

A COMPUTER PROGRAM FOR  
BEAM TRANSPORT CALCULATIONS

by

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT  
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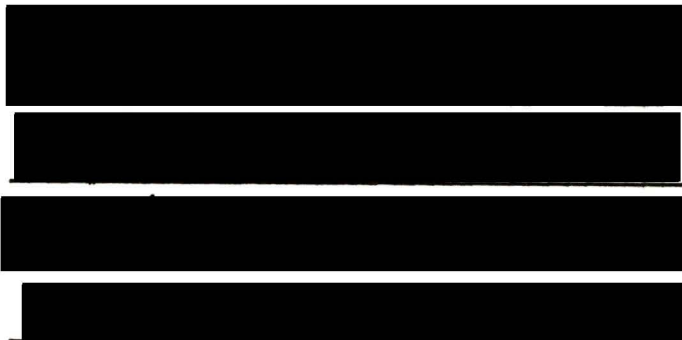
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## ABSTRACT

A computer program TRIUMF for the solution of beam transport problems is described. This program tracks, to first order, particle trajectories and phase space ellipses through any combination of up to 30 drift spaces, quadrupole magnets, and constant field bending magnets with rotated pole faces. Particle trajectories or the beam envelope may be plotted for points every 20 cm. along the optic axis. The phase space ellipse at the exit of any beam element may also be plotted. The program does trajectory matching and ellipse matching to a waist or to a specified ellipse. It will also attempt to create an 'identity system'. The matching routines are based on a method for finding the minimum of a function of  $N$  variables without calculating first derivatives due to Powell. The program has been written in Fortran II and will run without modification on the IBM/360 digital computer.

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## 1. INTRODUCTION

The high energy particle beams extracted from large accelerators must be transported over long distances to the experimental areas. In order to keep the lateral dimensions of the beam from growing it is necessary to have some system which will focus the particles in the horizontal and vertical directions perpendicular to the direction of travel of the beam. It was shown in 1952 by Courant et al that particles can be effectively focussed in both transverse directions by using quadrupole magnets. Since then it has become a standard technique to use combinations of quadrupole magnets (doublets, triplets, etc.) for controlling high energy beams. If the beam is required to be deflected as well as contained, then bending magnets are used. These magnets produce a uniform vertical field which bends the beam, enabling one to guide it to the appropriate experimental area. In addition such magnets can focus or defocus particles in the two transverse directions depending on the field configuration at the entrance and exit pole faces. Bending magnets also produce dispersion of the beam, i.e. particles with the same initial conditions except for different initial momenta will have different trajectories through the magnet. This dispersion can be used to obtain

a better momentum resolution of the beam. If it is assumed that magnet construction problems have been solved and that uniform fields and quadrupole fields are available then a matrix formalism developed (to first order) by Penner in 1961 can be used to calculate the effect of these magnets on charged particles. The calculations performed by the TRIUMF program are based on Penner's method.

Good introductions into the subject of beam transport are given by King (1964), Livingood (1961), Livingston and Blewett (1962) and Banford (1966). The notation used in this thesis and the two basic problems in beam transport theory are introduced below.

For any given beam transport system it is assumed that there is a central trajectory, the optic axis, which is the path followed by a particle with zero initial displacement and slope, and momentum  $p$ . The displacements and slopes of other trajectories are referred to it. The distance along the optic axis is called  $z$ . Relative displacements in the horizontal plane are denoted by  $x$  and in the vertical plane by  $y$ .

If for a given system the trajectory equations ( $x = x(z)$ ,  $y = y(z)$ ) are known for any particle in the beam then one has essentially accomplished what is known as 'tracking'. The trajectory equations are found by solving the equations of motion for the particle in each beam handling element (drift space, quadrupole magnet, constant

field bending magnet) and then computing the cumulative effect on the particle for any point  $z$  along the optic axis. The slopes of the trajectory equations,  $\frac{dx}{dz}$  and  $\frac{dy}{dz}$ , are denoted by  $x'$  and  $y'$  respectively.

A more difficult problem in beam transport theory is 'matching'. Here, starting from a guessed initial system one varies designated element parameters until the beam at the exit of the system is closest to having certain desired properties. The initial system is determined by intuition, experience or from approximate calculations using 'thin' lenses in place of quadrupole magnets. The chief difficulty in matching is that the action of the system on the beam is different in the two transverse directions and thus the solving of a matching problem usually entails simultaneous solution of two or more non-linear equations.

The rest of the material in this thesis describes the TRIUMF digital computer program and how it is used to solve tracking and matching problems.

Chapter 2 outlines the basic concepts and standard techniques used in beam transport theory which are needed to understand the content of the TRIUMF program. In section 2.1 the usefulness of matrices for beam transport calculations is shown. Section 2.2 describes the types of elements used in the program and section 2.3 discusses the mathematical analysis of particle beams.

Chapter 3 describes the TRIUMF program. Sections

3.1 and 3.2 give an outline of the facilities which have been incorporated into the program. In section 3.3 is a brief discussion of the method used to solve the matching problem and section 3.4 contains a description of the functions performed by the main program and by the various subroutines. Section 3.5 briefly compares the TRIUMF program with existing programs and suggests some possible improvements which could be made.

Appendices I to IV supply the mathematical background for a better understanding of Chapter 2. In Appendices V to VII the information needed to successfully use the TRIUMF program is given. A sample problem is solved in Appendix VIII. From this problem it is seen that with the speed of the 360 computer even a fairly long job should take less than 10 minutes for compilation and execution.

## 2. BASIC BEAM TRANSPORT CONCEPTS

### 2.1 Matrix Representation of Beam Handling Elements

It has been shown by Penner (1961) that the action of a beam transport system on a charged particle can be conveniently described by a matrix formalism. Each particle deflecting element in the system can be represented by two transfer matrices, one for the horizontal plane and one for the vertical plane.\* The transfer matrices for an entire system are computed by multiplying together the individual element matrices. The initial particle displacement, slope, and momentum deviation (all with respect to the central trajectory) can be taken as the components of a column vector. Multiplying this column vector whose components are the input conditions by the transfer matrix gives a vector whose components are the output conditions.

The transfer matrices are 3 x 3 and the passage of a particle through a system of elements is described in the horizontal plane by

$$\begin{pmatrix} x \\ x' \\ \frac{dp}{p} \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ \frac{dp}{p} \end{pmatrix} \quad (2.1.1)$$

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\*This is only possible if the equations of motion for the two transverse planes are independent as is the case for the elements used in the TRIUMF program. In a more general treatment an element is represented by a single 6 x 6 matrix (Moore et al, 1963).

where  $x_0$  is the initial displacement,  $x$  the final displacement,  $x'_0$  the initial slope and  $x'$  is the final slope.  $dp = p_0 - p$  is the deviation of the particle momentum  $p_0$  from  $p$  where  $p$  is the momentum of a particle travelling down the central trajectory. A similar matrix equation holds in the vertical plane.

To find the transfer matrix components for any given system element one has to integrate the equations of motion,

$$\frac{d(\overrightarrow{mv})}{dt} = e\overrightarrow{v} \times \overrightarrow{B}, \quad (2.1.2)$$

for a particle in the magnetic field of that element. The equations of motion for the displacements are made linear by neglecting second and higher order terms in the displacement and slope. The matrix components are readily obtained and are given in Appendix I, II, and III for a drift space, quadrupole magnet, and bending magnet respectively.

## 2.2 Types of Beam Handling Elements

The types of magnetic particle-deflecting elements used in the TRIUMF program are described below.

2.2.1 Drift space: This is the space between magnets and is considered to be a region of zero magnetic field. It is specified by one parameter, the length  $L$  along the central trajectory of the field free region.

2.2.2 Quadrupole magnet: A quadrupole lens consists of four iron pole pieces mounted on a common yoke and excited by current carrying coils arranged in the configuration shown in Fig. 1. The field can be approximately represented by  $\vec{B} = \text{grad } \Phi$  where  $\Phi$  is a magnetostatic potential given by  $\Phi = g(z) xy$  (Yagi, 1964). For  $g(z) = g$ , a constant, this is the field produced by the magnet shown in Fig. 2 where the poles are rectangular hyperbolae of infinite permeability and the magnet is infinitely long in the  $z$ -direction. The field of this theoretical quadrupole magnet is then given by

$$\begin{aligned} B_x &= \frac{\partial}{\partial x} (gxy) = gy \\ B_y &= \frac{\partial}{\partial y} (gxy) = gx \\ B_z &= \frac{\partial}{\partial z} (gxy) = 0 \end{aligned} \tag{2.2.1}$$

where

$$g = \frac{\partial B_x}{\partial y} = \frac{\partial B_y}{\partial x}$$

is the magnetic field gradient.

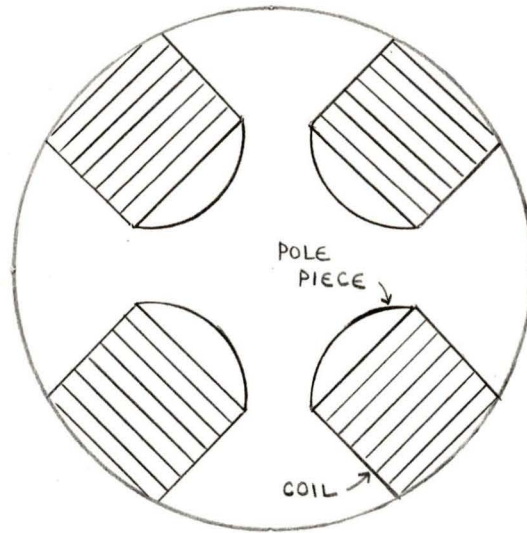


Fig. 1. The Quadrupole Magnet

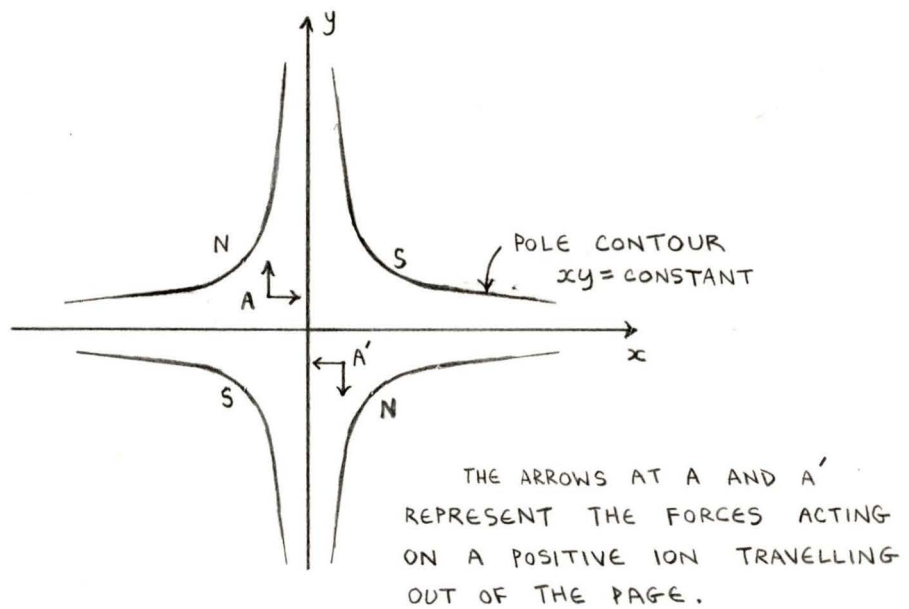


Fig. 2. The Ideal Quadrupole Magnet

For real quadrupoles the measured field gradient  $g(z)$  varies with  $z$  approximately as shown in Fig. 3. The simple potential equation  $\Phi = gxy$  can be retained if  $g(z)$  is replaced by the step function indicated by the dashed line in Fig. 3. This is called the hard edge model (Blackstein, 1967).

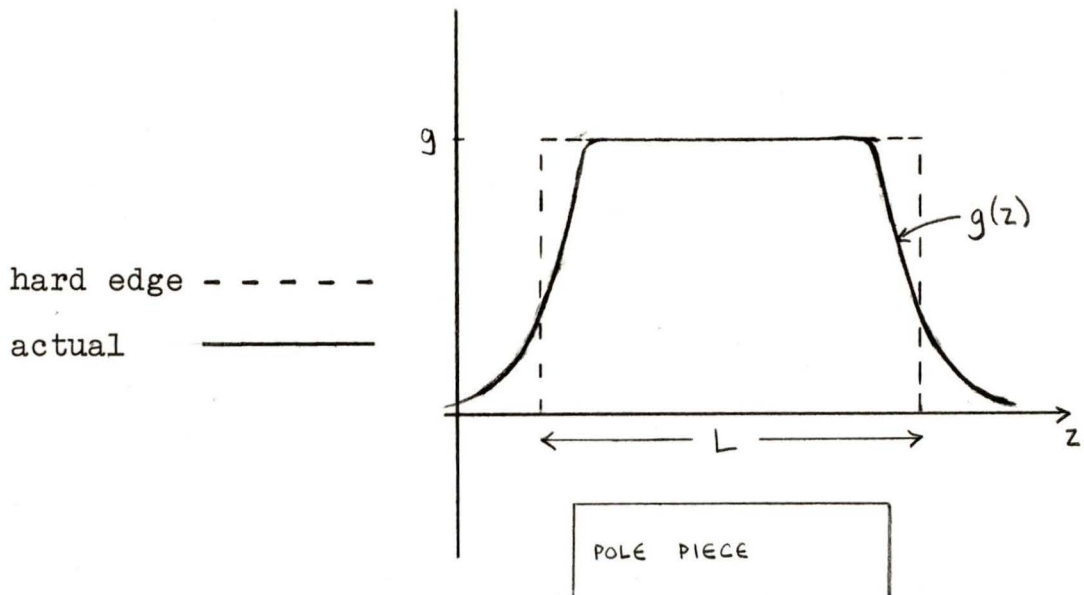


Fig. 3. The Hard Edge Quadrupole Model.

To replace the actual field by a fictitious hard edge model field the length  $L$  must be defined so that the magnet has approximately the same focussing action as does the real field. To first order accuracy  $L$  can be defined by

$$L = \frac{\int_{-\infty}^{+\infty} g(z) dz}{g}$$

where  $g(z)$  is the measured field gradient and  $g$  is the value of the gradient well within the magnet (Banford, 1961).

The force exerted on a particle carrying charge  $e$  and which is travelling parallel to the  $z$ -axis with velocity  $\dot{z} = v$  is given by the Lorentz force equation as

$$\vec{f} = e\vec{v} \times \vec{B}$$

or

$$f_x = e(\dot{y}B_z - \dot{z}B_y) = -ev B_y = -ev gx$$

and

(2.2.2)

$$f_y = e(\dot{z}B_x - \dot{x}B_z) = ev B_x = ev gy .$$

From this we see that for such particles the action of the field is focussing in the  $x$ -direction and defocussing in the  $y$ -direction, i.e. the force in the  $x$ -direction is directed towards the optic axis, the force in the  $y$ -direction away from it. In Fig. 2 these forces are shown for a positive ion travelling out of the paper at points  $A$  and  $A'$ .

If the magnet is rotated by  $90^\circ$  the north and south poles are exchanged and  $g$  becomes negative. The result is that the field now focusses in the  $y$ -direction and defocusses in the  $x$ -direction. A system of quadrupole magnets which is focussing in both directions can be obtained by alternating quadrupoles with positive and negative  $g$ -values (Courant et al, 1952).

2.2.3 Bending magnet (constant field): As in the case of quadrupoles a hard edge model can be used to represent a bending magnet. The field is taken as zero outside the pole faces and constant within. The approximate effective length  $L$  of the bending magnet can be defined

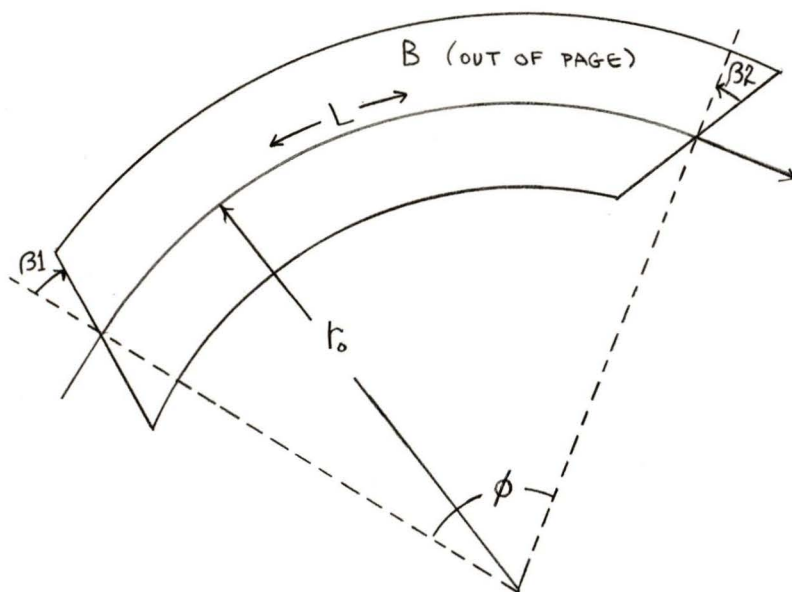
$$L = \frac{\int_{-\infty}^{+\infty} B_y(z) dz}{B}$$

where  $B_y(z)$  is the measured field,  $B$  is the field well within the magnet and the integral is taken over the curved central trajectory through the magnet.

Bending magnets deflect the central trajectory in the horizontal plane. Magnets which bend beams of positively charged particles to the right have a positive field. If the magnet bends such a beam to the left the field is negative.

Fig. 4(A) shows the plan view of a positive field bending magnet. The pole faces are not necessarily normal to the central trajectory but may be at an entrance angle  $\beta_1$  or an exit angle  $\beta_2$ . In both transverse planes these rotated pole faces have a focussing or defocussing effect on particles as explained in Appendix III. The parameters which specify a hard edge model constant field bending magnet are the effective length  $L$ , the field strength  $B$ , the entrance angle  $\beta_1$ , the exit angle  $\beta_2$  and the angle of bend  $\phi$  of the central trajectory. Only four of these

(A) Angle of Bend to Right



(B) Angle of Bend to Left

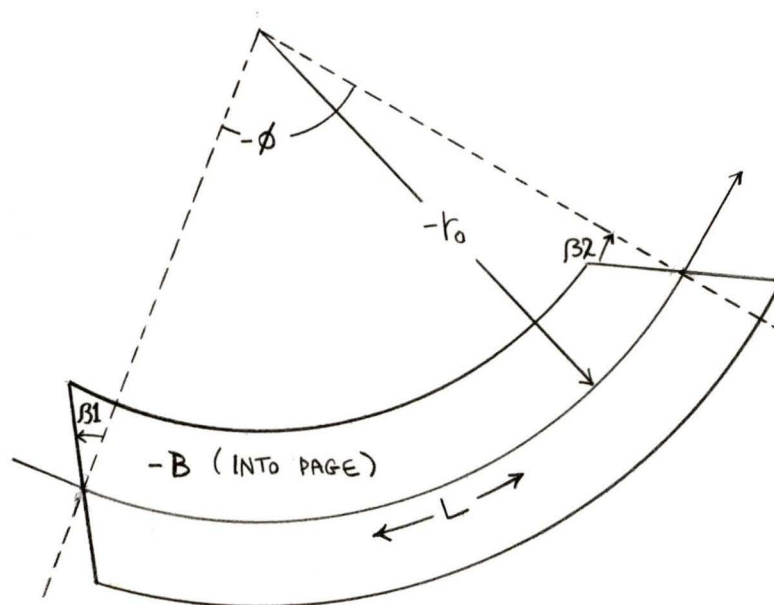


Fig. 4. Plan View of a Bending Magnet

are independent as  $L = r_0 \phi$  where  $r_0 = r_0(p, B)$  as is shown in Appendix III. The sign conventions for  $\phi$ ,  $\beta_1$ , and  $\beta_2$  are explained in Appendix III.

### 2.3 Phase Space Representation of a Particle Beam

Particles emerging from accelerators have a range of displacements and slopes such that it is impossible to bring all the particles to the optic axis or to have them all travelling parallel to the optic axis at one time. In order to treat such beams mathematically the concept of phase space is often used. As discussed by Banford (1961) the motion of each particle in time can be represented by the motion of a point in a six dimensional phase space with co-ordinates  $(x, y, z, p_x, p_y, p_z)$ . The total beam can be represented by a collection of points which for a beam of finite dimensions is contained within a finite six dimensional hypervolume in phase space. From statistical mechanics "Liouville's theorem" states that under the action of forces which can be derived from a Hamiltonian function the motion of a group of particles is such that the local density of representative points in the appropriate phase space remains constant. An alternate statement of Liouville's theorem is that the volume in phase space enclosing a given selection of points remains invariant. Appendix IV contains a derivation of Liouville's theorem.

For quadrupoles and bending magnets the coordinate axis can be chosen such that equations of motion in the x-, y- and z-directions are independent of each other. Liouville's theorem then simplifies to the statement that the areas in the  $(x, p_x)$ ,  $(y, p_y)$  and  $(z, p_z)$  planes which

contain the representative points are constant in time. Also, under the action of forces proportional to displacement such as are encountered in quadrupoles and bending magnets an elliptical contour in phase space remains an elliptical contour. It is convenient then to regard the particles of the beam as being confined within an elliptical phase space region.

For beams that are acted on only by static magnetic fields the average axial momentum  $p_z$  is constant and we have

$$p_x = m \frac{dx}{dt} = m \frac{dx}{dz} \frac{dz}{dt} = p_z x' .$$

Then the area in the space  $(x, x') = (x, p_x/p_z)$  is a constant times the area in the  $(x, p_x)$  phase space. Similarly  $y' = p_y/p_z$  and the same is true for the  $(y, y')$  space. The two dimensional phase spaces of invariant area can then be taken as  $(x, x')$  and  $(y, y')$ .

Following Steffen (1964) the phase space ellipse equation in the  $(x, x')$  plane can be written in a normalized form as

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \epsilon \quad (2.3.1)$$

where

$$\gamma\beta - \alpha^2 = 1 \quad (2.3.2)$$

and the "emittance"

$$\epsilon = \frac{\text{ellipse area}}{\pi} . \quad (2.3.3)$$

A similar equation holds in the  $(y, y')$  plane and the following discussion applies equally well to this plane. In fact any conclusions involving one transverse direction will be assumed valid for the other direction unless otherwise stated.

For any orientation of the ellipse the maximum displacement of any particle in the beam is given by  $x_m = (\epsilon\beta)^{\frac{1}{2}}$  and the maximum particle slope is

$$x'_m = (\epsilon\gamma)^{\frac{1}{2}} . \quad (2.3.4)$$

The beam envelope is defined as the plot of  $x_m$  as a function of  $z$ .

The emittance has been defined above for the  $(x, x')$  phase space and not the  $(x, p_x)$  phase space referred to in Liouville's theorem. The invariant emittance is then  $\epsilon_{in} = p_z \epsilon \sim p\epsilon$ .

