Qualitative Properties of the Anisotropic Manev Problem

by

Manuele Santoprete
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We accept this dissertation as conforming to the required standard.

Dr. F. N. Diacu, Supervisor (Department of Mathematics & Statistics)

Dr. R. Illner, Departmental Member (Department of Mathematics & Statistics)

Dr. R. Edwards, Departmental Member (Department of Mathematics & Statistics)

Dr. F.I. Cooperstock, Outside Member (Department of Physics & Astronomy)

Dr. E. Pérez Chavela, External Examiner (Departamento de Matemáticas, Universidad Autónoma Metropolitana-Iztapalapa)

Supervisor: Dr. F.N. Diacu.

Abstract

In this dissertation we study the anisotropic Manev problem that describes the motion of two point masses in an anisotropic space under the influence of a Newtonian force-law with a relativistic correction term. The dynamic of the system under discussion is very complicated and we use various methods to find a qualitative description of the flow.

One of the strategies we use is to study the collision and near collision orbits. In order to do that we utilize McGehee type transformations that lead to an equivalent analytic system with an analytic energy relation. In these new coordinates the collisions are replaced by an analytic two-manifold: the so called collision manifold. We focus our attention on the heteroclinic orbits connecting fixed points on the collision manifold and on the homoclinic orbit to the equator of the mentioned manifold. We prove that as the anisotropy is introduced only four heteroclinic orbits persist and we show the exixtence of infinitely many transversal homoclinic orbits using a suitable generalization of the Poincaré-Melnikov method.

Another strategy we apply is to study the symmetric periodic orbits of the system. To tackle this problem we follow two different approaches. First we apply the Poincaré continuation method and we find symmetric periodic orbits for small values of the anisotropy. Then we utilize a direct method of the calculus of variations, namely the lower semicontinuity method, and we prove the existence of symmetric periodic orbits for any value of the anisotropy parameter.

In the last chapter we use the Killing's equation in an unusual way to prove

that the anisotropic Kepler problem (that can be considered a particular case of the Manev) does not have first integrals linear in the momentum.

Examiners

Dr. F.N. Diacu, Supervisor (Department of Mathematics & Statistics)

Dr. R. Illner, Departmental Member (Department of Mathematics & Statistics)

Dr. R. Edwards, Departmental Member (Department of Mathematics & Statistics)

Dr. F.I. Cooperstock, Outside Member (Department of Physics & Astronomy)

Dr. E. Pérez Chavela, External Examiner (Departamento de Matemáticas, Universidad Autónoma Metropolitana-Iztapalapa)

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To Lisa and Bin

and

to my parents

We shall not cease from exploration
And the end of all our exploring
Will be to arrive where we started
And know the place for the first time.
Through the unknown, unremembered gate
When the last of earth left to discover
Is that which was the beginning;
At the source of the longest river
The voice of the hidden waterfall
And the children in the apple-tree
Not known, because not looked for
But heard, half-heard, in the stillness
Between two waves of the sea.

From *Little Gidding*
T.S. ELIOT

Nature and Nature's laws lay hid in night:
God said, Let Newton be! and all was light.

Epitaph on Isaac Newton
ALEXANDER POPE

Chapter 1

Introduction

Symmetry and anisotropy are very important features that characterize natural phenomena and the models that are used to describe them. Symmetry, on one hand, tends to simplify the models while anisotropy, on the other hand, complicates them. Anisotropic models describe many physical systems and their study is a very interesting problem.

For example the motion of particles and satellites around planets (i.e. in a multipolar gravitational field) is a very well known and important problem (see for example [30, 31]).

Furthermore motion of charged particles orbiting around black holes perturbed by electric and magnetic fields (see [65]) or gravitational waves [51] are other fascinating problems.

Galactic models described by galactic potential are also an interesting example of anisotropic problems (see for example [68] and references therein).

The type of anisotropic problems that we study in this dissertation have been introduced by Gutzwiller (see [34, 35, 36, 37, 38, 39, 40]) in the 1970s. His purpose was to understand better the relationship between classical and quantum mechanics. Gutzwiller analyzed the anisotropic Kepler problem that was later studied by

Devaney in [21, 22] and by Casasayas and Llibre in [8].

In this dissertation we analyze the anisotropic Manev problem that describes the motion of two point masses in an anisotropic configuration plane interacting with a potential of the form

$$U(\mathbf{q}) = \frac{1}{\sqrt{\mu x^2 + y^2}} + \frac{b}{\mu x^2 + y^2} \tag{1.1}$$

where $\mathbf{q} = (x, y)$.

Florin Diacu suggested the study of the anisotropic Manev problem in 1995, hoping to find connections between classical, quantum, and relativistic mechanics.

Recently another anisotropic problem, of the same kind of the one discussed here, has been introduced by Vasile Mioc, Ernesto Pérez-Chavela and Magda Stavinschi (see [56]). They analyzed the anisotropic Schwarzchild problem that presents some similarities and some differences with both the anisotropic Kepler problem and the anisotropic Manev problem. However we will not discuss this particular problem.

The origins of the Manev problem lie in the work of Newton, who introduced it in *Principia* aiming to understand the apsidal motion of the moon (see [19, 24]).

The Manev potential can be used to model various natural phenomena, in celestial mechanics, astrophysics and atomic physics. We refer the reader to [52] for a history of the problem and a summary of the main applications. We will just mention that, in the 1930s, Manev found that this potential allows a good theoretical justification of the perihelion advance of Mercury and of the other inner planets as well as of the motion of the Moon. Furthermore, in the 1970s Hagiara [42] pointed out that it describes the precession of the perihelion of Mercury with the same accuracy as general relativity. Also, another important application of the Manev problem is to the relativistic Hydrogen atom (see [73]). Indeed the classical (i.e. non quantistic) dynamics of this problem can be described using the Manev potential.

The anisotropic Manev problem can thus be considered as a relativistic version of the anisotropic Kepler problem, but can also be seen as describing some gravitational models with anisotropic gravitational constant (see [56, 77, 79], note however that these models do not seem to have great physical relevance).

In this dissertation, even if the discussion and the analysis of the physical applications of the anisotropic Manev problem is interesting, we are mainly concerned with the mathematical aspects of the model and not with its physical interpretations.

A number of interesting results were already obtained in [18] where the flow on and near the collision manifold was studied. In particular the authors found a positive-measure set of collisions formed by frontal homothetic, frontal nonhomothetic, spiralling and oscillatory collisions. Oscillatory collisions do not occur in any of the Kepler, anisotropic Kepler, Manev, or anisotropic Schwarzchild problems. This unintuitive type of motion is characteristic of the anisotropic Manev problem.

In this thesis we gain a better understanding of the complicated global dynamics of this problem. This is realized using various methods. On one hand we study the collision and near collision orbits, on the other we analyze the symmetric periodic orbits. Moreover we also describe some results concerning the orbits at infinity and the existence of linear integrals (this last result holds for $b = 0$, i.e. for the anisotropic Kepler problem).

The dissertation is organized as follows. In the next chapter we recall some known results on the Manev problem, and describ some new ones. Introducing McGehee coordinates we replace the collision singularity with a two dimensional analytic manifold and we study the flow on and near it. We also describe the behavior of the solutions at infinity studying the flow on and near the infinity manifold. Then we analyze the global flow and introduce suitable action angle variables. This chapter is introductory in nature and it is needed in the following in order to use perturbation theory.

In Chapter 3 we introduce the anisotropic Manev problem and some of its main features. In Section 3.2 we write the equations of motion and we mention some general properties. In the following section we show that, even if the rotational symmetry is broken by the presence of the anisotropy, the equations of motion have a discrete group of symmetry that is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and we describe some properties of the symmetric solutions that will be useful to find symmetric periodic orbits. Then in Section 3.4, using McGehee coordinates the collision manifold is introduced and the flow on it is studied following the approach used in [18]. In the subsequent section the infinity manifold is defined as in [18], however the flow on it is analyzed in more detail proving that there are no saddle connections. As a consequence of this, a theorem is proven to show that the flow on the infinity manifold is structurally stable. In the last section of this chapter we write the equations of motion in suitable action angle variables.

Chapter 4 is devoted to one of the main result of the thesis, that is the existence of infinitely many transversal homoclinic orbits (with possibly the appearance of chaos). After the Overview, in Section 4.2, we recall some (local) results, concerning the flow near the collision manifold, obtained in [18]. In the subsequent section we describe the heteroclinic orbits. Physically they correspond to ejection-collision orbits. In the Manev problem there is a continuum of heteroclinic orbits connecting fixed points of the collision manifolds, however as soon as a small anisotropy appears most of the heteroclinic orbits are destroyed. We prove that only four heteroclinic orbits persist for any value of the anisotropy. Section 4.4 is dedicated to the physical interpretation of the local result in [18], i.e. to the description of the different kinds of collision orbits. We show some collision orbits obtained numerically for each nontrivial class of collisions, in particular showing an example of oscillatory collision. In the following section we introduce a perturbative approach that will be used in the last two Sections of this Chapter. As remarked in [64], the perturbation analysis of [25, 64] cannot be used to study ejection-collision solutions. However, in this work this difficulty is surpassed with the help of McGehee-type coordinates, which allow us to view the anisotropic Manev problem as a perturbation of the classical Manev case. In this section we also find explicitly the equations describing the manifolds of orbits homoclinic to the periodic orbits on the equator of the collision manifold. In Section 4.6, using an approach inspired by [12, 13], which works in some degenerate cases, as for example those of unstable non-hyperbolic points or critical points located at infinity (see [14, 15, 16, 25]), we develop a suitable extension of the Poincaré-Melnikov method, which can be used to prove the existence of transversal homoclinic orbits to a periodic one. It is interesting to note that our result extends the one obtained in [14, 15, 16] for a non-Hamiltonian system that has negatively and positively asymptotic sets to a nonhyperbolic periodic orbit. In the present context the asymptotic sets are the stable and the unstable manifolds. In the last section of the Chapter we apply the Melnikov method we developed in the previous section. Computing the Melnikov integrals using the method of residues we find that there are infinitely many simple zeroes and therefore we prove the existence of infinitely many transversal homoclinic orbits to the periodic orbits on the equator of the collision manifold. This possibly implies the existence of a chaotic dynamics.

In Chapter 5 we study the symmetric periodic orbits using the Poincaré continuation method, that is a perturbative technique that was first introduced by Poincaré in his monumental work Les Méthods Nouvelles de la Mécanique Céleste [62]. The idea of this method is to use a known periodic solution (of the unperturbed system) and, by small changes of the parameter and of the initial conditions, continue the known solution. In this case we use a version of the continuation method similar to the ones used in [4, 55, 76], in which the continuation method is applied to find symmetric periodic orbits. In Section 5.2 we find symmetric periodic orbits of the "second kind"(i.e. the non-circular ones), using suitable action-angle coordinates that are discussed in the previous chapters. The use of suitable coordinates to study symmetric periodic orbits is inspired by the articles [4, 55]. In the subsequent section we prove the existence of orbits of the first kind (i.e. the circular ones) using Cartesian coordinates and adapting the techniques used in [76].

Chapter 6 is devoted to proving the existence of symmetric periodic orbits using variational techniques. The idea of using variational principles to obtain periodic orbits for n-body-type particle systems can be traced back to Poincaré [62]. We use the so-called lower semicontinuity method (developed mostly by Tonelli see [74]) that is a direct method of calculus of variations. This method has been recently used to obtain new (symmetric) periodic orbits in the classical n-body problem (see [9]). But unlike the Newtonian case, the Manev force is "strong" (as defined in [32]), so the variational method is easier to apply in our situation than in the Newtonian one. This is because in the Manev case we do not have to deal with the difficulty of avoiding collision orbits, since they have infinite action and therefore cannot be minimizers. In Section 6.2 some preliminary notations and definitions are introduced. In particular we show that the anisotropic Manev problem satisfies the strong force conditions and that the spaces of symmetric paths we discuss are Sobolev. Furthermore we introduce the winding number and we classify the paths according to it. In the following section the action principle and some lemmas that are needed in order to use the variational method are presented. In particular we make sure that the solutions we find using this variational technique are solutions in the classical sense. Moreover we exclude the possibility that the minimizers are found when the bodies are at infinite distance from each other and the possibility that minimizers are collision paths. Section 6.3 is dedicated to describing some properties of the lower semicontinuous functions, while the last section of the chapter contains the statement and the proof of the existence of symmetric periodic orbits. The theorem shows that for each period $T > 0$, for each space of symmetric paths and for each homotopy class, an absolute minimizer exists and it is a solution in the classical sense. The proof of the theorem is based on the idea of lower semicontinuity and on the properties discussed in the previous section. We also present some symmetric periodic orbits obtained numerically.

In Chapter 7 we present some results that hold for the anisotropic Kepler problem, considered as a particular case of the anisotropic Manev problem with $b = 0$. In Section 7.2 we introduce some preliminary notions. In particular we observe that showing that the Killing equations do not have nontrivial solutions is a way to prove the non-existence of linear integrals. In the subsequent section we use some transformations and we rewrite the initial system as a geodesic flow on a surface. In Section 7.4 we finally show that the Killing's equations do not have nontrivial solutions. This proves that there are no linear integrals in the momentum.

Either the well was very deep, or she fell very slowly, for she had plenty of time to look about her, and to wonder what was going to happen next.

Alice's Adventures in Wonderland
LEWIS CARROLL

Chapter 2

The Manev Problem

2.1 Overview

In this chapter we summarize some known facts about the Manev problem and we also add some new results. The purpose of this chapter is to give the basis for the work on the anisotropic Manev problem. This is because some of the techniques used in this thesis are borrowed from perturbation theory and thus understanding the unperturbed problem (i.e. the Manev problem) is of fundamental importance. Most of the material in this chapter can be found in [23, 19, 24]. Some results concerning the topology of the invariant sets of the problem under discussion can be found in [52]. For a detailed account of the general theory of the action-angle variables see [2], while for the application of the action-angle variables to the Manev problem the reader is referred to [25, 64].

In the next section we define the Manev problem and we write its equation of motion giving a general picture of the problem. In Section 2.3 we introduce the collision manifold and describe the flow on and near it. In Section 2.4 we analyze the flow at infinity introducing the infinity manifolds. In Section 2.5 we describe the flow on non-negative energy levels. Finally, in the last section we introduce action-angle variables for the problem under discussion.

2.2 The Equations of Motion

Consider two interacting bodies P_1 and P_2, of mass m_1 and m_2 respectively, such that the potential energy of the interaction depends only on the distance between them. If $\mathbf{q}_1$ and $\mathbf{q}_2$ are the position vectors of P_1 and P_2 respectively, then the Lagrangian of such a system can be written as

$$L_0 = \frac{1}{2}m_1\dot{\mathbf{q}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{q}}_2^2 - U_0(\|\mathbf{q}_1 - \mathbf{q}_2\|). \tag{2.1}$$

Let $\mathbf{q} \equiv \mathbf{q}_1 - \mathbf{q}_2$ be the relative position vector, and let the origin of the coordinate system be at the center of mass, i.e. $m_1\,\mathbf{q}_1 + m_2\,\mathbf{q}_2 = 0$. These two equations give

$$\mathbf{q}_1 = \frac{m_2\mathbf{q}}{m_1+m_2}, \qquad \mathbf{q}_2 = -\frac{m_1\mathbf{q}}{m_1+m_2}. \tag{2.2}$$

Substitution in the Lagrangian gives

$$L_0 = \frac{1}{2}m\dot{\mathbf{q}}^2 - U_0(\|\mathbf{q}\|) \tag{2.3}$$

where $m = m_1m_2/(m_1+m_2)$ is called *reduced mass*. The function (2.3) is formally identical with the Lagrangian of a particle of mass m moving in an external field $U_0(\|\mathbf{q}\|)$ which is symmetrical about a fixed origin. Thus the problem of the motion of two interacting particles is equivalent to that of the motion of one particle in a given external field $U_0(\|\mathbf{q}\|)$. From the solution $\mathbf{q}(t)$ of this problem the paths $\mathbf{q}_1(t)$ and $\mathbf{q}_2(t)$ of the two particles, relative to their common center of mass, can be obtained by means of (2.2).

Thus, setting $m = 1$, the Hamiltonian describing two bodies interacting with the Manev potential can be reduced to:

$$H_0 = \frac{1}{2}\mathbf{p}^2 + U_0(\|\mathbf{q}\|) \tag{2.4}$$

where H_0 is defined on $(\mathbb{R}^2 - \{0\}) \times \mathbb{R}^2$, $\mathbf{q} = (x, y)$ is the *configuration* of the system of two particles and $\mathbf{p} = (p_x, p_y)$ is the *momentum*. Moreover

$$U_0(\mathbf{q}) = -\frac{1}{r} - \frac{b}{r^2} \tag{2.5}$$

is the potential energy, where $r = \sqrt{x^2 + y^2} = \|\mathbf{q}\|$ and b is a positive constant. Then the Hamiltonian equations become:

$$\begin{cases} \dot{\mathbf{q}} = \mathbf{p} \\ \dot{\mathbf{p}} = -\nabla U_0 \end{cases} \tag{2.6}$$

where the dot denotes the derivative with respect to t.

These differential equations define the Manev problem.
The Hamiltonian function H_0 has the property $H_0(\mathbf{q}, \mathbf{p}) = h$ (constant), i.e. it is a first integral, called the integral of energy. There exists another first integral that is the angular momentum $\mathbf{K} = \mathbf{q(t)} \times \mathbf{p(t)}$. Moreover, since the two integrals are independent and in involution, i.e. $\{H_0, \mathbf{K}\} = 0$, the equations (2.6) are integrable by quadratures.[1] Indeed the Liouville-Arnold Theorem, applied to the Manev system says

Theorem 2.1 (Liouville-Arnold). *The Hamiltonian system (2.6) has the Hamiltonian H_0 and the angular momentum $\mathbf{K}$ as two independent first integrals in involution. Consider the level sets of the functions $H_0, \mathbf{K}$*

$$I_{hc} = \{(\mathbf{q}, \mathbf{p}) | H_0(\mathbf{q}, \mathbf{p}) = h, K = |\mathbf{K}| = c\}$$

Assume that the functions H_0, $\mathbf{K}$ are independent on I_{hc} (i.e., the 1-forms dH and $d\mathbf{K}$ are linearly independent at each point of I_{hc}). Then

(i) I_{hc} is a smooth invariant two-manifold under the phase flow with Hamiltonian H_0.

(ii) If the manifold I_{hc} is compact and connected, then it is diffeomorphic to the 2-torus $\mathbb{T}^2 = \{(\phi_1, \phi_2) mod\ 2\pi\}$.

[1] Integration by quadratures of a system of differential equations is the search for its solutions by a finite number of "algebraic" operations (including inversion of functions) and "quadratures", i.e., calculation of integrals of known functions.

(iii) The phase flow with Hamiltonian H_0 determines a conditionally periodic motion on I_{hc}, which means that, in angular coordinates (ϕ_1, ϕ_2), we have

$$\dot{\phi}_1 = \omega_1(H_0, K), \qquad \dot{\phi}_2 = \omega_2(H_0, K)$$

(iv) The canonical equations (2.6) are integrable by quadratures.

For more details about Hamiltonian systems and the proof of the previous theorem see [1, 2, 3, 49]. For an analysis of the topology of the invariant sets I_{hc} see [52].

Since $U : (\mathbb{R}^2 - \{0\}) \to \mathbb{R}$ is real analytic, standard results of differential-equation theory guarantee, for any initial data $(\mathbf{q}(0), \mathbf{p}(0)) \in (\mathbb{R}^2 - \{0\}) \times \mathbb{R}$, the existence and uniqueness of analytic solutions defined on a maximal interval (t^{*-}, t^{*+}), where $-\infty \leq t^{*-} < 0 < t^{*+} \leq \infty$. If either $t^{*-} > -\infty$ or $t^{*+} < \infty$, the solution is said to experience a *singularity.*

In the Manev problem all the singularities are due to collisions, as can be shown imitating the proof used in the classical Kepler problem [80].

A solution $(\mathbf{q}(t), \mathbf{p}(t))$ of Equations (2.6) is called a *collision* (resp. *ejection*) *solution* if there exist a t^{*+} such that $\mathbf{q}(t) \to 0$ as $t \to t^{*+}$ (resp. there exist a t^{*-} such that $\mathbf{q}(t) \to 0$ as $t \to t^{*-}$). Collision and escape solution are of particular interest because the whole qualitative structure of the phase space depends on their behavior.

There are other solutions that play a role similar to the collision and ejection orbits, i.e. they determine the qualitative behavior of the whole phase space. They are the escape and capture solutions. We say that a solution $(\mathbf{q}(t), \mathbf{p}(t))$ is an *escape* (resp. *capture*) *solution* if $\|\mathbf{q}\| \to \infty$ when $t \to \infty$ (resp. $t \to -\infty$)

2.3 The Collision Manifold

In order to remove the collision singularity and to study collision and near collision singularities (see [23, 19]) we introduce a change of variables developed by McGehee [53]. The idea is to blow-up the collision singularity, paste instead a manifold and extend the phase space to it. Of course such a manifold is fictitious in the sense that the flow on it does not represent orbits that have physical reality. However the flow on the collision manifold, due to the continuity of the solutions with respect to initial data, gives information on the flow near collisions. Consider the coordinate transformations

$$\begin{cases} r = |\mathbf{q}| \\ \theta = \arctan(y/x) \\ v = \dot{r}r = (xp_x + yp_y) \\ u = r^2\dot{\theta} = (xp_y - yp_x), \end{cases} \tag{2.7}$$

and the rescaling of time

$$d\tau = r^{-2}dt. \tag{2.8}$$

Composing these transformations, which are analytic diffeomorphisms in their respective domains, system (2.6) becomes (see [23] for more details)

$$\begin{cases} r' = rv \\ v' = 2r^2h + r \\ \theta' = u \\ u' = 0 \end{cases} \tag{2.9}$$

and the energy relation (2.4) takes the form

$$u^2 + v^2 - 2r - 2b = 2r^2h, \tag{2.10}$$

where the new variables $(r, v, \theta, u) \in (0, \infty) \times \mathbb{R} \times S^1 \times \mathbb{R}$ depend on the fictitious time τ and the prime denotes differentiation with respect to τ. Note that the change of variables is not canonical so system (2.9) is not Hamiltonian.

Now the vector field in (2.9) is analytic on the boundary $r = 0$, since r no longer occurs in the denominators of the vector field. The *collision manifold* is the

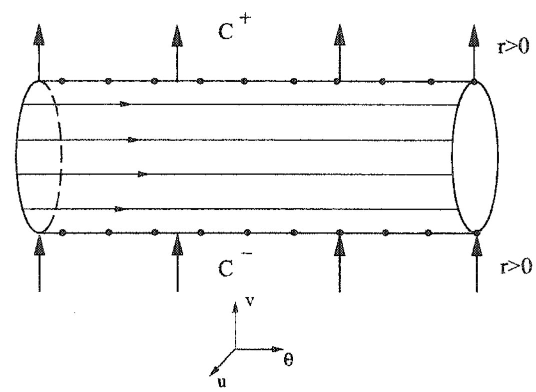


Figure 2.1: The flow on the collision manifold of the Manev problem.

following analytic manifold

$$C_0 = \{(r, \theta, v, u) : r = 0,\ u^2 + v^2 - 2r - 2b = 2r^2 h\} \tag{2.11}$$

that is a cylinder in the three-dimensional space (v, θ, u), and when $u \neq 0$ the flow on it is formed by solutions parallel to the axis of the cylinder (see Figure 2.1). If $u = 0$ then $v = \pm\sqrt{2b}$ and those lines consist of fixed points. Since $\theta \in [0, 2\pi]$, by identifying the caps of the cylinder we obtain a 2-dimensional torus. The flow on the torus is given by periodic orbits $P_v^+ = \{v = k \text{ (const.) } ,\theta \in [0, 2\pi), u > 0\}$, $P_v^- = \{v = k \text{ (const.) } ,\theta \in [0, 2\pi), u < 0\}$ for $v \neq \pm\sqrt{2b}$ and by a circle formed entirely by fixed points in each of the cases $v = \pm\sqrt{2b}$. Let us denote with C^+ the upper circle of fixed points and with C^- the lower circle.

Each of the points on the circle of fixed points C^+ has a one-dimensional local unstable manifold, while each on the point on C^- has a local one-dimensional stable one, since the linearization at a point $(r = 0, v = \pm\sqrt{2b}, \theta = \theta_0, u = 0)$ gives

$$\begin{pmatrix} \pm\sqrt{2b} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.12}$$

that has a non zero eigenvalue $\pm\sqrt{2b}$ and three zero ones. Moreover, as was shown in [23], from each of the fixed points on C^+ there emerge a single orbit (that is the global unstable manifold) and to each of the fixed points on C^- will tend one single orbit (that is the global stable manifold). Indeed, since $u = 0$ at $v = \pm\sqrt{2b}$ it follows from (2.9) that $\theta =$ constant. Consequently there exists an orbit emerging from every equilibrium on C^+ Similarly one can show that for each fixed point on C^- there is a single orbit tending to it.

One can also show that for every orbit $P_v^{\pm}$ with $0 < v < \sqrt{2b}$ there exists a local two dimensional unstable manifold $W^u_{loc}(P_v^{\pm})$ and that, for every orbit, with $-\sqrt{2b} < v < 0$ there exist a local two dimensional stable manifold $W^s_{loc}(P_v^{\pm})$. A proof of those properties for the anisotropic Manev problem can be found in [18]. The same proof applies here taking $\mu = 1$ in Equation (1.1). An alternative proof can be easily obtained applying Floquet theory [41, 33]. If $v = 0$ then for the periodic orbit $P_v^{\pm}$ all the eigenvalues of the monodromy operator are zero as can be easily checked. Consequently there are no stable and unstable manifolds. However, it can be proved that there exist two-dimensional negatively and positively asymptotic sets that play the same role as the stable and unstable manifold. In [18] this statement is proved for the anisotropic Manev problem.

2.4 The Infinity Manifolds

To analyze the asymptotic behavior at infinity, we need to apply suitable blow-up transformations. Essentially we need a transformation that brings the fictitious points $r = \infty$ into the so called infinity manifold. Since the potential is *quasihomogeneous* the transformations we use are slightly different from the ones for collisions since the term of degree -1 predominates when $t \to \infty$. Moreover these transformations also differ from the ones introduced by Lacomba and Simó [50] for the Kepler problem because of the term of degree -2.

Observe that if $h < 0$ the motion is bounded and thus r can only reach infinity when $h \geq 0$. Indeed, the following proposition holds

Proposition 2.1. *If $h < 0$ the motion is bounded by the zero velocity circle*

$$\left(r_0 = \frac{-1 - \sqrt{1 - 4hb}}{2h}, v = 0, \theta, u = 0 \right) \tag{2.13}$$

Proof. Consider the energy relation (2.10), then $u^2 + v^2 = 0$ implies that $u = 0$, $v = 0$ and $r' = 0$. On the other hand $u^2 + v^2 = 0$ when $2r^2h + 2r + 2b = 0$. The solutions of the last equation are

$$r = \frac{-1 \pm \sqrt{1 - 4hb}}{2h}. \tag{2.14}$$

Since $r \geq 0$ and $h < 0$ we consider only the solution with the minus sign. This proves that (2.13) is the zero velocity circle. Moreover if $r > r_0$ then $u^2 + v^2 < 0$. Consequently the motion is bounded by the zero velocity circle. $\square$

In order to study the infinity we will consider the energy levels $h = 0$ and $h > 0$ separately. Indeed we will first introduce a transformation to study the case $h = 0$. This transformation is the same as that used in [18] for the anisotropic Manev problem. Later we will introduce another change of variables and rescaling of time that will enable us to study the case $h > 0$.

First consider the case $h = 0$. Taking $h = 0$ and $\rho = 1/r$, (2.9) becomes

$$\begin{cases} \rho' = -\rho v \\ v' = \rho^{-1} \\ \theta' = u \\ u' = 0 \end{cases} \tag{2.15}$$

and the energy relation takes the form

$$\rho(u^2 + v^2) - 2 - 2b\rho = 0. \tag{2.16}$$

Rescaling the velocities by using the transformations $\overline{v} = \rho^{1/2}v$, $\overline{u} = \rho^{1/2}u$ and rescaling the time variable by defining the transformation $d\tau = \rho^{1/2}ds$ the equations of motion take the form

$$\begin{cases} \dot{\rho} = -\rho\overline{v} \\ \dot{\overline{v}} = -(1/2)\overline{v}^2 + 1 \\ \dot{\theta} = \overline{u} \\ \dot{\overline{u}} = -(1/2)\overline{v}\,\overline{u} \end{cases} \tag{2.17}$$

where the dot denotes differentiation with respect to the new time variable s. In the new coordinates the energy relation becomes

$$\overline{u}^2 + \overline{v}^2 - 2 - 2b\rho = 0 \tag{2.18}$$

From (2.17) $\rho = 0$ defines an invariant manifold under the flow. We call it the *infinity manifold* I_0 and it appears as the boundary manifold glued to the zero energy level. We remark that

$$I_0 = \{(\rho, \overline{v}, \theta, \overline{u}) : \rho = 0 \text{ and } \overline{u}^2 + \overline{v}^2 = 2, \theta \in S^1\} \tag{2.19}$$

is also a two-dimensional torus. The flow on I_0 is given by

$$\begin{cases} \dot{\overline{v}} = -(1/2)\overline{v}^2 + 1 = (1/2)\overline{u}^2 \\ \dot{\theta} = \overline{u} \\ \dot{\overline{u}} = -(1/2)\overline{u}\,\overline{v} \end{cases} \tag{2.20}$$

and it is the same as the flow on the collision manifold of the Kepler problem [8]. The flow on the infinity manifold has two circles $I_0^{\pm}$ of equilibrium points defined by $\overline{v} = \pm\sqrt{2},\ u = 0,\ \theta \in S^1$. Of course these are also equilibrium points of (2.17)

Solutions on I_0 move from the lower circle I_0^- to the upper one I_0^+. Each rest point ($\overline{v} = -\sqrt{2},\ \theta = \theta_0,\ 0$) has associated a unique two-dimensional unstable manifold and each ($\overline{v} = +\sqrt{2},\ \theta = \theta_0,\ 0$) has associated a two-dimensional stable manifold. If we consider the flow of (2.17) then we must add the coordinate ρ. The linearization computed at a point $(\rho = 0, \overline{v} = \pm\sqrt{2}, \theta = \theta_0, \overline{u} = 0)$ is

$$\begin{pmatrix} \mp\sqrt{2} & 0 & 0 & 0 \\ 0 & \mp\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \mp\sqrt{2}/2 \end{pmatrix} \tag{2.21}$$

and thus the points on I_0^+ have a three dimensional stable manifold while the points of I_0^- have a three dimensional unstable manifold.

