

**COMMUTATIVITY PRESERVING MAPPINGS OF  
VON NEUMANN ALGEBRAS**

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ABSTRACT

A map  $\theta : M \rightarrow N$  where  $M$  and  $N$  are rings is said to preserve commutativity in both directions if the elements  $a, b \in M$  commute if and only if  $\theta(a)$  and  $\theta(b)$  commute. In this paper we show that if  $M$  and  $N$  are von Neumann algebras with no central summands of type  $I_1$  or  $I_2$  and  $\theta$  is a bijective map which preserves commutativity in both directions then  $\theta(x) = c\varphi(x) + f(x)$  where  $c$  is an invertible element in  $Z_N$ , the center of  $N$ ,  $\varphi : M \rightarrow N$  is a Jordan isomorphism of  $M$  onto  $N$ , and  $f$  is an additive map of  $M$  into  $Z_N$ .

INTRODUCTION

By a commutativity preserving mapping of an algebra  $M$  into an algebra  $N$  we mean a mapping  $\theta : M \rightarrow N$  which maps commuting pairs of elements into commuting pairs. We say that  $\theta$  preserves commutativity in both directions if the elements  $a, b \in M$  commute if and only if  $\theta(a)$  and  $\theta(b)$  commute. The aim in the study of commutativity preserving mappings is to determine their structure. In this paper, we consider the case when  $M$  and  $N$  are von Neumann algebras. We shall prove

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**THEOREM 1.** *Let  $M$  and  $N$  be von Neumann algebras with no central summands of type  $I_1$  or  $I_2$ . Let  $\theta : M \rightarrow N$  be a bijective additive mapping. If  $\theta$  preserves commutativity in both directions then it is of the form*

$$\theta(x) = c\varphi(x) + f(x)$$

where  $c$  is an invertible element in  $Z_N$ ,  $\varphi : M \rightarrow N$  is a Jordan isomorphism of  $M$  onto  $N$ , and  $f$  is an additive mapping of  $M$  into  $Z_N$ .

It can be easily shown that Jordan isomorphisms of von Neumann algebras preserve commutativity in both directions (cf. [2, Theorem 3.4]). Thus, Theorem 1 characterizes bijective additive mappings preserving commutativity in both directions.

One usually assumes that a commutativity preserving mapping is linear. Our algebraical methods enable us to weaken this assumption and to assume only the additivity of the mapping. Also, mappings such as isomorphisms, anti-isomorphisms and Jordan isomorphisms will be considered in a ring sense -- for instance, by a Jordan isomorphism  $\varphi$  of  $M$  into  $N$  we shall mean an additive bijective mapping satisfying  $\varphi(x^2) = \varphi(x)^2$  for all  $x \in M$ .

A number of authors have characterized commutativity preserving mappings of various algebras. These characterizations are essentially the same as in Theorem 1, although in some algebras Jordan isomorphisms can be expressed in a more explicit form. It seems that the first result of that kind was given by Watkins in [16] where the form of bijective linear commutativity preserving mappings of  $M_n(F)$ , the algebra of all  $n \times n$  matrices,  $n \geq 4$ , over a field  $F$ , was determined. Also, by a simple counterexample it was shown that the situation in case  $n = 2$  is quite different (this justifies the assumption in Theorem 1 that von Neumann algebras must not contain central summands of type  $I_2$ ). The case when  $n = 3$  was

settled in [1] and [14]. In a series of papers [7,8,15] mappings preserving commutativity of symmetric matrices were discussed. The paper [8] of Choi, Jafarian, and Radjavi also contains some extensions of these results to the algebra of all bounded linear operators on an infinite dimensional Hilbert space. Subsequently, Omladič [13] described the structure of bijective linear mappings of  $\mathcal{B}(X)$ , the algebra of all bounded operators on a Banach space  $X$ ,  $\dim X \geq 3$ , which preserve commutativity in both directions. An analogous result for bijective  $*$ -linear mappings of von Neumann factors was obtained by the second named author [12] (note that in Theorem 1 we do not assume that  $\theta$  preserves adjoints). Finally, in [6] the first named author characterized linear bijective commutativity preserving mappings of prime algebras (satisfying some additional assumptions). Moreover, the assumption that  $\theta$  preserves commutativity was replaced by a weaker assumption that  $\theta(x)$  and  $\theta(x^2)$  commute for any element  $x$ . In this paper we use a similar approach as in [6], and, in fact, the assumption in Theorem 1 that  $\theta$  preserves commutativity in both directions can be replaced by a quite weaker one (see Theorem 3).

A mapping  $f$  of a ring  $M$  into itself is said to be commuting if  $f(x)$  commutes with  $x$  for every  $x$  in  $M$ . Additive commuting mappings of prime rings and von Neumann algebras were characterized in [4] and [5], respectively. A mapping  $q : M \rightarrow M$  is said to be a trace of a biadditive mapping if there exists a biadditive mapping  $B : M \times M \rightarrow M$  such that  $q(x) = B(x,x)$  for all  $x \in M$ . There is a simple connection between commuting traces of biadditive mappings and commutativity preserving mappings (see the proof of Step 2 of Theorem 3). The fundamental result in [6], upon which all the other results in [6] depend, determines the structure of commuting traces of biadditive mappings of certain prime rings. Following the procedure in [6], we will first obtain an analogous result for von Neumann algebras (Theorem 2).