When the ellipse coefficient  $\alpha$  is negative the ellipse is tilted forward and when it is positive it is tilted backwards. If  $\alpha$  is zero the ellipse is upright and the beam envelope has a minimum (narrow waist) or a maximum (broad waist) value.

At a waist the emittance is given by  $\epsilon = x'_m x_m$  and

$$\begin{aligned} \gamma &= \frac{x'_m}{x_m} \\ \beta &= \frac{x_m}{x'_m} = \frac{1}{\gamma} . \end{aligned} \quad (2.3.5)$$

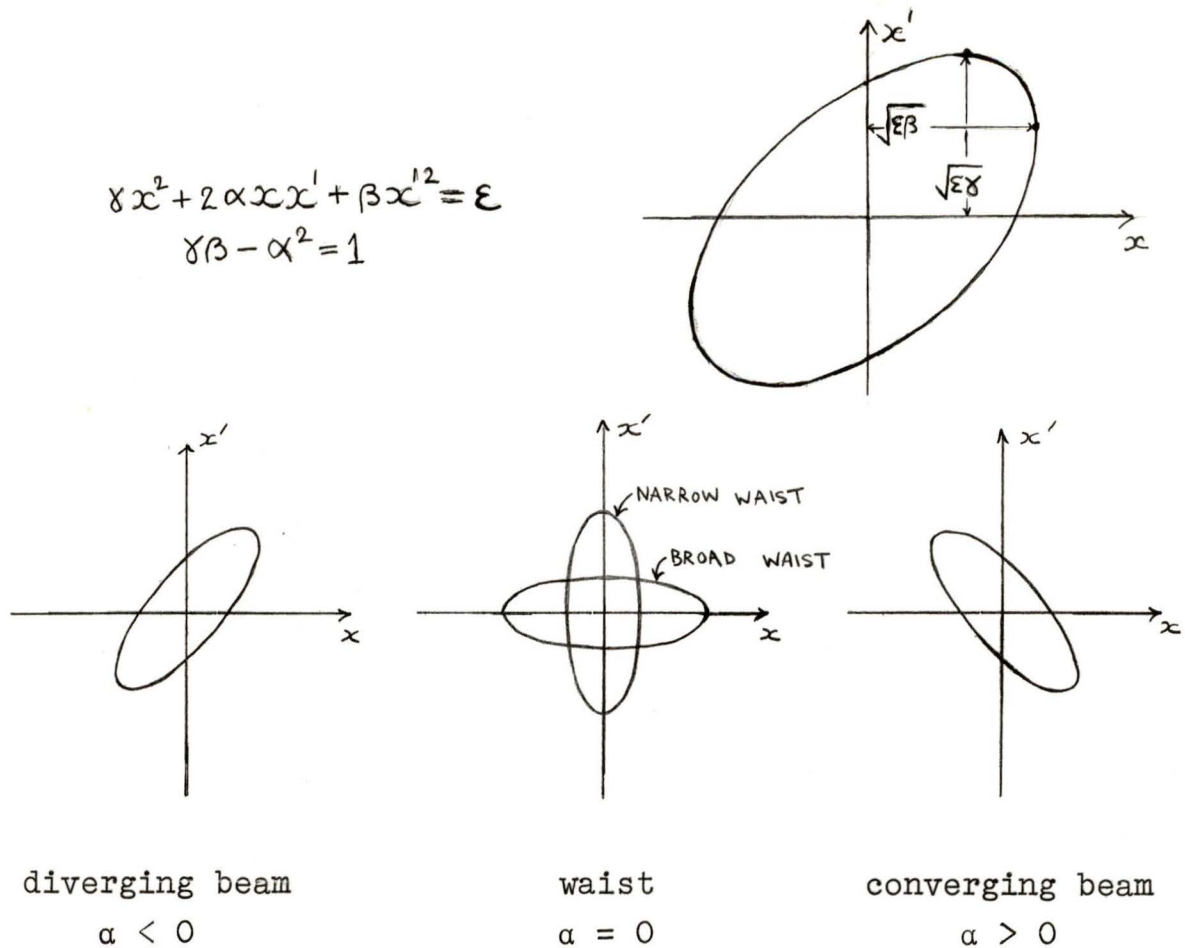


Fig. 5. Phase Space Ellipses

When the initial particle displacement  $x_0$  and divergence  $x_0'$  are changed due to the action of a beam transport device in accordance with equations 2.1.1 then in the case that  $dp/p = 0$  the initial ellipse parameters  $\gamma_0, \alpha_0, \beta_0$  become transformed by means of the equations

$$\begin{pmatrix} \gamma \\ \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} T_{22}^2 & -2 T_{21} T_{22} & T_{21}^2 \\ -T_{12} T_{22} & (T_{11} T_{22} + T_{12} T_{21}) & -T_{11} T_{21} \\ T_{12}^2 & -2 T_{11} T_{12} & T_{11}^2 \end{pmatrix} \begin{pmatrix} \gamma_0 \\ \alpha_0 \\ \beta_0 \end{pmatrix} \quad (2.3.6)$$

When Liouville's theorem holds, the transformation matrices of equation 2.1.1 have unit determinant. This fact is useful in checking numerical work.

A further discussion of equations 2.3.1 to 2.3.6 is given in Appendix IV.

### 3. DESCRIPTION OF TRIUMF TRACKING AND MATCHING PROGRAM

#### 3.1 Tracking Facilities

Tracking consists of finding the transfer matrices for points along the optic axis of some given beam transport system. If these are known and the initial beam parameters have been specified then equations 2.3.6 and 2.1.1 allow one to compute the phase space ellipse at any point along the optic axis or the trajectories of any particles in the beam.

3.1.1 The tracking subroutine of TRIUMF can compute by matrix multiplication the transfer matrix for the system at the exit of each element. There are then four options:

i) Particle tracking

The transfer matrices are used in equation 2.1.1 to find the position and direction of motion of the particle after passing through each element. This information is printed out along with the total distance down the optic axis.

ii) Ellipse tracking

The transfer matrix components are substituted into equations 2.3.6 and the ellipse coefficients alpha, beta and gamma are computed for points at the exit of each element. These values are printed out as well as the maximum particle slope and maximum particle displacement which are found from equations 2.3.4.

iii) Ellipse plotting

This option gives the same information as the above but also produces a graph of the ellipse on the line printer. Print outs do not occur after every element and one must specify the elements after which a graph of the ellipse is desired.

iv) Transfer matrix components

The transfer matrix components at the exit of each element are printed out.

3.1.2 For plotting purposes the subroutine TRACK can also compute the transfer matrices for positions inside magnets. The present program will upon request calculate the transfer matrix every 20 cm. along the optic axis. At these points the particle displacement and slope and the total distance down the optic axis are printed out. Plotting the particle displacement at each point yields the particle trajectory.

These same transfer matrices can be used to compute the ellipse parameters at every 20 cm. along the beam axis. Plotting the maximum particle displacement at each point yields a beam envelope trace.

### 3.2 Matching Routines

3.2.1 Trajectory Matching: For particle beams with a small emittance a narrow waist can be approximated by a focussed zero emittance beam (all particles have displacement  $x = 0$ ) and a broad waist can be approximated by a parallel zero emittance beam (all particles have slope  $x' = 0$ ). Then considering particle trajectories may lead to effective matching. For example, consider that the component  $T_{12}$  of the transfer matrix for a system in either transverse plane is zero. The final coordinates of a charged particle with initial displacement zero and zero momentum deviation will be from equations 2.1.1

$$\begin{pmatrix} x \\ x' \\ \frac{dp}{p} \end{pmatrix} = \begin{pmatrix} T_{11} & 0 & T_{13} \\ T_{21} & T_{22} & T_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ x'_0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ T_{22} x'_0 \\ 0 \end{pmatrix}$$

and thus  $x = 0$  for any value of  $x'_0$ . This transfer matrix will thus take any such particle and return it to the optic axis. This is termed the "focus to focus" condition.

Similarly we see that if  $T_{11} = 0.0$  and  $x'_0 = \frac{dp}{p} = 0$  then  $x = 0$  for all  $x_0$  and this situation is termed the "parallel to focus" condition. If  $T_{21} = 0$  and  $x'_0 = \frac{dp}{p} = 0$ , then  $x' = 0$  for all  $x_0$ . This is the "parallel to parallel" condition. If  $T_{22} = 0$  and  $x_0 = \frac{dp}{p} = 0$  then  $x' = 0$  for all  $x'_0$ . This is the "focus to parallel" condition.

For the case where  $T_{13} = T_{23} = 0.0$  we see that  $x$  and  $x'$  are independent of  $\frac{dp}{p}$  and the system is dispersionless. The components  $T_{13}$ ,  $T_{23}$  are, for the first order elements considered in this program, non-zero only for bending magnets in the horizontal plane and thus all beam transport systems dealt with in TRIUMF are dispersionless unless they contain bending magnets and then they can be non-dispersive only in the horizontal plane.

The trajectory matching routines used in TRIUMF are as shown in the table.

<u>Matching Routine</u>	<u>Routine Number</u>	<u>Mathematical Condition</u>
parallel-to-focus	1	sets $T_{11} = 0.0$
focus-to-focus	2	sets $T_{12} = 0.0$
parallel-to-parallel	3	sets $T_{21} = 0.0$
focus-to-parallel	4	sets $T_{22} = 0.0$
achromatic system (horizontal plane only)	0	sets $T_{13} = T_{23} = 0.0$

These routine numbers apply to either transverse plane. If a routine number of 0 is specified in the vertical plane no matching is done in that plane.

3.2.2 Ellipse Matching: TRIUMF has three routines grouped under the heading of ellipse matching:

i) Routine 6 attempts to set the ellipse parameter alpha equal to zero. This corresponds to a search for a waist with an unrestricted value for the final beam size.

ii) Routine 7 attempts to set both alpha and beta to

some specified value. Since the final gamma is determined through equation 2.3.2 and the emittance is invariant, this constitutes a search for a unique ellipse.

iii) Routine 5 attempts to create an identity section, that is it tries to set  $T_{11} = T_{12} = \pm 1$  and  $T_{12} = T_{21} = 0.0$ . Substitution of the above matrix components into equations 2.3.6 shows that such a transfer matrix leaves the ellipse coefficients of any input ellipse unchanged. Because the transfer matrices have unit determinant it is only necessary to set  $T_{12} = T_{21} = 0.0$  and  $|T_{11}| = 1$  in order to achieve an identity system.

The ellipse matching routines are listed in the table below.

<u>Matching Routine</u>	<u>Routine Number</u>	<u>Mathematical Condition</u>
Match to a waist	6	sets $\alpha = 0$
Match to a specified ellipse	7	sets $\alpha - \alpha_r = 0$ sets $\beta - \beta_r = 0$ where $\alpha_r, \beta_r$ are the requested final ellipse parameters
Match to an identity section	5	sets $ T_{11}  = 1$ sets $T_{12} = T_{21} = 0.0$

All matching routines may be used independently in both planes. One could do, for example, ellipse matching in the horizontal plane and trajectory matching in the vertical plane.

### 3.3 Method of Matching

For both trajectory matching and ellipse matching the same method is used. An error function is formed by taking the square root of the sum of the squares of the quantities which are desired to be set to zero. The minimum of this error function is searched for using a method developed by Powell (1964). This method finds an unconstrained minimum of a function of N variables without calculation of derivatives. It contains a variation of the method of minimizing a function of several values by changing one parameter at a time such that the convergence rate from a bad approximation to a minimum is always efficient.

The elements which are varied are those within the matching section which have been designated as variables in the input data.

The number of variable elements can range from one to twenty and it is suggested that one should try to leave at least one element variable for trajectory matching and two elements variable for ellipse matching or for generating dispersionless systems. No constraints have been imposed upon the element parameters except in the case of drift lengths which are restricted to being greater than zero.

### 3.4 General Description of Program

The TRIUMF program consists of a main program and 5 subroutines. It is written in Fortran II and is about 800 statements long. As can be seen by the program listing given in Appendix VII many 'comment' cards have been inserted to aid the reader. Further information is given below.

3.4.1 Main Program: This program is responsible for reading in all the input data. It reads in the design energy of the beam and computes the design momentum using equation VI-2. Next it reads in the initial system parameters for each beam handling element. It then calls subroutine ASSIGN which computes the horizontal and vertical plane matrix components for each beam handling element in the system. Then an instruction card is read in to determine the required job to be performed. Any information which is needed to complete the job and was not contained on the instruction card is also read in at this point. If the instruction card requests a tracking job the program calls subroutine TRACK. For a matching job subroutine MATCH is called. On a request for a system change the program makes the change, calls subroutine ASSIGN to find the new matrix components, and then reads in another instruction card. On returning from subroutine TRACK or subroutine MATCH the program determines the next job by reading in another instruction card. This process of reading an

instruction card to determine the next job continues until an instruction card requesting termination of the program is encountered.

3.4.2 Subroutine TRACK: This subroutine will do trajectory tracking or plotting, ellipse tracking or plotting, a beam envelope trace or a print out of transfer matrix components. It consists of a DO loop which is traversed NE times where NE is the number of elements in the system. On the 1<sup>th</sup> time through the loop the transfer matrix for the system up to the exit of the 1<sup>th</sup> element is computed. This matrix may be used to carry out any of the tracking jobs mentioned in section 3.1.1. If trajectory plotting or a beam envelope trace is required then another DO loop nested within the previous one computes the transfer matrices at points within each element so that either of the two tracking jobs mentioned in section 3.1.2 can be performed.

3.4.3 Subroutine ASSIGN: This subroutine takes the current values of the system element parameters and computes the horizontal and vertical plane matrix components for elements N to M inclusive where N and M must be specified before entering the subroutine. Quadrupole matrix components are computed using equations II-21, II-22 and bending magnet matrix components are calculated from equations III-26, III-27 (except that components T13, T23 have been multiplied by 100.0 because of the change in units as explained in Appendix VI). Drift space matrix components are found using equations I-5, I-6 except that L has been

replaced by `!L!`. This avoids matching solutions with negative drift lengths.

3.4.4 Subroutine VAO4A: This subroutine attempts to find the minimum of a function of `N` variables using an iterative procedure developed by M. J. Powell. For the complete details see the reference, "An Efficient Method for Finding the Minimum of a Function of Several Variables Without Calculating Derivatives", Applied Mathematics Group, Atomic Energy Research Establishment, Harwell, Berkshire, 1964. This subroutine was written by Powell and has been used without making any significant changes.

To use this subroutine a number of variables must first be specified. This is done in subroutine `MATCH` and they are set as indicated below:

`N` (no. of variables)

This is set equal to the number of elements in the matching section with variable parameters.

`ESCALE` (maximum step size during minimization)

This is set at 100000. Lower values were tried and it was found that too many function values were being calculated. Higher values have not been tried.

`IPRINT` (print out control)

If `IPRINT = 1` information is printed out after each function evaluation. If `IPRINT = 2` information is printed out only after each iteration. This is set at 2 as a setting of 1 prints out too much information. As shown on page 128 the print out consists of the number of

iterations completed, the number of function values calculated, and also the values of the function and the variables at the completion of the last iteration.

MAXIT (maximum number of iterations)

This is set at 40.

E(I) (accuracy to which variables are to be determined)

This is set as follows:

- for quadrupole field gradients                    0.1 Gauss/cm.
- for bending magnet fields                        0.1 K Gauss
- for drift lengths                                    0.001 meters .

If on successive iterations the variables do not change by more than the values given above then the iteration procedure stops.

X(I) (initial values of variables)

These are set to the variable element parameters of the initial guessed system.

3.4.5 Subroutine MATCH: This subroutine determines those elements within the matching section which are to have variable parameters. It then defines the variables listed above for VAO4A. Next subroutine VAO4A is called to carry out the matching job. On return from subroutine VAO4A the results of the matching attempt are printed out.

3.4.6 Subroutine CALCF: This subroutine takes the current values of the system element parameters and defines and calculates an error function which is to be minimized by subroutine VAO4A in solving a matching problem. From the matching routine numbers IA and IB one of the mathematical

conditions listed in the table in section 3.2.1 or 3.2.2 is chosen for the horizontal and vertical plane. The square root of the sum of the squares of those quantities which are to be set to zero constitutes the value of the error function. CALCF calculates this and then returns to subroutine VAO4A.

### 3.5 Conclusion

The digital computer program, TRIUMF, has been tested successfully on a problem involving 5 quadrupoles and 2 bending magnets which was studied by Paul (1961). The output from this problem appears in Appendix VIII.

Other programs such as TRAMP (Gardner and Whiteside) and TRANSPORT (Moore et al) have been written to solve tracking and matching problems. The TRIUMF program differs from existing programs mainly in that it uses the minimization subroutine of Powell (1961) to solve matching problems. The advantage gained by using this new method is that the iteration procedure will converge to a solution even if the initial "guessed" system is not close to the solution system.

The TRIUMF program could be made more versatile by using general 6 x 6 transfer matrices to represent beam handling elements as is done in the TRANSPORT program. This would enable ellipse tracking for the case  $\frac{dp}{p} \neq 0$  to be done and would also allow second order tracking to be incorporated in the program. Another program modification would be to include more types of beam transport elements. The velocity separator and non-constant field bending magnet as well as a general matrix read in by components (which could represent the effect of the fringing field of an accelerator) would be useful additions.

This program is to be used as an aid in the design of a beam transport system for the proposed "Tri-University Meson Facility" (Vogt et al, 1966).

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## APPENDIX I: DRIFT SPACE TRANSFER MATRIX COMPONENTS

The equations of motion for a particle in a region of zero magnetic field are from equation (2.1.2) with  $\vec{B} = 0$

$$m\ddot{x} = 0 \quad \text{and} \quad m\ddot{y} = 0 . \quad \text{I-1}$$

In the horizontal plane we have

$$\dot{x} = \frac{dx}{dt} = \frac{dx}{dz} \frac{dz}{dt} = x' \dot{z}$$

$$\ddot{x} = \frac{d^2x}{dz^2} \left(\frac{dz}{dt}\right)^2 + \frac{dx}{dz} \frac{d^2z}{dt^2} = x'' \dot{z}^2 + x' \ddot{z} . \quad \text{I-2}$$

Now  $\ddot{z} = 0$  as there are no forces in the z-direction and we have

$$m\ddot{x} = mx'' \dot{z}^2 = 0$$

or

$$x'' = 0 . \quad \text{I-3}$$

The solution is

$$x = Az + B .$$

If for  $z = 0$ ,  $x = x_0$  then  $B = x_0$ .

By differentiation  $x' = A$  and if at  $z = 0$ ,  $x = x_0$  then  $A = x'_0$ .

The effect of a drift space of length  $L$  on a particle is then

$$x = x'_0 L + x_0$$

$$x' = x'_0$$

or in matrix notation

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} .$$

A parallel procedure leads to the same matrix components for the vertical plane. Since a field free region is non-dispersive we can write

$$\begin{pmatrix} x \\ x' \\ \frac{dp}{p} \end{pmatrix} = \begin{pmatrix} 1 & L & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ \frac{dp}{p} \end{pmatrix} \quad \text{I-4}$$

and

$$\begin{pmatrix} y \\ y' \\ \frac{dp}{p} \end{pmatrix} = \begin{pmatrix} 1 & L & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_0 \\ y'_0 \\ \frac{dp}{p} \end{pmatrix} . \quad \text{I-5}$$

## APPENDIX II: QUADRUPOLE MAGNET MATRIX COMPONENTS

II.1. Magnetic Fields with Quadrupole Symmetry

Following the treatment given by Steffen (1964) the magnetostatic potential for a field which has the symmetries of a quadrupole magnet field is derived and compared with the potential for an ideal quadrupole field.

With the hard edge model the field in the z-direction is zero and using cylindrical coordinates  $r = (x^2 + y^2)^{\frac{1}{2}}$ ,  $\phi = \tan^{-1} y/x$  in the xy-plane we have  $\Phi = \Phi(r, \phi)$  where  $\Phi$  is a general potential which is independent of z and where the field is  $\vec{B} = \nabla \Phi$ . Then from Maxwell's equation

$$\nabla \cdot \vec{B} = 0 = \nabla^2 \Phi \quad \text{II-1}$$

we have in cylindrical coordinates

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \phi^2} = 0 \quad .$$

If we assume

$$\Phi(r, \phi) = u(r) f(\phi)$$

and substitute this in the above and then multiply by  $r^2/\Phi$  we get

$$\frac{r}{u} \frac{\partial}{\partial r} (ru) + \frac{1}{f} \frac{\partial^2 f}{\partial \phi^2} = 0 \quad ,$$

For this to be possible we must have

$$\frac{r}{u} \frac{\partial}{\partial r} (ru) = -\frac{1}{f} \frac{\partial^2 f}{\partial \phi^2} = m^2$$

$m = \text{constant}$  or

$$r \frac{d}{dr} \left( r \frac{du}{dr} \right) - m^2 u = 0$$

and

$$\frac{d^2 f}{d\phi^2} + m^2 f = 0 .$$

Two solutions are

$$u = ar^m + br^{-m} \quad \text{II-2}$$

and

$$f = c \cos m\phi + d \sin m\phi . \quad \text{II-3}$$

In equation II-2 as  $r$  approaches zero we must have  $b = 0$  if  $\phi$  is to remain finite ( $m > 0$ ).

Rotation of the poles in Fig. 1 by  $90^\circ$  gives the original pole configuration except that the polarities reversed and we have

$$\Phi(r, \phi) = -\Phi(r, \phi + \frac{\pi}{2}) . \quad \text{II-4}$$

Reflection of the poles across the x-axis produces the same results as above hence

$$\Phi(r, \phi) = -\Phi(r, -\phi) . \quad \text{II-5}$$

Equation II-5 applied to equation II-3 gives that

$$f(\phi) = -f(-\phi) \quad \text{or} \quad f(0) = 0$$

and in equation II-3 we must put  $c = 0$  .

Also equation II-4 implies that  $f(\phi + \frac{\pi}{2}) = -f(\phi)$

and from equation II-3 we must have

$$\sin m \left( \vartheta + \frac{\pi}{2} \right) = -\sin m\vartheta = \sin m\vartheta \cos \frac{m\pi}{2} + \cos m\vartheta \sin \frac{m\pi}{2} .$$

For this to be satisfied for all  $\vartheta$  we need

$$\cos \frac{m\pi}{2} = -1 \quad \text{or} \quad \frac{m\pi}{2} = (2n + 1)\pi$$

$$m = 2(2n + 1) \quad n = 0, 1, 2 \dots$$

and

$$\sin \frac{m\pi}{2} = 0 \quad \text{or} \quad \frac{m\pi}{2} = n\pi$$

$$m = 2n \quad n = 0, 1, 2 \dots$$

hence for both conditions to be satisfied we must take

$$m = 2(2n + 1) \quad n = 0, 1, 2 \dots \quad \text{II-6}$$

The general solution satisfying the above symmetry conditions is

$$\Phi(r, \vartheta) = \sum_{n=0} a_n r^{2(2n+1)} \sin 2(2n+1) \vartheta . \quad \text{II-7}$$

If we set  $a_0 = g/2$ ,  $a_n = 0$  for  $n > 0$  then we get the pure quadrupole field potential

$$\Phi = gr^2 \sin 2\vartheta = gr \sin \vartheta r \cos \vartheta = gxy .$$

In any physical quadrupole magnet the shape of the pole faces do not follow the hyperbolic equipotentials out to infinity. Because of the space required for exciting

coils, the pole shape must deviate from this theoretical shape and the potential will contain some higher harmonics.

