

Relative Equilibria in The Curved N -Body Problem

by

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B.Sc., King Abdulaziz University, 2008

M.Sc., New South Wales University, 2011

Graduate Certificate in Learning and Teaching in Higher Education, University of
Victoria, 2018

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ABSTRACT

We consider the curved N -body problem, $N > 2$, on a surface of constant Gaussian curvature $\kappa \neq 0$; i.e., on spheres \mathbb{S}_κ^2 , for $\kappa > 0$, and on hyperbolic manifolds \mathbb{H}_κ^2 , for $\kappa < 0$. Our goal is to define and study relative equilibria, which are orbits whose mutual distances remain constant during the motion. We find new relative equilibria in the curved N -body problem for $N = 4$, and see whether bifurcations occur when passing through $\kappa = 0$. After obtaining a criterion for the existence of quadrilateral configurations on the equator of the sphere, we study two restricted 4-body problems: One in which two bodies are massless, and the second in which only one body is massless. In the former we prove the existence for square-like relative equilibria, whereas in the latter we discuss the existence of kite-shaped relative equilibria.

We will further study the 5-body problem on surfaces of constant curvature. Four of the masses arranged at the vertices of a square, and the fifth mass at the north pole of \mathbb{S}_κ^2 , when the curvature is positive, it is shown that relative equilibria exist when the four masses at the vertices of the square are either equal or two of them are infinitesimal, such that they do not affect the motion of the remaining three masses. In the hyperbolic case \mathbb{H}_κ^2 , $\kappa < 0$, there exist two values for the angular velocity which produce negative elliptic relative equilibria when the masses at the vertices of the square are equal. We also show that the square pyramidal relative equilibria with non-equal masses do not exist in \mathbb{H}_κ^2 .

Based on the work of Florin Diacu on the existence of relative equilibria for 3-body problem on the equator of \mathbb{S}_κ^2 , we investigate the motion of more than three bodies. Furthermore, we study the motion of the negative curved 2- and 3-centre problems on the Poincaré upper semi-plane model. Using this model, we prove that the 2-centre problem is integrable, and we study the dynamics around the equilibrium point. Further, we analyze the singularities of the 3-centre problem due to the collision; i.e., the configurations for which at least two bodies have identical coordinates.

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DEDICATION

I dedicate my work to my beloved late mother Norah and the distinguished Professor Florin Diacu.

Chapter 1

Introduction

This dissertation deals with relative equilibria in the curved N -body problem of celestial mechanics when the parameter κ changes sign, where κ is the constant Gaussian curvature of the space in which the bodies are moving ($\kappa < 0$ corresponds to the hyperbolic space, $\kappa = 0$ to the Euclidean space, and $\kappa > 0$ to the elliptic space).

The classical N -body problem has a long history. Isaac Newton first proposed it in 1687 in his first edition of *Principia* in the context of the moon's motion. He assumed that universal gravitation acts between celestial bodies (reduced to point masses) in direct proportion with the product of the masses and in inverse proportion with the square of the distance. The study of the N -body problem was further advanced by Bernoulli, Lagrange, Laplace, Euler, Cauchy, Jacobi, Dirichlet, Poincaré and many physicists and mathematicians.

The idea of extending the gravitational force between point masses to spaces of constant curvature occurred soon after the discovery of hyperbolic geometry. In the 1830s, independently of each other, Bolyai and Lobachevsky realized that there must be an intimate connection between the laws of physics and the geometry of the universe, [4], [27], [23]. A few years earlier, Gauss had interpreted Newton's gravitational law as stating that the attracting force between bodies is inversely proportional with the area of the sphere of radius equal to the distance between the point masses (i.e. proportional to $1/r^2$, where r is the distance). Using this idea, Bolyai and Lobachevsky suggested that, should space be hyperbolic, the attracting force between bodies must be inversely proportional to the hyperbolic area of the corresponding hyperbolic sphere (i.e. proportional to $1/\sinh^2(|\kappa|^{1/2}r)$, where r is the distance and $\kappa < 0$ is the curvature of the hyperbolic space). This is equivalent to saying that, in hyperbolic space, the potential that describes the gravitational force is proportional to $\coth(|\kappa|^{1/2}r)$.

The above analytic expression of the potential was first introduced by Schering, [33], [34], and then extended to elliptic space by Killing, [19], [20], [21]. But with no ways of checking the validity of this generalization of the gravitational force, it was unclear whether the cotangent potential had any physical meaning, the more so since Lipschitz had proposed a different extension of the law, which turned out to be short lived, [26]. The breakthrough came at the dawn of the 20th century when Liebmann made two important discoveries, [24], [25]. He showed that two basic properties of the Newtonian potential are also satisfied by the cotangent potential: (1) in the Kepler problem, which studies the motion of one body around a fixed centre, the potential is a harmonic function (i.e. a solution of the Laplace equation in the Euclidean case, but of the Laplace-Beltrami equation in the non-flat case); (2) both in the flat and the non-flat cases, all bounded orbits of the Kepler problem are closed, a property discovered by Bertrand for the Newtonian law, [3]. These similarities between the flat and the curved problem convinced the scientific community that the cotangent potential was the natural way to express gravity in spaces of constant curvature.

The curved N -body problem became somewhat neglected after the birth of general relativity, but was revived after the discretization of Einstein's equation showed that an N -body problem in spaces of variable curvature is too complicated to be treated with analytical tools. In the 1990s, the Russian school of celestial mechanics considered both the curved Kepler and the curved 2-body problems, [22], [35]. After understanding that, unlike in the Euclidean case, these problems are not equivalent, the latter failing to be integrable, [35], the 2-body case was intensively studied by several researchers of this school. More recently, the work of Diacu, Santoprete, and Pérez-Chavela considered the curved N -body problem for $N > 2$ in a new framework, leading to many interesting results, [5], [8], [7], [10], [9], [13], [14], [15], [32]. Other researchers developed these ideas further, [17] [28], [29], [36], [38], and the problem is growing in popularity.

To establish the geometric nature of the physical space in the 17th century, Isaac Newton developed the equation of motion of the N -body problem in Euclidean space. This equation is not similar to the equations we use on a daily basis. The main point of his work was to use the connection between geometry and dynamics. In the later years, physicists agreed to approximate macrocosmic reality using constant Gaussian curvature. In this way, the understanding of the geometric nature of the universe becomes the process of finding (using astronomical observations) existence of the orbits that are mathematically proved to exist. Everything mentioned above is possible in the

case of extending Newton's gravitational law to 2-dimensional spheres and 2-dimensional hyperbolic manifolds altogether with relevant mathematical proof that the existence of solutions that are specific to only one of the negative, zero, or positive constant Gaussian curvature spaces, but not to the other two.

The Newtonian's N -body problem extended to a field of nonzero constant Gaussian curvature is very important for the understanding geometry of physical space. The better understanding of the dynamics of the classical Euclidean case may come as a consequence of analysis of this system when the curvature tends to zero. Many branches of mathematics use a similar approach in problem solving.

1.1 Summary and organization

This dissertation builds on the work of Florin Diacu and Ernesto Pérez-Chavela on curved spaces. In what follows, we will lay the mathematical background of the curved N -body problem and discuss in details the original results we have obtained, then end this dissertation with a proposal for future work. In Chapter 2, we will derive the equations of motion in curved space ($\kappa \neq 0$), and show how they can be written to include the flat case ($\kappa = 0$). Furthermore, we outline the bifurcations that occur for the integrals of motion when the curvature parameter passes through zero. Chapter 3 deals with the bifurcation of Lagrangian solutions in the in the 3-and 4-body problems. In this chapter, we also prove three results. The first is a criterion for the existence of quadrilateral relative equilibria on the equator of the sphere. The second shows that if two bodies are massless and the other two are equal, then square-like relative equilibria exists on spheres, but—surprisingly—not on hyperbolic spheres. The element of surprise arises from the fact that, in the general problem, square-like equilibria exist both on the hyperbolic sphere and on the sphere (except for the case when they are on the equator), [6]. In the third result we prove that if only one body is massless and the other three are equal, some kite-shaped relative equilibria exist on spheres, but not on hyperbolic spheres.

In Chapter 4, we first introduce the relative equilibria on the equator, then we discuss the existence of solutions for the 5-body problem, in which equilibrium solutions appear for particular geometrical configurations. We also show that the relative equilibria of the 5-body problem with exactly four equal masses always exist for the spaces corresponding to $\kappa > 0$, and $\kappa < 0$. Moreover, we prove that the square pyramidal

solutions in \mathbb{S}^2 for two pairs of equal masses do not exist. We consider the planetary problem when some bodies are massless. Then, we show that the square pyramidal solutions in \mathbb{S}^2 with two massless bodies do exist. However, we show that the relative equilibria for two pairs of equal masses do not exist in \mathbb{H}^2 .