Recall that a bijective additive mapping  $\theta$  of a ring  $M$  onto a ring  $N$  is called a Lie isomorphism if it preserves commutators, *i.e.*,  $\theta([x,y]) = [\theta(x),\theta(y)]$  for all  $x,y \in M$  where  $[u,v]$  denotes  $uv - vu$ . Obviously, these mappings preserve commutativity in both directions. Therefore, as a consequence of Theorem 1 we obtain a result concerning Lie isomorphisms of von Neumann algebras (Theorem 4). A similar result was obtained by the second named author in [10]. Comparing this result with Theorem 4 we see that in Theorem 4 we do not assume any continuity or  $*$ -linearity, but on the other hand, we have to exclude von Neumann algebras containing central summands of type  $I_1$  or  $I_2$ . Possibly Theorem 4 holds for arbitrary von Neumann algebras; however, to prove this one should have to use quite different methods.

The center of an algebra  $M$  will be denoted by  $Z_M$ . A ring is called semi-prime if  $aMa = 0$  implies  $a = 0$  for  $a \in M$ . Any  $C^*$ -algebra is semi-prime. We use [9] as a general reference for the theory of operator algebras.

## THE RESULTS

Our first goal is to determine the structure of all commuting traces of biadditive mappings on von Neumann algebras with no central summands of type  $I_1$  or  $I_2$ . For this purpose we need some preliminary results.

**LEMMA 1.** *Let  $M$  be a type I von Neumann algebra and let  $p \in M$  be a projection. There exist projections  $e, f_1, f_2$  in  $M$  such that  $p = e + f_1 + f_2$ ,  $e$  is abelian,  $f_1 \sim f_2$ ,  $f_1 \perp f_2$ , and  $e \perp f_1 + f_2$ .*

*Proof.* By considering the type I algebra  $pMp$  it suffices to assume  $p = 1$ ,

the identity of  $M$ . Since  $M$  is of type I,  $M = \bigoplus_{n \in K} M_n$  where  $K$  is a set

of distinct cardinals and  $M_n$  is a homogeneous algebra of type  $I_n$ . Now

$1 = \sum_{n \in K} p_n$  where  $p_n$  is the identity of  $M_n$  and is the sum of  $n$

orthogonal equivalent abelian projections. If  $n$  is finite and even then

$p_n = f_{1n} + f_{2n}$  where  $f_{1n} \sim f_{2n}$  and  $f_{1n} \perp f_{2n}$ . If  $n$  is finite and odd then

$p_n = e_n + f_{1n} + f_{2n}$  where  $e_n \neq 0$  is abelian,  $f_{1n} \sim f_{2n}$ ,  $f_{1n} \perp f_{2n}$ . If  $n$  is

infinite then by breaking up the set of  $n$  orthogonal, equivalent abelian projections

that sum to  $p_n$  into two subsets of the same cardinality we can write

$p_n = f_{1n} + f_{2n}$ ,  $f_{1n} \sim f_{2n}$ ,  $f_{1n} \perp f_{2n}$ . Set  $e = \sum e_n$ ,  $f_1 = \sum f_{1n}$ ,  $f_2 = \sum f_{2n}$ . Then

$e$  is abelian since it is a sum of abelian projections with mutually disjoint central

supports. Moreover  $f_1 \sim f_2$ ,  $f_1 \perp f_2$ ,  $e \perp f_1 + f_2$  and  $1 = \sum_{n \in K} p_n =$

$e + f_1 + f_2$ .

**LEMMA 2.** Let  $M$  be a von Neumann algebra with no type  $I_1$  or  $I_2$  summands.

Then the ideal  $I$  of  $M$  generated algebraically by  $\{[x^2, z]y[x, z] - [x, z]y[x^2, z] :$

$x, y, z \in M\}$  is equal to  $M$ .

*Proof.* If  $I \neq M$  then  $J = \bar{I} \neq M$  where  $\bar{I}$  is the uniform closure of  $I$ . Let

$N = M/J$ . Then  $N$  is semi-prime since it is a  $C^*$ -algebra, and  $N$  satisfies

$[x^2, z]y[x, z] = [x, z]y[x^2, z]$ . Standard polynomial identity theory for semi-prime rings

implies that  $[x, y]^2 \in Z_N$ . If  $p$  is in the continuous part of  $M$  then

$p = f_1 + f_2$  where  $f_1 \sim f_2$ ,  $f_1 \perp f_2$ . Hence there exists  $v \in M$  such that

$vv^* = f_1$ ,  $v^*v = f_2$  so that  $[v, v^*]^2 = (f_1 - f_2)^2 = f_1 + f_2$ . Hence

$\bar{p} = p + J \in Z_N$ . Let  $M_D$  be the type I part of  $M$  where  $D$  is a

projection in  $Z_M$ . If  $p \in M_D$  then, by Lemma 1,  $p = e + f_1 + f_2$  where  $e$  is

abelian,  $f_1 \sim f_2$ ,  $f_1 \perp f_2$  so that we can apply the previous argument to show that  $\overline{f_1 + f_2} = f_1 + f_2 + J \in Z_N$ . Let  $D = \bigoplus_{n \in K} D_n$  where  $D_n$  is a homogeneous summand of type  $n$  and  $K$  is a set of distinct cardinals. Then  $e_n = D_n e$  is an abelian projection in  $M_{D_n}$ . Since  $M$  has no summand of type  $I_1$  or  $I_2$  we can choose  $f_n, g_n$  in  $M_{D_n}$  such that  $\{e_n, f_n, g_n\}$  is a set of three pairwise orthogonal equivalent projections. Thus  $e = \Sigma e_n$ ,  $f = \Sigma f_n$ ,  $g = \Sigma g_n$  are pairwise orthogonal and equivalent. By the above argument,  $\bar{e} + \bar{f}$ ,  $\bar{f} + \bar{g}$ , and  $\bar{e} + \bar{g}$  are in  $Z_N$  so that  $\bar{e} \in Z_N$ . Thus  $N = M/J \subseteq Z_N$ . By [11, Lemma 2],  $Z_N = Z_M + J$  so that  $[M, M] \subseteq J$ . By [5, Lemma 2.6] the ideal generated by  $[M, M]$  is  $M$  so  $M = J$ , a contradiction. ■

A connection between Lemma 2 and commuting traces of biadditive mappings is indicated in the following lemma, which was proved in [6] (although it is not explicitly stated there, it is clear from the proof of [6, Theorem 1]).