## II.2. Matrix Components for an Ideal Quadrupole Magnet

The equation of motion for a point particle of mass  $m$  and charge  $e$  in a magnetic field is from equation (2.1.2)

$$\frac{d}{dt} (m\vec{v}) = e\vec{v} \times \vec{B} .$$

In the x-direction this is

$$\frac{d}{dt} (m\dot{x}) = e (\dot{y}B_z - \dot{z}B_y) \quad \text{or} \quad \ddot{x} = \frac{e}{m} (\dot{y}B_z - \dot{z}B_y) . \quad \text{II-8}$$

This is relativistically exact because

$$m = \left( \sqrt{1 - v^2/c^2} \right)^{-1} m_0$$

is a constant since  $|\vec{v}|$  does not change in a magnetic field. Similarly in the y- and z-directions

$$\ddot{y} = \frac{e}{m} (\dot{z}B_x - \dot{x}B_z) \quad \text{II-9}$$

$$\ddot{z} = \frac{e}{m} (\dot{x}B_y - \dot{y}B_x) . \quad \text{II-10}$$

If we change to  $z$  as the independent variable we have as in equations I-2

$$\dot{x} = x' \dot{z}$$

$$\ddot{x} = x'' \dot{z}^2 + x' \ddot{z} \quad \text{II-11}$$

and similarly in the vertical plane

$$\dot{y} = y' \dot{z}$$

$$\ddot{y} = y'' \dot{z}^2 + y' \ddot{z} \quad . \quad \text{II-12}$$

Putting equations II-11 in II-8 gives

$$x'' \dot{z}^2 + x' \ddot{z} = \frac{e}{m} (y' \dot{z} B_z - \dot{z} B_y)$$

and from II-10 we get

$$x'' \dot{z}^2 + x' \ddot{z} \frac{e}{m} (x' B_y - y' B_x) = \dot{z} \frac{e}{m} (y' B_z - B_y)$$

or

$$x'' = \frac{1}{\dot{z}} \frac{e}{m} (x' y' B_x - (1 + x'^2) B_y + y' B_z) \quad .$$

From equations II-11, II-12

$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{z}^2 (1 + x'^2 + y'^2) \quad \text{II-13}$$

and we get

$$x'' = \frac{e}{mv} (1 + x'^2 + y'^2)^{\frac{1}{2}} (x' y' B_x - (1 + x'^2) B_y + y' B_z)$$

and similarly making the same kind of substitutions in equation II-9 gives

$$y'' = \frac{-e}{mv} (1 + x'^2 + y'^2)^{\frac{1}{2}} (x' B_z + y' x' B_y - (1 + y'^2) B_x) \quad .$$

If we use the paraxial approximation that  $x' \ll 1$ ,  $y' \ll 1$  then products of  $x'$  and  $y'$  can be neglected and

$$x'' = \frac{e}{mv} (-B_y + y' B_z) \quad y'' = \frac{-e}{mv} (x' B_z - B_x) \quad .$$

Substituting for the pure quadrupole field  $B_y = gx$  ,  
 $B_x = gy$  ,  $B_z = 0$  yields

$$x'' = \frac{-e}{mv} gx = -K^2 x \quad \text{II-14}$$

where

$$K^2 = \frac{eg}{mv} = \frac{eg}{p} \quad \text{II-15}$$

and

$$y'' = \frac{e}{mv} gy = K^2 y = -i^2 K^2 y . \quad \text{II-16}$$

The solution to equation II-14 is

$$x = A \cos Kz + B \sin Kz . \quad \text{II-17}$$

By differentiation

$$x' = -AK \sin Kz + KB \cos Kz . \quad \text{II-18}$$

If at  $z = 0$  ,  $x = x_0$  then  $A = x_0$  and  
 if at  $z = 0$  ,  $x' = x'_0$  then  $KB = x'_0$  ,  $B = x'_0/K$   
 and substitution of  $A$  ,  $B$  into equations II-17 and II-18  
 gives

$$x = x_0 \cos Kz + \frac{x'_0}{K} \sin Kz$$

$$x' = -x_0 K \sin Kz + x'_0 \cos Kz .$$

The solution to equation II-16 is

$$y = A \cos iKz + B \sin iKz = A' \cosh Kz + B' \sinh Kz . \quad \text{II-19}$$

By differentiation

$$y' = KA' \sinh Kz + B'K \cosh Kz \quad . \quad \text{II-20}$$

If at  $z = 0$ ,  $y = y_0$  then  $A' = y_0$  and if at  $z = 0$ ,  $y' = y'_0$  then  $B'K = y'_0$ ,  $B = y'_0/K$  .

The equations II-19 and II-20 become

$$y = y_0 \cosh Kz + \frac{y'_0}{K} \sinh Kz$$

$$y' = +y_0 K \sinh Kz + y'_0 \cosh Kz \quad .$$

Since the central trajectory is not deflected by a quadrupole there is no dispersion to first order (Hansford and Aspley, 1967) and we can take  $T_{13} = T_{23} = 0.0$  . If the effective length of the quadrupole is  $L$  then in the horizontal plane we can write

$$\begin{pmatrix} x \\ x' \\ \frac{dp}{p} \end{pmatrix} = \begin{pmatrix} \cos KL & \frac{1}{K} \sin KL & 0 \\ -K \sin KL & \cos KL & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ \frac{dp}{p} \end{pmatrix} \quad \text{II-21}$$

and for the vertical plane

$$\begin{pmatrix} y \\ y' \\ \frac{dp}{p} \end{pmatrix} = \begin{pmatrix} \cosh KL & \frac{1}{K} \sinh KL & 0 \\ K \sinh KL & \cosh KL & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_0 \\ y'_0 \\ \frac{dp}{p} \end{pmatrix} \quad \text{II-22}$$

Equations II-21, II-22 agree with those given by Penner (1961).

### II.3 Thin Lens Approximation

Replacing ideal quadrupole magnets by approximately equivalent thin lenses simplifies the analysis considerably and can be used to arrive at a "guessed" system prior to matching. This section gives the matrix components in terms of the focal length of the equivalent thin lens. These relations are due to Penner (1961).

The action of a thin lens on a particle is to change the particle slope without affecting its displacement. Suppose as in Fig. 6 a thin lens at A brings all particles travelling parallel to the central trajectory at any initial displacement  $x_0$  to a focus at A'. The change in slope at A must be  $\Delta x'_0 = -x_0/f$  and the effect of a focussing thin lens on a particle is given by

$$x = x_0$$

$$x' = x'_0 + \Delta x'_0 = x'_0 - x_0/f$$

or in matrix form

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} .$$

For a defocussing thin lens  $f$  would be negative.

A thin lens does not deflect the central trajectory and we may write

$$\begin{pmatrix} x \\ x' \\ \frac{dp}{p} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1/f & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ \frac{dp}{p} \end{pmatrix}$$

II-23

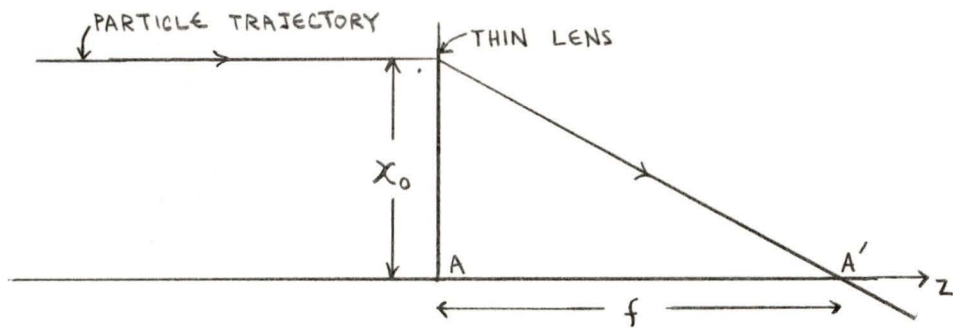


Fig. 6. The Thin Lens

The transfer matrix representing a drift space of length  $L_1$ , followed by a thin lens of focal length  $f$  and a drift space of length  $L_2$  is

$$\begin{pmatrix} 1 & L_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1/f & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-L_2/f & L_1+L_2-L_1L_2/f & 0 \\ -1/f & 1-L_1/f & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{II-24}$$

Comparison of equation II-24 with II-21 shows that a

focussing ideal quadrupole magnet has the same effect on a particle as a thin lens sandwiched between two drift spaces if the following equations can be satisfied.

$$1 - L_1/f = 1 - L_2/f = \cos KL$$

$$-1/f = -K \sin KL$$

$$L_1 + L_2 - L_1 L_2/f = \frac{\sin KL}{K}$$

The solution is

$$L_1 = L_2 = \frac{1 - \cos KL}{K \sin KL}$$

II-25

$$f = \frac{1}{K \sin KL} \quad .$$

Similarly comparison of equations II-24 with II-22 gives

$$1 - L_1/f = 1 - L_2/f = \cosh KL$$

$$-1/f = K \sinh KL$$

$$L_1 + L_2 - L_1 L_2/f = \frac{\sinh KL}{K}$$

with the solution

$$L_1 = L_2 = \frac{\cosh KL - 1}{\sinh KL}$$

II-26

$$f = \frac{-1}{K \sinh KL} \quad .$$

A power series expansion of the trigonometric functions yields for equations II-25

$$L_1 = L_2 = \frac{L}{2} \left( 1 + \frac{K^2 L^2}{12} + \dots \right)$$

II-27

$$1/f = K^2 L \left( 1 - \frac{K^2 L^2}{6} + \dots \right)$$

and similarly for equations II-26

$$L_1 = L_2 = \frac{L}{2} \left( 1 - \frac{K^2 L^2}{12} + \dots \right)$$

II-28

$$1/f = -K^2 L \left( 1 + \frac{K^2 L^2}{6} + \dots \right) .$$

In the thin lens approximation it is assumed that  $K^2 L^2 \ll 1$  and from equations II-27 and II-28

$$L_1 = L_2 \approx \frac{L}{2}$$

$$f \approx \frac{1}{K^2 L}$$

for a focussing lens and

$$f \approx \frac{-1}{K^2 L}$$

for a defocussing lens.

## APPENDIX III: BENDING MAGNET MATRIX COMPONENTS

III.1 Matrix Components for Bending Magnets with Normal Entrance and Exit Angles - Horizontal Plane

We consider the motion of an ion in a constant magnetic field  $B$ . Cylindrical coordinates  $(r, \phi, y)$  are chosen such that the  $y$ -axis is parallel to the field  $B$ , and  $r$  and  $\phi$  are measured in the plane of the trajectory.

In cylindrical coordinates the radial acceleration is  $a_r = \ddot{r} - r\dot{\phi}^2$ . The force in the radial direction is  $-e r \dot{\phi} B$ . If we use the approximation  $r \dot{\phi} = \dot{z} \cong v$  then the radial equations of motion are

$$m a_r = m \left( \ddot{r} - \frac{v^2}{r} \right) = -e v B \quad . \quad \text{III-1}$$

A possible solution to this equation is a circular path, i.e.  $r = r_0 = \text{const.}$  The term  $\ddot{r}$  disappears and we have

$$\frac{m v^2}{r_0} = e v B$$

or

$$r_0 = \frac{m v}{e B} = \frac{p}{e B} \quad . \quad \text{III-2}$$

Following the treatment of Livingood (1961) we consider now a particle with a radial displacement at  $r_0 + x$  where  $x \ll r_0$  and a small momentum deviation  $dp$

from the previous case represented by equation III-1.

A momentum of  $p + dp$  implies that the velocity and mass are slightly different and equation III-1 becomes

$$(m + dm) \frac{d}{dt^2} (r_0 + x) - \frac{(m + dm)(v + dv)^2}{(r_0 + x)} = -e (v + dv)B. \text{ III-3}$$

Using the approximation

$$\frac{1}{r_0 + x} = \frac{1}{r_0 (1 + x/r_0)} \sim \frac{1}{r_0} (1 - x/r_0)$$

and noting that

$$\frac{d^2 r_0}{dt^2} = 0,$$

we get if we neglect products of small terms  $x$ ,  $dm$ ,  $dv$ ,  
 $\ddot{x}$

$$\frac{m d^2 x}{dt^2} + \frac{mv^2}{r_0^2} x = \frac{v^2}{r_0} dm - eBdv + 2m \frac{v}{r_0} dv = \frac{v^2}{r_0} dm + \frac{mv}{r_0} dv \quad \text{III-4}$$

where equation III-2 has been used to reduce the last two terms.

Since  $p = mv$ ,  $dp = mdv + vdm$  and division by  $p$  gives

$$\frac{dp}{p} = \frac{dv}{v} + \frac{dm}{m}$$

and hence equation III-4 becomes

$$\frac{d^2 x}{dt^2} + \frac{v^2}{r_0^2} x = \frac{v^2}{r_0} \frac{dp}{p}.$$

Changing to  $z$  as independent variable, we have as in equations I-2

$$\ddot{x} = x' \dot{z}^2 + x'' \dot{z} \approx v^2 x''$$

and the above equation becomes

$$x'' + \frac{x}{r_0^2} = \frac{1}{r_0} \frac{dp}{p} \quad \text{III-5}$$

The solution to equation III-5 is  $x = A \cos \frac{z}{r_0} + B \sin \frac{z}{r_0} + \frac{dp}{p} r_0 (1 - \cos \frac{z}{r_0})$  and by differentiation

$$x' = -\frac{A}{r_0} \sin \frac{z}{r_0} + \frac{B}{r_0} \cos \frac{z}{r_0} + \frac{dp}{p} \sin \frac{z}{r_0} .$$

If for  $z = 0$ ,  $x = x_0$  then  $A = x_0$  and if for  $z = 0$ ,  $x' = x'_0$  the  $\frac{B}{r_0} = x'_0$  or  $B = \frac{x'_0}{r_0}$  and the equations

above are  $x = x_0 \cos \frac{z}{r_0} + r_0 x'_0 \sin \frac{z}{r_0} + \frac{dp}{p} r_0 (1 - \cos \frac{z}{r_0})$

$$x' = \frac{-x_0}{r_0} \sin \frac{z}{r_0} + x'_0 \cos \frac{z}{r_0} + \frac{dp}{p} \sin \frac{z}{r_0} \quad \text{III-6}$$

or with  $\phi = z/r_0 = L/r_0$  where  $L$  is the effective length of the magnet and the angle  $\phi$  is the angle of bend of the central trajectory this can be written conveniently as the matrix equation

$$\begin{pmatrix} x \\ x' \\ \frac{dp}{p} \end{pmatrix} = \begin{pmatrix} \cos \phi & +r_0 \sin \phi & r_0(1 - \cos \phi) \\ -\frac{1}{r_0} \sin \phi & \cos \phi & \sin \phi \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ \frac{dp}{p} \end{pmatrix} \quad \text{III-7}$$

Note: The deflection of the beam is concealed in the output as the coordinate system is attached to the central trajectory which is deflected the same amount.

### III.2 Effect of Rotated Pole Edges - Horizontal Plane

Fig. 7 shows the exit face of a magnet with the pole face rotated by an angle  $\beta_2 > 0$ .

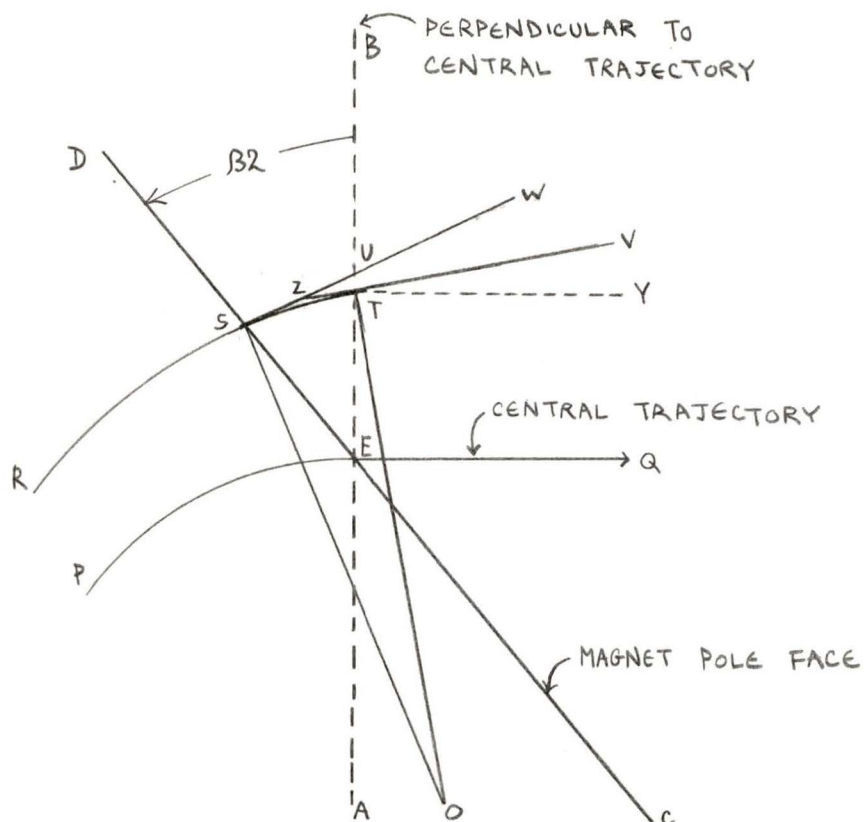


Fig. 7.

Construction for the Calculation of the Effects of Pole Edge Rotation on the Radial Trajectories.

The displacement and slope with non-normal exit of the central trajectory are calculated as a separate transfer matrix operating on the displacement and slope vector for normal exit. Referring to Fig. 7 AEB is perpendicular to the optic axis at the exit point. The curve RSTV represents the trajectory of a particle with initial displacement and slope  $x_0$  and  $x'_0$ . For a magnet with normal exit for the central trajectory  $\beta_2 = 0$  and the segment RST inside such a magnet is the arc of a circle of radius  $r_0 + \Delta r_0$  centered at O. The final displacement of this trajectory at E is  $x_f = ET$  and final slope  $x'_f$ , the angle YTV. For the magnet with  $\beta_2 \neq 0$  (non-normal exit), the segment SUW of the displaced trajectory RSUW lies outside the magnetic field, the final displacement and slope corresponding to position E on the optic axis are

$$x = x_f + TU \quad x' = x'_f + WZV \quad .$$

Since the sum of the interior angles of the polygon OSZT is  $2\pi$  and angles  $OSZ = ZTO = \pi/2$ , then  $TOS + SZT = \pi$ . Since  $SZT + WZV = \pi$ , angle  $WZV =$   
 $TOS = \frac{\text{arc ST}}{r_0 + \Delta r_0} \quad .$

If we use the approximations that the arc  $ST = ET \tan \beta_2 = x_f \tan \beta_2$  and  $\frac{1}{r_0 + \Delta r_0} \sim \frac{1}{r_0}$

then

$$x' = x'_f + x_f \frac{\tan \beta_2}{r_o} .$$

If we further consider TU to be sufficiently small that it may be neglected we have  $x = x_o$  and the approximate effect of the rotated pole edge on the particle is given by

$$x = x_f$$

$$x' = x'_f + \frac{x_f \tan \beta_2}{r_o}$$

or in matrix notation

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{\tan \beta_2}{r_o} & 1 \end{pmatrix} \begin{pmatrix} x_f \\ x'_f \end{pmatrix} . \quad \text{III-8}$$

In the same manner a similar matrix can be derived to represent the effect on a particle due to crossing a rotated entrance face. We have then

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{\tan \beta_1}{r_o} & 1 \end{pmatrix} \begin{pmatrix} x_o \\ x'_o \end{pmatrix} \quad \text{III-9}$$

where  $\beta_1$  is the angle of rotation of the entrance pole face. Equations III-8 and III-9 agree with those derived by Penner (1961).

### III.3 Effect of Rotated Pole Edges - Vertical Plane

At the edge of a constant field magnet there is a fringing field approximately as shown in Fig. 8(A).

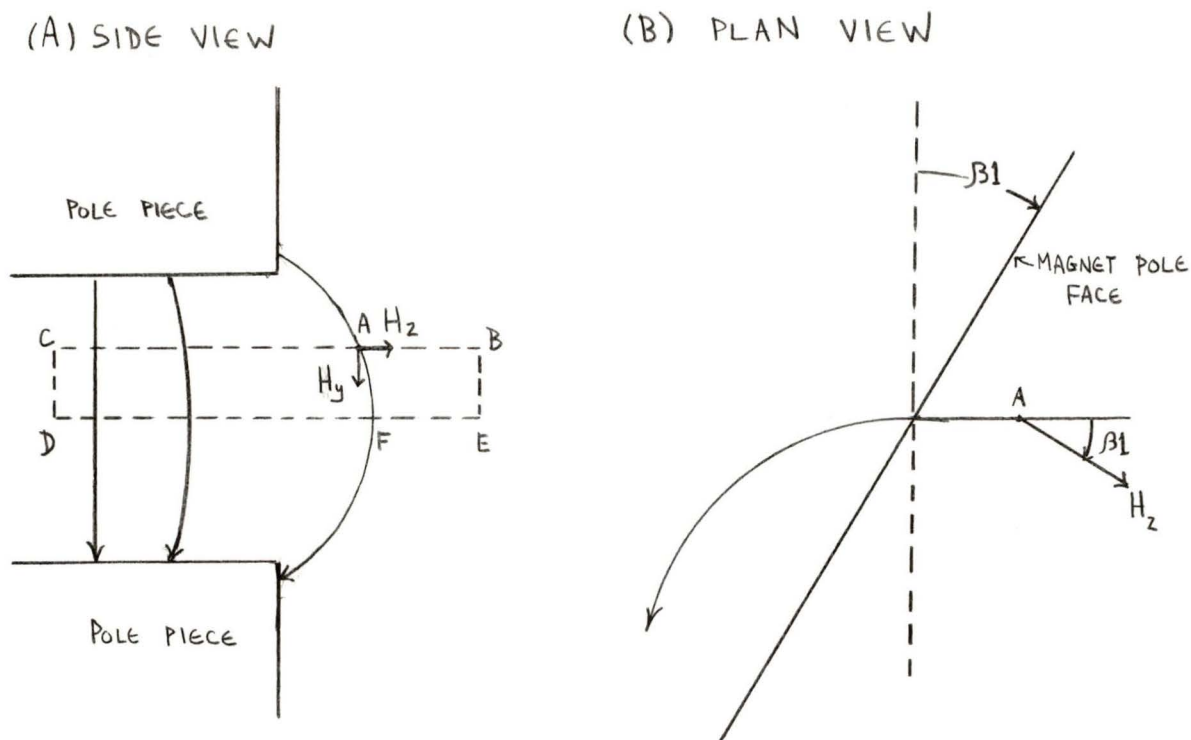


Fig. 8.