In Chapter 5, we derive a criterion for the existence of hexagonal relative equilibria on the equator of the sphere. Then, we recover the kite-shaped relative equilibrium for the case of three equal masses and three massless bodies in certain shapes.

In Chapter 6, we derive the equations of motion for the 2-and 3-centre problems corresponding to the upper half plane model. Then, we study the dynamics around the equilibrium point of the 2-center problem. Finally, we classify the collisions in the 3-center problem, whereas Chapter 7 describes the conclusion and future work.

Chapter 2

Equations of Motion of The Curved N-Body Problem

In this chapter, we introduce the equations of motion of the curved N -body problem on the surfaces of constant curvature. Then, we provide the equations of motion of the flat case in the context of curved space.

2.0.1 The potential

We introduce the potential function on spaces of constant curvature. Consider $N \geq 2$ point masses $m_1, \dots, m_N > 0$ moving on the 3-sphere of constant Gaussian curvature $\kappa > 0$,

$$\mathbb{S}_\kappa^3 := \{(x, y, z, w) \mid x^2 + y^2 + z^2 + w^2 = \kappa^{-1}\},$$

embedded in \mathbb{R}^4 , or on the hyperbolic 3-sphere of curvature $\kappa < 0$,

$$\mathbb{H}_\kappa^3 := \{(x, y, z, w) \mid x^2 + y^2 + z^2 - w^2 = \kappa^{-1}\},$$

embedded in the Minkowski space $\mathbb{R}^{3,1}$. The Minkowski space has four spatial components instead of one temporal and three spatial ones, as understood in general relativity. Generically, we can merge these two manifolds into

$$\mathbb{M}_\kappa^3 := \{(w, x, y, z) \mid w^2 + x^2 + y^2 + \sigma z^2 = \kappa^{-1}, \text{ with } z > 0 \text{ for } \kappa < 0\}.$$

The notation $\mathbb{R}^{3,1}$ expresses the signature $(+, +, +, -)$ of the inner product, which is defined below.

Let $\mathbf{q}_i = (x_i, y_i, z_i, w_i)$ be the coordinates of the point mass m_i and they satisfy the constraint

$$x_i^2 + y_i^2 + z_i^2 + \sigma w_i^2 = \kappa^{-1},$$

where σ is the signum function

$$\sigma := \begin{cases} +1 & \text{for } \kappa \geq 0 \\ -1 & \text{for } \kappa < 0. \end{cases}$$

The inner product between the vectors $\mathbf{q}_i = (x_i, y_i, z_i, w_i)$ and $\mathbf{q}_j = (x_j, y_j, z_j, w_j)$ is given by

$$q^{ij} = \mathbf{q}_i \cdot \mathbf{q}_j := x_i x_j + y_i y_j + z_i z_j + \sigma w_i w_j.$$

We define the distance between the bodies m_i and m_j in \mathbb{S}_κ^3 , \mathbb{R}^3 and \mathbb{H}_κ^3 as

$$d(\mathbf{q}_i, \mathbf{q}_j) := \begin{cases} \kappa^{-\frac{1}{2}} \cos^{-1}(\kappa q^{ij}) & \kappa > 0 \\ |\mathbf{q}_i - \mathbf{q}_j| & \kappa = 0 \\ (-\kappa)^{-\frac{1}{2}} \cosh^{-1}(\kappa q^{ij}) & \kappa < 0 \end{cases}$$

and the gradient operator as

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \sigma \frac{\partial}{\partial w} \right).$$

We further introduce the trigonometric κ -functions, which unify the circular and hyperbolic trigonometry. The κ -sine functions defined as

$$sn_\kappa(x) := \begin{cases} \kappa^{-\frac{1}{2}} \sin(\kappa^{\frac{1}{2}} x) & \kappa > 0 \\ x & \kappa = 0 \\ (-\kappa)^{-\frac{1}{2}} \sinh((-\kappa^{\frac{1}{2}}) x) & \kappa < 0 \end{cases}$$

the κ -cosine functions are defined as

$$csn_\kappa(x) := \begin{cases} \cos(\kappa^{\frac{1}{2}} x) & \kappa > 0 \\ 1 & \kappa = 0 \\ \cosh((-\kappa^{\frac{1}{2}}) x) & \kappa < 0 \end{cases}$$

where the κ -tangent functions and the κ -contangent functions are defined as

$$tn_\kappa(x) := \frac{sn_\kappa(x)}{c sn_\kappa(x)} \quad \text{and} \quad ct n_\kappa(x) := \frac{c sn_\kappa(x)}{sn_\kappa(x)}$$

respectively. These trigonometric κ -functions are continuous with respect to κ . Let now m_1, \dots, m_N be N point masses, $N \geq 2$, with corresponding position vectors $\mathbf{q}_1, \dots, \mathbf{q}_N$, and denote by $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_N)$ the configuration of the system. Then, as long as $\kappa \neq 0$, the cotangent potential is given by $-U_\kappa$, where

$$U_\kappa(\mathbf{q}) = \begin{cases} \sum_{1 \leq i < j \leq N} \coth(d(m_i, m_j)) & \kappa < 0 \\ \sum_{1 \leq i < j \leq N} \cot(d(m_i, m_j)) & \kappa > 0 \end{cases}$$

is the potential function. For $\kappa = 0$, potential $U_\kappa(\mathbf{q})$ tends to the Newtonian force function. Thus, the potential $U_\kappa(\mathbf{q})$ varies continuously with the curvature κ , [6].

Straightforward computations show that

$$U_\kappa(\mathbf{q}) = \sum_{1 \leq i < j \leq N} \frac{m_i m_j |\kappa|^{1/2} \kappa \mathbf{q}_i \cdot \mathbf{q}_j}{[(\kappa \mathbf{q}_i \cdot \mathbf{q}_i)(\kappa \mathbf{q}_j \cdot \mathbf{q}_j) - (\kappa \mathbf{q}_i \cdot \mathbf{q}_j)^2]^{1/2}} \quad (2.1)$$

for $\kappa \neq 0$.

2.0.2 Euler's classical theorem on homogeneous functions

We will state Euler's classical theorem on homogeneous functions and show how it applies to the curved potential.

Theorem 1. A function $F: \mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_n) \in \mathbb{R}^m \rightarrow F(\mathbf{q}) \in \mathbb{R}$, where m is a positive integer, is called homogeneous of degree $\alpha \in \mathbb{R}$ if for every $\eta \neq 0$ and $\mathbf{q} \in \mathbb{R}^m$ we have $F(\eta \mathbf{q}) = \eta^\alpha F(\mathbf{q})$. Then

$$\mathbf{q} \cdot \nabla F(\mathbf{q}) = \alpha F(\mathbf{q})$$

Since U_κ is a homogeneous function of degree zero, Euler's theorem

$$\mathbf{q} \cdot \nabla F(\mathbf{q}) = \alpha F(\mathbf{q})$$

leads to the conclusion that

$$\mathbf{q} \cdot \nabla_{\mathbf{q}_i} U_\kappa(\mathbf{q}) = 0.$$

This feature was proposed by Diacu, Pérez-Chavela and Reyes Victoria.

2.0.3 Derivation of the Equations of Motion

We will use the theory of constrained Lagrangian dynamics in order to derive the equations of motion of the curved N -body problem. The Lagrangian of the curved N -body problem is expressed as

$$L_\kappa(\mathbf{q}, \dot{\mathbf{q}}) = T_\kappa(\mathbf{q}, \dot{\mathbf{q}}) + U_\kappa(\mathbf{q})$$

where $T_\kappa(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} \sum_{i=1}^N m_i (\dot{\mathbf{q}} \odot \dot{\mathbf{q}})$ is the kinetic energy, where \odot is the inner product. Then, according to the theory of constrained Lagrangian dynamics, the equations of motion has the form

$$\frac{d}{dt} \left(\frac{\partial L_\kappa}{\partial \dot{\mathbf{q}}_i} \right) - \frac{\partial L_\kappa}{\partial \mathbf{q}_i} - \lambda_\kappa^i(t) \frac{\partial f_\kappa^i}{\partial \mathbf{q}_i} = 0, \quad i = 1, \dots, N,$$

where $f_\kappa^i = (\mathbf{q}_i \odot \mathbf{q}_i) - \kappa^{-1}$ is the function that characterizes the constraints $f_\kappa^i = 0$, $i = 1, 2, \dots, N$, and λ_κ^i are the Lagrange multipliers given by $\lambda_\kappa^i = -\kappa m_i (\dot{\mathbf{q}} \odot \dot{\mathbf{q}})$, $i = 1, \dots, N$.