Observe that even if the flow on the infinity manifold is the same as the flow on the collision manifold of the Kepler problem the flow outside is very different. Indeed the upper and lower circle in the Kepler problem are normally hyperbolic [21].

The flow on and near the infinity manifold is depicted in Figure 2.2.

Now, we introduce a different change of coordinates and rescaling of time that allows us to study the behavior at infinity for $h \geq 0$. Consider the following transformation

$$\begin{cases} R = \frac{1}{r^{2/3}} \\ V = R^{3/2} v \\ U = R^{3/2} u \\ d\tau = R^{3/2} d\eta \end{cases} \tag{2.22}$$

From (2.9) and (2.10) it follows that

$$\begin{cases} \dot{R} = -(2/3)\ RV \\ \dot{V} = -V^2 + 2h + R^{3/2} \\ \dot{\theta} = U \\ \dot{U} = -UV \end{cases} \tag{2.23}$$

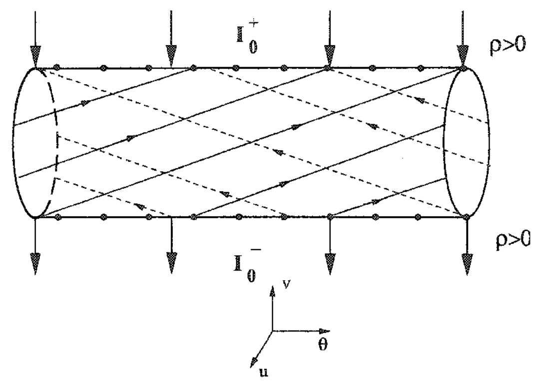


Figure 2.2: The flow on and near the infinity manifold I_0 of the Manev problem.

where, now, with an abuse of notation, the dot indicates differentiation with respect to η.

The energy relation goes to the following C^1 function

$$U^2 + V^2 - 2R^{3/2} - 2bR^3 = 2h. \tag{2.24}$$

The flow for any fixed h is confined on the three-dimensional invariant manifold defined by the energy, namely

$$N_h = \{(R, V, \theta, U) | U^2 + V^2 = 2R^{3/2} + 2bR^3 + 2h, \theta \in [0, 2\pi)\}. \tag{2.25}$$

From system (2.23) it follows that for each $h \geq 0$ fixed, the manifold $R = 0$ is invariant under the flow. In these new coordinates we can introduce the *infinity manifold*, denoted by I_h, where $h > 0$, as the restriction of the energy manifold N_h to $R = 0$, i.e.

$$I_h = \{(R, V, \theta, U) | R = 0, \theta \in [0, 2\pi), \text{ and } U^2 + V^2 = 2h\}. \tag{2.26}$$

Unlike the collision manifold, which is uniquely defined and independent from the energy level, the infinity manifolds depend continuously on h. For each $h > 0$ fixed, the corresponding infinity manifold I_h is a two-dimensional torus which becomes a circle of degenerate equilibria as $h \to 0+$.

The flow on I_h is given by

$$\begin{cases} \dot{V} = 2h - V^2 = U^2 \\ \dot{\theta} = U \\ \dot{U} = -UV \end{cases} \tag{2.27}$$

We see that using the change of variables $(V, \theta, U) = 1/2(\overline{V}, \overline{\theta}, \overline{U}))$ it is immediate that the expressions of the equations are the same as in the case of I_0, but they are not equivalent because they are defined in different spaces.

For each $h > 0$ fixed, the infinity manifold posseses two circles of equilibria defined by $V = \pm\sqrt{h}$, $U = 0$, $\theta \in S^1$. and all orbits are strictly increasing with respect to V. As for I_0 each rest point $(\overline{V} = -\sqrt{2h}, \theta = \theta_0, U = 0)$ has associated a unique two-dimensional unstable manifold and each $(\overline{V} = \sqrt{2h}, \theta = \theta_0, U = 0)$ has associated a two-dimensional stable manifold. The flow on and near the infinity manifold I_h is illustrated in Figure 2.3.

2.5 The Flow for Negative Energy

The flow outside the collision and infinity manifolds and in particular the stable and unstable manifolds can, in general, only be described locally. However, as we have already remarked, the Manev problem is an integrable system and thus the global stable and unstable manifolds and the asymptotic sets can be found explicitly and thus the global flow can be characterized precisely. To describe the flow outside the collision manifold it is convenient to exploit the fact that θ does not appear explicitly in the equations (2.9). Indeed this allows us to reduce the four-dimensional phase space to dimension three by factorizing the flow to S^1. Consequentely, exploiting

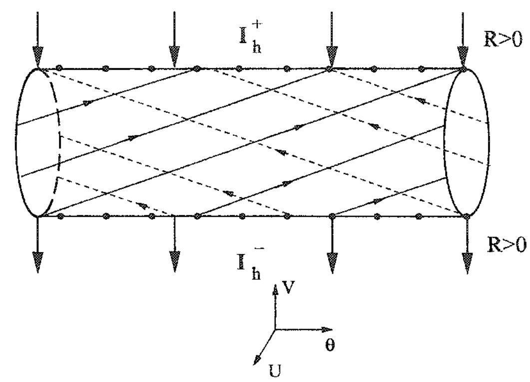


Figure 2.3: The flow on and near the infinity manifold I_h of the Manev problem.

the symmetry, we will obtain clear pictures of the flow in phase space. Factorizing the collision manifold to S^1, the torus becomes a circle.

Consider the negative-energy case, i.e. $h < 0$ and observe that, in the reduced phase space, every energy level is a two-dimensional ellipsoid, as can be deduced from the energy relation written in the following form:

$$u^2 + v^2 - 2h(r + 1/(2h))^2 = 2b - 1/(2h). \tag{2.28}$$

In the reduced space (r, v, u) with $r \geq 0$ the collision manifold reduces to the circle $r = 0, u^2 + v^2 = 2b$ (see Figure 2.4) and an analysis of the equations (2.9) allows to describe the flow on the negative energy levels [19].

There are two equilibria outside the collision manifold, located at $r = -1/(2h), v = 0, u = \pm\sqrt{2b - 1/(2h)}$. Moreover, since u is a first integral, all the solutions lie on parallel planes $u =$ constant. If $|u| < \sqrt{2b}$ the orbits are heteroclinic and they form the stable and unstable manifolds of the orbits $P_v^{\pm}$. In particular if $u = 0$ then the corresponding heteroclinic orbit connects a point on the upper circle C^+ of the

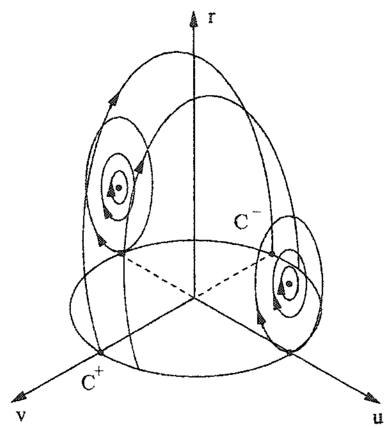


Figure 2.4: The flow on each negative energy level.

collision manifold with a point on C^-. If $u = \pm\sqrt{2b}$ the orbits are homoclinic, and they form a manifold homoclinic to the two periodic orbits $P_v^{\pm}$ with $v = 0$. Physically the heteroclinic and the homoclinic solutions correspond to solutions ejecting from collisions and then tending to collisions. In particular the heteroclinic solutions lying on the plane $u = 0$ correspond to rectilinear ejection-collision orbits, while all the other heteroclinic and homoclinic solutions correspond to spiralling ejection-collision orbits. If $\sqrt{2b} < |u| < \sqrt{2b - 1/(2h)}$ then the orbits are periodic. In the full phase space these orbits correspond to a linear flow on the torus $S^1 \times S^1$, where each torus is filled either with periodic or with quasiperiodic orbits. Finally, the two equilibria outside the collision manifold at $r = -1/(2h), v = 0, u = \pm\sqrt{2b - 1/(2h)}$ correspond, in the full phase space, to periodic solutions that are circular orbits in the physical space.

2.6 The Flow for Non-Negative Energy

When $h = 0$, i.e. on the zero-energy manifold the energy relation takes the form

$$u^2 + v^2 = 2r + 2b \tag{2.29}$$

which implies that the zero-energy level is a paraboloid with the cap removed (see Figure 2.5).

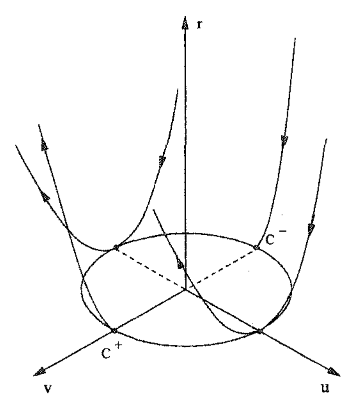


Figure 2.5: The flow for $h = 0$.

The cap is removed because $r \geq 0$. The first integral $u =$ constant foliates the zero-energy level into curves lying in parallel planes. The curves are parabolas for $|u| \geq \sqrt{2b}$, or arcs of parabolas, for $|u| < \sqrt{2b}$. Moreover there are no equilibria outside the collision manifold. In phyisical space the arcs of the parabola tending to C^- or emerging from C^+ correspond to rectilinear orbits (with zero angular momenum) that tend to (emerge from) infinity with asymptotic velocity zero. The other arcs of parabolas that tend to (emerge from) the collision circle correspond, in physical space, to orbits that spiral at collision (ejection) and have asymptotic velocity zero.

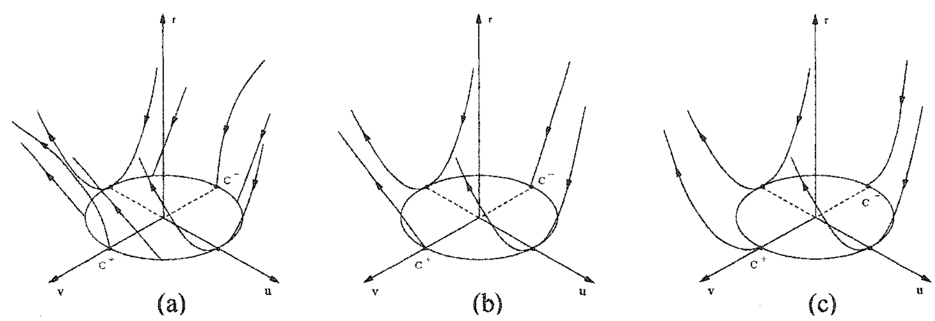


Figure 2.6: (a) The flow for $h > 1/(4b)$. (b) The flow for $h = 1/(4b)$. (c) The flow for $h < 1/(4b)$

In the positive energy case, the energy relation takes the form

$$u^2 + v^2 - 2h(r + 1/(2h))^2 = 2b - 1/(2h) \tag{2.30}$$

Depending on the relationship between h and b three possibilities arise.

If $h > 1/(4b)$ the energy relation describes an hyperboloid of one sheet intersected with $r \geq 0$ (see Figure 2.6(a)). Since u is a first integral all solutions are, again, represented by curves lying on parallel planes $u =$ constant. Such curves are branches of hyperbolas or arcs of branches of hyperbolas and two pairs of half-lines. The physical interpretation is similar to that of the zero-energy case, except that parabolas are replaced by branches of hyperbolas and the asymptotic velocity at infinity is not zero but positive.

If $h = 1/(4b)$ the energy relation is a cone intersected with $r \geq 0$ (see Figure 2.6(b)). The first integral $u =$ constant foliates the surface in branches of hyperbolas or arcs of branches of hyperbolas and two half-lines (corrseponding to C^+ and C^-). The physical interpretation is similar to the one of the case $h > 1/(4b)$.

If $h < 1/(4b)$ the energy relation is an hyperboloid of two sheets intersected with $r \geq 0$ (see Figure 2.6(c)). In this case the curves lying on the planes $u =$ constant are branches of hyperbolas and arc of branches of hyperbolas. The physical interpretation is similar to the one of the previous cases.

2.7 Action Angle Variables

In Section 2.2 we studied the level sets of the integrals $I_{hc} = \{(\mathbf{q}, \mathbf{p}) | H_0(\mathbf{q}, \mathbf{p}) = h, \mathbf{K} = c\}$ when they are compact and connected. We found that I_{hc} is diffeomorphic to a 2-torus invariant under the flow. We chose some angle coordinates ϕ_i on I_{hc} such that the phase flow generated by H_0 is of the form

$$\frac{d\phi_i}{dt} = \omega_i(h, c), \quad \phi_i(t) = \phi_i(0) + \omega_i t, \quad \text{for } i = 1, 2. \tag{2.31}$$

We now look at a neighborhood of the two-dimensional manifold I_{hc} in four-dimensional phase space.

In the coordinates (H_0, K, ϕ_1, ϕ_2) the phase flow can be written in a very simple form as the following system of four ordinary differential equations

$$\begin{cases} \frac{dH}{dt} = 0 \\ \frac{dK}{dt} = 0 \\ \frac{d\phi_1}{dt} = \omega_1(H_0, K) \\ \frac{d\phi_2}{dt} = \omega_2(H_0, K) \end{cases} \tag{2.32}$$

which is easily integrated:

$$\begin{cases} H_0(t) = H_0(0) \\ K(t) = K(0) \\ \phi_1(t) = \phi_1(0) + \omega_1(H_0, K)t \\ \phi_2(t) = \phi_2(0) + \omega_2(H_0, K)t. \end{cases}$$

Thus, in order to integrate explicitly the original canonical system it is sufficient to find the variables ϕ_i in explicit form. It turns out that this can be done using only quadratures.

Let us remark that the variables (H_0, K, ϕ_1, ϕ_2) are not symplectic. However one can find some functions of H_0 and K denoted by I_1, I_2 such that I_1, I_2, ϕ_1, ϕ_2 are symplectic. The variables I_1, I_2 are called *action variables* and with the variables ϕ_1, ϕ_2 form the so called *action-angle variables.*

From the general theory of action angle variables [2] one can construct such variables in the following way. Let γ_1, γ_2 be a basis for one-dimensional cycles on the torus I_{hc} (the increase of the coordinate ϕ_i on the cycle γ_j is equal to 2π if $i = j$ and 0 if $i \neq j$). We set

$$I_i = \frac{1}{2\pi} \int_{\gamma_i} \mathbf{p} d\mathbf{q} \tag{2.33}$$

The quantities I_i given by the formula (2.33) are called action variables. Therefore the action variables for the Manev problem, already introduced in [25, 73], are given by

$$\begin{cases} I = I_1 = \frac{1}{2\pi} \oint p_r dr = -\sqrt{K^2 - 2b} + \frac{1}{2}\sqrt{\frac{2}{|h|}} \\ K = I_2 = xp_y - yp_x \end{cases} \tag{2.34}$$

where h is the energy constant and K is the angular momentum. These variables are defined for $h < 0$ and $K^2 > 2b$, $I > 0$, to avoid collision orbits as well as circular orbits. The related frequencies are

$$\begin{cases} \omega_I = \frac{1}{(I+\sqrt{K^2-2b})^3} \\ \omega_K = \frac{K}{\sqrt{K^2-2b}(I+\sqrt{K^2-2b})^3}, \end{cases}$$

and $\Theta = \phi_1$ and $\Phi = \phi_2$ are the angle variables associated to K and I respectively.

The unperturbed Hamiltonian in the new variables can be written as

$$H_0 = -\frac{1}{2(I + \sqrt{K^2 - 2b})^2}.$$

Now we can consider new variables that are linear combinations of the previous ones. They are defined by the following canonical transformation

$$\begin{cases} L = K + I \\ G = -I \\ l = \Theta \\ g = \Theta - \Phi \end{cases} \tag{2.35}$$

Where l is the mean anomaly (where $l(t) = \omega_L(t - t_0)$ and t_0 is the time of pericenter passage), g is the longitude of pericenter as they are defined for the Manev problem in [67]. Moreover also the action variables can be written in terms of the orbital elements of the Manev problem. If we set

$$a = \frac{1}{2|h|} \qquad \text{and} \qquad e = \sqrt{1 - 2(K^2 - 2b)|h|}$$

as in [25, 67] then

$$G = -a^{1/2}\left[1 - (1 - e^2)^{1/2}\right] \quad \text{and} \quad L = -G \pm \sqrt{a(1 - e^2) + 2b}$$

where a is the pseudo-semimajor axis, e is the pseudo-eccentricity, and the sign + (resp. -) holds for $K > 0$ (resp. < 0). The conditions to avoid collision orbits and circular orbits, on which g becomes meaningless, can be written in terms of the orbital elements as $a > 0$ and $0 < e < 1$. The new unperturbed Hamiltonian is

$$H_0 = -\frac{1}{2(-G + \sqrt{(G + L)^2 - 2b})^2} \tag{2.36}$$

so the equations of motion in action angle variables are

$$\begin{cases} \dot{L} = -\frac{\partial(H_0)}{\partial l} = 0 \\ \dot{G} = -\frac{\partial(H_0)}{\partial g} = 0 \\ \dot{l} = \frac{\partial(H_0)}{\partial L} = \omega_L \\ \dot{g} = \frac{\partial(H_0)}{\partial G} = \omega_G \end{cases} \tag{2.37}$$

where

$$\begin{cases} \omega_L = \omega_K = \frac{G+L}{(-G+\sqrt{(G+L)^2-2b})^3\sqrt{(G+L)^2-2b}} \\ \omega_G = \omega_K - \omega_I = \frac{G+L-\sqrt{(G+L)^2-2b}}{(-G+\sqrt{(G+L)^2-2b})^3\sqrt{(G+L)^2-2b}}. \end{cases}$$

Chapter 3

The Anisotropic Manev Problem

3.1 Overview

The main objective of this chapter is to introduce the anisotropic Manev problem and study some of its main features.

In the next section we introduce the anisotropic Manev problem and its equations of motion.

In Section 3.3 we analyze the discrete symmetry group of the anisotropic problem and in particular we outline some properties of the symmetric orbits in general and of the symmetric periodic orbits in particular.

Then, in Section 3.4 we examine, using McGehee coordinates, the collision manifold and the flow on it [18].

In the following section we study the infinity manifolds, recalling some known results (see [18]) and strengthening the previous understanding of the flow on the infinity manifold to the point that we can prove that it is structurally stable.

In the last section we write the equations of motion in the action-angle variables that were introduced in Section 2.7 Such equations will be important in Chapter 5 to prove the existence of periodic orbits.

3.2 The Equations of Motion

The (planar) anisotropic Manev problem is described by the Hamiltonian

$$H = \frac{1}{2}\mathbf{p}^2 + U(\mathbf{q}). \tag{3.1}$$

where $\mathbf{q} = (x, y)$ is the position of one body with respect to the other considered fixed at the origin of the Cartesian coordinate system, $\mathbf{p} = (p_x, p_y)$ is the momentum of the moving particle and

$$U(\mathbf{q}) = -\frac{1}{\sqrt{\mu x^2 + y^2}} - \frac{b}{\mu x^2 + y^2} \tag{3.2}$$

is the potential energy. The parameters $\mu > 0$ and $b > 0$ are constants and μ measures the strength of the anisotropy. If $\mu < 1$ the attraction is weakest in the direction of the x-axis and strongest in that of the y-axis. The situation is reversed if $\mu > 1$. If $\mu = 1$ the space is isotropic and we recover the classical Manev problem described in Chapter 2.

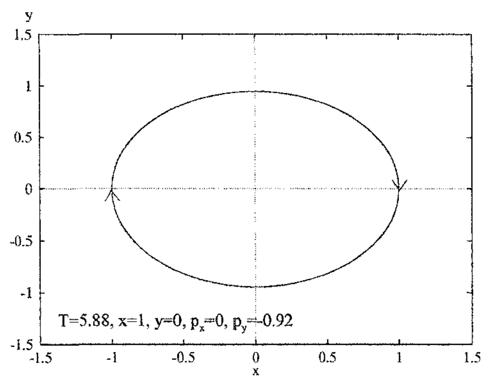
Since both remaining cases have a weakest-force and a strongest-force direction, we can assume, without loss of generality, that $\mu > 1$. The equations of motion form a system of ordinary differential equations that can be expressed as

$$\begin{cases} \dot{\mathbf{q}} = \mathbf{p} \\ \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}} \end{cases}. \tag{3.3}$$

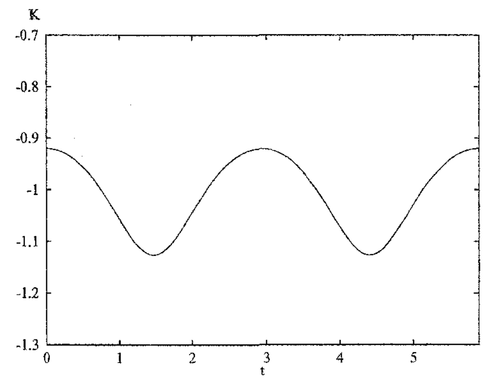
The Hamiltonian function provides the first integral

$$H(\mathbf{p}(t), \mathbf{q}(t)) = h, \tag{3.4}$$

where h is a real constant. However, since the force $-\nabla U$ is not central for $\mu > 1$ (i.e. the rotational invariance of the Hamiltonian function is destroyed) the angular momentum $\mathbf{K}(t) = \mathbf{p}(t) \times \mathbf{q}(t)$ is not a first integral of the system, as is illustrated by an example in Figure 3.1. Figure 3.1 (a) depicts a periodic orbit of the anisotropic



(a)



(b)

Figure 3.1: (a) A periodic orbit of the anisotropic Manev problem with $\mu = 1.5$, $b = 0.1$. (b) The angular momentum K of the periodic orbit as a function of time.

Manev problem while Figure 3.1 (b) represent its angular momentum K as a function of time, showing that it is not constant but that it oscillates. For $\mu = 1$ we recover the Manev problem and the angular momentum is a constant of motion for such a system (see Chapter 2).

The fact that the angular momentum is not a constant of motion for the problem under discussion, is a strong clue that the anisotropic Manev problem has very complicated, and possibly chaotic, dynamics. An example of an orbit that behaves erratically, obtained numerically for $\mu = 15$ is depicted in Figure 3.2.

Now consider weak anisotropies, i.e. choose the parameter $\mu > 1$ close to 1. Introducing the notation $r = \sqrt{x^2 + y^2}$, $\theta = \arctan(y/x)$ and $\epsilon = \mu - 1$ with $\epsilon \ll 1$ we can expand the Hamiltonian (3.1) in powers of ϵ and obtain

$$H = \frac{1}{2}\mathbf{p}^2 - \frac{1}{r} - \frac{b}{r^2} + \epsilon\left(\frac{1}{2r} + \frac{b}{r^2}\right)\cos^2\theta \equiv H_0 + \epsilon W(r,\theta). \tag{3.5}$$

It should be pointed out that the term $W(r,\theta)$ becomes unbounded as $r \to 0$ so that a perturbation analysis is not correct on the ejection-collision orbits. This means

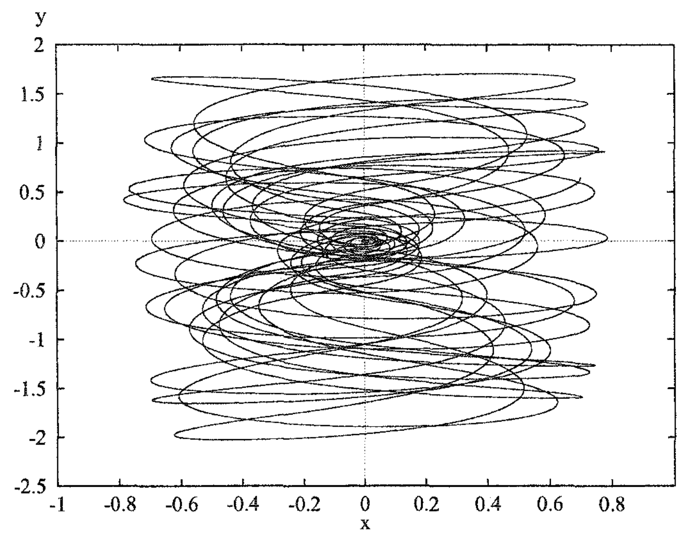


Figure 3.2: A ”‘chaotic”’ orbit that behaves erratically obtained numerically for $\mu = 15$.

that the global dynamics of the anisotropic Manev problem cannot be completely described by perturbations to the Manev problem even at the limit $\epsilon \to 0$. However many interesting results concerning the Hamiltonian (3.1) for weak anisotropies (i.e. $\epsilon \ll 1$) can be found studying the Hamiltonian (3.5), some of which are presented in this thesis.

In the next section we describe the symmetries of the anisotropic Manev problem and we find some properties that will be useful to find symmetric periodic orbits.

3.3 Symmetries of the Anisotropic Manev Problem

Symmetries play a very important role in studying differential equations. Indeed continuous symmetries allow reduction of the order of differential equations, and are closely related to the existence of first integrals and to the integrability [58, 2]. In particular the Liouville-Arnold theorem, stated in Section 2.2 for the Manev problem, is closely related to the existence of two continuous symmetries: the homogeneity of time and isotropy of space. In this section we are interested in studying the discrete symmetries of the anisotropic Manev problem, because this will be essential in finding periodic orbits. The importance of the role of discrete symmetries in finding periodic orbits was already observed by many authors (see, for example, [4, 55]).

In order to introduce the symmetries of the anisotropic Manev problem we need to recall some standard definitions and known facts about symmetries.

Let $\mathbf{f}$ be a vector field on a phase space U.

Definition 3.1. We call a *symmetry of the field* $\mathbf{f}$ a diffeomorphism $\mathbf{g} : U \to U$ such that

$$\mathbf{f}(\mathbf{g}(\mathbf{u})) = \frac{\partial \mathbf{g}}{\partial \mathbf{u}} \mathbf{f}(\mathbf{u}). \tag{3.6}$$

The field $\mathbf{f}$ is called *invariant* under the diffeomorphism $\mathbf{g}$.

Definition 3.2. We call a *symmetry of a slope field* a diffeomorphism of the extended phase space that sends the field into itself. The direction field is then called *invariant* under the diffeomorphism.

As an example consider a differential equation $\dot{u} = f(u)$ that has the slope field depicted in Figure 3.3. The slope field is invariant under translations along the t axis.

Definition 3.3. The system of differential equations $\dot{\mathbf{u}} = \mathbf{f}(\mathbf{u})$ (respectively $\dot{\mathbf{u}} = \mathbf{f}(\mathbf{u}, t)$) is *invariant* (or *covariant*) under a diffeomorphism $\mathbf{g}$ of the phase space (respectively, of the extended phase space) if the vector field (respectively, slope field) is invariant under $\mathbf{g}$. Moreover the diffeomorphism $\mathbf{g}$ is called a *symmetry* of this system of differential equations.

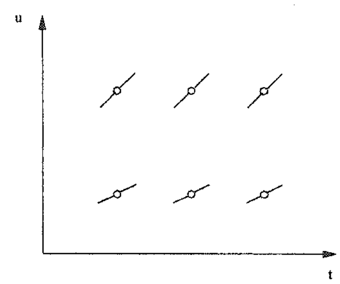


Figure 3.3: An example of a translation invariant slope field

Another important fact is that the symmetries of a field form a group. Furthermore it is clear that if $\gamma(t)$ is a solution of $\dot{\mathbf{u}} = \mathbf{f}(\mathbf{u})$ then also $\mathbf{g}(\gamma(t))$ is a solution.

The symmetries of the anisotropic Manev problem have been examined in [18] and, as is easy to see, are defined by the following diffeomorphisms in the extended phase space:

$$\begin{array}{rl} Id: & (x,y,p_x,p_y,t) \longrightarrow (x,y,p_x,p_y,t) \\ S_0: & (x,y,p_x,p_y,t) \longrightarrow (x,y,-p_x,-p_y,-t) \\ S_1: & (x,y,p_x,p_y,t) \longrightarrow (x,-y,-p_x,p_y,-t) \\ S_2: & (x,y,p_x,p_y,t) \longrightarrow (-x,y,p_x,-p_y,-t) \\ S_3: & (x,y,p_x,p_y,t) \longrightarrow (-x,-y,-p_x,-p_y,t) \\ S_4: & (x,y,p_x,p_y,t) \longrightarrow (-x,y,-p_x,p_y,t) \\ S_5: & (x,y,p_x,p_y,t) \longrightarrow (x,-y,p_x,-p_y,t) \\ S_6: & (x,y,p_x,p_y,t) \longrightarrow (-x,-y,p_x,p_y,-t) \end{array} \tag{3.7}$$

where Id is the identity.

The invariance under these symmetries implies that if $\gamma(t)$ is a solution of (3.3), then also $S_i(\gamma(t))$ is a solution for $i \in \{0,1,2,3,4,5,6\}$. In figure 3.4 we draw all the solutions $S_i(\gamma(t))$ for $i = 0,1,2,3,4,5,6$. For $i \in \{0,1,2,3,4,5,6\}$ the orbit $\gamma(t)$ will be called symmetric if and only if $S_i(\gamma(t)) = \gamma(t)$.

Let us remark that the symmetries in (3.7), together with the composition of functions, denoted by $\circ$, form an abelian group, that we shall denote $\mathcal{G}$, in which the operation acts according to Table 3.1.