Fringing Field at the Edge of a Bending Magnet.

For points not on the median plane ED the field has a component  $H_z$  normal to the magnet pole face. If a particle enters the fringe field above or below the median plane it is acted on by this field which can be resolved

as shown in Fig. 8(B) into a component along the trajectory  $H_z \cos \beta l$  and a component normal to the trajectory  $H_z \sin \beta l$ . From the Lorentz force equation the vertical force on a particle at point A is

$$F_y = ev H_z \sin \beta l . \quad \text{III-10}$$

The total change in vertical momentum for a particle passing into the magnet through point A would be given by

$$\Delta p_y = \int_B^C F_y dt \quad \text{III-11}$$

where B is some point sufficiently far from the magnet so that  $H_z = 0$  and C is a point inside the magnet where there is no fringe field. Combining equations III-10 and III-11 gives

$$\Delta p_y = e \sin \beta l \int_B^C H_z v dt = e \sin \beta l \int_B^C H_z ds \quad \text{III-12}$$

where  $ds$  is an element of distance along the trajectory. For the static current free region of field that we are considering Maxwell's equation

$$\text{curl } \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t} = 0$$

and from Stokes' theorem

$$\oint \vec{H} \cdot d\vec{l} = \int \text{curl } \vec{H} \cdot d\vec{s} = 0 . \quad \text{III-13}$$

To evaluate the integral in equation III-12 we

take the line integral around the closed path BCDEB.

Along CB,  $H_y$  is perpendicular to the path of integration and does not contribute to the integral. We get

$$\int_C^B H_z \cos \beta l \, dl \cong \int_C^B H_z \cos \beta l \, ds \quad \text{III-14}$$

where we have made the approximation that path CB represents the particle trajectory. Along CD,  $H_z$  is zero as we are sufficiently far away from the fringe field and the integral becomes

$$\int_C^D H \, dy = -y_0 (-H) = y_0 H \quad \text{III-15}$$

where  $y_0$  is the displacement of the particle from the median plane and we have used the same approximation as in equation III-14. Along DE,  $H_z$  is zero as we are in the median plane and  $H_y$  is perpendicular to the path of integration, hence the contribution to the line integral is zero. Along EB we are assumed sufficiently far from the magnet so that  $H$  is zero and there is also no contribution from this segment of the path. For the closed circuit the net result is from III-13

$$\int_C^D H_z \cos \beta l \, ds + y_0 H = 0$$

or

$$\int_C^B H_z \cos \beta l \, ds = \frac{-y_0 H}{\cos \beta l} \quad \text{III-16}$$

Substituting equation III-16 in III-12

Substituting equation III-16 in III-12 yields

$$\Delta p_y = \frac{-e \sin \beta l y H}{\cos \beta l} = -y_o e H \tan \beta l .$$

The change in slope of the particle trajectory is given by

$$\Delta y'_o = \frac{\Delta p_y}{p_o} = \frac{-y_o e H \tan \beta l}{mv}$$

and from  $\frac{1}{r_o} = \frac{eH}{mv}$ ,

$$\Delta y'_o = -y_o \frac{\tan \beta l}{r_o} . \quad \text{III-17}$$

In this approximation the effect of the rotated pole face is given by

$$y = y_o$$

$$y' = y'_o + \Delta y'_o = y_o \frac{\tan \beta l}{r_o}$$

or in matrix form

$$\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{-\tan \beta l}{r_o} & 1 \end{pmatrix} \begin{pmatrix} y_o \\ y'_o \end{pmatrix} . \quad \text{III-18}$$

A similar effect occurs at the exit face and we can write

$$\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{-\tan \beta 2}{r_o} & 1 \end{pmatrix} \begin{pmatrix} y_o \\ y'_o \end{pmatrix} . \quad \text{III-19}$$

These matrix components are derived by Banford (1966).

### III.4 Matrix Components for Bending Magnets with Rotated Pole Faces

Since the rotated pole faces do not bend the central trajectory there is no first order dispersion and we can write from equations III-9 and III-18

$$\begin{pmatrix} x \\ x' \\ \frac{dp}{p} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{+\tan \beta l}{r_o} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_o \\ x'_o \\ \frac{dp}{p} \end{pmatrix} \quad \text{III-20}$$

and

$$\begin{pmatrix} y \\ y' \\ \frac{dp}{p} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{-\tan \beta l}{r_o} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_o \\ y'_o \\ \frac{dp}{p} \end{pmatrix} \quad \text{III-21}$$

and similar equation exists for the exit faces with  $\beta 1$  replaced by  $\beta 2$ .

#### Horizontal Plane

The net transfer matrix for a constant field bending magnet with rotated pole faces is then for the horizontal plane

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{\tan \beta l}{r_o} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \phi & r_o \sin \phi & r_o(1-\cos \phi) \\ \frac{-\sin \phi}{r_o} & \cos \phi & \sin \phi \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \frac{\tan \beta l}{r_o} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\left( \begin{array}{l} \cos \phi + \sin \phi \tan \beta_1 \\ \frac{\tan \beta_2}{r_0} (\cos \phi + \sin \phi \tan \beta_1) - \frac{\sin \phi}{r_0} + \frac{\cos \phi \tan \beta_1}{r_0} \\ 0 \end{array} \right.$$

III-22

$$\left. \begin{array}{ll} r_0 \sin \phi & r_0 (1 - \cos \phi) \\ \sin \phi \tan \beta_2 + \cos \phi & \sin \phi + (1 - \cos \phi) \tan \beta_2 \\ 0 & 1 \end{array} \right)$$

Using trigonometric identities some of these components can be put in a simplified form.

$$\begin{aligned} \text{e.g. } T_{11} &= \cos \phi + \sin \phi \tan \beta_1 \\ &= \cos \phi + \sin \phi \frac{\sin \beta_1}{\cos \beta_1} \end{aligned}$$

Now using

$$\cos (\phi - \beta_1) = \cos \phi \cos \beta_1 + \sin \phi \sin \beta_1$$

gives

$$T_{11} = \cos \phi + \frac{1}{\cos \beta_1} (\cos (\phi - \beta_1) - \cos \phi \cos \beta_1)$$

III-23

$$= \frac{\cos (\phi - \beta_1)}{\cos \beta_1}$$

and similarly

$$T_{22} = \cos \frac{(\phi - \beta_2)}{\cos \beta_2} \cdot$$

III-24

Also

$$\begin{aligned}
 T_{21} &= \frac{1}{r_0} (\cos \phi \tan \beta_2 + \sin \phi \tan \beta_1 \tan \beta_2 - \\
 &\quad \sin \phi + \cos \phi \tan \beta_1) \\
 &= \frac{-1}{r_0} (\sin \phi (1 - \tan \beta_1 \tan \beta_2) - \cos \phi \\
 &\quad \tan (\beta_1 + \beta_2) (1 - \tan \beta_1 \tan \beta_2))
 \end{aligned}$$

where we have used

$$\tan (\beta_1 + \beta_2) = \frac{\tan \beta_1 + \tan \beta_2}{1 - \tan \beta_1 \tan \beta_2} .$$

This leads to

$$T_{21} = \frac{-1}{r_0} \frac{(1 - \tan \beta_1 \tan \beta_2)}{\cos (\beta_1 + \beta_2)} \sin (\phi - (\beta_1 + \beta_2)) \quad \text{III-25}$$

where we have used

$$\sin (\phi - (\beta_1 + \beta_2)) = \sin \phi \cos (\beta_1 + \beta_2) - \cos \phi \sin (\beta_1 + \beta_2) .$$

Putting equations III-23, III-24 and III-25 in equations III-22 gives the matrix  $H_R$  for the horizontal plane for a magnet which bends the central trajectory to the right. It is

$$H_R = \begin{pmatrix} \frac{\cos (\phi - \beta_1)}{\cos \beta_1} \\ \frac{-1}{r_0} \frac{(1 - \tan \beta_1 \tan \beta_2)}{\cos (\beta_1 + \beta_2)} \sin (\phi - (\beta_1 + \beta_2)) \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} r_0 \sin \phi & r_0 (1 - \cos \phi) \\ \frac{\cos (\phi - \beta 2)}{\cos \beta 2} & \sin \phi + (1 - \cos \phi) \tan \beta 2 \\ 0 & 1 \end{pmatrix} \quad \text{III-26}$$

### Vertical Plane

Since the y-axis was chosen to be parallel to the field B , the Lorentz force equation gives that  $f_y = 0$  and the equation of motion in the y-direction is  $m\ddot{y} = 0$  . The magnet thus acts in the vertical plane as a drift space of effective length  $L = r_0 \phi$  .

Taking into account the effects due to the rotated pole faces described by equations III-20 and III-21 we get the matrix  $V_R$  representing the bending magnet in the vertical plane to be

$$V_R = \begin{pmatrix} 1 & 0 & 0 \\ \frac{-\tan \beta 2}{r_0} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & r_0 \phi & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \frac{-\tan \beta 1}{r_0} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \quad \text{III-27}$$

$$\begin{pmatrix} 1 - \phi \tan \beta 1 & r_0 \phi & 0 \\ -\frac{\tan \beta 1}{r_0} - \frac{\tan \beta 2}{r_0} + \frac{\phi \tan \beta 2 \tan \beta 1}{r_0} & 1 - \phi \tan \beta 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The matrix components given in equations III-26 and III-27 agree with equations (30) and (33) given by Penner (1961) except for component  $T_{12}$  in the vertical plane where Penner fails to include the term  $\phi \tan \beta_1 \tan \beta_2$ .

Plotting at points inside bending magnets gives the correct values for the slope only at the exit of the magnet. This is because the matrix components given in equations III-26 and III-27 assume the action of the two rotated pole faces occur simultaneously whereas the effect of the exit edge should not occur until the exit of the magnet has been reached. If accurate values of particle slope were required inside bending magnets the program could be modified by splitting the present matrix into 3 matrices, one representing the effect of entrance pole face rotation, one for a magnet with normal entry and exit, and one for the exit face rotation. If these matrices were used in the above order the correct trajectory would result.

### III.5 Sign Convention for Bending Magnets

For a bending magnet which bends positively charged particles to the right (looking in the direction of beam motion) the angle of bend  $\phi$ , and the magnetic field are taken as positive. The radius of curvature  $r_0$  from equation III-2 is then also positive.

Comparison of equations III-20 and III-21 with equation II-23 shows that the effect of rotating a pole

face through an angle  $\beta$  is approximately equivalent to that of having a thin lens of focal length  $f = \frac{-r_0}{\tan \beta}$  in the horizontal plane and of  $f = \frac{+r_0}{\tan \beta}$  in the vertical plane where as defined above  $r_0 > 0$  for a bend to the right. The sign convention for  $\beta$  is that if the effect of the rotated edge is to produce horizontal defocussing or vertical focussing  $\beta > 0$  and if the effect of the lens is to produce horizontal focussing or vertical defocussing then  $\beta < 0$ . Hence in Fig. 3(A)  $\beta_1 > 0$  and  $\beta_2 < 0$ . To find the sign conventions for a magnet which bends positively charged particles to the left we note from Fig. 3 that a magnet that bends the optic axis to the left can be obtained from one that bends to the right by rotation about the central trajectory by  $180^\circ$ . As shown by Penner (1961) the effect of a left bend magnet on a particle can be found by rotating the particle coordinates and left bend magnet through  $180^\circ$ , using the known matrix components of a right bend magnet to compute the particle position and slope at the exit of this magnet and then rotating the particle coordinates through  $180^\circ$  again to arrive at the final position and slope. This procedure yields for the horizontal plane

$$H_L = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times H_R \times \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{\cos(\phi - \beta_1)}{\cos \beta_1} \\ \frac{-1}{r_0} \frac{(1 - \tan \beta_1 \tan \beta_2)}{\cos(\beta_1 + \beta_2)} \sin(\phi - (\beta_1 + \beta_2)) \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} r_0 \sin \phi & -r_0 (1 - \cos \phi) \\ \frac{\cos(\phi - \beta_2)}{\cos \beta_2} & -\sin \phi - (1 - \cos \phi) \tan \beta_2 \\ 0 & 1 \end{pmatrix} \quad \text{III-28}$$

and for the vertical plane

$$V_L = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times V_R \times \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 - \phi \tan \beta_1 \\ \frac{1}{r_0} (\phi \tan \beta_1 \tan \beta_2 - \tan \beta_1 - \tan \beta_2) \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} r_0 \phi & 0 \\ 1 - \phi \tan \beta_2 & 0 \\ 0 & 1 \end{pmatrix} = V_R \quad \text{III-29}$$

Since  $\cos(-x) = \cos x$ ,  $\sin(-x) = -\sin x$  and  $\tan(-x) = -\tan x$  we can reproduce the above matrices by adopting the convention that for magnets which bend the central trajectory to the left  $B$ ,  $\theta$ ,  $\beta_1$ ,  $\beta_2$  are taken as having signs opposite of those for magnets which bend the optic axis to the right. Thus in Fig. 3(B) we have that  $\beta_1 < 0$  and  $\beta_2 > 0$ .

The table below shows explicitly the sign convention for the 8 possible cases.

Pole Face	Deflection of Central Trajectory	Effect of Pole Face		Convention
		Horizontal Plane	Vertical Plane	
entrance	to right	focussing	defocussing	$\beta_1 < 0$
"	" "	defocussing	focussing	$\beta_1 > 0$
"	to left	focussing	defocussing	$\beta_1 > 0$
"	" "	defocussing	focussing	$\beta_1 < 0$
exit	to right	focussing	defocussing	$\beta_2 < 0$
"	" "	defocussing	focussing	$\beta_2 > 0$
"	to left	focussing	defocussing	$\beta_2 > 0$
"	" "	defocussing	focussing	$\beta_2 < 0$

## APPENDIX IV: LIOUVILLE'S THEOREM AND PHASE SPACE ELLIPSES

IV.1 Liouville's Theorem

The conservation of the number of particles in phase space is represented by the equation of continuity

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{U}) = 0 \quad \text{IV-1}$$

where  $\vec{U} = (\dot{x}, \dot{y}, \dot{z}, \dot{p}_x, \dot{p}_y, \dot{p}_z) = (\dot{x}_i, \dot{p}_i)$  is a particle current vector and

$$\rho = \rho(x, y, z, p_x, p_y, p_z) = \rho(x_i, p_i)$$

is the phase space density. From IV-1 we have

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \sum_i \left( \frac{\partial}{\partial x_i} (\rho \dot{x}_i) + \frac{\partial}{\partial p_i} (\rho \dot{p}_i) \right) &= 0 \\ &= \frac{\partial \rho}{\partial t} + \sum_i \left( \frac{\partial \rho}{\partial x_i} \dot{x}_i + \rho \frac{\partial \dot{x}_i}{\partial x_i} + \frac{\partial \rho}{\partial p_i} \dot{p}_i + \rho \frac{\partial \dot{p}_i}{\partial p_i} \right) . \end{aligned} \quad \text{IV-2}$$

Now consider the terms

$$\sum_i \rho \left( \frac{\partial \dot{x}_i}{\partial x_i} + \frac{\partial \dot{p}_i}{\partial p_i} \right) . \quad \text{IV-3}$$

If Hamilton's equations of motion apply then

$$\dot{x}_i = \frac{\partial H}{\partial p_i}$$

and

$$\dot{p}_i = - \frac{\partial H}{\partial x_i}$$

and therefore the terms of IV-3 are

$$\sum_i \rho \left( \frac{\partial}{\partial x_i} \left( \frac{\partial H}{\partial p_i} \right) + \frac{\partial}{\partial p_i} \left( - \frac{\partial H}{\partial x_i} \right) \right) = 0$$

and from equation IV-2

$$\frac{d\rho}{dt} + \sum_i \left( \frac{\partial \rho}{\partial x_i} \dot{x}_i + \frac{\partial \rho}{\partial p_i} \dot{p}_i \right) = \frac{d\rho}{dt} = 0 \quad . \quad \text{IV-4}$$

Thus in general if the forces acting on a particle can be derived from a Hamiltonian function the local density of the representative points in phase space is constant in time. This is valid for both conservative and non-conservative systems provided a Hamiltonian exists for the system (Lobb, 1963). The relativistic Hamiltonian function for a particle of charge  $e$ , rest mass  $m_0$ , in an external magnetic field exists and is well known to be

$$H = c (m_0^2 c^2 + (\vec{P} - e\vec{A})^2)^{\frac{1}{2}} \quad \text{IV-5}$$

where the canonical momentum is

$$\vec{P} = m\vec{v} + e\vec{A}$$

and

$$\vec{B} = \text{curl } \vec{A}$$

(Panofsky and Phillips, 1955). From the derivation of Liouville's theorem given above we see that the coordinates used must be a generalized coordinate and its conjugate

momentum. In a drift space  $\vec{A} = 0$  and the canonical momentum is then the usual momentum  $m\vec{v}$ .

For the case of particle motion in a magnetic field it can also be shown that both the terms

$$\sum_i \frac{\partial \dot{x}_i}{\partial x_i} \quad \text{and} \quad \sum_i \frac{\partial \dot{p}_i}{\partial p_i}$$

occurring in IV-3 are zero separately when  $\vec{p} = m\vec{v}$  is the normal momentum.

Thus from

$$\vec{v} = \frac{\vec{p}}{m} = \frac{c\vec{p}}{E/c} = \frac{c\vec{p}}{(p^2 + m_0^2 c^2)^{\frac{1}{2}}}$$

or

$$\dot{x}_i = \frac{cp_i}{(p^2 + m_0^2 c^2)^{\frac{1}{2}}} \quad \text{IV-7}$$

where

$$p^2 = \sum_i p_i p_i$$

we have

$$\sum_i \frac{\partial \dot{x}_i}{\partial x_i} = \sum_i \frac{\partial}{\partial x_i} \left( \frac{cp_i}{(p^2 + m_0^2 c^2)^{\frac{1}{2}}} \right) = 0$$

and from

$$\vec{f} = \dot{\vec{p}} = e\vec{v} \times \vec{B}$$

we have

$$\begin{aligned} \sum_i \frac{\partial \dot{p}_i}{\partial p_i} &= \sum_i \frac{\partial}{\partial p_i} \left( \sum_{j,k} e \epsilon_{ijk} \dot{x}_j B_k \right) \\ &= \sum_{i,j,k} e \epsilon_{ijk} \left( \dot{x}_j \frac{\partial B_k}{\partial p_i} + B_k \frac{\partial \dot{x}_j}{\partial p_i} \right) \end{aligned}$$

The first term is zero as the magnetic field does not depend on the particle momentum and we have from equation IV-7

$$\begin{aligned} &= \sum_{i,j,k} e \epsilon_{ijk} B_k \frac{\partial}{\partial p_i} \left( \frac{c p_j}{(p^2 + m_0^2 c^2)^{\frac{1}{2}}} \right) \\ &= \sum_{i,j,k} e B_k \epsilon_{ijk} c \left( \frac{-p_j p_i}{(p^2 + m_0^2 c^2)^{3/2}} + \frac{\delta_{ij}}{(p^2 + m_0^2 c^2)^{\frac{1}{2}}} \right) \end{aligned}$$

Since the term in brackets is symmetric with respect to interchange of  $i$  and  $j$  whereas the term  $\epsilon_{ijk}$  is anti-symmetric with respect to those indices, the sum adds to zero and this gives the desired result

$$\sum_i \frac{\partial \dot{p}_i}{\partial p_i} = 0 .$$

We can thus use the six dimensional phase space with coordinates  $x, y, z, m\dot{x}, m\dot{y}, m\dot{z}$ .

Another form of Liouville's theorem is useful in beam transport theory. Consider a region of phase space  $\delta V$  taken small enough so that the density can be regarded

constant over its extension. The number of points within this region is  $\delta N = \rho \delta V$ . If we follow the motion of this region through the phase space allowing the boundaries of the region to be determined by points originally within the region, then

$$\frac{d}{dt} (\delta N) = 0 \quad \text{IV-6}$$

as no points are created or destroyed due to their correlations with mechanical systems and no points can cross the boundaries because of the unambiguous determination of mechanical motions. Differentiating equation IV-6 gives

$$\rho \frac{d}{dt} (\delta V) + \delta V \frac{d\rho}{dt} = 0 \quad \text{IV-7}$$

and we get that

$$\frac{d}{dt} (\delta V) = 0 .$$

Since we can combine small elements this expression applies in the case of large phase space volumes providing their boundaries are determined by the same selection of representative points. If the equations of motion in each plane are independent of each other then the density function can be written as a product  $\rho = \rho_x \rho_y \rho_z$  and Liouville's theorem holds for each plane. The above statement of Liouville's theorem then becomes that the area in each plane is an invariant of the motion.\*

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\*Discussions of Liouville's theorem and its relationship to beam transport theory are given by Steffen (1961), Banford (1966) and Lobb (1963). A general treatment by Tolman (1934) was also consulted.

IV.2 Ellipse Calculations

A standard form for writing the equation for a central ellipse is

$$ax^2 + 2bxx' + c(x')^2 = 1 \quad \text{IV-8}$$

where  $a > 0$ ,  $c > 0$ ,  $b^2 - ac > 0$ .

This ellipse is centered on the origin because if  $(x_0, x_0')$  satisfies the equation so also does  $(-x_0, -x_0')$ .