Then, the equation of motion are given by the system of differential equations

$$m_i \ddot{\mathbf{q}}_i = \nabla_{\mathbf{q}_i} U_\kappa(\mathbf{q}) - m_i \kappa (\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i) \mathbf{q}_i, \quad \mathbf{q}_i \cdot \mathbf{q}_i = \kappa^{-1}, \quad \mathbf{q}_i \cdot \dot{\mathbf{q}}_i = 0, \quad i = 1, \dots, N.$$

where $\kappa \neq 0$ and

$$\nabla_{\mathbf{q}_i} U_\kappa(\mathbf{q}) = \sum_{\substack{j=1 \\ j \neq i}}^N \frac{m_i m_j |\kappa|^{3/2} (\kappa \mathbf{q}_j \cdot \mathbf{q}_j) [(\kappa \mathbf{q}_i \cdot \mathbf{q}_i) \mathbf{q}_j - (\kappa \mathbf{q}_i \cdot \mathbf{q}_j) \mathbf{q}_i]}{[\sigma(\kappa \mathbf{q}_i \cdot \mathbf{q}_i) (\kappa \mathbf{q}_j \cdot \mathbf{q}_j) - \sigma(\kappa \mathbf{q}_i \cdot \mathbf{q}_j)^2]^{3/2}}. \quad (2.2)$$

To keep the bodies on the manifolds, we have to assume that, at some initial time, the position vectors and the velocity vectors satisfy the $2N$ constraints

$$\kappa \mathbf{q}_i \cdot \mathbf{q}_i = 1, \quad \mathbf{q}_i \cdot \dot{\mathbf{q}}_i = 0.$$

Using these constraints we can write that

$$\nabla_{\mathbf{q}_i} U_\kappa(\mathbf{q}) = \sum_{j=1, j \neq i}^N \frac{m_i m_j |\kappa|^{3/2} [\mathbf{q}_j - (\kappa \mathbf{q}_i \cdot \mathbf{q}_j) \mathbf{q}_i]}{[\sigma - \sigma(\kappa \mathbf{q}_i \cdot \mathbf{q}_j)^2]^{3/2}}, \quad \kappa \neq 0, \quad i = 1, \dots, N.$$

Note that for $\kappa = 0$, the equations of motion become undetermined.

2.1 Extension to the flat case

The inconvenience of having to use two different sets of equations, one for $\kappa \neq 0$ and another for $\kappa = 0$, if we want to consider the problem for any $\kappa \in \mathbb{R}$, has been overcome by F. Diacu in [10]. He first noticed that

$$2q^{ij} = q_i^2 + q_j^2 - q_{ij}^2,$$

where

$$q_{ij} := \begin{cases} [(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 + (w_i - w_j)^2]^{1/2} & \kappa > 0 \\ [(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2]^{1/2} & \kappa = 0 \\ [(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 - (w_i - w_j)^2]^{1/2} & \kappa < 0 \end{cases}$$

is the Euclidean distance between m_i and m_j in \mathbb{R}^4 for $\kappa > 0$, but in \mathbb{R}^3 for $\kappa = 0$, and the corresponding distance in the Minkowski space $\mathbb{R}^{3,1}$ for $\kappa < 0$. All of them taken between the point masses m_i and m_j . Notice that the Minkowski distance does not satisfy the triangle inequality $q_{ik} \leq q_{ij} + q_{jk}$, but is always non-negative.

Writing the potential function U_κ using the above relationship we obtain a new expression of the force function,

$$V_\kappa(\mathbf{q}) = \sum_{1 \leq i < j \leq N} \frac{m_i m_j (\kappa q_i^2 + \kappa q_j^2 - \kappa q_{ij}^2)}{[2(\kappa q_i^2 + \kappa q_j^2) q_{ij}^2 - \kappa (q_i^2 - q_j^2)^2 - \kappa q_{ij}^4]^{1/2}},$$

which can be put into the form

$$V_\kappa(\mathbf{q}) = \sum_{1 \leq i < j \leq N} \frac{m_i m_j \left(1 - \frac{\kappa q_{ij}^2}{2}\right)}{q_{ij} \left(1 - \frac{\kappa q_{ij}^2}{4}\right)^{1/2}}. \quad (2.3)$$

Notice that the formula of U_κ in (2.1) cannot be extended to the flat case, but the right hand side of (2.3) makes sense for $\kappa = 0$. Then for $\kappa = 0$, we recover the classical Newtonian force function

$$V_0(\mathbf{q}) = \sum_{1 \leq i < j \leq N} \frac{m_i m_j}{q_{ij}}.$$

Definition 1. Consider a family of homogeneous functions $F_\kappa: \mathbb{R}^m \rightarrow \mathbb{R}$, $\kappa \in \mathbb{R}$, that is continuous relative to κ . The family F_κ experiences no change in homogeneity if the

value of α does not depend on κ , where α is the degree of F_κ . If the value of α depends on κ , then the family experiences a change in homogeneity at the points κ where the value of α changes.

Proposition 1. The force function V_κ given in (2.3) is continuous in κ for all $\kappa \in \mathbb{R}$ and the family has a change in homogeneity at $\kappa = 0$, namely it provides homogeneous functions of degree 0 for $\kappa \neq 0$, but a homogeneous function of degree -1 at $\kappa = 0$.

Proof. The continuity of V_κ in κ is obvious from (2.3). For $\kappa \neq 0$, the force function U_κ in (2.1) is the same as V_κ in (2.3). Using the expression of $\nabla_{\mathbf{q}_i} U_\kappa(\mathbf{q})$ in (2.2), it follows that

$$\mathbf{q} \cdot \nabla_{\mathbf{q}_i} U_\kappa(\mathbf{q}) = \sum_{\substack{j=1 \\ j \neq i}}^N \frac{m_i m_j |\kappa|^{3/2} (\kappa \mathbf{q}_j \cdot \mathbf{q}_j) [(\kappa \mathbf{q}_i \cdot \mathbf{q}_i)(\mathbf{q}_j \cdot \mathbf{q}_i) - (\kappa \mathbf{q}_i \cdot \mathbf{q}_j)(\mathbf{q}_i \cdot \mathbf{q}_i)]}{[\sigma(\kappa \mathbf{q}_i \cdot \mathbf{q}_i)(\kappa \mathbf{q}_j \cdot \mathbf{q}_j) - \sigma(\kappa \mathbf{q}_i \cdot \mathbf{q}_j)^2]^{3/2}} = 0$$

since the brackets vanish in all the numerators. When $\kappa \neq 0$, the curved force function U_κ experiences no change in homogeneity. We conclude from (2.1) that the degree of homogeneity is 0. For $\kappa = 0$, we obtain that

$$\mathbf{q} \cdot \nabla U_0(\mathbf{q}) = -U_0(\mathbf{q}).$$

Therefore, according to Euler's formula, the degree of homogeneity of U_0 is -1 , a fact that completes the proof. \square

The equations of motion take the form

$$\ddot{\mathbf{q}}_i = \sum_{j=1, j \neq i}^N \frac{m_j \left[\mathbf{q}_j - \left(1 - \frac{\kappa q_{ij}^2}{2}\right) \mathbf{q}_i \right]}{\left(1 - \frac{\kappa q_{ij}^2}{4}\right)^{3/2} q_{ij}^3} - \kappa (\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i) \mathbf{q}_i, \quad i = 1, \dots, N.$$

Further Introducing the transformations

$$\omega_i = w_i - |\kappa|^{-1/2}, \quad i = 1, \dots, N,$$

which shift the origin of the coordinate system to the North Pole of the spheres and making the notations

$$\mathbf{R} = (0, 0, 0, \sigma |\kappa|^{-1/2}), \quad \mathbf{r}_i = (x_i, y_i, z_i, \omega_i), \quad r_{ij} = q_{ij}, \quad i = 1, \dots, N,$$

the equations of motion take the form

$$\ddot{\mathbf{r}}_i = \sum_{j=1, j \neq i}^N \frac{m_j \left[\mathbf{r}_j - \left(1 - \frac{\kappa r_{ij}^2}{2}\right) \mathbf{r}_i + \frac{\mathbf{R} r_{ij}^2}{2} \right]}{\left(1 - \frac{\kappa r_{ij}^2}{4}\right)^{3/2} r_{ij}^3} - (\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i) (\kappa \mathbf{r}_i + \mathbf{R}), \quad i = 1, \dots, N,$$

and the $2N$ constraints become

$$\kappa \mathbf{r}_i^2 + 2|\kappa|^{1/2} \omega_i = 0, \quad \kappa \mathbf{r}_i \cdot \dot{\mathbf{r}}_i + |\kappa|^{1/2} \dot{\omega}_i = 0$$

