SPREAD OF A SYMMETRIC RANDOM WALK

(Amer. Math. Monthly Advanced Problem 6665)

by

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6665. *Proposed by Jose Luis Palacios, New Jersey Institute of Technology, Newark, NJ, and Dennis P. Sandell, Swedish University of Agricultural Sciences, Garpenberg, Sweden.*

Let S_n $(n \geq 0)$ be a simple symmetric random walk, i.e., $S_0 = 0$ and $S_n = X_1 + X_2 + \cdots + X_n$ for $n > 0$, where the X_i are independent identically distributed random variables with $P(X_i = 1) = P(X_i = -1) = \frac{1}{2}$. Let N be an arbitrary positive integer and let T be the first time that the difference between the maximum and minimum of the random walk is N, i.e., let

$$T = \min\left\{n: \max_{0\leq k\leq n} S_k - \min_{0\leq k\leq n} S_k = N\right\}.$$

Find the expected value of T.

Solution by Bruce R. Johnson, University of Victoria, Victoria, B.C., Canada. We will show that $E(T) = N(N+1)/2$.

Let $\max_{0\leq k\leq n} S_k - \min_{0\leq k\leq n} S_k$ be called the *spread* of the random walk after n steps. For $i \in \{1,2,\ldots,N\}$ define

Y_i = the number of steps after the spread first reaches $i-1$ until the spread first reaches i.

When the spread first reaches $i-1$, the random walk is at either a current minimum or maximum and is $i-1$ units away from the other extreme. Thus, Y_i has the same probability distribution as the *duration of play* in the classical gambler's ruin problem where

(a) the gambler is just as likely to win one unit as lose one unit on each play,

(b) the gambler has initial capital 1 unit, and

(c) the game continues until the gambler's capital is either reduced to zero or increased to $i+1$.

Therefore, from gambler's ruin theory[1] (See pp. 245–246 of K.L. Chung's *Elementary Probability Theory with Stochastic Processes*, 3rd. ed., Springer–Verlag, New York, 1979; or pp. 348–349 of W. Feller's *An Introduction to Probability Theory and its Applications*, Vol. 1, 3rd. ed., Wiley, New York, 1968.)

$E(Y_i)$ = expected duration of play $= (1)((i+1)-1) = i$.

Hence,

$$E(T) = E(Y_1 + Y_2 + \cdots + Y_N) = E(Y_1) + E(Y_2) + \cdots + E(Y_N)$$
$$= 1 + 2 + \cdots + N = N(N+1)/2.$$

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[1]Letting e_k denote the expected duration of play for a gambler with initial capital k units, we condition on the outcome of the first play to obtain the difference equations

$$e_k = (e_{k-1}+1)\tfrac{1}{2} + (e_{k+1}+1)\tfrac{1}{2} \quad \text{for} \quad k = 1, 2, \cdots, i.$$

Expressing these equations in the form

$$e_k - e_{k-1} = e_{k+1} - e_k + 2$$

and equating the telescoping sum of left–hand–sides to the telescoping sum of right–hand–sides, we obtain

$$e_i - e_0 = e_{i+1} - e_1 + 2i.$$

The boundary conditions $e_0 = 0 = e_{i+1}$ and the symmetry condition $e_1 = e_i$ now yield the desired result $e_1 = i$.