The area  $A$  of this ellipse is from calculus

$$A = \frac{\pi}{(ac - b^2)^{\frac{1}{2}}} .$$

If we define

$$\epsilon = \frac{A}{\pi} = \frac{1}{(ac - b^2)^{\frac{1}{2}}}$$

then multiplying equation IV-8 by  $\epsilon$  gives

$$\frac{a}{(ac - b^2)^{\frac{1}{2}}} x^2 + 2 \frac{b}{(ac - b^2)^{\frac{1}{2}}} xx' + \frac{c}{(ac - b^2)^{\frac{1}{2}}} x'^2 = \epsilon$$

and if we define

$$\gamma = \frac{a}{(ac - b^2)^{\frac{1}{2}}} \quad \alpha = \frac{b}{(ac - b^2)^{\frac{1}{2}}} \quad \beta = \frac{c}{(ac - b^2)^{\frac{1}{2}}} \quad \text{IV-9}$$

then we have that

$$\gamma\beta - \alpha^2 = \frac{a}{(ac - b^2)^{\frac{1}{2}}} \frac{c}{(ac - b^2)^{\frac{1}{2}}} - \frac{b^2}{ac - b^2} = 1 .$$

A normalized form for the ellipse equation is then

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \epsilon \quad \text{IV-10}$$

where

$$\gamma\beta - \alpha^2 = 1 \quad \text{IV-11}$$

and

$$\epsilon = \frac{A}{\pi} . \quad \text{IV-12}$$

The maximum value of  $x$  is found by differentiating equation IV-10 with respect to  $x'$  giving

$$\gamma 2x \frac{dx}{dx'} + 2\alpha (x + x' \frac{dx}{dx'}) + \beta 2x' = 0 ,$$

At  $x_m$ ,

$$\frac{dx}{dx'} = 0$$

and we have

$$2\alpha x_m + \beta 2x' = 0$$

or

$$x' = -\frac{\alpha}{\beta} x_m$$

and substituting in equation IV-10

$$\gamma x_m^2 + 2\alpha x_m \left(-\frac{\alpha}{\beta} x_m\right) + \beta \left(-\frac{\alpha}{\beta} x_m\right)^2 = \epsilon$$

$$x_m^2 = \frac{\epsilon}{\left(\gamma - 2\frac{\alpha^2}{\beta} + \frac{\alpha^2}{\beta}\right)} = \frac{\beta\epsilon}{(\beta\gamma - \alpha^2)} = \beta\epsilon$$

$$x_m = (\epsilon\beta)^{\frac{1}{2}}$$

The maximum value of  $x'$  is found by differentiating IV-10 with respect to  $x$  and setting  $\frac{dx'}{dx} = 0$ . This gives  $x'_m = \frac{-\gamma}{\alpha} x$  and substituting in equation IV-8 gives

$$x'_m = (\epsilon\gamma)^{\frac{1}{2}}. \quad \text{IV-14}$$

The value  $x'_i$  of  $x'$  when  $x = 0$  is easily seen from equation IV-10 to be

$$x'_i = \pm (\frac{\epsilon}{\beta})^{\frac{1}{2}} \quad \text{IV-15}$$

and similarly when  $x' = 0$  we have

$$x_i = \pm (\frac{\epsilon}{\gamma})^{\frac{1}{2}}. \quad \text{IV-16}$$

These results are given by Steffen (1964).

For an upright ellipse ( $\alpha = 0$ ), the absolute values of  $x'_i$  and  $x_i$  become the semi-major and semi-minor axis for the ellipse. Also from equation IV-9 we have  $\gamma\beta = 1$  and

$$x'_i = (\epsilon\gamma)^{\frac{1}{2}} = x'_m$$

$$x_i = (\epsilon\beta)^{\frac{1}{2}} = x_m.$$

Also for an upright ellipse the ellipse area is given by  $\pi$  times the product of the semi-major and the semi-minor axis and the emittance becomes

$$\epsilon = \frac{A}{\pi} = \frac{\pi x'_i x_i}{\pi} = x'_m x_m \quad \text{IV-17}$$

with

$$\gamma = \frac{x'_m{}^2}{\epsilon} = \frac{x'_m{}^2}{x'_m x_m} = \frac{x'_m}{x_m} \quad \text{IV-18}$$

and

$$\beta = \frac{x_m{}^2}{\epsilon} = \frac{x_m{}^2}{x'_m x_m} = \frac{x_m}{x'_m} \quad \text{IV-19}$$

### IV.3 Unit Determinant of Transfer Matrix

In the horizontal plane an initial phase space ellipse may be written in normalized form as

$$\gamma_0 x_0^2 + 2\alpha_0 x_0 x'_0 + \beta_0 x'_0{}^2 = \epsilon$$

or

$$(x_0 \ x'_0) \begin{pmatrix} \gamma_0 & \alpha_0 \\ \alpha_0 & \beta_0 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} = \epsilon$$

or

$$X_0^T E_0 X_0 = \epsilon \quad \text{IV-20}$$

(where  $X_0$ ,  $E_0$  represent matrices and where  $X_0^T$  is the transpose of  $X_0$ ).

Also

$$\det E_0 = \gamma_0 \beta_0 - \alpha_0^2 = 1 \quad \text{IV-21}$$

and from equation IV-12,  $\pi\epsilon$  is the ellipse area.

If the transfer matrix for a system is given and  $\frac{dp}{p} = 0$  the final displacement  $x$  and slope  $x'$  of a

particle in the beam as computed from equation 2.1.1 are

$$\begin{pmatrix} x \\ x' \\ \frac{dp}{p} \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_o \\ x'_o \\ 0 \end{pmatrix} .$$

No information is lost if we write this as

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} x_o \\ x'_o \end{pmatrix} , \quad \frac{dp}{p} = 0$$

or in matrix form

$$X = TX_o , \quad \frac{dp}{p} = 0 . \quad \text{IV-22}$$

We have then

$$X_o = T^{-1} X \quad \text{IV-23}$$

and

$$X_o^T = X^T (T^{-1})^T \quad \text{IV-24}$$

where  $X^T$  is the transpose of  $X$  .

To find the final ellipse equation we substitute equations IV-23 and IV-24 into equation IV-20 and get

$$X^T (T^{-1})^T E_o T^{-1} X = \epsilon$$

or

$$X^T E X = \epsilon$$

or

$$(XX^T) \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix} = \epsilon \quad \text{IV-25}$$

where the matrix representing the final ellipse has been taken as

$$E = (T^{-1})^T E_0 T^{-1} . \quad \text{IV-26}$$

We see that the final ellipse area will still be equal to  $\pi\epsilon$  if once again the ellipse equation is in normalized form or  $\det E = 1$ . Thus for

$$\begin{aligned} 1 &= \det E = \det (T^{-1})^T E_0 T^{-1} \\ &= \det (T^{-1})^T \det E_0 \det T^{-1} \\ &= \det (T^{-1})^T \det T^{-1} \\ &= \det (T^{-1}) \det (T^{-1}) \\ &= \frac{1}{\det T} \frac{1}{\det T} \end{aligned}$$

and we must have

$$\begin{aligned} (\det T)^2 &= 1 \\ \det T &= \pm 1 . \end{aligned}$$

To see which of these two possibilities holds we observe that the transfer matrix  $T$  must reduce to the unit matrix as  $z$  approaches zero. The determinant of the unit matrix is  $+1$  and since  $\det T$  cannot change discontinuously we must have

$$\det T = +1 . \quad \text{IV-27}$$

#### IV.4 Transformation of Ellipse Coefficients

From the previous section the final matrix representing the ellipse  $E$  is given in terms of the initial ellipse  $E_0$  through the matrix equation IV-26

$$E = (T^{-1})^T E_0 (T^{-1}).$$

Since

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

and from equation IV-27

$$\det T = 1$$

then

$$T^{-1} = \begin{pmatrix} T_{22} & -T_{12} \\ -T_{21} & T_{11} \end{pmatrix}$$

and

$$(T^{-1})^T = \begin{pmatrix} T_{22} & -T_{21} \\ -T_{12} & T_{11} \end{pmatrix}$$

We get

$$\mathbf{E} = \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} = \begin{pmatrix} T_{22} & -T_{21} \\ -T_{12} & T_{11} \end{pmatrix} \begin{pmatrix} \gamma_0 & \alpha_0 \\ \alpha_0 & \beta_0 \end{pmatrix} \begin{pmatrix} T_{22} & -T_{12} \\ -T_{21} & T_{11} \end{pmatrix} = \begin{pmatrix} (T_{22}^2 \gamma_0 - 2T_{21} T_{22} \alpha_0 + T_{21}^2 \beta_0) & (-T_{22} T_{12} \gamma_0 + (T_{11} T_{22} + T_{21} T_{12}) \alpha_0 - T_{11} T_{21} \beta_0) \\ (-T_{12} T_{22} \gamma_0 + (T_{12} T_{21} + T_{11} T_{22}) \alpha_0 - T_{11} T_{21} \beta_0) & (T_{12}^2 \gamma_0 - 2T_{11} T_{12} \alpha_0 + T_{11}^2 \beta_0) \end{pmatrix}$$

or

$$\gamma = T_{22}^2 \gamma_0 - 2T_{21} T_{22} \alpha_0 + T_{21}^2 \beta_0$$

$$\alpha = -T_{12} T_{22} \gamma_0 + (T_{12} T_{21} + T_{11} T_{22}) \alpha_0 - T_{11} T_{21} \beta_0$$

$$\beta = T_{12}^2 \gamma_0 - 2T_{11} T_{12} \alpha_0 + T_{11}^2 \beta_0$$

or in matrix form

$$\begin{pmatrix} \gamma \\ \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} T_{22}^2 & -2 T_{21} T_{22} & T_{21}^2 \\ -T_{12} T_{22} & T_{12} T_{21} + T_{11} T_{22} & -T_{11} T_{21} \\ T_{12}^2 & -2T_{11} T_{12} & T_{11}^2 \end{pmatrix} \begin{pmatrix} \gamma_0 \\ \alpha_0 \\ \beta_0 \end{pmatrix} \quad \text{IV-28}$$

This matrix equation agrees with the result given by Steffen (1964).

## APPENDIX V: INPUT AND OUTPUT CONVENTIONS

V.1 Input Conventions

V.1.1 All input data is read in by the main program. The first card of any job contains four numbers specifying the design kinetic energy for the system, the rest energy for the type of particle under consideration, the total number of elements in the system, and a parameter which indicates whether the drift lengths (which are read in later) are to be taken as measured between magnet centers or edges. Thus, the card reads, in format (2F10.5, 2I3).

design kinetic energy (Mev)	particle rest energy (Mev)	total number of elements	MMM
--------------------------------	-------------------------------	-----------------------------	-----

where if

MMM = 0 drift lengths are taken as being measured from magnet edges.

= 1 drift lengths are taken as being measured from magnet centers.

V.1.2 Next, in the proper sequence, a card for each element in the system is read in. Each card specifies the type of element and certain element parameters in format (F4.1, F11.4, F11.5). The detailed scheme for reading in the  $I^{\text{th}}$  element is shown in the table below.

<u>Element Type</u>	<u>Element Type No. ET(I)</u>	<u>First Parameter D(I)</u>	<u>Second Parameter C(I)</u>
drift space	1.0	length (meters)	
quadrupole magnet	2.0	effective length (meters)	field gradient (gauss/cm.)
bending magnet	3.0	bending angle (degrees)	field (K gauss)

Note: A card representing a bending magnet ( $ET(I) = 3.0$ ) must be followed immediately by a card specifying the entrance and exit angles of the rotated pole faces. This card is read in format (2F10.5) and the angles are assumed to be in degrees. For normal entry and exit this may be a blank card.

From equation (2.2.2) for positively charged particles horizontally focussing quadrupoles have positive field gradients and horizontally defocussing quadrupoles have negative field gradients.

V.1.3 After the initial system parameters have been read in, the program reads an instruction card to determine which job it is required to perform. This instruction card contains up to six numbers in format (3I3, 3F10.5) and gives directions in accordance with the table below.

INST Tracking	IA Type of Tracking	IB Plane	U,V,W Initial Conditions
1	0 trajectory tracking	+1 vertical plane	for IA = 0, 1 U = initial displacement (cm) V = initial slope (rad/100) W = momentum deviation
	1 trajectory plotting	-1 horizontal plane	
	2 ellipse tracking		for IA = 2, 3, 4
	3 ellipse plotting		U = alpha V = beta (cm/ $\frac{\text{rad}}{100}$ )
	4 beam envelope trace		W = emittance (cm $\frac{\text{rad}}{100}$ )
	5 matrix components		

INST Matching	IA Horizontal Plane Matching Routine	IB Vertical Matching Routine	U,V,W Matching Section
2	0 'achromatic system' 1 parallel to focus 2 focus to focus 3 parallel to parallel 4 focus to parallel 5 identity system 6 match to 'waist' 7 match to specified ellipse	0 no routine specified 1 parallel to focus 2 focus to focus 3 parallel to parallel 4 focus to parallel 5 identity system 6 match to 'waist' 7 match to specified ellipse	U = first element V = last element
System Changes			
3	0 change to complete new system N change N <sup>th</sup> system element -1 change to new design momentum	- - -	for IA = -1 U = new design momentum (Gev/c)
Termination of program			
4	-	-	-

Additional Notes: Some jobs require more information than that which is read in on the instruction card. These cases are listed below, along with a description of the additional information required.

- a) INST = 1, IA = 3: An instruction card requesting ellipse plotting must be followed immediately by a card specifying the total number of ellipses which are to be plotted and the numbers of the elements after which plotting is requested. The card is in format (30I2).
- b) INST = 2, IA = 6 or IB = 6: An ellipse matching to a waist instruction card must be followed immediately by a data card containing the initial ellipse coefficients alpha, beta and the beam emittance in format (3F10.5).
- c) INST = 2, IA = 7 or IB = 7: An instruction card requesting ellipse matching must be followed immediately by a data card containing the initial ellipse coefficients alpha, beta and the beam emittance as well as the final requested alpha and beta. This data card is read in format (5F10.5).
- d) INST = 2, IA = 6 or 7 and IB = 6 or 7: An instruction card requesting ellipse matching in both planes must be followed immediately by two data cards of the type described in b) and c) with the card for the horizontal plane preceding that for the vertical plane.
- e) INST = 3, IA = N: An instruction card which requests a change in the  $N^{\text{th}}$  element of the system must be followed immediately by a card representing the new  $N^{\text{th}}$  element.

This card is read in according to the same scheme as was outlined in section V.1.2.

- f) INST = 3, IA = 0: An instruction card requesting a change to an entirely new beam transport system must be followed immediately by a set of data cards representing the new system. These cards are read in according to the schemes outlined in sections V.1.1 and V.1.2.

V.1.4 Designation of variable parameters: If matching is to be done for a section of the beam transport system then some of the element parameters within the section must be designated as variable. The convention adopted was to set the fraction part of the element type number ET(I) to a non-zero value for those elements whose parameters were to be varied. The variable parameter for a drift length is its length, for a quadrupole magnet its field gradient and for a bending magnet its magnet its magnetic field.

A possible modification of the program would be to arrange it so certain designated parameters could be varied together. This could be useful, for example, in designing systems containing 'symmetric' quadrupole triplets.

## V.2 Output Conventions

V.2.1 The initial beam transport system is always printed out according to the format shown on page 120. The calculated particle momentum corresponding to the design kinetic energy for the system is printed out on the second line. For bending magnets the effective length  $L = r_0 \phi$  is calculated and printed out in addition to the input parameters  $\phi$ ,  $B$ ,  $\beta_1$ ,  $\beta_2$ . The column on the left numbers the elements.

V.2.2 The output from a trajectory tracking job (INST = 1, IA = 0) is shown on pages 123, 124 and 135. The first column gives the number of the element and the second column gives the distance in meters down the central trajectory to the exit of that element. The next two columns give the particle displacement (cm.) and slope (rad./100) at the distance specified in column 2.

A similar print out is given for trajectory plotting (INST = 1, IA = 1) and on the right appears the particle trajectory. The calculated particle displacement is multiplied by 4.0 and the nearest integer below this number is plotted as an asterisk as shown on page 125. For displacements greater than 10.5 cm. in absolute value no asterisk is plotted. As seen from column 2 displacements are plotted at every 20 cm. along the optic axis. This increment could be adjusted by changing the value of DI in subroutine TRACK.

The ellipse tracking output (INST = 1, IA = 2) is shown on page 126. The first column specifies the element

number and the next 5 columns give the values of

$$\gamma(\frac{\text{rad.}}{100}/\text{cm.}) , \alpha, \beta(\text{cm.}/\frac{\text{rad.}}{100}), x_m(\text{cm.}) \text{ and } x'_m(\frac{\text{rad.}}{100})$$

at the exit of this element.

The print out from a beam envelope trace is shown on page 127. Columns 1 and 2 give the element number and distance in meters along the optic axis. The next columns contain the ellipse parameters  $\gamma$ ,  $\alpha$ ,  $\beta$  and  $x_m$  and  $x'_m$ . On the right is a graph of the beam envelope. The maximum displacement  $x_m$  is multiplied by 10.0 and the nearest integer below this number is plotted as an asterisk as shown.

A plot of a phase space ellipse is shown on page 132. This is done by solving equation 2.3.1 for  $x$  as a double valued function of  $x'$ , i.e.

$$x = - \frac{\alpha x' \pm (\alpha^2 x'^2 - \gamma (\beta x'^2 - \epsilon))^{\frac{1}{2}}}{\gamma}$$

and then letting  $x'$  run through its range of values. Each value of  $x$  obtained is rounded to the nearest integer below it and plotted as an asterisk.

V.2.3 The output from a matching problem is shown on pages 133 and 134. The print out from subroutine VAO4A on page 128 depends on the parameter IPRINT as described in section 3.4.4. If the iteration procedure has been unsuccessful because the function being minimized has

not decreased on successive iterations the subroutine prints out 'VAO4A ACCURACY LIMITED BY ERRORS IN F' or if it fails because the maximum number of iterations allowed, MAXIT, has been reached it prints out the value of MAXIT and next to it 'ITERATIONS COMPLETED BY VAO4A'. The format for the print out of the results of a matching attempt is shown on page 129. On the first line the matching routine numbers are given with the number for the horizontal plane preceding that for the vertical plane. The final values of the variable element parameters and transfer matrix components are always printed out as shown. For ellipse matching to a waist the initial and final phase space ellipse parameters are printed out and for matching to a specified ellipse the parameters for the requested ellipse are also printed out. An example of this is shown on page 133.

## APPENDIX VI: UNITS

The input units for the average beam kinetic energy are conveniently taken as Mev. Relativistic mechanics allows us to compute the momentum  $p$  in terms of energy.

From

$$E^2 = c^2 p^2 + m_0^2 c^4 = c^2 p^2 + E_0^2$$

where  $E_0$  is the particle rest energy, we get

$$p = \frac{1}{c} (E^2 - E_0^2)^{\frac{1}{2}} . \quad \text{VI-1}$$

Expressing the total particle energy as the sum of the rest energy and kinetic energy  $T$  so that  $E = T + E_0$  and substituting in equation VI-1 gives

$$p = \frac{1}{c} (T^2 + 2T E_0)^{\frac{1}{2}} .$$

If  $T$  and  $E_0$  are expressed in Mev we have

$$p = (T^2 + 2T E_0)^{\frac{1}{2}} \frac{\text{Mev}}{c} = \frac{(T^2 + 2T E_0)^{\frac{1}{2}}}{10^3} \frac{\text{Gev}}{c} . \quad \text{VI-2}$$

Momentum  $p$  is calculated by the program and stored in units of  $\frac{\text{Gev}}{c}$  and is used subsequently in computing the quadrupole constant  $K$  and the radius of curvature  $r_0$  for bending magnets. Since the effective length of a quadrupole  $L$  is read in meters  $K$  must

be in inverse meters. We have from Appendix II, equation II-15

$$K^2 = \frac{eg}{p} = \frac{1.6 \times 10^{-19} \text{ [coul]} \text{ g} \frac{\text{[gauss]}}{\text{[cm]}}}{p \text{ [Gev/c]}} \quad \text{VI-3}$$

and we know

$$10^{-4} \left[ \frac{\text{webers}}{\text{m}^2} \right] = 1 \text{ [gauss]}$$

thus

$$10^{-2} \left[ \frac{\text{webers}}{\text{m}^2} \frac{1}{\text{m}} \right] = 1 \left[ \frac{\text{gauss}}{\text{cm}} \right] \quad \text{VI-4}$$

and

$$\begin{aligned} 1 \left[ \frac{\text{Gev}}{\text{c}} \right] &= \frac{10^9 \text{ ev} \cdot 1.6 \times 10^{-19} \left[ \frac{\text{joules}}{\text{ev}} \right]}{2.9978 \times 10^8 \left[ \frac{\text{m}}{\text{sec}} \right]} \\ &= \frac{1.6 \times 10^{-10}}{2.9978 \times 10^8} \left[ \frac{\text{kg} \frac{-\text{m}^2}{\text{sec}}}{\frac{\text{m}}{\text{sec}}} \right] \\ &= \frac{1.6 \times 10^{-10}}{2.9978 \times 10^8} \left[ \text{kg} \frac{-\text{m}}{\text{sec}} \right] \quad \text{VI-5} \end{aligned}$$

Putting equations VI-4 and VI-5 in VI-3

$$K^2 = \frac{1.6 \times 10^{-19} \text{ [coul]} \text{ g} \cdot 10^{-2} \left[ \frac{\text{webers}}{\text{m}^2} \frac{1}{\text{m}} \right]}{p \frac{1.6 \times 10^{-10}}{2.9978 \times 10^8} \left[ \frac{\text{kg} \cdot \text{m}}{\text{sec}} \right]} =$$

$$= .0029978 \frac{\text{g}}{\text{p}} \left[ \text{coul} \frac{\frac{\text{webers}}{\text{m}^2} \frac{1}{\text{m}}}{\frac{\text{kg} - \text{m}}{\text{sec}}} \right] . \quad \text{VI-6}$$

From the Lorentz force law we find that

$$\left[ \frac{\text{weber}}{\text{m}^2} \right] = \left[ \frac{\text{newton}}{\text{coul} \frac{\text{m}}{\text{sec}}} \right] = \left[ \frac{\text{kg} \frac{-\text{m}}{\text{sec}^2}}{\text{coul} \frac{\text{m}}{\text{sec}}} \right] = \left[ \frac{\text{kg}}{\text{coul} \text{ sec}} \right] \quad \text{VI-7}$$

and putting equations II-7 in VI-6 gives

$$K^2 = .0029978 \frac{\text{g}}{\text{p}} - \left[ \frac{1}{\text{m}^2} \right] . \quad \text{VI-8}$$

Also the radius of curvature for bending magnets is assumed to be in meters. We have from Appendix III, equation III-2

$$\frac{1}{r_0} = \frac{eB}{p} = \frac{1.6 \times 10^{-19} [\text{coul}] B [\text{K gauss}]}{p [\text{Gev/c}]}$$

and from equations VI-4 and VI-5

$$= \frac{1.6 \times 10^{-19} [\text{coul}] B 10^{-1} \left[ \frac{\text{webers}}{\text{m}^2} \right]}{p \frac{1.6 \times 10^{-10}}{2.9978 \times 10^8} \left[ \frac{\text{kg-m}}{\text{sec}} \right]}$$

$$= .029978 \frac{\text{B}}{\text{p}} \left[ \text{coul} \frac{\frac{\text{webers}}{\text{m}^2}}{\text{kg} - \frac{\text{m}}{\text{sec}}} \right]$$

and from equation VI-7

$$= .029978 \frac{B}{p} \left[ \frac{1}{m} \right] .$$

Since the particle slope is physically always small we can use the approximation that  $x' = \tan \theta \cong \theta$  (where  $\theta$  is the angle the trajectory makes with the optic axis) and measure slopes in radians. The units for equations 2.1.1 are as shown below.

$$\begin{pmatrix} m \\ \text{rad} \\ - \end{pmatrix} = \begin{pmatrix} - & \frac{m}{\text{rad}} & m \\ \frac{\text{rad}}{m} & - & \text{rad} \\ \frac{1}{m} & \frac{1}{\text{rad}} & - \end{pmatrix} \begin{pmatrix} m \\ \text{rad} \\ - \end{pmatrix}$$

Because of the small displacements and slopes it is convenient to use units of cm. and rad/100. This entails no changing of the numerical value of matrix components  $T_{12}$ ,  $T_{21}$  as  $\frac{1 \text{ meter}}{1 \text{ rad}} = \frac{1 \text{ cm}}{1 \text{ rad}/100}$ . However the components  $T_{13}$ ,  $T_{23}$  must be multiplied by 100.0 to convert from meters to cm. and from radians to rad/100 respectively.