On components, these equations can be written as

$$\left\{ \begin{array}{l} \ddot{x}_i = \sum_{j=1, j \neq i}^N \frac{m_j \left[x_j - \left(1 - \frac{\kappa r_{ij}^2}{2}\right) x_i \right]}{\left(1 - \frac{\kappa r_{ij}^2}{4}\right)^{3/2} r_{ij}^3} - \kappa (\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i) x_i \\ \ddot{y}_i = \sum_{j=1, j \neq i}^N \frac{m_j \left[y_j - \left(1 - \frac{\kappa r_{ij}^2}{2}\right) y_i \right]}{\left(1 - \frac{\kappa r_{ij}^2}{4}\right)^{3/2} r_{ij}^3} - \kappa (\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i) y_i \\ \ddot{z}_i = \sum_{j=1, j \neq i}^N \frac{m_j \left[z_j - \left(1 - \frac{\kappa r_{ij}^2}{2}\right) z_i \right]}{\left(1 - \frac{\kappa r_{ij}^2}{4}\right)^{3/2} r_{ij}^3} - \kappa (\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i) z_i \\ \ddot{\omega}_i = \sum_{j=1, j \neq i}^N \frac{m_j \left[\omega_j - \left(1 - \frac{\kappa r_{ij}^2}{2}\right) \omega_i \right]}{\left(1 - \frac{\kappa r_{ij}^2}{4}\right)^{3/2} r_{ij}^3} - [\kappa \omega_i + (\sigma \kappa)^{1/2}] (\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i) \omega_i \end{array} \right.$$

For $\kappa = 0$ we recover the classical Newtonian equations of the N -body problem,

$$\ddot{\mathbf{r}}_i = \sum_{j=1, j \neq i}^N \frac{m_j (\mathbf{r}_j - \mathbf{r}_i)}{r_{ij}^3}, \quad i = 1, \dots, N.$$

When $\kappa = 0$, the position vectors $\mathbf{r}_i = (x_i, y_i, z_i, 0)$ are free of constraints and the velocities $\dot{\mathbf{r}}_i = (\dot{x}_i, \dot{y}_i, \dot{z}_i, 0)$, which is like thinking that the motion in \mathbb{R}^3 takes place in a hyperplane of \mathbb{R}^4 .

2.2 Bifurcation of the integrals of motion

The integrals of motion are functions of the phase-space coordinates that are constant along orbits. Let us investigate the integrals of motion for the above system of equations.

2.2.1 The integrals of the centre of mass and the linear momentum

Summing up $m_i \ddot{\mathbf{r}}_i$ in the equations of motion from $i = 1$ to $i = N$ we obtain the identity

$$\sum_{i=1}^N m_i \ddot{\mathbf{r}}_i = \sum_{i=1}^N \sum_{j=1, j \neq i}^N \frac{m_j m_i \frac{r_{ij}^2}{2} (\kappa \mathbf{r}_i + \mathbf{R})}{\left(1 - \frac{\kappa r_{ij}^2}{4}\right)^{3/2} r_{ij}^3} - \sum_{i=1}^N m_i (\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i) (\kappa \mathbf{r}_i + \mathbf{R}).$$

For $\kappa = 0$, we have $\sum_{i=1}^N m_i \ddot{\mathbf{r}}_i = 0$. Integrating this identity we obtain the three integrals of the linear momentum in the flat case,

$$\sum_{i=1}^N m_i \dot{\mathbf{r}}_i = \mathbf{a}$$

where $\mathbf{a} = (a_1, a_2, a_3)$ is a constant vector. Integrating again, we obtain

$$\sum_{i=1}^N m_i \mathbf{r}_i = \mathbf{a}t + \mathbf{b}$$

where $\mathbf{b} = (b_1, b_2, b_3)$ is another constant vector.

By taking the origin of the coordinate system at the centre of mass, we obtain

$$\sum_{i=1}^N m_i \dot{\mathbf{r}}_i = 0 \quad \text{and} \quad \sum_{i=1}^N m_i \mathbf{r}_i = 0,$$

which means the centre of mass is fixed relative to the coordinate system. Hence, the forces acting on the centre of mass cancel each other.

Obviously, these integrals do not show up when $\kappa \neq 0$, thus we encounter a bifurcation at $\kappa = 0$. Clearly, this bifurcation means a change in the phase-space portrait.

2.2.2 The integral of energy

Taking $m_i \ddot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i$ and summing up from $i = 1$ to $i = N$, we have

$$\sum_{i=1}^N m_i \ddot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i = \sum_{i=1}^N \dot{\mathbf{q}}_i \cdot \nabla_{\mathbf{q}_i} V_\kappa(\mathbf{q}) - \sum_{i=1}^N m_i (\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i) (\kappa \mathbf{q}_i \cdot \dot{\mathbf{q}}_i) = \frac{d}{dt} V_\kappa(\mathbf{q}).$$

By integration we obtain the energy integral

$$H_\kappa(\mathbf{q}, \dot{\mathbf{q}}) = T_\kappa(\mathbf{q}, \dot{\mathbf{q}}) - \mathbf{V}_\kappa(\mathbf{q}) = h,$$

where h is a constant, H_κ is the Hamiltonian function, and T_κ is the kinetic energy. After applying the transformations $\omega_i = w_i - |\kappa|^{-1/2}$, the kinetic energy becomes

$$T_\kappa(\mathbf{r}, \dot{\mathbf{r}}) = \frac{1}{2} \sum_{i=1}^N m_i [\kappa r_i^2 + 2(\sigma\kappa)^{1/2} \omega_i + 1] (\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i).$$

Then the integral of energy for the system takes the form

$$\frac{1}{2} \sum_{i=1}^N m_i [\kappa r_i^2 + 2(\sigma\kappa)^{1/2} \omega_i + 1] (\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i) - \sum_{1 \leq i < j \leq N} \frac{m_i m_j \left(1 - \frac{\kappa r_{ij}^2}{2}\right)}{r_{ij} \left(1 - \frac{\kappa r_{ij}^2}{4}\right)^{1/2}} = h.$$

For $\kappa = 0$, we obtain the integral of the Newtonian equations:

$$\frac{1}{2} \sum_{i=1}^N m_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) - \sum_{1 \leq i < j \leq N} \frac{m_i m_j}{r_{ij}} = h.$$

Thus no bifurcations occur in this case.

2.2.3 The integrals of the total angular momentum

We define the total angular momentum as

$$\sum_{i=1}^N m_i \mathbf{q}_i \wedge \dot{\mathbf{q}}_i$$

where \wedge is the exterior (wedge) product in the Grassman algebra over \mathbb{R}^4 , [10]. The total angular momentum measures the rotation of the system relative to the six planes given by every two of the four axes that form the coordinate system of \mathbb{R}^4 . This quantity is conserved for the equations of motion as shown in [10],

$$\sum_{i=1}^N m_i \mathbf{q}_i \wedge \dot{\mathbf{q}}_i = \mathbf{c}$$

where

$$\mathbf{c} = c_{wx}\mathbf{e}_w \wedge \mathbf{e}_x + c_{wy}\mathbf{e}_w \wedge \mathbf{e}_y + c_{wz}\mathbf{e}_w \wedge \mathbf{e}_z + c_{xy}\mathbf{e}_x \wedge \mathbf{e}_y + c_{xz}\mathbf{e}_x \wedge \mathbf{e}_z + c_{yz}\mathbf{e}_y \wedge \mathbf{e}_z,$$

with $c_{wx}, c_{wy}, c_{wz}, c_{xy}, c_{xz}, c_{yz} \in \mathbb{R}$, and the vectors of the standard basis of \mathbb{R}^4 being

$$\mathbf{e}_x = (1, 0, 0, 0), \mathbf{e}_y = (0, 1, 0, 0), \mathbf{e}_z = (0, 0, 1, 0), \mathbf{e}_w = (0, 0, 0, 1).$$

On components, we have six scalar integrals,

$$\begin{aligned} \sum_{i=1}^N m_i(x_i \dot{y}_i - \dot{x}_i y_i) &= c_{xy}, & \sum_{i=1}^N m_i(x_i \dot{z}_i - \dot{x}_i z_i) &= c_{xz}, & \sum_{i=1}^N m_i(y_i \dot{z}_i - \dot{y}_i z_i) &= c_{yz}, \\ \sum_{i=1}^N m_i(w_i \dot{x}_i - \dot{w}_i x_i) &= c_{wx}, & \sum_{i=1}^N m_i(w_i \dot{y}_i - \dot{w}_i y_i) &= c_{wy}, & \sum_{i=1}^N m_i(w_i \dot{z}_i - \dot{w}_i z_i) &= c_{wz}. \end{aligned}$$

Using the transformation $\omega_i = w_i - |\kappa|^{-1/2}$, these integrals take the form

$$\begin{aligned} \sum_{i=1}^N m_i(x_i \dot{y}_i - \dot{x}_i y_i) &= c_{xy}, & \sum_{i=1}^N m_i(x_i \dot{z}_i - \dot{x}_i z_i) &= c_{xz}, & \sum_{i=1}^N m_i(y_i \dot{z}_i - \dot{y}_i z_i) &= c_{yz}, \\ \sum_{i=1}^N m_i \dot{x}_i + (\sigma\kappa)^{1/2} \sum_{i=1}^N m_i(w_i \dot{x}_i - \dot{w}_i x_i) &= (\sigma\kappa)^{1/2} c_{wx}, \\ \sum_{i=1}^N m_i \dot{y}_i + (\sigma\kappa)^{1/2} \sum_{i=1}^N m_i(w_i \dot{y}_i - \dot{w}_i y_i) &= (\sigma\kappa)^{1/2} c_{wy}, \\ \sum_{i=1}^N m_i \dot{z}_i + (\sigma\kappa)^{1/2} \sum_{i=1}^N m_i(w_i \dot{z}_i - \dot{w}_i z_i) &= (\sigma\kappa)^{1/2} c_{wz}. \end{aligned}$$

Notice that, for $\kappa = 0$, the last three integrals become linear momentum integrals, so an interesting kind of bifurcation occurs. From the physical point of view, this appears natural, given that a ‘‘curved’’ dimension, which conveys the conservation of a rotational component of the motion, ‘‘straightens up’’ when κ goes from nonzero values to zero, and thus gets transformed into a linear component of the motion.