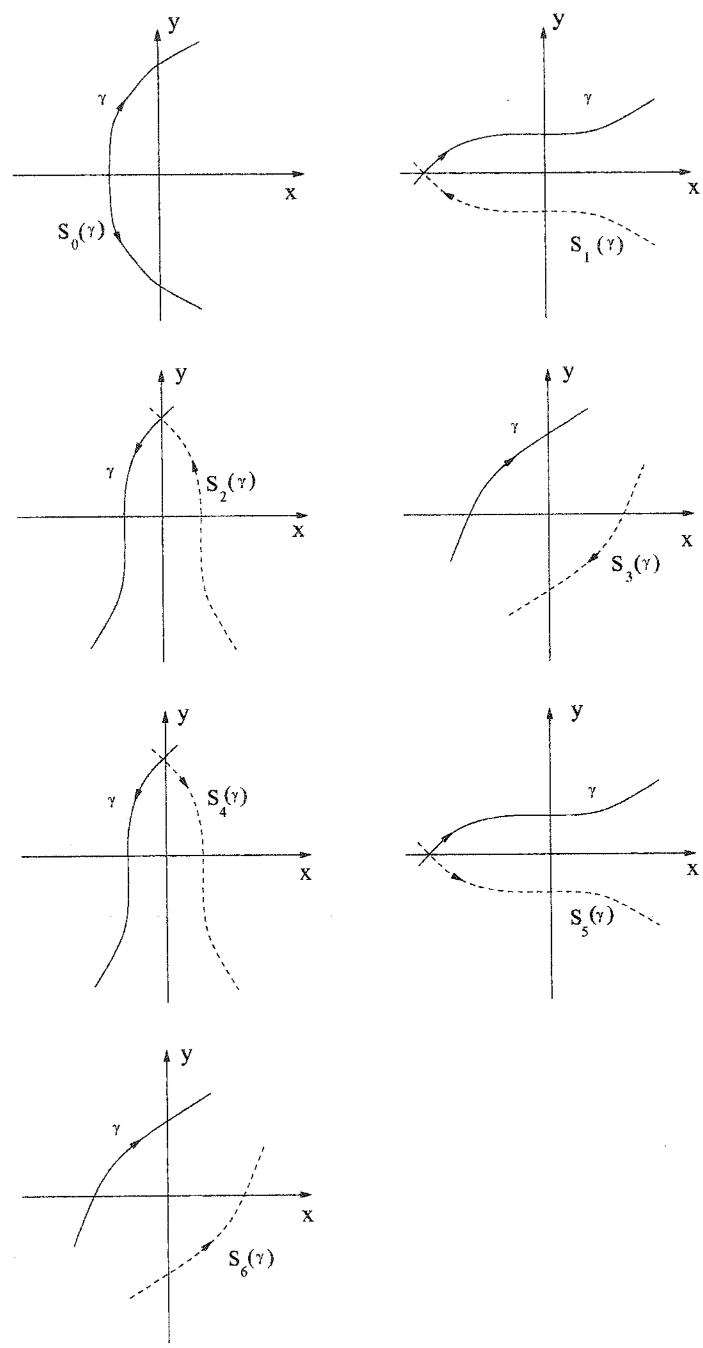


Figure 3.4: The symmetric orbits of $\gamma(t)$: $S_i(\gamma(t))$ for i=0,1,2,3,4,5,6.

$\circ$	Id	S_0	S_1	S_2	S_3	S_4	S_5	S_6
Id	Id	S_0	S_1	S_2	S_3	S_4	S_5	S_6
S_0	S_0	Id	S_5	S_4	S_6	S_2	S_1	S_3
S_1	S_1	S_5	Id	S_3	S_2	S_6	S_0	S_4
S_2	S_2	S_4	S_3	Id	S_1	S_0	S_6	S_5
S_3	S_3	S_6	S_2	S_1	Id	S_5	S_4	S_0
S_4	S_4	S_2	S_6	S_0	S_5	Id	S_3	S_1
S_5	S_5	S_1	S_0	S_6	S_4	S_3	Id	S_2
S_6	S_6	S_3	S_4	S_5	S_0	S_1	S_2	Id

Table 3.1: Cayley table of the symmetry group of the anisotropic Manev problem

From the Cayley table of the symmetry group it is easy to deduce the following

Proposition 3.1. *The symmetries of the anisotropic Manev problem form an elementary abelian group $\mathcal{G}$ of order eight, i.e. a group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. S_0, S_1 and S_2 are the generators of $\mathcal{G}$.*

The discrete group of symmetry described above appears in many Hamiltonian systems, as for instance the anisotropic Kepler problem [8] or the collinear three body problem [17]. The symmetries in (3.7), (except Id and S_6) are very useful to find symmetric periodic orbits, especially by means of the continuation method and of variational techniques. Some important properties of the symmetric orbits, summarized in [8] , are expressed in the following lemma:

Lemma 3.2. *(i) For $i = 1$ (resp. $i = 2$) we have that an orbit $\gamma(t)$ is S_i-symmetric if and only if it crosses the x-axis (resp. y-axis) orthogonally.*

(ii) An orbit $\gamma(t)$ is S_0-symmetric if and only if it has a point on the zero velocity curve.

(iii) For $i = 4, 5$ if an orbit $\gamma(t)$ is S_i-symmetric then it is S_0-symmetric.

(iv) All the S_3-symmetric orbits are periodic.

Proof. (i) Using the uniqueness theorem of a solution of an ordinary differential equation, it follows that $\gamma(t)$ is an S_1-symmetric solution if and only if $\gamma(t)$ intersects the plane $y = 0$, $p_x = 0$ at least in one point. The same reasoning works for S_2-symmetric solutions.

(ii) Using again the uniqueness theorem it follows that $\gamma(t)$ is S_0-symmetric if and only if $\gamma(t)$ intersects the plane $p_x = 0$, $p_y = 0$ at least in one point, or in other words if and only if $\gamma(t)$ has a point on the zero velocity curve.

(iii) An S_4-symmetric orbit must intersect the plane $x = 0$, $p_x = 0$ at least in one point. If the solution lies on the y-axis then it must be an ejection-collision solution and then $p_x = p_y = 0$ at one point, i.e. the orbit intersect the zero velocity curve. On the other hand if the orbit does not lie on the y-axis then, if $p_x(t_0) = 0$ and $x(t_0) = 0$ then clearly $p_y(t_0+\epsilon) = -p_y(t_0-\epsilon)$ for every $\epsilon \geq 0$, since it is S_4-symmetric. This proves that S_4-symmetric orbits have a point on the zero velocity curve. Similarly one can show that S_5-symmetric orbits have a point on the zero velocity curve. Consequently S_i-symmetric orbits for $i = 4, 5$ are S_0-symmetric.

(iv) Consider a point in phase space $(\mathbf{q}, \mathbf{p})$ and assume there is an S_3-symmetric orbit $\Phi(t, (\mathbf{q}, \mathbf{p}))$ passing through it. Then we have the following property for the flow:

$$\Phi(t, -(\mathbf{q}, \mathbf{p})) = -\Phi(t, (\mathbf{q}, \mathbf{p})).$$

Furthermore the solution $\Phi(t, (\mathbf{q}, \mathbf{p}))$ satisfies the equation

$$\Phi\left(\frac{\tau}{2}, (\mathbf{q}, \mathbf{p})\right) = -(\mathbf{q}, \mathbf{p}) \tag{3.8}$$

for some time τ. Then by the uniqueness of solution of ordinary differential equations we have

$$\Phi\left(t + \frac{\tau}{2}, (\mathbf{q}, \mathbf{p})\right) = -\Phi(t, (\mathbf{q}, \mathbf{p}))$$

for all t. By the equivariance of the flow $\Phi(t, \mathbf{q}, \mathbf{p}))$ and the above property, it follows that

$$\Phi(t + \tau, (\mathbf{q}, \mathbf{p})) = \Phi(t, (\mathbf{q}, \mathbf{p})) \tag{3.9}$$

which means that $\Phi(t, \mathbf{q}, \mathbf{p}))$ is a periodic solution.

□

The properties of the S_i-symmetric orbits were first studied by Birkhoff [7] for the restricted three body problem and later by many other authors. In particular Casasayas and Llibre (see [8]) state a proposition that gives a technique useful to obtain symmetric periodic orbits with respect to S_0, S_1,S_2 for the anisotropic Kepler problem that are verified also for the problem under discussion in this work:

Proposition 3.3. *(i) For $i = 1$ (resp. $i = 2$) we have that an orbit $\gamma(t)$ is an S_i-symmetric periodic orbit if and only if it crosses the x-axis (resp. y-axis) orthogonally at two distinct points.*

(ii) An orbit $\gamma(t)$ is an S_0-symmetric periodic orbit if and only if it meets the zero velocity curves at two distinct points.

(iii) An orbit $\gamma(t)$ is an S_1 and S_2 symmetric periodic orbit if and only if it crosses the x-axis and the y-axis orthogonally.

(iv) For $i = 1, 2$ an orbit $\gamma(t)$ is an S_0 and S_i-symmetric periodic orbit if and only if it meets the zero velocity curve and crosses the x, respectively y-axis orthogonally.

(v) *For* $i = 4, 5$, *if an orbit* $\gamma(t)$ *is* S_i*-symmetric then it is* S_0*-symmetric and periodic.*

Proof. We will prove only the first statement, since the proof of the other ones is similar.

(i) From Lemma 3.2 it follows immediately that an S_1-symmetric solution is periodic if and only if it intersects the y-axis transversally at two points. The proof for the case of S_2-symmetric solutions is analogous.

□

3.4 The Collision Manifold

Since our first goal is to study collision and near collision solutions, it is helpful to transform system (3.3) using, as we did in Section 2.3, a method developed by McGehee [53]. Thus consider the transformation of coordinates (2.7) and the rescaling of time (2.8). Composing these transformations, which are analytic diffeomorphisms in their respective domains, system (3.3) becomes

$$\begin{cases} r' = rv \\ v' = 2r^2h + r\Delta^{-1/2} \\ \theta' = u \\ u' = (1/2)(\mu - 1)(r\Delta^{-3/2} + 2b\Delta^{-2})\sin 2\theta \end{cases} \tag{3.10}$$

and the energy relation (3.4) takes the form

$$u^2 + v^2 - 2r\Delta^{-1/2} - 2b\Delta^{-1} = 2r^2h, \tag{3.11}$$

where $\Delta = \mu\cos^2\theta + \sin^2\theta$ and the new variables $(r, v, \theta, u) \in (0, \infty) \times \mathbb{R} \times S^1 \times \mathbb{R}$ depend on the fictitious time τ. The prime denotes differentiation with respect to

τ. The generators S_0, S_1, S_2 of the symmetry group $\mathcal{G}$ in the new coordinates are changed to $\tilde{S}_0, \tilde{S}_1, \tilde{S}_2$, where

$$\begin{aligned} \tilde{S}_0(r,v,\theta,u,\tau) &\to (r,-v,\theta,-u,-\tau) \\ \tilde{S}_1(r,v,\theta,u) &\to (r,-v,-\theta,u,-\tau) \\ \tilde{S}_2(r,v,\theta,u) &\to (r,-v,\pi-\theta,u,-\tau). \end{aligned} \tag{3.12}$$

The set

$$C = \{(r,v,\theta,u)|r = 0 \text{ and the energy relation (3.11) holds}\} \tag{3.13}$$

is the *collision manifold*, which replaces the set of singularities $\{(\mathbf{q},\mathbf{p})|\mathbf{q} = \mathbf{0}\}$. This 2-dimensional manifold, embedded in $\mathbb{R}^3 \times S^1$, is homeomorphic to a torus and it is given by the equations

$$r = 0 \quad \text{and} \quad u^2 + v^2 = 2b\Delta^{-1}. \tag{3.14}$$

Now consider a fixed *constant energy surface*

$$\mathcal{E} = \{(r,v,\theta,u)|r > 0 \text{ and the energy relation (3.11) holds}\}.$$

The system (3.10) does not have singularities on $\mathcal{E}_h \cup C$. The restriction of the equation in (3.10) to C yields the system:

$$\begin{cases} v' = 0 \\ \theta' = u \\ u' = b(\mu - 1)\Delta^{-2}\sin 2\theta \end{cases} \tag{3.15}$$

The flow on the collision manifold was studied in detail in [18]. Here we will recall its features.

For $\mu = 1$ we have the usual Manev problem with two circles of equilibria $C^{\pm}$ on C_0. When $\mu > 1$, each of these circles breaks up into four distinct equilibria: two centers and two hyperbolic saddles. We single this fact out as a proposition:

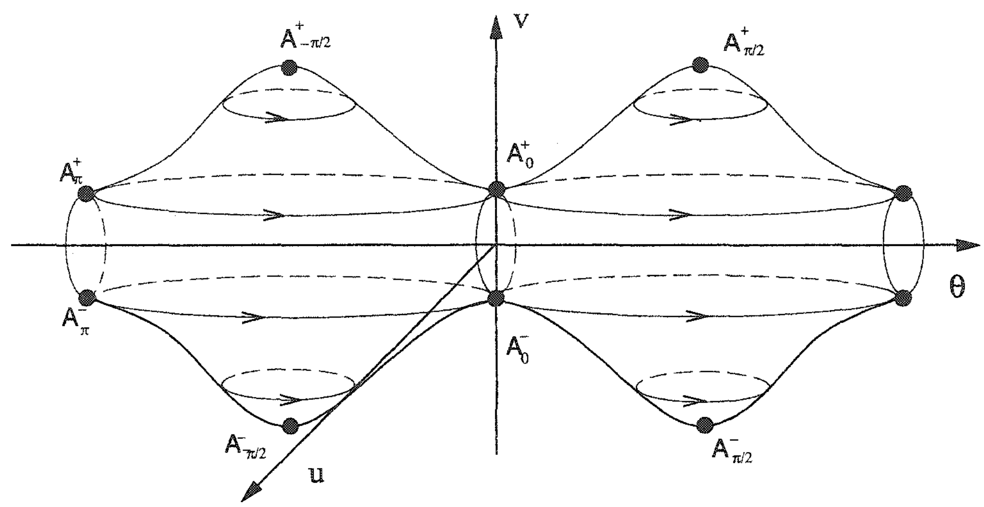


Figure 3.5: The flow on the collision manifold, which is formed by periodic orbits, eight equilibria, and eight heteroclinic orbits.

Proposition 3.4. *If $\mu > 1$ the system of differential equations (3.10) admits exactly eight equilibrium solutions. The locations as well as the characteristic exponents of these equilibria are displayed in Table 3.2.*

Proof. To see that these are the only equilibria on C, first note that $r' = 0$ and $v' = 0$ on C. On the other hand $\theta' = 0$ if and only if $u = 0$. Furthermore $u' = 0$ if and only if $(1/2)(\mu - 1)(2b\Delta^{-2})\sin 2\theta = 0$. Since $\mu > 1$ it follows that $u' = 0$ if and only if $\theta = 0, \pm\pi/2, \pi$ with $\theta \in (-\pi, \pi]$. This yields the result.

To compute the characteristic exponents we consider the linearization at the various equilibria.

First consider the following four equilibria: $A_0^{\pm} = (0, \pm\sqrt{2b/\mu}, 0, 0)$ and $A_\pi^{\pm} = (0, \pm\sqrt{2b/\mu}, \pi, 0)$. At these points the linearized system has the matrix

$$\begin{pmatrix} \pm\sqrt{2b/\mu} & 0 & 0 & 0 \\ 1/\sqrt{\mu} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2b(\mu-1)/\mu^2 & 0 \end{pmatrix} \tag{3.16}$$

Equilibrium point	Characteristic Exponents	Type on C
$A^-_{-\pi/2} = (0, -\sqrt{2b}, -\pi/2, 0)$	$-\sqrt{2b}, 0, \pm\sqrt{2b(1-\mu)}$	Center
$A^-_0 = (0, -\sqrt{2b/\mu}, 0, 0)$	$-\sqrt{2b/\mu}, 0, \pm\sqrt{2b(\mu-1)}/\mu$	Saddle
$A^-_{\pi/2} = (0, -\sqrt{2b}, \pi/2, 0)$	$-\sqrt{2b}, 0, \pm\sqrt{2b(1-\mu)}$	Center
$A^-_\pi = (0, -\sqrt{2b/\mu}, \pi, 0)$	$-\sqrt{2b/\mu}, 0, \pm\sqrt{2b(\mu-1)}/\mu$	Saddle
$A^+_{-\pi/2} = (0, \sqrt{2b}, -\pi/2, 0)$	$+\sqrt{2b}, 0, \pm\sqrt{2b(1-\mu)}$	Center
$A^+_0 = (0, \sqrt{2b/\mu}, 0, 0)$	$+\sqrt{2b/\mu}, 0, \pm\sqrt{2b(\mu-1)}/\mu$	Saddle
$A^+_{\pi/2} = (0, \sqrt{2b}, \pi/2, 0)$	$+\sqrt{2b}, 0, \pm\sqrt{2b(1-\mu)}$	Center
$A^+_\pi = (0, \sqrt{2b/\mu}, \pi, 0)$	$+\sqrt{2b/\mu}, 0, \pm\sqrt{2b(\mu-1)}/\mu$	Saddle

Table 3.2: Characteristic exponents of the equilibrium points on C

the corresponding eigenvalues being real and taking the values $\pm\sqrt{2b/\mu}, 0$ and $\pm\sqrt{2b(1-\mu)}/\mu$. The other four equilibria are $A^{\pm}_{\pm\pi/2} = (0, \pm\sqrt{2b}, \pm\pi/2, 0)$ and the linearized system at these points is given by the matrix

$$\begin{pmatrix} \pm\sqrt{2b} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2b(1-\mu) & 0 \end{pmatrix} \tag{3.17}$$

and the corresponding eigenvalues are $\pm\sqrt{2b}, 0$ and $\pm\sqrt{2b(1-\mu)}$, where the last two eigenvalues are purely imaginary since $\mu > 1$. $\square$

To complete the analysis of the flow on C one can remark that, since $v' = 0$, the solutions of (3.15) lie on the level curves $v =$ constant of the torus C. Consequently there are eight heteroclinic orbits (i.e. orbits connecting two distinct equilibria) which lie in the level sets $v = \pm\sqrt{2b/\mu}$. All the other solutions are periodic. The flow is sketched in Figure 3.5.

It is important to remark that the flow on the collision manifold is fictitious, i.e. it doesn't have a physical meaning, however, due to the continuity of the solutions with respect to the initial data, its structure gives informations about the behavior of collision and near-collision orbits. The structure of the flow of the anisotropic Manev Problem is fairly simple but it differs from that of the anisotropic Kepler problem (compare with [8]).

3.5 The Infinity Manifolds

In this section we study the infinity manifolds for $h = 0$ and for $h > 0$ (recall that for $h < 0$ the motion is bounded and therefore there is not infinity manifold). Precisely, we have the following

Proposition 3.5. *If $h < 0$ the motion is bounded by the zero velocity curve*

$$\left(r_0 = \frac{-\Delta^{-1/2} - \sqrt{\Delta^{-1} - 4hb\Delta^{-1}}}{2h}, v = 0, \theta, u = 0 \right) \tag{3.18}$$

Proof. Using the energy relation (3.11) we see that $u = 0$ and $v = 0$ if and only if $u^2 + v^2 = 0$ and $v = 0$ implies $r' = 0$. On the other hand $u^2 + v^2 = 0$ if and only if $2r^2h + 2r\Delta^{-1/2} + 2b\Delta^{-1} = 0$. This second order equation has solutions

$$r = \frac{-\Delta^{-1/2} \pm \sqrt{\Delta^{-1} - 4hb\Delta^{-1}}}{2h}. \tag{3.19}$$

Since $r \geq 0$ and $h < 0$ we only take the solution with the minus sign. This shows that r_0 is the zero velocity curve. The fact that the motion is bounded by this curve follows from the remark that $r > r_0$ implies $u^2 + v^2 < 0$. □

In order to describe the flow on the infinity manifolds it is convenient to use the same transformations that we introduced in Section 2.4 for the Manev problem.

First consider the case $h = 0$, studied in [18]. Taking $h = 0$ and $\rho = 1/r$ the system of equations 3.10 becomes:

$$\begin{cases} \rho' = -\rho v \\ v' = \rho^{-1}\Delta^{-1/2} \\ \theta' = u \\ u' = [(\mu-1)/2](\rho^{-1}\Delta^{-3/2} + 2b\Delta^{-2})\sin 2\theta \end{cases} \tag{3.20}$$

and the energy relation (3.11) takes the form

$$\rho(u^2+v^2) - 2\Delta^{-1/2} - 2b\rho\Delta^{-1} = 0. \tag{3.21}$$

Taking $\overline{v} = \rho^{1/2}v, \overline{u} = u\rho^{1/2}$, and rescaling the time variable by using $d\tau = \rho^{1/2}ds$, the system of equations in (3.20) takes the form

$$\begin{cases} \dot{\rho} = -\rho\overline{v} \\ \dot{\overline{v}} = -(1/2)\overline{v}^2 + \Delta^{-1/2} \\ \dot{\theta} = \overline{u} \\ \dot{\overline{u}} = -(1/2)\overline{v}\ \overline{u} + [(\mu-1)/2](\Delta^{-3/2} + 2b\rho\Delta^{-2})\sin 2\theta \end{cases} \tag{3.22}$$

where the dot denotes differentiation with respect to the new time variable s. In the new coordinates the energy relation takes the form

$$\overline{u}^2 + \overline{v}^2 - 2\Delta^{-1/2} - 2b\rho\Delta^{-1} = 0. \tag{3.23}$$

The generators S_0, S_1, S_2 of the symmetry group of the anisotropic Manev problem, in the new coordinates are changed to $\overline{S}_0, \overline{S}_1, \overline{S}_2$ where

$$\begin{aligned} \overline{S}_0(\rho,\overline{v},\theta,\overline{u},s) &\to (\rho,-\overline{v},\theta,-\overline{u},-s) \\ \overline{S}_1(\rho,\overline{v},\theta,\overline{u},s) &\to (\rho,-\overline{v},-\theta,\overline{u},-s) \\ \overline{S}_2(\rho,\overline{v},\theta,\overline{u},s) &\to (\rho,-\overline{v},\pi-\theta,\overline{u},-s). \end{aligned} \tag{3.24}$$

We can now define the infinity manifold for the anisotropic Manev problem as

$$\overline{I}_0 = \{(\rho,\overline{v},\theta,\overline{u}) | \rho = 0 \text{ and } \overline{u}^2 + \overline{v}^2 = 2\Delta^{-1/2}\} \tag{3.25}$$

which is homeomorphic to a torus (see Figure 3.6).

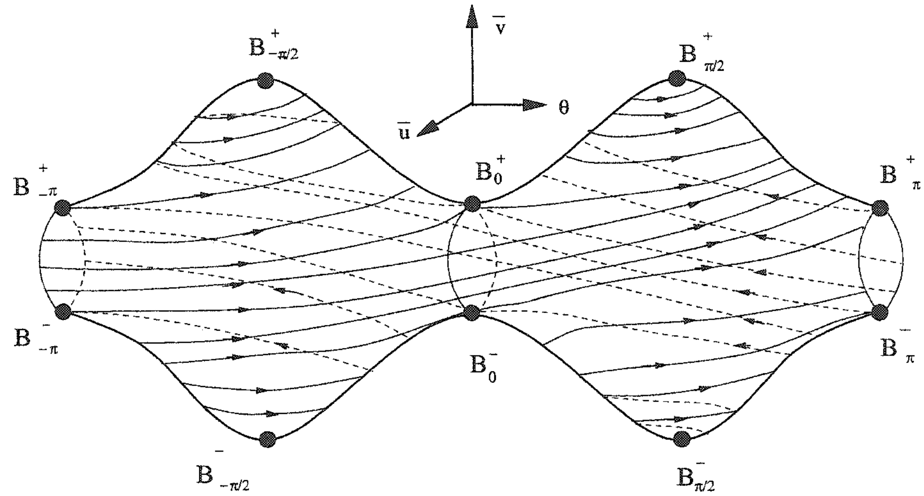


Figure 3.6: The flow on the infinity manifold

The flow on $\overline{I}_0$ is described by the following system of differential equations

$$\begin{cases} \dot{\overline{v}} = (1/2)\overline{u}^2 \\ \dot{\theta} = \overline{u} \\ \dot{\overline{u}} = -(1/2)\overline{v}\ \overline{u} + [(\mu - 1)/2]\Delta^{-3/2}\sin 2\theta. \end{cases} \tag{3.26}$$

For $\mu = 1$ we have the usual Manev problem with two circles of equilibria $\overline{I}_0^{\pm}$ on $\overline{I}_0$. When $\mu > 1$, each of these circles breaks up into four distinct equilibria: two sources (or sinks) and two hyperbolic saddle points. More precisely we have the following

Proposition 3.6. *The flow in (3.26) admits exactly eight equilibrium points. The location as well as the characteristic exponents of these equilibria are as displayed in Table 3.3, where* $D = (4\mu)^{-1/2} + 4(\mu - 1)\mu^{-3/2}$ *and* $E = -9/2 + 4\mu$

Proof. To see that these are the only equilibria on $\overline{I}_0$, note that $\dot{\theta} = 0$ if and only if $\overline{u} = 0$ and that $\rho = 0$ on $\overline{I}_0$. Hence $\dot{\overline{u}} = [(\mu - 1)/2]\Delta^{-3/2}\sin 2\theta$ that is zero only when $\sin 2\theta = 0$. The solutions of the previous equation in the interval $(-\pi, \pi]$ are $0, \pm\pi/2, \pi$. Correspondingly for $\theta = 0, \pi$ we have $\dot{\overline{v}} = -(1/2)\overline{v}^2 + \mu^{-1/2}$ which gives

$\overline{v} = \pm\sqrt{2\mu^{-1/2}}$. On the other hand, when $\theta = \pm\pi/2$ we get $\dot{\overline{v}} = -(1/2)\overline{v}^2 + 1$ which gives $\overline{v} = \pm\sqrt{2}$.

To compute the characteristic exponents of the various equilibria, one simply calculates the linearization of the system at the equilibria. The linearization computed at $B_0^{\pm} = (0, \pm\sqrt{2\mu^{-1/2}}, 0, 0)$ and $B_{\pi}^{\pm} = (0, \pm\sqrt{2\mu^{-1/2}}, \pi, 0)$ is

$$\begin{pmatrix} \mp\sqrt{2\mu^{-1/2}} & 0 & 0 & 0 \\ 0 & \mp\sqrt{2\mu^{-1/2}} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & (\mu-1)/\mu^{3/2} & \mp(1/2)\sqrt{2\mu^{-1/2}} \end{pmatrix}. \tag{3.27}$$

The eigenvalues of this matrix are:

$$\begin{aligned} \lambda_0 &= \mp\sqrt{2\mu^{-1/2}} \\ \lambda_1 &= \mp\sqrt{2\mu^{-1/2}} \\ \lambda_2 &= \mp(4\mu)^{-1/4}/2 + (1/2)\sqrt{D} \\ \lambda_3 &= \mp(4\mu)^{-1/4}/2 - (1/2)\sqrt{D} \end{aligned}$$

where $D = (4\mu)^{-1/2} + 4(\mu - 1)\mu^{-3/2}$. Finally, the linearization at the equilibria $B_{\pm\pi/2}^{\pm} = (0, \pm\sqrt{2}, \pm\pi/2, 0)$ is

$$\begin{pmatrix} \mp\sqrt{2} & 0 & 0 & 0 \\ 0 & \mp\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & (1-\mu) & \mp(1/2)\sqrt{2} \end{pmatrix} \tag{3.28}$$

with eigenvalues

$$\begin{aligned} \lambda_0 &= \mp\sqrt{2} \\ \lambda_1 &= \mp\sqrt{2} \\ \lambda_2 &= \mp\sqrt{2}/4 + (i/2)\sqrt{E} \\ \lambda_3 &= \mp\sqrt{2}/4 - (i/2)\sqrt{E} \end{aligned}$$

where $E = -9/2 + 4\mu$. □

Equilibrium point	Characteristic Exponents	Type on $\overline{I}_0$
$B^-_{-\pi/2} = (0, -\sqrt{2}, -\pi/2, 0)$	$+\sqrt{2}, +\sqrt{2},\ \sqrt{2}/4 \pm (i/2)\sqrt{E}$	Source
$B^-_0 = (0, -\sqrt{2\mu^{-1/2}}, 0, 0)$	$+\sqrt{2\mu^{-1/2}}, +\sqrt{2\mu^{-1/2}},\ (4\mu)^{-1/4}/2 \pm (1/2)\sqrt{D}$	Saddle
$B^-_{\pi/2} = (0, -\sqrt{2\mu^{-1/2}}, \pi/2, 0)$	$+\sqrt{2}, +\sqrt{2},\ \sqrt{2}/4 \pm (i/2)\sqrt{E}$	Source
$B^-_{\pi} = (0, -\sqrt{2\mu^{-1/2}}, \pi, 0)$	$+\sqrt{2\mu^{-1/2}}, +\sqrt{2\mu^{-1/2}},\ (4\mu)^{-1/4}/2 \pm (1/2)\sqrt{D}$	Saddle
$B^+_{-\pi/2} = (0, \sqrt{2}, -\pi/2, 0)$	$-\sqrt{2}, -\sqrt{2},\ -\sqrt{2}/4 \pm (i/2)\sqrt{E}$	Sink
$B^+_0 = (0, \sqrt{2\mu^{-1/2}}, 0, 0)$	$-\sqrt{2\mu^{-1/2}}, -\sqrt{2\mu^{-1/2}},\ -(4\mu)^{-1/4}/2 \pm (1/2)\sqrt{D}$	Saddle
$B^+_{\pi/2} = (0, \sqrt{2\mu^{-1/2}}, \pi/2, 0)$	$-\sqrt{2}, -\sqrt{2},\ -\sqrt{2}/4 \pm (i/2)\sqrt{E}$	Sink
$B^+_{\pi} = (0, \sqrt{2\mu^{-1/2}}, \pi, 0)$	$-\sqrt{2\mu^{-1/2}}, -\sqrt{2\mu^{-1/2}},\ -(4\mu)^{-1/4}/2 \pm (1/2)\sqrt{D}$	Saddle

Table 3.3: Characteristic exponents of the equilibrium points on $\overline{I}_0$

Note that the characteristic exponents of both the sinks and the sources have non-zero imaginary parts. This means that the orbits tend to spiral into and away from corresponding sinks and sources.

To have a complete qualitative picture of the flow on $\overline{I}_0$ we need to study the ultimate behavior of the stable and unstable manifolds of the saddle points. In our case the stable and unstable manifolds are analytic curves that consist precisely of two orbits tending toward or away from the equilibrium. We want to show that, for most values of μ, there are no saddle connections for the flow on $\overline{I}_0$. In order to do that we need several additional facts.