The equation for the phase space ellipse is

$$\gamma x^2 + 2\alpha xx' + \beta x'^2 = \epsilon$$

with the displacements  $x$  in cm., the slopes  $x'$  in rad/100. and where the emittance is taken to have units of cm. rad/100. The units of the ellipse coefficients are then

$$\gamma \left[ \frac{\text{rad}}{100/\text{cm}} \right]$$

$$\alpha \quad -$$

$$\beta \left[ \frac{\text{cm}}{\text{rad}/100} \right]$$

## APPENDIX VII: PROGRAM LISTING

```

C   MAIN PROGRAM
      DIMENSION ET(30),D(30),C(30),H11(30),H12(30),H21(30),H22(30),
1V11(30),V12(30),V21(30),V22(30),H13(30),H23(30),XO(10),SO(10)
      DIMENSION BETA1(30),BETA2(30),W(440),X(20),E(20),T(10)
      DIMENSION NEL(30),IPL0T(85)
      COMMON ET,D,C,H11,H12,H21,H22,V11,V12,V21,V22,H13,H23,AN,W,X,E
      COMMON VG,VGI,VGF,VA,VAI,VAF,VB,VBI,VBF,HG,HGI,HGF,HA,HAI,HAF,HB
      COMMON T,SO,HBI,HBF,HEPI,VEPI,BETA1,BETA2,ESCALE,F,P,U,V,Z
      COMMON N,M,NN,MM,MMM,ME,INST,IA,IB,IPRINT,MAXIT,NEL
      COMMON NE,ITERC,IE
C   TRACKING AND MATCHING PROGRAM
C
C   SETTING UP INITIAL BEAM HANDLING SYSTEM
98 READ 61,ENERGY,REN,NE,MMM
61 FORMAT(2F10.5,2I3)
C   CALCULATION OF PARTICLE MOMENTUM
      P=SQRT(ENERGY**2+2.0*ENERGY*REN)/1000.0
      PRINT 62,ENERGY,REN,P,NE
62 FORMAT(14H1DESIGN ENERGY,F12.5,4H MEV/21H PARTICLE REST ENERGY,F10
1.5,4H MEV,9H MOMENTUM,F10.5,6H GEV/C/16H NO. OF ELEMENTS,I3)
      PRINT 5
      5 FORMAT(26H INITIAL SYSTEM PARAMETERS)
      PRINT 50
50 FORMAT(6H UNITS/14H LENGTH-METERS/24H FIELD GRADIENT-GAUSS/CM/16H
1FIELD-KILOGAUSS/14H ANGLE-DEGREES)
C   READING IN INITIAL SYSTEM PARAMETERS
      L=1
      K=NE
99 DO 9 I=L,K
      READ 13,ET(I),D(I),C(I)
13 FORMAT(F4.1,F11.4,F11.5)
      IV=ET(I)
      GO TO(31,32,34),IV
31 PRINT 41,I,D(I)
41 FORMAT(1X,I3,15H DRIFT SPACE      ,10H LENGTH      ,F10.4)
      C(I)=D(I)
      GO TO 45
32 IF(C(I))33,52,52
52 PRINT 42,I,D(I),C(I)
42 FORMAT(1X,I3,15H QUADRUPOLE-FH ,10H EF LENGTH,F10.4,15H FIELD GRAD
1IENT,F12.5)
      GO TO 45
33 PRINT 43,I,D(I),C(I)
43 FORMAT(1X,I3,15H QUADRUPOLE-DH ,10H EF LENGTH,F10.4,15H FIELD GRAD
1IENT,F12.5)
      GO TO 45
34 RAD=P/(.029978*C(I))
      ANG=D(I)
      D(I)=RAD*D(I)*.017453

```

```

      PRINT 44,I,D(I),C(I),ANG
44  FORMAT(1X,I3,15H BENDING MAGNET,10H EF LENGTH,F10.4,15H FIELD
      1      ,F12.5,14H BENDING ANGLE,F10.5)
48  READ 49,BETA1(I),BETA2(I)
49  FORMAT(2F10.5)
46  PRINT 51,BETA1(I),BETA2(I)
51  FORMAT(4X,15H ENTRANCE ANGLE,F10.5/4X,15H EXIT ANGLE      ,F10.5)
45  CONTINUE
      9  CONTINUE
77  N=L
      M=K
C    COMPUTATION OF BEAM ELEMENT MATRIX COMPONENTS
      IF(MMM)64,22,64
64  PRINT 63
63  FORMAT(45H DRIFT LENGTHS MEASURED TO CENTERS OF MAGNETS)
22  CONTINUE
      CALL ASSIGN
      6  MMM=0
      IE=1
C
C    READING INSTRUCTION CARD TO DETERMINE JOB
C    FOR INST=1,TRACKING
C    FOR INST=2,MATCHING
C    FOR INST=3, SYSTEM CHANGE
C    FOR INST=4,TERMINATION OF PROGRAM
      READ7,INST,IA,IB,U,V,Z
      7  FORMAT(3I2,3F10.5)
      GO TO(1,2,3,4),INST
C
      1  IF(IA-3)400,401,400
C    J ELLIPSES ARE TO BE PLOTTED AFTER ELEMENTS NEL)K*
401  DO 500 I=1,NE
500  NEL(I)=0
      READ 1398,J,(NEL(K),K=1,J)
1398  FORMAT(30I2)
400  CALL TRACK
      GO TO 6
C
      2  IF(IA-6)310,301,302
C    READING IN ELLIPSE PARAMETERS FOR ELLIPSE MATCHING
C    HORIZONTAL PLANE
301  READ 305,HAI,HBI,HEPI
305  FORMAT(3F10.5)
      HAF=0.0
      GO TO 300
302  READ 306,HAI,HBI,HEPI,HAF,HBF
306  FORMAT(5F10.5)
300  HGI=(1.0+HAI**2)/HBI
      IE=0

```

```
C  VERTICAL PLANE
310 IF (IB-6) 210, 201, 202
201 READ 205, VAI, VBI, VEPI
205 FORMAT(3F10.5)
    VAF=0.0
    GO TO 200
202 READ 206, VAI, VBI, VEPI, VAF, VBF
206 FORMAT(5F10.5)
200 VGI=(1.0+VAI**2)/VBI
    IE=0
210 CONTINUE
    CALL MATCH
    GO TO 6

C
  3 IF (IA) 94, 98, 89
89 PRINT 97
97 FORMAT(14H1SYSTEM CHANGE)
93 L=IA
    K=IA
    GO TO 99
94 P=U
    PRINT 92, P
92 FORMAT(13H1NEW MOMENTUM, F10.5)
    L=1
    K=NE
    GO TO 77

C
  4 CALL EXIT
    END
```



```

EK=SQRT(.0029978*CK/P)
XK=EK*D(K)
CX=COS(XK)
SX=SIN(XK)
EX=EXP(XK)
REX=1.0/EX
CHX=0.5*(EX+REX)
SHX=0.5*(EX-REX)
IF(C(K))104,103,103
C   QUADRUPOLE-FOCUSING IN HORIZONTAL PLANE
103 H11(K)=CX
    H12(K)=SX/EK
    H21(K)=-EK*SX
    H22(K)=CX
    H13(K)=0.0
    H23(K)=0.0
    V11(K)=CHX
    V12(K)=SHX/EK
    V21(K)=EK*SHX
    V22(K)=CHX
    GO TO 106
C   QUADRUPOLE-DEFOCUSING IN HORIZONTAL PLANE
104 H11(K)=CHX
    H12(K)=SHX/EK
    H21(K)=EK*SHX
    H22(K)=CHX
    H13(K)=0.0
    H23(K)=0.0
    V11(K)=CX
    V12(K)=SX/EK
    V21(K)=-EK*SX
    V22(K)=CX
    GO TO 106
C
C   CALCULATING BENDING MAGNET MATRIX COMPONENTS
105 B1=.017453*BETA1(K)
    B2=.017453*BETA2(K)
    RAD=P/(.029978*C(K))
    ANG=D(K)/RAD
    CA=COS (ANG)
    SA=SIN (ANG)
    SB1=SIN (B1)
    CB1=COS (B1)
    SB2=SIN (B2)
    CB2=COS (B2)
    TB1=SB1/CB1
    TB2=SB2/CB2
    H11(K)=COS (ANG-B1)/CB1
    H12(K)=RAD*SA

```

H21(K)=-1.0\*(1.0-TB1\*TB2)\*SIN(ANG-B1-B2)/(RAD\*COS(B1+B2))

H22(K)=COS (ANG-B2)/CB2

H13(K)=RAD\*(1.0-CA)\*100.0

H23(K)=(SA+(1.0-CA)\*TB2)\*100.0

V11(K)=1.0-ANG\*TB1

V12(K)=RAD\*ANG

V21(K)=-1.0\*(TB1+TB2-ANG\*TB1\*TB2)/RAD

V22(K)=1.0-ANG\*TB2

106 CONTINUE

RETURN

END

```

SUBROUTINE TRACK
  DIMENSION ET(30),D(30),C(30),H11(30),H12(30),H21(30),H22(30),
  1V11(30),V12(30),V21(30),V22(30),H13(30),H23(30),X0(10),SO(10)
  DIMENSION BETA1(30),BETA2(30),W(440),X(20),E(20),T(10)
  DIMENSION NEL(30),IPL0T(85)
  COMMON ET,D,C,H11,H12,H21,H22,V11,V12,V21,V22,H13,H23,AN,W,X,E
  COMMON VG,VGI,VGF,VA,VAI,VAF,VB,VBI,VBF,HG,HGI,HGF,HA,HAI,HAF,HB
  COMMON T,SO,HBI,HBFI,HEPI,VEPI,BETA1,BETA2,ESCALE,F,P,U,V,Z
  COMMON N,M,NN,MM,MMM,ME,INST,IA,IB,IPRINT,MAXIT,NEL
  COMMON NE,ITERC,IE

C
  DATA INUL,IDOT,ISTAR/' ','.',',','*','/'

C
C PRINTING HEADINGS FOR TRACKING RESULTS
  DI=0.0
  IP=1
  IF(IB)232,232,233
232 PRINT 203
203 FORMAT(17H1HORIZONTAL PLANE)
  GO TO 2000
233 PRINT 204
204 FORMAT(15H1VERTICAL PLANE)
2000 CONTINUE
C IA=0 FOR TRAJECTORY TRACKING
C IA=1 FOR TRAJECTORY PLOTTING
C IA=2 FOR ELLIPSE TRACKING
C IA=3 FOR ELLIPSE PLOTTING
C IA=4 FOR BEAM ENVELOPE TRACE
  IG=IA+1
  GO TO(10,11,12,12,12,13),IG
  10 PRINT 201,Z
201 FORMAT(29H TRAJECTORY TRACKING AT DP/P=,F7.3)
  PRINT 208
208 FORMAT(//,38H          DISTANCE  DISPLACEMENT          SLOPE)
  PRINT 42,DI,U,V
  42 FORMAT(4X,F10.5,4X,2F10.5)
  GO TO 241
  11 PRINT 200,Z
200 FORMAT(29H TRAJECTORY PLOTTING AT DP/P=,F7.3)
  PRINT 198
198 FORMAT(//,38H          DISTANCE  DISPLACEMENT          SLOPE,35X,15HTRAJECT
  1ORY PLOT)
  IP=0
  GO TO 241
  12 EP=Z
  Z=(1.0+U**2)/V
  XM=SQRT(EP*V)
  SM=SQRT(EP*Z)
  IF(IA-3)22,241,24

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22 PRINT 309,EP
309 FORMAT(17H ELLIPSE TRACKING/11H EMITTANCE ,F10.5)
PRINT 1382
1382 FORMAT(57H          GAMMA      ALPHA      BETA      DMAX      SMA
1X)
25 PRINT 1301,Z,U,V,XM,SM
1301 FORMAT(9X,6F10.5)
GO TO 241
24 PRINT1606,EP
1606 FORMAT(20H BEAM ENVELOPE TRACE/11H EMITTANCE ,F10.5)
PRINT 1600
1600 FORMAT(95H          GAMMA      ALPHA      BETA      DMAX      SMA
1X          BEAM ENVELOPE)
IP=0
GO TO 241
13 PRINT 1700
1700 FORMAT(27H TRANSFER MATRIX COMPONENTS/61H          T11      T12
1 T21      T22      T13      T23)
C
C INITIALIZING MATRIX MULTIPLICATION BY SETTING UP UNIT MATRIX
241 D11=1.0
D12=0.0
D21=0.0
D22=1.0
D13=0.0
D23=0.0
DI=.20
EI=.0
DIST=0.0
DIS=-DI
KM=1
C THIS LOOP COMPUTES TRANSFER MATRIX, THEN DOES REQUESTED TRACKING JOB
DO257 L=1,NE
IF(IP)271,249,271
C IP=0 FOR PLOTTING AT POINTS INTERIOR TO MAGNETS
249 DD=D(L)
IV=ET(L)
DO270 K=1,100
B=K
EL=(B-1.0)*DI+EI
OV=EL-DD
KM=K
IF(OV)270,271,271
270 CONTINUE
271 DO272 J=1,KM
IF(IP)266,101,266
101 RR=J
IF(J-KM)220,221,221
220 D(L)=(RR-1.0)*DI+EI

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```

DIS=DIS+DI
GO TO 222
221 D(L)=DD
222 IV=ET(L)
IF(IV-1)8887,8889,8887
8889 C(L)=D(L)
8887 CONTINUE
N=L
M=L
CALL ASSIGN
266 IF(IB)244,244,248
C MATRIX MULTIPLICATION-HORIZONTAL PLANE
244 C11=H11(L)*D11+H12(L)*D21
C12=H11(L)*D12+H12(L)*D22
C21=H21(L)*D11+H22(L)*D21
C22=H21(L)*D12+H22(L)*D22
C13=H11(L)*D13+H12(L)*D23+H13(L)
C23=H21(L)*D13+H22(L)*D23+H23(L)
GO TO 250
C MATRIX MULTIPLICATION-VERTICAL PLANE
248 C11=V11(L)*D11+V12(L)*D21
C12=V11(L)*D12+V12(L)*D22
C21=V21(L)*D11+V22(L)*D21
C22=V21(L)*D12+V22(L)*D22
C13=0.0
C23=0.0
250 CONTINUE
IF(IP)104,290,104
290 IF(J-KM)297,272,272
297 D0291 I=1,85
291 IPLOT(I)=INUL
104 IF(IA-1)1250,1250,1300
1300 IF(IA-5)1353,1800,1800
C
C TRANSFER MATRIX COMPONENTS
1800 PRINT 1801,L,C11,C12,C21,C22,C13,C23
1801 FORMAT(1X,I2,6F10.5)
GO TO 272
C
C CALCULATION OF ELLIPSE PARAMETERS
1353 GAMMA=C22**2*Z-2.0*C21*C22*U+C21**2*V
ALPHA=-C12*C22*Z+(C11*C22+C12*C21)*U-C11*C21*V
BETA=C12**2*Z-2.0*C11*C12*U+C11**2*V
XM=SQRT(EP*BETA)
SM=SQRT(EP*GAMMA)
IF(IA-3)390,398,1500
C
C BEAM ENVELOPE PLOTTING
1500 XM=10.0*XM

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      IF(XM-30.0)1390,1390,172
1390 NX=XM
      GO TO 173
172 NX=80
173 XM=.1*XM
      IPLOT(1)=IDOT
      IPLOT(NX)=ISTAR
1391 PRINT 1392,L,DIS,GAMMA,ALPHA,BETA,XM,SM,(IPLOT(KK),KK=1,30)
1392 FORMAT(1X,I3,F5.1,5F10.5,20X,30A1)
      IPLOT(NX)=INUL
      GO TO 272

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```

C
C ELLIPSE PLOTTING
398 CONTINUE
      DO 1399 II=1,NE
      IF(L-NEL(II))1399,1397,1399
1/399 CONTINUE
      GO TO 265
1397 CONTINUE
      N2=1
      N1=1
      DO 350 I=1,85
350 IPLOT(I)=INUL
      IF(XM-SM)322,322,323
322 TM=SM
      GO TO 324
323 TM=XM
324 TM=2.0*TM
      V1=TM/25.0
      H1=TM/42.0
      PRINT 120,L,GAMMA,ALPHA,BETA,EP,XM,SM
120 FORMAT(1H1,I3,8H GAMMA,F10.5,8H ALPHA,F10.5,7H BETA,F10.5/10
1H EMITTANCE,F10.5,18H MAX. DISPLACEMENT,F10.5,11H MAX. SLOPE,F10.5
2)
      PRINT 325,H1,V1
325 FORMAT(6H UNITS/23H ABSCISSA-ONE DIVISION ,F10.5,3H CM/23H ORDINAT
1E-ONE DIVISION ,F10.5,8H RAD/100)
      DO 399 K=1,51
      IPLOT(43)=IDOT
      IF(26-K)307,308,307
308 DO 319 J=1,85
319 IPLOT(J)=IDOT
307 RK=26-K
      S=(TM*RK)/25.0
      Q=4.0*ALPHA**2*S**2-4.0*GAMMA*(BETA*S**2-EP)
      IF(Q)301,392,392
392 Q=SQRT (Q)
      X1=(-2.0*ALPHA*S+Q)/(2.0*GAMMA)
      X2=(-2.0*ALPHA*S-Q)/(2.0*GAMMA)

```

```

N1=X1/TM*42.0+.5
N2=X2/TM*42.0+.5
N1=43+N1
N2=43+N2
I PLOT(N1)=I STAR
I PLOT(N2)=I STAR
301 PRINT 305,(I PLOT(JJ),JJ=1,85)
305 FORMAT(1X,85A1)
I PLOT(N1)=I NUL
I PLOT(N2)=I NUL
IF(26-K)399,362,399
362 D0363 J=1,85
363 I PLOT(J)=I NUL
399 CONTINUE
GO TO 272

C
C / ELLIPSE TRACKING
390 PRINT 315,L,GAMMA,ALPHA,BETA,XM,SM
315 FORMAT(1X,I3,5X,5F10.5)
GO TO 272

C
C CALCULATION OF PARTICLE DISPLACEMENT AND SLOPE
1250 R=C11*U+C12*V+C13*Z
S=C21*U+C22*V+C23*Z
IF(IP)252,400,252

C
C TRAJECTORY TRACKING
C CALCULATION OF SYSTEM LENGTH
252 DIST=DIST+D(L)
PRINT 274,L,DIST,R,S
274 FORMAT(1X,I3,F10.5,4X,2F10.5)
GO TO 272

C
C TRAJECTORY PLOTTING
400 I PLOT(43)=I DOT
Y=ABS(R)
IF(Y-10.5)1200,1200,272
1200 R=4.0*R
NX=R+43.0
R=.25*R
1202 I PLOT(NX)=I STAR
PRINT 292,L,DIS,R,S,(I PLOT(KK),KK=1,85)
292 FORMAT(1X,I3,F10.5,4X,2F10.5,85A1)
I PLOT(NX)=I NUL
272 CONTINUE
EI=OV
265 D11=C11
D12=C12
D21=C21

```

D22=C22  
D13=C13  
D23=C23  
257 CONTINUE  
RETURN  
END

```

SUBROUTINE MATCH
DIMENSION ET(30),D(30),C(30),H11(30),H12(30),H21(30),H22(30),
1V11(30),V12(30),V21(30),V22(30),H13(30),H23(30),XO(10),SO(10)
DIMENSION BETA1(30),BETA2(30),W(440),X(20),E(20),T(10)
DIMENSION NEL(30),IPLOT(85)
COMMON ET,D,C,H11,H12,H21,H22,V11,V12,V21,V22,H13,H23,AN,W,X,E
COMMON VG,VGI,VGF,VA,VAI,VAF,VB,VBI,VBF,HG,HGI,HGF,HA,HAI,HAF,HB
COMMON T,SO,HBI,HBF,HEPI,VEPI,BETA1,BETA2,ESCALE,F,P,U,V,Z
COMMON N,M,NN,MM,MMM,ME,INST,IA,IB,IPRINT,MAXIT,NEL
COMMON NE,ITERC,IE
C   MATCHING IS TO BE DONE FOR ELEMENT NN THROUGH TO ELEMENT MM
400 NN=U
    MM=V
C   DETERMINATION OF VARIABLE PARAMETERS
    ME=0
    DO410 I=NN,MM
        IV=ET(I)
        ES=IV
        EEE=ET(I)-ES
        IF(EEE)401,410,401
401 ME=ME+1
    NEL(ME)=I
410 CONTINUE
C
C   DEFINING X(I) AND E(I),IPRINT,ESCALE,MAXIT,N FOR VAO4A
    DO 1402 I=1,ME
        K=NEL(I)
C   COMPONENTS OF X(I) ARE THE INITIAL SYSTEM VARIABLE PARAMETERS
        X(I)=C(K)
        IV=ET(K)
        IF(IV-1)499,498,499
C   ERROR ALLOWANCE FOR DRIFT LENGTHS
498 E(I)=.001
        GO TO 1402
C   ERROR ALLOWANCE FOR QUADRUPOLES
499 E(I)=.1
1402 CONTINUE
        IPRINT=2
        ESCALE=100000.0
        MAXIT=40
        N=ME
C
C   VAO4A ATTEMPTS TO FIND MINIMUM OF ERROR FUNCTION(DEFINED IN CALCF)
    CALL VAO4A
C
C   PRINT OUT OF RESULTS OF MATCHING
    PRINT 9999,IA,IB
9999 FORMAT(17H1MATCHING ROUTINE,2I3)
    PRINT 496,NN,MM

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496 FORMAT(30H MATCHING SECTION FROM ELEMENT,I3,3H TO,I3)
PRINT 1401,(NEL(KK),KK=1,ME)
1401 FORMAT(18H VARIABLE ELEMENTS,9I3)
PRINT 900
900 FORMAT(36H FINAL VALUES OF VARIABLE PARAMETERS)
DO 1403 I=1,ME
K=NEL(I)
IK=ET(K)
IF(IK-1)200,200,202
202 C(K)=X(I)
GO TO 1403
200 C(K)=ABS(X(I))
1403 PRINT 1404,K,C(K)
1404 FORMAT(1X,I3,F10.3)
PRINT 1111,SO(1),SO(2),SO(9),SO(3),SO(4),SO(10)
1111 FORMAT(24H FINAL MATRIX COMPONENTS/17H HORIZONTAL PLANE/3F15.5//3F
115.5)
PRINT 1112,SO(5),SO(6),SO(7),SO(8)
1112 FORMAT(15H VERTICAL PLANE/2F15.5//2F15.5)
IF(IE)70,71,70
71 PRINT 72
72 FORMAT(19H ELLIPSE PARAMETERS,18X,48H
1HA BETA EM)
GAMMA ALP
70 CONTINUE
IF(IA-6)600,601,601
601 PRINT 602,HGI,HAI,HBI,HEPI,HG,HA,HB
602 FORMAT(34H HORIZONTAL PLANE INITIAL,14X,4F10.5/26X,6H FIN
1AL,16X,3F10.5)
IF(IA-7)600,700,600
700 HGF=(1.0+HAF**2)/HBF
PRINT 701,HGF,HAF,HBF
701 FORMAT(26X,10H REQUESTED,12X,3F10.5)
600 IF(IB-6)603,604,604
604 PRINT 605,VGI,VAI,VBI,VEPI,VG,VA,VB
605 FORMAT(34H VERTICAL PLANE INITIAL,14X,4F10.5/26X,6H FIN
1AL,16X,3F10.5)
IF(IB-7)300,800,300
800 VGF=(1.0+VAF**2)/VBF
PRINT 801,VGF,VAF,VBF
801 FORMAT(26X,10H REQUESTED,12X,3F10.5)
603 CONTINUE
300 CONTINUE
ITERC=ITERC-1
PRINT 1000,ITERC,F
1000 FORMAT(1X,11H ITERATIONS,I3,6H ERROR,F12.8)
RETURN
END