**LEMMA 3.** *Let  $M$  be any ring admitting the operator  $\frac{1}{2}$  (i.e., the mapping  $x \rightarrow 2x$  is bijective). If  $q : M \rightarrow M$  is a commuting trace of a biadditive mapping, then there exist mappings  $g_1 : M \times M \times M \rightarrow M$  and  $g_2, g_3 : M \times M \times M \times M \rightarrow M$  such that*

$$\gamma(x, y, z)uq(w) = g_1(x, y, z)uw^2 + g_2(x, y, z, w)uw + g_3(x, y, z, w)u \quad (1)$$

for all  $x, y, z, w, u \in M$ , where

$$\gamma(x, y, z) = [x^2, z]y[x, z] - [x, z]y[x^2, z].$$

Moreover,  $g_2$  is additive in the last argument.

We will need the following simple lemma, which is a special case of [2, Lemma 1.2].

**LEMMA 4.** *Let  $G$  be an additive group and  $M$  be a semiprime ring. Suppose that additive mappings  $S$  and  $T$  of  $G$  into  $M$  satisfy  $S(x)MT(x) = \{0\}$  for all  $x \in G$ . Then  $S(x)MT(y) = \{0\}$  for all  $x, y \in G$ .*

We are now in a position to prove

**THEOREM 2.** *Let  $M$  be a von Neumann algebra with no central summands of type  $I_1$  or  $I_2$ . Let  $q : M \rightarrow M$  be a trace of a biadditive mapping. If  $q$  is commuting then it is of the form*

$$q(x) = \lambda x^2 + \mu(x)x + \nu(x), \quad x \in M,$$

where  $\lambda \in Z_M$ ,  $\mu$  and  $\nu$  are mappings of  $M$  into  $Z_M$ , and  $\mu$  is additive.

*Proof.* Replacing  $u$  by  $uv$  in (1), and then comparing the relation so obtained by (1), we obtain

$$\gamma(x,y,z)u[v,q(w)] = g_1(x,y,z)u[v,w^2] + g_2(x,y,z,w)u[v,w]. \quad (2)$$

Let  $1$  be the identity element of  $M$ . By Lemma 2 there exist  $t_i, x_i, y_i, z_i, u_i \in M$ ,  $i = 1, \dots, n$ , such that

$$\sum_{i=1}^n t_i \gamma(x_i, y_i, z_i) u_i = 1.$$

Using (2), we then see that for any  $v, w \in M$  we have

$$\begin{aligned}
[v, q(w)] &= 1[v, q(w)] \\
&= \left\{ \sum_{i=1}^n t_i \gamma(x_i, y_i, z_i) u_i \right\} [v, q(w)] \\
&= \sum_{i=1}^n t_i \{ \gamma(x_i, y_i, z_i) u_i [v, q(w)] \} \\
&= \sum_{i=1}^n t_i g_1(x_i, y_i, z_i) u_i [v, w^2] + \sum_{i=1}^n t_i g_2(x_i, y_i, z_i, w) u_i [v, w].
\end{aligned}$$

Thus

$$[v, q(w)] = \lambda[v, w^2] + \mu(w)[v, w], \quad v, w \in M \quad (3)$$

for some  $\lambda \in M$  and some map  $\mu : M \rightarrow M$ ; note that  $\mu$  is additive since, by Lemma 3,  $g_2$  is additive in the last argument. Our intention is to show that  $\lambda \in Z_M$  and that  $\mu$  maps  $M$  into  $Z_M$ .

Substituting  $vy$  for  $v$  in (3) we obtain

$$[v, q(w)]y + v[y, q(w)] = \lambda[v, w^2]y + \lambda v[y, w^2] + \mu(w)[v, w]y + \mu(w)v[y, w].$$

On the other hand, (3) shows that

$$[v, q(w)]y + v[y, q(w)] = \lambda[v, w^2]y + \mu(w)[v, w]y + v\lambda[y, w^2] + v\mu(w)[y, w].$$

Comparing the last two relations we get

$$[\lambda, v][y, w^2] + [\mu(w), v][y, w] = 0, \quad v, y, w \in M. \quad (4)$$

Replacing  $v$  by  $xv$  in (4), it follows that

$$x[\lambda, v][y, w^2] + [\lambda, xv][y, w^2] + x[\mu(w), v][y, w] + [\mu(w), xv][y, w] = 0.$$

By (4), the sum of the first and the third summands equals zero. Hence

$$[\lambda, xv][y, w^2] + [\mu(w), xv][y, w] = 0, \quad x, v, y, w \in M. \quad (5)$$

In particular,

$$[\lambda, xv](v[y, w]z)[y, w^2] + [\mu(w), xv](v[y, w]z)[y, w] = 0.$$

But on the other hand, (5) yields

$$([\mu(w), xv][y, w])z[y, w] = -[\lambda, xv][y, w^2]z[y, w].$$

Comparing the last two relations we arrive at

$$[\lambda, xv](v[y, w]z[y, w^2] - [y, w^2]z[y, w]) = 0.$$

Lemma 2 implies that  $[\lambda, xv]M = 0$  for all  $x \in M$ , and therefore,  $\lambda \in Z_M$ .

Now, (5) reduces to

$$[\mu(w),x]M[y,w] = \{0\}, \quad x,y,w \in M. \quad (6)$$

Now fix  $x,y \in M$  and introduce additive mappings  $S$  and  $T$  of  $M$  by  $S(w) = [\mu(w),x]$ ,  $T(w) = [y,w]$ . By (6), we have  $S(w)MT(w) = \{0\}$  for all  $w \in M$ , so it follows from Lemma 4 that  $S(w)MT(z) = \{0\}$  for all  $w,z \in M$ . Thus  $[\mu(w),x]\nu[y,z] = 0$  holds for any  $w,x,v,y,z \in M$ . In particular,  $[\mu(w),x]\nu[\mu(w),x] = 0$ ,  $w,x \in M$ , which shows that  $\mu(w) \in Z_M$ ,  $w \in M$ . By (2) we now see that  $\nu(w) = q(w) - \lambda w^2 - \mu(w)w$  lies in  $Z_M$  as well. With this the theorem is proved. ■

Our next aim is to consider commutativity preserving maps of von Neumann algebras. We need two preliminary results.

**LEMMA 5.** *Let  $M$  be a von Neumann algebra with no central summands of type  $I_1$ . If  $c \in Z_M$  is such that  $cM \subseteq Z_M$ , then  $c = 0$ .*

*Proof.* We have  $[cx,y] = 0$ , and therefore,  $c[x,y] = 0$  for all  $x,y \in M$ . Thus  $cI = \{0\}$  where  $I$  is the ideal of  $M$  generated by all commutators in  $M$ . But  $I = M$  [5, Lemma 2.6], and so  $c$  must be zero. ■

Recall that a ring  $M$  is said to be torsion-free if  $nx = 0$ , where  $x \in M$  and  $n$  is any positive integer, implies  $x = 0$ .

**LEMMA 6.** *Let  $M$  be a semiprime torsion-free ring and  $G$  be an additive group. Suppose that mappings  $\epsilon : G \times G \rightarrow M$  and  $\tau : G \times G \times G \rightarrow M$  are additive in each argument. If  $\epsilon(x,x)M\tau(x,x,x) = \{0\}$  for every  $x \in G$ , then  $\epsilon(y,y)M\tau(x,x,x) = \{0\}$  for all  $x,y \in G$ .*

*Proof.* We have  $\epsilon(x,x)r\tau(x,x,x) = 0$ . Note that the substitution  $x + ny$  for  $x$ , where  $x,y \in G$  and  $n$  is an integer, yields

$$\begin{aligned} n\{(\epsilon(x,y) + \epsilon(y,x))r\tau(x,x,x) + \epsilon(x,x)r(\tau(x,x,y) + \tau(x,y,x) + \tau(y,x,x))\} \\ + n^2z_2 + n^3z_3 + n^4z_4 = 0 \end{aligned}$$

for some elements  $z_2, z_3, z_4 \in M$  depending on  $x, y$  and  $r$ . Since  $n$  is an arbitrary integer and  $M$  is torsion-free, it follows easily that

$$(\epsilon(x,y) + \epsilon(y,x))r\tau(x,x,x) + \epsilon(x,x)r(\tau(x,x,y) + \tau(x,y,x) + \tau(y,x,x)) = 0.$$

Multiplying from the right by  $s\tau(x,x,x)$ , since  $\epsilon(x,x)M\tau(x,x,x) = \{0\}$ , we arrive at

$$(\epsilon(x,y) + \epsilon(y,x))r\tau(x,x,x)s\tau(x,x,x) = 0.$$

Since  $r$  and  $s$  are arbitrary elements in  $M$ , the semiprimeness of  $M$  implies that

$$(\epsilon(x,y) + \epsilon(y,x))M\tau(x,x,x) = \{0\} \tag{7}$$

for all  $x,y \in G$ . In this relation, replace  $x$  by  $x + nz$  with  $x,z \in G$  and  $n$  an integer. Arguing similarly as above, one obtains easily that

$$(\epsilon(z,y) + \epsilon(y,z))r\tau(x,x,x) + (\epsilon(x,y) + \epsilon(y,x))r(\tau(z,x,x) + \tau(x,z,x) + \tau(x,x,z)) = 0.$$

Multiplying from the right by  $s\tau(x,x,x)$ , and then using (7), we get

$$(\epsilon(z,y) + \epsilon(y,z))\Gamma\tau(x,x,x)\delta\tau(x,x,x) = 0.$$

Since  $R$  is semiprime, it follows that  $(\epsilon(z,y) + \epsilon(y,z))M\tau(x,x,x) = \{0\}$ . A special case of this relation, where  $z = y$ , gives the assertion of the lemma.

We now come to the central theorem of this paper; note that this theorem includes Theorem 1.

**THEOREM 3.** *Let  $M$  and  $N$  be von Neumann algebras with no central summands of type  $I_1$  or  $I_2$ . Let  $\theta : M \rightarrow N$  be a bijective additive mapping such that  $\theta(Z_M) = Z_N$ ,  $[\theta(x^2), \theta(x)] = 0$  for all  $x \in M$ , and  $[\theta^{-1}(wy), \theta^{-1}(y)] = 0$  for all  $y \in N$  and  $w \in Z_N$ . Then  $\theta$  is of the form*

$$\theta(x) = c\varphi(x) + f(x)$$

where  $c$  is an invertible element in  $Z_N$ ,  $f$  is an additive mapping of  $M$  into  $Z_N$ , and  $\varphi$  is a Jordan isomorphism of  $M$  onto  $N$ .