Proposition 3.7. *The flow on $\overline{I}_0$ is gradient-like*[1] *with respect to the $\overline{v}$-coordinate.*

Proof. From the first of the equations in (3.26) it is clear that $\dot{\overline{v}} \geq 0$. If $\dot{\overline{v}} = 0$ then $\overline{u} = 0$ and $\ddot{\overline{v}} = \overline{u}\,\dot{\overline{u}} = 0$. Moreover $\dddot{\overline{v}} = \dot{\overline{u}}^2 + \overline{u}\,\ddot{\overline{u}} = \dot{\overline{u}}^2$, which is 0 only at the

[1] The flow is called *gradient-like* with respect to one of the coordinates, if every nonequilibrium solution increases on that coordinate

equilibria and is positive otherwise. □

Corollary 3.8. *If we exclude the fixed points there are no closed or recurrent orbits*[2] *on* $\overline{I}_0$.

Proof. If a periodic orbit $\gamma(t)$ of period T existed then $\gamma(t) = \gamma(t+T)$. Consequently $\overline{v}(\gamma(t)) = \overline{v}(\gamma(t+T))$. Rolle's theorem implies that there exists $\xi \in]t, t+T[$ such that $\dot{\overline{v}}(\gamma(\xi)) = 0$. This contradicts the fact that the flow is gradient-like with respect to $\overline{v}$.

Now, suppose a recurrent orbit $\gamma(t)$ exists and consider $T > 0$ such that $\overline{v}(\gamma(T)) - \overline{v}(\gamma(0)) > \delta$ for some $\delta > 0$. Furthermore consider a neighborhood U of $\gamma(0)$ such that $|\overline{v} - \overline{v}(\gamma(0))| < \delta/2$. Then by definition of recurrent orbit there is a $t > T$ such that $\gamma(t) \in U$ and hence $|\overline{v}(\gamma(t)) - \overline{v}(\gamma(0)| < \delta/2$. On the other hand, since the flow is gradient-like with respect to $\overline{v}$ one has that $\overline{v}(\gamma(t)) > \overline{v}(\gamma(T))$ and thus $|\overline{v}(\gamma(t)) - \overline{v}(\gamma(0))| > \delta$. This is a contradiction. □

As a consequence all orbits on the infinity manifold must tend toward one of the equilibria. Figure 3.6 shows the flow on $\overline{I}_0$. The stable manifolds at B_0^- and B_π^- must emanate directly from the sources, while the unstable manifolds at B_0^+ and B_π^+ must fall directly into the sinks for all $\mu > 1$. We now want to show that there are no saddle connections for the flow. In order to do that it is sufficient to prove that the remaining invariant manifolds do not match up. This is true for most values of μ. Observe that the two branches of $W^s(B_\pi^-)$ each emanate from distinct sources. This follows immediately by the fact that the flow is invariant under the symmetry

[2] An orbit $\gamma(t)$ is said to be recurrent if for any point $x \in \gamma(t)$, for every neighborhood U of x, and for any $T > 0$ there exists $t > T$ such that $\gamma(t) \in U$. Periodic orbits are a particular case of recurrent orbits.

$\overline{S}_5(\rho, \overline{v}, \theta, \overline{u}, s) = (\rho, \overline{v}, -\theta, -\overline{u}, s)$ (where $\overline{S}_5 = \overline{S}_0 \circ \overline{S}_1$). A similar result holds for the other saddle points.

It is now convenient to introduce different variables on $\overline{I}_0$. Indeed we observed that $\overline{I}_0$ is diffeomorphic to a torus. Consequently we can describe the flow on the infinity manifold using angle variables as was observed when we stated the Liouville-Arnold theorem (see Section 2.2) and when we introduced angle variables (see 2.7). Introducing a new angle variable ψ defined by

$$\begin{cases} \overline{u} = \frac{\sqrt{2}}{\Delta^{1/4}} \sin\psi \\ \overline{v} = \frac{\sqrt{2}}{\Delta^{1/4}} \cos\psi \end{cases} \tag{3.29}$$

we can rewrite the flow on the collision manifold as:

$$\begin{cases} \dot{\psi} = \frac{d}{ds}\left(\frac{\sqrt{2}}{\Delta^{1/4}}\right) \frac{\Delta^{1/4}}{\sqrt{2}} \cot\psi + \frac{1}{\sqrt{2}\Delta^{1/4}} \sin\psi \\ \dot{\theta} = \frac{\sqrt{2}}{\Delta^{1/4}} \sin\psi \end{cases} \tag{3.30}$$

The equilibrium points in the new variables (θ, ψ) take the following form: $B^-_{\pm\pi/2} = (\pm\pi/2, \pi)$, $B^+_{\pm\pi/2} = (\pm\pi/2, 0)$, $B^-_0 = (0, \pi)$, $B^+_0 = (0, 0)$, $B^-_\pi = (\pi, \pi)$ and $B^+_\pi = (\pi, 0)$.

Proposition 3.9. *For an open and dense set of real numbers $\mu > 1$, the unstable manifolds at B^-_π and B^-_0 miss the stable manifolds at B^+_0 and B^+_π.*

Proof. First consider $W^u(B^-_\pi) = W^u(B^-_{-\pi})$. Then from (3.30) we get

$$\frac{d\psi}{d\theta} = \frac{d}{ds}\left(\frac{\sqrt{2}}{\Delta^{1/4}}\right) \frac{\Delta^{1/2}}{2} \frac{\cos\psi}{\sin^2\psi} - \frac{1}{2} = F(\theta, \psi, \epsilon) \tag{3.31}$$

where $\epsilon = \mu - 1$ and $\Delta = 1 + \epsilon\cos^2\theta$. When $\epsilon = 0$, $W^s(B^+_\pi)$ matches exactly with $W^u(B^-_{-\pi})$. Consider the branch of $W^u(B^-_{-\pi})$ which contains the point $(0, \pi/2)$. From (3.31) it can be easily shown that this curve lies along the line

$$-2\psi - \theta = -\pi \tag{3.32}$$

since, for $\epsilon = 0$, $-\theta - 2\psi$ is constant along the orbits.

When ϵ increases above 0, this branch of the unstable manifold varies smoothly in $\overline{I}_0$. Let $\zeta(\theta, \epsilon)$ denote the ψ-coordinate of this curve, with $\zeta(-\pi, \epsilon) = \pi$. Below we prove :

Lemma 3.10. $\frac{\partial}{\partial \epsilon}\zeta(0,0) = -\pi/4 < 0$.

Consequently $\zeta(0, \epsilon) < \pi/2$ for ϵ small and positive. Furthermore $\overline{v}(0, \zeta(0, \epsilon)) \approx \sqrt{2\mu^{-1/2}}\cos(\pi/2 - \epsilon\pi/4) > 0$ for ϵ small. Now Equations (3.30) are reversed by the transformation

$$(\theta, \psi) \to (-\theta, \pi - \psi).$$

In particular the unstable manifold through $B^-_{-\pi}$ is mapped onto the stable manifold through B^+_π by this map. Hence the stable manifold goes through some point $(0, \psi_0)$ with $\psi_0 > \pi/2$. At this point $\overline{v}(0, \psi_0) < 0$. Thus, the unstable manifold $W^s(B^-_{-\pi})$ misses the stable manifold $W^u(B^+_\pi)$ at least for ϵ small and positive. Similar arguments show that all unstable manifolds of the saddle points miss the stable manifolds for $\epsilon > 0$ small. Moreover, these stable and unstable manifolds intersect only for a discrete set of ϵ, since they vary analytically with ϵ. This completes the proof with the exception of Lemma 3.10 □

Proof of Lemma 3.10. Observe that ζ satisfies the equation:

$$\zeta(\theta, \epsilon) = \int_{-\pi}^{\theta} F(\eta, \zeta(\eta), \epsilon) d\eta \tag{3.33}$$

where F is given by (3.31). For ϵ small we can write

$$\zeta = \zeta_0(\theta) + \epsilon\zeta_1(\theta) + O(\epsilon^2). \tag{3.34}$$

We also have that

$$\zeta_0(\theta) = -(1/2)\theta + \pi/2 \tag{3.35}$$

Now to compute $\zeta_1(\theta)$ we can use the Taylor expansion of (3.33) with respect to ϵ and we find

$$\zeta_1(\theta) = \int_{-\pi}^{\theta} \left(\frac{\partial}{\partial \epsilon} F(\eta, \zeta_0(\eta), 0) + \frac{\partial}{\partial \psi} F(\eta, \zeta_0(\eta), 0)\zeta_1(\eta) \right) d\eta. \tag{3.36}$$

Standard computations show that

$$\frac{\partial}{\partial \epsilon} F(\eta, \zeta_0(\eta), \epsilon) = \frac{1}{2} \frac{\cos(\eta)\sin(\eta)\cos(\zeta_0(\eta))}{\sin(\zeta_0(\eta))} + O(\epsilon) \tag{3.37}$$

and that

$$\frac{\partial}{\partial \psi} F(\eta, \zeta_0(\eta), \epsilon) = O(\epsilon). \tag{3.38}$$

We can now compute $\zeta_1(\theta)$:

$$\begin{aligned}
\zeta_1(\theta) &= \int_{-\pi}^{\theta} \left(\frac{\partial}{\partial \epsilon} F + \frac{\partial}{\partial \psi} F \zeta_1 \right) d\eta \\
&= \frac{1}{2} \int_{-\pi}^{\theta} \frac{\cos(\eta)\sin(\eta)\cos(\zeta_0(\eta))}{\sin(\zeta_0(\eta))} d\eta \\
&= \frac{1}{2} \int_{-\pi}^{\theta} \frac{\cos(\eta)\sin(\eta)\cos(-\frac{1}{2}\eta + \frac{\pi}{2})}{\sin(-\frac{1}{2}\eta + \frac{\pi}{2})} d\eta \\
&= \frac{1}{2} \int_{-\pi}^{\theta} \frac{\cos(\eta)\sin(\eta)\cos(\frac{1}{2}\eta)}{\sin(\frac{1}{2}\eta)} d\eta \\
&= \frac{3}{2} \cos\left(\frac{1}{2}\theta\right) \sin\left(\frac{1}{2}\theta\right) - \frac{1}{4}\theta - \left(\cos\left(\frac{1}{2}\theta\right)\right)^3 \sin\left(\frac{1}{2}\theta\right) - \frac{1}{4}\pi.
\end{aligned} \tag{3.39}$$

Thus when $\theta = 0$, we have

$$\zeta_1(0) = -\pi/4 = \frac{\partial}{\partial \epsilon} \zeta(0, 0). \tag{3.40}$$

this completes the proof of the Lemma. □

Using the fact that the flow is gradient-like with respect to $\overline{v}$ and Proposition 3.9 we have

Theorem 3.1. *For an open and dense set of $\mu > 1$ the flow on the infinity manifold $\overline{I}_0$ satisfies: all stable manifolds of saddle points emanate from sources, and all unstable manifolds of saddle points die in sinks.*

A similar statement is verified for the flow on the collision manifold of the anisotropic Kepler problem (see [21]). Comparing the properties of our flow on $\overline{I}_0$ with the flow on the total collision manifold studied in [21] for the anisotropic Kepler problem it is immediate to prove

Proposition 3.11. *For an open dense set of $\mu \geq 1$ the flow on the collision manifold $\overline{I}_0$ of the anisotropic Manev problem is topologically equivalent to the flow on the collision manifold of the anisotropic Kepler problem.*

Another interesting property of the flow on $\overline{I}_0$ is that it is structurally stable. In order to prove this we need the famous Peixoto's Theorem for Two-Dimensional Flows

Theorem 3.2. *A C^r vector field on a compact two-dimensional manifold M^2 is structurally stable if and only if:*

(i) the number of fixed points and closed orbits is finite and each is hyperbolic;

(ii) there are no orbits connecting saddle points;

(iii) the nonwandering set consists of fixed points and periodic orbits alone.

Moreover if M^2 is orientable, the set of structurally stable vector fields is open-dense in the set of all vector fields on two-dimensional manifolds.

For a proof of the theorem see [61]. We are now ready to prove the following

Theorem 3.3. *The flow on the infinity manifold $\overline{I}_0$ is structurally stable for an open and dense set of $\mu > 1$.*

Proof. To prove the theorem we only need to show that the hypotheses of Theorem 3.2 are verified. Firstly observe that $\overline{I}_0$ is a compact two-manifold since it is diffeomorphic to a torus. (i) is verified because by Proposition 3.7 there are eight hyperbolic equilibrium points, and by Corollary 3.8 there are no periodic orbits. (ii) is verified because, by Theorem 3.1, there are no saddle connections for an open dense set of $\mu > 1$. (iii) is verified because from the qualitative description of the flow, and in particular from Corollary 3.8, it follows that the nonwandering set consists of fixed points alone. This concludes the proof. □

Remark 3.1. The previous theorem also implies that the flow on the collision manifold of the anisotropic Kepler problem is structurally stable.

Remark 3.2. For $\mu = 1$ the flow on I_0 is obviously not structurally stable because there are saddle connections.

Now let us consider the case $h > 0$. We make the same change of variables (2.22) used for the Manev problem. From (3.10) we obtain

$$\begin{cases} \dot{R} = -(2/3)RV \\ \dot{V} = -V^2 + 2h + R^{3/2}\Delta^{-1/2} \\ \dot{\theta} = U \\ \dot{U} = -UV + [(\mu-1)/2][R^{3/2}\Delta^{-3/2} + 2bR^3\Delta^{-2}]\sin 2\theta \end{cases} \tag{3.41}$$

where the dot denotes differentiation with respect to s. From (3.11) the energy relation becomes

$$U^2 + V^2 - 2R^{3/2}\Delta^{-1/2} - 2bR^3\Delta^{-2} = 2h \tag{3.42}$$

Again $R = 0$ is an invariant manifold under the flow given by (3.41), defined by

$$\overline{I}_h = \{(R, V, \theta, U) | R = 0 \text{ and } U^2 + V^2 = 2h, \theta \in S^1\}. \tag{3.43}$$

The flow on $\overline{I}_h$ is given by

$$\begin{cases} \dot{V} = -V^2 + 2h \\ \dot{\theta} = U \\ \dot{U} = -UV \end{cases} \tag{3.44}$$

and it is immediate to see that this is exactly the flow on the infinity manifold I_h of the Manev problem. This shows that the perturbation does not change the infinity manifold for $h > 0$. The flow on I_h is sketched in Figure 2.3.

3.6 Equations of Motion in Action Angle Variables

In Section 2.7 we introduced action angle variables for the Manev Problem. In Section 3.2 we showed that for weak anisotropies, i.e. $\epsilon = \mu - 1 \ll 1$, the problem under discussion in this Chapter can be considered as a perturbation of the Manev one. It is often convenient to write the equation of motion of the perturbed system in terms of action angle variables of the unperturbed one. In particular, in our case the variables introduced in equation (2.35) are very important because, in Chapter 5, they will enable us to find symmetric periodic orbits using the Poincaré continuation method. Recall that the Hamiltonian of the Manev problem is

$$H_0 = -\frac{1}{2(-G + \sqrt{(G+L)^2 - 2b})^2} \tag{3.45}$$

and the action angle variable are defined by the equations (2.34) and (2.35). Consequently the perturbed equations of motion become of the form

$$\begin{cases} \dot{L} = -\frac{\partial(H_0+\epsilon W)}{\partial l} = -\epsilon\frac{\partial W}{\partial l} \\ \dot{G} = -\frac{\partial(H_0+\epsilon W)}{\partial g} = -\epsilon\frac{\partial W}{\partial g} \\ \dot{l} = \frac{\partial(H_0+\epsilon W)}{\partial L} = \omega_L + \epsilon\frac{\partial W}{\partial L} \\ \dot{g} = \frac{\partial(H_0+\epsilon W)}{\partial G} = \omega_G + \epsilon\frac{\partial W}{\partial G} \end{cases} \tag{3.46}$$

where $W = W(L, G, l, g)$ is expressed in the new variables and

$$\begin{cases} \omega_L = \omega_K = \frac{G+L}{(-G+\sqrt{(G+L)^2-2b})^3\sqrt{(G+L)^2-2b}} \\ \omega_G = \omega_K - \omega_I = \frac{G+L-\sqrt{(G+L)^2-2b}}{(-G+\sqrt{(G+L)^2-2b})^3\sqrt{(G+L)^2-2b}}. \end{cases}$$

Chapter 4

The Flow on Negative Energy Levels

4.1 Overview

This chapter is devoted to analyzing the flow on the negative energy levels. As was shown in Section 3.5 the motion for $h < 0$ is bounded, and the dynamics on the negative energy levels is usually the most interesting. This is because on one hand one is mostly interested to know what are the dynamics of bodies the motion of which remains bounded. If one is considering celestial bodies one is usually interested in studying the motion of the planets and the asteroids, or if considering atomic systems one is interested in the bound states. On the other hand the dynamics on the negative energy levels are more complex and rich compared to the dynamics for $h \geq 0$.

In the next section we recall some results concerning the flow near the collision manifold that was studied in detail in [18].

In the following section we analyze some properties of the heteroclinic orbits connecting the equilibria on the collision manifold.

In Section 4.4 we give a physical interpretation of the flow near the collision manifold and we describe the collision orbits. In particular we introduce three different kinds of collisions: the *frontal collisions*, the *spiraling collisions* and the *oscillatory collisions*. We also depict, for each of the three classes mentioned above, an orbit obtained performing a numerical integration.

In Sections 4.5 and 4.6 we develop a perturbation technique that is an extension of the Poincaré-Melnikov method. This method allows us to study how the positively and negatively asymptotic sets of the periodic orbit on the equator of the collision manifold intersect each other.

In the last Section we prove that the positively and negatively asymptotic sets of the periodic orbit intersect transversally in infinitely many points, possibly giving rise to chaotic dynamics.

4.2 The Flow Near The Collision Manifold

In this section we want to complete the study of the flow near the collision manifold introducing some local results. The flow near the collision manifold was studied in detail in [18]. Let us recall some facts that summarize the behavior of the flow near the collision manifold.

Proposition 4.1. *On the collision manifold C the equilibria $A_0^{\pm}$ and $A_\pi^{\pm}$ are saddles whereas the equilibria $A_{\pm\pi/2}^{\pm}$ are centers. In phase space the equilibria A_0^{+} and A_π^{+} , have a 2-dimensional unstable analytic manifold whereas $A_{\pm\pi/2}^{+}$ have a 1-dimensional unstable manifold. The equilibria A_0^{-} and A_π^{-}, have a 2-dimensional stable analytic manifold, whereas $A_{\pm\pi/2}^{-}$ have a 1-dimensional stable manifold.*

Proof. The proof follows immediately from Table 3.2. □

Denote by p_η the periodic orbit on C having $v = \eta$. Notice that for each $\eta \in (\sqrt{2b/\mu}, \sqrt{2b}) \cup (-\sqrt{2b}, -\sqrt{2b/\mu})$ there are two periodic orbits whose angular coordinate θ varies in different domains. Moreover for each $\eta \in (-\sqrt{2b/\mu}, \sqrt{2b/\mu})$

there are two periodic orbits one "internal" to the collision manifold and one "external". However we will denote each of them by the same p_η. It is natural to ask what are the properties of the stable and unstable manifolds of such orbits. The following proposition was proved in [18].

Proposition 4.2. *Each periodic orbit* η *on* C *with* $v = \eta > 0 (v = \eta < 0)$ *has a 2-dimensional local unstable analytic manifold, while the periodic orbit with* $v = 0$ *has both a 2-dimensional local unstable and a 2-dimensional local stable manifold (see Figure 4.1).*

4.3 Heteroclinic Orbits

In this section we prove some global results that extend the understanding of the problem under discussion. Specifically we want to consider special orbits of the anisotropic Manev problem which begin and end in collision with the origin. The change of time scale has the effect of slowing such orbits down so that they tend asymptotically toward or away from C. The orbits which begin and end at collision are called *bi-collision orbits.* For the Manev problem, all these orbits are well understood. For negative total energy there is a two parameter family of such orbits. They lie on the "cylinders" $u = \text{const}$. Some of them connect the upper and lower circles $C^{\pm}$ of the collision manifold C_0 and others are in the intersection of the stable and unstable manifold of periodic orbits on C_0. When $\mu > 1$ many of these bi-collision orbits are destroyed. Some, however, persist for all μ. In particular there are four *primary bi-collision orbits* that persist and are given by the proposition below.

Proposition 4.3. *There are four bi-collision orbits for the anisotropic problem which persist for all* μ*. Each orbit leaves the origin and travels along the positive or negative* x *or* y *axis to the zero velocity curve and then returns to* C*.*

Proof. The original differential equation is invariant under the symmetry S_5. Hence the orbits passing through points of the form $(x, y, p_x, p_y) = (x, 0, p_x, 0)$ are trapped in the (x, p_x) plane; they project to orbits which travel along the x-axis in configuration space. Furthermore, for negative energy, Proposition 3.5 implies that such orbits are bounded by

$$|x| \leq \frac{(-1 - \sqrt{1 - 4hb})}{2h}.$$

Thus it follows that, for each negative energy level, there are exactly two bi-collision orbits trapped on the x-axis. Each orbit leaves the origin of the coordinate system with infinite velocity and travels along either the positive or negative $x-$axis until reaching $x = (-1 - \sqrt{1 - 4hb})/2h$. At that point, the particle momentarily has zero velocity, and then falls back toward the origin. We denote these bi-collision orbits by γ_0, γ_π.

The differential equation is also invariant under S_4. This implies the existence of two additional bi-collision orbits for each negative energy level trapped on the $y-$axis. We denote these orbits by $\gamma_{\pm\pi/2}$. □

The change of time scale (2.8) has the effect of slowing down the primary bi-collision orbits so that they approach C asymptotically in both time directions. Furthermore, from the change of coordinates (2.7) it is easy to see that each such orbit is asymptotic to an equilibrium point on C. We summarize this data as follows

Proposition 4.4. *Let $W^s(q)$ and $W^u(q)$ denote the stable and unstable manifolds at the equilibrium point q. Then*

(i) $\gamma_{-\pi/2} = W^u(A^+_{-\pi/2}) \cap W^s(A^-_{-\pi/2})$

(ii) $\gamma_{\pi/2} = W^u(A^+_{\pi/2}) \cap W^s(A^-_{\pi/2})$

(iii) $\gamma_0 \subset W^u(A^+_0) \cap W^s(A^-_0)$

(iv) $\gamma_\pi \subset W^u(A^+_\pi) \cap W^s(A^-_\pi)$

Proof. To conclude the proof it is enough to show that the equality holds in (i) and (ii) and not in (iii) and (iv). From Table 3.2, both $W^u(A^+_{\pm\pi/2})$ and $W^s(A^-_{\pm\pi/2})$ are one-dimensional; this accounts for the equality in (i) and (ii). In contrast the dimensions of the remaining stable and unstable manifolds: $W^u(A^+_0)$, $W^u(A^+_\pi)$ and $W^s(A^+_0)$ $W^s(A^+_\pi)$ are all two. □

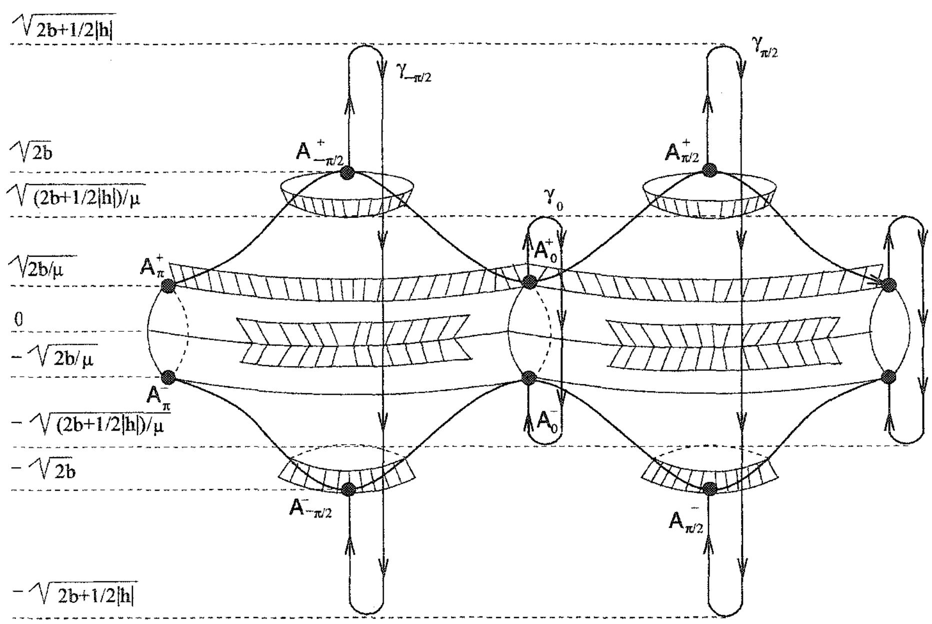


Figure 4.1: The flow can reach the collision manifold at the equilibria or at any of the periodic orbits. There are four heteroclinic orbits $\gamma_{-\pi/2}$,γ_0 ,$\gamma_{\pi/2}$,γ_π connecting respectively $A^+_{-\pi/2}$ with $A^-_{-\pi/2}$, A^+_0 with A^-_0 , $A^+_{\pi/2}$ with $A^-_{\pi/2}$, and A^+_π with A^-_π.

4.4 Physical Interpretation

In this section we give physical interpretations of the results mentioned in the two previous sections. Since the motion on the collision manifold is fictitious we will consider only the orbits outside the collision manifold. In particular we are interested in analyzing the collision orbits. We can divide the orbits tending to (emerging from) the collision manifold into three classes:

(i) the orbits on the stable manifold (unstable manifold) of one of the equilibria

(ii) the orbits on the stable manifold (unstable manifold) of one of the periodic orbits p_v for $v \in (-\sqrt{2b/\mu}, \sqrt{2b/\mu})$

(iii) the orbits on the stable manifold (unstable manifold) of one of the periodic orbits p_v for $v \in (-\sqrt{2b}, -\sqrt{2b/\mu})$ or $v \in (\sqrt{2b/\mu}, \sqrt{2b})$ (i.e. one of the periodic orbits on the bumps).

In the case (i) the solutions tending to (emerging from) the equilibria represent collisions (ejections) which have limiting zero angular momentum. We call those solutions *frontal collisions* (*ejections*) or just *frontal collisions* for short. There are two kinds of frontal collisions the *homotetic ones* and the *nonhomotetic* ones.

The first ones coincide with the orbits $\gamma_{\pm\pi/2}, \gamma_0, \gamma_\pi$ that we introduced in the previous sections. Therefore in the physical space the homotetic orbits are straight lines that lie on the x or y-axis.

The nonhomotetic orbits have a different behavior. For example, Figure 4.2 depicts a (doubly asymptotic) nonhomotetic collision orbit that was found numerically. Taking an appropriate initial condition on the y-axis the corresponding solution departs from the y-axis and then collides at the origin coming from the negative part of the x-axis. The orbit is tangent to the x-axis at the collision point. Integrating backwards one finds that the orbit collides coming from the positive part of the x-axis.

Let us now describe case (ii). The orbits on the stable manifold (unstable manifold) (of those periodic orbits) are collision (ejection) orbits that spiral around

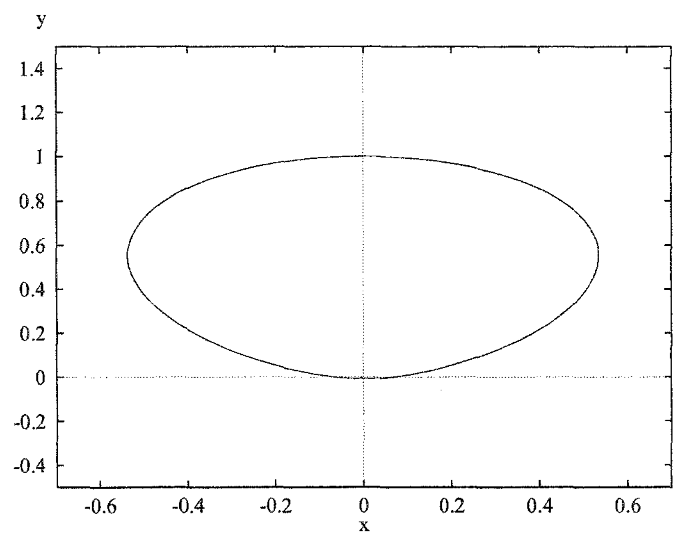


Figure 4.2: A nonhomotetic collision orbit obtained numerically for $\mu = 3$, $b = 0.1$

the origin infinitely many times. Their angular momentum does not vanish. They are called *spiraling collisions* and an example of a spiraling collision is represented in Figure 4.3.

Finally the orbits of case (iii) have an oscillatory behavior. This is because these orbits are on the stable (unstable) manifolds of the cycles on the "bumps" and therefore they oscillate about one of the coordinate axes (see Figure 4.4). Consequently they are called *oscillatory* collisions. The angular momentum of these orbits also oscillates about zero since it is clear that the variable u changes sign. An example of an oscillatory orbit is depicted in Figure 4.4. This orbit was obtained numerically for $\mu = 1000$ and $b = 0.1$.

Remark 4.1. The behavior described above is only local and therefore the collision (ejection) orbit can look remarkably complicated. However the numerical examples depicted in Figure 4.2-4.4 are relatively simple.