```

```

SUBROUTINE VAO4A
DIMENSION ET(30),D(30),C(30),H11(30),H12(30),H21(30),H22(30),
1V11(30),V12(30),V21(30),V22(30),H13(30),H23(30),X0(10),S0(10)
DIMENSION BETA1(30),BETA2(30),W(440),X(20),E(20),T(10)
DIMENSION NEL(30),IPLDT(85)
COMMON ET,D,C,H11,H12,H21,H22,V11,V12,V21,V22,H13,H23,AN,W,X,E
COMMON VG,VGI,VGF,VA,VAI,VAF,VB,VBI,VBF,HG,HGI,HGF,HA,HAI,HAF,HB
COMMON T,S0,HBI,HBF,HEPI,VEPI,BETA1,BETA2,ESCALE,F,P,U,V,Z
COMMON N,M,NN,MM,MMM,ME,INST,IA,IB,IPRINT,MAXIT,NEL
COMMON NE,ITERC,IE
PRINT 2001,N,IPRINT,MAXIT,ESCALE
2001 FORMAT(2H1N,I3,7H IPRINT,I3,6H MAXIT,I3,7H ESCALE,F12.1)
DDMAG=0.1*ESCALE
SCER=0.05/ESCALE
JJ=N*N+N
AAAA=3.
K=N+1
NFCC=1
DO 1 I=1,N
DO 2 J=1,N
W(K)=0.
IF(I-J)4,3,4
3 W(K)=E(I)
W(I)=ESCALE
4 K=K+1
2 CONTINUE
1 CONTINUE
ITERC=1
ISGRAD=2
CALL CALCF
5 ITONE=1
FP=F
SUM=0.
IXP=JJ
DO 6 I=1,N
IXP=IXP+1
W(IXP)=X(I)
6 CONTINUE
IDIRN=N+1
ILINE=1
7 DMAX=W(ILINE)
DACC=DMAX*SCER
GO TO (70,71),ITONE
70 DMAG=AMIN1(DDMAG,0.1*DMAX)
DMAG=AMAX1(DMAG,20.0*DACC)
DL=0.
G=DMAG
FPREV=F

```

```
IS=5
DDMAX=10.*DMAG
FA=F
DA=DL
8 DD=G-DL
DL=G
58 K=IDIRN
DO 9 I=1,N
X(I)=X(I)+DD*W(K)
K=K+1
9 CONTINUE
CALL CALCF
NFCC=NFCC+1
GO TO (10,11,12,13,14,96),IS
14 IF(F-FA)15,16,24
16 IF(G-DMAX)17,17,18
17 G=G+G
GO TO 8
18 PRINT 19
19 FORMAT(5X,44HVA04A MAXIMUM CHANGE DOES NOT ALTER FUNCTION)
GO TO 20
15 FB=F
DB=G
GO TO 21
24 FB=FA
DB=DA
FA=F
DA=G
21 GO TO (83,23),ISGRAD
23 G=DB+DB-DA
IS=1
GO TO 8
83 G=0.5*(DA+DB-(FA-FB)/(DA-DB))
IS=4
IF((DA-G)*(G-DB))25,8,8
25 IS=1
IF(ABS(G-DB)-DDMAX)8,8,26
26 G=DB+SIGN(DDMAX,DB-DA)
IS=1
DDMAX=DDMAX+DDMAX
IF(DDMAX-DMAX)8,8,27
27 DDMAX=DMAX
GO TO 8
13 IF(F-FA)28,23,23
28 FC=FB
DC=DB
29 FB=F
DB=G
```

```
GO TO 30
12 IF(F-FB)28,28,31
31 FA=F
DA=G
GO TO 30
11 IF(F-FB)32,10,10
32 FA=FB
DA=DB
GO TO 29
71 DL=1.
DDMAX=5.
FA=FP
DA=-1.
FB=FHOLD
DB=0.
G=1.
10 FC=F
DC=G
30 AZ=(DB-DC)*(FA-FC)
B=(DC-DA)*(FB-FC)
IF((AZ+B)*(DA-DC))33,33,34
33 FA=FB
DA=DB
FB=FC
DB=DC
GO TO 26
34 G=0.5*(AZ*(DB+DC)+B*(DA+DC))/(AZ+B)
DI=DB
FI=FB
IF(FB-FC)44,44,43
43 DI=DC
FI=FC
44 IF (ABS(G-DI)-DACC) 41,41,93
93 IF (ABS(G-DI)-0.03*ABS(G)) 41,41,45
45 IF ((DA-DC)*(DC-G)) 47,46,46
46 FA=FB
DA=DB
FB=FC
DB=DC
GO TO 25
47 IS=2
IF((DB-G)*(G-DC))48,8,8
48 IS=3
GO TO 8
41 F=FI
G=DI-DL
DD=SQRT((DC-DB)*(DC-DA)*(DA-DB)/(AZ+B))
DO 49 I=1,N
```

```

X(I)=X(I)+G*W(IDIRN)
W(IDIRN)=DD*W(IDIRN)
IDIRN=IDIRN+1
49 CONTINUE
W(ILINE)=W(ILINE)/DD
ILINE=ILINE+1
IF(IPRINT-1)51,50,51
50 PRINT52,ITERC,NFCC,F,(X(I),I=1,N)
52 FORMAT(1X,9HITERATION,I5,I15,16H FUNCTION VALUES,
110X,3HF =,E21.14/(5E24.14))
GO TO(51,53),IPRINT
51 GO TO (55,38),ITONE
55 IF(FPREV-F-SUM)94,95,95
95 SUM=FPREV-F
JIL=ILINE
94 IF(IDIRN-JJ)7,7,84
84 FHOLD=F
IS=6
IXP=JJ
DO 59 I=1,N
IXP=IXP+1
W(IXP)=X(I)-W(IXP)
59 CONTINUE
DD=1.
GO TO 58
96 G=0.5/(FP+F-2.*FHOLD)
IF(G)92,92,57
57 IF(SUM*G*(FP-F)**2-(FP-SUM-FHOLD)**2)37,92,92
92 J=JIL*N+1
IF(J-JJ)60,60,61
60 DO 62 I=J,JJ
K=I-N
W(K)=W(I)
62 CONTINUE
DO 97 I=JIL,N
W(I-1)=W(I)
97 CONTINUE
61 IDIRN=IDIRN-N
ITONE=2
ISGRAD=2
K=IDIRN
IXP=JJ
AAA=0.
DO 65 I=1,N
IXP=IXP+1
W(K)=W(IXP)
IF(AAA-ABS(W(K)/E(I)))66,67,67
66 AAA=ABS(W(K)/E(I))

```

```
67 K=K+1
65 CONTINUE
   DDMAG=1.
   W(N)=ESCALE/AAA
   ILINE=N
   GO TO 7
37 IXP=JJ
   AAA=0.
   F=FHOLD
   DO 99 I=1,N
     IXP=IXP+1
     X(I)=X(I)-W(IXP)
     IF(AAA*E(I)-ABS(W(IXP)))98,99,99
98 AAA=ABS(W(IXP)/E(I))
99 CONTINUE
   GO TO 72
38 AAA=AAA*(1.+DI)
72 IF(IPRINT-2)53,50,50
53 ITCR=ITCR+1
   IF(AAA-2.)89,89,76
89 IF(AAAA-2.)20,20,76
20 RETURN
76 IF(F-FP)35,78,78
78 PRINT 80
80 FORMAT (5X,37HVA04A ACCURACY LIMITED BY ERRORS IN F)
   GO TO 20
35 DDMAG=0.4*SQRT(FP-F)
   ISGRAD=1
   AAAA=AAA
   IF(ITCR-MAXIT)5,5,81
81 PRINT 82,MAXIT
82 FORMAT(15,30H ITERATIONS COMPLETED BY VA04A)
   GO TO 20
   END
```

```

SUBROUTINE CALCF
DIMENSION ET(30),D(30),C(30),H11(30),H12(30),H21(30),H22(30),
1V11(30),V12(30),V21(30),V22(30),H13(30),H23(30),XO(10),SO(10)
DIMENSION BETA1(30),BETA2(30),W(440),X(20),E(20),T(10)
DIMENSION NEL(30),IPL0T(85)
COMMON ET,D,C,H11,H12,H21,H22,V11,V12,V21,V22,H13,H23,AN,W,X,E
COMMON VG,VGI,VGF,VA,VAI,VAF,VB,VBI,VBF,HG,HGI,HGF,HA,HAI,HAF,HB
COMMON T,SO,HBI,HBF,HEPI,VEPI,BETA1,BETA2,ESCALE,F,P,U,V,Z
COMMON N,M,NN,MM,MMM,ME,INST,IA,IB,IPRINT,MAXIT,NEL
COMMON NE,ITERC,IE
C
C CALCF DEFINES AND COMPUTES ERROR FUNCTION TO BE MINIMIZED BY VA04A
C
C CALCULATION OF MATRIX COMPONENTS FOR VARIABLE ELEMENTS USING
C CURRENT VALUES OF VARIABLE PARAMETERS
DO 635 JJ=1,ME
K=NEL(JJ)
C(K)=X(JJ)
625 N=K
M=K
CALL ASSIGN
635 CONTINUE
N=ME
C
C MATRIX MULTIPLICATION OF MATRICES IN MATCHING SECTION
XO(1)=1.0
XO(2)=0.0
XO(3)=0.0
XO(4)=1.0
XO(5)=1.0
XO(6)=0.0
XO(7)=0.0
XO(8)=1.0
XO(9)=0.0
XO(10)=0.0
D0416 L=NN,MM
SO(1)=H11(L)*XO(1)+H12(L)*XO(3)
SO(2)=H11(L)*XO(2)+H12(L)*XO(4)
SO(3)=H21(L)*XO(1)+H22(L)*XO(3)
SO(4)=H21(L)*XO(2)+H22(L)*XO(4)
SO(5)=V11(L)*XO(5)+V12(L)*XO(7)
SO(6)=V11(L)*XO(6)+V12(L)*XO(8)
SO(7)=V21(L)*XO(5)+V22(L)*XO(7)
SO(8)=V21(L)*XO(6)+V22(L)*XO(8)
SO(9)=H11(L)*XO(9)+H12(L)*XO(10)+H13(L)
SO(10)=H21(L)*XO(9)+H22(L)*XO(10)+H23(L)
D0430 J=1,10
430 XO(J)=SO(J)
416 CONTINUE

```

```

NC=1
C
C   DEFINING ERROR FUNCTION FOR MATCHING ROUTINE SPECIFIED BY IA,IB
C   COMPONENTS OF T VECTOR ARE QUANTITIES THAT ARE TO BE SET TO ZERO
      IF(IA-5)300,301,302
300 IF(IA)10,11,10
C   FOR IA=0 TRIES TO FIND DISPERSIONLESS SYSTEM
      11 T(NC)=SO(9)
         NC=NC+1
         T(NC)=SO(10)
         GO TO 305
C   FOR IA=1,2,3,4 TRAJECTORY MATCHING
      10 T(NC)=SO(IA)
         GO TO 305
C   FOR IA=5 TRIES TO FIND IDENTITY SYSTEM
      301 T(NC)=SO(2)
         NC=NC+1
         T(NC)=SO(3)
         NC=NC+1
         T(NC)=1.0-ABS(SO(1))
         GO TO 305
C   CALCULATION OF ELLIPSE PARAMETERS FOR ELLIPSE MATCHING
      302 HG=XO(3)**2*HBI-2.0*XO(3)*XO(4)*HAI+XO(4)**2*HGI
         HA=-XO(1)*XO(3)*HBI+(XO(1)*XO(4)+XO(2)*XO(3))*HAI-XO(2)*XO(4)*VGI
         HB=XO(1)**2*HBI-2.0*XO(1)*XO(2)*HAI+XO(2)**2*HGI
         IF(IA-6)303,303,304
C   FOR IA=6 TRIES TO SET ALPHA TO HAF=0.0
C   FOR IA=7 TRIES TO SET ALPHA TO HAF,BETA TO HBF
      304 T(NC)=HB-HBF
         NC=NC+1
      303 T(NC)=HA-HAF
C   IF IB=0 NO MATCHING IS DONE IN VERTICAL PLANE
      305 IF(IB)12,405,12
         12 NC=NC+1
            IF(IB-5)400,401,402
C   FOR IB=1,2,3,4 TRAJECTORY MATCHING
      400 T(NC)=SO(IB+4)
         GO TO 405
C   FOR IB=5 TRIES TO FIND IDENTITY SYSTEM
      401 T(NC)=SO(6)
         NC=NC+1
         T(NC)=SO(7)
         NC=NC+1
         T(NC)=1.0-ABS(SO(5))
         GO TO 405
C   CALCULATION OF ELLIPSE PARAMETERS FOR ELLIPSE MATCHING
      402 VG=XO(7)**2*VBI-2.0*XO(7)*XO(8)*VAI+XO(8)**2*VGI
         VA=-XO(5)*XO(7)*VBI+(XO(5)*XO(8)+XO(6)*XO(7))*VAI-XO(6)*XO(8)*VGI
         VB=XO(5)**2*VBI-2.0*XO(5)*XO(6)*VAI+XO(6)**2*VGI

```

```
      IF (IB=6) 403,403,404
C     FOR IB=6 TRIES TO SET ALPHA TO VAF=0.0
C     FOR IB=7 TRIES TO SET ALPHA TO VAF, BETA TO VBF
404  T(NC)=VB-VBF
      NC=NC+1
403  T(NC)=VA-VAF
405  CONTINUE

C
C     ERROR FUNCTION IS THE LENGTH OF THE T VECTOR
626  ER=0.0
      DO 627 JL=1,NC
          A=T(JL)
627  ER=ER+A**2
      ER=SQRT(ER)
      F=ER
      RETURN
      END
```

## APPENDIX VIII: SAMPLE PROBLEM

The sample problem chosen illustrates all of the facilities available in the TRIUMF tracking and matching program. A list of the input data cards appears on page 118 .

Data cards 1 to 17 specify a 14 element  $30^\circ$  dispersionless bending system described by A. C. Paul (1964). The information contained on these cards is read in according to the schemes outlined in sections V.1.1 and V.1.2 and is printed out on page 120.

Data cards 18 and 19 request a print out of the transfer matrix components at the exit of each of the 14 elements in the vertical and horizontal planes respectively. The results are shown on pages 121 and 122. Inspection of the matrix components for the vertical plane shows that the system satisfies the parallel-to-parallel condition ( $T_{21} = -0.00086$ ) at the exit of element 14 and of course is dispersionless in this plane ( $T_{13} = T_{23} = 0.0$ ).

These matrix components agree to 3 significant figures with those given by Paul as is shown below.

Component	TRIUMF Results	Paul's Results
T11	-0.99558	-0.996
T12	10.21587 $\frac{\text{cm/rad}}{100}$	0.402 in/mr = 10.21 $\frac{\text{cm/rad}}{100}$
T21	-0.00086 $\frac{\text{rad}}{100/\text{cm}}$	-0.018 mr/in = 0.001 $\frac{\text{rad}}{100/\text{cm}}$
T22	-0.99558	-0.996

In the horizontal plane we see that the system is dispersionless to a high degree of accuracy as  $T_{13} = 0.03839$  and  $T_{23} = 0.01048$ .

Data card 20 requests trajectory tracking in the horizontal plane for a particle with initial displacement .5 cm. and initial slope .5 rad./100 and zero momentum deviation. The output is shown on page 123. The second column gives the distance in meters along the optic axis to the exit of each element in the system so that we see the total length of the system is about 7.88 meters.

Data card 21 requests the same job as above except that  $dp/p = 0.01$ . The results are shown on page 124. Comparison of the final displacement and slope ( $-1.57012$  cm.,  $-0.97872 \frac{\text{rad.}}{100}$ ) with those on page 123 ( $-1.57050$  cm.,  $-0.97883 \frac{\text{rad.}}{100}$ ) shows that the system is effectively dispersionless for this momentum deviation.

Data card 22 requests a plot of the same trajectory as above and the output is shown on page 125.

Data card 23 requests ellipse tracking in the vertical plane for a phase space ellipse with initial coefficients  $\alpha_0 = 0.0$ ,  $\beta_0 = 33.866$  cm./rad./100, and an emittance of .01493 cm. The output is on page 126. Columns 5 and 6 give the maximum displacement DMAX (cm.) and maximum slope SMAX (rad./100) of any particle in the beam. These results agree with those given by Paul for the exit of the system. His results are a final maximum displacement of 0.29 in = 0.74 cm. and final maximum slope of

0.21 mr = 0.021 rad./100 as compared to 0.73648 cm. and 0.02091 rad./100 for the TRIUMF program.

Data card 24 requests a beam envelope trace for the same initial ellipse as above and this is shown on page 127 .

Data card 25 requests matching to a dispersionless system in the horizontal plane and matching to a parallel-to-parallel condition to the vertical plane. The print out from subroutine VAO4A which always precedes the print out of the matching results is shown on page 128. Page 129 shows the results of the matching attempt. Because the initial beam transport system essentially already satisfied the matching requirements the iteration procedure quickly converged (3 iterations) to the solution shown.

Data card 26 requests a change to a completely new beam transport system.

Data cards 27 to 36 specify an initial guessed system of fixed length 40 meters and of design energy 500 Mev as shown on page 130 .

Data card 37 requests matching to an identity system. The result on page 131 shows that the iteration procedure found the solution minus one times the unit matrix in both planes.

Data cards 38 and 39 request the graphing of one ellipse at the exit of the 9<sup>th</sup> (last) element of the system. The initial ellipse coefficient are  $\alpha_0 = -2.0$ ,  $\beta_0 = 2.0$  cm./ $\frac{\text{rad.}}{100}$  , and emittance  $.022$  cm. $\frac{\text{rad.}}{100}$  . As expected for

an identity system the results on page 132 show that the final ellipse is practically identical with the initial ellipse.

Data cards 40, 41 and 42 request ellipse matching to a waist or bust in the horizontal plane and ellipse matching in the vertical plane to an ellipse with requested coefficients  $\alpha_r = 2.0$  ,  $\beta_r = 2.0 \text{ cm.}/\frac{\text{rad.}}{100}$  . In both planes the initial ellipse is taken to have coefficients  $\alpha_o = 0.0$ ,  $\beta_o = 4.0 \text{ cm.}/\frac{\text{rad.}}{100}$  and an emittance of  $.022 \text{ cm.}/\frac{\text{rad.}}{100}$  . Thus with the identity system as an initial guessed system the program after 7 iterations arrived at the solution shown on page 133 .

Data card 43 requests that the design momentum of the system be changed to  $0.95423 \text{ Gev./c}$  (corresponding to an energy of 400 Mev.).

Data cards 44 to 51 request that the drift lengths between the quadrupole magnets be changed to 7.0 meters so that the total system length becomes 30 meters.

Data card 52 requests matching to an identity system in both planes. The result shown on page 134 is that the program created a 30 meter identity system for 400 Mev. particles.

Data card 53 requests trajectory tracking in the vertical plane. The result is shown on page 135.

Data card 54 requests termination of the program. The time for compilation and execution of the program is given as 5.446 minutes.