Moreover, there exist central projections  $p \in M$  and  $q \in N$  such that the restriction of  $\varphi$  to  $pM$  is an isomorphism of  $pM$  onto  $qN$ , and the restriction of  $\varphi$  to  $(1-p)M$  is an anti-isomorphism of  $(1-p)M$  onto  $(1-q)N$ .

*Proof.* The proof is broken up into a series of steps.

**STEP 1.** *There is an isomorphism  $\alpha : Z_M \rightarrow Z_N$  such that*

$$\begin{aligned} \theta(zx) - \alpha(z)\theta(x) &\in Z_N \text{ for all } z \in Z_M, x \in M, \text{ and} \\ \theta^{-1}(wy) - \alpha^{-1}(w)\theta^{-1}(y) &\in Z_M \text{ for all } w \in Z_N, y \in N. \end{aligned}$$

*Proof of Step 1.* Take  $x \in M$  and  $z \in Z_M$ . As  $\theta$  maps  $Z_M$  into  $Z_N$ , a substitution  $x + z$  for  $x$  in  $[\theta(x^2), \theta(x)] = 0$  gives  $[\theta(zx), \theta(x)] = 0$ . Denoting  $\theta(x)$  by  $y$ , we thus have  $[\theta(z\theta^{-1}(y)), y] = 0$  for arbitrary  $z \in Z_M$  and  $y \in N$ . That is, for any  $z \in Z_M$ ,  $y \rightarrow \theta(z\theta^{-1}(y))$  is a commuting additive mapping of a von Neumann algebra  $N$ . By [5, Theorem 2.1] it follows that there exists an element  $w$  in  $Z_N$  (depending on  $z$ ) such that  $\theta(z\theta^{-1}(y)) - wy \in Z_N$  for all  $y \in N$ ; or equivalently,  $\theta(zx) - w\theta(x) \in Z_N$  for all  $x \in M$ . We set  $w = \alpha(z)$ , and claim that the mapping  $z \rightarrow \alpha(z)$  is an isomorphism of  $Z_M$  onto  $Z_N$ . Our key relation is

$$\theta(zx) - \alpha(z)\theta(x) \in Z_N \quad \text{for all } x \in M, z \in Z_M. \quad (8)$$

Let us first prove that  $\alpha$  is additive. Take  $z_1, z_2 \in Z_M$ . According to (8), for any  $x \in M$  we have

$$\theta((z_1+z_2)x) \in \alpha(z_1+z_2)\theta(x) + Z_N.$$

On the other hand,

$$\theta((z_1+z_2)x) = \theta(z_1x) + \theta(z_2x) \in \alpha(z_1)\theta(x) + \alpha(z_2)\theta(x) + Z_N.$$

Comparing we get  $(\alpha(z_1+z_2) - \alpha(z_1) - \alpha(z_2))\theta(x) \in Z_N$ . Since  $\theta$  is onto, Lemma 5 implies that  $\alpha(z_1+z_2) = \alpha(z_1) + \alpha(z_2)$ .

Next, let us show that  $\alpha$  is multiplicative. On the one hand, for  $z_1, z_2 \in Z_M$ ,  $x \in M$ , we have

$$\theta(z_1z_2x) \in \alpha(z_1z_2)\theta(x) + Z_N,$$

while on the other hand,

$$\begin{aligned}
\theta(z_1(z_2x)) &\in \alpha(z_1)\theta(z_2x) + Z_N \\
&\subseteq \alpha(z_1)(\alpha(z_2)\theta(x) + Z_N) + Z_N \\
&= \alpha(z_1)\alpha(z_2)\theta(x) + Z_N.
\end{aligned}$$

Hence  $(\alpha(z_1z_2) - \alpha(z_1)\alpha(z_2))\theta(x) \in Z_N$ , and so  $\alpha(z_1z_2) = \alpha(z_1)\alpha(z_2)$  by Lemma 5.

Suppose that  $\alpha(z) = 0$  for some  $z \in Z_M$ . By (8), we then have  $\theta(zx) \in Z_N$  for every  $x \in M$ . Since we assumed that  $\theta(Z_M) = Z_N$  this implies  $zx \in Z_M$ ,  $x \in M$ , and so Lemma 5 yields  $z = 0$ . Thus  $\alpha$  is one-to-one.

Let us show that  $\alpha$  is onto. Take  $w \in Z_N$ . By assumption, we have  $[\theta^{-1}(wy), \theta^{-1}(y)] = 0$  for all  $y \in N$ . Writing  $y$  as  $\theta(x)$ , we get  $[\theta^{-1}(w\theta(x)), x] = 0$ . That is,  $x \rightarrow \theta^{-1}(w\theta(x))$  is a commuting additive mapping of  $M$ . By [5, Theorem 2.1] there is an element  $\beta(w) \in Z_M$  such that  $\theta^{-1}(w\theta(x)) - \beta(w)x \in Z_M$  for all  $x \in M$ ; or equivalently,

$$\theta^{-1}(wy) - \beta(w)\theta^{-1}(y) \in Z_M \quad \text{for all } y \in N. \quad (9)$$

As  $\theta(Z_M) = Z_N$  it follows that  $wy - \theta(\beta(w)\theta^{-1}(y)) \in Z_N$ ,  $y \in N$ . In view of (8),  $\theta(\beta(w)\theta^{-1}(y)) \in \alpha(\beta(w))y + Z_N$ , and therefore  $(w - \alpha(\beta(w)))y \in Z_N$ ,  $y \in N$ . But then  $w = \alpha(\beta(w))$  by Lemma 5. This implies that  $\alpha$  is onto. Of course,  $\beta = \alpha^{-1}$ , and so, according to (9), the assertion of Step 1 is proved.