Remark 4.2. Observe that as μ increases the Lebesgue measure (computed on

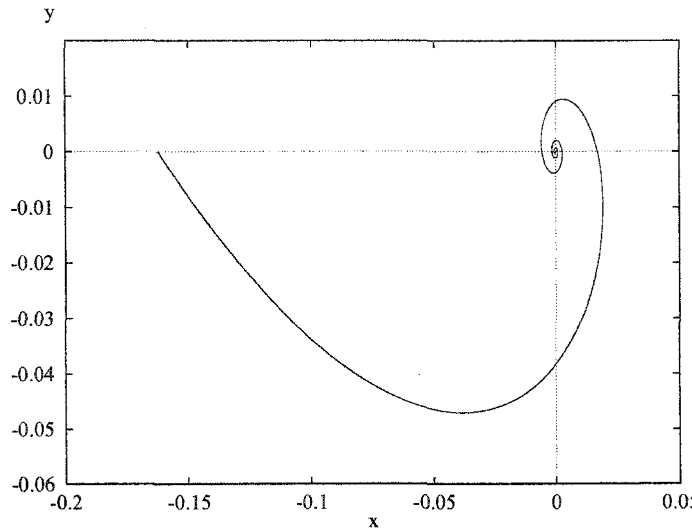


Figure 4.3: A spiraling collision obtained numerically for $\mu = 1.5$, $b = 0.1$ and initial conditions $x = -0.1621258$, $y = 2.736857 \times 10^{-6}$, $p_x = 3.003735$, $p_y = -2.080168$

the three dimensional energy level) of the oscillatory collisions increases while the measure of the spiraling collisions decreases. Indeed the orbits on the bumps have $v \in (-\sqrt{2b}, -\sqrt{2b/\mu})$ or $v \in (\sqrt{2b/\mu}, \sqrt{2b})$ while the other periodic orbits have $v \in (-\sqrt{2b/\mu}, \sqrt{2b/\mu})$. Consequently as $\mu \to \infty$ the measure of the spiraling collisions tends to zero.

4.5 A Perturbative Approach

We will now write the anisotropic Manev problem as a perturbation of the classical Manev case. Consider weak anisotropies, i.e., choose the parameter μ close to 1. Introducing the notation $\mu - 1 = \epsilon > 0$ with $\epsilon \ll 1$, we can expand the equation of

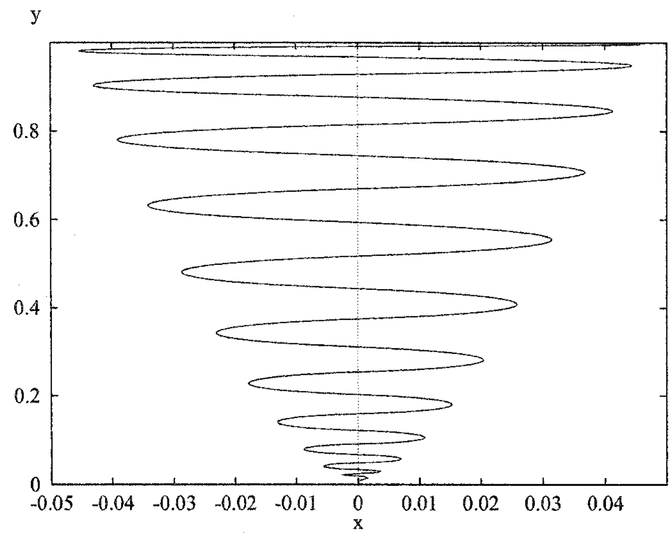


Figure 4.4: An oscillatory collision obtained numerically for $\mu = 1000$, $b = 0.1$ and initial conditions $x = 0,\ y = 1,\ p_x = -1,\ p_y = 0$

motion in powers of ϵ to obtain

$$\begin{cases} r' = rv \\ v' = 2r^2h + r - \epsilon(r/2\cos^2\theta) \\ \theta' = u \\ u' = \epsilon/2(r + 2b)\sin 2\theta. \end{cases} \tag{4.1}$$

The energy relation becomes

$$u^2 + v^2 - 2r - 2b + \epsilon(r + 2b)\cos^2\theta = 2r^2h. \tag{4.2}$$

For $\epsilon = 0$, system (4.1) and equation (4.2) yield the Manev problem. We now recall some important facts from Section 2.3. The collision manifold is the set of solutions given by

$$r = 0, \qquad u^2 + v^2 = 2b. \tag{4.3}$$

and the collision manifold is a cylinder in the three-dimensional space of coordinates (u, θ, v) that can be identified with a torus. Furthermore the flow on the collision manifold is formed almost exclusively by non-hyperbolic periodic orbits, except for the upper and lower circles of the torus given by $r = 0, \quad u = 0, \quad v = \pm\sqrt{2b}$, which consist of equilibrium points. Moreover for every periodic orbit $P_v^\pm$ on the collision manifold with $0 < v < \sqrt{2b}$ there exists a manifold of orbits, lying on a cylinder, which emerge from $P_v^\pm$. On the other hand, for every orbit $P_v^\pm$, with $-\sqrt{2b} < v < 0$, there exists a manifold of orbits, lying on a cylinder, which tend to $P_v^\pm$.

If $v = 0$ both types of manifolds exist, so the two orbits $P_0^\pm$ have a homoclinic manifold. Indeed, the equations that describe the manifold can be found explicitly: they have $u = \pm\sqrt{2b}$. From the energy relation we get

$$v = \pm\sqrt{2r^2h + 2r}, \tag{4.4}$$

and using the equation of motion we obtain

$$r' = \pm r\sqrt{2r^2h + 2r}. \tag{4.5}$$

By integrating equation (4.5) it is easy to find that

$$R(\tau - \tau_0) = \frac{2}{2|h| + (\tau - \tau_0)^2}, \quad R' = -\frac{4(\tau - \tau_0)}{(2|h| + (\tau - \tau_0)^2)^2} \tag{4.6}$$

and

$$V(\tau - \tau_0) = \frac{R'}{R} = -\frac{2(\tau - \tau_0)}{2|h| + (\tau - \tau_0)^2}. \tag{4.7}$$

Furthermore

$$U(\tau - \tau_0) = \pm\sqrt{2b} = \omega \quad \text{and} \quad \vartheta(\tau - \tau_0, \theta_0) = \Theta(\tau - \tau_0) - \theta_0, \tag{4.8}$$

where $\Theta(\tau-\tau_0) = \omega(\tau-\tau_0)$. As τ_0 and θ_0 vary, equations (4.6-4.8) describe the entire 2-dimensional homoclinic manifold. The capital letters are introduced to denote the homoclinic orbits. An orbit lying on the homoclinic manifold is represented in Figure 4.5. Such an orbit is obtained by choosing $\theta_0 = 0$; it emerges from the equator of the

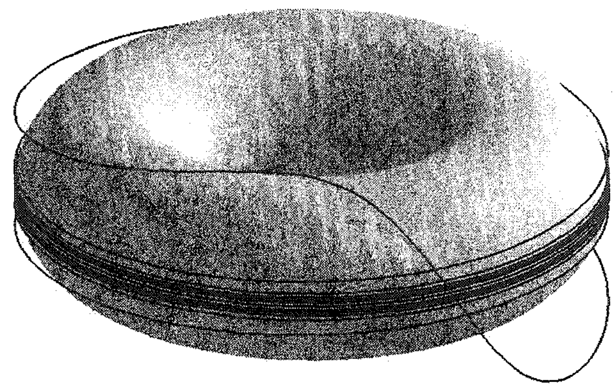


Figure 4.5: An homoclinic orbit to P_0^+ lying on the homoclinic manifold. This orbit spirals out of the equator of the collision manifold and then spirals back to it.

collision manifold, spiraling around it and moving upwards, then changes directions, goes downwards and upwards again, spiraling towards the periodic orbit P_0^+. The homoclinic manifold plays an important role in the following section and is necessary for developing the generalization of the Melnikov technique.

4.6 A Generalized Melnikov Method

Let $\chi = (R(\tau), V(\tau), \Theta(\tau), U(\tau))$ be the homoclinic orbit selected when we choose $\tau_0 = 0$ and $\theta_0 = 0$. Consider solutions of the form

$$\begin{cases} r(\tau, \tau_0) = R(\tau - \tau_0) + \tilde{r}(\tau, \tau_0) \\ v(\tau, \tau_0) = V(\tau - \tau_0) + \tilde{v}(\tau, \tau_0) \\ \theta(\tau, \tau_0, \theta_0) = \Theta(\tau - \tau_0) - \theta_0 + \tilde{\theta}(\tau, \tau_0) \\ u(\tau, \tau_0) = U(\tau - \tau_0) + \tilde{u}(\tau, \tau_0) \end{cases} \tag{4.9}$$

where, in the first line of the previous system of equations, $\tilde{r}(\tau, \tau_0)$ is the difference between the solution of the anisotropic Manev problem $r(\tau, \tau_0)$ and the homoclinic solution of the unperturbed problem $R(\tau - \tau_0)$. The explanation for the other lines of the system above is similar. Let $\tilde{\mathbf{z}} = (\tilde{r}, \tilde{v}, \tilde{\theta}, \tilde{u})$, then the variational equation is

$$\tilde{\mathbf{z}}' = A(\tau)\tilde{\mathbf{z}} + \tilde{\mathbf{b}}(\tilde{\mathbf{z}}, \chi, \tau, \tau_0, \theta_0, \epsilon), \tag{4.10}$$

where

$$A(\tau) = \begin{pmatrix} V & R & 0 & 0 \\ 1+4Rh & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.11}$$

and

$$\tilde{\mathbf{b}}(\tilde{\mathbf{z}}, \chi, \tau, \tau_0, \theta_0, \epsilon) = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} \tilde{r}(\tau-\tau_0)\tilde{v}(\tau-\tau_0) \\ -\epsilon\left(\frac{(R+\tilde{r})}{2}\cos^2(\Theta-\theta_0+\tilde{\theta})\right) \\ 0 \\ \frac{\epsilon}{2}\left((R+\tilde{r})+2b\right)\sin 2(\Theta-\theta_0+\tilde{\theta}) \end{pmatrix}. \tag{4.12}$$

The general solution of the variational equation (4.10) is

$$\tilde{\mathbf{z}} = \Phi(\tau)\int_{\tau_0}^{\tau} \Phi^{-1}(s)\tilde{\mathbf{b}}\, ds, \tag{4.13}$$

(see [41]), where Φ is the fundamental matrix. If we let $c = \Phi^{-1}\tilde{\mathbf{b}}$, the previous equation becomes

$$\tilde{z}_i(\tau) = \Phi_{ij}\int_{\tau_0}^{\tau} c_j(s)\, ds, \tag{4.14}$$

where $c_j = det\ D_j(t)/(det\Phi)(t)$ and D_j is the matrix obtained replacing the j-th column of Φ with $\tilde{\mathbf{b}}$. Furthermore the following formula for the trace holds:

$$\det\Phi(\tau) = Ce^{\int_{\tau_0}^{\tau} TrA(s)\, ds}. \tag{4.15}$$

One solution of the homogeneous part of the variational equation is given by

$$\chi'(\tau-\tau_0, \theta_0) = (R'(\tau-\tau_0), V'(\tau-\tau_0), \Theta'(\tau-\tau_0), U'(\tau-\tau_0)), \tag{4.16}$$

where

$$\begin{aligned} R' &= -\tfrac{4(\tau-\tau_0)}{(2|h|+(\tau-\tau_0)^2)^2} \\ V' &= -\tfrac{2}{2|h|+(\tau-\tau_0)^2} + \tfrac{4(\tau-\tau_0)^2}{(2|h|+(\tau-\tau_0)^2)^2} \\ \Theta' &= \pm\sqrt{2b}. \\ U' &= 0. \end{aligned} \tag{4.17}$$

It is easy to check that two other independent solutions are $(0,0,1,0)$ and $(0,0,\tau,1)$. Knowing three independent solutions of a linear system, it is possible to find a fourth independent solution ψ. This is achieved through the following lemma, which will be used to estimate how fast ψ diverges.

Lemma 4.5. *Let $\tilde{\mathbf{z}}' = A\tilde{\mathbf{z}}$ be the homogeneous part of (4.10). Given the three independent solutions above, a fourth is defined by*

$$\begin{cases} \psi_1 = -4\dfrac{\tau^4 - 4\tau_0\tau^3 + (12|h| + 6\tau_0^2)\tau^2 + (-12|h|\tau_0 - 3\tau_0^3)\tau - 12|h|^2}{(2|h| + (\tau - \tau_0)^2)^2} \\ \psi_2 = \dfrac{\psi_1' - V\psi_1}{R} \\ \psi_3 = 1 \\ \psi_4 = 0. \end{cases} \tag{4.18}$$

Proof. Observe that the first two and the second two equations of the homogeneous part of 4.10 are completely independent. Hence we can analyze the first two equations independently from the others. They can be written as a system:

$$\begin{cases} \tilde{r}' = V\tilde{r} + R\tilde{v} \\ \tilde{v}' = (1 + 4Rh)\tilde{r} \end{cases} \tag{4.19}$$

or as a second order linear differential equation

$$\tilde{r}'' = 2V\tilde{r}' + (V' - V^2 + R(1 + 4Rh))\tilde{r} \tag{4.20}$$

where $\tilde{v}$ is

$$\tilde{v} = \frac{\tilde{r}' - V\tilde{r}}{R}. \tag{4.21}$$

Obviously R' is a solution of the differential equation. To find another solution we use the so called *reduction of the order* and we look for solutions to (4.20) of the form $f(\tau)R'$. We now substitute $f(\tau)R'$ in (4.20). Upon collecting terms that involve derivatives of f of the same order, we have

$$R'f'' + (2R'' - 2VR')f' + ((R''' - 2VR'') + (-V' + V^2 - R(1 + 4Rh))R')f = 0. \tag{4.22}$$

The coefficient of f vanishes , since R' is a solution of equation (4.20). Therefore the differential equation for f has the form

$$R'f'' + (2R'' - 2VR')f' + (R''' - 2VR'') = 0. \tag{4.23}$$

Since this equation has no term involving f itself it can be regarded as an equation of first order for the derivative f'. Solving for f' and then integrating again one finds the general solution

$$f(\tau) = A + \frac{B}{(\tau - \tau_0)}(\tau^4 - 4\tau_0\tau^3 + (12|h| + 6\tau_0^2)\tau^2 + (-12|h|\tau_0 - 3\tau_0^3)\tau - 12|h|^2) \tag{4.24}$$

If we choose $A = 0, B = 1$ we obtain a solution of (4.20), independent from the one we already knew, that has the form

$$\psi_1 = -4\frac{\tau^4 - 4\tau_0\tau^3 + (12|h| + 6\tau_0^2)\tau^2 + (-12|h|\tau_0 - 3\tau_0^3)\tau - 12|h|^2}{(2|h| + (\tau - \tau_0)^2)^2}. \tag{4.25}$$

Furthermore one sees immediately that

$$\psi_2 = \frac{\psi_1' - V\psi_1}{R}. \tag{4.26}$$

To complete the proof of the Lemma, since the second two differential equations are independent from the first two, we can set

$$\psi_3 = 1 \quad , \quad \psi_4 = 0 \tag{4.27}$$

$\square$

Remark 4.3. To find a fourth independent solution it would have been natural to use the technique of "reduction to a smaller system" that can be found in the book of Hartman [41]. However the formulas given in [41] are only "local", i.e. they are, in general, applicable only to subintervals of the domain of definition of the system of differential equations and they vary from subinterval to subinterval. If we apply this "reduction" method to find a fourth independent solution of the homogeneous part of equation 4.10 we may obtain, after considerably long computations, solutions that are only defined on subintervals of the real line. Consequently such solutions are not suitable for our purpouses. Indeed, in the following, we will need to use solutions defined on the real line.

To obtain necessary and sufficient conditions such that the negatively and positively asymptotic sets intersect transversely, we first obtain conditions for the existence of solutions bounded on $\mathbb{R}$ for the non-homogeneous linear variational equation around χ.

For this, let $\mathcal{B}(\mathbb{R}) = \{\tilde{\mathbf{b}} : \mathbb{R} \to \mathbb{R} \times \mathbb{R} \times S^1 \times \mathbb{R}$ bounded, continuous$\}$ with $\|\tilde{\mathbf{b}}\| = \sup_{\tau \in \mathbb{R}} \|\tilde{\mathbf{b}}(\tau)\|$ for $\tilde{\mathbf{b}} \in \mathcal{B}(\mathbb{R})$. Then we have the following version of the Fredholm alternative for solutions bounded on $\mathbb{R}$ (see [12, 13, 15] for a similar approach).

Lemma 4.6. *Let $\tilde{\mathbf{z}} \in \mathbb{R} \times \mathbb{R} \times S^1 \times \mathbb{R}$ and assume that $\tilde{\mathbf{z}} \equiv 0$ in the expression of the function $\tilde{\mathbf{b}}$. Then the variational equation*

$$\tilde{\mathbf{z}}' = A(\tau)\tilde{\mathbf{z}} + \tilde{\mathbf{b}}(\tilde{\mathbf{z}}, \chi, \tau, \tau_0, \theta_0, \epsilon) \tag{4.28}$$

has a bounded solution if and only if

$$\int_{-\infty}^{+\infty} e^{-\int_{\tau_0}^{\tau} Tr A(s) ds} \; R'(\tau - \tau_0) \; b_2(\chi, \tau, \tau_0, \theta_0, \epsilon) \; d\tau = 0. \tag{4.29}$$

The solution is unique and continuous and has the form $\tilde{\mathbf{z}} = \mathbf{L}(\tilde{\mathbf{b}}) + \mathbf{w}$, where $\mathbf{L}$ is a bounded linear operator, $\mathbf{w} = (0, 0, \tilde{\theta}(\tau_0), \tilde{u}(\infty))$, when $\tilde{r}(\tau_0)R'(\tau_0) + \tilde{v}(\tau_0)V'(\tau_0)) = 0$, and b_4 satisfies the relation below,

$$\int_{-\infty}^{+\infty} b_4(\chi, \tau, \tau_0, \theta_0, \epsilon) \; d\tau = 0. \tag{4.30}$$

Proof. Using Lemma 4.5 it is easy to determine the behavior of ψ as $\tau \to \pm\infty$, precisely,

$$\tau \to \pm\infty \begin{cases} \psi_1 \sim \text{const.} \\ \psi_2 \sim \tau \\ \psi_3 \sim \text{const.} \\ \psi_4 \sim \text{const.} \end{cases} \tag{4.31}$$

Using (4.14) and (4.15), the general solution of the complete (non-homogeneous) equation (4.10) can be written in integral form as

$$\begin{aligned}
\tilde{r} &= R'\left(A - \int_{\tau_0}^{\tau} e^{-\int_{s_0}^{s} TrA(\eta)d\eta}(\psi_1 b_2 - \psi_2 b_1)\, ds\right) \\
&\quad + \psi_1\left(B + \int_{\tau_0}^{\tau} e^{-\int_{s_0}^{s} TrA(\eta)d\eta}(R' b_2 - V' b_1)\, ds\right) \\
\tilde{v} &= V'\left(A - \int_{\tau_0}^{\tau} e^{-\int_{s_0}^{s} TrA(\eta)d\eta}(\psi_1 b_2 - \psi_2 b_1)\, ds\right) \\
&\quad + \psi_2\left(B + \int_{\tau_0}^{\tau} e^{-\int_{s_0}^{s} TrA(\eta)d\eta}(R' b_2 - V' b_1)\, ds\right) \\
\tilde{\theta} &= \pm\sqrt{2b}\left(A - \int_{\tau_0}^{\tau} e^{-\int_{s_0}^{s} TrA(\eta)d\eta}(\psi_1 b_2 - \psi_2 b_1)\, ds\right) \\
&\quad + \psi_3\left(B + \int_{\tau_0}^{\tau} e^{-\int_{s_0}^{s} TrA(\eta)d\eta}(R' b_2 - V' b_1)\, ds\right) \\
&\quad + C - \int_{\tau_0}^{\tau} e^{-\int_{s_0}^{s} TrA(\eta)d\eta}\left[(-V'\psi_3 \pm \sqrt{2b}\psi_2)b_1 + (R'\psi_3 \pm \sqrt{2b}\psi_1)b_2\right] \\
\tilde{u} &= D + \int_{\tau_0}^{\tau} b_4\, ds,
\end{aligned} \tag{4.32}$$

where, for notational convenience, we failed to mention the dependence on $\tilde{\mathbf{z}}$, χ, τ_0, etc.

Consider now the linearization of the problem (4.32) around the solution $\tilde{\mathbf{z}}(\tau) \equiv 0$; in particular this amounts to deleting the high-order terms in the expression of $\tilde{\mathbf{b}}$ (i.e. $b_1 = 0$, etc.). Taking also into account the different behavior of the different solutions given in Lemma 4.5, it is easy to see that to have bounded solutions we need to require that

$$\psi_i\left(A - \int_{\tau_0}^{\tau} e^{-\int_{s_0}^{s} TrA(\eta)d\eta}\psi_1 b_2(\chi, s, \tau_0, \theta_0, \epsilon)\, ds\right) \qquad \text{for } i = 1, \ldots, 4, \tag{4.33}$$

remains bounded as $\tau \to \pm\infty$. More precisely $\tilde{\mathbf{z}}$ is bounded on $[\tau_0, \infty)$ if and only if

$$A = \int_{\tau_0}^{\infty} e^{-\int_{s_0}^{s} TrA(\eta)d\eta} \psi_1 b_2 \, ds \tag{4.34}$$

and bounded on $(-\infty, \tau_0]$ if and only if

$$A = -\int_{-\infty}^{\tau_0} e^{-\int_{s_0}^{s} TrA(\eta)d\eta} \psi_1 b_2 \, ds. \tag{4.35}$$

We also require

$$\tilde{u}(\pm\infty) = \lim_{\tau \to \pm\infty} \tilde{u}(\tau) = D + \lim_{\tau \to \pm\infty} \int_{\tau_0}^{\tau} b_4(\chi, s, \tau_0, \theta_0, \epsilon) \, ds, \tag{4.36}$$

where, obviously, $\tilde{u}(\infty) = \tilde{u}(-\infty)$. The latter condition is not needed for the boundedness of the solution, but its role will be clear later when analyzing some properties of the negatively and positively asymptotic sets. It is easy to see that the above conditions are simultaneously satisfied both at $\tau = -\infty$ and at $\tau = +\infty$ if for some τ_0 the following Melnikov-type conditions:

$$\begin{aligned} &\int_{-\infty}^{+\infty} e^{-\int_{\tau_0}^{\tau} TrA(\eta)\, d\eta} \psi_1 b_2(\chi, s, \tau_0, \theta_0, \epsilon) \, ds = 0 \\ &\int_{-\infty}^{+\infty} b_4(\chi, s, \tau_0, \theta_0, \epsilon) \, ds = 0 \end{aligned} \tag{4.37}$$

are fulfilled. Thus we can rewrite the general solution (4.32) using (4.37) and, by neglecting to mention the dependence on χ, s, etc., we obtain

$$\begin{aligned} \tilde{r} &= -R' \int_{\infty}^{\tau} e^{-\int_{s_0}^{s} TrA(\eta)d\eta} \psi_1 b_2 \, ds + \psi_1 \left(B + \int_{\tau_0}^{\tau} e^{-\int_{s_0}^{s} TrA(\eta)d\eta} R' b_2 \, ds \right) \\ \tilde{v} &= -V' \int_{\infty}^{\tau} e^{-\int_{s_0}^{s} TrA(\eta)d\eta} \psi_1 b_2 \, ds + \psi_2 \left(B + \int_{\tau_0}^{\tau} e^{-\int_{s_0}^{s} TrA(\eta)d\eta} R' b_2 \, ds \right) \\ \tilde{\theta} &= \mp\sqrt{2b} \int_{\infty}^{\tau} e^{-\int_{s_0}^{s} TrA(\eta)d\eta} \psi_1 b_2 \, ds + \psi_3 \left(B + \int_{\tau_0}^{\tau} e^{-\int_{s_0}^{s} TrA(\eta)d\eta} R' b_2 \, ds \right) \\ &\quad + C - \int_{\tau_0}^{\tau} e^{-\int_{s_0}^{s} TrA(\eta)d\eta} (R'\psi_3 \pm \sqrt{2b}\psi_1) b_2 \\ \tilde{u} &= \tilde{u}(\infty) + \int_{\infty}^{\tau} b_4 \, ds. \end{aligned} \tag{4.38}$$

To obtain $\tilde{r}(\tau_0)R'(\tau_0) + \tilde{v}(\tau_0)V'(\tau_0)) = 0$ we must have

$$B = \frac{(R'^2(\tau_0) + V'^2(\tau_0))}{\psi_1(\tau_0)R'(\tau_0) + \psi_2(\tau_0)V'(\tau_0)} \int_{\infty}^{\tau_0} \psi_1(s)b_2(s)\ ds. \tag{4.39}$$

Moreover we also get

$$C = \tilde{\theta}(\tau_0) \pm \sqrt{2b} \int_{\infty}^{\tau_0} e^{-\int_{s_0}^{s} TrA(\eta)d\eta} \psi_1 b_2\ ds \tag{4.40}$$

and

$$D = \tilde{u}(\infty) + \int_{\infty}^{\tau_0} b_4\ ds. \tag{4.41}$$

This uniquely defines B, $C - \tilde{\theta}(\tau_0)$, and $D - \tilde{u}(\infty)$ as continuous linear functionals on $\mathcal{B}(\mathbb{R})$. From (4.38) we observe that the corresponding solution is of the form $\tilde{\mathbf{z}} = \mathbf{L}(\tilde{\mathbf{b}}) + \mathbf{w}$, where $\mathbf{L}$ is a bounded linear operator. It follows that this operator is continuous and hence the solution $\tilde{\mathbf{z}} = \mathbf{L}(\tilde{\mathbf{b}}) + \mathbf{w}$ is continuous on $\mathcal{B}(\mathbb{R})$. This completes the proof. □

To obtain necessary and sufficient conditions that the negatively and positively asymptotic sets intersect, let us first consider all the solutions of (4.10) which are bounded as $\tau \to -\infty$ and such that their angles remain close to the ones on the periodic orbit. The solution $\tilde{\mathbf{z}}$ is given by (4.32) satisfying (4.35) and (4.30) with negative sign. In particular the solutions of the variational equation that are bounded as $\tau \to -\infty$ (i.e. which remain in a sufficiently small neighborhood of the periodic orbit as $\tau \to -\infty$) and with perturbed angles that do not drift but remain near the angles on the periodic orbit, must be on the negatively asymptotic set. In the same way, we obtain the positively invariant set from the solutions that remain bounded as $\tau \to \infty$ and whose angles stay close to the one of the periodic orbit, which was in fact the reason why we required that condition (4.30) be satisfied.

Moreover it is important to remark that the solutions we found are not only bounded but also such that $\tilde{r} \to 0$, $\tilde{v} \to 0$ as $\tau \to \infty$ and this is important since, on the collision manifold we have many periodic orbits and this condition is needed to show that the orbits are actually asymptotic to the equator.

With the preparations above, we can now prove the following result.

Theorem 4.1. *System (4.1) has transversal homoclinic solutions if and only if there exist τ_0^* and a θ_0^* such that*

$$\tilde{M}_1(\tau_0^*, \theta_0^*) = \tilde{M}_2(\tau_0^*, \theta_0^*) = 0 \quad \textit{and} \quad \left. \frac{\partial \tilde{M}_1}{\partial \tau_0}\frac{\partial \tilde{M}_2}{\partial \theta_0} - \frac{\partial \tilde{M}_1}{\partial \theta_0}\frac{\partial \tilde{M}_2}{\partial \tau_0} \right|_{\substack{\tau_0=\tau_0^* \\ \theta_0=\theta_0^*}} \neq 0, \tag{4.42}$$

where

$$\begin{aligned} \tilde{M}_1(\tau_0, \theta_0) &= \int_{-\infty}^{+\infty} e^{-\int_{\tau_0}^{\tau} TrA(s)ds}\, R'\, b_2(\tilde{\mathbf{z}}^*, \tau, \tau_0, \theta_0, \epsilon)\ d\tau, \\ \tilde{M}_2(\tau_0, \theta_0) &= \int_{-\infty}^{+\infty} b_4(\tilde{\mathbf{z}}^*, \tau, \tau_0, \theta_0, \epsilon)\ d\tau, \end{aligned} \tag{4.43}$$

and $\tilde{\mathbf{z}}^*$ is a solution of $\tilde{\mathbf{z}} = \mathbf{L}(\tilde{\mathbf{b}}(\tilde{\mathbf{z}}, \tau, \tau_0, \theta_0, \epsilon)) + \mathbf{w}$. Moreover if the perturbation is periodic we get infinitely many intersections.

Proof. The stable and unstable manifolds intersect if and only if the solution (4.32) satisfies the Melnikov-like conditions (4.29) and (4.30) of Lemma 4.6. This was already proved in the case when $\tilde{\mathbf{b}}$ did not implicitly depend on $\tilde{\mathbf{z}}$. But because of this implicit dependence we need to apply the implicit function theorem, which states that given $\tilde{\mathbf{z}} = \mathbf{L}(\tilde{\mathbf{b}}(\tilde{\mathbf{z}}, \tau, \tau_0, \theta_0, \epsilon)) + \mathbf{w}$ with $\tilde{\mathbf{z}} - \mathbf{w} = \mathbf{L}(0, \tau, \tau_0, \theta_0, 0) = 0$, there exist a δ and a unique solution $\tilde{\mathbf{z}}^*(\epsilon, \tau_0, \theta_0)$ (that has continuous derivatives up to order 2 in $\tau_0, \theta_0, \epsilon$) such that $\epsilon < \delta$, $|\tilde{\mathbf{z}}| < \delta$ if the linearized operator $\tilde{\mathbf{z}} = \mathbf{L}(\tilde{\mathbf{b}}(0, \tau, \tau_0, \theta_0, \epsilon)) + \mathbf{w}$ is invertible. But Lemma 4.6 proved that such an operator is invertible. Moreover the homoclinic solutions are transversal if and only if the integrals (4.43) have simple zeroes, as functions of τ_0 and θ_0 (see [12, 13]). This concludes the proof. □

Unfortunately the Melnikov integrals of Theorem 4.1 are difficult to compute explicitly. To overcome this difficulty we need to rewrite these integrals to the first order approximation in ϵ. Hence if we let $\tilde{\mathbf{z}}^* = \epsilon\chi$ and $\tilde{\mathbf{b}} = \epsilon\mathbf{d}$ with $\mathbf{d} = (d_1, d_2, d_3, d_4)$, the next result follows immediately.

