Generating function approach for the effective degree SIR Model

by

Kurtis Manke
B.Sc., Thompson Rivers University, 2017

A Thesis Submitted in Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

in the Department of Mathematics and Statistics

Generating function approach for the effective degree SIR Model

by

Kurtis Manke
B.Sc., Thompson Rivers University, 2017

Supervisory Committee

Dr. Junling Ma, Co-Supervisor
(Department of Mathematics and Statistics)

Dr. Slim Ibrahim, Co-Supervisor
(Department of Mathematics and Statistics)

Supervisory Committee

Dr. Junling Ma, Co-Supervisor
(Department of Mathematics and Statistics)

Dr. Slim Ibrahim, Co-Supervisor
(Department of Mathematics and Statistics)

ABSTRACT

The effective degree model has been applied to both SIR and SIS type diseases (those which confer permanent immunity and those which do not, respectively) with great success. The original model considers a large system of ODEs to keep track of the number of infected and susceptible neighbours of an individual. In this thesis, we use a generating function approach on the SIR effective degree model to transform the system of ODEs into a single PDE. This has the advantage of allowing the consideration of infinite networks. We derive existence and uniqueness of solutions to the PDE. Furthermore, we show that the linear stability of the PDE is governed by the same disease threshold derived by the ODE model, and we also show the nonlinear instability of the PDE agrees with the same disease threshold.

Contents

List of Figures

ACKNOWLEDGEMENTS

I would like to thank:

- Junling Ma and Slim Ibrahim, for their mentoring and endless patience.
- Karen Manke and Kyle Erickson, for countless breakfasts, lunches, and dinners.
- My roommates, for keeping my spirits high.
- Lastly, to my loving girlfriend, Solenn, for listening to me moan and groan, and supporting me through it all.

Thank you.

Chapter 1

Introduction

Agenda or Thesis outline

Chapter 1 contains an overview of disease dynamics on contact networks. The overview will focus specifically on the Effective Degree model and the Volz model.

Chapter 2 includes a derivation of the partial differential equation effective degree model and a proves the existence and uniqueness of solutions to the PDE. This chapter also shows that our model can be reduced to the Volz model.

Chapter 3 handles the stability of the disease-free equilibria. We look at both the stability of the linearized problem and the non-linear instability (using an initial condition that reduces our model to the Volz model).

Chapter 4 is a concluding chapter. It restates the claims and results of the thesis, and provides ideas for future work and further development.

1.1 ODE models

The classic compartmental epidemic model is a special case of a model introduced by Kermack and McKendrick in 1927 [2] called the Susceptible-Infectious-Recovered (SIR) model. In this model the population is divided into three compartments or classes: the number of those susceptible to infection S, the number of those who are infected and currently transmitting the disease I, and the number of those who have recovered and are no longer susceptible to reinfection R. Individuals within classes are considered to be identical. The following system of ODEs governs the transfer rates between compartments

$$\begin{aligned}\frac{\mathrm{d}S}{\mathrm{d}t} &= -\beta SI \\ \frac{\mathrm{d}I}{\mathrm{d}t} &= \beta SI - \gamma I \\ \frac{\mathrm{d}R}{\mathrm{d}t} &= \gamma I\end{aligned}$$

where β is the infection rate of the disease per individual, which moves individuals from S to I; and γ is the recovery rate of infected individuals, moving them from I to R. Once recovered (in class R), individuals no longer contribute to the disease dynamics, and for this reasion the equation for R is often dropped since it is determined by S and I. Assuming that the probability of infection is governed by the product SI is usually referred to as the principle of "mass-action", which is well studied in its use for chemical reactions, see for example [3], and assumes that the population is "well-mixed" or homogenous in the number of contact rates between individuals.

There are many different ways that one could construct compartments based on what they want to study: the SIR model studies diseases that confer immunity after infection, and the shorthand "SIR" is often used to refer to modelling these types of diseases. Alternatively, there exist similar compartmental models such as the Susceptible-Infectious-Susceptible model (SIS), where individuals who recover from being infectious move from I back into S (for more examples see [4]).

While these models make some unrealistic simplifying assumptions, they exhibit a *disease threshold condition*: when introducing the disease to an entirely susceptible population, the disease will not invade (no epidemic occurs) unless the parameters satisfy the disease threshold condition (an epidemic occurs). The Kermack-McKendrick SIR model shows the existence of a disease threshold condition, and many more complicated models aim to show its existence as well [5, 6, 1].

1.2 Contact Network Models

The Kermack-McKendrick SIR model has the advantage of being simple to understand, is analytically solvable [7], and for large, well-mixed populations it can provide at least a qualitative understanding of the disease dynamics. However, at the beginning of an epidemic, there are relatively few infected individuals and disease dynamics

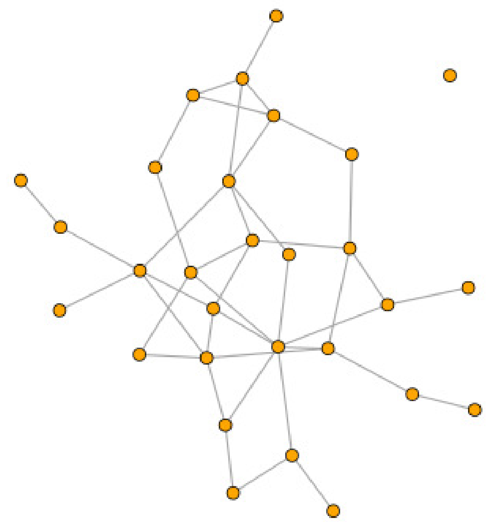

Figure 1.1: An example of a contact network. The nodes represent individuals and the edges represent transmission pathways.

depend on the interactions between these individuals and the susceptible members of the population. Furthermore, evidence suggests that there are members of the population who infect many others, while most members of the population spread it to relatively few [8]. This heterogeneity of contacts should be captured by a model if we want it to be more realistic.

One way to capture the heterogeneity is by modeling the spread of disease on a *contact network*. Networks are an obvious choice when a system has a number of interacting parts. Figure 1.1 shows an example of a network represented as a graph: the network has nodes representing units, and edges representing interactions between units. In the context of this thesis, the nodes represent individuals (in one of the S,I, or R compartments, for example), and the edges represent interactions or contacts between them that can transmit disease.

This thesis will mainly focus on an extension of the Effective Degree model, and we also present the Volz model and show that it is a reduction of the former.

1.2.1 Effective Degree Model

The Effective Degree model [1] compartmentalizes the network based on the state of the nodes (susceptible, S; infected, I; recovered, R) and by the number of neighbours that it has in each state. Hence, S_{sir} is the fraction of the population that is

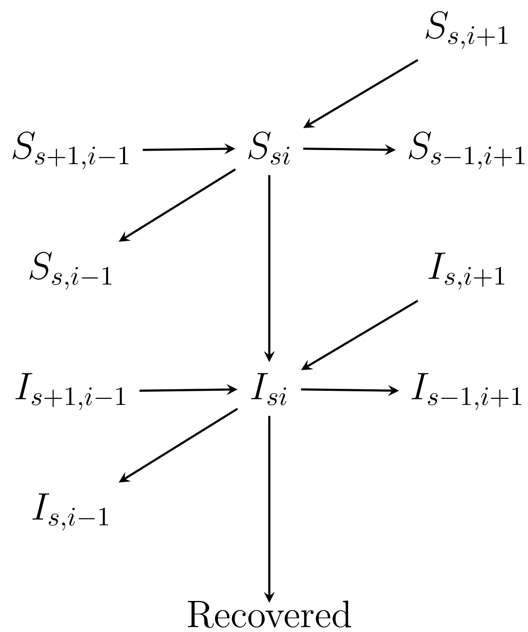

Figure 1.2: The flow between compartments for the Effective Degree SIR model. This figure has been adapted from [1].

susceptible, having s susceptible, i infected, and r recovered neighbours.

The nodes can change compartments in a number of ways: the node itself could change states, or one of its neighbours could. As an example, a node could change from susceptible to infected $S_{sir} \to I_{sir}$, or if one of its neighbours changes states instead $S_{sir} \to S_{s-1,i+1,r}$, etc. The recovered individuals cannot transmit the disease or be infected, and so instead of tracking them, we only keep track of the susceptible and infected neighbours, and allow a nodes degree to change. By only keeping track of a node's s susceptible and i infected neighbours, we say $s+i$ is the *effective degree* of the node. The flow chart for the SIR system is shown in figure 1.2. The disease dynamics of the effective degree SIR model satisfy [1]

Case 1 If $\sum\limits_{s\geq 0, i\geq 0} sS_{si} \neq 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}S_{si} = -\beta i S_{si} + \gamma[(i+1)S_{s,i+1} - iS_{si}] + \frac{\sum_{s,i} \beta si S_{si}}{\sum_{s,i} sS_{si}}[(s+1)S_{s+1,i-1} - sS_{si}] \quad (1.1)$$

Case 2 If $\sum\limits_{s\geq 0, i\geq 0} sS_{si} = 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t} S_{si} = -\beta i S_{si} + \gamma[(i+1)S_{s,i+1} - iS_{si}] \tag{1.2}$$

The transmission rate along an edge is β, and γ is the recovery rate of an infectious node. Both these parameters are positive, as well as the proportions of nodes are non-negative i.e. $\beta, \gamma > 0$ and $S_{si} \geq 0$. The notation $\sum_{si}$ represents the double sum over all s and i, i.e. $\sum_{si} = \sum_{s=0}^{\infty} \sum_{i=0}^{\infty}$, and it is assumed that $S_{si} = 0$ if either index is negative.

We need to distinguish between the two cases because the third term in (2.9) would be undefined if the denominator is zero. We can also see that $\sum_{s,i} sS_{si} = 0$ implies that for all $s > 0$, $S_{si} = 0$ becuase $S_{si} \geq 0$ by definition. The condition that $S_{si} = 0$ for $s > 0$ describes a network where no two susceptible nodes are connected by an edge (i.e. neighbours). Since the third term in (2.9) represents the flow into S_{si} due to the infection of a neighbour, and case 2 involves a network in which no susceptible node has susceptible neighbours, we understand that this term must be zero in case 2.

Case 2 is linear, and solvable (by the method of characteristics, for example), so we won't focus on it in this thesis.

There is a similar system of differential equations describing the infected nodes I_{si}, however the total rate of change of the proportion of infected nodes is

$$I' = \beta \sum_{s,i} iS_{si} - \gamma I \tag{1.3}$$

which can be obtained by computing $I = \sum_{si} I_{si}$ in [1]. Similarly for R. Since the I equation is completely determined by the S equations, we need only to focus on S.

Remark. *Modeling a disease with non-permanent immunity (SIRS) would require keeping track of the recovered neighbours, and the I equations could not be simplified as in* (1.3)*. Furthermore, we would need to keep track of the recovered nodes in a similar manner, i.e. $R_{s,i,r}$.*

1.2.2 Volz Model

The Volz model [6] assumes a given network with degree distribution such that the probability of a node having degree k is p_k, and the compartments for the network nodes are SIR. The Volz model assumes that a susceptible node has an infectious neighbour with uniform probability, and from this assumption derives 3 coupled, nonlinear differential equations that track the degree distributions of the susceptible and infected nodes.

The system of differential equations derived in the Volz model are

$$\boxed{\begin{aligned} \theta'(t) &= -\beta p_I(t)\theta(t) \\ p_I'(t) &= \beta p_S p_I \theta \frac{g''(\theta)}{g'(\theta)} - \beta p_I(1-p_I) - \gamma p_I \\ p_S'(t) &= \beta p_S p_I \left(1 - \theta \frac{g''(\theta)}{g'(\theta)}\right) \\ \tilde{S}(t) &= g(\theta) \\ \tilde{I}'(t) &= \beta p_I \theta g'(\theta) - \gamma I \end{aligned}} \tag{1.4}$$

where

$\tilde{S}(t) :=$ The total fraction of suscpetible nodes,

$\tilde{I}(t) :=$ The total fraction of infected nodes,

$\theta(t) :=$ The fraction of degree one nodes that remain susceptible at time t,

$g(x) :=$ The generating function of the degree distribution,

$p_S(t) :=$ The probability of a susceptible node having a susceptible neighbour at time t,

$p_I(t) :=$ The probability of a susceptible node having an infected neighbour at time t.

The generating function $g(x)$ here is defined as

$$g(x) = \sum_k p_k x^k$$

where p_k is the probability of a node having degree k.

In the Volz model, the author defines u_k as the fraction of degree k nodes that remain susceptible at time t, and shows that u_k is given as

$$u_k(t) = \exp\left\{\int_0^t -\beta p_I(\tau)\ d\tau\right\}^k.$$

By definition then,

$$\theta = u_1$$

and

$$u_k(t) = \theta^k.$$

Thus the total fraction of suscpetible nodes is

$$\begin{aligned}\tilde{S} &= \sum_k p_k u_k \\ &= g(\theta).\end{aligned}$$

The Volz model is interesting because it reduces the problem of tracking nodes and neighbours found in the Effective Degree model to tracking the probabilty distributions θ, p_S, and p_I, effectively reducing the system to three ordinary differential equations (ODEs).

We can show that the Volz model exhibits a disease threshold condition by considering the linear stability of the disease free equilibrium. Define $X(t) \coloneqq (\theta, p_S, p_I)$ and F such that

$$X'(t) = F(X).$$

The disease-free equilibrium is $\bar{Y} = (1, 1, 0)$. To linearize about $\bar{Y}$, let $X = \bar{Y} + Y$. Then

$$Y' = X' = F(X).$$

Use a linear approximation of F about $\bar{Y}$

$$\begin{aligned}Y' &\approx F(\bar{Y}) + J\Big|_{\bar{Y}}(X - \bar{Y}) \\ &= J\Big|_{\bar{Y}} Y\end{aligned}$$

because $F(\bar{Y}) = 0$ since $\bar{Y}$ is an equilibrium point. The Jacobian at this equilibrium is given by

$$J = \begin{bmatrix} 0 & 0 & -\beta \\ 0 & 0 & \beta - \beta\frac{g''(1)}{g'(1)} \\ 0 & 0 & \beta\frac{g''(1)}{g'(1)} - \beta - \gamma \end{bmatrix},$$

and

$$J - \lambda I = \begin{bmatrix} -\lambda & 0 & -\beta \\ 0 & -\lambda & \beta - \beta\frac{g''(1)}{g'(1)} \\ 0 & 0 & \beta\frac{g''(1)}{g'(1)} - \beta - \gamma - \lambda \end{bmatrix}.$$

The characteristic equation for the eigenvalues is

$$\lambda^2\left(\beta\frac{g''(1)}{g'(1)} - \beta - \gamma - \lambda\right) = 0,$$

thus the non-zero eigenvalue and its associated eigenvector are

$$\lambda = \beta\frac{g''(1)}{g'(1)} - \beta - \gamma$$

$$v = \begin{bmatrix} -\beta \\ -\lambda - \gamma \\ \lambda \end{bmatrix}$$

Hence the stability of $\bar{Y}$ is determined by the sign of λ, and furthermore we can see that $\lambda > 0$ implies

$$\mathcal{R}_0 = \frac{\beta}{\beta + \gamma}\frac{g''(1)}{g'(1)} > 1,$$

in agreement with the basic reprouction number $\mathcal{R}_0$ from percolation theory [5].

Chapter 2

The Model

The ODE effective degree model of the previous section only considers finite degree distributions, hence extending the model to an infinite distribution would require a different approach. In this chapter, we derive a partial differential equation (PDE) from the effective degree model using a generating function approach. The S_{si} compartments are probabilities, and the use of probability generating functions to represent a discrete variable by a continuous function is a standard technique in probability theory, see for example [9, chapter 1.2].