Chapter 3

Bifurcation of The Lagrangian Orbits

This chapter has been published in RELATIVE EQUILIBRIA IN CURVED RESTRICTED 4-BODY PROBLEMS in The Canadian Mathematical Bulletin [2].

In this chapter we consider the bifurcation of solutions of the equations of motion on spheres, \mathbb{S}_κ^2 and hyperbolic spheres \mathbb{H}_κ^2 . We deal with the Lagrangian orbits, which are equilateral triangles in the 3-body problem. We consider the motion of four bodies on 2-dimensional surfaces of constant curvature κ . For $\kappa > 0$ we use as a model the spheres of radius $1/\sqrt{\kappa}$. This sphere is embedded in \mathbb{R}^3 with the Euclidean metric, we denote it by \mathbb{S}_κ^2 . For $\kappa = 0$ we take the Euclidean plane \mathbb{R}^2 , and for $\kappa < 0$ we take the upper part of the hyperboloid

$$x^2 + y^2 - z^2 = -1/\sqrt{-\kappa}, \quad (3.1)$$

embedded in the Minkowski space $\mathbb{R}^{2,1}$, that is \mathbb{R}^3 endowed with the Lorenz inner product (for $a, b \in \mathbb{R}^3$, $a \odot b = a_x b_x + a_y b_y - a_z b_z$). This space is known as the hyperbolic sphere or the pseudo sphere, and it is denoted by \mathbb{H}_κ^2 .

Now, we will arrange these objects in \mathbb{R}^3 , maintaining the different metric for the sphere and the pseudo sphere, such that they all have a common point at which lie all the north poles of the spheres and the vertices of the hyperbolic spheres, to all of which the plane \mathbb{R}^2 is tangent. We fix the origin of the new coordinate system at this point. In other words, we translate the origin to the north pole of the sphere and the pseudo sphere, abusing notation we keep the same notation for these objects, then we can write

$$\mathbb{S}_\kappa^2 := \{(x, y, z) \mid \kappa(x^2 + y^2 + z^2) + 2\kappa^{\frac{1}{2}}z = 0\} \text{ for } \kappa > 0,$$

$$\mathbb{H}_\kappa^2 := \{(x, y, z) \mid \kappa(x^2 + y^2 - z^2) + 2|\kappa|^{\frac{1}{2}}z = 0, z \geq 0\} \text{ for } \kappa < 0.$$

Consider now four point masses, $m_i > 0$, $i = 1, 2, 3, 4$, whose position vectors, velocities, and accelerations are given by

$$\mathbf{r}_i = (x_i, y_i, z_i), \quad \dot{\mathbf{r}}_i = (\dot{x}_i, \dot{y}_i, \dot{z}_i), \quad \ddot{\mathbf{r}}_i = (\ddot{x}_i, \ddot{y}_i, \ddot{z}_i), \quad i = 1, 2, 3, 4.$$

Then, as shown in [9], the equations of motion take the form

$$\begin{cases} \ddot{x}_i = \sum_{j=1, j \neq i}^N \frac{m_j \left[x_j - \left(1 - \frac{\kappa r_{ij}^2}{2} \right) x_i \right]}{\left(1 - \frac{\kappa r_{ij}^2}{4} \right)^{3/2} r_{ij}^3} - \kappa (\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i) x_i \\ \ddot{y}_i = \sum_{j=1, j \neq i}^N \frac{m_j \left[y_j - \left(1 - \frac{\kappa r_{ij}^2}{2} \right) y_i \right]}{\left(1 - \frac{\kappa r_{ij}^2}{4} \right)^{3/2} r_{ij}^3} - \kappa (\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i) y_i \\ \ddot{z}_i = \sum_{j=1, j \neq i}^N \frac{m_j \left[z_j - \left(1 - \frac{\kappa r_{ij}^2}{2} \right) z_i \right]}{\left(1 - \frac{\kappa r_{ij}^2}{4} \right)^{3/2} r_{ij}^3} - (\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i) (\kappa z_i + \sigma |\kappa|^{1/2}), \quad i = 1, 2, 3, 4, \end{cases} \quad (3.2)$$

where $\sigma = 1$ for $\kappa \geq 0$, $\sigma = -1$ for $\kappa < 0$, and

$$r_{ij} := \begin{cases} [(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2]^{1/2} & \text{for } \kappa \geq 0 \\ [(x_i - x_j)^2 + (y_i - y_j)^2 - (z_i - z_j)^2]^{1/2} & \text{for } \kappa < 0 \end{cases}$$

for $i, j \in \{1, 2, 3, 4\}$. The above system has eight constraints, namely

$$\kappa(x_1^2 + y_1^2 + \sigma z_1^2) + 2|\kappa|^{1/2} z_1 = 0,$$

$$\kappa \mathbf{r}_i \cdot \dot{\mathbf{r}}_i + |\kappa|^{1/2} \dot{z}_i = 0, \quad i = 1, 2, 3, 4.$$

If satisfied at an initial instant, these constraints are satisfied for all time because the sets \mathbb{S}_κ^2 , \mathbb{R}^2 , and \mathbb{H}_κ^2 are invariant for the equations of motion, [6]. Notice that for $\kappa = 0$ we recover the classical Newtonian equations of the 4-body problem on the Euclidean plane, namely

$$\ddot{\mathbf{r}}_i = \sum_{j=1, j \neq i}^N \frac{m_j (\mathbf{r}_j - \mathbf{r}_i)}{r_{ij}^3},$$

where $\mathbf{r}_i = (x_i, y_i, 0)$, $i = 1, 2, 3, 4$.

3.1 Relative Equilibria

It is well known that in the curved N -body problem, the total energy and the angular momentum are first integrals, but the linear momentum is no longer a constant of motion, which is a big difference with the Euclidean case [11]. The goal of this section is to describe particular solutions for this problem, the simplest ones, called relative equilibria. The formal definition is the following.

Definition 2. Relative equilibria are solutions of the curved N -body problem in which the mutual distances among the particles remain constant for all time $t \in \mathbb{R}$. That is the particles move like a rigid body.

So in order to study relative equilibria we must analyze all isometries for both the sphere \mathbb{S}_κ^2 and the pseudo-sphere \mathbb{H}_κ^2 . According to the above definition, the relative equilibria will be the solutions of the equations of motion which are invariant under the action of the isometry groups for the respective surfaces of positive and negative curvature.

3.1.1 Relative equilibria for positive κ

This is the simplest case, because we know that all isometries in \mathbb{R}^3 are rotations; and The Principal Axis Theorem states that any rotation in \mathbb{R}^3 , is around a fixed axis [16]. So in this case, without loss of generality we can assume that the rotation is around the z -axis, and we have that, a relative equilibrium is a solution of the equations of motion that is invariant under the action of the isometry given by a rotation matrix

$$A(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (3.3)$$

3.1.2 Relative equilibria for negative κ

We first consider the 2-dimensional Weierstrass model, which is built on one of the sheets of hyperboloid of two sheets represented by (3.1), in the Minkowski space $(\mathbb{R}^{2,1}, \odot)$ for $\kappa < 0$. The lorentz inner product between the vectors $\mathbf{q}_i = (x_i, y_i, z_i)$ and $\mathbf{q}_j = (x_j, y_j, z_j)$ is given by

$$\mathbf{q}_i \odot \mathbf{q}_j := x_i x_j + y_i y_j - z_i z_j.$$

This surface can be represented by the upper sheet of the hyperbolic sphere with $z > 0$. Equivalently, this surface being as a pseudosphere of imaginary radius (iR) , such that $(iR)^2 = \kappa^{-1}$.