Corollary 4.7. *System (4.1) has transversal homoclinic solutions if and only if there exist τ_0^* and a θ_0^* such that*

$$M_1(\tau_0^*, \theta_0^*) = M_2(\tau_0^*, \theta_0^*) = 0 \quad \textit{and} \quad \left.\frac{\partial M_1}{\partial \tau_0}\frac{\partial M_2}{\partial \theta_0} - \frac{\partial M_1}{\partial \theta_0}\frac{\partial M_2}{\partial \tau_0}\right|_{\substack{\tau_0=\tau_0^* \\ \theta_0=\theta_0^*}} \neq 0, \tag{4.44}$$

where

$$\begin{aligned} M_1(\tau_0, \theta_0) &= \int_{-\infty}^{+\infty} e^{-\int_{\tau_0}^{\tau} TrA(s)ds}\ R'(\tau - \tau_0)\ b_2(\chi(\tau - \tau_0), \Theta(\tau - \tau_0) - \theta_0)\ d\tau, \\ M_2(\tau_0, \theta_0) &= \int_{-\infty}^{+\infty} b_4(\chi(\tau - \tau_0), \Theta(\tau - \tau_0) - \theta_0)\ d\tau. \end{aligned} \tag{4.45}$$

Moreover if the perturbation is periodic we get infinitely many intersections.

Corollary 4.7 generalizes the Melnikov integrals obtained in [44, 78] to nonhyperbolic whiskered tori (periodic orbits) in non-Hamiltonian systems. We remark that the second integral in (4.45) converges only conditionally. This is not a new feature of this non-Hamiltonian system since the same nuisance was present in [44, 78]. However some authors, more recently, found a way to write the Melnikov conditions for hyperbolic whiskered tori in Hamiltonian systems using only convergent integrals see [20, 75]. It would be interesting to generalize those results to nonhyperbolic tori in non-Hamiltonian systems and to apply the newly developed technique to the problem under discussion in this paper. But this is not a project we aim to develop here.

4.7 The Melnikov Integrals

Now we would like to apply Corollary ?? to our problem. The Melnikov conditions take the form

$$M_1(\tau_0,\theta_0) = \int_{-\infty}^{+\infty} \Big[e^{-\frac{1}{2}\int_{\tau_0}^{\tau} V(s)ds} R(\tau-\tau_0) R'(\tau-\tau_0) \times \cos^2(\omega(\tau-\tau_0)-\theta_0) \Big]\, d\tau = 0 \tag{4.46}$$

and

$$M_2(\tau_0,\theta_0) = \frac{1}{2}\int_{-\infty}^{+\infty} (R(\tau-\tau_0)+2b)\sin(2(\omega(\tau-\tau_0)-\theta_0))\, d\tau = 0. \tag{4.47}$$

Let $\tilde{\theta}_0 = -\theta_0 - \omega\tau_0$. With this assumption we can rewrite the first Melnikov condition as

$$M_1 = \cos^2\tilde{\theta}_0 I_1^a + \sin^2\tilde{\theta}_0 I_1^b - \sin 2\theta_0 I_1^c, \tag{4.48}$$

where

$$\begin{cases} I_1^a = \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\int_{\tau_0}^{\tau} V(s)ds} RR' \cos^2\omega\tau\, d\tau \\ I_1^b = \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\int_{\tau_0}^{\tau} V(s)ds} RR' \sin^2\omega\tau\, d\tau \\ I_1^c = \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\int_{\tau_0}^{\tau} V(s)ds} RR' \sin\omega\tau\cos\omega\tau\, d\tau. \end{cases} \tag{4.49}$$

The second Melnikov condition can be expressed as

$$M_2 = \cos 2\tilde{\theta}_0 I_2^a + \sin 2\tilde{\theta}_0 I_b^2, \tag{4.50}$$

where

$$\begin{cases} I_2^a = \frac{1}{2}\int_{-\infty}^{+\infty} (R+2b)\sin 2\omega\tau\, d\tau \\ I_2^b = \frac{1}{2}\int_{-\infty}^{+\infty} (R+2b)\cos 2\omega\tau\, d\tau. \end{cases} \tag{4.51}$$

All the integrals above can be computed using the method of residues. Straightforward computations give

$$I_1^a = -I_1^b = \frac{-1}{|h|}\int_{-\infty}^{+\infty} \frac{(\tau-\tau_0)\cos 2\omega\tau}{(2|h|+(\tau-\tau_0)^2)^2}\, d\tau = \frac{\pi\sin(2\omega\tau_0)e^{-2\omega\sqrt{2|h|}}}{|h|\sqrt{2|h|}}\omega \tag{4.52}$$

and

$$I_1^c = \frac{-1}{|h|} \int_{-\infty}^{+\infty} \frac{(\tau - \tau_0) \sin \omega\tau \cos \omega\tau}{(2|h| + (\tau - \tau_0)^2)^2} \, d\tau = -\frac{\pi \cos(2\omega\tau_0) e^{-2\omega\sqrt{2|h|}}}{|h|\sqrt{2|h|}} \omega. \tag{4.53}$$

Particular care is needed when integrating I_2^a and I_2^b since they converge only conditionally. To obtain computational convergence, we choose the limits in I_2^a such that

$$\begin{aligned} I_2^a &= \lim_{N \to \infty} \int_{-N\pi/2\omega}^{N\pi/2\omega} \left(b + \frac{1}{2|h| + (\tau - \tau_0)^2} \right) \sin 2\omega\tau \, d\tau \\ &= \frac{\pi \sin(2\omega\tau_0) e^{-2\omega\sqrt{2|h|}}}{\sqrt{2|h|}}. \end{aligned} \tag{4.54}$$

The integral was also computed using the method of residues. Similarly, for I_2^b, we have

$$\begin{aligned} I_2^b &= \lim_{N \to \infty} \int_{-N\pi/2\omega}^{N\pi/2\omega} \left(b + \frac{1}{2|h| + (\tau - \tau_0)^2} \right) \cos 2\omega\tau \, d\tau \\ &= \frac{\pi \cos(2\omega\tau_0) e^{-2\omega\sqrt{2|h|}}}{\sqrt{2|h|}} \end{aligned} \tag{4.55}$$

and thus

$$M_1 = M_2\omega = \sin(2(\omega\tau_0 + \tilde{\theta}_0)) \frac{\pi e^{-2\omega\sqrt{2|h|}}}{\sqrt{2|h|}} \omega. \tag{4.56}$$

We therefore have only one independent condition; this is clearly a consequence of the energy relation.

We can find simple zeroes when $\sin(2(\omega\tau_0 + \tilde{\theta}_0)) = 0$, i.e., for $-(\omega\tau_0 + \tilde{\theta}_0) = \theta_0 = \pm k\pi/2$ for $k = 0, 1, 2, \ldots$.

Hence, by Corollary **??**, we have proved the existence of an infinite sequence of intersections on the Poincaré section of the negatively and positively asymptotic sets of the periodic orbit and the existence of homoclinic orbits leaving the equator of the collision manifold and going back to it. This situation is clearly reminiscent of the chaotic dynamics described by the Poincaré-Birkhoff-Smale theorem in terms of symbolic dynamics and the Smale horseshoe. Unfortunately this theorem cannot be directly applied, nor can the theorems proved in [6], since the Poincaré-Birkhoff-Smale theorem considers hyperbolic fixed points while the arguments in [6] apply to area-preserving diffeomorphisms. However the arguments contained in those theorems strongly suggest the occurrence of chaotic dynamics.

Moreover it is easy to verify, and interesting to remark, that the orbits we found above are not $\overline{S}_0$-symmetric, where the $\overline{S}_0$ symmetry is defined by $\overline{S}_0(r, v, \theta, u, \tau) = (r, -v, \theta, -u, -\tau)$ (see [18]) and an orbit $\gamma(\tau)$ is said to be $\overline{S}_0$-symmetric if $\overline{S}_0(\gamma(\tau)) = \gamma(\tau)$. Indeed an orbit is $\overline{S}_0$-symmetric if and only if it has a point on the zero velocity curve, i.e., if there is a $\overline{\tau}$ such that $v(\overline{\tau}) = u(\overline{\tau}) = 0$ (see [64]). But this cannot happen in our problem because the unperturbed solution verifies $u \equiv \pm\sqrt{2b}$. Thus for ϵ small enough the perturbed orbit can never have $u = 0$.

We can now summarize the above discussion as follows:

Theorem 4.2. *Let us consider the anisotropic Manev problem given by the equation of motion (3.10) with the energy relation (3.11). Then there is an infinite sequence of intersections in the Poincaré section of the negatively and positively asymptotic sets of the periodic orbits at the equator of the collision manifold (possibly giving rise to chaotic dynamics). Furthermore there exist the homoclinic non $\overline{S}_0$-symmetric orbits to the periodic orbit described above.*

Chapter 5

Symmetric Periodic Solutions Via the Continuation Method

5.1 Overview

In this chapter we study the existence of symmetric periodic solutions in the anisotropic Manev problem via the analytic continuation method. The idea of this technique is to use a known periodic solution and, by small changes of the parameters and of the initial conditions, continue analytically the known solution. This allows us to find periodic solutions of the anisotropic Manev problem for small perturbations, i.e. for weak anisotropies. Most of the material of this chapter can be found in [64]. In the next section we study the periodic orbits of the "second kind" (i.e. the non-circular ones). The proof of the existence of the periodic orbits is realized using the variables (2.35) that put the system in a simple form, particularly suitable to study periodic orbits. The idea of the proof is based on the work of various authors that studied symmetric periodic orbits in Hamiltonian systems. In particular, while the main ideas of the continuation method are already contained in the work of Poincaré, especially enlightening is the work of Barrar (see [4]) and Milani (see [55]) that study symmetric periodic solutions of the restricted circular three body problem. Barrar observes that the problem of finding periodic solutions can be done more readily using particular variables. In his problem Barrar uses the Delaunay variables and suggests that the Poincaré variables are equally suitable. Milani in [55] uses Poincaré variables. Similarly we use the variables (2.35) and we prove the existence of S_i-symmetric non-circular orbits for $i = 1, 2, 3$. In Section 5.3.1 we study the periodic orbits of the "first kind", i.e. the circular ones. The proof in this case is more complicated and we follow a different approach. We use the continuation method described in the book of Siegel and Moser [66].

5.2 S_i-Symmetric Periodic Solutions with $i = 1, 2$

In this section we want to study the S_i-symmetric periodic solutions of the anisotropic Manev problem using the Poincaré continuation method. In order to do that we will use the canonical variables (2.35) that were obtained from the action angle variables. Such variables transform the system under discussion into a simple form that is especially suitable for studying periodic orbits. The importance of choosing the right variables to study periodic orbits was already clear from the work of Barrar [4] that uses the Delaunay variables to prove the existence of periodic orbits of the second kind in the restricted circular three body problem. The next result proves the existence of orbits of the "second kind" (i.e. non circular ones).

Theorem 5.1. *Let $\gamma(t)$ be an S_i-symmetric periodic orbit of the Manev problem with $i = 1, 2$. Let the period be τ and set $\epsilon = \mu - 1$ with $\epsilon \ll 1$. Then there exists a τ-periodic solution of the Anisotropic Manev problem $\gamma_\epsilon(t)$ such that $\gamma_\epsilon(t) = \gamma(t) + O(\epsilon)$.*

Proof. Let's consider an S_1-symmetric orbit of period $\tau = 2\pi m/k$ (m, k relatively prime integers). We remark that, since the equations of motion are autonomous, we need only study the symmetric orbits that have either the pericenter or the apocenter on the positive x-axis at $t = 0$.

If at $t = 0$, $\epsilon = 0$, the pericenter of this orbit is on the positive x-axis, it is crossing the x-axis perpendicularly, and we have

$$g(0) = 0 \quad \text{and} \quad l(0) = 0. \tag{5.1}$$

Since the periodic orbit is S_1-symmetric, by Proposition 3.3, at the half period one has

$$g(\tau/2) = m\pi \qquad l(\tau/2) = k\pi \tag{5.2}$$

which follows from the solution of (3.46) for $\epsilon = 0$:

$$\begin{array}{ll} L = \text{const.} & G = \text{const.} \\ l = \omega_L t & g = \omega_G t \end{array} \tag{5.3}$$

Now if, for $\epsilon \neq 0$ we consider only S_1-symmetric solutions of (3.46), it follows from the implicit function theorem that if the functional determinant

$$D = det \begin{pmatrix} \partial l/\partial L & \partial l/\partial G \\ \partial g/\partial L & \partial g/\partial G \end{pmatrix} \neq 0 \tag{5.4}$$

at

$$t = \tau/2 \qquad \epsilon = 0 \tag{5.5}$$

then (5.2) would be satisfied for $\epsilon > 0$. To compute the determinant we can by analyticity substitute (5.2) into (5.3) to find out at the time $t = \tau/2$ that

$$D = \frac{6b(\tau/2)^2}{(-G + \sqrt{(G+L)^2 - 2b})^7((G+L)^2 - 2b)^{3/2}} \neq 0 \tag{5.6}$$

Thus the existence of S_1-symmetric periodic orbits of period τ obtained from the τ-periodic S_1-symmetric solutions of the unperturbed problem, that at $t = 0$ have the pericenter on the positive x-axis, is readily established.

On the other hand, if at $t = 0$, $\epsilon = 0$, the apocenter is on the positive x-axis, and it is crossing the x-axis perpendicularly, we have

$$g(0) = \pi/\lambda \quad \text{and} \quad l(0) = -\pi/\lambda \tag{5.7}$$

where $\lambda = (\omega_L - \omega_G)/\omega_L$. By Proposition 3.3, at the half period we have

$$g(\tau/2) = (m + 1/\lambda)\pi \qquad l(\tau/2) = (-1/\lambda + k)\pi. \tag{5.8}$$

Instead of computing the functional determinant directly, in this case, it is easier to consider the new variables given by the relations,

$$\begin{cases} \tilde{L} = L \\ \tilde{G} = G \\ \tilde{l} = l + \pi/\lambda_0 \\ \tilde{g} = g - \pi/\lambda_0 \end{cases} \tag{5.9}$$

that define a family of canonical transformations parametrized by $\lambda_0(L_0, G_0)$. For each orbit choose a different transformation from the family (5.9), where $\lambda_0 = \lambda$ is a fixed quantity defined by the value of the action variables along the periodic orbit under consideration.

The equations (5.8), expressed in the new variables, are of the same form as in (5.2). Thus the functional determinant, in the new variables, is exactly D, and the existence of the remaining S_1-symmetric τ-periodic orbits follows.

Now the proof for the S_2-symmetric orbits can be done along the same lines. Consider an S_2-symmetric periodic orbit of period $\tau = 2\pi m/k$. If at $t = 0$, $\epsilon = 0$ the pericenter of the orbit is on the positive y-axis and it is crossing the y-axis perpendicularly, we have

$$g(0) = \pi/2 \quad \text{and} \quad l(0) = 0 \tag{5.10}$$

Since the periodic orbit is S_2-symmetric one has, at the half period

$$g(\tau/2) = m\pi + \pi/2 \qquad\qquad l(\tau/2) = k\pi. \tag{5.11}$$

Now we consider only S_2-symmetric solutions of (3.46) for $\epsilon \neq 0$ again it follows from the implicit function theorem that if the determinant D computed at $t = \tau/2$ for $\epsilon = 0$ is non-zero then (5.11) would be satisfied for $\epsilon > 0$. It is trivial to see from (5.6) that $D \neq 0$, and hence we found S_2-symmetric periodic orbits for the perturbed problem.

For the S_2-symmetric orbits having the apocenter on the positive x-axis at $t = 0$ the canonical transformation (5.9) can be used. Again we find the same expression for the functional determinant and hence, by the implicit function theorem, the existence of the remaining S_2-symmetric periodic orbits is proved. □

Remark 5.1. Note that the theorem above also proves the existence of noncircular S_3-symmetric orbits, since $S_3 = S_1 \circ S_2$.

Remark 5.2. It is worth mentioning that the Theorem above and its proof can be easily extended to consider any S_i-symmetric perturbation with $i = 1, 2$ and a very general class of nondegenerate integrable Hamiltonians, however such a generalization is trivial and not strictly related to the problem under consideration and hence it will not be discussed any further.

Remark 5.3. We can also observe that for $b = 0$, i.e. for the Kepler problem, the determinant in (5.6) is zero. Thus in the case of the Anisotropic Kepler Problem, the continuation theorem proved above cannot be applied, and the existence of symmetric periodic orbits of the "second kind" (for weak anisotropies) remains unclear. On the other hand the continuation theorem that we prove in the next section (for the circular orbits) can be applied to the Anisotropic Kepler Problem [76] and hence at least the existence of symmetric periodic orbits of the first kind is a well established fact.

5.3 The Circular Orbits

To retain the same notation as in [64] consider the following canonical transformation:

$$\begin{cases} X = y \\ Y = x \\ P_X = p_y \\ P_Y = p_x \end{cases} \tag{5.12}$$

and with an abuse of notation denote the new coordinates with (x, y, p_x, p_y). Then the new Hamiltonian is of the form

$$\mathcal{H} = \frac{1}{2}\mathbf{p}^2 - \frac{1}{\sqrt{x^2 + \mu y^2}} - \frac{b}{x^2 + \mu y^2} \tag{5.13}$$

which is the same Hamiltonian studied in [64]. Clearly studying the circular solutions of the Hamilton equations with Hamiltonian (5.13) is equivalent to studying the original equations (3.3). Therefore we will study the circular orbits of the Hamiltonian system with Hamiltonian (5.13). Take the parameter μ close to 1 as in the previous section. Let $\Phi(t, (\mathbf{r}, \dot{\mathbf{r}}), \mu)$ be the flow of the equations of motion. In this section we prove the following theorem:

Theorem 5.2. *Let $\mathbf{r}^0(t)$ be an S_3-symmetric periodic orbit of the Manev problem, i.e. a circular one. Set $\epsilon = \mu - 1$, and let τ be the period of $\mathbf{r}^0(t)$. Then there exists a τ-periodic solution $\Phi(t, (\mathbf{r}(\epsilon), \dot{\mathbf{r}}(\epsilon)), \epsilon)$ of the Anisotropic Manev problem such that $\Phi(t, (\mathbf{r}(0), \dot{\mathbf{r}}(0)), 0) = (\mathbf{r}^0(t), \dot{\mathbf{r}}^0(t))$.*

5.3.1 The equation of motion

Now using the same notation as in [76] let $\mathbf{r^0(t)}$ be a circular solution of the Manev problem which corresponds to $\mu = 1$ in the xy-plane, ω its angular speed and a its radius. For $\epsilon = \mu - 1 \neq 0$ we set,

$$\mathbf{r}(t, \epsilon) = \mathbf{r}^0(t) + \epsilon\ \mathbf{s}(t, \epsilon). \tag{5.14}$$

Expanding ∇H in powers of $\mu - 1$ sufficiently small, after substituting the expression for $\mathbf{r}$ given above, considering the notation $\mathbf{r}^0(t) = x_0(t) + iy_0(t)$ and $\mathbf{s} = u + iv$ we have that $\mathbf{r}(t,\epsilon)$ is a solution of the equation of motion defined by (3.1) if, and only if, $\mathbf{s}(t,\epsilon)$ is a solution of the equations

$$\begin{aligned} \ddot{u} &= -\left(\tfrac{1}{a^3} - \tfrac{3x_0^2}{a^5} - \tfrac{8bx_0^2}{a^6} + \tfrac{2b}{a^4}\right) u + \left(\tfrac{3x_0y_0}{a^5} + \tfrac{8bx_0y_0}{a^6}\right) v + \eta(t) + O(\epsilon) \\ \ddot{v} &= \left(\tfrac{3x_0y_0}{a^5} + \tfrac{8bx_0y_0}{a^6}\right) u - \left(\tfrac{1}{a^3} - \tfrac{8by_0^2}{a^6} - \tfrac{3y_0^2}{a^5} + \tfrac{2b}{a^4}\right) v + \xi(t) + O(\epsilon) \end{aligned} \tag{5.15}$$

where

$$\begin{aligned} \eta(t) &= \tfrac{3x_0y_0^2}{a^5} + \tfrac{4bx_0y_0^2}{a^6} \\ \xi(t) &= \tfrac{3y_0^2}{2a^5} - \tfrac{y_0}{a^3} + \tfrac{4by_0^3}{a^6} - \tfrac{2by_0}{a^4} \end{aligned}$$

Consider the orthonormal frame in $\mathbf{R}^2$, $\mathbf{e}_1(t)$ and $\mathbf{e}_2(t)$ defined by

$$\mathbf{e}_1 = \frac{\mathbf{r}^0}{|\mathbf{r}^0|} = e^{i\omega t} = \cos\omega t + i\sin\omega t, \qquad \mathbf{e}_2 = i\mathbf{e}_1$$

for which we have

$$\dot{\mathbf{e}}_1 = \omega\mathbf{e}_2, \quad \dot{\mathbf{e}}_2 = -\omega\mathbf{e}_1. \tag{5.16}$$

Setting

$$\mathbf{s} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2, \qquad \dot{\mathbf{s}} = y_1\mathbf{e}_1 + y_2\mathbf{e}_2 \tag{5.17}$$

or equivalently,

$$\begin{aligned} u &= x_1\cos\omega t - x_2\sin\omega t, \qquad & v &= x_1\sin\omega t + x_2\cos\omega t \\ \dot{u} &= y_1\cos\omega t - y_2\sin\omega t, \qquad & \dot{v} &= y_1\sin\omega t + y_2\cos\omega t \end{aligned} \tag{5.18}$$

equations (5.15) can be written in an equivalent form as:

$$\dot{\mathbf{z}} = A_0(t) + A\mathbf{z} + O(\epsilon), \tag{5.19}$$

where $\mathbf{z} = (x_1, x_2, y_1, y_2)^T$, and

$$A_0 = \begin{pmatrix} 0 \\ 0 \\ \alpha(t) \\ \beta(t) \end{pmatrix} \qquad A = \begin{pmatrix} 0 & \omega & 1 & 0 \\ -\omega & 0 & 0 & 1 \\ 2\,\omega^2 + 2\,\frac{b}{a^4} & 0 & 0 & \omega \\ 0 & -\omega^2 & -\omega & 0 \end{pmatrix}$$

where

$$\begin{aligned} \alpha(t)\cos\omega t - \beta(t)\sin\omega t &= \eta(t) \\ \alpha(t)\sin\omega t + \beta(t)\cos\omega t &= \xi(t) \end{aligned}$$

or equivalently,

$$\begin{aligned} \alpha(t) &= \sin^2\omega t\left(\tfrac{1}{2a^2} + \tfrac{2b}{a^3}\right) \\ \beta(t) &= -\sin\omega t\ \cos\omega t\left(\tfrac{1}{a^2} + \tfrac{2b}{a^3}\right) \end{aligned} \tag{5.20}$$

The eigenvalues of A are 0, with multiplicity two, $\frac{i}{a^{3/2}}$ and $-\frac{i}{a^{3\ 2}}$. One of the two eigenvalues vanishes because the system is autonomous, and the second due to the presence of the first integral H.

Now consider the real Jordan form J of A. The matrix J is defined by the relation $J = \mathcal{T}^{-1}A\mathcal{T}$ where $\mathcal{T}$ is

$$\mathcal{T} = \begin{pmatrix} 2\,\omega^2 a^3 & 0 & \frac{\omega^2 a^4 + 2\,b}{a} & 0 \\ 0 & -\frac{\omega\left(3\,\omega^2 a^4 + 2\,b\right)}{a} & 0 & -2\,\frac{\omega a^2\left(\omega^2 a^4 + 2\,b\right)}{(a)^{3/2}} \\ 0 & \frac{1}{2}\,\frac{4\,a\left(\omega^2 a^4 + b\right) + 2\left(\omega^2 a^4 + 2\,b\right)^2}{a^5} & 0 & \frac{\left(\omega^2 a^4 + 2\,b\right)^2}{a^{7/2}} \\ -\frac{\omega\left(\omega^2 a^4 + 2\,b\right)}{a} & 0 & -\frac{\omega\left(\omega^2 a^4 + 2\,b\right)}{a} & 0 \end{pmatrix}$$

and the columns of $\mathcal{T}$ are the generalized eigenvectors of A.

The vector $J_0 = \mathcal{T}^{-1}A_0$ and the matrix J are:

$$J_0 = \begin{pmatrix} j_1(t) \\ j_2(t) \\ j_3(t) \\ j_4(t) \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{a}}{a^2} \\ 0 & 0 & -\frac{\sqrt{a}}{a^2} & 0 \end{pmatrix}$$

where the fact that $j_1(t) = (2\omega^3 a^2 - \frac{\omega(\omega^2 a^4+2b)}{a^2})^{-1}\beta(t)$ is the only information about J_0 that we need to retain. Furthermore we remark that $\omega^2 a^4 - a - 2b = 0$ gives the relation between a and ω and solving this equation gives only one positive solution (for $b > 0$).

Letting $\mathbf{z} = \mathcal{T}\zeta$, the equation of motion becomes

$$\dot{\zeta} = J_0(t) + J\zeta + O(\epsilon), \tag{5.21}$$

and its flow is given by

$$\psi(t,\zeta,\epsilon) = \gamma(t) + e^{Jt} + O(\epsilon) \tag{5.22}$$

where by the variation of constants

$$\gamma(t) = e^{Jt}\int_0^t e^{-Js}J_0(s)ds \tag{5.23}$$

Therefore we have

$$e^{Jt} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t & 1 & 0 & 0 \\ 0 & 0 & \cos\frac{\sqrt{a}}{a^2}t & \sin\frac{\sqrt{a}}{a^2}t \\ 0 & 0 & -\sin\frac{\sqrt{a}}{a^2}t & \cos\frac{\sqrt{a}}{a^2}t \end{pmatrix}$$

and from (5.23) we obtain

$$\gamma(t) = \begin{pmatrix} \gamma_1(t) \\ \gamma_2(t) \\ \gamma_3(t) \\ \gamma_4(t) \end{pmatrix} \tag{5.24}$$

where we retain only the information that

$$\gamma_1(t) = (2\omega^3 a^2 - \frac{\omega(\omega^2 a^4 + 2b)}{a^2})^{-1}\int_0^t \beta(s)ds. \tag{5.25}$$

5.3.2 The periodicity equation

Since the Hamiltonian H of the anisotropic Manev problem is S_3-symmetric, as we have shown, we can write the periodicity equation as in [76],

$$\Phi\left(\frac{\tau}{2},(\mathbf{r},\dot{\mathbf{r}}),\epsilon\right)=-(\mathbf{r},\dot{\mathbf{r}}). \tag{5.26}$$

Then it is easy to check that $\Phi(t,(\mathbf{r},\dot{\mathbf{r}}),\epsilon)$ is a periodic solution of the equation of motion with period τ. To find periodic solutions we have to verify that (5.26) is satisfied for a family of initial conditions. Equation (5.26) in ζ coordinates is

$$\psi\left(\frac{\tau}{2},\zeta,\epsilon\right)-\zeta=0 \tag{5.27}$$

where $\psi(t,\zeta,\epsilon)$ is the flow of (5.21). Let us denote by $\mathcal{P}(\zeta,\epsilon)$ the left hand side of the periodicity equation (5.26), that is, let

$$\mathcal{P}(\zeta,\epsilon)=\psi\left(\tau/2,\zeta,\epsilon\right)-\zeta=\gamma(\tau/2)+\left(e^{J\frac{\tau}{2}}-I\right)=0. \tag{5.28}$$

Using (5.22) we notice that the requirement

$$\mathcal{P}(\zeta^*,0)=\gamma(\tau/2)+\left(e^{J\frac{\tau}{2}}-I\right)\zeta^*=0, \tag{5.29}$$

imposes the restrictions

$$\gamma_1(\tau/2)=0,\quad \zeta_1^*=-\frac{2}{\tau}\gamma_2(\tau/2),\quad \zeta_2^*=\text{arbitrary} \tag{5.30}$$

and

$$\begin{aligned}\zeta_3^*&=\tfrac{1}{2(1-\cos\alpha^*)}\left(-\gamma_3(\tau/2)(\cos\alpha^*-1)+\gamma_4(\tau/2)\sin\alpha^*\right)\\ \zeta_4^*&=\tfrac{-1}{2(1-\cos\alpha^*)}(\gamma_3(\tau/2)\sin\alpha^*+\gamma_4(\tau/2)(\cos\alpha^*-1))\end{aligned} \tag{5.31}$$

where $\alpha^*=\pi(1+2b/a)^{-1/2}$. It easy to see from (5.20) and (5.25) that $\gamma_1(\tau/2)=0$, therefore, we take

$$\zeta^*=(\zeta_1^*,\zeta_2^*,\zeta_3^*,\zeta_4^*)^T, \tag{5.32}$$

with ζ_2^* arbitrary, for the moment. Now using the flow (5.22), we determine that the Jacobian matrix of $\mathcal{P}$ with respect to the variables ζ evaluated at the point $(\zeta^*, 0)$ is given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ \tau/2 & 0 & 0 & 0 \\ 0 & 0 & \cos\alpha^* - 1 & \sin\alpha^* \\ 0 & 0 & -\sin\alpha^* & \cos\alpha^* - 1 \end{pmatrix}. \tag{5.33}$$

Consider the system of three equations formed by those in (5.28) corresponding to the indices i=2,3,4 and fix the variable $\zeta_2 = \zeta_2^*$. Its Jacobian matrix has determinant $\tau(1 - \cos\alpha^*)$, that is always positive since $0 < \pi(1 + 2b/a)^{-1/2} \leq \pi$. Therefore the implicit function theorem guarantees the existence of analytic functions $\zeta_i = \zeta_i(\epsilon)$, $i = 1, 3, 4$ in a neighborhood of $\epsilon = 0$, satisfying the equations

$$\mathcal{P}_i(\zeta, \epsilon) = 0, \quad (i = 2, 3, 4) \tag{5.34}$$

where

$$\zeta(\epsilon) = (\zeta_1(\epsilon), \zeta_2^*, \zeta_3(\epsilon), \zeta_4(\epsilon)) \tag{5.35}$$

and such that

$$\zeta_i(0) = \zeta_i^* \quad (i = 1, 2, 3, 4). \tag{5.36}$$

It remains to show, in order to have periodicity, that also the remaining equation

$$\mathcal{P}_1(\zeta(\epsilon), \nu(\epsilon), \epsilon) = 0, \tag{5.37}$$

is satisfied in a possibly smaller neighborhood of $\epsilon = 0$. That will be done employing a first integral of the system under discussion, i.e. the Hamiltonian.