**STEP 2.** *There exist an element  $\lambda \in Z_N$ , an additive mapping  $\mu : M \rightarrow Z_N$  and a mapping  $\nu : M \rightarrow Z_N$  such that*

$$\theta(x^2) = \lambda\theta(x)^2 + \mu(x)x + \nu(x) \quad \text{for all } x \in M. \quad (10)$$

*Proof of Step 2.* The relation  $[\theta(x), \theta(x^2)] = 0$ ,  $x \in M$ , can be written in the form  $[y, \theta(\theta^{-1}(y)^2)] = 0$ ,  $y \in N$ . Thus,  $q(y) = \theta(\theta^{-1}(y)^2)$  is a commuting mapping of  $N$ . Since  $q$  is a trace of a biadditive mapping  $B(y, z) = \theta(\theta^{-1}(y)\theta^{-1}(z))$ , Theorem 2 can be applied. Hence there are  $\lambda \in Z_N$ , an additive mapping  $\mu_1 : N \rightarrow Z_N$  and a mapping  $\nu_1 : N \rightarrow Z_N$  such that

$$\theta(\theta^{-1}(y)^2) = \lambda y^2 + \mu_1(y)y + \nu_1(y)$$

for all  $y \in N$ . Note that this implies (10) where  $\mu = \mu_1\theta$  and  $\nu = \nu_1\theta$ .

**STEP 3.**  *$\lambda$  is invertible.*

*Proof of Step 3.* We have

$$x^2 = \theta^{-1}(\lambda\theta(x)^2 + \mu(x)\theta(x) + \nu(x)).$$

Applying Step 1 and the assumption that  $\theta^{-1}$  maps  $Z_N$  into  $Z_M$  it follows that

$$x^2 - \alpha^{-1}(\lambda)\theta^{-1}(\theta(x)^2) - \alpha^{-1}(\mu(x))x \in Z_M.$$

Consequently

$$[x^2, u] = \alpha^{-1}(\lambda)[\theta^{-1}(\theta(x)^2), u] - \alpha^{-1}(\mu(x))[x, u]$$

holds for all  $x, u \in M$ . From this relation we see that

$$\begin{aligned} [x^2, u]v[x, u] - [x, u]v[x^2, u] \\ = \alpha^{-1}(\lambda)\{[\theta^{-1}(\theta(x)^2), u]v[x, u] - [x, u]v[\theta^{-1}(\theta(x)^2), u]\} \end{aligned}$$

for all  $x, u, v \in M$ . Since the ideal generated by elements of the form  $[x^2, u]v[v, u] - [x, u]v[x^2, u]$  is equal to  $M$  (Lemma 2), it follows that  $1 \in \alpha^{-1}(\lambda)M$ , which means that  $\alpha^{-1}(\lambda)$  is invertible. But then  $\lambda$  is invertible.

**STEP 4.** A mapping  $\varphi : M \rightarrow N$ , defined by

$$\varphi(x) = \lambda\theta(x) + \frac{1}{2}\mu(x)$$

is a Jordan homomorphism.

*Proof of Step 4.* We will argue similarly as in the proof of [6, Theorem 2]. We have

$$\begin{aligned} \varphi(x^2) &= \lambda\theta(x^2) + \frac{1}{2}\mu(x^2) \\ &= \lambda^2\theta(x)^2 + \lambda\mu(x)\theta(x) + \lambda\nu(x) + \frac{1}{2}\mu(x^2), \end{aligned}$$

and

$$\begin{aligned} \varphi(x)^2 &= (\lambda\theta(x) + \frac{1}{2}\mu(x))^2 \\ &= \lambda^2\theta(x)^2 + \lambda\mu(x)\theta(x) + \frac{1}{4}\mu(x)^2. \end{aligned}$$

Comparing these two relations we get

$$\varphi(x^2) - \varphi(x)^2 \in Z_N \quad \text{for all } x \in M. \quad (11)$$

Define the mapping  $\epsilon : N \times N \rightarrow N$  by

$$\epsilon(x,y) = \varphi(xy+yx) - \varphi(x)\varphi(y) - \varphi(y)\varphi(x).$$

Obviously,  $\epsilon$  is biadditive and it satisfies  $\epsilon(x,y) = \epsilon(y,x)$  for all  $x,y \in M$ .

Replacing  $x$  by  $x + y$  in (11) we see that  $\epsilon$  in fact maps into  $Z_N$ . In order to show that  $\varphi$  is a Jordan homomorphism we must prove that  $\epsilon = 0$ .

Note that  $\varphi(x^2) = \varphi(x)^2 + \frac{1}{2}\epsilon(x,x)$ . Next, we have

$$\begin{aligned} \varphi(x^3) &= \frac{1}{2}\varphi(x^2x + xx^2) \\ &= \frac{1}{2}\{\varphi(x^2)\varphi(x) + \varphi(x)\varphi(x^2) + \epsilon(x^2,x)\} \\ &= \frac{1}{2}\{(\varphi(x)^2 + \frac{1}{2}\epsilon(x,x))\varphi(x) + \varphi(x)(\varphi(x)^2 + \frac{1}{2}\epsilon(x,x)) + \epsilon(x^2,x)\} \\ &= \varphi(x)^3 + \frac{1}{2}\epsilon(x,x)\varphi(x) + \frac{1}{2}\epsilon(x^2,x). \end{aligned}$$

Hence

$$\begin{aligned} \varphi(x^4) &= \frac{1}{2}\varphi(xx^3 + x^3x) \\ &= \frac{1}{2}\{\varphi(x)\varphi(x^3) + \varphi(x^3)\varphi(x) + \epsilon(x^3,x)\} \\ &= \frac{1}{2}\{\varphi(x)(\varphi(x)^3 + \frac{1}{2}\epsilon(x,x)\varphi(x) + \frac{1}{2}\epsilon(x^2,x)) + (\varphi(x)^3 + \frac{1}{2}\epsilon(x,x)\varphi(x) \\ &\quad + \frac{1}{2}\epsilon(x^2,x))\varphi(x) + \epsilon(x^3,x)\} \\ &= \varphi(x)^4 + \frac{1}{2}\epsilon(x,x)\varphi(x)^2 + \frac{1}{2}\epsilon(x^2,x)\varphi(x) + \frac{1}{2}\epsilon(x^3,x). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \varphi(x^4) &= \varphi((x^2)^2) = \varphi(x^2)^2 + \frac{1}{2}\epsilon(x^2,x^2) \\ &= (\varphi(x)^2 + \frac{1}{2}\epsilon(x,x))^2 + \frac{1}{2}\epsilon(x^2,x^2) \\ &= \varphi(x)^4 + \epsilon(x,x)\varphi(x)^2 + \frac{1}{4}\epsilon(x,x)^2 + \frac{1}{2}\epsilon(x^2,x^2). \end{aligned}$$

Comparing the two relations, so obtained for  $\varphi(x^4)$ , we get

$$\epsilon(x,x)\varphi(x)^2 - \epsilon(x^2,x)\varphi(x) \in Z_N$$

where  $x$  is an arbitrary element in  $M$ . This implies that

$$\epsilon(x,x)[\varphi(x)^2,u] = \epsilon(x^2,x)[\varphi(x),u]$$

for all  $x \in M$ ,  $u \in N$ , and therefore,

$$\epsilon(x,x)([\varphi(x)^2,u]y[\varphi(x),u] - [\varphi(x),u]y[\varphi(x)^2,u]) = 0$$

for all  $x \in M$ ,  $y,u \in N$ . Now pick  $y,u \in N$  and define  $\tau : M \times M \times M \rightarrow N$  by

$$\tau(x_1,x_2,x_3) = [\varphi(x_1)\varphi(x_2),u]y[\varphi(x_3),u] - [\varphi(x_1),u]y[\varphi(x_2)\varphi(x_3),u]$$

and note that  $\epsilon(x,x)\tau(x,x,x) = 0$  for all  $x \in M$ . Since  $\epsilon$  maps in the center of  $N$ , we also have  $\epsilon(x,x)M\tau(x,x,x) = \{0\}$ . Thus, the mappings  $\epsilon$  and  $\tau$  satisfy the requirements of Lemma 6, and so

$$\epsilon(v,v)N\tau(x,x,x) = \{0\} \text{ for all } x,v \in M.$$

Using the definition of  $\varphi$ , we see that

$$\tau(x,x,x) = \lambda^3([\theta(x)^2,u]y[\theta(x),u] - [\theta(x),u]y[\theta(x)^2,u]).$$

Thus, since  $\lambda$  is invertible and  $\theta$  is onto, we have

$$\epsilon(v,v)N([s^2,u]y[s,u] - [s,u]y[s^2,u]) = \{0\}$$

for all  $v \in M$  and  $s,y,u \in N$ . Applying Lemma 2 we see that  $\epsilon(v,v)$  must be zero. This proves that  $\varphi$  is a Jordan homomorphism.

We set  $c = \lambda^{-1}$  and  $f(x) = -\frac{1}{2}\lambda^{-1}\mu(x)$ , so we have  $\theta(x) = c\varphi(x) + f(x)$ .

**STEP 5.**  $\varphi$  is one-to-one and onto.

*Proof of Step 5.* Suppose  $\varphi(a) = 0$  for some  $a \in M$ . Then  $\theta(a) = f(a) \in Z_N$ . By assumption, this yields  $a \in Z_M$ . Therefore,  $\varphi(ax) = \frac{1}{2}\varphi(ax + xa) = \frac{1}{2}(\varphi(a)\varphi(x) + \varphi(x)\varphi(a)) = 0$  for every  $x \in M$ . As above, this implies that  $ax \in Z_M$ . But then  $a = 0$  by Lemma 5.

Let us show that the restriction of  $\varphi$  to  $Z_M$  is equal to  $\alpha$ . Take  $z \in Z_M$ . Then  $\theta(z) \in Z_N$ , and therefore,  $\varphi(z) \in Z_N$ . Hence

$$\varphi(zx) = \frac{1}{2}\varphi(zx + xz) = \frac{1}{2}(\varphi(z)\varphi(x) + \varphi(x)\varphi(z)) = \varphi(z)\varphi(x)$$

holds for any  $x$  in  $M$ . Consequently

$$\theta(zx) = c\varphi(zx) + f(zx) = c\varphi(z)\varphi(x) + f(zx).$$

Thus  $\theta(zx) = c\varphi(z)\varphi(x) \in Z_N$  for all  $x \in M$ . On the other hand, by Step 1 we have  $\theta(zx) - \alpha(z)\theta(x) \in Z_N$ , that is,  $\theta(zx) - c\alpha(z)\varphi(x) \in Z_N$ . Comparing we get  $(\varphi(z) - \alpha(z))c\varphi(x) \in Z_N$ , and therefore, as  $c\varphi(x) = \theta(x) - f(x)$ , we have  $(\varphi(z) -$

$\alpha(z))\theta(x) \in Z_N$ . By Lemma 5 it follows that  $\varphi(z) = \alpha(z)$ .