A linear transformation $T : \mathbb{R}^{2,1} \rightarrow \mathbb{R}^{2,1}$ is orthogonal if $T(x) \odot T(x) = x \odot x$ for $x \in \mathbb{R}^{2,1}$. The set of these transformations with the Lorentz inner product form the orthogonal group $O(\mathbb{R}^{2,1}) = \{T(x) \odot T(x) = x \odot x, x \in \mathbb{R}^{2,1}\}$, given by matrices with determinant ± 1 .

The subgroup of $SO(\mathbb{R}^{2,1}) = \{T \in O(\mathbb{R}^{2,1}), \det(T) = 1\}$ is called the special orthogonal group with determinant $+1$. The subset $G(\mathbb{R}^{2,1})$ is another subgroup of $O(\mathbb{R}^{2,1})$ which is formed by the transformation T that leaves \mathbb{H}_κ^2 invariant. Moreover, $G(\mathbb{R}^{2,1})$ has the closed Lorentz subgroup, $Lor(\mathbb{R}^{2,1}) := G(\mathbb{R}^{2,1}) \cap SO(\mathbb{R}^{2,1})$.

Let $Lor(\mathbb{H}_\kappa^2, \odot)$ be the group of all orthogonal transformations of determinant 1 that maintain the upper part of the hyperboloid invariant (the Lorentz group formed by all isometries of \mathbb{H}_κ^2) (see [11, 13, 32] for more details). Applying the corresponding Principal Axis Theorem [30] to $Lor(\mathbb{H}_\kappa^2, \odot)$, which states that any 1-parameter subgroup of $Lor(\mathbb{L}^2, \odot)$ can be written, in a proper basis, as

$$A(t) = P \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} P^{-1},$$

or

$$A(t) = P \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh t & \sinh t \\ 0 & \sinh t & \cosh t \end{pmatrix} P^{-1},$$

or

$$A(t) = P \begin{pmatrix} 1 & -t & t \\ t & 1 - t^2/2 & t^2/2 \\ t & -t^2/2 & 1 + t^2/2 \end{pmatrix} P^{-1},$$

where $P \in Lor(\mathbb{H}_\kappa^2, \odot)$. Then, any isometry of $Lor(\mathbb{H}_\kappa^2, \odot)$ can be written as a composition of some of the above three transformations, called respectively, elliptic, hyperbolic and parabolic transformations. So in this case, the relative equilibria on the

pseudo sphere are the solutions of the equations of motion which are invariant under all isometry of $Lor(\mathbb{H}_\kappa^2, \odot)$.

3.2 The case of positive curvature: \mathbb{S}_κ^2

By the previous discussion, in this case we have to find the initial conditions which lead to solutions invariant under the action of the isometry given by the rotation matrix defined by equation (3.3).

First, we introduce spherical coordinates (φ, ω) , which were originally used in [9] for the case $N = 3$, to detect relative equilibria on and near the equator of \mathbb{S}_κ^2 , where φ measures the angle from the x -axis in the xy -plane, while ω is the height on the vertical z -axis. In these new coordinates, the 8 constraints for the original equations of motion (3.2) become

$$x_i^2 + y_i^2 + \omega_i^2 + 2\kappa^{-1/2}\omega_i = 0, \quad i = 1, 2, 3, 4, 5, \dots, N. \quad (3.4)$$

With the notation,

$$\Omega_i = x_i^2 + y_i^2 = -\kappa^{-1/2}\omega_i(\kappa^{1/2}\omega_i + 2) \geq 0, \quad \omega_i \in [-2\kappa^{-1/2}, 0], \quad i = 1, 2, 3, 4, \dots, N$$

where equality occurs when the body is at the North or the South Pole of the sphere, the (φ, ω) -coordinates are given by the transformations

$$x_i = \Omega_i^{1/2} \cos \varphi_i, \quad y_i = \Omega_i^{1/2} \sin \varphi_i.$$

Thus the equations of motion (3.2) take the form

$$\begin{aligned} \ddot{\varphi}_i &= \Omega_i^{-1/2} \sum_{j=1, j \neq i}^N \frac{m_j \Omega_j^{1/2} \sin(\varphi_j - \varphi_i)}{\rho_{ij}^3 (1 - \frac{\kappa \rho_{ij}^2}{4})^{3/2}} - \frac{\dot{\varphi}_i \dot{\Omega}_i}{\Omega_i}, \\ \ddot{\omega}_i &= \Omega_i^{-1/2} \sum_{j=1, j \neq i}^N \frac{m_j \left[\omega_j + \omega_i + \frac{\kappa \rho_{ij}^2}{2} (\omega_i + \kappa^{-1/2}) \right]}{\rho_{ij}^3 \left(1 - \frac{\kappa \rho_{ij}^2}{4} \right)^{3/2}} \\ &\quad - (\kappa \omega_i + \kappa^{1/2}) \left(\frac{\dot{\Omega}_i^2}{4\Omega_i} + \dot{\varphi}_i^2 \Omega_i + \dot{\omega}_i^2 \right), \end{aligned} \quad (3.5)$$

where

$$\dot{\Omega}_i = -2\kappa^{-1/2}\dot{\omega}_i(\kappa^{1/2}\omega_i + 1)$$

$$\rho_{ij}^2 = \Omega_i + \Omega_j - 2\Omega_i^{1/2}\Omega_j^{1/2} \cos(\varphi_i - \varphi_j) + (\omega_i - \omega_j)^2, \quad i, j = 1, 2, 3, 4, \dots, N \quad i \neq j.$$

3.2.1 Relative equilibria on the equator for $N = 3$

By first seeking relative equilibria on the equator, we take $\omega = -\kappa^{-1/2}$, so we have

$$\omega_i = -\kappa^{-1/2}, \quad \dot{\omega}_i = 0, \quad \Omega_i = \kappa^{-1}, \quad \dot{\Omega}_i = 0,$$

$$\rho_{ij}^2 = -2\kappa^{-1}[1 - \cos(\varphi_i - \varphi_j)], \quad i, j = 1, 2, 3, \quad i \neq j.$$

Substituting these values into the equations of motion we are led to the system

$$\ddot{\varphi}_i = \kappa^{3/2} \sum_{j=1, j \neq i}^3 \frac{m_j \sin(\varphi_j - \varphi_i)}{|\sin(\varphi_j - \varphi_i)|^3}, \quad i = 1, 2, 3.$$

For relative equilibria, the angular velocity is the same constant for all masses on the equator. Denoting this velocity by $\alpha \neq 0$, the change of the angles can be represented as

$$\varphi_1 = \alpha t + a_1, \quad \varphi_2 = \alpha t + a_2, \quad \varphi_3 = \alpha t + a_3,$$

where t is the time and a_1, a_2, a_3 are real constants. Thus

$$\ddot{\varphi}_i = 0.$$

Using the notations

$$s_1 := \frac{\kappa^{3/2} \sin(\varphi_1 - \varphi_2)}{|\sin(\varphi_1 - \varphi_2)|^3}, \quad s_2 := \frac{\kappa^{3/2} \sin(\varphi_2 - \varphi_3)}{|\sin(\varphi_2 - \varphi_3)|^3}, \quad s_3 := \frac{\kappa^{3/2} \sin(\varphi_3 - \varphi_1)}{|\sin(\varphi_3 - \varphi_1)|^3},$$

we are led to the algebraic system

$$\begin{cases} m_1 s_1 - m_3 s_2 = 0 \\ -m_2 s_1 + m_3 s_3 = 0 \\ m_2 s_2 - m_1 s_3 = 0, \end{cases}$$

which has infinitely many solutions,

$$s_1 = \frac{m_3}{m_2} \gamma, \quad s_2 = \frac{m_1}{m_2} \gamma, \quad s_3 = \gamma,$$

where $\gamma \neq 0$ is a real parameter. Hence we have proved the following result.

Theorem 2. For every acute scalene triangle inscribed in the equator of \mathbb{S}_κ^2 , we can find a class of masses, $m_1, m_2, m_3 > 0$, which if placed at the vertices of the triangle form a relative equilibrium that rotates on the equator with any chosen nonzero angular velocity.

The bifurcation of these solutions when κ varies was studied in [9] and we will not present it further. But we will use the ideas introduced above to consider the curved N -body problems in various contexts for $N = 4, 5, 6$.