2.1 Derivation of the Model

The discrete system of ODEs (1.1) is typically very large and can be difficult to analyze. It is often the case that when studying infectious diseases on networks, the use of infinite degree distributions often makes analysis easier, e.g. Poisson, power law, etc. However, the ODE model analysis requires a finite dimensional system. To analayze the infinite dimensional system, we need a different approach. Look for solutions S in the form of generating functions

$$S(t,x,y) = \sum_{si} x^s y^i S_{si}(t), \tag{2.1}$$

where (x, y) are in the unit square, i.e. $(x, y) \in [0, 1]^2$. Take a derivative of S with respect to time

$$\frac{\partial S}{\partial t} = \sum_{si} x^s y^i \frac{\mathrm{d}}{\mathrm{d}t} S_{si}, \tag{2.2}$$

and substitute (1.1) into (2.2)

$$\begin{aligned}\frac{\partial S}{\partial t} = & -\beta\sum_{si} x^s y^i iS_{si} + \gamma\sum_{si} x^s y^i[(i+1)S_{s,i+1} - iS_{si}] \\ & + \frac{\sum_{si}\beta siS_{si}}{\sum_{si} sS_{si}}\sum_{si} x^s y^i[(s+1)S_{s+1,i-1} - sS_{si}] \\ = & -\beta\sum_{si} x^s y^i iS_{si} + \gamma\sum_{si}\left(x^s y^{i-1} iS_{si} - ix^s y^i S_{si}\right) \\ & + \frac{\sum_{si}\beta siS_{si}}{\sum_{si} sS_{si}}\sum_{si}\left(x^{s-1}y^{i+1}sS_{si} - sx^s y^i S_{si}\right).\end{aligned}$$

We can simplify by taking derivatives of S (2.1) with respect to x and y

$$\begin{aligned}S_y &= \sum_{si} x^s y^{i-1} iS_{si} \\ S_x &= \sum_{si} x^{s-1} y^i sS_{si} \\ S_{xy} &= \sum_{si} x^{s-1} y^i siS_{si},\end{aligned}$$

which gives

$$\frac{\partial S}{\partial t} = -\beta y S_y + \gamma[1-y]S_y + \frac{S_{xy}(1,1)}{S_x(1,1)}\beta[y-x]S_x.$$

That is,

$$\boxed{\frac{\partial S}{\partial t} = -(\beta+\gamma)\left(y - \frac{\gamma}{\beta+\gamma}\right)S_y + \frac{S_{xy}(t,1,1)}{S_x(t,1,1)}\beta(y-x)S_x.} \tag{2.3}$$

Remarks.

- In section 2.2.2 we prove that the characteristics have the nice property that they flow out of the unit square. We will see that this allows us to avoid the lack of boundary conditions.
- Recall that we are focusing on the case where $\frac{S_{xy}(t,1,1)}{S_x(t,1,1)} \neq 0$, at least initiallly, as this corresponds to the more interesting ODE model (1.1).
- If $S_x(t,1,1) = 0$, define $S_{xy}(t,1,1)/S_x(t,1,1) = 0$, because this corresponds to case 2 of the ODE model (1.2).

Definition 2.1.1. A biologically relevant solution to (2.3) has power series with strictly non-negative coefficients. The coefficients S_{si} of S from (2.1) represent *pro-*

portions of the population that are susceptible, hence negative values would be nonsense.

2.1.1 Steady-State Solutions

The purpose of this section is to characterize disease-free solutions. More precisely, we have the following lemma.

Lemma 1. *Any function* $S = S^*(x)$*, i.e. strictly a function of* x*, is a steady-state solution to* (2.3)*. Furthermore, these comprise the entire family of biologically relevant steady-state solutions.*

Proof. Suppose $S = S(x, y)$ is a steady-state solution to (2.3). Then $S_{xy}(1,1)/S_x(1,1) = A$, a constant. Furthermore, S biologically relevant implies that $A \geq 0$, as both S_{xy} and S_x have power series with strictly non-negative coefficients.
Equation (2.3) becomes

$$-(\beta+\gamma)\left(y - \frac{\gamma}{\beta+\gamma}\right)S_y + A\beta(y-x)S_x = 0, \tag{2.4}$$

which can be solved via the method of characteristics. Notice that if $A = 0$, then we immediately get the result $S_y = 0$, i.e. $S = S^*(x)$. Assume that $A > 0$, and solve (2.4) along the characteristic curves parametrised by s satisfying

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}s} &= A\beta(y-x) \\ \frac{\mathrm{d}y}{\mathrm{d}s} &= -(\beta+\gamma)\left(y - \frac{\gamma}{(\beta+\gamma)}\right) \\ \frac{\mathrm{d}S}{\mathrm{d}s} &= 0 \end{aligned} \tag{2.5}$$

Solve the equation for y using the integrating factor

$$e^{-(\beta+\gamma)s}.$$

That is,

$$\boxed{y(s) = \left(y_0 - \frac{\gamma}{\beta+\gamma}\right)e^{-(\beta+\gamma)s} + \frac{\gamma}{\beta+\gamma},}$$

where $y(0) = y_0$. Substitute $y(s)$ into the x equation,

$$\frac{\mathrm{d}x}{\mathrm{d}s} + A\beta x = A\beta\left(y_0 - \frac{\gamma}{\beta+\gamma}\right)e^{-(\beta+\gamma)s} + A\beta\frac{\gamma}{\beta+\gamma},$$

which can be solved with the integrating factor

$$e^{A\beta s}.$$

That is,

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[e^{A\beta s}x\right] = A\beta\left(y_0 - \frac{\gamma}{\beta+\gamma}\right)e^{A\beta s-(\beta+\gamma)s} + A\beta\frac{\gamma}{\beta+\gamma}e^{A\beta s}.$$

Integrate from 0 to s

$$\begin{aligned} e^{A\beta s}x = x_0 &+ \frac{A\beta}{A\beta-\beta-\gamma}\left(y_0 - \frac{\gamma}{\beta+\gamma}\right)e^{A\beta s-(\beta+\gamma)s} + \frac{\gamma}{\beta+\gamma}e^{A\beta s} \\ &- \frac{A\beta}{A\beta-\beta-\gamma}\left(y_0 - \frac{\gamma}{\beta+\gamma}\right) - \frac{\gamma}{\beta+\gamma}, \end{aligned}$$

and multiply both sides by $e^{-A\beta s}$

$$\begin{aligned} x = x_0e^{-A\beta s} &+ \frac{A\beta}{A\beta-\beta-\gamma}\left(y_0 - \frac{\gamma}{\beta+\gamma}\right)e^{-(\beta+\gamma)s} + \frac{\gamma}{\beta+\gamma} \\ &- \frac{A\beta}{A\beta-\beta-\gamma}\left(y_0 - \frac{\gamma}{\beta+\gamma}\right)e^{-A\beta s} - \frac{\gamma}{\beta+\gamma}e^{-A\beta s}. \end{aligned}$$

That is,

$$\boxed{x = x_0e^{-A\beta s} + \frac{A\beta}{A\beta-\beta-\gamma}\left(y_0 - \frac{\gamma}{\beta+\gamma}\right)\left(e^{-(\beta+\gamma)s} - e^{-A\beta s}\right) + \frac{\gamma}{\beta+\gamma}\left(1 - e^{-A\beta s}\right)}$$

The third characteristic from (2.5) is

$$\frac{\mathrm{d}S}{\mathrm{d}s} = 0,$$

thus the solution is constant along the curve $(x(s), y(s))$, i.e.,

$$S(x(s), y(s)) = c$$

In particular,

$$S(x(s), y(s)) = S(x_0, y_0)$$

where $x(s), y(s)$ and x_0, y_0 lie along the same characteristic curve. Fix x, y, s, and write x_0 and y_0 in terms of those variables, i.e.

$$S(s, x, y) = S(x_0(s, x, y), y_0(s, x, y)) \tag{2.6}$$

where

$$\begin{aligned} x_0 &= xe^{A\beta s} - \frac{A\beta}{A\beta - \beta - \gamma}\left(y - \frac{\gamma}{\beta+\gamma}\right)\left(e^{A\beta s} - e^{(\beta+\gamma)s}\right) - \frac{\gamma}{\beta+\gamma}\left(e^{A\beta s} - 1\right) \\ y_0 &= \left(y - \frac{\gamma}{\beta+\gamma}\right)e^{(\beta+\gamma)s} + \frac{\gamma}{\beta+\gamma}. \end{aligned}$$

Take derivatives in x and y of (2.6)

$$\begin{aligned} S_x(s, x, y) &= \frac{\partial S}{\partial x_0}\frac{\partial x_0}{\partial x} \\ S_{xy}(s, x, y) &= \frac{\partial S}{\partial x_0}\frac{\partial^2 x_0}{\partial x \partial y} + \left(\frac{\partial^2 S}{\partial x_0^2}\frac{\partial x_0}{\partial y} + \frac{\partial^2 S}{\partial x_0 \partial y_0}\frac{\partial y_0}{\partial y}\right)\frac{\partial x_0}{\partial x}. \end{aligned}$$

By assumption we have

$$\begin{aligned} A &= \frac{S_{xy}(s, 1, 1)}{S_x(s.1, 1)} \\ &= \left.\frac{\frac{\partial^2 S}{\partial x_0^2}\frac{\partial x_0}{\partial y} + \frac{\partial^2 S}{\partial x_0 \partial y_0}\frac{\partial y_0}{\partial y}}{\frac{\partial S}{\partial x_0}}\right|_{x=y=1} \\ &= \left.\frac{\frac{\partial^2 S}{\partial x_0^2}\frac{A\beta}{A\beta-\beta-\gamma}\left(e^{(\beta+\gamma)s} - e^{A\beta s}\right) + \frac{\partial^2 S}{\partial x_0 \partial y_0}e^{(\beta+\gamma)s}}{\frac{\partial S}{\partial x_0}}\right|_{x=y=1} \end{aligned} \tag{2.7}$$

Equation (2.7) must be true for all s, and so taking the limit $s \to -\infty$ gives $A = 0$, a contradiction since A assumed $A > 0$.

This implies that $A = 0$ is the only case, which implies $S_y = 0$ and

$$S = S(x).$$

□

Definition 2.1.2. $S = S^*(x)$ is a disease-free equilibrium of the PDE (2.3) for all biologically relevant $S^*(x)$. A function strictly of x has power series coefficients $S_{si} = 0$ for all $i > 0$. Hence, the network is disease-free.

2.1.2 Change of Variables

Some aspects of the PDE (2.3) are easier to study if we consider a change of variables (in particular, the linear stability analysis in chapter 3 is greatly simplified): let $\Gamma = \frac{\gamma}{\beta+\gamma}$, and define

$$\begin{aligned} z &= y - \Gamma \\ w &= x - \Gamma \end{aligned}$$

(since (x, y) are in the unit square, $(w, z) \in [-\Gamma, 1-\Gamma]^2$).

Consider the power series $S(t, x, y)$ given by (2.1) with the new variables w and z

$$\begin{aligned} S(t, x, y) &= S(t, w + \Gamma, z + \Gamma) \\ &= \sum_{s\geq 0, i\geq 0} (w + \Gamma)^s (z + \Gamma)^i S_{si}(t) \end{aligned}$$

and center the series at $w = z = 0$

$$\begin{aligned} \sum_{s\geq 0, i\geq 0} (w + \Gamma)^s (z + \Gamma)^i S_{si}(t) &= \sum_{s\geq 0, i\geq 0} \left(\sum_{m\leq s, n\leq i} \binom{s}{m} w^m \Gamma^{s-m} \binom{i}{n} z^n \Gamma^{i-n} S_{si}(t) \right) \\ &= \sum_{m\geq 0, n\geq 0} w^m z^n \left(\sum_{s\geq m, i\geq n} \binom{s}{m} \Gamma^{s-m} \binom{i}{n} \Gamma^{i-n} S_{si}(t) \right) \\ &= \sum_{m\geq 0, n\geq 0} w^m z^n s_{mn}(t) \end{aligned}$$

where

$$s_{mn}(t) := \sum_{s\geq m, i\geq n} \binom{s}{m} \binom{i}{n} \Gamma^{s+i-m-n} S_{si}(t). \tag{2.8}$$

Define $\tilde{S} = \tilde{S}(t, w, z)$ as the power series

$$\boxed{\tilde{S}(t, w, z) = \sum_{m\geq 0, n\geq 0} w^m z^n s_{mn}(t)}$$

and we can see that $S(t, x, y) = \tilde{S}(t, w, z)$, where the coefficients of each series are related by (2.8).

Remark. *If $S(t,x,y)$ has non-negative coefficients, then $\tilde{S}(t,w,z)$ has non-negative coefficients. By definition 2.1.1, we also see that this implies that if $S(t,x,y)$ is biologically relevant, then $\tilde{S}(t,w,z)$ has non-negative coefficients.*

This is clear by observing equation (2.8): if $S_{si} \geq 0$, then $s_{mn} \geq 0$.

Substitute the change of variables into the PDE (2.3) to see that $\tilde{S}$ solves

$$\boxed{\begin{gathered}\frac{\partial \tilde{S}}{\partial t} = -(\beta+\gamma) z \tilde{S}_z + \beta \frac{\tilde{S}_{wz}(t,1-\Gamma,1-\Gamma)}{\tilde{S}_w(t,1-\Gamma,1-\Gamma)}(z-w)\tilde{S}_w. \\ \tilde{S}_0(w,z) = \tilde{S}(0,w,z)\end{gathered}} \tag{2.9}$$

Consider solutions in the space X, defined as the set of functions

$$\tilde{S}(t,w,z) = \sum_{m,n} w^m z^n s_{mn}(t)$$

equipped with norm

$$\left\|\tilde{S}\right\|_X = \sum_{m,n} (1+m)|s_{mn}| < \infty. \tag{2.10}$$

We will be considering functions belonging to X to the problem (2.9).

Claim 2. *Any function $\bar{S}(w) \in X$, i.e. not a function of z, is a steady-state solution to* (2.9).

Proof. This is easy to see, as $\bar{S}_z(w) = 0 \implies \bar{S}_{wz}(w) = 0$, hence the RHS of (2.9) is zero. □

Remark. *The power series* (2.9) *must be converted back into x and y variables to get the coefficients which correspond to the compartments of the ODE model* (1.1).

2.2 Well Posedness of the Problem

For the problem to be well-posed, we aim to not only show the existence and uniqueness of solutions, but to show that those solutions make biological sense. First we are going to use the method of characteristics to solve the PDE (2.9), and second we will show the uniqueness of the characteristics. Lastly, we show that the solutions are biologically relevant.

Remark. *From this point onward, we will be considering the PDE* (2.9) *with w and z variables, but we will be dropping the "tilde" notation on S, i.e.*

$$S(t, w, z) = \sum_{m,n} w^m z^n s_{mn}.$$

2.2.1 General Solution via Characteristics

Here we present two slightly different approaches for solving the characteristics. The first involves choosing $t_0 = 0$, and the second leaves the initial time undetermined until the end. The two methods are equivalent.

Method 1

In equation (2.9), treat $\frac{S_{wz}(t,1-\Gamma,1-\Gamma)}{S_w(t,1-\Gamma,1-\Gamma)}$ as an undetermined function of time, leaving us with a linear PDE. That is,

$$S_t(t, w, z) + (\beta + \gamma)zS_z(t, w, z) + \beta(w - z)\varphi'(t)S_w(t, w, z) = 0$$

where

$$\varphi(t) := \int_0^t \frac{S_{wz}(s, 1 - \Gamma, 1 - \Gamma)}{S_w(s, 1 - \Gamma, 1 - \Gamma)}\, ds,$$

so that

$$\varphi'(t) = \frac{S_{wz}(t, 1 - \Gamma, 1 - \Gamma)}{S_w(t, 1 - \Gamma, 1 - \Gamma)}.$$

Note by definition we have $\varphi(0) = 0$, and recall that we are focusing on the case where $\frac{S_{wz}(t,1-\Gamma,1-\Gamma)}{S_w(t,1-\Gamma,1-\Gamma)} \neq 0$ initially (i.e. $\varphi'(0) \neq 0$).

Once we find a solution using characteristics, $\varphi'(t)$ will impose a compatibility condition, i.e., the characteristics will give us an implicit solution for S, which we can then take derivatives S_{wz} and S_w, evaluate at $w = z = 1 - \Gamma$, and make the LHS and RHS agree.