Since  $\alpha$  is onto, there is  $c_1 \in Z_M$  such that  $c = \alpha(c_1) = \varphi(c_1)$ .

Similarly, for every  $x \in M$  there is  $f_1(x) \in Z_M$  such that  $\varphi(f_1(x)) = f(x)$ .

Thus  $\theta(x) = \varphi(c_1)\varphi(x) + \varphi(f_1(x))$ . As shown above, we have  $\varphi(c_1)\varphi(x) = \varphi(c_1x)$ , which gives  $\theta(x) = \varphi(c_1x + f_1(x))$ . Thus, since  $\theta$  is onto,  $\varphi$  is onto as well.

It remains to prove

**STEP 6.** *There exist central projections  $p \in M$  and  $q \in N$  such that the restriction of  $\varphi$  to  $pM$  is an isomorphism of  $pM$  onto  $qN$ , and the restriction of  $\varphi$  to  $(1-p)M$  is an anti-isomorphism of  $(1-p)M$  onto  $(1-q)N$ .*

*Proof of Step 6.* This assertion follows immediately from [3, Theorem 1]. Namely, this theorem tells us that if  $\varphi$  is a Jordan isomorphism of a ring  $M$  onto a 2-torsion-free semiprime ring  $N$  in which the annihilator of any ideal is a direct summand (i.e., for any ideal  $I$  in  $N$ , we have  $N = \text{Ann}(I) \oplus J$  for some ideal  $J$  of  $N$  -- von Neumann algebras certainly satisfy this condition), then there exist ideals  $U$  and  $V$  of  $M$  and ideals  $U'$  and  $V'$  of  $N$  such that  $U \oplus V = M$ ,  $U' \oplus V' = N$ , the restriction of  $\varphi$  to  $U$  is an isomorphism of  $U$  onto  $U'$ , and the restriction of  $\varphi$  to  $V$  is an anti-isomorphism of  $V$  to  $V'$ . By standard arguments one shows that in case  $M$  and  $N$  are von Neumann algebra, these ideals must be of the form  $U = pM$ ,  $v = (1-p)M$  for some central projection  $p$  in  $M$ , and  $U' = qN$ ,  $V' = (1-q)N$  for some central projection  $q$  in  $N$ .

The proof of the theorem is thereby completed.

Our last goal is to determine the structure of Lie isomorphisms of von Neumann algebras. For this purpose we need a refinement of Lemma 5.

**LEMMA 7.** *Let  $M$  be a von Neumann algebra with no central summands of type  $I_1$ . If  $c \in Z_M$  is such that  $c[x,y] \in Z_M$  for all  $x,y \in M$ , then  $c = 0$ .*

*Proof.* We have  $c[[x,y],u] = 0$  for all  $x,y,u \in M$ . Replacing  $y$  by  $yx$  it follows that

$$0 = c[[x,y]x,u] = c[x,y][x,u] + c[[x,y],u]x = c[x,y][x,u].$$

Thus  $c[x,y][x,u] = 0$  for all  $x,y,u \in M$ . Substituting  $uv$  for  $u$  and then using the relation  $[x,uv] = [x,u]v + u[x,v]$ , we then get  $c[x,y]M[x,v] = \{0\}$  for all  $x,y,v \in M$ . Note that Lemma 4 implies that  $c[x,y]M[u,v] = \{0\}$  for all  $x,y,u,v \in M$ . Using the fact that the ideal generated by all commutators in  $M$  is equal to  $M$  [5, Lemma 2.6], it follows easily that  $c = 0$ .

**THEOREM 4.** *Let  $M$  and  $N$  be von Neumann algebras with no central summands of type  $I_1$  or  $I_2$ . If  $\theta : M \rightarrow N$  is a Lie isomorphism then it is of the form  $\theta = \psi + f$  where  $f$  is an additive mapping of  $M$  into  $Z_N$  sending commutators to zero, and, for some central projections  $p \in M$  and  $q \in N$ , the restriction of  $\psi$  to  $pM$  is an isomorphism of  $pM$  onto  $qN$  and the restriction of  $\psi$  to  $(1-p)M$  is a negative of an anti-isomorphism of  $(1-p)M$  onto  $(1-q)N$ .*

*Proof.* Clearly,  $\theta$  satisfies the requirements of Theorem 3, and it is, therefore, of the form described in the statement of Theorem 3.

Take  $x,y \in pM$ . Since the restriction of  $\varphi$  to  $pM$  is a homomorphism, we have

$$\theta([x,y]) = c\varphi([x,y]) + f([x,y]) = c[\varphi(x),\varphi(y)] + f([x,y]).$$

On the other hand,

$$\theta([x,y]) = [\theta(x),\theta(y)] = c^2[\varphi(x),\varphi(y)].$$

Comparing we get  $(c^2-c)[\varphi(x),\varphi(y)] = f([x,y]) \in Z_N$ . Since  $\varphi$  maps  $pM$  onto  $qN$  it follows that  $(c^2-c)q = 0$ , and therefore, since  $c$  is invertible,  $cq = q$ . Note that this implies that  $f([x,y]) = 0$  for all  $x,y \in pM$ .

Similarly, by computing  $\theta([x,y])$ ,  $x,y \in (1-p)M$ , in two ways, one shows that  $(c^2+c)[\varphi(x),\varphi(y)] = f([x,y]) \in Z_N$ . This yields  $c(1-q) = -(1-q)$  and  $f([x,y]) = 0$ ,  $x,y \in (1-p)M$ .

Now it can be easily shown that the mapping  $\psi(x) = c\varphi(x)$  satisfies the desired conclusions.

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