3.2.2 Relative equilibria on the equator for $N = 4$

If we restrict the motion of the four bodies to the equator of \mathbb{S}_κ^2 , then

$$\omega_i = -\kappa^{-1/2}, \quad \dot{\omega}_i = 0, \quad \Omega_i = \kappa^{-1}, \quad i = 1, 2, 3, 4,$$

and the equations of motion (3.5) take the simple form

$$\ddot{\varphi}_i = \kappa^{3/2} \sum_{j=1, j \neq i}^4 \frac{m_j \sin(\varphi_j - \varphi_i)}{|\sin(\varphi_j - \varphi_i)|^3}, \quad i = 1, 2, 3, 4. \quad (3.6)$$

For the relative equilibria, the angular velocity is the same constant for all masses, so we denote this velocity by $\alpha \neq 0$ and take

$$\varphi_1 = \alpha t + a_1, \quad \varphi_2 = \alpha t + a_2, \quad \varphi_3 = \alpha t + a_3, \quad \varphi_4 = \alpha t + a_4,$$

where a_1, a_2, a_3, a_4 are real constants, so

$$\ddot{\varphi}_i = 0, \quad i = 1, 2, 3, 4.$$

Using the notation

$$s_1 := \frac{\kappa^{3/2} \sin(\varphi_1 - \varphi_2)}{|\sin(\varphi_1 - \varphi_2)|^3}, \quad s_2 := \frac{\kappa^{3/2} \sin(\varphi_2 - \varphi_3)}{|\sin(\varphi_2 - \varphi_3)|^3}, \quad s_3 := \frac{\kappa^{3/2} \sin(\varphi_3 - \varphi_1)}{|\sin(\varphi_3 - \varphi_1)|^3},$$

$$s_4 := \frac{\kappa^{3/2} \sin(\varphi_4 - \varphi_1)}{|\sin(\varphi_4 - \varphi_1)|^3}, \quad s_5 := \frac{\kappa^{3/2} \sin(\varphi_2 - \varphi_4)}{|\sin(\varphi_2 - \varphi_4)|^3}, \quad s_6 := \frac{\kappa^{3/2} \sin(\varphi_3 - \varphi_4)}{|\sin(\varphi_3 - \varphi_4)|^3},$$

then the theorem can be expressed as follows;

Theorem 3. A necessary condition that the quadrilateral inscribed in the equator of \mathbb{S}_κ^2 , with the four masses $m_1, m_2, m_3, m_4 > 0$ at its vertices, forms a relative equilibrium is that

$$s_1 s_6 + s_3 s_5 = s_2 s_4.$$

Proof. We obtain from the equations of motion corresponding to $\ddot{\varphi}_i$ that

$$\begin{cases} -m_2 s_1 + m_3 s_3 + m_4 s_4 = 0 \\ m_1 s_1 - m_3 s_2 - m_4 s_5 = 0 \\ -m_1 s_3 + m_2 s_2 - m_4 s_6 = 0 \\ -m_1 s_4 + m_2 s_5 + m_3 s_6 = 0. \end{cases} \quad (3.7)$$

To have other solutions of the masses than $m_1 = m_2 = m_3 = m_4 = 0$, the determinant of the above system must vanish, which is equivalent to

$$s_1 s_6 + s_3 s_5 = s_2 s_4.$$

This remark completes the proof. \square

3.2.3 Equivalent equations of motion

In this subsection, we obtain another form of equations of motion in which the action of the isometry groups that define the relative equilibria is conserved. Let us now introduce some equivalent equations of motion that are suitable for the kind of solutions we are seeking. First, by eliminating ω_i from the constraints given by equation (3.4) we get

$$\kappa(x_i^2 + y_i^2) + (|\kappa|^{1/2} z_i + 1)^2 = 1, \quad (3.8)$$

and solving explicitly for z_i , we obtain

$$z_i = |\kappa|^{-1/2} [\sqrt{1 - \kappa(x_i^2 + y_i^2)} - 1]. \quad (3.9)$$

The idea here is to eliminate the four equations involving z_1, z_2, z_3, z_4 , but they still appear in the terms r_{ij}^2 in the form $\sigma(z_i - z_j)^2$ as

$$\sigma(z_i - z_j)^2 = \frac{\kappa(x_i^2 + y_i^2 - x_j^2 - y_j^2)^2}{\left[\sqrt{1 - \kappa(x_i^2 + y_i^2)} + \sqrt{1 - \kappa(x_j^2 + y_j^2)} \right]^2}. \quad (3.10)$$

The case of physical interest is when κ is not far from zero, so the above expression exists even for small $\kappa > 0$ under this assumption. Then the equations of motion become

$$\begin{aligned}\ddot{x}_i &= \sum_{j=1, j \neq i}^N \frac{m_j \left[x_j - \left(1 - \frac{\kappa \rho_{ij}^2}{2} \right) x_i \right]}{\left(1 - \frac{\kappa \rho_{ij}^2}{4} \right)^{3/2} \rho_{ij}^3} - \kappa (\dot{x}_i^2 + \dot{y}_i^2 + \kappa B_i) x_i \\ \ddot{y}_i &= \sum_{j=1, j \neq i}^N \frac{m_j \left[y_j - \left(1 - \frac{\kappa \rho_{ij}^2}{2} \right) y_i \right]}{\left(1 - \frac{\kappa \rho_{ij}^2}{4} \right)^{3/2} \rho_{ij}^3} - \kappa (\dot{x}_i^2 + \dot{y}_i^2 + \kappa B_i) y_i,\end{aligned}\quad (3.11)$$

where

$$\begin{aligned}\rho_{ij}^2 &= (x_i - x_j)^2 + (y_i - y_j)^2 + \frac{\kappa (A_i - A_j)^2}{(\sqrt{1 - \kappa A_i} + \sqrt{1 - \kappa A_j})^2}, \\ A_i &= x_i^2 + y_i^2, \\ B_i &= \frac{(x_i \dot{x}_i + y_i \dot{y}_i)^2}{1 - \kappa (x_i^2 + y_i^2)}, \quad i = 1, 2, 3, 4.\end{aligned}$$

It is obvious that for $\kappa = 0$ we recover the classical Newtonian equations of motion of the planar 4-body problem. Also, since the relative equilibria for Newtonian equations are invariant under the action of the rotation matrix given by (3.3), and the other terms in equation (3.11) depend essentially on mutual distances and its derivatives, then the corresponding relative equilibria for the new system is conserved by (3.3).

3.2.4 The case of two massless bodies for $N = 4$

We now consider the case when two out of the four given masses are massless, $m_3 = m_4 = 0$. Then the equations of motion become

$$\left\{ \begin{aligned}\ddot{x}_1 &= \frac{m_2 \left[x_2 - \left(1 - \frac{\kappa \rho_{12}^2}{2} \right) x_1 \right]}{\left(1 - \frac{\kappa \rho_{12}^2}{4} \right)^{3/2} \rho_{12}^3} - \kappa (\dot{x}_1^2 + \dot{y}_1^2 + \kappa B_1) x_1 \\ \ddot{y}_1 &= \frac{m_2 \left[y_2 - \left(1 - \frac{\kappa \rho_{12}^2}{2} \right) y_1 \right]}{\left(1 - \frac{\kappa \rho_{12}^2}{4} \right)^{3/2} \rho_{12}^3} - \kappa (\dot{x}_1^2 + \dot{y}_1^2 + \kappa B_1) y_1\end{aligned}\right.\quad (3.12a)$$

$$\left\{ \begin{array}{l} \ddot{x}_2 = \frac{m_1 \left[x_1 - \left(1 - \frac{\kappa \rho_{12}^2}{2} \right) x_2 \right]}{\left(1 - \frac{\kappa \rho_{12}^2}{4} \right)^{3/2} \rho_{12}^3} - \kappa (\dot{x}_2^2 + \dot{y}_2^2 + \kappa B_2) x_2 \\ \ddot{y}_2 = \frac{m_1 \left[y_1 - \left(1 - \frac{\kappa \rho_{12}^2}{2} \right) y_2 \right]}{\left(1 - \frac{\kappa \rho_{12}^2}{4} \right)^{3/2} \rho_{12}^3} - \kappa (\dot{x}_2^2 + \dot{y}_2^2 + \kappa B_2) y_2 \end{array} \right. \quad (3.12b)$$

$$\left\{ \begin{array}{l} \ddot{x}_3 = \frac{m_1 \left[x_1 - \left(1 - \frac{\kappa \rho_{13}^2}{2} \right) x_3 \right]}{\left(1 - \frac{\kappa \rho_{13}^2}{4} \right)^{3/2} \rho_{13}^3} + \frac{m_2 \left[x_2 - \left(1 - \frac{\kappa \rho_{32}^2}{2} \right) x_3 \right]}{\left(1 - \frac{\kappa \rho_{32}^2}{4} \right)^{3/2} \rho_{23}^3} - \kappa (\dot{x}_3^2 + \dot{y}_3^2 + \kappa B_3) x_3 \\ \ddot{y}_3 = \frac{m_1 \left[y_1 - \left(1 - \frac{\kappa \rho_{13}^2}{2} \right) y_3 \right]}{\left(1 - \frac{\kappa \rho_{13}^2}{4} \right)^{3/2} \rho_{13}^3} + \frac{m_2 \left[y_2 - \left(1 - \frac{\kappa \rho_{32}^2}{2} \right) y_3 \right]}{\left(1 - \frac{\kappa \rho_{32}^2}{4} \right)^{3/2} \rho_{32}^3} - \kappa (\dot{x}_3^2 + \dot{y}_3^2 + \kappa B_3) y_3 \end{array} \right. \quad (3.12c)$$