We look for a solution by introducing the characteristic equations parametrized by s

$$
\begin{aligned}
\frac{\mathrm{d}t}{\mathrm{d}s} &= 1 \\
\frac{\mathrm{d}w}{\mathrm{d}s} &= \beta(w-z)\varphi'(t) \\
\frac{\mathrm{d}z}{\mathrm{d}s} &= (\beta+\gamma)z \\
\frac{\mathrm{d}S}{\mathrm{d}s} &= 0
\end{aligned}
$$

Since $\frac{\mathrm{d}t}{\mathrm{d}s} = 1$, $t = s + c_1$, thus we can actually parametrize by t (setting $c_1 = 0$):

$$
\begin{aligned}
\frac{\mathrm{d}w}{\mathrm{d}t} &= \beta\varphi'(t)w - \beta\varphi'(t)z \\
\frac{\mathrm{d}z}{\mathrm{d}t} &= (\beta+\gamma)z \\
\frac{\mathrm{d}S}{\mathrm{d}t} &= 0
\end{aligned}
\tag{2.11}
$$

The z characteristic is easily solved for

$$
\boxed{z(t) = z_0 e^{(\beta+\gamma)t}} \tag{2.12}
$$

where $z(0) = z_0$. Substitute z into the equation for w

$$
\frac{\mathrm{d}w}{\mathrm{d}t} = \beta\varphi'(t)w - \beta\varphi'(t)z_0 e^{(\beta+\gamma)t}
$$

which can be solved with the integrating factor

$$
e^{-\beta\varphi(t)}
$$

i.e.,

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{-\beta\varphi(t)}w\right] = -\beta\varphi'(t)e^{-\beta\varphi(t)}z_0 e^{(\beta+\gamma)t}
$$

Integrate from 0 to t

$$
e^{-\beta\varphi(t)}w - w_0 = \int_0^t -\beta\varphi'(s)e^{-\beta\varphi(s)}z_0 e^{(\beta+\gamma)s}\, ds.
$$

We can simplify by doing the integral by parts

$$\int_0^t -\beta\varphi'(t)e^{-\beta\varphi(t)}z_0 e^{(\beta+\gamma)t}\ ds = e^{(\beta+\gamma)s}e^{-\beta\varphi(s)}\Big|_0^t - (\beta+\gamma)\int_0^t e^{(\beta+\gamma)s-\beta\varphi(s)}\ ds$$

Thus,

$$e^{-\beta\varphi(t)}w - w_0 = z_0\left(e^{(\beta+\gamma)t}e^{-\beta\varphi(t)} - 1 - (\beta+\gamma)\int_0^t e^{(\beta+\gamma)s-\beta\varphi(s)}\ ds\right)$$

Solve for w

$$\boxed{w = w_0 e^{\beta\varphi(t)} + z_0\left(e^{(\beta+\gamma)t} - e^{\beta\varphi(t)} - (\beta+\gamma)e^{\beta\varphi(t)}\int_0^t e^{(\beta+\gamma)s-\beta\varphi(s)}\ ds\right)} \tag{2.13}$$

The third characteristic equation from (2.11) is

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 0$$

which means that S is a constant along the characteristic curve $(w(t), z(t))$, given by (2.12) and (2.13) respectively. In particular

$$\boxed{S(t,w,z) = S_0(w_0(t,w,z), z_0(z,t))} \tag{2.14}$$

where $S_0 : [-\Gamma, 1-\Gamma]^2 \to \mathbb{R}$ is given by

$$S_0 = \sum_{s,i} w^s z^i s_{mn}(0).$$

Equation (2.14) tells us that to evaluate S at some fixed (t, w, z), we need to evaluate S_0 at (w_0, z_0), where (w_0, z_0) satisfy the characteristic equations (2.12) and (2.13)

$$\boxed{\begin{aligned} z_0(t,z) &= ze^{-(\beta+\gamma)t} \\ w_0(t,w,z) &= we^{-\beta\varphi(t)} + z\eta(t,\varphi(t)) \end{aligned}} \tag{2.15}$$

where

$$\boxed{\eta(t,\varphi(t)) := -e^{-\beta\varphi(t)} + e^{-(\beta+\gamma)t} + (\beta+\gamma)e^{-(\beta+\gamma)t}\int_0^t e^{(\beta+\gamma)s-\beta\varphi(s)}\ ds} \tag{2.16}$$

for the same fixed (t, w, z).

Method 2

Again, in equation (2.9), treat $\frac{S_{wz}(t,1-\Gamma,1-\Gamma)}{S_w(t,1-\Gamma,1-\Gamma)}$ as an undetermined function of time, leaving us with the linear PDE

$$S_t(t, w, z) + (\beta + \gamma)zS_z(t, w, z) + \beta(w - z)\varphi'(t)S_w(t, w, z) = 0$$

where

$$\varphi(t; t_0) := \int_{t_0}^{t} \frac{S_{wz}(s, 1-\Gamma, 1-\Gamma)}{S_w(s, 1-\Gamma, 1-\Gamma)}\, ds, \tag{2.17}$$

so that

$$\varphi'(t; t_0) = \frac{S_{wz}(t, 1-\Gamma, 1-\Gamma)}{S_w(t, 1-\Gamma, 1-\Gamma)}.$$

Note by definition we have $\varphi(t_0; t_0) = 0$.

The characteristic equations are the same as in method 1 (equation (2.11), however the solutions are found by integrating from t_0 to t, instead of from 0 to t. Hence, the z characteristic is

$$\boxed{z(t) = z_0 e^{(\beta+\gamma)(t-t_0)}}$$

where $z(t_0) = z_0$. Substitute z into the equation for w

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \beta\varphi'(t; t_0)w - \beta\varphi'(t; t_0)z_0 e^{(\beta+\gamma)(t-t_0)}$$

with integrating factor $e^{-\beta\varphi(t;t_0)}$, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{-\beta\varphi(t;t_0)}w\right] = -\beta\varphi'(t; t_0)e^{-\beta\varphi(t;t_0)}z_0 e^{(\beta+\gamma)(t-t_0)}$$

Integrate from t_0 to t

$$e^{-\beta\varphi(t;t_0)}w - w_0 e^{-\beta\varphi(t_0;t_0)} = \int_{t_0}^{t} -\beta\varphi'(s; t_0)e^{-\beta\varphi(s;t_0)}z_0 e^{(\beta+\gamma)(s-t_0)}\, ds.$$

Recall that $\varphi(t_0; t_0) = 0$, and simplify by integrating by parts

Thus,

$$e^{-\beta\varphi(t;t_0)}w - w_0 = z_0\left(e^{(\beta+\gamma)(t-t_0)-\beta\varphi(t;t_0)} - 1 - (\beta+\gamma)\int_{t_0}^{t} e^{(\beta+\gamma)(s-t_0)-\beta\varphi(s;t_0)}\, ds\right).$$

Solve for w

$$\boxed{w(t) = w_0 e^{\beta\varphi(t;t_0)} + z_0\left(e^{(\beta+\gamma)(t-t_0)} - e^{\beta\varphi(t;t_0)} - (\beta+\gamma)e^{\beta\varphi(t;t_0)}\int_{t_0}^{t} e^{(\beta+\gamma)(s-t_0)-\beta\varphi(s;t_0)}\, ds\right).}$$

The third characteristic equation from (2.11) is unchanged, and so the S characteristic gives

$$S(t, w, z) = constant,$$

so in particular, at $t = 0$ we have

$$S(t, w, z) = S(0, w(0), z(0)),$$

i.e.

$$S(t, w, z) = S_0(w(0), z(0)).$$

Set the initial point as

$$(t_0, w_0, z_0) = (t, w, z)$$

and trace backward to $t = 0$ along the characteristic curves. The equation for w is

$$w(0) = we^{\beta\varphi(0;t)} + z\left(e^{-(\beta+\gamma)t} - e^{\beta\varphi(0;t)} - (\beta+\gamma)e^{\beta\varphi(0;t)}\int_{t}^{0} e^{(\beta+\gamma)(s-t)-\beta\varphi(s;t)}\, ds\right)$$

From the definition of $\varphi(t; t_0)$ (2.17), notice that

$$\varphi(t;t_0) = \int_{t_0}^{t} \frac{S_{wz}(s, 1-\Gamma, 1-\Gamma)}{S_w(s, 1-\Gamma, 1-\Gamma)}\, ds = -\varphi(t_0; t)$$

and that

$$\varphi(s;t) + \varphi(t;0) = \varphi(s;0).$$

Then we find

$$\boxed{\begin{aligned} z(0) &= ze^{-(\beta+\gamma)t} \\ w(0) &= we^{-\beta\varphi(t;0)} + z\left(e^{-(\beta+\gamma)t} - e^{-\beta\varphi(t;0)} + (\beta+\gamma)e^{-(\beta+\gamma)t}\int_0^t e^{(\beta+\gamma)s-\beta\varphi(s;0)}\,ds\right) \end{aligned}}$$

Comparing to (2.15), we see methods 1 and 2 are equivalent. We still need to show

- The existence of unique solutions. That is, each (t, w, z) gives exactly one initial point (w_0, z_0) to evaluate S_0 at.
- The initial condition S_0 is defined on the square $[-\Gamma, 1-\Gamma] \times [-\Gamma, 1-\Gamma]$, and we cannot be sure (yet) that for all $t \geq 0$ and $(w, z) \in [-\Gamma, 1-\Gamma] \times [-\Gamma, 1-\Gamma]$ that $w_0(t, w, z)$ and $z_0(t, z)$ also lie in the square.

2.2.2 Compatibility Condition

The solution S given by the characteristics (2.14) depends on the function $\varphi(t)$ (as $w_0 = w_0(t, w, z, \varphi)$), but recall the definition of φ

$$\varphi(t) := \int_0^t \frac{S_{wz}(s, 1-\Gamma, 1-\Gamma)}{S_w(s, 1-\Gamma, 1-\Gamma)}\,ds,$$

which means that S is implicitly defined. We can find a compatibility condition by recognizing that

$$\varphi'(t) = \frac{S_{wz}(t, 1-\Gamma, 1-\Gamma)}{S_w(t, 1-\Gamma, 1-\Gamma)},$$

and then taking w and z derivatives of S given by (2.14). Compute S_w

$$S_w = \frac{\partial S_0}{\partial w_0}\frac{\partial w_0}{\partial w}$$

$$\begin{aligned} S_{wz} &= \frac{\partial^2 S_0}{\partial w_0 \partial z}\frac{\partial w_0}{\partial w} + \frac{\partial S_0}{\partial w_0}\frac{\partial^2 w_0}{\partial w \partial z} \\ &= \frac{\partial w_0}{\partial w}\left(\frac{\partial^2 S_0}{\partial w_0^2}\frac{\partial w_0}{\partial z} + \frac{\partial^2 S_0}{\partial w_0 \partial z_0}\frac{\partial z_0}{\partial z}\right) + \frac{\partial S_0}{\partial w_0}\frac{\partial^2 w_0}{\partial w \partial z} \end{aligned}$$

Now, $\frac{\partial^2 w_0}{\partial w \partial z} = 0$, so we have

$$\begin{aligned}
\varphi'(t) &= \frac{\frac{\partial w_0}{\partial w}\left(\frac{\partial^2 S_0}{\partial w_0 \partial z_0}\frac{\partial z_0}{\partial z} + \frac{\partial^2 S_0}{\partial w_0^2}\frac{\partial w_0}{\partial z}\right)}{\frac{\partial S_0}{\partial w_0}\frac{\partial w_0}{\partial w}}\Bigg|_{w=z=1-\Gamma} \\
&= \frac{\frac{\partial^2 S_0}{\partial w_0 \partial z_0}\frac{\partial z_0}{\partial z}}{\frac{\partial S_0}{\partial w_0}}\Bigg|_{w=z=1-\Gamma} + \frac{\frac{\partial^2 S_0}{\partial w_0^2}\frac{\partial w_0}{\partial z}}{\frac{\partial S_0}{\partial w_0}}\Bigg|_{w=z=1-\Gamma} \\
&= f(t,\varphi,\eta)\frac{\partial z_0}{\partial z}\Bigg|_{w=z=1-\Gamma} + h(t,\varphi,\eta)\frac{\partial w_0}{\partial z}\Bigg|_{w=z=1-\Gamma},
\end{aligned} \tag{2.18}$$

where

$$\begin{aligned}
f(t,\varphi,\eta) &= \frac{\frac{\partial^2 S_0}{\partial w_0 \partial z_0}}{\frac{\partial S_0}{\partial w_0}}(w_0(t,w,z), z_0(t,z))\Bigg|_{w=z=1-\Gamma} \\
h(t,\varphi,\eta) &= \frac{\frac{\partial^2 S_0}{\partial w_0^2}}{\frac{\partial S_0}{\partial w_0}}(w_0(t,w,z), z_0(t,z))\Bigg|_{w=z=1-\Gamma}.
\end{aligned}$$

Note $f : [-\Gamma, 1-\Gamma]^2 \to \mathbb{R}$, similarly for h.
Simplify by evaluating equation (2.15) at $w = z = 1-\Gamma$

$$\boxed{\begin{aligned}
f(t,\varphi,\eta) &= \frac{\frac{\partial^2 S_0}{\partial w_0 \partial z_0}}{\frac{\partial S_0}{\partial w_0}}\big((1-\Gamma)(e^{-\beta\varphi}+\eta), (1-\Gamma)e^{-(\beta+\gamma)t}\big) \\
h(t,\varphi,\eta) &= \frac{\frac{\partial^2 S_0}{\partial w_0^2}}{\frac{\partial S_0}{\partial w_0}}\big((1-\Gamma)(e^{-\beta\varphi}+\eta), (1-\Gamma)e^{-(\beta+\gamma)t}\big).
\end{aligned}} \tag{2.19}$$

To write an ODE system for φ and η, compute

$$\begin{aligned}
\frac{\partial w_0}{\partial z} &= \eta \\
\frac{\partial z_0}{\partial z} &= e^{-(\beta+\gamma)t}.
\end{aligned}$$

and substitute into (2.18)

$$\varphi'(t) = f(t,\varphi,\eta)e^{-(\beta+\gamma)t} + h(t,\varphi,\eta)\eta. \tag{2.20}$$

Take a time derivative of η given by (2.16)

$$\begin{aligned}
\eta'(t) &= \beta\varphi' e^{-\beta\varphi} - (\beta+\gamma)e^{-(\beta+\gamma)t} - (\beta+\gamma)^2 e^{-(\beta+\gamma)t}\int_0^t e^{-(\beta+\gamma)(t-s)-\beta\varphi(s)}\, ds \\
&\quad + (\beta+\gamma)e^{-(\beta+\gamma)t}\big(e^{(\beta+\gamma)t-\beta\varphi(t)}\big) \\
&= \beta\varphi' e^{-\beta\varphi} - (\beta+\gamma)\left(-e^{-\beta\varphi(t)} + e^{-(\beta+\gamma)t} + (\beta+\gamma)e^{-(\beta+\gamma)t}\int_0^t e^{(\beta+\gamma)s-\beta\varphi(s)}\, ds\right) \\
&= +\beta\varphi' e^{-\beta\varphi} - (\beta+\gamma)\eta. \qquad (2.21)
\end{aligned}$$

Replace φ' in (2.21) with (2.20), and the compatibility condition can be written as the following initial value problem

$$\boxed{\begin{array}{l}\begin{pmatrix}\varphi'(t)\\ \eta'(t)\end{pmatrix} = \begin{pmatrix} f(t,\varphi,\eta)e^{-(\beta+\gamma)t} + h(t,\varphi,\eta)\eta \\ f(t,\varphi,\eta)\beta e^{-(\beta+\gamma)t-\beta\varphi} + [h(t,\varphi,\eta)\beta e^{-\beta\varphi} - \beta - \gamma]\eta \end{pmatrix} \\ \varphi(0)=0,\ \eta(0)=0.\end{array}} \qquad (2.22)$$

We have the solution $S(t,w,z) = S_0(w_0(t,w,z), z_0(z,t))$, where the existence and uniqueness of φ will guarantee the existence and uniqueness of S, since $S_0(w_0,z_0)$ depends on φ.

However, before we prove the existence and uniqueness of φ, we want to show that the solution S is well defined. That is, we want to show that for all $t \geq 0$ and $(w,z) \in [-\Gamma, 1-\Gamma]^2$ that $(w_0(t,w,z), z_0(t,z))$ lie in the same square. This is because the initial condition S_0 is defined on the square $[-\Gamma, 1-\Gamma]^2$, hence we can't evaluate S_0 anywhere else.

Claim 3. *If $S_0(w_0,z_0)$ has a power series with all non-negative coefficients, then $f(t,\varphi), h(t,\varphi) \geq 0$.*

Proof. Suppose the power series of $S_0(w_0,z_0)$ is

$$S_0(w_0,z_0) = \sum_{mn} A_{mn} w_0^m z_0^n$$

and $A_{mn} \geq 0$. Then the coefficients of its derivatives are also positive. For example

$$\frac{\partial^2 S_0}{\partial w_0 \partial z_0}(w_0, z_0) = \sum_{mn} mnA_{mn} w_0^{m-1} z_0^{n-1}$$
$$= \sum_{mn} (m+1)(n+1)A_{m+1,n+1} w_0^m z_0^n$$

is a series with coefficients $(m+1)(n+1)A_{m+1,n+1} \geq 0$ since $A_{mn} \geq 0$.

In the definition of f and h, equation (2.19), notice that the ratio of derivatives is evaluated at the point

$$(w_0^*, z_0^*) = \left((1-\Gamma)(e^{-(\beta+\gamma)t} + (\beta+\gamma)e^{-(\beta+\gamma)t} \int_0^t e^{(\beta+\gamma)s - \beta\varphi(s)}\ ds), (1-\Gamma)e^{-(\beta+\gamma)t}\right)$$

which has $w_0^*, z_0^* \geq 0$. Thus, f and h are ratios of power series with non-negative coefficients, evaluated at a positive point, hence $f, h \geq 0$. □

Claim 4. *The function $\varphi'(t)$ defined in* (2.20) *is non-negative whenever $f, h \geq 0$.*

Proof. Assume $h(t, \varphi, \eta) \geq 0$ and $f(t, \varphi, \eta) \geq 0$. If $h = 0$, the proof is trivial. Assume $h(t, \varphi, \eta) > 0$. From the system (2.22), it suffices to prove that $\eta(t) \geq 0$. If $f = 0$, then

$$\eta(t) = \exp\left\{\int_0^t \beta h(s) e^{-\beta\varphi(s)} - \beta - \gamma\ ds\right\} > 0.$$

For $f(t, \varphi, \eta) > 0$, notice that whenever $\eta(c) = 0$ for some point c, we have that $\eta'(c) > 0$. This bounds $\eta(t) \geq 0$ since $\eta(0) = 0$. Thus we have shown that $\eta(t) \geq 0$, which implies $\varphi'(t) \geq 0$. □

Lemma 5. *If $S_0(w_0, z_0)$ is biologically relevant, then $\varphi'(t) \geq 0$.*

Proof. Let $\tilde{S}_0(x, y)$ be biologically relevant according to definition 2.1.1, i.e.

$$\tilde{S}_0(x, y) = \sum_{s,i} x^s y^i S_{si}$$

where $S_{si} \geq 0$. Recall in section 2.1.2 we asserted that if $\tilde{S}_0(x, y)$ has non-zero coefficients, then $S_0(w, z) = \tilde{S}_0(x, y)$ also has non-negative coefficients. Hence, an application of Claims 3 and 4 together assert that $\varphi'(t) \geq 0$. □

Lemma 6. *The characteristic equations given by* (2.11) *flow out of the square* $[-\Gamma, 1-\Gamma]^2$. *This implies that for all* $(w, z) \in [-\Gamma, 1-\Gamma]^2$, $(w_0, z_0) \in [-\Gamma, 1-\Gamma]^2$.