DATA CARDS FOR SAMPLE PROBLEM					
550.0	938.2	14	0		1
3.0	14.9	15.7			2
.0	.0				3
1.0	.3048				4
2.0	.4318	-548.622			5
1.0	.3048				6
2.2	.4318	400.197			7
1.0	1.3208				8
2.1	.4064	422.4409			9
1.0	1.3208				10
2.2	.4318	400.197			11
1.0	.3048				12
2.0	.4318	-548.622			13
1.0	.3048				14
3.0	14.9	15.7			15
.0	.0				16
1.0	.6096				17
1 5 1					18
1 5-1					19
1 0-1	.5	.5	.0		20
1 0-1	.5	.5	.01		21
1 1-1	.5	.5	.01		22
1 2 1	.0	33.866	.01493		23
1 4 1	.0	33.866	.01493		24
2 0 3	1.0	14.0			25
3					26
500.0	938.2	9	0		27
2.3	.25	-100.0			28
1.0	9.5				29
2.2	.50	100.0			30
1.0	9.5				31
2.2	.50	-100.0			32
1.0	9.5				33
2.2	.50	100.0			34
1.0	9.5				35
2.3	.25	-100.0			36
2 5 5	1.0	9.0			37
1 3 1	-2.0	2.0	.022		38
1 9					39
2 6 7	1.0	9.0			40
0.0	4.0		.022		41
0.0	4.0		.022	2.0	42
3-1	.95423			2.0	43
3 2					44
1.0	7.0				45
3 4					46
1.0	7.0				47
3 6					48

1.0	7.0			49
3 8				50
1.0	7.0			51
2 5 5	1.0	9.0		52
1 0 1	.5	.5	.0	53
4				54

DESIGN ENERGY 550.00000 MEV  
PARTICLE REST ENERGY 938.19995 MEV MOMENTUM 1.15521 GEV/C  
NO. OF ELEMENTS 14

INITIAL SYSTEM PARAMETERS  
UNITS

LENGTH-METERS  
FIELD GRADIENT-GAUSS/CM  
FIELD-KILOGAUSS  
ANGLE-DEGREES

1	BENDING MAGNET	EF LENGTH	0.6383	FIELD	15.70000	BENDING ANGLE	14.90000
	ENTRANCE ANGLE	0.0					
	EXIT ANGLE	0.0					
2	DRIFT SPACE	LENGTH	0.3048				
3	QUADRUPOLE-DH	EF LENGTH	0.4318	FIELD GRADIENT	-548.62183		
4	DRIFT SPACE	LENGTH	0.3048				
5	QUADRUPOLE-FH	EF LENGTH	0.4318	FIELD GRADIENT	400.19678		
6	DRIFT SPACE	LENGTH	1.3208				
7	QUADRUPOLE-FH	EF LENGTH	0.4064	FIELD GRADIENT	422.44067		
8	DRIFT SPACE	LENGTH	1.3208				
9	QUADRUPOLE-FH	EF LENGTH	0.4318	FIELD GRADIENT	400.19678		
10	DRIFT SPACE	LENGTH	0.3048				
11	QUADRUPOLE-DH	EF LENGTH	0.4318	FIELD GRADIENT	-548.62183		
12	DRIFT SPACE	LENGTH	0.3048				
13	BENDING MAGNET	EF LENGTH	0.6383	FIELD	15.70000	BENDING ANGLE	14.90000
	ENTRANCE ANGLE	0.0					
	EXIT ANGLE	0.0					
14	DRIFT SPACE	LENGTH	0.6096				

VERTICAL PLANE  
TRANSFER MATRIX COMPONENTS

	T11	T12	T21	T22	T13	T23
1	1.00000	0.63829	0.0	1.00000	0.0	0.0
2	1.00000	0.94309	0.0	1.00000	0.0	0.0
3	0.87019	1.23361	-0.58791	0.31574	0.0	0.0
4	0.69099	1.32985	-0.58791	0.31574	0.0	0.0
5	0.49685	1.60147	-0.32579	0.96258	0.0	0.0
6	0.06654	2.87285	-0.32579	0.96258	0.0	0.0
7	-0.06377	3.53997	-0.32518	2.36990	0.0	0.0
8	-0.49327	6.67013	-0.32518	2.36990	0.0	0.0
9	-0.68679	8.38307	-0.58558	5.69164	0.0	0.0
10	-0.86528	10.11788	-0.58558	5.69164	0.0	0.0
11	-0.99477	11.15480	-0.00086	-0.99558	0.0	0.0
12	-0.99503	10.85134	-0.00086	-0.99558	0.0	0.0
13	-0.99558	10.21587	-0.00086	-0.99558	0.0	0.0
14	-0.99611	9.60897	-0.00086	-0.99558	0.0	0.0

HORIZONTAL PLANE  
TRANSFER MATRIX COMPONENTS

	T11	T12	T21	T22	T13	T23
1	0.96638	0.63112	-0.10476	0.96638	8.25263	25.71284
2	0.93445	0.92567	-0.10476	0.96638	16.08989	25.71284
3	1.01397	1.48726	0.48123	1.69206	29.87360	39.53630
4	1.16065	2.00300	0.48123	1.69206	41.92426	39.53630
5	1.25123	2.51946	-0.06845	0.66137	54.45650	17.57063
6	1.16082	3.39300	-0.06845	0.66137	77.66377	17.57063
7	1.03032	3.35119	-0.56404	-0.86402	77.66563	-17.56155
8	0.28533	2.20999	-0.56404	-0.86402	54.47034	-17.56155
9	0.02238	1.63330	-0.63417	-1.74106	41.94055	-39.53407
10	-0.17091	1.10762	-0.63417	-1.74106	29.89056	-39.53407
11	-0.48022	0.47241	-0.83000	-1.26587	16.11015	-25.69937
12	-0.73320	0.08658	-0.83000	-1.26587	8.27699	-25.69937
13	-1.23238	-0.71525	-0.72528	-1.23238	0.03200	0.01048
14	-1.67451	-1.46650	-0.72528	-1.23238	0.03839	0.01048

HORIZONTAL PLANE  
TRAJECTORY TRACKING AT DP/P= 0.0

	DISTANCE	DISPLACEMENT	SLOPE
	0.0	0.50000	0.50000
1	0.63829	0.79875	0.43081
2	0.94309	0.93006	0.43081
3	1.37489	1.25061	1.08664
4	1.67969	1.58182	1.08664
5	2.11149	1.88535	0.29646
6	3.43229	2.27691	0.29646
7	3.83869	2.19075	-0.71403
8	5.15948	1.24766	-0.71403
9	5.59128	0.83034	-1.18762
10	5.89608	0.46835	-1.18762
11	6.32788	-0.00390	-1.04793
12	6.63268	-0.32331	-1.04793
13	7.27097	-0.97381	-0.97883
14	7.88057	-1.57050	-0.97883

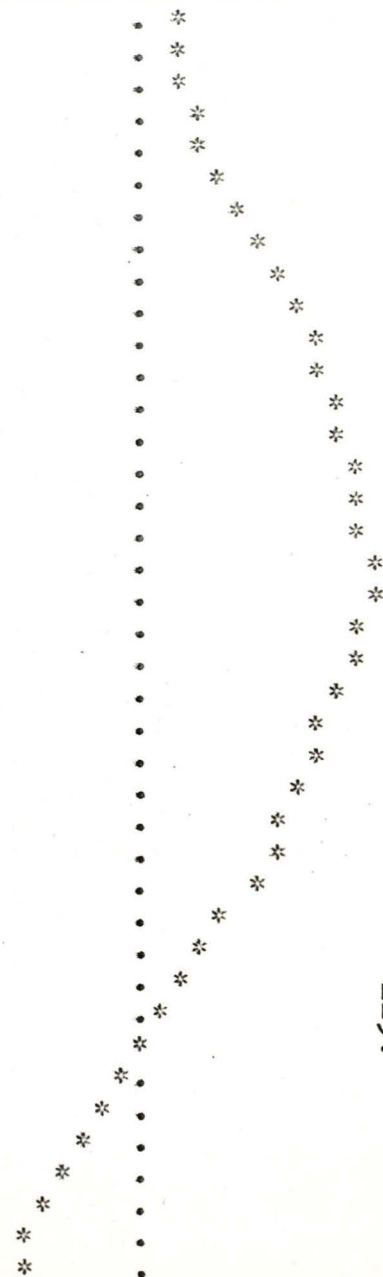
HORIZONTAL PLANE  
TRAJECTORY TRACKING AT DP/P= 0.010

	DISTANCE	DISPLACEMENT	SLOPE
	0.0	0.50000	0.50000
1	0.63829	0.88127	0.68794
2	0.94309	1.09096	0.68794
3	1.37489	1.54935	1.48201
4	1.67969	2.00106	1.48201
5	2.11149	2.42991	0.47216
6	3.43229	3.05355	0.47216
7	3.83869	2.96741	-0.88965
8	5.15948	1.79236	-0.88965
9	5.59128	1.24974	-1.58296
10	5.89608	0.76726	-1.58296
11	6.32788	0.15720	-1.30493
12	6.63268	-0.24054	-1.30493
13	7.27097	-0.97349	-0.97872
14	7.88057	-1.57012	-0.97872

HORIZONTAL PLANE  
 TRAJECTORY PLOTTING AT DP/P= 0.010

	DISTANCE	DISPLACEMENT	SLOPE
1	0.0	0.50000	0.50000
1	0.20000	0.60637	0.56315
1	0.40000	0.72501	0.62257
1	0.60000	0.85513	0.67786
2	0.80000	0.99252	0.68794
3	1.00000	1.13265	0.77799
3	1.20000	1.32214	1.12582
4	1.40000	1.58657	1.48201
4	1.60000	1.88297	1.48201
5	1.80000	2.16390	1.22148
5	2.00000	2.36172	0.74985
6	2.20000	2.47170	0.47216
6	2.40000	2.56614	0.47216
6	2.60000	2.66057	0.47216
6	2.80000	2.75500	0.47216
6	3.00000	2.84943	0.47216
6	3.20000	2.94387	0.47216
6	3.40000	3.03830	0.47216
7	3.60000	3.08537	-0.09362
7	3.80000	2.99938	-0.76310
8	4.00000	2.82390	-0.88965
8	4.20000	2.64597	-0.88965
8	4.40000	2.46804	-0.88965
8	4.60000	2.29011	-0.88965
8	4.80000	2.11218	-0.88965
8	5.00000	1.93425	-0.88965
9	5.20000	1.75480	-0.96428
9	5.40000	1.52696	-1.30628
10	5.60000	1.23595	-1.58296
10	5.80000	0.91936	-1.58296
11	6.00000	0.60825	-1.48134
11	6.20000	0.32657	-1.34888
12	6.39999	0.06310	-1.30493
12	6.59999	-0.19789	-1.30493
13	6.79999	-0.45244	-1.22711
13	6.99999	-0.68795	-1.12664
13	7.19999	-0.90260	-1.01869
14	7.39999	-1.09977	-0.97872
14	7.59999	-1.29552	-0.97872
14	7.79999	-1.49126	-0.97872

TRAJECTORY PLOT



VERTICAL PLANE  
ELLIPSE TRACKING  
EMITTANCE 0.01493

	GAMMA	ALPHA	BETA	DMAX	SMAX
	0.02953	0.0	33.86600	0.71107	0.02100
1	0.02953	-0.01885	33.87802	0.71120	0.02100
2	0.02953	-0.02785	33.89226	0.71134	0.02100
3	11.70820	17.31395	25.68909	0.61930	0.41809
4	11.70820	13.74530	16.22223	0.49214	0.41809
5	3.62187	5.43631	8.43582	0.35489	0.23254
6	3.62187	0.65254	0.39367	0.07666	0.23254
7	3.74694	-0.95002	0.50776	0.08707	0.23652
8	3.74694	-5.89898	9.55392	0.37768	0.23652
9	12.56945	-15.02893	18.04921	0.51911	0.43320
10	12.56945	-18.86008	28.37857	0.65092	0.43320
11	0.02929	0.29886	37.18683	0.74512	0.02091
12	0.02929	0.28993	37.00737	0.74332	0.02091
13	0.02929	0.27124	36.64917	0.73971	0.02091
14	0.02929	0.25338	36.32936	0.73648	0.02091

VERTICAL PLANE  
 BEAM ENVELOPE TRACE  
 EMITTANCE 0.01493

		GAMMA	ALPHA	BETA	DMAX	SMAX	BEAM ENVELOPE
1	0.0	0.02953	-0.0	33.86600	0.71107	0.02100	. *
1	0.2	0.02953	-0.00591	33.86717	0.71108	0.02100	. *
1	0.4	0.02953	-0.01181	33.87071	0.71112	0.02100	. *
1	0.6	0.02953	-0.01772	33.87662	0.71118	0.02100	. *
2	0.8	0.02953	-0.02362	33.88489	0.71127	0.02100	. *
3	1.0	0.24706	2.70844	33.73946	0.70974	0.06073	. *
3	1.2	4.40162	11.60447	30.82127	0.67835	0.25635	. *
4	1.4	11.70820	17.01991	24.82686	0.60882	0.41809	. *
4	1.6	11.70820	14.67829	18.48723	0.52537	0.41809	. *
5	1.8	8.67023	10.69053	13.29694	0.44556	0.35979	. *
5	2.0	5.05209	6.96966	9.81300	0.38276	0.27464	. *
6	2.2	3.62187	5.11573	7.50183	0.33467	0.23254	. *
6	2.4	3.62187	4.39135	5.60042	0.28916	0.23254	. *
6	2.6	3.62187	3.66698	3.98875	0.24403	0.23254	. *
6	2.8	3.62187	2.94261	2.66683	0.19954	0.23254	. *
6	3.0	3.62187	2.21823	1.63467	0.15622	0.23254	. *
6	3.2	3.62187	1.49386	0.89225	0.11542	0.23254	. *
6	3.4	3.62187	0.76949	0.43958	0.08101	0.23254	. *
7	3.6	3.50323	-0.00067	0.28545	0.06528	0.22870	. *
7	3.8	3.67333	-0.78647	0.44061	0.08111	0.23419	. *
8	4.0	3.74694	-1.55445	0.91176	0.11667	0.23652	. *
8	4.2	3.74694	-2.30384	1.68342	0.15854	0.23652	. *
8	4.4	3.74694	-3.05323	2.75483	0.20280	0.23652	. *
8	4.6	3.74694	-3.80261	4.12600	0.24820	0.23652	. *
8	4.8	3.74694	-4.55200	5.79692	0.29419	0.23652	. *
8	5.0	3.74694	-5.30139	7.76760	0.34054	0.23652	. *
9	5.2	4.26717	-6.47347	10.05485	0.38745	0.25241	. *
9	5.4	7.65094	-10.04281	13.31313	0.44583	0.33798	. *
10	5.6	12.56945	-15.13844	18.31203	0.52287	0.43320	. *
10	5.8	12.56945	-17.65230	24.87018	0.60935	0.43320	. *
11	6.0	7.47189	-15.42061	31.95912	0.69076	0.33400	. *
11	6.2	1.14467	-6.37730	36.40340	0.73723	0.13073	. *
12	6.4	0.02929	0.29675	37.14388	0.74469	0.02091	. *
12	6.6	0.02929	0.29089	37.02635	0.74351	0.02091	. *
13	6.8	0.02929	0.28503	36.91116	0.74235	0.02091	. *
13	7.0	0.02929	0.27918	36.79832	0.74121	0.02091	. *
13	7.2	0.02929	0.27332	36.68784	0.74010	0.02091	. *
14	7.4	0.02929	0.26746	36.57967	0.73901	0.02091	. *
14	7.6	0.02929	0.26160	36.47385	0.73794	0.02091	. *
14	7.8	0.02929	0.25574	36.37039	0.73689	0.02091	. *

```
N 3 IPRINT 2 MAXIT 40 ESCALE 100000.0
ITERATION 1 43 FUNCTION VALUES F = 0.72526372969151E-02
0.40053466796875E 03 0.42243969726562E 03 0.40020263671875E 03
ITERATION 2 59 FUNCTION VALUES F = 0.14537223614752E-02
0.40034033203125E 03 0.42242651367187E 03 0.40035058593750E 03
ITERATION 3 70 FUNCTION VALUES F = 0.11519445106387E-02
0.40035766601562E 03 0.42242480468750E 03 0.40035034179687E 03
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MATCHING ROUTINE 0 3  
MATCHING SECTION FROM ELEMENT 1 TO 14  
VARIABLE ELEMENTS 5 7 9  
FINAL VALUES OF VARIABLE PARAMETERS

5 400.358  
7 422.425  
9 400.350

FINAL MATRIX COMPONENTS

HORIZONTAL PLANE

-1.67501            -1.46821            -0.00143

-0.72533            -1.23279            0.00023

VERTICAL PLANE

-0.99546            9.61308

-0.00099            -0.99500

ITERATIONS 3 ERROR 0.00115194

DESIGN ENERGY 500.00000 MEV  
PARTICLE REST ENERGY 938.19995 MEV MOMENTUM 1.09004 GEV/C  
NO. OF ELEMENTS 9

INITIAL SYSTEM PARAMETERS  
UNITS

LENGTH-METERS

FIELD GRADIENT-GAUSS/CM

FIELD-KILOGAUSS

ANGLE-DEGREES

1	QUADRUPOLE-DH	EF LENGTH	0.2500	FIELD GRADIENT	-100.00000
2	DRIFT SPACE	LENGTH	9.5000		
3	QUADRUPOLE-FH	EF LENGTH	0.5000	FIELD GRADIENT	100.00000
4	DRIFT SPACE	LENGTH	9.5000		
5	QUADRUPOLE-DH	EF LENGTH	0.5000	FIELD GRADIENT	-100.00000
6	DRIFT SPACE	LENGTH	9.5000		
7	QUADRUPOLE-FH	EF LENGTH	0.5000	FIELD GRADIENT	100.00000
8	DRIFT SPACE	LENGTH	9.5000		
9	QUADRUPOLE-DH	EF LENGTH	0.2500	FIELD GRADIENT	-100.00000

MATCHING ROUTINE 5 5  
MATCHING SECTION FROM ELEMENT 1 TO 9  
VARIABLE ELEMENTS 1 3 5 7 9  
FINAL VALUES OF VARIABLE PARAMETERS

1 -104.302  
3 104.606  
5 -104.607  
7 104.607  
9 -104.914

FINAL MATRIX COMPONENTS

HORIZONTAL PLANE

-1.00003 0.00016 0.0

-0.00001 -0.99996 0.0

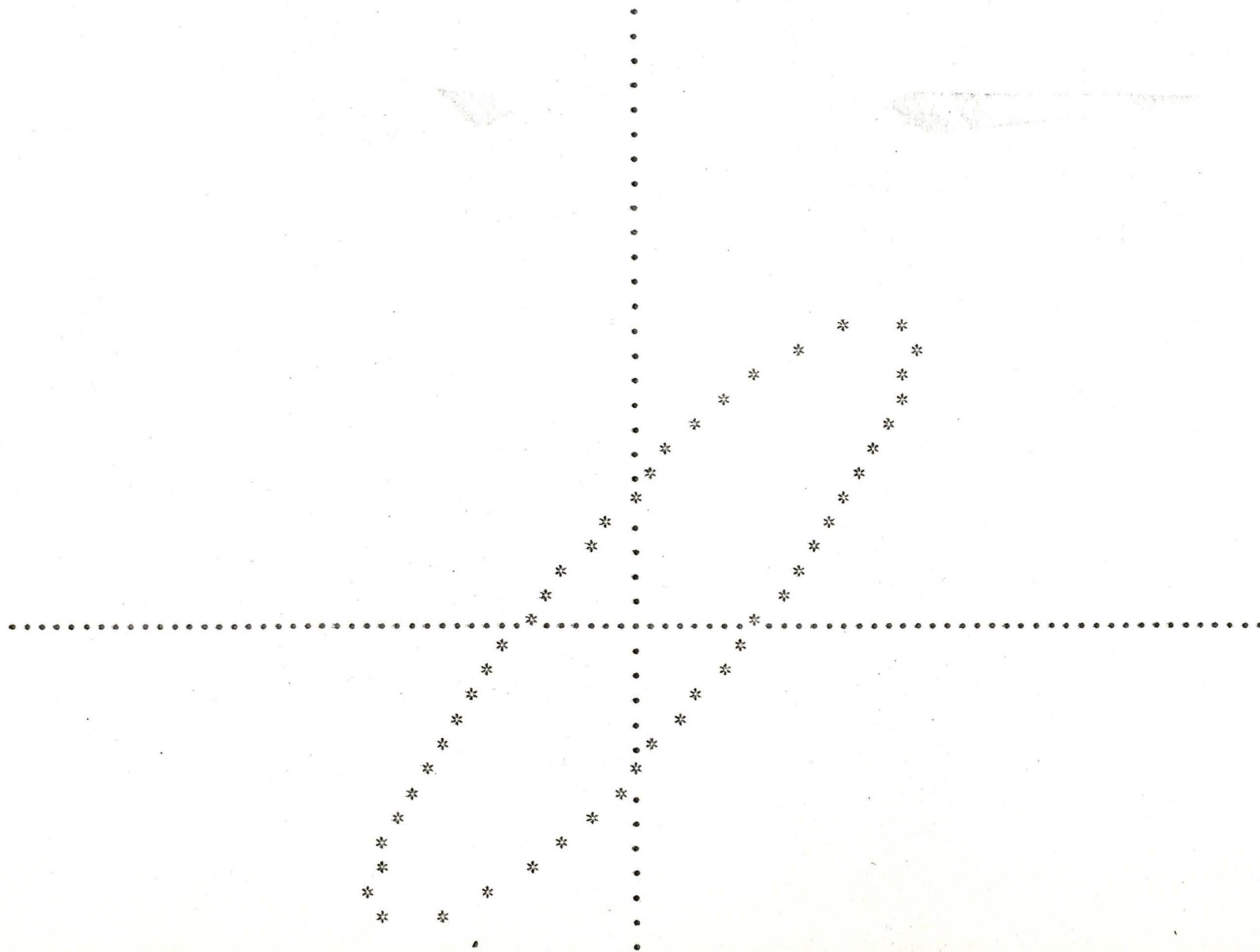
VERTICAL PLANE

-0.99995 0.00075

0.00000 -1.00005

ITERATIONS 17 ERROR 0.00009036

9 GAMMA 2.50023 ALPHA -1.99812 BETA 1.99681  
EMITTANCE 0.02200 MAX. DISPLACEMENT 0.20959 MAX. SLOPE 0.23453  
UNITS  
ABSCISSA-ONE DIVISION 0.01117 CM  
ORDINATE-ONE DIVISION 0.01876 RAD/100



MATCHING ROUTINE 6 7  
 MATCHING SECTION FROM ELEMENT 1 TO 9  
 VARIABLE ELEMENTS 1 3 5 7 9  
 FINAL VALUES OF VARIABLE PARAMETERS

1 -679.314  
 3 94.690  
 5 -104.075  
 7 101.994  
 9 -883.232

FINAL MATRIX COMPONENTS

HORIZONTAL PLANE

-0.01224 2.09062 0.0

-0.47806 -0.04463 0.0

VERTICAL PLANE

-0.67661 -0.82419

0.77960 -0.52833

ELLIPSE PARAMETERS

HORIZONTAL PLANE

INITIAL

GAMMA	ALPHA	BETA	EM
0.25000	0.0	4.00000	0.02200
0.91468	-0.00007	1.09327	

FINAL

0.25000 0.0

VERTICAL PLANE

INITIAL

0.25000	0.0	4.00000	0.02200
2.50087	2.00107	2.00102	

FINAL

2.50087 2.00107 2.00102

REQUESTED

2.50000 2.00000 2.00000

ITERATIONS 7 ERROR 0.00013674

MATCHING ROUTINE 5 5  
MATCHING SECTION FROM ELEMENT 1 TO 9  
VARIABLE ELEMENTS 1 3 5 7 9  
FINAL VALUES OF VARIABLE PARAMETERS

1 -123.001  
3 122.810  
5 -122.808  
7 122.806  
9 -122.663

FINAL MATRIX COMPONENTS

HORIZONTAL PLANE

-1.00005 0.00005 0.0

-0.00004 -0.99995 0.0

VERTICAL PLANE

-1.00002 0.00055

0.00004 -0.99998

ITERATIONS 20 ERROR 0.00010545

VERTICAL PLANE  
TRAJECTORY TRACKING AT DP/P= 0.0

	DISTANCE	DISPLACEMENT	SLOPE
	0.0	0.50000	0.50000
1	0.25000	0.61847	0.44587
2	7.25000	3.73954	0.44587
3	7.75000	4.14787	1.20059
4	14.75000	12.55197	1.20059
5	15.25000	12.54217	-1.23946
6	22.25000	3.86596	-1.23946
7	22.75000	3.42416	-0.54191
8	29.75000	-0.36924	-0.54191
9	30.00000	-0.49973	-0.49997

/8

ELAPSED TIME = 0005.446 MINUTES

Surname: TAUTZ Given Names: MAURICE FRANCIS

Place of Birth: VICTORIA, B.C. Date of Birth: OCTOBER 29, 1941

Educational Institutions Attended, with Dates of Entering and Leaving:

UNIVERSITY OF VICTORIA 1960 to 1964

Degrees, Diplomas, Etc., Awarded, with Dates and Names of Institutions:

B.Sc. 1964 UNIVERSITY OF VICTORIA, VICTORIA

Honours and Awards:

Publications: