      REAL*8 Y,HN,HP                                                     GRN42300
      COMMON/A0/ IA,IB,IC,ROOT2,PID2,PID4,PID8,D2,D8,SJ,NP,NN,NDIMA,NRHSGRN42400
      COMMON/A1/ Y(75),DIC(75),KKK(75),XX(75),YY(75),YP(75),ZZ(75)       GRN42500
      COMMON/A2/ DEL,SIG,DFREQ,OMEG,I,IPW,O,W,T,PI,M1,MM,M2,ML,M3,MK,JERGRN42600
      COMMON/A3/ ZR(75,75),ZS(75,75),QQ(75,75)                           GRN42700
                                                                         GRN42800
      COMMON/A4/ SN,TN,SP,TP                                             GRN42900
      COMMON/A6/ ZUL(75),ZUR(75),ZK0L(75),ZK0R(75)                       GRN43000
      COMMON/WS/ ZHH(400),G(400),S(400),DN(400),DP(400),D3(400)          GRN43100
     1           ,IN(75),IP(75),HN(75),HP(75)                            GRN43200
C                                                                        GRN43300
C --- CALCULATION OF INTEGRALS HN AND HP (N=NEGATIVE,P=POSITIVE).        GRN43400
      DO 10 M = M2,ML                                                    GRN43500
      IF(Y(M)) 1,2,1                                                     GRN43600
    1 HN(M) = -(Y(M)+Y(M1)*DLOG(W-Y(M)/Y(M1)))/(Y(M)*Y(M)*Y(M1))         GRN43700
      HP(M) =  (Y(M)+Y(MM)*DLOG(W-Y(M)/Y(MM)))/(Y(M)*Y(M)*Y(MM))         GRN43800
      GOTO 10                                                            GRN43900
    2 HN(M) =  W/(T*Y(M1)*Y(M1))                                         GRN44000
      HP(M) = -W/(T*Y(MM)*Y(MM))                                         GRN44100
   10 CONTINUE                                                           GRN44200
C                                                                        GRN44300
C --- CALCULATION OF INTEGRALS IN AND IP (N=NEGATIVE,P=POSITIVE).        GRN44400
      NPP = NP - 1                                                       GRN44500
      ZC = .25D0*(I-W)                                                   GRN44600
      ZD = .5D0*(I-W)                                                    GRN44700
      DO 3 J = 1,NP                                                      GRN44800
      S(J) = SJ*J                                                        GRN44900
      D = S(J)                                                           GRN45000
      Z = IPW*D                                                          GRN45100
      D3(J) = W/(D*D*D)                                                  GRN45200
      CALL INTK0 (D,Z,ZH,ZI)                                             GRN45300
    3 ZHH(J) = ZH                                                        GRN45400
      DO 4 J = 1,NPP                                                     GRN45500
    4 G(J) = ZHH(J+1) - ZHH(J)                                           GRN45600
C                                                                        GRN45700
      DO 30 M = M2,ML                                                    GRN45800
      IN(M) = 0                                                          GRN45900
      D = Y(M) - Y(M1)                                                   GRN46000
      Z = IPW*D                                                          GRN46100
      IF(D.GE.S(NP)) GOTO 31                                             GRN46200
      DO 32 J = 2,NP                                                     GRN46300
      N = J                                                              GRN46400
      IF(D.GT.S(J-1).AND.D.LE.S(J)) GOTO 33                              GRN46500
   32 CONTINUE                                                           GRN46600
   33 IF(Y(M)) 34,134,34                                                 GRN46700
   34 DO 35 J = N,NP                                                     GRN46800
   35 DN(J) = W/(S(J)-Y(M))**2                                           GRN46900
      IF(D.EQ.S(N)) GOTO 36                                              GRN47000
      CALL INTK0(D,Z,ZH,ZI)                                              GRN47100
      IN(M) = (DN(N)+W/Y(M1)**2)*(ZHH(N)-ZH)                             GRN47200
      IF(N.EQ.NP) GOTO 31                                                GRN47300
   36 DO 37 J = N,NPP                                                    GRN47400
   37 IN(M) = IN(M) + (DN(J+1)+DN(J))*G(J)                               GRN47500
   31 IF(Y(M).EQ.0) GOTO 131                                             GRN47600
      IN(M) = (ZUL(M)-ZK0L(M)/Y(M1)+ZC*IN(M))/Y(M)                       GRN47700
      GOTO 30                                                            GRN47800
  134 IF(D.EQ.S(N)) GOTO 136                                             GRN47900
```

```
      CALL INTKO(D,Z,ZH,ZI)                                              GRN47900
      IN(M) = (D3(N) + W/(D*D*D))*(ZHH(N)-ZH)                            GRN48000
  136 IF(N.EQ.NP) GOTO 131                                               GRN48100
      DO 137 J = N,NPP                                                   GRN48200
  137 IN(M) = IN(M) + (D3(J+1)+D3(J))*G(J)                               GRN48300
  131 IN(M) = ZKOL(M)/(D*D)+ZD*IN(M)                                     GRN48400
   30 CONTINUE                                                           GRN48500
C                                                                        GRN48600
      DO 50 M = M2,ML                                                    GRN48700
      IP(M) = 0                                                          GRN48800
      D = Y(MM) - Y(M)                                                   GRN48900
      Z = IPW*D                                                          GRN49000
      IF(D.GE.S(NP)) GOTO 51                                             GRN49100
      DO 52 J = 2,NP                                                     GRN49200
      N = J                                                              GRN49300
      IF(D.GT.S(J-1).AND.D.LE.S(J)) GOTO 53                              GRN49400
   52 CONTINUE                                                           GRN49500
   53 IF(Y(M)) 54,154,54                                                 GRN49600
   54 DO 55 J = N,NP                                                     GRN49700
   55 DP(J) = W/(S(J)+Y(M))**2                                           GRN49800
      IF(D.EQ.S(N)) GOTO 56                                              GRN49900
      CALL INTKO(D,Z,ZH,ZI)                                              GRN50000
      IP(M) = (DP(N)+W/Y(MM)**2)*(ZHH(N)-ZH)                             GRN50100
   56 IF(N.EQ.NP) GOTO 51                                                GRN50200
      DO 57 J = N,NPP                                                    GRN50300
   57 IP(M) = IP(M) + (DP(J+1)+DP(J))*G(J)                               GRN50400
   51 IF(Y(M).EQ.0) GOTO 151                                             GRN50500
      IP(M) = (ZUR(M)+ZKOR(M)/Y(MM)+ZC*IP(M))/Y(M)                       GRN50600
      GOTO 50                                                            GRN50700
  154 IF(D.EQ.S(N)) GOTO 156                                             GRN50800
      CALL INTKO(D,Z,ZH,ZI)                                              GRN50900
      IP(M) = (D3(N) + W/(D*D*D))*(ZHH(N)-ZH)                            GRN51000
  156 IF(N.EQ.NP) GOTO 151                                               GRN51100
      DO 157 J = N,NPP                                                   GRN51200
  157 IP(M) = IP(M) + (D3(J+1)+D3(J))*G(J)                               GRN51300
  151 IP(M) = -ZKOR(M)/(D*D)-ZD*IP(M)                                    GRN51400
   50 CONTINUE                                                           GRN51500
C                                                                        GRN51600
C --- RE-EVALUATION OF COEFFICIENTS Q AND R TO INCLUDE NEW BOUNDARY      GRN51700
C           CONDITIONS.                                                  GRN51800
      SN = (Y(M2)-Y(M1))/(Y(M2)-T*Y(M1))                                 GRN51900
      SP = (Y(MM)-Y(ML))/(T*Y(MM)-Y(ML))                                 GRN52000
      TN = Y(M1)/(T*Y(M1)-Y(M2))                                         GRN52100
      TP = Y(MM)/(T*Y(MM)-Y(ML))                                         GRN52200
C                                                                        GRN52300
      DO 20 M = M2,ML                                                    GRN52400
      QQ(M,M2) = QQ(M,M2)     + TN*(QQ(M,M1)-Y(M1)*HH(M))                GRN52500
      QQ(M,M1) = QQ(M,M1)*SN + TN*Y(M1)*HN(M)                            GRN52600
      QQ(M,ML) = QQ(M,ML)     + TP*(QQ(M,MM)-Y(MM)*HP(M))                GRN52700
   20 QQ(M,MM) = QQ(M,MM)*SP + TP*Y(MM)*HP(M)                            GRN52800
C                                                                        GRN52900
      DO 60 M = M2,ML                                                    GRN53000
      ZR(M,M2) = ZR(M,M2) + TN*(ZR(M,M1)-ZUL(M)+Y(M1)*IN(M))             GRN53100
      ZR(M,M1) = ZUL(M)   + SN*(ZR(M,M1)-ZUL(M)) - Y(M1)*TN*IN(M)        GRN53200
      ZR(M,ML) = ZR(M,ML) + TP*(ZR(M,MM)-ZUR(M)+Y(MM)*IP(M))             GRN53300
   60 ZR(M,MM) = ZUR(M)   + SP*(ZR(M,MM)-ZUR(M)) - Y(MM)*TP*IP(M)        GRN53400
C                                                                        GRN53500
      RETURN                                                             GRN53600
      END                                                                GRN53700
C                                                                        GRN53800
```

```
C------------------------------------ BPOL ------------------------------------GRN53900
C --- CALCULATION OF B-POLARIZATION FIELDS.  (SEE SECTION 4.3)                  GRN54000
C                                                                               GRN54100
      SUBROUTINE BPOL                                                           GRN54200
      IMPLICIT COMPLEX*16 (E,F,G,I,X,Y,Z),REAL*8 (A-D,K,O-W)                    GRN54300
      REAL*8 Y                                                                  GRN54400
      COMMON/A0/ IA,IB,IC,ROOT2,PID2,PID4,PID8,D2,D8,SJ,NP,NN,NDIMA,NRHSGRN54500
      COMMON/A1/ Y(75),BIC(75),KKK(75), X(75),FF(75),PP(75),GG(75)      GRN54600
      COMMON/A2/ DEL,SIG,DFREQ,OMEG,I,IPW,O,W,T,PI,M1,MM,M2,ML,M3,MK,JERGRN54700
      COMMON/A3/ ZR(75,75),ZS(75,75),ZZ(75,75)                          GRN54800
      COMMON/A5/ A(75),B(75),C(75),P(75),Q(75),R(75),AA(75),KK(75)      GRN54900
      COMMON/WS/ ZA(73,73),ZB(73)                                       GRN55000
C                                                                       GRN55100
      XL = W/(W+IPW*KKK(M1))                                            GRN55200
      XR = W/(W+IPW*KKK(MM))                                            GRN55300
      XRR = XR-XL                                                       GRN55400
      FL = -.5D0*IPW*XL                                                 GRN55500
      FR = -.5D0*IPW*XR                                                 GRN55600
      FRR = FR - FL                                                     GRN55700
      ZC = T*PI*I                                                       GRN55800
C                                                                       GRN55900
      DO 10 M = M2,ML                                                   GRN56000
      L = M - M1                                                        GRN56100
      DO 20 J = M2,ML                                                   GRN56200
      N = J - M1                                                        GRN56300
   20 ZA(L,N) = ZR(M,J) + ZS(M,J)                                       GRN56400
      ZA(L,L) = ZA(L,L) + ZC*KKK(M)                                     GRN56500
   10 ZB(L) = ZC*(KKK(M1)-KKK(M))*FL - (ZR(M,MM)+ZS(M,MM))*FRR          GRN56600
C                                                                       GRN56700
      CALL LINSYS(JER,LHS(ZA,'COMPLEX',NN,NDIMA),PRECIS('DOUBLE'),      GRN56800
     1      RHS(ZB,NN,NRHS))                                            GRN56900
      DO 40 M = M2,ML                                                   GRN57000
   40 FF(M) = ZB(M-M1)                                                  GRN57100
      FF(M1) = O                                                        GRN57200
      FF(MM) = FRR                                                      GRN57300
C                                                                       GRN57400
      DO 30 M = M2,ML                                                   GRN57500
   30 X(M) = T*KKK(M)*FF(M) + IPW*(KKK(M1)-KKK(M))*XL                   GRN57600
      X(M1) = O                                                         GRN57700
      X(MM) = XRR                                                       GRN57800
C                                                                       GRN57900
      DO 60 M = M1,ML                                                   GRN58000
   60 GG(M) = -.5D0*(X(M+1)-X(M))*AA(M+1)                               GRN58100
      GG(MM) = O                                                        GRN58200
C                                                                       GRN58300
      DO 70 J = M1,MM                                                   GRN58400
      X(J) = X(J) + XL                                                  GRN58500
   70 FF(J) = FF(J) + FL                                                GRN58600
C                                                                       GRN58700
      RETURN                                                            GRN58800
      END                                                               GRN58900
C                                                                       GRN59000
```

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C------------------------------------ EPOL ------------------------------GRN59100
C --- CALCULATION OF E-POLARIZATION FIELDS.  (SEE SECTION 4.2)            GRN59200
C                                                                         GRN59300
      SUBROUTINE EPOL(JCOR)                                               GRN59400
      IMPLICIT COMPLEX*16 (E,F,G,I,X,Y,Z),REAL*8(A-D,K,O-W)               GRN59500
      REAL*8 Y                                                            GRN59600
      COMMON/A0/ IA,IB,IC,ROOTZ,PID2,PID4,PID8,D2,D8,SJ,NP,NN,NDIMA,NRHSGRN59700
      COMMON/A1/ Y(75),DIC(75),KKK(75), E(75),YM(75),YP(75),ZZ(75)        GRN59800
      COMMON/A2/ DEL,SIG,DFREQ,OMEG,I,IPW,O,W,T,PI,M1,MM,M2,ML,M3,MK,JERGRN59900
      COMMON/A3/ ZR(75,75),ZS(75,75),QQ(75,75)                            GRN60000
      COMMON/A4/ SN,TN,SP,TP                                              GRN60100
      COMMON/A5/ A(75),B(75),C(75),P(75),Q(75),R(75),AA(75),KK(75)        GRN60200
      COMMON/WS/ ZA(73,73),ZB(73)                                         GRN60300
C                                                                         GRN60400
      ZC = I*T*PI                                                         GRN60500
      ZE = I/PI                                                           GRN60600
      EL = (IPW/T)/(W+IPW*KKK(M1))                                        GRN60700
      ER = (IPW/T)/(W+IPW*KKK(MM))                                        GRN60800
      YPL = (W-1)*EL                                                      GRN60900
      ERR = ER - EL                                                       GRN61000
C                                                                         GRN61100
      DO 10 M = M2,ML                                                     GRN61200
      L = M - M1                                                          GRN61300
      DO 20 J = M2,ML                                                     GRN61400
      N = J - M1                                                          GRN61500
   20 ZA(L,N) = ZR(M,J) + QQ(M,J)                                         GRN61600
      ZA(L,L) = ZA(L,L) + ZC*KKK(M)                                       GRN61700
   10 ZB(L) = ZC*(KKK(M1)-KKK(M))*EL - (ZR(M,MM)+QQ(M,MM))*ERR            GRN61800
C                                                                         GRN61900
      CALL LINSYS(JER,LHS(ZA,'COMPLEX',NN,NDIMA),PRECIS('DOUBLE'),        GRN62000
     1     RHS(ZB,NN,NRHS))                                               GRN62100
      DO 40 M = M2,ML                                                     GRN62200
   40 E(M) = ZB(M-M1)                                                     GRN62300
      E(M1) = O                                                           GRN62400
      E(MM) = ERR                                                         GRN62500
      IF(JCOR.NE.1) GOTO 30                                               GRN62600
      E(M1) = TN*E(M2)                                                    GRN62700
      E(MM) = TP*E(ML) + SP*ERR                                           GRN62800
C                                                                         GRN62900
   30 DO 50 M = M2,ML                                                     GRN63000
      ZC = ZR(M,MM)*ERR                                                   GRN63100
      ZK = QQ(M,MM)*ERR                                                   GRN63200
      DO 60 J = M2,ML                                                     GRN63300
      ZC = ZC + ZR(M,J)*E(J)                                              GRN63400
   60 ZK = ZK + QQ(M,J)*E(J)                                              GRN63500
      YM(M) = ZK*ZE                                                       GRN63600
   50 YP(M) = -ZC*ZE                                                      GRN63700
C                                                                         GRN63800
      DO 70 M = M1,ML                                                     GRN63900
   70 ZZ(M) = -I*AA(M+1)*(E(M+1)-E(M))                                    GRN64000
C                                                                         GRN64100
      YM(M1) = O                                                          GRN64200
      YM(MM) = O                                                          GRN64300
      YP(M1) = O                                                          GRN64400
      YP(MM) = (W-I)*ERR                                                  GRN64500
C                                                                         GRN64600
      DO 80 J = M1,MM                                                     GRN64700
      E(J) = E(J) + EL                                                    GRN64800
      YM(J) = YM(J) + W                                                   GRN64900
   80 YP(J) = YP(J) + YPL                                                 GRN65000
      RETURN                                                              GRN65100
      END                                                                 GRN65200
```

```
C                                                                       GRN65300
C------------------------------- RESULT -------------------------------GRN65400
C --- PRINT OUT OF OPTIONS AND FIELD VALUES.                            GRN65500
C                                                                       GRN65600
      SUBROUTINE RESULT(JP,JHP,JEP,JC,JCP,JT,JN,JDIM,JM,JCOR)          GRN65700
      IMPLICIT COMPLEX*16(E,F,G,I,X,Y,Z),COMPLEX(C),REAL*8(A,D,K,O-W)  GRN65800
      COMMON/A1/ S(75),DIC(75),KKK(75),XX(75),YY(75),YP(75),ZZ(75)     GRN65900
      COMMON/A2/ DEL,SIG,DFREQ,OMEG,I,IPW,O,W,T,PI,M1,MM,M2,ML,M3,MK,JERGRN66000
      COMMON/WS/ CX(75),CY(75),CP(75),CZ(75),H(75),HM(75),HIC(75),HK(75)GRN66100
                                                                        GRN66200
    5 FORMAT(' ',I5,3E15.5)                                             GRN66300
    6 FORMAT('1'//3X,'ERROR',3X,'FREQUENCY',8X,'OMEGA',10X,'SIGMA',10X, GRN66400
     1      'DELTA',7X,'JHP',4X,'JEP',4X,'JC',3X,'JCP',4X,             GRN66500
     1       'JT',4X,'JN',3X,'JDIM',3X,'JM',3X,'JCOR'/)                 GRN66600
    7 FORMAT('0',I5,4D15.5,9I6)                                         GRN66700
    8 FORMAT('1'/5X,'J',8X,'GRID',10X,'ICOND',12X,'KKK'//)              GRN66800
    9 FORMAT(I3,F10.3,2E14.5,6E13.4,F10.3)                              GRN66900
   10 FORMAT('1'/2X,'M',6X,'Y(J)',15X,'X',26X,'F',51X,'G',15X,          GRN67000
     1      'Y(J+1/2)'//)                                              GRN67100
   11 FORMAT('1'/ 2X,'M',6X,'Y(J)',6X,'MOD X', 8X,'PHASE X',7X,'MOD F', GRN67200
     1      7X,'PHASE F',32X,'MOD G',8X,'PHASE G',6X,'Y(J+1/2)'//)      GRN67300
   12 FORMAT('1'/ 2X,'M',5X,'Y(J)',15X,'E',26X,'Y-',24X,'Y+',25X,'Z',   GRN67400
     1      14X,'Y(J+1/2)'//)                                          GRN67500
   13 FORMAT('1'/2X,'M',5X,'Y(J)',6X,'MOD E',9X,'PHASE E',7X,'MOD Y-',  GRN67600
     1      6X,'PHASE Y-',6X,'MOD Y+',6X,'PHASE Y+',6X,'MOD Z',7X,     GRN67700
     1      'PHASE Z',6X,'Y(J+1/2)'//)                                 GRN67800
   14 FORMAT(11A4,2I2)                                                  GRN67900
   15 FORMAT(I2,2I4,2D15.5,2A8,6A4)                                     GRN68000
C                                                                       GRN68100
      HSCALE = DEL*1.E-3                                                GRN68200
      DO 1 J = M1,MM                                                    GRN68300
      HIC(J) = DIC(J)                                                   GRN68400
      HK(J)  = KKK(J)                                                   GRN68500
      H(J)   = S(J)                                                     GRN68600
      CX(J) = XX(J)                                                     GRN68700
      CY(J) = YY(J)                                                     GRN68800