Proof. Define the characteristic flow p through the boundary of the square $[-\Gamma, 1-\Gamma]^2$ as

$$p = \vec{n} \cdot \left(\frac{\mathrm{d}w}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t} \right),$$

where $\vec{n}$ is the outward unit normal.
Along $z = -\Gamma$, $w \in [-\Gamma, 1-\Gamma]$ we have

$$\begin{aligned} p &= (0, -1) \cdot \left(\frac{\mathrm{d}w}{\mathrm{d}t}, (\beta + \gamma)(-\Gamma) \right) \\ &= \gamma > 0 \end{aligned}$$

(recall that $\Gamma = \frac{\gamma}{\beta+\gamma}$). Similarly, along $z = 1 - \Gamma$, $w \in [-\Gamma, 1-\Gamma]$ we have

$$\begin{aligned} p &= (0, 1) \cdot \left(\frac{\mathrm{d}w}{\mathrm{d}t}, (\beta + \gamma)(1 - \Gamma) \right) \\ &= \beta > 0 \end{aligned}$$

Along $w = -\Gamma$, $z \in (-\Gamma, 1-\Gamma)$ we have

$$\begin{aligned} p &= (-1, 0) \cdot (\beta\varphi'(t)(-\Gamma) - \beta\varphi'(t)z, \frac{\mathrm{d}z}{\mathrm{d}t}) \\ &= \beta\varphi'(t)(z + \Gamma) \geq 0 \end{aligned}$$

because $\varphi' \geq 0$ by Claim 4 and $z + \Gamma > 0$ since $z > -\Gamma$. If $p = 0$ for all time t, then $\varphi'(t) \equiv 0$. However, recall that we're considering the case where $\varphi'(t) \not\equiv 0$ for all time. Thus, along this boundary $p > 0$.
Similarly, along $w = 1 - \Gamma$, $z \in [-\Gamma, 1-\Gamma]$ we have

$$\begin{aligned} p &= \beta\varphi'(t)(1 - \Gamma - z) \\ &\geq 0. \end{aligned}$$

Thus we have shown that the characteristics $(w(t), z(t))$ move out of the square $[-\Gamma, 1-\Gamma]^2$. Tracing *backward* along the characteristic leads to (w_0, z_0) inside the square. Thus, for all $(w, z) \in [-\Gamma, 1-\Gamma]^2$, we have $(w_0, z_0) \in [-\Gamma, 1-\Gamma]^2$. □

Recall equation (2.14), $S(t, w, z) = S_0(w_0, z_0)$. Since the domain of S_0 is the square, Lemma 6 asserts that the solutions given by the characteristics are well defined for (w, z) in the square.

Theorem 7. *A solution of the form* (2.14) *with a biologically relevant initial condition remains biologically relevant for all time t that the solution is defined.*

The proof of this theorem involves reverting back to the x and y variables. To avoid confusion, the proof is contained in Appendix A

Next we want to show the existence and uniqueness of solutions to the compatibility condition (2.22).

2.2.3 Existence and Uniqueness for the Compatibility Condition

From [10] we present the Picard-Lindelöf existence and uniqueness theorem and extension theorems.

Theorem 8. *Let $F : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ be a function continuous on $R : t_0 \leq t \leq t_0 + a,\ |y - y_0| \leq b$ and Lipschitz continuous with respect to y, for any $t_0 \in \mathbb{R}$ and $y_0 \in \mathbb{R}^d$. Let M be a bound for $|F(t, y)|$ on R; $\alpha = \min(a, b/M)$. Then*

$$y' = F(t, y), \quad y(t_0) = y_0 \tag{2.23}$$

has a unique solution $y = y(t)$ on $[t_0, t_0 + \alpha]$ [10].

Definition 2.2.1. Let $F(t, y)$ be continuous on a (t, y)-set E and let $y = y(t)$ be a solution of

$$y' = F(t, y) \tag{2.24}$$

on an interval J. The interval J is called a *right maximal interval of existence* for y if there does not exist an extension of $y(t)$ over an interval J_1 so that $y = y(t)$ remains a solution of (2.24). J is a proper subset of J_1, and both intervals have different right endpoints. A *left maximal interval of existence* for y is defined similarly. A maximal interval of existence is an interval which is both a left and right maximal interval [10].

Theorem 9. *Let $F(t, y)$ be continuous on an open (t, y)-set E and let $y(t)$ be a solution of* (2.24) *on some interval. Then $y(t)$ can be extended as a solution over a maximal interval of existence (T_-, T_+). Also, if (T_-, T_+) is a maximal interval of existence, then $y(t)$ tends to the boundary ∂E of E as $t \to T_-$ and $t \to T_+$* [10].

Corollary 10. *Let $F(t, y)$ be continuous on the closure $\bar{E}$ of an open (t, y)-set E and let* (2.23) *possess a solution $y = y(t)$ on a maximal right interval J. Then either*

$J = [t_0, \infty)$, or $J = [t_0, \delta]$ with $\delta < \infty$ and $(\delta, y(\delta)) \in \partial E$, or $J = [t_0, \delta)$ with $\delta < \infty$ and $|y(t)| \to \infty$ as $t \to \delta$ [10].

A sufficient condition for the existence and uniqueness of solutions to (2.22) is presented below

Theorem 11. *If $S_0(w, z)$ is biologically relevant, S_0 is smooth enough that $\frac{\partial S_0}{\partial w}$ and $\frac{\partial^2 S_0}{\partial w \partial z}$ are continuously differentiable, and $\frac{\partial S_0}{\partial w} \neq 0$, then the system* (2.22) *has a unique solution defined for all time $t \in [0, \infty)$.*

Proof. Recall the system (2.22)

$$\begin{pmatrix} \varphi'(t) \\ \eta'(t) \end{pmatrix} = \begin{pmatrix} f(t,\varphi,\eta)e^{-(\beta+\gamma)t} + h(t,\varphi,\eta)\eta \\ f(t,\varphi,\eta)\beta e^{-(\beta+\gamma)t-\beta\varphi} + [h(t,\varphi,\eta)\beta e^{-\beta\varphi} - \beta - \gamma]\eta \end{pmatrix}$$
$$\varphi(0) = 0, \;\; \eta(0) = 0.$$

as well as f and h from (2.19)

$$f(t,\varphi,\eta) = \frac{\frac{\partial^2 S_0}{\partial w_0 \partial z_0}}{\frac{\partial S_0}{\partial w_0}} \left((1-\Gamma)(e^{-\beta\varphi} + \eta), (1-\Gamma)e^{-(\beta+\gamma)t}\right)$$
$$h(t,\varphi,\eta) = \frac{\frac{\partial^2 S_0}{\partial w_0^2}}{\frac{\partial S_0}{\partial w_0}} \left((1-\Gamma)(e^{-\beta\varphi} + \eta), (1-\Gamma)e^{-(\beta+\gamma)t}\right).$$

and η from (2.16)

$$\eta(t,\varphi(t)) = -e^{-\beta\varphi(t)} + e^{-(\beta+\gamma)t} + (\beta+\gamma)e^{-(\beta+\gamma)t} \int_0^t e^{(\beta+\gamma)s - \beta\varphi(s)} \, ds$$

Since S_0 is smooth enough in both variables, then f and h are quotients of continously differentiable functions (with $\frac{\partial S_0}{\partial w} \neq 0$), and thus are themselves continuously differentiable in t, φ, and η by function composition. Furthermore, this implies that the RHS of the system (2.22) satisfies the conditions of the Picard-Lindelöf theorem (Theorem 8), hence we have existence and uniqueness of a solution $\begin{pmatrix} \varphi \\ \eta \end{pmatrix}$ on *some* interval $[0, \alpha]$, $(\alpha > 0)$.

To extend the solution to $t \in [0, \infty)$, we want to apply Corollary 10 by showing that $\left| \begin{pmatrix} \varphi \\ \eta \end{pmatrix} \right| < \infty$ for all $t \in [0, \infty)$. From Lemma 6, we have that f and h are evaluated at a point in $[-\Gamma, 1-\Gamma]^2$, and since S_0, $\frac{\partial S_0}{\partial w}$, and $\frac{\partial^2 S_0}{\partial w \partial z}$ are continuous and defined on a compact set, f and h are bounded. Let $M := \max(|f|, |h|)$ and consider the second component of (2.22)

$$\begin{aligned}\eta' &= f(t,\varphi,\eta)\beta e^{-(\beta+\gamma)t-\beta\varphi} + [h(t,\varphi,\eta)\beta e^{-\beta\varphi} - \beta - \gamma]\eta \\ &\le M\beta(e^{-(\beta+\gamma)t-\beta\varphi} + e^{-\beta\varphi}\eta)\end{aligned}$$

Since S_0 is biologically relevant, Lemma 4 applies, so $\varphi' \ge 0$, and since $\varphi(0) = 0$, we have $\varphi(t) \ge 0$. Hence

$$\eta' \le M\beta(\eta + 1)$$

and since $\eta(0) = 0$ we have

$$\eta(t) \le e^{M\beta t} - 1.$$

Thus $\eta(t) < \infty$ for all $t \in [0, \infty)$. Consider the first component

$$\begin{aligned}\varphi' =& f(t,\varphi,\eta)e^{-(\beta+\gamma)t} + h(t,\varphi,\eta)\eta \\ &\le M(1+\eta) \\ &\le Me^{M\beta t}\end{aligned}$$

and since $\varphi(0) = 0$ we have

$$\varphi(t) \le \frac{e^{M\beta t}}{\beta}$$

and so $\varphi(t) < \infty$ for all $t \in [0, \infty)$. Thus from Corollary 10, the interval of existence is $[0, \infty)$. □

The characteristics (2.15) define the solution S given by (2.14). These characteristics are determined by the function $\varphi(t)$, hence a unique φ implies unique characteristic curves, guaranteeing the uniqueness of S.

2.3 Reduction to the Volz Model

The goal of this section is to show that by assuming solutions given by a multinomial distribution, the effective degree model reduces to the Volz model.

Assume a generating function solution to (2.9) of the form (see appendix B to motivate this choice)

$$S(t,w,z) = \sum_k p_k S_k(t)(p_S(t)(w+\Gamma-1)+p_I(t)(z+\Gamma-1)+1)^k \tag{2.25}$$

where $S_k(t)$ is the proportion of suscpetible nodes having degree k at time t, $k=s+i$ is the effective degree of a node, and $p_S(t)$ and $p_I(t)$ are the probabilities of a susceptible node having a susceptible/infected neighbour at time t, respectively.

Our goal now is to show that there exist solutions of the form (2.25) satisfying the PDE (2.9). Substitute (2.25) into (2.9),

$$\begin{aligned}
&\sum_{k=0}^{\infty} p_k k S_k (p_S(w+\Gamma-1)+p_I(z+\Gamma-1)+1)^{k-1}((w+\Gamma-1)p_S' + (z+\Gamma-1)p_I') \\
&+ p_k S_k'(p_S(w+\Gamma-1)+p_I(z+\Gamma-1)+1)^k \\
=& -(\beta+\gamma)z\sum_{k=0}^{\infty} k p_k S_k (p_S(w+\Gamma-1)+p_I(z+\Gamma-1)+1)^{k-1} p_I \\
&+ \beta\frac{\sum_{k=0}^{\infty} k(k-1)p_k S_k p_I p_S}{\sum_{k=0}^{\infty} k p_k S_k p_S}(z-w)\sum_{k=0}^{\infty} k p_k S_k (p_S(w+\Gamma-1)+p_I(z+\Gamma-1)+1)^{k-1} p_S
\end{aligned}$$

Compare coefficients of $(p_S(w+\Gamma-1)+p_I(z+\Gamma-1)+1)^{k-1}$.

$$\begin{aligned}
&p_k k S_k(p_S'(w+\Gamma-1)+p_I'(z+\Gamma-1)) + p_k S_k'(p_S(w+\Gamma-1)+p_I(z+\Gamma-1)+1) \\
=& p_k\Big(-(\beta+\gamma)zkS_k p_I + \beta\frac{\sum_{k=0}^{\infty} k(k-1)p_k S_k}{\sum_{k=0}^{\infty} k p_k S_k}(z-w)kS_k p_S p_I\Big).
\end{aligned}$$

That is,

$$\begin{aligned}
&(kS_k p_S' + S_k' p_S)w + (kS_k p_I' + S_k' p_I)z + (\Gamma-1)kS_k(p_S'+p_I') + (\Gamma-1)S_k'(p_S+p_I) + p_k S_k' \\
=& -(\beta+\gamma)zkS_k p_I + \beta\frac{\sum_{k=0}^{\infty} k(k-1)S_k}{\sum_{k=0}^{\infty} kS_k}(z-w)kS_k p_S p_I.
\end{aligned}$$

Now compare coefficients of w, z, and constant terms

$$\begin{aligned}
&kS_k p_S' + S_k' p_S = -\beta \frac{\sum_{k=0}^{\infty} k(k-1) p_k S_k}{\sum_{k=0}^{\infty} k p_k S_k} k S_k p_S p_I \\
&kS_k p_I' + S_k' p_I = -(\beta+\gamma) k S_k p_I + \beta \frac{\sum_{k=0}^{\infty} k(k-1) p_k S_k}{\sum_{k=0}^{\infty} k p_k S_k} k S_k p_S p_I \\
&(\Gamma - 1) S_k' (p_S + p_I) + (\Gamma - 1) k S_k (p_S' + p_I') + S_k' = 0
\end{aligned} \tag{2.26}$$

Multiply the third equation by $\dfrac{1}{1-\Gamma}$, and add it to the first two equations in (2.26) to get an ODE for S_k

$$\frac{S_k'}{1-\Gamma} = -(\beta+\gamma) k S_k p_I.$$

That is

$$\boxed{S_k'(t) = -\beta k S_k(t) p_I(t)} \tag{2.27}$$

which has a solution

$$S_k(t) = \exp\left\{-\int_0^t \beta p_I(\tau)\ d\tau\right\}^k,$$

or

$$\boxed{S_k = S_1^k.} \tag{2.28}$$

Replacing S_k' in (2.26) with (2.27) and S_k with (2.28) gives

$$\boxed{\begin{aligned}
&p_I' = \beta p_I^2 - (\beta+\gamma) p_I + \beta \frac{\sum_{k=0}^{\infty} k(k-1) p_k S_1^k}{\sum_{k=0}^{\infty} k p_k S_1^k} p_S p_I \\
&p_S' = \beta p_I p_S - \beta \frac{\sum_{k=0}^{\infty} k(k-1) p_k S_1^k}{\sum_{k=0}^{\infty} k p_k S_1^k} p_S p_I
\end{aligned}}$$

Comparison with (1.4) shows that we have reduced the PDE effective degree model to the Volz model.

Chapter 3

Stability Analysis

The first section of this chapter begins by linearizing the full nonlinear problem (2.9) about a disease free equilibrium, and then proceed to solve the eigenvalue problem. The eigenvalues completely determine the stability of the linear system.

In the second section, we show that perturbing the disease free equilibrium of the nonlinear system with an initial condition given by a multinomial distribution (i.e. a reduction to the Volz model) results in an unstable perturbation. This shows nonlinear instability, however we cannot guarantee nonlinear stability in this way.