5.3.3 Integral of motion

Since the Hamiltonian is a integral of motion of the problem under discussion we can apply the same analysis as in [76, 66]. In particular using the same notation as in [76] we can define

$$H_\epsilon(\mathbf{z}, t) = H(\mathbf{r}, \dot{\mathbf{r}}, \epsilon),$$

where $H_\epsilon(\mathbf{z},t)$ is a time-dependent, τ-periodic first integral for system (5.19). The above integral satisfies the following relation

$$H_\epsilon(\mathbf{z},t+\tau/2) = H_\epsilon(\mathbf{z},t) \tag{5.38}$$

for all t, since $H(-\mathbf{r},-\dot{\mathbf{r}}) = H(\mathbf{r},\dot{\mathbf{r}})$, $\mathbf{r}(t) = \mathbf{r}^0(t) + \epsilon\, \mathbf{s}(t)$ and

$$\mathbf{r}^0(t+\tau/2) = -\mathbf{r}^0(t) \quad , \quad \mathbf{s}(\mathbf{z},t+\tau/2) = -\mathbf{s}(\mathbf{z},t).$$

Performing a change of coordinates we can define $\mathcal{H}_\epsilon(\zeta,t) = H_\epsilon(\mathcal{T}\zeta,t)$, hence (5.38) can be written as

$$\mathcal{H}_\epsilon(\zeta,t+\tau/2) = \mathcal{H}_\epsilon(\zeta,t). \tag{5.39}$$

Moreover since $\mathcal{H}_\epsilon$ is an integral of motion it verifies that

$$\mathcal{H}_\epsilon(\phi(\zeta,\epsilon,t)) = H_\epsilon(\zeta,0). \tag{5.40}$$

Thus applying equations (5.39-5.40) it follows that

$$\mathcal{H}_\epsilon(\psi(\tau/2,\zeta,\epsilon),0) = \mathcal{H}_\epsilon(\zeta,0)$$

and by means of the Mean Value Theorem we obtain

$$\nabla_\zeta \mathcal{H}_\epsilon(\tilde{\zeta},0)\cdot \mathcal{P}(\zeta,\epsilon) = 0, \tag{5.41}$$

where $\nabla_\zeta \mathcal{H}_\epsilon$ is the gradient of $\mathcal{H}_\epsilon$ with respect to ζ, and $\tilde{\zeta}$ is a point on the segment joining ζ to $\psi(\tau/2,\zeta,\epsilon)$.

Expanding $\Psi(\epsilon) = \psi(\tau/2,\zeta,\epsilon)$ in powers of ϵ sufficiently small it is easy to show (see [76]) that $\Psi(\epsilon) = \zeta^* + O(\epsilon)$ and consequently

$$\tilde{\zeta} = s\zeta(\epsilon) + (1-s)\Psi(\epsilon) = \zeta^* + O(\epsilon)$$

for some $s \in (0,1)$. Moreover if we also expand the Hamiltonian $H_\epsilon(\mathbf{z},0)$ in powers of ϵ we get

$$H_\epsilon(\mathbf{z},0) = H_0 + \epsilon(H_1 + H_2\cdot \mathbf{z}) + O(\epsilon^2)$$

or, in ζ coordinates

$$\mathcal{H}_\epsilon(\zeta, 0) = \mathcal{H}_0 + \epsilon(\mathcal{H}_1 + \mathcal{H}_2 \cdot \zeta) + O(\epsilon^2), \tag{5.42}$$

where $\mathcal{H}_0 = H_0 = \left(\frac{1}{2}\omega^2 a^2 - \frac{1}{a} - \frac{b}{a^2}\right)$, $\mathcal{H}_1 = H_1$ and $\mathcal{H}_2 = \mathcal{T}^T H_2 = \mathcal{T}^T(a^{-2} + 2ba^{-3}, 0, 0, a\omega) = (a\omega^2\zeta_1, 0, 0, 0)$. Hence we obtain

$$\frac{1}{\epsilon}\nabla_\zeta \mathcal{H}_\epsilon(\tilde{\zeta}, 0) = \mathcal{H}_2 + O(\epsilon). \tag{5.43}$$

With these preparations equation (5.41) reduces to the equation in the unknown $\mathcal{P}_1$

$$[a\omega^2 + O(\epsilon)]\mathcal{P}_1 = 0, \tag{5.44}$$

since, for small ϵ, we already found in Section 5.3.2 that $\mathcal{P}_i = 0$ for $i = 2, 3, 4$.

It is easy to see that for $\epsilon = 0$ the equation above has solution $\mathcal{P}_1 = 0$. Thus, by continuity, $[a\omega^2 + O(\epsilon)]$ is different from zero for ϵ sufficiently small. Therefore for such values of ϵ this equation has a unique solution that is the trivial one. Consequently the remaining equation

$$\mathcal{P}_1(\zeta(\epsilon), \epsilon) = 0,$$

is also satisfied in a possibly smaller neighborhood of $\epsilon = 0$. Hence all the equations of the periodicity system (5.28) are satisfied when $\zeta = \zeta(\epsilon)$, as long as ϵ is sufficiently small. This completes the proof of Theorem 5.2.

Chapter 6

Symmetric Periodic Solutions Via Variational Techniques

6.1 Overview

In this chapter we prove the existence of symmetric periodic solutions of the anisotropic Manev problem. To find the periodic orbits we use cartesian coordinates, the symmetries S_1, S_2 and S_3, that we defined in Section 3.3, and the variational principle according to which extremum values of the action integral yield periodic solutions of the equations (3.3). The techniques used in this chapter were recently applied to the three body problem to find a remarkable periodic solution: the so called figure eight orbit [9]. However these techniques are not completely new, indeed already Poincaré realized the importance of variational methods in connection with periodic orbits:

> La théorie des solutions périodiques peut, dans certains cas, se rattacher au principe de moindre action.

In the next section we give some preliminary definitions and Lemmas. We introduce the definition of strong force (to which the variational methods apply) and we prove that the anisotropic Manev problem is strong. We recall some notation and definitions and we prove that the spaces of symmetric $T-$periodic loops defined below are Sobolev. Finally we classify the loops according to their winding number.

In Section 6.3 we show that finding a minimizer of the action, among the symmetric closed loops of period T, is equivalent to find solutions of the Hamilton's equations. Moreover we find a way to exclude the possibility that the minimizers are obtained when the bodies are at infinite distance from each other or when they are collision orbits.

In the following section we recall some properties of lower semicontinuous functions that will be needed to prove the main result of this chapter.

In the last section we state and prove the main result of this chapter: i.e. the existence of symmetric periodic orbits.

6.2 Preliminaries

The variational methods that we use here do not apply (directly) to every potential, for example they do not apply in the case of the Newtonian potential, as Poincaré was well aware [63]. Therefore we now wish to introduce a class of potentials for which we know variational techniques apply directly and then to verify that the anisotropic Manev potential is in this class. We will use Gordon's general definition of *strong force* [32]. Consider a general Hamiltonian system with Hamiltonian

$$\overline{H} = 1/2\eta^2 + U(\xi) \tag{6.1}$$

where $\xi = (\xi_1, ..., \xi_N)$ denotes a general point of $\mathbb{R}^N$, $\eta = (\eta_1, ..., \eta_N)$ denotes the canonical momenta and U is a real valued function of $\mathbb{R}^N$. We assume U to be of class C^2 everywhere on $\mathbb{R}^N$ except at a closed nonempty set S at which U has infinitely deep wells (i.e. $U \to -\infty$ as $\xi \to S$). We can now introduce the following

Definition 6.1. The Hamiltonian system with Hamiltonian $\overline{H} = 1/2\eta^2 + U(\xi)$ is said to satisfy the strong force condition if and only if there exist a neighborhood

N of S and a C^2 function J on $N - S$ such that $J(\xi) \to -\infty$ as $\xi \to S$ and $-U(\xi) \geq |\nabla J(\xi)|^2$ for all ξ in $N - S$.

We can now verify that the anisotropic Manev potential satisfies the strong force condition

Lemma 6.1. *The anisotropic Manev problem satisfies the strong force condition.*

Proof. In this particular case $\xi = \mathbf{q} = (x, y)$ and $\eta = \mathbf{p} = (p_x, p_y)$ and the set S consists of a single point, the origin. Set $J(\xi) = \sqrt{b}/(2\sqrt{\mu})\ \ln(\mu x^2 + y^2)$; clearly $J \to -\infty$ as $\mathbf{q} \to 0$. Furthermore

$$|\nabla J|^2 = b\frac{\mu x^2 + y^2/\mu}{(\mu x^2 + y^2)^2} \leq \frac{b}{(\mu x^2 + y^2)}. \tag{6.2}$$

Accordingly we can verify that $-U(\mathbf{q}) \geq |\nabla J(\mathbf{q})|^2$. Indeed we have that

$$-U = \frac{1}{\sqrt{\mu x^2 + y^2}} + \frac{b}{\mu x^2 + y^2} \geq \frac{b}{(\mu x^2 + y^2)} \geq |\nabla J(\mathbf{q})|^2 \tag{6.3}$$

since the previous inequality is equivalent to

$$\sqrt{\mu x^2 + y^2} \geq 0 \tag{6.4}$$

for every $x, y \in \mathbb{R} - \{0\}$. □

We can now introduce some notation and preliminary definitions. Let $C^\infty([0,T], \mathbb{R}^2)$ be the space of T-periodic C^∞ cycles $f : [0,T] \to \mathbb{R}^2$. Define the inner products

$$\begin{aligned} \langle f, g\rangle_{L^2} &= \textstyle\int_0^T f(t) \cdot g(t)dt, \\ \langle f, g\rangle_{H^1} &= \langle f, g\rangle_{L^2} + \langle \dot{f}, \dot{g}\rangle_{L^2}, \end{aligned} \tag{6.5}$$

and let $\|\cdot\|_{L^2}$, $\|\cdot\|_{H^1}$ be the corresponding norms. Then the completion of $C^\infty([0,T], \mathbb{R}^2)$ with respect to the norm $\|\cdot\|_{L^2}$ is denoted by L^2 and it is the space of square integrable functions. The completion with respect to $\|\cdot\|_{H^1}$ is denoted by H^1 and is the

Sobolev space of all absolutely continuous T-periodic paths that have L^2 derivatives defined almost everywhere (see [32]).

Let $\Sigma_i([0,T],\mathbb{R}^2)$ denote the subset of H^1 formed by the S_i-symmetric paths, with $i \in \{0,1,2,3,4,5,6\}$. It is easy to see that each Σ_i is a subspace of H^1; in fact they are Sobolev spaces. Let us now prove the following result.

Lemma 6.2. *Let H^1 be defined as above, then the subspaces Σ_i of S_i-symmetric paths with $i = 0,1,2,3,4,5,6$ are complete with respect to the norm $\|\cdot\|_{H^1}$, and therefore they are Sobolev spaces.*

Proof. To prove that Σ_i is complete it is enough to show that every absolutely convergent series in Σ_i converges. Let $\sum_{k=1}^{\infty}\gamma_k$ be an absolutely convergent series. Accordingly, since H^1 is complete, we have that $\sum_{k=1}^{\infty}\|\gamma_k\|_{H^1} = \|\gamma\|_{H^1} < \infty$ and $\sum_{k=1}^{\infty}\gamma_k$ converges to an element γ of H^1. Thus for every $\epsilon > 0$ there exists an N such that for every $n > N$, $\|\sum_{k=1}^{n}\gamma_k - \gamma\|_{H^1} < \epsilon$. Clearly we have that

$$\left\|\sum_{k=1}^{n}\gamma_k - \gamma\right\|_{H^1} = \left\|S_i(\sum_{i=1}^{n}\gamma_i - \gamma)\right\|_{H^1} = \left\|\sum_{k=1}^{n}S_k(\gamma_k) - S_i(\gamma)\right\|_{H^1} < \epsilon. \tag{6.6}$$

since it is easy to verify that $\|g\|_{H^1} = \|S_i(g)\|_{H^1}$. Let's verify this statement explicitly for the symmetry S_i with $i = 0$. The other cases are analogous. Consider $g = (g_x, g_y)$, then

$$\begin{aligned} \|S_0(g)\|_{H^1} &= \left(\int_0^T g_x(-t)^2 + g_y(-t)^2 dt\right)^{1/2} \\ &+ \left(\int_0^T (-\dot{g}_x(-t))^2 + (-\dot{g}_y(-t))^2 dt\right)^{1/2} \end{aligned} \tag{6.7}$$

which with the substitution $\eta = -t$ can be written as

$$\begin{aligned}\|S_0(g)\|_{H^1} &= \left(\int_{-T}^{0} g_x(\eta)^2 + g_y(\eta)^2 d\eta\right)^{1/2} \\ &+ \left(\int_{-T}^{0} (-\dot{g}_x(\eta))^2 + (-\dot{g}_y(\eta))^2 d\eta\right)^{1/2} \\ &= \left(\int_{0}^{T} g_x(\eta)^2 + g_y(\eta)^2 d\eta\right)^{1/2} \\ &+ \left(\int_{0}^{T} (-\dot{g}_x(\eta))^2 + (-\dot{g}_y(\eta))^2 d\eta\right)^{1/2} = \|g\|_{H^1}\end{aligned} \tag{6.8}$$

where the penultimate equality holds because $g(t)$ is a periodic path of period T. Letting $g = \sum_{i=1}^{n} \gamma_i - \gamma$ one proves equation (6.6).

On the other hand if $\gamma_k \in \Sigma_i$ then $S_i(\gamma_k) = \gamma_k$. Consequently $\sum_{k=1}^{n} \gamma_k = \sum_{k=1}^{n} S_i(\gamma_k)$. From this last observation and from equation (6.6) it follows immediately that $\gamma = S_i(\gamma)$ and thus $\gamma \in \Sigma_i$. This concludes the proof since every absolutely convergent series in Σ_i converges to an element of Σ_i. $\square$

We now need to present some definitions. Introducing the complex variable $z = x + iy$ we have

Definition 6.2. The winding number of a contour γ about a point z_0, denoted $n(\gamma, z_0)$ is

$$n(\gamma, z_0) = \frac{1}{2\pi i} \oint_{\gamma} \frac{dz}{z - z_0} \tag{6.9}$$

and gives the number of times γ passes (counterclockwise) around a point.

Counterclockwise winding is assigned a positive winding number, while clockwise winding is assigned a negative winding number.

Definition 6.3. A path in Σ_i is of class L_n, $n = 0, \pm 1, \pm 2, \pm 3, \ldots$, if its winding number about the origin of the coordinate system is n.

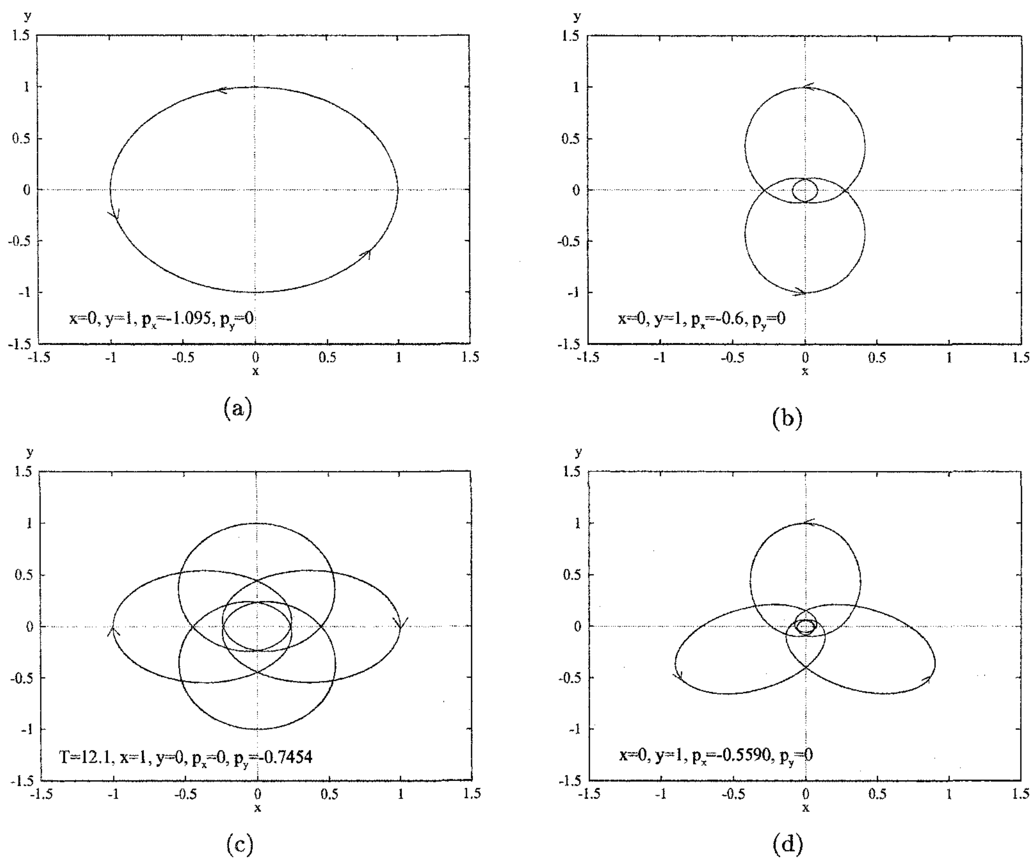


Figure 6.1: Four periodic orbits of the Manev Problem. (a) S_3-symmetric orbit of class L_1 (b) S_1-symmetric orbit of class L_3 (c) S_3-symmetric orbit of class L_{-5} (d) S_2-symmetric orbit of class L_5

For instance Figures 6.1(a), 6.1(b), 6.1(c) and 6.1(d) depict a path of class L_1, L_3, L_{-5} and L_5 respectively. The paths represented in Figure 6.1 are symmetric periodic orbits of the isotropic Manev problem ($\mu = 1$) and were found numerically. Consider the sets $\overline{\Sigma}_i([0,T], \mathbb{R}^2\backslash\{0\})$. Notice that they are open submanifolds of the spaces $\Sigma_i([0,T], \mathbb{R}^2)$ and that the family $(L_n)_{n\in\mathbf{Z}}$ provides a partition of those spaces into homotopy classes, also called components.

6.3 The Variational Principles

The Lagrangian $L(\mathbf{q},\dot{\mathbf{q}}) = T(\dot{\mathbf{q}}) + U(\mathbf{q})$ of the anisotropic Manev problem given by system (3.3) has the expression

$$L(x,y,\dot{x},\dot{y}) = \frac{1}{2}(\dot{x}^2+\dot{y}^2) + \frac{1}{\sqrt{\mu x^2+y^2}} + \frac{b}{\mu x^2+y^2}, \tag{6.10}$$

and the action integral along a path f from time 0 to time T, whose Euclidean coordinate representation is $\mathbf{q} = \mathbf{q}(t) = (x(t), y(t))$, takes the form

$$A_T(f) = \int_0^T L(\mathbf{q}(t),\dot{\mathbf{q}}(t))dt.$$

According to Hamilton's principle, the extremals of the functional A_T are solutions of the equations (3.3). Hence we want to obtain periodic solutions of (3.3) by finding extremals of the functional A. For this we will use a direct method of the calculus of variations, namely the lower-semicontinuity method (see [69]). In preparation for a satisfactory theory of existence, the notion of an admissible function has to be relaxed since the extremals we obtain belong to a Sobolev space. Therefore the above method provides only "weak" solutions of our problem. To show that the paths are regular enough to be classical solutions, we need the following result, proved in [32].

Lemma 6.3. *The critical points of $A_T|_{\overline{\Sigma}_i([0,T],\mathbb{R}^2\setminus\{0\})}$ are T-periodic solutions of equations (3.3).*

In particular it is well known that if f is a minimizer of the action A_T in the space $H^1([0,T],\mathbb{R}^2)$ and if f has no collisions, then f is a T-periodic solution to (3.3). Collisions have to be excluded because equations (3.3) break down at collisions and because the action is not differentiable at paths with collisions. In this chapter we are interested in restricting ourselves to the spaces Σ_i of S_i-symmetric paths for $i = 1, 2, 3$.

Now it is not obvious that a collisionless minimizer in Σ_i is a periodic solution of system (3.3). However, according to the principle of "symmetric criticality" (see for example [10, 60]) this is actually true. Indeed, it can be proved that if f is a collision-free path with $dA_t(f)(h) = 0$ for every $f \in \Sigma_i$, then $dA_T(f)(h) = 0$ for all $f \in H^1([0,T], \mathbb{R}^2)$ and thus f is a critical point in the bigger loop space H^1 (see [10]).

The only obstacle left for applying the direct method is the "noncompactness" of the configuration space. Indeed we want to exclude the possibility that the minimizers are obtained when the bodies are at infinite distance from each other or when they are collision paths. The first problem is solved by restricting ourselves to non-simple cycles, i.e., to cycles that are not homotopic to a point and thus are not in the homotopy class L_0. The second problem is solved by the following result.

Lemma 6.4. *Any family Γ of non-simple homotopic cycles in $\overline{\Sigma}_i([0,T], \mathbb{R}^2\backslash\{0\})$ for $i = 1, 2, 3$ on which $J(f) = \int_0^T \frac{1}{2}|\dot{\mathbf{q}}(t)|^2 dt$ and $E(f) = \int_0^T U(\mathbf{q}(t))dt$ are bounded, is bounded away from the origin.*

The proof of this result follows from [32] if we remark that the anisotropic Manev potential is "strong" according to Gordon's definition and that the Lagrangian is positive.

6.4 Some Properties of Lower Semicontinuous Functions

To apply the direct method of the calculus of variations we still need to recall some properties of lower semicontinuous (l.s.c) functions.

Definition 6.4. Let $\mathcal{F} : X \to (-\infty, \infty]$ be a real valued function on a topological space X. Then $\mathcal{F}$ is l.s.c. if and only if $\mathcal{F}^{-1}(-\infty, a]$ is closed for every $a \in \mathbb{R}$

From the previous definition the proposition below follows immediately

Proposition 6.5. *Suppose $\mathcal{F} : X \to \mathbb{R}$ is a real-valued function on an Hausdorff space X and*

$$\mathcal{F}^{-1}(-\infty, b] \text{ is compact for every real } b.$$

Then $\mathcal{F}$ is l.s.c., bounded below, and attains its infimum value on X.

Proof. Since X is Hausdorff then compact sets are necessarily closed and thus $\mathcal{F}$ is l.s.c by definition. Therefore $\mathcal{F}$ is bounded below and it attains its infimum on every compact subset of X. □

Another useful definition is the one of weakly sequential lower semicontinuity:

Definition 6.5. A functional $Q : H^1 \to [0, +\infty]$ is weakly sequentially lower semicontinuous in H^1 if the lower semicontinuity inequality

$$Q(x) \leq \liminf_n Q(x_n) \tag{6.11}$$

holds for all $x \in H^1$ and $\{x_n\}$ weakly converging to x in H^1.

We will also need the following property of the norm:

Lemma 6.6. *The norm $\|\cdot\|_{H^1}$ is weakly sequentially lower semicontinuous*[1].

Proof. If $f_n \to f$ weakly in H^1 we have $\langle g, f_n\rangle_{H^1} \to \langle g, f\rangle_{H^1}$ for every $g \in H^1$. In particular one can choose $g = f$. Consequently $\|f\|_{H^1} = \lim_{n\to\infty}\langle f_n, f\rangle_{H^1} = \lim_{n\to\infty}|\langle f_n, f\rangle_{H^1}|$. On the other hand $|\langle f_n, f\rangle_{H^1}| \leq \|f_n\|_{H^1}\|f\|_{H^1}$ by the Schwarz inequality. This implies $\|f\|^2_{H^1} = \liminf|\langle f_n, f\rangle_{H^1}| \leq \liminf\|f_n\|_{H^1}\|f\|_{H^1}$. Therefore, if $\|f\|_{H^1} \neq 0$ we conclude that $\|f\|_{H^1} \leq \liminf\|f_n\|_{H^1}$. If $\|f\|_{H^1} = 0$ then the inequality is trivially true, since $\liminf\|f_n\|_{H^1} \geq 0$. □

[1] Clearly a much more general statement is true. Indeed the proof of the Lemma holds true for any Hilbert space.

6.5 Main Result: the Existence of Symmetric Periodic Orbits

We can now prove the main result of this chapter

Theorem 6.1. *For any $T > 0$ and any $n = \pm 1, \pm 2, \pm 3, \dots$, there is at least one S_i-symmetric ($i = 1, 2, 3$) periodic orbit of the anisotropic Manev problem that has period T and winding number n (i.e., belongs to the homotopy class L_n).*

Proof. Let X be a component of $\overline{\Sigma}_i([0,T], \mathbb{R}^2 \backslash \{0\})$ for $i = 0, 1, 2, 3$, that consists of non-simple cycles. Endow X with the weak topology it inherits from $\Sigma_i([0,T], \mathbb{R}^2)$. Then X is a subset of an Hilbert space and it is weakly compact if and only if it is weakly closed and bounded.

We wish to apply Proposition 6.5 with $\mathcal{F} = A_T$ and thus we have to show that $X \cap A_T^{-1}(-\infty, b]$ is a bounded and weak-closed subset of $\Sigma_i([0,T], \mathbb{R}^2)$.

Since $J = A_T - E$ and $U > 0$, we have $E > 0$ and therefore

$$
\begin{aligned}
J \leq b \quad &\text{on} \quad {A_T}^{-1}(-\infty, b] = {A_T}^{-1}[0, b], \\
E = A_T - J \leq b \quad &\text{on} \quad {A_T}^{-1}[0, b].
\end{aligned}
\tag{6.12}
$$

Since $J \leq b$ the elements of X are bounded in arc length, and from Lemma 6.4 it follows that the elements of X are bounded away from the origin. Moreover the elements of X are non-simple and thus bounded in the C^0 norm and hence in the L^2 norm. This last fact combined with $J \leq b$ shows that X is bounded in the $\|\cdot\|_{H^1}$ norm. Thus also $X \cap {A_T}^{-1}(-\infty, b]$ is bounded in the H^1 norm.

Now suppose that $\{f_n\} = \{(f_n^1, f_n^2)\}$ is a sequence in $X \cap {A_T}^{-1}[0, b]$ that converges weakly to a cycle $f \in \Sigma_i([0,T], \mathbb{R}^2)$ for $i = 0, 1, 2$. From general principles, $\|f_n\|_{H^1}$ is bounded and $\|f_n\|_{L^2} \to \|f\|_{L^2}$ because weak Σ_i-convergence implies C^0-convergence. Since $J(f_n) = 1/2\|f_n\|_{H^1}^2 - 1/2\|f_n\|_{L^2}^2$ it means that $J(f_n)$ is bounded and since $E \leq b$ on ${A_T}^{-1}[0, b]$ it follows that $\{E(f_n)\}$ is bounded. Moreover, Lemma

6.4 guarantees that the functions f_n are bounded away from the origin so that f is homotopic to the f_n in $\mathbb{R}^2\backslash\{0\}$. Therefore $f \in X$.

To complete the proof we have to show that $f \in {A_T}^{-1}[0, b]$. We know that $E(f_n) \to E(f)$ since weak convergence in Σ_i implies C^0-convergence. For each n let

$$g_n(t) = \frac{1}{\sqrt{\mu(f_n^1(t))^2 + (f_n^2(t))^2}} + \frac{1}{\mu(f_n^1(t))^2 + (f_n^2(t))^2}$$

and denote

$$g(t) = \frac{1}{\sqrt{\mu(f^1(t))^2 + \mu(f^2(t))^2}} + \frac{1}{\mu(f^1(t))^2 + (f^2(t))^2}.$$

Each g_n is of class L^1 since $A_T(f_n) < \infty$. This implies that the set of all t for which $f_n(t) = 0$ has zero measure, otherwise the integral of $g_n(t)$ would be unbounded. So $g_n(t) \to g(t)$ almost everywhere. Also $\int_0^T g_n(t)\, dt < A_T(f_n) \le b$. By Fatou's lemma it follows that g is L^1 and that

$$\int_0^T g(t)\, dt = \int_0^T \liminf g_n(t)\, dt \le \liminf \int_0^T g_n(t)\, dt.$$

(see [69]), thus

$$\|\dot{f}\|_{L^2}^2 = \|f\|_{H^1}^2 - \|f\|_{L^2}^2 \le \liminf \|f_n\|_{H^1}^2 - \|f\|_{L^2}^2 = \liminf \|\dot{f}_n\|_{L^2}^2,$$

where the last equality holds since $\{f_n\}$ converges strongly to f in L^2. Consequently

$$\begin{aligned} A_T(f) &= \frac{1}{2}\|\dot{f}\|_{L^2}^2 + \int_0^T g(t)\, dt \\ &\le \liminf \frac{1}{2}\|\dot{f}_n\|_{L^2}^2 + \liminf \int_0^T g_n(t)\, dt \le \liminf A_T(f_n) \le b. \end{aligned} \tag{6.13}$$

Relation (6.13) now implies that $f \in {A_T}^{-1}[0, b]$. This completes the proof. □

Recall now that two intersections of every S_1-symmetric (S_2-symmetric) orbit with the x-axis (y-axis) must be orthogonal. To distinguish them from accidental

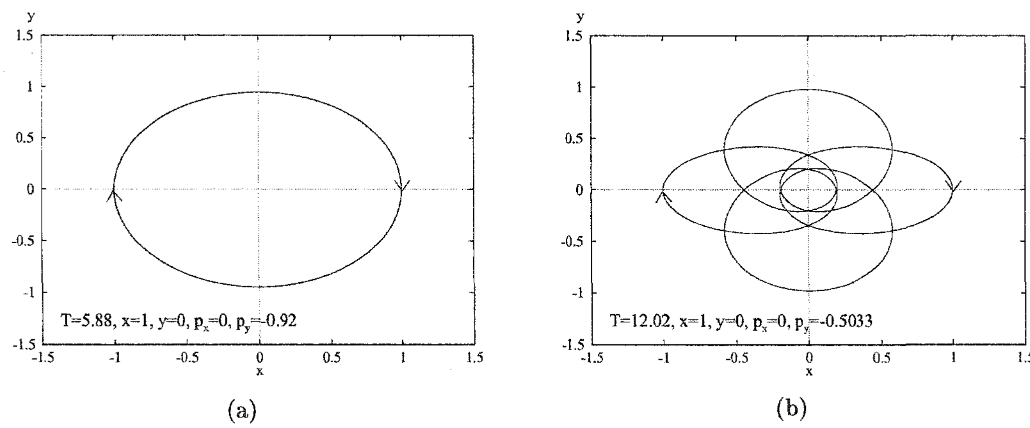


Figure 6.2: Two symmetric periodic orbits of the anisotropic Manev Problem with $\mu = 1.5$, $b = 0.1$. (a) An S_3-symmetric periodic orbit of class L_{-1}. (b) An S_3-symmetric periodic orbit of class L_{-5}.

orthogonal intersections, which do not follow because of the symmetry, we will call them *essential orthogonal intersections.* From the proof of Theorem 6.1 and obvious index theory considerations, the following result follows (see also Figure 6.1).