      CP(J) = YP(J)                                                     GRN68900
    1 CZ(J) = ZZ(J)                                                     GRN69000
      IF(JDIM.NE.0) GOTO 3                                              GRN69100
      DO 2 J = M1,MM                                                    GRN69200
    2 H(J) = H(J)*HSCALE                                                GRN69300
    3 HM(MM) = 0                                                        GRN69400
      DO 4 J = M1,ML                                                    GRN69500
    4 HM(J) = .5 * (H(J+1)+H(J))                                        GRN69600
      WRITE(6,6)                                                        GRN69700
      WRITE(6,7) JER,DFREQ,OMEG,SIG,DEL,JHP,JEP,JC,JCP,JT,JN,JDIM,JM,   GRN69800
     1     JCOR                                                         GRN69900
      WRITE(6,8)                                                        GRN70000
      WRITE(6,5) (J,H(J),HIC(J),HK(J),J=M1,MM)                          GRN70100
      IF(JP.EQ.1) WRITE(6,10)                                           GRN70200
      IF(JP.EQ.2) WRITE(6,12)                                           GRN70300
      WRITE(6,9) (J,H(J),CX(J),CY(J),CP(J),CZ(J),HM(J),J=M1,MM)         GRN70400
      IF(JC.EQ.1) WRITE(7,15) JP,M1,MM,DFREQ,SIG,OMEG,DEL               GRN70500
      IF(JC.EQ.1) WRITE(7,14) (H(J),HM(J),HIC(J),CX(J),CY(J),CP(J),     GRN70600
     1     CZ(J),JP,J,J=M1,MM)                                          GRN70700
      CALL MODULO(CX,CX)                                                GRN70800
      CALL MODULO(CY,CY)                                                GRN70900
      CALL MODULO(CP,CP)                                                GRN71000
      CALL MODULO(CZ,CZ)                                                GRN71100
      IF(JP.EQ.1) WRITE(6,11)                                           GRN71200
      IF(JP.EQ.2) WRITE(6,13)                                           GRN71300
      WRITE(6,9) (J,H(J),CX(J),CY(J),CP(J),CZ(J),HM(J),J=M1,MM)         GRN71400
      IF(JC.EQ.1) WRITE(7,14) (H(J),HM(J),HIC(J),CX(J),CY(J),CP(J),     GRN71500
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     1     CZ(J),JP,J,J=M1,MM)                                          GRN71500
           RETURN                                                       GRN71600
           END                                                          GRN71700
C                                                                       GRN71800
C---------------------------------- MODULO ------------------------------GRN71900
C --- CALCULATION OF MODULUS AND PHASE OF FIELDS.                       GRN72000
C                                                                       GRN72100
           SUBROUTINE MODULO  (CRI,CMP)                                 GRN72200
           REAL CABS,MOD                                                GRN72300
           COMPLEX CRI(75),CMP(75),I                                    GRN72400
           COMMON/A2/ A(24),M1,MM                                       GRN72500
C                                                                       GRN72600
           O = 0.                                                       GRN72700
           I = (0.,1.)                                                  GRN72800
           DO 1 M = M1,MM                                               GRN72900
           R = REAL(CRI(M))                                             GRN73000
           AI = AIMAG(CRI(M))                                           GRN73100
           MOD = CABS(CRI(M))                                           GRN73200
           PHASE = O                                                    GRN73300
           IF(R.NE.O.AND.AI.NE.O) PHASE = ATAN2(AI,R)                   GRN73400
     1     CMP(M) = MOD * I*PHASE                                       GRN73500
           RETURN                                                       GRN73600
           END                                                          GRN73700
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VITA

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Title of Thesis:

The Two-Dimensional Theory of Electromagnetic Induction in Thin Sheets with Applications to the Earth.

Author [redacted]

Signature

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April, 1978