3.1 Linear Stability

Determining the linear stability of (2.9) first requires linearizing the system. Let $\bar{S}(w) = S^*(x)$ be a disease free equilibrium solution of (2.9), and $V(t, w, z)$ be a deviation from it such that S is given by

$$S = \bar{S}(w) + V(t, w, z). \tag{3.1}$$

Substitute (3.1) into (2.9)

$$\begin{aligned} V_t = & -(\beta+\gamma)zV_z + \beta(z-w)\frac{V_{wz}(t,1-\Gamma,1-\Gamma)}{\bar{S}'(1-\Gamma)+V_w(t,1-\Gamma,1-\Gamma)}(\bar{S}_w+V_w) \\ = & -(\beta+\gamma)zV_z \\ & + \beta(z-w)\frac{V_{wz}(t,1-\Gamma,1-\Gamma)}{\bar{S}'(1-\Gamma)}(\bar{S}_w+V_w) \\ & \times \left[1-\frac{V_w(t,1-\Gamma,1-\Gamma)}{\bar{S}'(1-\Gamma)}+\left(\frac{V_w(t,1-\Gamma,1-\Gamma)}{\bar{S}'(1-\Gamma)}\right)^2-\dots\right] \end{aligned},$$

since

$$\frac{1}{1+x} = 1-x+x^2-x^3+\dots$$

Treating V as a dummy variable and replacing it with S gives the linearized PDE about the disease free equilibrium

$$S_t = -(\beta+\gamma)zS_z + \beta(z-w)\frac{\bar{S}'(w)}{\bar{S}'(1-\Gamma)}S_{wz}(1-\Gamma,1-\Gamma).$$

Its eigenvalue problem is

$$\boxed{\lambda S = -(\beta+\gamma)zS_z + \beta(z-w)\frac{\bar{S}'(w)}{\bar{S}'(1-\Gamma)}S_{wz}(1-\Gamma,1-\Gamma).} \tag{3.2}$$

Written as an operator, we have

$$(L-\lambda\mathbf{1})S = 0$$

where

$$L[S] = -(\beta+\gamma)zS_z + \beta(z-w)\frac{\bar{S}'(w)}{\bar{S}'(1-\Gamma)}S_{wz}(1-\Gamma,1-\Gamma). \tag{3.3}$$

Remark. *It is assumed that* $S_{wz}(1-\Gamma,1-\Gamma)\neq 0$ *because this is the more interesting case related to the ODE model* (1.1).

The domain of L is a subspace of X, that is

$$D(L)\subset X.$$

Let the domain $D(L)$ be defined as the functions in X having finite domain norm

$$\|S\|_{\text{Dom}} = \sum_{m,n} (1+m)(1+n)|s_{mn}| < \infty$$

Definition 3.1.1. A linear operator $L : X \to Y$ is between Banach spaces X and Y is said to be bounded if there exists M such that for all $x \in X$ [11, Section 12.5.1.1]

$$\|Lx\|_Y \leq M\|x\|_X.$$

Definition 3.1.2. Let X be a Banach space and $L :\ X \to X$ be a linear operator on X defined on domain $D(L) \subset X$. A complex number λ is said to be in the resolvent set, that is, the complement of the spectrum of a linear operator

$$L :\ D(L) \subset X \to X$$

if the operator

$$L - \lambda \mathbf{1} :\ D(L) \to X$$

has a bounded inverse. That is, the inverse

$$(L - \lambda \mathbf{1})^{-1} : X \to D(L)$$

exists, and is bounded [12, Page 566].

Lemma 12. *The set* $\lambda = \{-k(\beta+\gamma), 0, \beta\frac{\bar{S}''(1)}{\bar{S}'(1)} - (\beta+\gamma)\}$ *for* $k = 1, 2, \ldots,$ *are in the spectrum of the operator* L *defined by equation* (3.3)*.*

Proof. By definition 3.1.2, the set λ is in the spectrum of L if $(L - \lambda\mathbf{1})$ is noninvertible. This is equivalent to showing that non-trivial solutions exist to

$$(L - \lambda \mathbf{1})S = 0.$$

The eigenvalue problem (3.2) can be treated as an ODE of $S(y)$ with x being a parameter

$$(\beta+\gamma)zS_z + \lambda S = \beta(z-w)\frac{\bar{S}'(w)}{\bar{S}'(1-\Gamma)}S_{wz}(1-\Gamma, 1-\Gamma) \tag{3.4}$$

and thus,

$$S_z + \frac{\lambda S}{(\beta+\gamma)z} = \frac{\beta}{\beta+\gamma}\left(1 - \frac{w}{z}\right)\frac{\bar{S}'(w)}{\bar{S}'(1-\Gamma)}S_{wz}(1-\Gamma, 1-\Gamma)$$

With integrating factor

$$z^{\frac{\lambda}{\beta+\gamma}}$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[z^{\frac{\lambda}{\beta+\gamma}}S\right] = \frac{\beta}{\beta+\gamma}\left(z^{\frac{\lambda}{\beta+\gamma}} - wz^{\frac{\lambda}{\beta+\gamma}-1}\right)\frac{\bar{S}'(w)}{\bar{S}'(1-\Gamma)}S_{wz}(1-\Gamma, 1-\Gamma) \tag{3.5}$$

We can solve (3.5) by integrating with respect to z, which gives rise to the following cases.

Case 1. If $\lambda = 0$, then (3.5) is

$$S_z = \frac{\beta}{\beta+\gamma}\left(1 - wz^{-1}\right)\frac{\bar{S}'(w)}{\bar{S}'(1-\Gamma)}S_{wz}(1-\Gamma, 1-\Gamma)$$

integrate with respect to z

$$S(w,z) = \frac{\beta}{\beta+\gamma}[z - w\ln(z)] \cdot \frac{\bar{S}'(w)}{\bar{S}'(1-\Gamma)}S_{wz}(1-\Gamma, 1-\Gamma) + C(w)$$

where $C(w)$ is an undetermined function in X. Because $\ln(z) \notin X$ (as well as $w\ln(z)$), for S to be in X it is required that

$$\frac{\bar{S}'(w)}{\bar{S}'(1-\Gamma)}S_{wz}(1-\Gamma, 1-\Gamma) = 0.$$

Thus

$$\boxed{S(w,z) = C(w)}$$

So any single variable function in X is a solution when $\lambda = 0$.

Case 2. If $\frac{\lambda}{\beta+\gamma} = -1$, equation (3.5) is

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[z^{-1}S\right] = \frac{\beta}{\beta+\gamma}\left(z^{-1} - wz^{-2}\right)\frac{\bar{S}'(w)}{\bar{S}'(1-\Gamma)}S_{wz}(1-\Gamma, 1-\Gamma)$$

Integrate with respect to z

$$z^{-1}S = \frac{\beta}{\beta+\gamma}\left(\ln(z) + wz^{-1}\right)\frac{\bar{S}'(w)}{\bar{S}'(1-\Gamma)}S_{wz}(1-\Gamma, 1-\Gamma) + C(w)$$

that is,

$$S = \frac{\beta}{\beta+\gamma}(z\ln(z) + w)\frac{\bar{S}'(w)}{\bar{S}'(1-\Gamma)}S_{wz}(1-\Gamma, 1-\Gamma) + zC(w)$$

Again, $z\ln(z) \notin X$, so for $S \in X$, we require $\frac{\bar{S}'(w)}{\bar{S}'(1-\Gamma)}S_{wz}(1-\Gamma, 1-\Gamma) = 0$. Hence

$$S_{wz}(1-\Gamma, 1-\Gamma) = 0.$$

With $S_{wz}(1-\Gamma, 1-\Gamma) = 0$, note that

$$S(w,z) = zC(w)$$

and so

$$S_{wz} = C'(w)$$

Requiring $S_{wz}(1-\Gamma, 1-\Gamma) = 0$ gives

$$C'(1-\Gamma) = 0$$

Thus the solution for $\lambda = -\beta - \gamma$ is

$$\boxed{\begin{array}{l} S(w,z) = zC(w) \\ C'(1-\Gamma) = 0. \end{array}}$$

where $C(w) \in X$.

Case 3. If λ otherwise, solve equation (3.5) by integrating with respect to y

$$z^{\frac{\lambda}{\beta+\gamma}}S = \frac{\beta}{\beta+\gamma}\left(\frac{\beta+\gamma}{\lambda+\beta+\gamma}z^{\frac{\lambda}{\beta+\gamma}+1} - \frac{\beta+\gamma}{\lambda}wz^{\frac{\lambda}{\beta+\gamma}}\right)\frac{\bar{S}'(w)}{\bar{S}'(1-\Gamma)}S_{wz}(1-\Gamma, 1-\Gamma) + C(w)$$

that is,

$$\boxed{S(w,z) = C(w)z^{-\frac{\lambda}{\beta+\gamma}} + \left[\frac{\beta}{\lambda+\beta+\gamma}z - \frac{\beta}{\lambda}w\right]\frac{\bar{S}'(w)}{\bar{S}'(1-\Gamma)}S_{wz}(1-\Gamma,1-\Gamma)} \tag{3.6}$$

We can show that this solves the ODE (3.4),

$$\begin{aligned}
&(\beta+\gamma)zS_z + \lambda S \\
&= (\beta+\gamma)z\left\{-\frac{\lambda}{\beta+\gamma}C(w)z^{-\frac{\lambda}{\beta+\gamma}-1} + \frac{\beta}{\lambda+\beta+\gamma}\frac{\bar{S}'(w)}{\bar{S}'(1-\Gamma)}S_{wz}(1-\Gamma,1-\Gamma)\right\} \\
&\quad + \lambda\left\{C(w)z^{-\frac{\lambda}{\beta+\gamma}} + \left[\frac{\beta}{\lambda+\beta+\gamma}z - \frac{\beta}{\lambda}w\right]\frac{\bar{S}'(w)}{\bar{S}'(1-\Gamma)}S_{wz}(1-\Gamma,1-\Gamma)\right\} \\
&= \frac{\bar{S}'(w)}{\bar{S}'(1-\Gamma)}S_{wz}(1-\Gamma,1-\Gamma)\left[\frac{\beta(\beta+\gamma)z}{\lambda+\beta+\gamma} + \frac{\beta\lambda z}{\lambda+\beta+\gamma} - \lambda\frac{\beta}{\lambda}w\right] \\
&\quad + \left[\lambda z^{-\frac{\lambda}{\beta+\gamma}} - (\beta+\gamma)z\frac{\lambda}{\beta+\gamma}z^{-\frac{\lambda}{\beta+\gamma}-1}\right]C(w) \\
&= \beta(z-w)\frac{\bar{S}'(w)}{\bar{S}'(1-\Gamma)}S_{wz}(1-\Gamma,1-\Gamma).
\end{aligned}$$

That is

$$(\beta+\gamma)zS_z + \lambda S = \beta(z-w)\frac{\bar{S}'(w)}{\bar{S}'(1-\Gamma)}S_{wz}(1-\Gamma,1-\Gamma)$$

as desired.

Subcase 3.1. Let $C(w) = 0$, in which case

$$S(w,z) = \left[\frac{\beta}{\lambda+\beta+\gamma}z - \frac{\beta}{\lambda}w\right]\frac{\bar{S}'(w)}{\bar{S}'(1-\Gamma)}S_{wz}(1-\Gamma,1-\Gamma)$$

To determine the eigenvalue λ, note that

$$S_z = \frac{\beta}{\lambda+\beta+\gamma}\frac{\bar{S}'(w)}{\bar{S}'(1-\Gamma)}S_{wz}(1-\Gamma,1-\Gamma)$$

and

$$S_{zw} = \frac{\beta}{\lambda+\beta+\gamma}\frac{\bar{S}''(w)}{\bar{S}'(1-\Gamma)}S_{wz}(1-\Gamma,1-\Gamma)$$

At $w = z = 1 - \Gamma$,

$$S_{wz}(1-\Gamma, 1-\Gamma) = \frac{\beta}{\lambda + \beta + \gamma} \frac{\bar{S}''(1-\Gamma)}{\bar{S}'(1-\Gamma)} S_{wz}(1-\Gamma, 1-\Gamma) \tag{3.7}$$

$$\frac{\beta}{\lambda + \beta + \gamma} \frac{\bar{S}''(1-\Gamma)}{\bar{S}'(1-\Gamma)} = 1.$$

Solving equation (3.7) for the eigenvalue λ

$$\lambda = \beta \frac{\bar{S}''(1-\Gamma)}{\bar{S}'(1-\Gamma)} - \beta - \gamma,$$

Thus, λ must be real, and $\lambda > 0$ implies

$$\frac{\beta}{\beta + \gamma} \frac{\bar{S}''(1-\Gamma)}{\bar{S}'(1-\Gamma)} > 1.$$

Similarly, $\lambda < 0$ reverses the inequality.

In equation (3.7), $S_{wz}(1-\Gamma, 1-\Gamma)$ is a free parameter, which was assumed non-zero. Set $S_{wz}(1-\Gamma, 1-\Gamma) = 1$, then the eigenfunction is

$$\boxed{S(w,z) = \left[\frac{\beta}{\lambda + \beta + \gamma} z - \frac{\beta}{\lambda} w\right] \frac{\bar{S}'(w)}{\bar{S}'(1-\Gamma)}} \tag{3.8}$$

Definition 3.1.3. The *basic reproduction number* $\mathcal{R}_0$ from [1] and equivalently definined in [5], is the expected number of secondary infections from an infectious individual introduced into a susceptible population:

$$\boxed{\mathcal{R}_0 = \frac{\beta}{\beta + \gamma} \frac{(S^*)''(1)}{(S^*)'(1)}.}$$

Note: recall S^* is a disease free equilibrium, and is a function of x, not w.

Changing variables back to x and y, we can view the eigenvalue in terms of the reproduction number

$$\mathcal{R}_0 = \frac{\lambda}{\beta + \gamma} + 1.$$

Subcase 3.2. Suppose $C(w) \neq 0$. Then for the solution (3.6) to be in X, it is required that $C(w) \in X$ and

$$\lambda = -k(\beta + \gamma)$$

for $k = 2, 3, \ldots$. Thus the solution is

$$S(w,z) = C(w)z^k + \left[\frac{\beta}{\lambda+\beta+\gamma}z - \frac{\beta}{\lambda}w\right]\frac{\bar{S}'(w)}{\bar{S}'(1-\Gamma)}S_{wz}(1-\Gamma, 1-\Gamma)$$

Take derivatives in z and w to find

$$S_{wz}(w,z) = kC'(w)z^{k-1} + \frac{\beta}{\lambda+\beta+\gamma}\frac{\bar{S}''(w)}{\bar{S}'(1-\Gamma)}S_{wz}(1-\Gamma, 1-\Gamma)$$

Evaluate at $(w, z) = (1-\Gamma, 1-\Gamma)$, and gather like terms

$$\left[1 - \frac{\beta}{\lambda+\beta+\gamma}\frac{\bar{S}''(1-\Gamma)}{\bar{S}'(1-\Gamma)}\right]S_{wz}(1-\Gamma, 1-\Gamma) = k(1-\Gamma)^{k-1}C'(1-\Gamma)$$

If the LHS is non-zero, then $S_{wz}(1-\Gamma, 1-\Gamma)$ is an arbitrary constant because $C(w)$ is an arbitrary function in X. If the LHS is zero, $S_{wz}(1-\Gamma, 1-\Gamma)$ is still arbitrary, hence set it to equal one, and the eigenfunction is

$$\boxed{\begin{aligned} S(w,z) &= C(w)z^k + \left[\frac{\beta}{\lambda+\beta+\gamma}z - \frac{\beta}{\lambda}w\right]\frac{\bar{S}'(w)}{\bar{S}'(1-\Gamma)}, \\ C'(1-\Gamma) &= \frac{1}{k(1-\Gamma)^{k-1}}\left[1 - \frac{\beta}{\lambda+\beta+\gamma}\frac{\bar{S}''(1-\Gamma)}{\bar{S}'(1-\Gamma)}\right] \end{aligned}}$$

□

From definition 3.1.2, we need to show that for $\lambda \notin \{-k(\beta+\gamma), 0, \beta\frac{\bar{S}''(1)}{\bar{S}'(1)} - (\beta+\gamma)\}$, that for $L - \lambda\mathbf{1}$ as defined in (3.3) has a bounded inverse.

Lemma 13. *The operator* $(L - \lambda I) : D(L) \to X$ *is invertible if* $\lambda \notin \{-k(\beta + \gamma), 0, \beta\frac{\bar{S}''(1-\Gamma)}{\bar{S}'(1-\Gamma)} - \beta - \gamma\}$, *for* $k = 1, 2, \ldots$.

Proof. For invertible, we need to show that we have unique solutions $S \in D(L)$ to

$$(L - \lambda\mathbf{1})S = u(w,z)$$

where $u(w,z) \in X$ defined by 2.10. If we can show this, and that the inverse operator is bounded, then it shows that the spectrum of L is the discrete set of eigenvalues, and then the stability is determined by those.

Let $u(w,z) \in X$, then we want to find $S \in D(L)$ such that

$$(L - \lambda I)S = u.$$

We have

$$-(\beta+\gamma)zS_z + \beta(z-w)\frac{\bar{S}'(w)}{\bar{S}'(1-\Gamma)}S_{wz}(1-\Gamma,1-\Gamma) - \lambda S = u.$$

Rearrange to write it as an ODE of z

$$(\beta+\gamma)zS_z + \lambda S = \beta(z-w)\frac{\bar{S}'(w)}{\bar{S}'(1-\Gamma)}S_{wz}(1-\gamma,1-\Gamma) - u(w,z).$$

Divide through by $(\beta+\gamma)z$

$$S_z + \frac{\lambda}{(\beta+\gamma)z}S = \frac{\beta(z-w)}{(\beta+\gamma)z}\frac{\bar{S}'(w)}{\bar{S}'(1-\Gamma)}S_{wz}(1-\Gamma,1-\Gamma) - \frac{u(w,z)}{(\beta+\gamma)z}. \tag{3.9}$$

This problem has integrating factor

$$z^\alpha$$

where

$$\alpha := \lambda/(\beta+\gamma).$$

Since $\lambda \notin \{-k(\beta+\gamma), 0, \beta\frac{\bar{S}''(1-\Gamma)}{\bar{S}'(1-\Gamma)} - \beta - \gamma\}$, $(k = 1,2,\ldots)$ we have that

$$\alpha \notin \{-k, 0, \frac{\beta}{\beta+\gamma}\frac{\bar{S}''(1-\Gamma)}{\bar{S}'(1-\Gamma)} - 1\}$$

for the same k. Then the ODE (3.9) can be rewritten with the integrating factor

$$\frac{\mathrm{d}}{\mathrm{d}z}[z^\alpha S] = z^\alpha\frac{\beta(z-w)}{(\beta+\gamma)z}\frac{\bar{S}'(w)}{\bar{S}'(1-\Gamma)}S_{wz}(1-\Gamma,1-\Gamma) - z^{\alpha-1}\frac{u(w,z)}{(\beta+\gamma)}$$

Integrate both sides with respect to z, then multiply both sides by $z^{-\alpha}$ (which is well defined since $\alpha \neq 0$)

$$S = z^{-\alpha} \int z^{\alpha} \frac{\beta(z-w)}{(\beta+\gamma)z} \frac{\bar{S}'(w)}{\bar{S}'(1-\Gamma)} S_{wz}(1-\Gamma, 1-\Gamma)\ dz - z^{-\alpha} \int z^{\alpha-1} \frac{u(w,z)}{(\beta+\gamma)}\ dz$$

Define

$$\begin{aligned} v_1 &= z^{-\alpha} \int z^{\alpha} \frac{\beta(z-w)}{(\beta+\gamma)z} \frac{\bar{S}'(w)}{\bar{S}'(1-\Gamma)} S_{wz}(1-\Gamma, 1-\Gamma)\ dz \\ v_2 &= z^{-\alpha} \int z^{\alpha-1} \frac{u(w,z)}{(\beta+\gamma)}\ dz, \end{aligned}$$

We want to show $v_1, v_2 \in D(L)$. For the first integral, it is easy to see that

$$\begin{aligned} v_1 &= z^{-\alpha} \int z^{\alpha} \frac{\beta(z-w)}{(\beta+\gamma)z} \frac{\bar{S}'(w)}{\bar{S}'(1-\Gamma)} S_{wz}(1-\Gamma, 1-\Gamma)\ dz \\ &= \left[\frac{\beta}{\beta+\gamma} \frac{\bar{S}'(w)}{\bar{S}'(1-\Gamma)} S_{wz}(1-\Gamma, 1-\Gamma) \right] z^{-\alpha} \int z^{\alpha} \frac{z-w}{z}\ dz \\ &= \left[\frac{\beta}{\beta+\gamma} \frac{\bar{S}'(w)}{\bar{S}'(1-\Gamma)} S_{wz}(1-\Gamma, 1-\Gamma) \right] z^{-\alpha} \int z^{\alpha} - w z^{\alpha-1}\ dz \\ &= \left[\frac{\beta}{\beta+\gamma} \frac{\bar{S}'(w)}{\bar{S}'(1-\Gamma)} S_{wz}(1-\Gamma, 1-\Gamma) \right] \left(\frac{z}{1+\alpha} - \frac{w}{\alpha} \right) \end{aligned}$$

This uses the fact that α is not a negative integer because of the choice of λ. By the proof of a claim, we can see that $v_1 \in D(L)$ if $\bar{S}'(w) \in D(L)$, hence

$$\|v_1\|_{\text{Dom}} < \infty.$$

Claim 14. *If a function f is in the domain $D(L)$, then so are wf and zf.*

Proof. Expand f as its power series

$$f(w,z) = \sum_{m,n} f_{mn} w^m z^n.$$

Since $f \in D(L)$ we have

$$\|f\|_{\text{Dom}} = \sum_{m,n} (1+m)(1+n)|f_{mn}| < \infty.$$

The power series for wf is

$$wf(w,z) = \sum_{m,n} f_{m-1,n} w^m z^n$$

(defining $f_{m,n} = 0$ for any negative subscripts) and its domain norm is

$$\begin{aligned}
\|wf\|_{\text{Dom}} &= \sum_{m,n} (1+m)(1+n)|f_{m-1,n}| \\
&= \sum_{m,n} (2+m)(1+n)|f_{mn}| \\
&\leq 2\|f\|_{\text{Dom}} \\
&< \infty.
\end{aligned}$$

Hence $wf \in D(L)$. The proof for zf is identical. □

For the second integral, v_2. Expand u as a power series

$$\begin{aligned}
v_2 &= z^{-\alpha} \int z^{\alpha-1} \frac{u(w,z)}{(\beta+\gamma)}\, dz \\
&= \frac{z^{-\alpha}}{\beta+\gamma} \int z^{\alpha-1} \sum_{m,n} u_{mn} w^m z^n\, dz \\
&= \sum_{m,n} \frac{u_{mn}}{\beta+\gamma} w^m z^{-\alpha} \int z^{n+\alpha-1}\, dz \\
&= \sum_{m,n} \frac{u_{mn}}{(\beta+\gamma)(\alpha+n)} w^m z^{-\alpha} z^{n+\alpha} \\
&= \frac{1}{\beta+\gamma} \sum_{m,n} \frac{u_{mn}}{\alpha+n} w^m z^n
\end{aligned}$$

Again, $\frac{1}{\alpha+n}$ is well defined as α is not a negative integer. Then we want to show that

$$\sum_{m,n} (1+m)(1+n) \left| \frac{u_{mn}}{\alpha+n} \right| < \infty$$

Notice that because α is not a negative integer, $\max_n \left|\frac{1+n}{\alpha+n}\right|$ is well defined. Hence

$$\begin{aligned}\sum_{m,n}(1+m)(1+n)\left|\frac{u_{mn}}{\alpha+n}\right| &\leq \max_n \left|\frac{1+n}{\alpha+n}\right| \sum_{m,n}(1+m)|u_{mn}| \\ &= \max_n \left|\frac{1+n}{\alpha+n}\right| \|u\|_X \\ &< \infty\end{aligned}$$

Hence $S \in D(L)$. We also have the solution $S = v_1 + v_2$ as

$$\boxed{S = \left[\frac{\beta}{\beta+\gamma}\frac{\bar{S}'(w)}{\bar{S}'(1-\Gamma)}S_{wz}(1-\Gamma,1-\Gamma)\right]\left(\frac{z}{1+\alpha}-\frac{w}{\alpha}\right)+\frac{1}{\beta+\gamma}\sum_{m,n}\frac{u_{mn}}{\alpha+n}w^m z^n} \tag{3.10}$$

This solution assumes from the beginning that $S_{wz}(1-\Gamma,1-\Gamma)$ is well defined. We can derive a compatibility condition by taking derivatives of (3.10). Take derivatives in w and z

$$S_{wz}(w,z) = \left[\frac{\beta}{\beta+\gamma}\frac{\bar{S}''(w)}{\bar{S}'(1-\Gamma)}S_{wz}(1-\Gamma,1-\Gamma)\right]\left(\frac{1}{1+\alpha}\right)+\frac{1}{\beta+\gamma}\sum_{m,n}mn\frac{u_{mn}}{\alpha+n}w^{m-1}z^{n-1}$$

and evaluate at $(w,z) = (1-\Gamma, 1-\Gamma)$

$$S_{wz}(1-\Gamma,1-\Gamma) = \left[\frac{\beta}{\beta+\gamma}\frac{\bar{S}''(1-\Gamma)}{\bar{S}'(1-\Gamma)}S_{wz}(1-\Gamma,1-\Gamma)\right]\left(\frac{1}{1+\alpha}\right)+\frac{1}{\beta+\gamma}\sum_{m,n}mn\frac{u_{mn}}{\alpha+n}(1-\Gamma)^{m+n-2}.$$

Solve for $S_{wz}(1-\Gamma,1-\Gamma)$ on the LHS

$$S_{wz}(1-\Gamma,1-\Gamma)\left[1-\frac{\beta}{(1+\alpha)(\beta+\gamma)}\frac{\bar{S}''(1-\Gamma)}{\bar{S}'(1-\Gamma)}\right] = \frac{1}{\beta+\gamma}\sum_{m,n}mn\frac{u_{mn}}{\alpha+n}(1-\Gamma)^{m+n-2} \tag{3.11}$$

Then compatibility condition is

$$S_{wz}(1-\Gamma,1-\Gamma) = \left[\beta+\gamma-\frac{\beta}{1+\alpha}\frac{\bar{S}''(1-\Gamma)}{\bar{S}'(1-\Gamma)}\right]^{-1}\sum_{m,n}mn\frac{u_{mn}}{\alpha+n}(1-\Gamma)^{m+n-2}$$

and the solution (3.10) is

$$S = \frac{\left[\frac{\beta}{\beta+\gamma}\frac{\bar{S}'(w)}{\bar{S}'(1-\Gamma)}\right]}{\left[\beta+\gamma-\frac{\beta}{1+\alpha}\frac{\bar{S}''(1-\Gamma)}{\bar{S}'(1-\Gamma)}\right]}\left(\sum_{m,n} mn\frac{u_{mn}}{\alpha+n}(1-\Gamma)^{m+n-2}\right)\left(\frac{z}{1+\alpha}-\frac{w}{\alpha}\right)$$
$$+\frac{1}{\beta+\gamma}\sum_{m,n}\frac{u_{mn}}{\alpha+n}w^m z^n$$

which simplifies to

$$\boxed{\begin{aligned} S = {} & \frac{\beta}{\beta+\gamma}\left[\frac{\bar{S}'(w)}{(\beta+\gamma)\bar{S}'(1-\Gamma)-\frac{\beta}{1+\alpha}\bar{S}''(1-\Gamma)}\right]\left(\sum_{m,n} mn\frac{u_{mn}}{\alpha+n}(1-\Gamma)^{m+n-2}\right)\left(\frac{z}{1+\alpha}-\frac{w}{\alpha}\right) \\ & +\frac{1}{\beta+\gamma}\sum_{m,n}\frac{u_{mn}}{\alpha+n}w^m z^n \end{aligned}} \tag{3.12}$$

Remark. *If* $\left[1-\frac{\beta}{(1+\alpha)(\beta+\gamma)}\frac{\bar{S}''(1-\Gamma)}{\bar{S}'(1-\Gamma)}\right]=0$ *there is no solution, but this cannot happen due to the restrictions on* α*.*

The requirement for no solution to exist is that in (3.11)

$$\frac{\beta}{(1+\alpha)(\beta+\gamma)}\frac{\bar{S}''(1-\Gamma)}{\bar{S}'(1-\Gamma)}=1$$

or

$$\alpha = \frac{\beta}{\beta+\gamma}\frac{\bar{S}''(1-\Gamma)}{\bar{S}'(1-\Gamma)}-1.$$

Recall that α is restricted by the choice of λ and

$$\alpha \notin \{-k, 0, \frac{\beta}{\beta+\gamma}\frac{\bar{S}''(1)}{\bar{S}'(1)}-1\}$$

Hence a solution S to the inverse problem,

$$(L-\lambda\mathbf{1})^{-1}S = u$$

exists and is compliant with the compatibility condition. □

Lemma 15. *The inverse operator* $(L-\lambda I)^{-1}: X \to D(L)$ *is bounded if* $\lambda \notin \{-k(\beta+\gamma), 0, \beta\frac{\bar{S}''(1-\Gamma)}{\bar{S}'(1-\Gamma)}-\beta-\gamma\}$*, for* $k=1,2,\ldots$*.*

Proof. We want to show that there exists $M > 0$ such that for any $u \in X$

$$\left\|(L-\lambda I)^{-1}u\right\|_{\text{Dom}} \leq M\|u\|_X.$$

Let $S = (L-\lambda I)^{-1}u$, and since $\lambda \notin \{-k(\beta+\gamma), 0, \beta\frac{\bar{S}''(1-\Gamma)}{\bar{S}'(1-\Gamma)} - \beta - \gamma\}$, S is given by equation (3.10). Then we want to show

$$\|S\|_{\text{Dom}} \leq M\|u\|_X.$$

Take the domain norm of equation (3.12)

$$\begin{aligned}
&\|S\|_{\text{Dom}} \\
&\leq \left\|\frac{\beta}{\beta+\gamma}\left[\frac{\bar{S}'(w)}{(\beta+\gamma)\bar{S}'(1-\Gamma) - \frac{\beta}{1+\alpha}\bar{S}''(1-\Gamma)}\right]\left(\sum_{m,n} mn\frac{u_{mn}}{\alpha+n}(1-\Gamma)^{m+n-2}\right)\left(\frac{z}{1+\alpha} - \frac{w}{\alpha}\right)\right\|_{\text{Dom}} \\
&\quad + \left\|\frac{1}{\beta+\gamma}\sum_{m,n}\frac{u_{mn}}{\alpha+n}w^m z^n\right\|_{\text{Dom}}
\end{aligned}$$

$$\begin{aligned}
&\leq \left|\sum_{m,n} mn\frac{u_{mn}}{\alpha+n}(1-\Gamma)^{m+n-2}\right| \cdot \left\|\frac{\beta}{\beta+\gamma}\left[\frac{\bar{S}'(w)}{(\beta+\gamma)\bar{S}'(1-\Gamma) - \frac{\beta}{1+\alpha}\bar{S}''(1-\Gamma)}\right]\left(\frac{z}{1+\alpha} - \frac{w}{\alpha}\right)\right\|_{\text{Dom}} \\
&\quad + \frac{1}{\beta+\gamma}\sum_{m,n}(1+m)(1+n)\left|\frac{u_{mn}}{\alpha+n}\right|
\end{aligned}$$

Define

$$c = \left\|\frac{\beta}{\beta+\gamma}\left[\frac{\bar{S}'(w)}{(\beta+\gamma)\bar{S}'(1-\Gamma) - \frac{\beta}{1+\alpha}\bar{S}''(1-\Gamma)}\right]\left(\frac{z}{1+\alpha} - \frac{w}{\alpha}\right)\right\|_{\text{Dom}},$$

then we have

$$
\begin{aligned}
\|S\|_{\text{Dom}} &\leq c\left|\sum_{m,n} mn\frac{u_{mn}}{\alpha+n}(1-\Gamma)^{m+n-2}\right| + \frac{1}{\beta+\gamma}\sum_{m,n}(1+m)(1+n)\left|\frac{u_{mn}}{\alpha+n}\right| \\
&\leq c\sum_{m,n} mn\left|\frac{u_{mn}}{\alpha+n}(1-\Gamma)^{m+n-2}\right| + \frac{1}{\beta+\gamma}\sum_{m,n}(1+m)(1+n)\left|\frac{u_{mn}}{\alpha+n}\right| \\
&\leq c\sum_{m,n} m(1+n)\left|\frac{u_{mn}}{\alpha+n}(1-\Gamma)^{m+n-2}\right| + \frac{1}{\beta+\gamma}\sum_{m,n}(1+m)(1+n)\left|\frac{u_{mn}}{\alpha+n}\right| \\
&\leq c\sum_{m,n} m\left|\frac{1+n}{\alpha+n}\right||u_{mn}| + \frac{1}{\beta+\gamma}\sum_{m,n}(1+m)\left|\frac{1+n}{\alpha+n}\right||u_{mn}| \\
&\leq \left(c+\frac{1}{\beta+\gamma}\right)\sum_{m,n}(1+m)\left|\frac{1+n}{\alpha+n}\right||u_{mn}| \\
&\leq \left(c+\frac{1}{\beta+\gamma}\right)\max_n\left|\frac{1+n}{\alpha+n}\right|\sum_{m,n}(1+m)|u_{mn}|
\end{aligned}
$$

By defining $N_\alpha := \max_n \left|\frac{1+n}{\alpha+n}\right|$ we have

$$
\begin{aligned}
\|S\|_{\text{Dom}} &\leq \left(c+\frac{1}{\beta+\gamma}\right)N_\alpha\sum_{m,n}(1+m)|u_{mn}| \\
&= \left(c+\frac{1}{\beta+\gamma}\right)N_\alpha\|u\|_X
\end{aligned}
$$

Hence

$$\|S\|_{\text{Dom}} \leq M\|u\|_X$$

where $M = \left(c+\frac{1}{\beta+\gamma}\right)N_\alpha$. □

Theorem 16. *The spectrum of the linear operator* $(L-\lambda\mathbf{1})$ *defined in* (3.3) *is its set of eigenvalues*

$$\lambda = \{-k(\beta+\gamma), 0, \beta\frac{\bar{S}''(1-\Gamma)}{\bar{S}'(1-\Gamma)} - (\beta+\gamma)\}$$

for $k = 1, 2, \ldots$, with corresponding eigenfunctions

$zC(w)$, $C'(1-\Gamma) = 0$	$\lambda = -(\beta+\gamma)$
$\left[\frac{\beta}{\lambda+\beta+\gamma}z - \frac{\beta}{\lambda}w\right]\frac{\bar{S}'(w)}{\bar{S}'(1-\Gamma)} + C(w)z^k$, $C'(1-\Gamma) = \frac{1}{k(1-\Gamma)^{k-1}}\left[1 - \frac{\beta}{\lambda+\beta+\gamma}\frac{\bar{S}''(1-\Gamma)}{\bar{S}'(1-\Gamma)}\right]$	$\lambda = -n(\beta+\gamma),\ n = 2, 3, \ldots$
$C(w)$	$\lambda = 0$
$\left[\frac{\beta}{\lambda+\beta+\gamma}z - \frac{\beta}{\lambda}w\right]\frac{\bar{S}'(w)}{\bar{S}'(1-\Gamma)}$	$\lambda = \beta\frac{\bar{S}''(1-\Gamma)}{\bar{S}'(1-\Gamma)} - (\beta+\gamma)$

Proof. From lemma 12, we know that the set above are in the spectrum according to definition 3.1.2 because for those λ, the operator is non-invertable.

Similarly, lemma 13 shows that for all other λ, the operator $(L - \lambda\mathbf{1})$ is invertable, and lemma 15 shows that its inverse is bounded. Again, by definition 3.1.2, all other λ are in the resolvent set, hence the spectrum is the set of eigenvalues. □

Lemma 17. *The basic reproduction number $\mathcal{R}_0$ (definition 3.1.3) determines the linear stability of L.*

Proof. From theorem 16, the spectrum of the operator L defined in (3.3) is given by its set of eigenvalues and the only eigenvalue that could be positive is

$$\lambda = \beta\frac{\bar{S}''(1-\Gamma)}{\bar{S}'(1-\Gamma)} - (\beta+\gamma) = (\beta+\gamma)(\mathcal{R}_0 - 1).$$

Hence, L has a positive eigenvalue if and only if $\mathcal{R}_0 > 1$. This is the same threshold condition derived in [1]. To see this, recall $\bar{S}(w) = S^*(x)$, a disease free equilibrium. When $w = 1-\Gamma$, $x = 1$, then for example

$$\bar{S}'(1-\Gamma) = (S^*)'(1).$$

□

3.2 Nonlinear Instability

In chapter 2 we showed that the effective degree PDE model (2.9) could be reduced to the Volz model (1.4) by assuming a multinomial generating function solution (2.25). We can use this fact to show the nonlinear instability of our PDE by taking an initial condition that reduces to the Volz model, and we will show that the instability of the Volz model causes instability of the effective degree PDE.

Lemma 18. *An initial condition for the Volz model* (1.4) *that is a perturbation of the disease-free equilibrium along the eigenvector associated to the positive eigenvalue* λ*, is equivalent to an initial condition for the effective degree model* (2.9) *that is a perturbation of the disease-free equilibrium along the eigenfunction* (3.8)*.*

Proof. Assume an initial condition to the effective degree model (2.9)

$$S_0(w,z) = \sum_k p_k \theta^k(0)(p_S(0)(w+\Gamma-1) + p_I(0)(z+\Gamma-1)+1)^k. \tag{3.13}$$

Let the initial condition of the Volz model be an ε-perturbation from the disease-free equilibrium along the eigenvector v associated to λ such that

$$\begin{aligned} (\theta(0), p_S(0), p_I(0)) &= (1,1,0) + \varepsilon v \\ v &= (-\beta, -\lambda-\gamma, \lambda) \\ \lambda &= \beta\frac{\bar{S}''(1-\Gamma)}{\bar{S}'(1-\Gamma)} - \beta - \gamma \end{aligned} \tag{3.14}$$

Substitution of (3.14) into (3.13) gives

$$\begin{aligned} S_0(w,z) =& \sum_k p_k(1-\varepsilon\beta)^k((1-\varepsilon(\lambda+\gamma))(w+\Gamma-1) + \varepsilon\lambda(z+\Gamma-1)+1)^k \\ =& \sum_k p_k\left[w+\Gamma+\varepsilon(\lambda z - w(\beta+\gamma+\lambda)) + O(\varepsilon^2)\right]^k \\ =& \sum_k p_k\left[(w+\Gamma)^k + \varepsilon k(w+\Gamma)^{k-1}(\lambda z - w(\beta+\gamma+\lambda)) + O(\varepsilon^2)\right] \\ =& \bar{S}(w+\Gamma) + \varepsilon\bar{S}'(w+\Gamma)(\lambda z - w(\beta+\gamma+\lambda)) + O(\varepsilon^2). \end{aligned}$$

Note that $\bar{S}'(w+\Gamma)(\lambda z - w(\beta+\gamma+\lambda))$ is a constant multiple of the eigenfunction (3.8). □

Define

$$\begin{aligned} Y(t) &\coloneqq (\theta(t), p_S(t), p_I(t)), \\ \bar{Y} &= (1, 1, 0). \end{aligned}$$

where Y solves the Volz model (1.4) and $\bar{Y}$ is the disease free equilibrium. Assume a multinomial generating function solution (2.25) to (2.9).

Definition 3.2.1. We say that a steady state solution $\bar{S}(w)$ of (2.9) is stable if for all $\varepsilon > 0$, there exists $\delta > 0$ such that, for all time t,

$$\left\| S(t, w, z) - \bar{S}(w) \right\|_X < \varepsilon$$

whenever

$$\left\| S(0, w, z) - \bar{S}(w) \right\|_X < \delta.$$

where $\|\cdot\|_X$ is the norm defined in (2.10). Otherwise we say that $\bar{S}(w)$ is an unstable equilibrium.

Before we prove the instability of $\bar{S}$, we need to prove a quick claim.

Claim 19. *The dot product* $(Y(T) - \bar{Y}) \cdot (w + \Gamma, w - 1, z - 1) = 0$ *for all* $(w, z) \in [-\Gamma, 1 - \Gamma]^2$ *if and only if* $Y(T) = \bar{Y}$.

Proof. The backward direction is trivial, as $0 \cdot v = 0$ for all vectors v. For the forward direction, consider $(Y(T) - \bar{Y}) \cdot (w + \Gamma, w - 1, z - 1) = 0$ for all w, z in the square. Expanding the bracket and computing the dot product gives

$$w(\theta(T) + p_S(T) - 2) + zp_I(T) + 1 + \Gamma\theta(T) - \Gamma - p_S(T) - p_I(T) = 0$$

Since this must be true for all $(w, z) \in [-\Gamma, 1-\Gamma]^2$, each coefficient ($w$, z, and constant terms) must be zero, hence the system to solve is

$$\begin{aligned} \theta(T) + p_S(T) - 2 &= 0 \\ p_I(T) &= 0 \\ 1 + \Gamma\theta(T) - \Gamma - p_S(T) - p_I(T) &= 0 \end{aligned}$$

Substituting the first two equations into the last gives

$$\begin{aligned} 0 &= 1 + \Gamma(2 - p_S(T)) - \Gamma - p_S(T) \\ &= (1+\Gamma) - (1+\Gamma)p_S(T) && = 0 \end{aligned}$$

Hence $p_S(T) = 1$, and the solution to this system is $Y(T) = \bar{Y}$. □

Claim 20. *For $\bar{S}(w)$ a biologically relevant equilibrium, $\left\|\bar{S}(w+c)\right\|_X \geq \left\|\bar{S}(w)\right\|_X$ for all $c > 0$.*

Proof. Expand $\bar{S}(w+c)$

$$\begin{aligned} \bar{S}(w+c) &= \sum_{m\geq 0} s_m (w+c)^m \\ &= \sum_{m\geq 0}\sum_{n\leq m} \binom{m}{n} s_m w^n c^{m-n} \\ &= \sum_{n\geq 0} w^n \sum_{m\geq n} \binom{m}{n} s_m c^{m-n} \\ &= \sum_{n\geq 0} w^n \left[s_n + \sum_{m\geq n+1} \binom{m}{n} s_m c^{m-n} \right] \\ &= \bar{S}(w) + \sum_{n\geq 0} w^n \left[\sum_{m\geq n+1} \binom{m}{n} s_m c^{m-n} \right] \end{aligned}$$

The coefficients $s_m \geq 0$ becaue $\bar{S}$ is biologically relevant, and so

$$\left\|\bar{S}(w+c)\right\|_X = \left\|\bar{S}\right\|_X + \left\| \sum_{m\geq n+1} \binom{m}{n} s_m c^{m-n} \right\|_X .$$

The last step is true because if f and g are power series with non-negative coefficients,

$$\begin{aligned} \|f+g\|_X &= \sum_n |f_n + g_n| \\ &= \sum_n f_n + \sum_n g_n \\ &= \|f\|_X + \|g\|_X \end{aligned}$$

□

Theorem 21. *If $\mathcal{R}_0 > 1$, then the disease-free equilibrium $\bar{S}(w+\Gamma)$ of the nonlinear problem* (2.9) *is unstable.*

Proof. The stability analysis and linearization of the Volz model is given in chapter 2. Let v be the unstable eigenvector of the Volz model, given by (3.14). Since the equilibrium $\bar{Y}$ is unstable when $\mathcal{R}_0 > 1$, with the initial condition $Y(0) = \bar{Y} + \varepsilon v$, there exists an $\varepsilon > 0$, such that, for all $\delta > 0$, there exists a time T such that

$$\left\|Y(0) - \bar{Y}\right\|_2 < \delta$$

and

$$\left\|Y(T) - \bar{Y}\right\|_2 = \varepsilon.$$

where $\|\cdot\|_2$ is the Euclidean norm. We want to show that

$$\left\|S(T, w, z) - \bar{S}(w+\Gamma)\right\|_X \geq C\varepsilon.$$

Take the initial condition (3.13). Then from (2.25) with $S_k = \theta^k$ we have

$$S(T, w, z) = \sum_k p_k \theta(T)^k (p_S(T)(w + \Gamma - 1) + p_I(T)(z + \Gamma - 1) + 1)^k$$

Expand $S(T, w, z)$ with respect to Y about the equilibrium $\bar{Y}$

$$S(T, w, z) = S(T, w, z)\bigg|_{\bar{Y}} + \frac{\partial S}{\partial \theta}\bigg|_{\bar{Y}} (\theta(T) - 1) + \ldots + o(Y - \bar{Y}).$$

Calculating $S(T, w, z)\big|_{\bar{Y}}$ from the above gives

$$\begin{aligned} S(T, w, z)\bigg|_{\bar{Y}} &= \sum_k p_k (1)^k ((1)(w + \Gamma - 1) + (0)(z + \Gamma - 1) + 1)^k \\ &= \sum_k p_k (w + \Gamma)^k \\ &= \bar{S}(w + \Gamma) \end{aligned}$$

then,

$$
\begin{aligned}
&\left\|S(T,w,z)-\bar{S}(w+\Gamma)\right\|_X \\
&=\left\|\frac{\partial S}{\partial \theta}\bigg|_{\bar{Y}}(\theta(T)-1)+\frac{\partial S}{\partial p_S}\bigg|_{\bar{Y}}(p_S(T)-1)+\frac{\partial S}{\partial p_I}\bigg|_{\bar{Y}}p_I(T)+o(Y(T)-\bar{Y})\right\|_X \\
&=\Bigg\|(\theta(T)-1)\sum_k p_k k(w+\Gamma)^k+(p_S(T)-1)\sum_k p_k k(w+\Gamma)^{k-1}(w-1) \\
&\qquad+p_I(T)\sum_k p_k k(w+\Gamma)^{k-1}(z-1)+o(Y(T)-\bar{Y})\Bigg\|_X \\
&=\Bigg\|\bar{S}'(w+\Gamma)\Big((\theta(T)-1)(w+\Gamma)+(p_S(T)-1)(w-1)+p_I(T)(z-1)\Big) \\
&\qquad+o(Y(T)-\bar{Y})\Bigg\|_X \\
&=\left\|\bar{S}'(w+\Gamma)(Y(T)-\bar{Y})\cdot(w+\Gamma,w-1,z-1)+o(Y-\bar{Y})\right\|_X \\
&\geq\left\|\bar{S}'(w+\Gamma)(Y(T)-\bar{Y})\cdot(w+\Gamma,w-1,z-1)\right\|_X-\left\|o(Y(T)-\bar{Y})\right\|_X
\end{aligned}
$$

That is,

$$
\begin{aligned}
\left\|S(T,w,z)-\bar{S}(w+\Gamma)\right\|_X \geq &\left\|\bar{S}'(w+\Gamma)(Y(T)-\bar{Y})\cdot(w+\Gamma,w-1,z-1)\right\|_X \\
&-\left\|o(Y(T)-\bar{Y})\right\|_X
\end{aligned}
$$

Claim 19 asserts that the dot product is non-zero, and since this inequality is true for all w and z,

$$
\begin{aligned}
\left\|S(T,w,z)-\bar{S}(w)\right\|_X &\geq \left\|(Y(T)-\bar{Y})\cdot(w+\Gamma,w-1,z-1)\right\|_\infty\left\|\bar{S}'(w+\Gamma)\right\|_X-\left\|o(Y(T)-\bar{Y})\right\|_X \\
&\geq \varepsilon\|\|(w+\Gamma,w-1,z-1)\|\cos\alpha\|_\infty\left\|\bar{S}'(w)\right\|_X-\left\|o(Y(T)-\bar{Y})\right\|_X \\
&= C\varepsilon-\left\|o(Y(T)-\bar{Y})\right\|_X \\
&\geq \frac{C\varepsilon}{2}
\end{aligned}
$$

where $C := \|\|(w+\Gamma,w-1,z-1)\|\cos\alpha\|_\infty\left\|\bar{S}'(w+\Gamma)\right\|_X$, α is the angle between $(Y(T)-\bar{Y})$ and $(w+\Gamma,w-1,z-1)$, and by definition of $o(\cdot)$ we have $o(Y(T)-\bar{Y})\leq\frac{C}{2}\varepsilon$, when ε sufficiently small. $\square$

Chapter 4

Conclusion

In this thesis, we have developed the PDE Effective Degree model from the ODE model. The PDE derived here is very atypical as it does not have, nor does it require, boundary conditions.

First we have shown the existence of solutions via the method of characteristics, and then the uniqueness of these solutions for properly chosen initial data.

Importantly, we have shown that our model retains the same threshold condition for linear stability, as well as the threshold condition for the nonlinear instability. Future work would include the analysis of the nonlinear stability, and comparative simulations of the original large system of ODES and the PDE.

It is our hope that the method of deriving the PDE for the SIR effective degree model can be used to derive a PDE model for the SIS effective degree model, which differs in that the disease does not confer immunity, and is a more complicated system.

Appendix A

Proof of Theorem 7

Restatement of the theorem: *A solution of the form* (2.14) *with a biologically relevant initial condition remains biologically relevant for all time t that the solution is defined.*

Proof. The goal of this proof is to show that if the PDE (2.9) has a biologically relevant initial condition, then the solution $\tilde{S}(t, w, z)$ stays biologically relevant for all time. This will be accomplished by consideration of the PDE (2.3) in x and y variables, and the relation $S(t, x, y) = \tilde{S}(t, w, z)$. That is, we only need to show that $S(t, x, y)$ is biologically relevant if it has a biologically relevant initial condition.

Let $S(t, x, y)$ solve (2.3), and its power series coefficients be given by $S_{si}(t)$, i.e.

$$S(t, x, y) = \sum_{s\geq 0, i\geq 0} S_{si}(t) x^s y^i.$$

solving

$$\frac{\partial S}{\partial t} = -(\beta+\gamma)\left(y - \frac{\gamma}{\beta+\gamma}\right)S_y + \frac{S_{xy}(t,1,1)}{S_x(t,1,1)}\beta(y-x)S_x. \tag{A.1}$$

Step 1: Solve the PDE for S

First, we need to solve (A.1). Treat $\frac{S_{xy}(t,1,1)}{S_x(t,1,1)}$ as an undetermined function of time, and rewrite (A.1) as a linear PDE. That is,

$$S_t + \beta(x-y)\varphi'(t)S_x + [(\beta+\gamma)y - \gamma]S_y = 0$$

where

$$\varphi(t) := \int_0^t \frac{S_{xy}(s,1,1)}{S_x(s,1,1)}\, ds,$$

so that

$$\varphi'(t) = \frac{S_{xy}(t,1,1)}{S_x(t,1,1)}.$$

Note by definition we have $\varphi(0) = 0$. Introduce the characteristic equations parametrized by t

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= \beta\varphi'(t)x - \beta\varphi'(t)y \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= (\beta+\gamma)y - \gamma \\ \frac{\mathrm{d}S}{\mathrm{d}t} &= 0\end{aligned} \tag{A.2}$$

Solve for y using the integrating factor

$$e^{-(\beta+\gamma)t},$$

such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{-(\beta+\gamma)t}y\right] = -\gamma e^{-(\beta+\gamma)t}.$$

Integrate from 0 to t to get

$$\boxed{y(t) = \left(y_0 - \frac{\gamma}{\beta+\gamma}\right)e^{(\beta+\gamma)t} + \frac{\gamma}{\beta+\gamma}} \tag{A.3}$$

where $y(0) = y_0$. Substituting y into the equation for x gives

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \beta\varphi' x - \beta\varphi'\left(y_0 - \frac{\gamma}{\beta+\gamma}\right)e^{(\beta+\gamma)t} - \frac{\beta\gamma}{\beta+\gamma}\varphi',$$

which can be solved with the integrating factor

$$e^{-\beta\varphi(t)}$$

i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{-\beta\varphi(t)}x\right] = -\beta\varphi'\left(y_0 - \frac{\gamma}{\beta+\gamma}\right)e^{(\beta+\gamma)t-\beta\varphi(t)} - \frac{\beta\gamma}{\beta+\gamma}\varphi' e^{-\beta\varphi(t)}.$$

Integrate from 0 to t

$$e^{-\beta\varphi(t)}x - x_0 = \frac{\gamma}{\beta+\gamma}\left(e^{-\beta\varphi(t)} - 1\right) - \beta\left(y_0 - \frac{\gamma}{\beta+\gamma}\right)\int_0^t \varphi'(s)e^{(\beta+\gamma)s-\beta\varphi(s)}\,ds.$$

We can simplify by doing the last integral by parts

$$-\beta \int_0^t \varphi'(s) e^{(\beta+\gamma)s-\beta\varphi(s)}\, ds = e^{(\beta+\gamma)t-\beta\varphi(t)} - 1 - (\beta+\gamma)\int_0^t e^{(\beta+\gamma)s-\beta\varphi(s)}\, ds.$$

Thus,

$$\begin{aligned} e^{-\beta\varphi(t)}x - x_0 = {} & \frac{\gamma}{\beta+\gamma}\big(e^{-\beta\varphi(t)} - 1\big) \\ & + \left(y_0 - \frac{\gamma}{\beta+\gamma}\right)\left(e^{(\beta+\gamma)t-\beta\varphi(t)} - 1 - (\beta+\gamma)\int_0^t e^{(\beta+\gamma)s-\beta\varphi(s)}\, ds\right) \end{aligned}$$

Solve for x, and recall η defined in (2.16)

$$\boxed{\begin{aligned} x(t) = {} & x_0 e^{\beta\varphi(t)} + \frac{\gamma}{\beta+\gamma}\big(1 - e^{\beta\varphi(t)}\big) \\ & - \left(y_0 - \frac{\gamma}{\beta+\gamma}\right)\left(e^{(\beta+\gamma)t} - e^{\beta\varphi(t)} - (\beta+\gamma)e^{\beta\varphi(t)}\int_0^t e^{(\beta+\gamma)s-\beta\varphi(s)}\, ds\right). \end{aligned}} \tag{A.4}$$

The third characteristic equation from (A.2) is

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 0$$

which means that S is a constant along the characteristic curve $(x(t), y(t))$, given by (A.3) and (A.4) respectively. In particular

$$S(t, x(t), y(t)) = S(0, x(0), y(0))$$

i.e.,

$$\boxed{S(t, x, y) = S_0(x_0(t, x, y), y_0(t, y))} \tag{A.5}$$

where S_0 is the power series of the initial distribution of susceptibles

$$S_0(x, y) = \sum_{s,i} x^s y^i S_{si}(0).$$

Rearrange (A.3) to solve for $y_0(t,y)$, and then substitute y_0 into (A.4) to solve for $x_0(t,x,y)$

$$x_0 = xe^{-\beta\varphi(t)} + \frac{\gamma}{\beta+\gamma}\left(1 - e^{-\beta\varphi(t)}\right) - \left(y - \frac{\gamma}{\beta+\gamma}\right)\left(e^{-\beta\varphi(t)} - e^{-(\beta+\gamma)t} - (\beta+\gamma)\int_0^t e^{-(\beta+\gamma)(t-s)-\beta\varphi(s)}\,ds\right)$$

Define $\eta(t)$ as

$$\eta(t) := -e^{-\beta\varphi(t)} + e^{-(\beta+\gamma)t} + (\beta+\gamma)\int_0^t e^{-(\beta+\gamma)(t-s)-\beta\varphi(s)}\,ds. \tag{A.6}$$

In summary, x_0 and y_0 are

$$x_0 = xe^{-\beta\varphi(t)} + \frac{\gamma}{\beta+\gamma}\left(1 - e^{-\beta\varphi(t)}\right) + \left(y - \frac{\gamma}{\beta+\gamma}\right)\eta \tag{A.7}$$

$$y_0 = \left(y - \frac{\gamma}{\beta+\gamma}\right)e^{-(\beta+\gamma)t} + \frac{\gamma}{\beta+\gamma} \tag{A.8}$$

Substitute x_0 and y_0 into (A.5)

$$S(t,x,y) = S_0\left(xe^{-\beta\varphi(t)} + \frac{\gamma}{\beta+\gamma}\left(1 - e^{-\beta\varphi(t)}\right) + \left(y - \frac{\gamma}{\beta+\gamma}\right)\eta, \left(y - \frac{\gamma}{\beta+\gamma}\right)e^{-(\beta+\gamma)t} + \frac{\gamma}{\beta+\gamma}\right)$$

As in section 2.2.2, solving for the compatibility condition ($\varphi'(t) = \frac{S_{xy}(1,1)}{S_x(1,1)}$) gives the system

$$\begin{pmatrix}\varphi'(t)\\ \eta'(t)\end{pmatrix} = \begin{pmatrix} f(t,\varphi,\eta)e^{-(\beta+\gamma)t} + h(t,\varphi,\eta)\eta \\ f(t,\varphi,\eta)\beta e^{-(\beta+\gamma)t-\beta\varphi} + [h(t,\varphi,\eta)\beta e^{-\beta\varphi} - \beta - \gamma]\eta \end{pmatrix}$$
$$\varphi(0) = 0,\ \ \eta(0) = 0. \tag{A.9}$$

but where f and h are given by

$$f(t,\varphi) = \frac{\frac{\partial^2 S_0}{\partial x_0 \partial y_0}}{\frac{\partial S_0}{\partial x_0}}\left(\frac{\beta e^{-\beta\varphi(t)} + \beta\eta + \gamma}{\beta+\gamma}, \frac{\beta e^{-(\beta+\gamma)t} + \gamma}{\beta+\gamma}\right)$$

$$h(t,\varphi) = \frac{\frac{\partial^2 S_0}{\partial x_0^2}}{\frac{\partial S_0}{\partial x_0}}\left(\frac{\beta e^{-\beta\varphi(t)} + \beta\eta + \gamma}{\beta+\gamma}, \frac{\beta e^{-(\beta+\gamma)t} + \gamma}{\beta+\gamma}\right).$$

The claims below are equivalent to Claims 3 and 4, and so are restated in this context without proof.

Claim 22. *If $S_0(x_0, y_0)$ has a power series with all non-negative coefficients, then $f(t, \varphi, \eta), h(t, \varphi, \eta) \geq 0$.*

Claim 23. *The functions $\varphi'(t)$ and $\eta(t)$, defined in* (A.9) *and* (A.6) *respectively, are non-negative whenever $f, h \geq 0$.*

Step 2: Show that the solution is biologically relevant

Let $S_0(x, y)$ be biologically relevant, with coefficients $S_{si} \geq 0$. Then the series expansion of the solution given by (A.5) is

$$S(t, x, y) = \sum_{s,i} (x_0(t, x, y))^s (y_0(t, y))^i S_{si}$$

From equations (A.7) and (A.8)

$$\begin{aligned} x_0 &= ax + b_1 y + c_1 \\ y_0 &= b_2 y + c_2 \end{aligned}$$

where

$$\begin{aligned} a &= e^{-\beta\varphi(t)} \\ b_1 &= \left(e^{-(\beta+\gamma)t} - e^{-\beta\varphi(t)} + (\beta + \gamma) \int_0^t e^{-(\beta+\gamma)(t-s)-\beta\varphi(s)} \, ds \right) \\ b_2 &= e^{-(\beta+\gamma)t} \\ c_1 &= \frac{\gamma}{\beta + \gamma} \left(1 - e^{-(\beta+\gamma)t} - (\beta + \gamma) \int_0^t e^{-(\beta+\gamma)(t-s)-\beta\varphi(s)} \, ds \right) \\ c_2 &= \frac{\gamma}{\beta + \gamma} \left(1 - e^{-(\beta+\gamma)t} \right) \end{aligned} \tag{A.10}$$

Then we have

$$S(t, x, y) = \sum_{s,i} (ax + b_1 y + c_1)^s (b_2 y + c_2)^i S_{si}. \tag{A.11}$$

If for all s, i, we have $a, b_1, b_2, c_1, c_2 \geq 0$, then by the multinomial theorem, expand (A.11) as

$$\sum_{j,k} x^j y^k A_{jk}$$

where $A_{jk} \geq 0$. Clearly, $a, b_2, c_2 \geq 0$ by definition. Notice that $b_1 = \eta$ from (A.6), and $\eta \geq 0$ by Claim 23. It remains to show that $c_1 \geq 0$. We have

$$c_1 = \frac{\gamma}{\beta+\gamma}\left(1 - e^{-(\beta+\gamma)t} - (\beta+\gamma)\int_0^t e^{-(\beta+\gamma)(t-s)-\beta\varphi(s)}\ ds\right) \geq 0$$

because

$$\begin{aligned}(\beta+\gamma)\int_0^t e^{-(\beta+\gamma)(t-s)-\beta\varphi(s)}\ ds &\leq (\beta+\gamma)\int_0^t e^{-(\beta+\gamma)(t-s)}\ ds \\ &= \left[1 - e^{-(\beta+\gamma)t}\right]\end{aligned}$$

and so

$$\begin{aligned}1 - e^{-(\beta+\gamma)t} - (\beta+\gamma)\int_0^t e^{-(\beta+\gamma)(t-s)-\beta\varphi(s)}\ ds &\geq 1 - e^{-(\beta+\gamma)t} - 1 + e^{-(\beta+\gamma)t} \\ &= 0\end{aligned}$$

i.e.

$$c_1 \geq 0.$$

Since all the terms in (A.10) are non-negative, we can use the multinomial theorem to write (A.11) as

$$S(t,x,y) = \sum_{j,k} x^j y^k A_{jk}$$

where $A_{jk} \geq 0$, and thus, $S(t,x,y)$ is a biologically relevant solution. □

Appendix B

Motivating the Reduction to the Volz Model

The model (2.3) only tracks the amount of nodes with a given number of infected and susceptible neighbours to reduce the complexity of the system, whereas the full model presented in this section also tracks the recovered neighbours of the nodes. The so-called "full" effective degree SIR Model is

$$\frac{\mathrm{d}}{\mathrm{d}t}S_{sir} = -\beta i S_{sir} + \gamma[(i+1)S_{s,i+1,r-1} - iS_{sir}] + \frac{\sum_{sir}\beta si S_{sir}}{\sum_{sir} sS_{sir}}[(s+1)S_{s+1,i-1,r} - sS_{sir}].$$

Let

$$S(t,x,y,z) = \sum_{sir} x^s y^i z^r S_{sir}. \tag{B.1}$$

Then,

$$\begin{aligned}
\frac{\partial S}{\partial t} &= \sum_{sir} x^s y^i z^r \frac{\mathrm{d}S_{sir}}{\mathrm{d}t} \\
&= -\beta \sum_{sir} x^s y^i z^r i S_{sir} + \gamma \sum_{sir}[(i+1)S_{s,i+1,r-1}x^s y^i z^r - iS_{sir}x^s y^i z^r] \\
&\quad + \left(\frac{\sum_{sir}\beta si S_{sir}}{\sum_{sir} sS_{sir}}\right)\sum_{sir}[(s+1)S_{s+1,i-1,r}x^s y^i z^r - sS_{sir}x^s y^i z^r] \\
&= -\beta y S_y + \gamma(z-y)S_y + \frac{S_{xy}(1,1)}{S_x(1,1)}(y-x)S_x.
\end{aligned}$$

That is,

$$\boxed{\frac{\partial S}{\partial t} = -\beta y S_y + \gamma(z-y)S_y + \beta \frac{S_{xy}(1,1)}{S_x(1,1)}(y-x)S_x}$$

Notice that $z = 1$ gives the model (2.3).

Assume that the number of susceptible, infected, and recovered neighbours of a node follows a multinomial distribution:

$$(s, i, r) \sim Multinomial\Big(s + i + r, p_S(t), p_I(t), p_R(t)\Big)$$

where $p_S(t)$ is the probability of a susceptible node having a susceptible neighbour at time t, p_I is the probability that a susceptible node has an infected neighbour at time t, etc., and

$$p_S(t) + p_I(t) + p_R(t) = 1.$$

Then

$$S_{sir} = p_k S_k(t) \binom{k}{s, i, r} p_S^s p_I^i p_R^r$$

where p_k is the network degree distribution, $S_k(t)$ is the proportion of suscpetible nodes having degree k at time t, and $k = s + i + r$ is the degree of a node.

The generating function (B.1) is

$$\begin{aligned}
S(t, x, y, z) &= \sum_{sir} x^s y^i z^r S_{sir} \\
&= \sum_{k=0}^{\infty} \sum_{s+i+r=k} x^s y^i z^r p_k S_k \binom{k}{s, i, r} p_S^s p_I^i p_R^r \\
&= \sum_{k=0}^{\infty} p_k S_k \sum_{s+i+r=k} \binom{k}{s, i, r} (p_S x)^s (p_I y)^i (p_R z)^r \\
&= \sum_{k=0}^{\infty} p_k S_k (p_S x + p_I y + p_R z)^k \quad \text{(by the multinomial theorem)}
\end{aligned}$$

that is,

$$S(t, x, y, z) = \sum_{k=0}^{\infty} p_k S_k (x p_S + y p_I + z p_R)^k$$

Taking $z = 1$ and $p_R = 1 - p_S - p_I$ gives

$$S(t, x, y) = \sum_{k=0}^{\infty} p_k S_k (p_S(x-1) + p_I(y-1) + 1)^k$$

Making the substitution $x = w + \Gamma$, and $y = z + \Gamma$ motivates the choice of (2.25).

Bibliography

[1] J.H. Lindquist, J. Ma, P. van den Driessche, and F.H. Willeboordse. Effective degree network disease models. *Journal of Mathematical Biology*, 62:143–164, 2011.

[2] W.O. Kermack and A.G. McKendrick. A contribution to the mathematical theory of epidemics. *Proc. R. Soc. Lond. A*, 115:700–721, 1927.

[3] P. Érdi and J. Tóth. In *Mathematical Models of Chemical Reactions: Theory and Applications of Deterministic and Stochastic Models*, chapter 1. Manchester University Press, 1989.

[4] F. Brauer. Compartmental models in epidemiology. In *Mathematical Epidemiology*, pages 19–79. Springer Berlin Heidelberg, Berlin, Heidelberg, 2008.

[5] M.E.J. Newman. Spread of epidemic disease on networks. *Phys. Rev. E*, 66:016128, 2002.

[6] E. Volz. SIR dynamics in random networks with heterogeneous connectivity. *Journal of Mathematical Biology*, 56:293–310, 2008.

[7] T. Harko, F.S.N. Lobo, and M.K. Mak. Exact analytical solutions of the Susceptible-Infected-Recovered (SIR) epidemic model and of the SIR model with equal death and birth rates. *Applied Mathematics and Computation*, 236:184–194, 2014.

[8] S. Riley, C. Fraser, C.A. Donnelly, et al. Transmission dynamics of the etiological agent of SARS in Hong Kong: impact of public health interventions. *Science*, 300:1961–1966, 2003.

[9] N. Johnson, A. Kemp, and S. Kotz. *Univariate Discrete Distributions*. John Wiley & Sons, 3rd edition, 2005.

[10] P. Hartman. *Ordinary Differential Equations.* Birkhäuser, Boston, Basel, Stuttgart, 2nd edition, 1982.

[11] I.N. Bronshtein, K.A. Semendyayev, G. Musiol, and H. Muehlig. *Handbook of Mathematics.* Springer-Verlag Berlin Heidelberg, 5th edition, 2007.

[12] N. Dunford and J.T. Schwartz. *Linear Operators. Part I: General Theory.* Interscience Publishers, New York, 1958.