Corollary 6.7. *If the essential orthogonal intersections with the x-axis (y-axis) of an S_1-symmetric (S_2-symmetric) periodic orbit lie on the same side of the axis with respect to the origin of the coordinate system, then the orbit has an even winding number. If the essential orthogonal intersections are on opposite sides with respect to the origin, then the periodic orbit has an odd winding number.*

Theorem 6.1 allows us to find one S_i-symmetric periodic orbit, with $i = 1, 2, 3$, for every $T > 0$ and $n \in \mathbb{Z} - \{0\}$. This result generalizes Theorem 5.1 and Theorem 5.2 of the previous chapter, where symmetric periodic orbits were found only for small perturbations of the Manev problem.

The results of Theorem 6.1 can be verified integrating numerically the equations of motion and looking for symmetric periodic orbits. A numerical integration

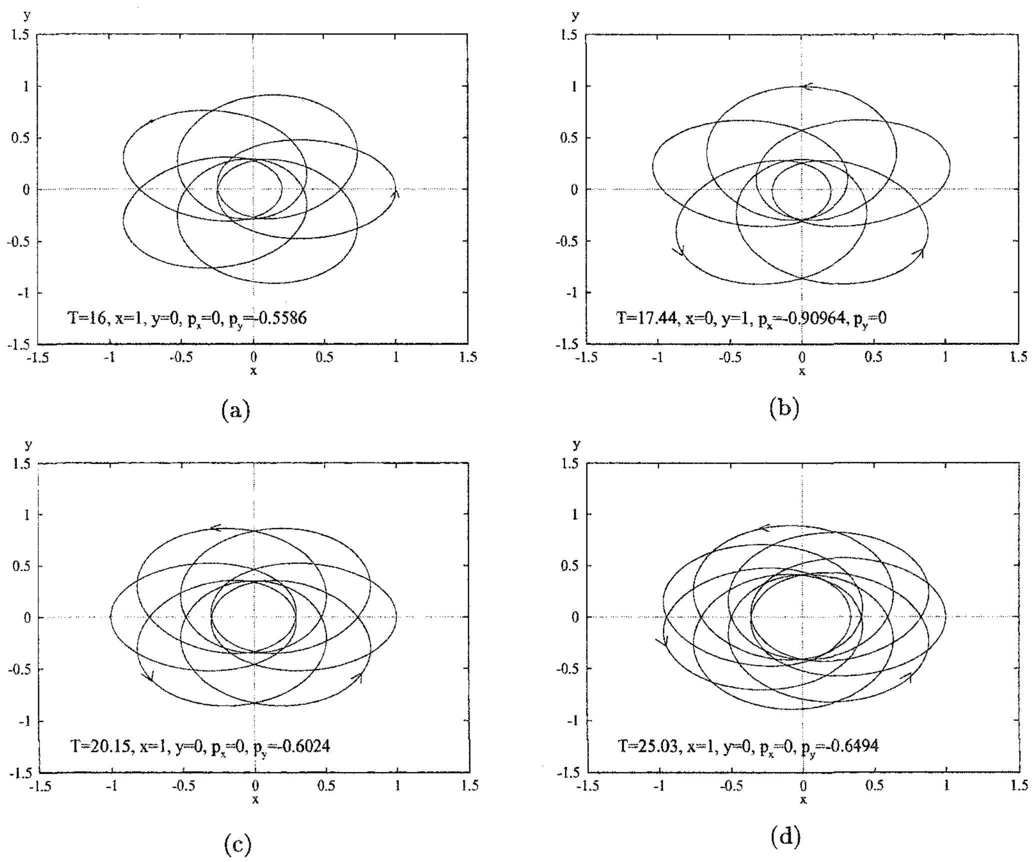


Figure 6.3: Four symmetric periodic orbits of the anisotropic Manev Problem with $\mu = 1.5$, $b = 0.1$. (a) An S_1-symmetric periodic orbit of class L_6. (b) An S_2-symmetric periodic orbit of class L_6. (c) An S_3-symmetric periodic orbit of class L_7. (d) An S_1-symmetric periodic orbit of class L_8.

was performed choosing $b = 0.1$ and $\mu = 1.5$. Some of the periodic orbits found numerically are depicted in Figure 6.2 and Figure 6.3.

It is interesting to note that considering the anisotropy parameter as a perturbation and the anisotropic Manev problem as a perturbation of the isotropic case, the S_i-symmetric ($i = 0, 1, 2$) periodic orbits of the isotropic problem are deformed but not destroyed by introducing the anisotropy, no matter how large its size. This shows that the S_i symmetries ($i = 1, 2, 3$) play an important role in understanding the system since some important properties of the system remain for any value of the anisotropy.

Remark 6.1. It would be interesting to investigate the existence of non-symmetric orbits. However this is a very difficult task because, in all the techniques we applied, the symmetry plays a very important role. More accessible would be a numerical analysis of the non-symmetric orbits. Nevertheless a numerical analysis would not provide a proof of the existence of non-symmetric orbits and thus it would not be a completely satisfactory approach.

Chapter 7

Some Remarks On the Anisotropic Kepler Problem

7.1 Overview

In the previous chapters we analyzed the anisotropic Manev problem with $b > 0$ and most of the results we proved are true only in that case. In this chapter we consider the anisotropic Kepler problem, which can be considered a particular case of the Manev case (with $b = 0$).

As we have already mentioned the Anisotropic Kepler Problem (AKP) was introduced by Gutzwiller and was presented as a classical mechanical system which approximates a quantum mechanical system describing the bound states of an electron near a donor impurity of a semiconductor (for more details on the physics of the problem see [34, 35, 36, 37, 38, 40]). Various authors studied many qualitative features of the Anisotropic Kepler Problem analyzing features that are of interest in the theory of differential equations. Many results and a detailed account of the qualitative properties of the Anisotropic Kepler Problem can be found in the work of Gutzwiller [38, 39, 40], in the papers of Devaney [21, 22] and in the work of Casasayas and Llibre [8].

In this chapter we want to consider a different approach to the AKP, i.e. we want to study the problem as a geodesic flow on a manifold. In particular we wish to study the integrability of the system. We will prove the non-existence of linear integrals in the Anisotropic Kepler Problem.

Let M be a manifold and let TM and T^*M be the tangent and cotangent bundles of M, respectively. Moreover let $g_{ij}(x)$ be a Riemannian metric on M. The corresponding flow on TM is given by the Lagrangian $\mathcal{L} = (1/2)g_{ij}(x)\dot{x}^i\dot{x}^j$. The resulting Hamiltonian system on T^*M has Hamiltonian $H = (1/2)g^{ij}(x)p_ip_j$.

Such systems are very important examples of Hamiltonian systems, since many problems in mechanics reduce to the study of geodesic flows on Riemann and Finsler metrics [48] . For example, with a suitable transformation, the Kepler problem can be described as the geodesic flow on a sphere [57, 59].

We recall that an Hamiltonian system with n degrees of freedom is *integrable* if there exist n first integrals $I_1, \ldots, I_n$ with the property that $\{I_i, I_j\} = 0$ for all i and j, and such that they are functionally independent. In particular the definition above can be used when the Hamiltonian describes the geodesic flow on a manifold.

If M is an analytic manifold and the metric $g_{i,j}$ is analytic then the flow is said to be *analytically integrable* if the indicated integrals $I_1, \cdots, I_n$ exist and can be chosen to be analytic.

In [48, 47] it was proved that the geodesic flow of any analytic metric on an analytic compact two-dimensional manifold of genus $g > 1$ is not analytically integrable and in [71] the discussion was generalized to n -dimensional manifolds. However it is clear that the obstructions to integrability are not only of topological origin. This is because different Riemannian metrics can be associated with the same manifold, even if the topology (of the manifold) imposes restrictions to the possible metrics. Indeed Donnay [26] and Burns and Gerber [5] have constructed real analytic metrics on the sphere whose geodesic flows are Bernoulli. Knieper and Weiss [45] constructed real analytic metrics with positive curvature and positive topological entropy. Thus, even if the sphere has genus $g = 0$ they found that there are metrics on the sphere such that the geodesic flow is not analytically integrable. Hence to refine the criterion of Kozlov (and Taimanov) it is necessary to find a geometrical approach that takes into account the peculiar metric defined on the manifold. Moreover the topological considerations of Kozlov and Taimanov can be applied only to compact manifolds and it would be useful to find a technique applicable in the non-compact case. The main purpose of this chapter is to consider such a geometrical approach that both takes into account the peculiar metric and can be applied to non-compact manifolds and apply it to the anisotropic Kepler problem.

The approach we consider relies on the simple idea that if you have an integrable system and you add a perturbation that destroys the initial symmetry then, unless a new symmetry appears, you obtain a non-integrable system. Examples of this situation are discussed in [25] and [65], where suitable generalizations of the Poicaré-Melnikov method (see for example [14, 15] and references therein) are applied to the perturbed systems. In [25] the anisotropic Manev problem is discussed for weak anisotropies, and resorting to the Melnikov method the occurrence of chaos on the zero energy manifold is proven. In [25] the broken symmetry is the rotational invariance. Similarly in [65], where we discussed black holes immersed in uniform electric and magnetic fields, the broken symmetry was the rotational invariance.

To formalize this intuitive idea, in the next section some results of Riemannian geometry are recalled, in particular the isometries, the infinitesimal isometrics and some of their properties are discussed. Then we observe that the equations of Killing can be used to find the symmetries that preserve the equation of motion and hence if those equations have only the zero solution no first integral can be found other than the Hamiltonian.

In Section 7.3 we introduce the anisotropic Kepler problem and we show that with a suitable canonical transformation it can be reduced to the geodesic flow on a Riemannian surface with non-constant curvature.

The last section of this chapter is devoted to showing that the equations of Killing do not have any non-trivial solutions for the AKP and hence that, by the results of Section 7.2, the Anisotropic Kepler Problem has no linear integrals.

7.2 Isometries and Infinitesimal Isometries: The Killing's Field

We will now recall some results and known facts from Riemannian geometry that we need in order to prove the main result of the chapter; for a more detailed discussion see, for example, [27, 46, 29] .

Let M be an n-dimensional manifold with a Riemannian metric g_{ij}. An isometry of M is a transformation of M which leaves the metric invariant. More precisely, given the coordinates $x_1, \ldots, x_n$ on M, a transformation $x_i = x_i(\overline{x}_1, \ldots, \overline{x}_n)$ is an isometry (or motion of the metric) if

$$\overline{g}_{ij}(\overline{x}_1, \ldots, \overline{x}_n) = g_{ij}(x_1(\overline{x}), \ldots, x_n(\overline{x})). \tag{7.1}$$

It is clear that such motions form a group, which is often called the group of motion. Moreover a vector field ξ^i is called an infinitesimal isometry (or a Killing vector field) if the local 1-parameter group of local transformations generated by ξ^i in a neighborhood of each point of M consists of local isometries. The set of infinitesimal isometries of M, denoted by $i(M)$, forms a Lie algebra. Now a transformation is an infinitesimal motion of M into itself, i.e. a transformation that preserves the metric, if the Lie derivative vanishes, that is, if the equations of Killing

$$L_\xi g_{ij} = \xi^k \frac{\partial g_{ij}}{\partial x^k} + g_{ik}\frac{\partial \xi^k}{\partial x^j} + g_{jk}\frac{\partial \xi^k}{\partial x^i} = 0 \tag{7.2}$$

are satisfied [29]. In order that there may be a non-null motion we must have the additional condition

$$g_{ij}\xi^i\xi^j \neq 0. \tag{7.3}$$

The first fact to verify is that the infinitesimal isometries are all the (linear) transformations that preserve the variational problem [1] defined by the Lagrangian $\mathcal{L} = (1/2)g_{ij}\dot{x}^i\dot{x}^j$, so that we can reduce the study of the symmetries preserving the equation of motion to the study of the isometries. This problem is solved by the following

Lemma 7.1. *An infinitesimal transformation linear in the velocities is an isometry for M with the metric g_{ij} if and only if it preserves the variational problem defined by the Lagrangian $\mathcal{L} = (1/2)g_{ij}\dot{x}^i\dot{x}^j$ on TM.*

Proof. Let us remark that a transformation is an infinitesimal isometry if and only if the Lie derivative of the metric is zero, i.e. if the Killing's equations (7.2) are satisfied. On the other hand a transformation preserves the variational problem defined by the Lagrangian (7.13) when the Lie derivative with respect to the field ξ^i is zero,

$$L_\xi(\mathcal{L}) = \xi^i \frac{\partial \mathcal{L}}{\partial x^i} + \frac{\partial \xi^i}{\partial x^j}\dot{x}^j \frac{\partial \mathcal{L}}{\partial \dot{x}^i} = 0 \tag{7.4}$$

which after a few manipulations takes the form

$$L_\xi(\mathcal{L}) = \frac{1}{2}\left\{\xi^s \frac{\partial g_{ij}}{\partial x^s} + \frac{\partial \xi^k}{\partial x^i} g_{kj} + \frac{\partial \xi^k}{\partial x^j} g_{ik}\right\} \dot{x}^j \dot{x}^i = 0. \tag{7.5}$$

It is trivial to see that (7.2) implies (7.5). The converse is also true since x^i is arbitrary and the tensor in the braces is symmetric. This concludes the proof. □

[1] Such transformations are called *variational symmetries* (see [58]). Observe that, in general, a variational symmetry is a symmetry of the Euler-Lagrange equations, however it is not true that every symmetry of the Euler-Lagrange equations is a variational symmetry. For example consider the Lagrangian $\mathcal{L} = (1/2)m\dot{x}^2 - gx$. The Euler-Lagrange equation is $\ddot{x} = -g$. Clearly the translation $x \to x + a$ is a symmetry of such equations. However translations are not a variational symmetry, since $L_\xi(\mathcal{L}) = -g \neq 0$, where $\xi = \partial(x+a)/\partial a = 1$

It is now easy to prove the following

Theorem 7.1. *Let M be an analytic n-manifold with Riemannian metric g_{ij} and $\mathcal{L} = (1/2)g_{ij}x^i x^j$ be the Lagrangian defined on TM. If the Killing's equations for M have m independent solutions, where $m \leq n$ then the geodesic flow has m linear integrals.*

Proof. The proof follows immediately from a fundamental property of the Killing vectors. Indeed the inner product of a Killing vector with a tangent vector to the geodesic of the manifold M is constant along the geodesics. Therefore contracting a Killing vector with a tangent vector one obtains a first integral (linear in the velocity) of the geodesic equation (see [29]). □

Using the celebrated Noether theorem (see [58] for details) we easily obtain the following

Proposition 7.2. *If the only solution of the Killing's equation is $\xi^i \equiv 0$, the corresponding geodesic flow does not have any linear integrals.*

Proof. The Noether theorem says that there is a one-to-one correspondence between equivalence classes of non-trivial conservation laws of the Euler-Lagrange equations and equivalence classes of variational symmetries[2]. Therefore if there are no non-trivial solutions of the Killing's equations it follows that there are no linear integrals.

□

We now want to consider some more properties of the infinitesimal isometries. A classical result [46, 29] is the following

[2] Note that two variational symmetries are equivalent provided that they differ by a trivial symmetry, meaning one whose characteristic vanishes on all solutions of the Euler-Lagrange equations. Moreover two conservation laws are equivalent if they differ by a trivial conservation law.

Proposition 7.3. *The Lie algebra of infinitesimal isometries $i(M)$ of a connected Riemannian manifold M is of dimension at most $\frac{1}{2}n(n+1)$, where $n = dimM$. If $dim\ i(M) = \frac{1}{2}n(n+1)$, then M is a space of constant curvature.*

This Proposition gives the maximum number of infinitesimal isometries and hence also the maximum number of first integrals for a geodesic flow on a manifold of dimension n. In particular in the two dimensional case we have

Corollary 7.4. *If M is a two-dimensional connected Riemannian manifold of non-constant curvature then the Lie algebra of infinitesimal isometries $i(M)$ is of dimension at most 1.*

One example that illustrates the results above is the Kepler problem on the plane. Such problem can be described as the geodesic flow on a sphere S^2, hence the isometries form a Lie algebra isomorphic to $SO(3)$, and there are three first integrals other than the Hamiltonian (one component of the angular momentum and two components of the Runge-Lenz vector). On the other hand, if we consider, for instance, an homogeneous potential, other than the harmonic oscillator and the Kepler one, we can describe the motion only as a geodesic flow on a surface that has an $SO(2)$ symmetry.

7.3 The Anisotropic Problems as Geodesic Flow on a Surface

The Hamiltonian function for the Anisotropic Kepler Problem on the plane can be written as

$$H_K(\mathbf{q}, \mathbf{p}) = \frac{\mathbf{p}^2}{2} - \frac{a}{(x^2 + \mu y^2)^{1/2}} \tag{7.6}$$

where $\mu > 1$ is a constant and $\mathbf{q} = (x, y)$ is the position of one body with respect to the other, considered fixed at the origin of the coordinate system, and $\mathbf{p} = (p_x, p_y)$

is the momentum of the moving particle. The constant measures the strength of the anisotropy and for $\mu = 1$ we recover the classical Kepler Problem.

With some canonical transformations[3] we want to write the problem as the geodesic flow on a surface. To accomplish this task we need the following Lemma

Lemma 7.5. *Let $(\mathbf{q}(t), \mathbf{p}(t))$ be a solution of the Hamiltonian system with Hamiltonian $H(\mathbf{q}, \mathbf{p})$ which lies on the level surface $H = 0$. Let us change the time variable t to τ along this trajectory according to the formula $d\tau/dt = 1/G(\mathbf{q}(t), \mathbf{p}(t)) \neq 0$. Then $(\mathbf{q}(\tau), \mathbf{p}(\tau)) = (\mathbf{q}(t(\tau)), \mathbf{p}(t(\tau)))$ is a solution of the Hamiltonian system (with respect to the same symplectic structure) with Hamiltonian $\tilde{H} = HG$. If $G = 2(H + \alpha)$ with $\alpha = const$, one can also put $\tilde{H} = (H + \alpha)^2$.*

The statement above can be found in [3] and the proof is based on the idea of extended phase space (see [27]). If we perform the change of time $d\tau/dt = (x^2 + \mu y^2)^{1/2}$ on the submanifold $H_K = h$, according to Lemma 7.5, we obtain the new Hamiltonian

$$(x^2 + \mu y^2)^{1/2}(H_K - h) = (x^2 + \mu y^2)^{1/2}\left(\frac{\mathbf{p}^2 - 2h}{2}\right) - a. \tag{7.7}$$

Then again change the time τ to s by $ds/d\tau = 2((x^2 + \mu y^2)^{1/2}(H_K - h) + a)$ on the same level surface $H_K = h$. By the second part of Lemma 7.5 we obtain the Hamiltonian system with Hamiltonian

$$\tilde{H}_K = (x^2 + \mu y^2)\frac{(\mathbf{p}^2 - 2h)^2}{4}. \tag{7.8}$$

Finally, applying Legendre's transformation, regarding $\mathbf{p}$ as coordinates and $\mathbf{q}$ as the canonically conjugate momenta, we can write

---

[3] In this context canonical transformations are transformations that preserve the canonical form of the Hamilton-Jacobi equations. However according to a different definition the canonical transformations are those transformations that preserve the differential 2-form $\sum_{i=1}^{n} dp_i \wedge dq_i$. The first definition is more general. For more details see [72]

$$\begin{cases} p_x' = \frac{\partial \tilde{H}_K}{\partial x} = \frac{x}{2}(\mathbf{p}^2 - 2h)^2 \\ p_y' = \frac{\partial \tilde{H}_K}{\partial y} = \frac{y\mu}{2}(\mathbf{p}^2 - 2h)^2 \end{cases} \tag{7.9}$$

where the prime indicates the derivative with respect to s. Consequently

$$\begin{cases} x = \frac{2p_x'}{(\mathbf{p}^2-2h)^2} \\ y = \frac{2p_y'}{\mu(\mathbf{p}^2-2h)^2} \end{cases} . \tag{7.10}$$

Moreover since we have that

$$\mathcal{L}_K = p_x' x(\mathbf{p}, p_x') + p_y' y(\mathbf{p}, p_y') - \tilde{H}_K(x(\mathbf{p}, p_x'), y(\mathbf{p}, p_y'), p_x, p_y) \tag{7.11}$$

we obtain a natural Lagrangian system with Lagrangian

$$\mathcal{L}_K = \frac{1}{(\mathbf{p}^2 - 2h)^2}\left(p_x'^2 + \frac{p_y'^2}{\mu}\right) \tag{7.12}$$

and if we let $p_x = u, p_y = v$ and $x^i = (u, v)$ we obtain

$$\mathcal{L}_K = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j \tag{7.13}$$

where

$$g_{ij} = \frac{2}{(u^2 + v^2 - 2h)^2}\begin{pmatrix} 1 & 0 \\ 0 & 1/\mu \end{pmatrix}. \tag{7.14}$$

The Lagrangian above defines a Riemann metric of Gaussian curvature given by

$$K = -(1-\mu)(u^2 - v^2) - 2h(1+\mu) \tag{7.15}$$

that for $\mu = 1$ describes a metric of constant Gaussian curvature (positive for $h < 0$ and negative for $h > 0$) [57, 59, 2]. It is interesting to remark that, by Corollary 7.4, for $\mu \neq 1$ we have at most one infinitesimal isometry (and hence also at most one first integral), since the curvature is not constant.

7.4 Nonexistence of Integrals Linear in the Momentum

In the previous section we reduced the Anisotropic Kepler Problem to a geodesic flow on a two-dimensional surface, and hence we can apply the results obtained in Section 7.2. This section is devoted to proving the non-existence of linear integrals in the AKP. Thus we are well on our way to proving the following

Theorem 7.2. *The Anisotropic Kepler Problem does not have any linear first integral.*

Proof. The only fact that remains to be proved is that the Killing's equations does not have any non-null solution. The equations of Killing for the metric (7.14) can be written as

$$\begin{cases} \frac{\partial \xi^1}{\partial v} + \frac{1}{\mu}\frac{\partial \xi^2}{\partial u} = 0 \\ -4\frac{(u\xi^1+v\xi^2)}{(u^2+v^2-2h)} + 2\frac{\partial \xi^1}{\partial u} = 0 \\ -4\frac{(x\xi^1+v\xi^2)}{(u^2+v^2-2h)} + 2\frac{\partial \xi^2}{\partial v} = 0 \end{cases} \tag{7.16}$$

Subtracting the third equation from the second we get

$$\frac{\partial \xi^1}{\partial u} = \frac{\partial \xi^2}{\partial v}, \tag{7.17}$$

and combining (7.17) with the first equation in system (7.16) we obtain

$$\frac{\partial^2 \xi^1}{\partial u^2} = -\mu \frac{\partial^2 \xi^1}{\partial v^2} \quad , \quad \frac{\partial^2 \xi^2}{\partial v^2} = -\frac{1}{\mu}\frac{\partial^2 \xi^2}{\partial u^2}. \tag{7.18}$$

Now multiplying the second equation in system (7.16) by (u^2+v^2-2h) (assume $h < 0$), differentiating with respect to u and using equation (7.17) and the first relation in (7.16) we get the following second order differential equation

$$2\mu(u^2+v^2-2h)\frac{\partial^2 \xi^1}{\partial v^2} - 4\mu v \frac{\partial \xi^1}{\partial v} + 4\xi^1 = 0. \tag{7.19}$$

Similarly from the third equation in (7.16) we get another second order differential equation

$$\frac{2}{\mu}(u^2+v^2-2h)\frac{\partial^2\xi^2}{\partial u^2}-\frac{4}{\mu}u\frac{\partial\xi^2}{\partial u}+4\xi^2=0. \tag{7.20}$$

It is easy to see that if $h<0$, $v=0$ is an ordinary point since the functions $-2v/(u^2+v^2-2h)$ and $2/(\mu(u^2+v^2-2h))$ are analytical in v at $v=0$. Let each of these functions be represented by its Taylor series at $v=0$ on the real line, then every solution of (7.19) on $\mathbb{R}$ is analytic at $v=0$ and can be represented on $\mathbb{R}$ by its Taylor series at $v=0$. Similarly every solution of (7.20) can be represented on $\mathbb{R}$ by its Taylor series at $u=0$ since $u=0$ is an ordinary point. Therefore we can look for solutions of equations (7.19) of the form $\xi^1=\sum_{n=0}^{\infty}a_n(u)v^n$ and solutions of (7.20) of the form $\xi^2=\sum_{n=0}^{\infty}b_n(v)u^n$. Routine calculations show that, from (7.19) we must have

$$a_2=-\frac{a_0}{\mu(u^2-2h)} \quad , \quad a_3=\frac{(\mu-1)a_1}{3\mu(u^2-2h)} \tag{7.21}$$

and the following recurrence relation

$$a_{n+2}=-\frac{2+\mu n(n-3)}{\mu(u^2-2h)(n+2)(n+1)}a_n \tag{7.22}$$

holds for $n>1$. From equation (7.20) we get

$$b_2=-\frac{\mu b_0}{(v^2-2h)} \quad , \quad b_3=-\frac{(\mu-1)b_1}{3(v^2-2h)} \tag{7.23}$$

and the recurrence relation

$$b_{n+2}=-\frac{2\mu+n(n-3)}{(v^2-2h)(n+2)(n+1)}b_n \tag{7.24}$$

for $n>1$. With the preparations above we can find the general solution of (7.19)

$$\begin{aligned}\xi^1 &= a_0(u)\left(1-\tfrac{1}{\mu(u^2-2h)}v^2+\tfrac{(1-\mu)}{6\mu^2(u^2-2h)^2}v^4+\dots\right)+\\ &+a_1(u)\left(v-\tfrac{\mu-1}{3\mu(u^2-2h)}v^3+\tfrac{\mu-1}{30\mu^2(u^2-2h)^2}v^5+\dots\right)\end{aligned} \tag{7.25}$$

and the general solution of (7.20)

$$\begin{aligned}\xi^2 = b_0(v)\left(1 - \tfrac{\mu}{(u^2-2h)}u^2 + \tfrac{(\mu-1)\mu}{6(u^2-2h)^2}u^4 + \dots\right) + \\ +b_1(v)\left(u - \tfrac{\mu-1}{3(u^2-2h)}u^3 + \tfrac{(\mu-1)\mu}{30(u^2-2h)^2}u^5 + \dots\right)\end{aligned} . \tag{7.26}$$

We will not need the explicit form of the solutions, but we will retain the recurrence relations. Now we need to check that the solutions we found above satisfy the "consistency" relations (7.18). Consider a solution of (7.19) written as a power series $\sum_{n=0}^{\infty} a_n(u)v^2$, using the first of equations (7.18) we find the relations

$$\frac{d^2 a_n}{du^2} = -2\mu a_{n+2} \tag{7.27}$$

since, by analyticity, we can exchange the sums with the derivatives. It is easy to check that choosing $n = 0$ the previous relation reduces to

$$\frac{d^2 a_0}{du^2} = -2\mu a_2 = \frac{2a_0}{(u^2-2h)} \tag{7.28}$$

which is solved by $a_0 = (u^2 - 2h)$. For $n = 2$ instead we have

$$\frac{d^2 a_2}{du^2} = -2\mu a_4 = \frac{1-\mu}{3(u^2-2h)}a_2. \tag{7.29}$$

From the first of equations (7.21) with $a_0 = (u^2 - 2h)$ we get that $a_2 = -1/\mu$ does not satisfy (7.29) if $\mu \neq 1$. Thus the consistency conditions impose that $a_0 = 0$ and consequently $a_{2n} = 0$ for every integer n.

Now, to show that the odd terms of the series also vanish, we consider equation (7.27) with $n = 1$ and we get

$$\frac{d^2 a_1}{du^2} = 2\mu a_3 = \frac{2(\mu-1)}{3(u^2-2h)}a_1. \tag{7.30}$$

Also for $n = 3$ we obtain

$$\frac{d^2 a_3}{du^2} = -2\mu a_5 = \frac{a_3}{5(u^2-2h)}. \tag{7.31}$$

Differentiating the second of equations (7.21) twice and combining the result with equation (7.30) and (7.21) we find the following differential equation

$$3(u^2-2h)\frac{d^2a_3}{du^2}+12u\frac{da_3}{du}+2(2+\mu)a_3=0 \tag{7.32}$$

Combining (7.31) and (7.32) we can eliminate the second order derivatives and we obtain the following first order differential equation

$$\frac{da_3}{du}=-\frac{2(2+\mu)+3/5}{12u}a_3=-\frac{\gamma}{u}a_3 \tag{7.33}$$

where $\gamma=\frac{2(2+\mu)+3/5}{12}$ and the general solution is $a_3=Cu^{-\gamma}$ and satisfies equation (7.31) if and only if $C=0$. Consequently $a_3=0$ and hence $a_{2n+1}=0$ for every integer n. Therefore $\xi^1\equiv 0$.

Similarly we can apply the same reasoning to prove that $\xi^2\equiv 0$. Indeed it is enough to consider again equations (7.28-7.33) exchanging u and v, replacing μ with $1/\mu$ and a_n with b_n. Therefore we can conclude that there is no non-null motion, i.e. $g_{ij}\xi^i\xi^j=0$ consequently, by Corollary 7.2, the geodesic flow does not have linear integrals in the velocity and this implies that the Anisotropic Kepler Problem does not have linear integrals. This completes the proof of Theorem 7.2. □

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The king's a beggar, now the play is done:
All is well ended, if this suit be won,
That you express content; which we will pay,
With strife to please you, day exceeding day:
Ours be your patience then, and yours our parts;
Your gentle hands lend us, and take our hearts.

All's well that ends well
WILLIAM SHAKESPEARE