$$\left\{ \begin{array}{l} \ddot{x}_4 = \frac{m_1 \left[x_1 - \left(1 - \frac{\kappa \rho_{14}^2}{2} \right) x_4 \right]}{\left(1 - \frac{\kappa \rho_{14}^2}{4} \right)^{3/2} \rho_{14}^3} + \frac{m_4 \left[x_4 - \left(1 - \frac{\kappa \rho_{42}^2}{2} \right) x_2 \right]}{\left(1 - \frac{\kappa \rho_{42}^2}{4} \right)^{3/2} \rho_{42}^3} - \kappa (\dot{x}_4^2 + \dot{y}_4^2 + \kappa B_4) x_4 \\ \ddot{y}_4 = \frac{m_1 \left[y_1 - \left(1 - \frac{\kappa \rho_{14}^2}{2} \right) y_4 \right]}{\left(1 - \frac{\kappa \rho_{14}^2}{4} \right)^{3/2} \rho_{14}^3} + \frac{m_4 \left[y_4 - \left(1 - \frac{\kappa \rho_{42}^2}{2} \right) y_2 \right]}{\left(1 - \frac{\kappa \rho_{42}^2}{4} \right)^{3/2} \rho_{42}^3} - \kappa (\dot{x}_4^2 + \dot{y}_4^2 + \kappa B_4) y_4, \end{array} \right. \quad (3.12d)$$

where $\rho_{ij}^2 = \rho_{ji}^2$, $i \neq j$,

$$\rho_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2 + \frac{\kappa (x_i^2 + y_i^2 - x_j^2 - y_j^2)^2}{[\sqrt{1 - \kappa (x_i^2 + y_i^2)} + \sqrt{1 - \kappa (x_j^2 + y_j^2)}]^2}.$$

We can now show that when $m_1 = m_2 =: m > 0$ and $m_3 = m_4 = 0$, square-like relative equilibria, i.e. equilateral equiangular quadrilaterals, always exist on \mathbb{S}_κ^2 .

Theorem 4. In the curved 4-body problem, assume that $m_1 = m_2 =: m > 0$ and $m_3 = m_4 = 0$. Then, in \mathbb{S}_κ^2 , there are two circles of radius $0 < r < \kappa^{-1/2}$, parallel with the equator, such that a square configuration inscribed in this circle, with m_1, m_2 at the opposite ends of one diagonal and m_3, m_4 at the opposite ends of the other diagonal, forms a relative equilibrium.

Proof. Observe that the variable r defined below is related with the height from the equator to the plane containing the configuration inscribed on the circle of radius r in positive (northern hemisphere) or negative sense (southern hemisphere). So, we can assume, without loss of generality, that the bodies are in the northern hemisphere. Then

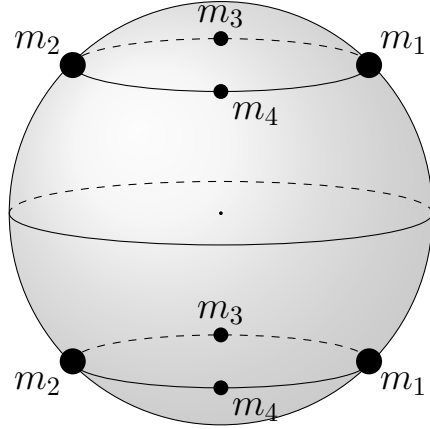


Figure 3.1: The case of 2 equal masses and 2 massless bodies in the northern or in the southern hemisphere

we must check the existence of a solution of the form

$$\mathbf{q} = (q_1, q_2, q_3, q_4) \in \mathbb{S}_\kappa^2, \quad \mathbf{q}_i = (x_i, y_i), \quad i = 1, 2, 3, 4.$$

$$x_1 = r \cos \alpha t, \quad y_1 = r \sin \alpha t,$$

$$x_2 = -r \cos \alpha t, \quad y_2 = -r \sin \alpha t,$$

$$x_3 = r \cos(\alpha t + \pi/2) = -r \sin \alpha t, \quad y_3 = r \sin(\alpha t + \pi/2) = r \cos \alpha t,$$

$$x_4 = -r \cos(\alpha t + \pi/2) = r \sin \alpha t, \quad y_4 = -r \sin(\alpha t + \pi/2) = -r \cos \alpha t,$$

where

$$x_i^2 + y_i^2 = r^2, \quad \rho^2 = \rho_{13}^2 = \rho_{14}^2 = \rho_{23}^2 = \rho_{24}^2 = 2r^2, \quad \rho_{12}^2 = \rho_{34}^2 = 4r^2.$$

Substituting these expressions into the system (3), the first four equations lead us to

$$\alpha^2 = \frac{m}{4r^3(1 - \kappa r^2)^{3/2}},$$

whereas the last four equations yield

$$\alpha^2 = \frac{2m(1 - \frac{\kappa \rho^2}{2})}{\rho^3(1 - \frac{\kappa \rho^2}{4})^{3/2}(1 - \kappa r^2)}.$$

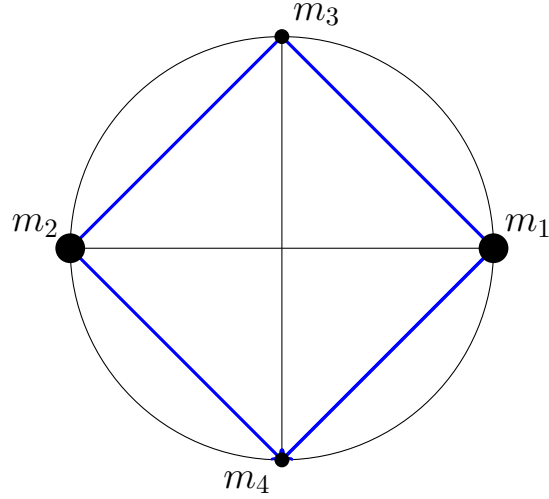


Figure 3.2: The case of two equal masses and two massless bodies.

So, to have a solution, the equation

$$\frac{m}{4r^3(1 - \kappa r^2)^{3/2}} = \frac{2m(1 - \frac{\kappa \rho^2}{2})}{\rho^3(1 - \frac{\kappa \rho^2}{4})^{3/2}(1 - \kappa r^2)}$$

must be satisfied. This equation is equivalent to

$$\frac{1}{8r^3(1 - \kappa r^2)^{3/2}} = \frac{1}{2\sqrt{2}r^3(1 - \frac{\kappa r^2}{2})^{3/2}},$$

which leads to

$$3\kappa r^2 = 2.$$

For \mathbb{S}_{κ}^2 , it leads to

$$r = \sqrt{2/3}\kappa^{-1/2}.$$

Since $r < \kappa^{-1/2}$, such a solution always exists in \mathbb{S}_{κ}^2 . □

3.2.5 The case of one massless body for $N = 4$

Let $m_1, m_2, m_3 = m > 0$ and assume that $m_4 = 0$. Then the equations of motion take the form

$$\left\{ \begin{array}{l} \ddot{x}_1 = \frac{m_2 \left[x_2 - \left(1 - \frac{\kappa \rho_{12}^2}{2} \right) x_1 \right]}{\left(1 - \frac{\kappa \rho_{12}^2}{4} \right)^{3/2} \rho_{12}^3} + \frac{m_3 \left[x_3 - \left(1 - \frac{\kappa \rho_{31}^2}{2} \right) x_1 \right]}{\left(1 - \frac{\kappa \rho_{31}^2}{4} \right)^{3/2} \rho_{31}^3} - \kappa (\dot{x}_1^2 + \dot{y}_1^2 + \kappa B_1) x_1 \\ \ddot{y}_1 = \frac{m_2 \left[y_2 - \left(1 - \frac{\kappa \rho_{12}^2}{2} \right) y_1 \right]}{\left(1 - \frac{\kappa \rho_{12}^2}{4} \right)^{3/2} \rho_{12}^3} + \frac{m_3 \left[y_3 - \left(1 - \frac{\kappa \rho_{31}^2}{2} \right) y_1 \right]}{\left(1 - \frac{\kappa \rho_{31}^2}{4} \right)^{3/2} \rho_{31}^3} - \kappa (\dot{x}_1^2 + \dot{y}_1^2 + \kappa B_1) y_1 \end{array} \right. \quad (3.13a)$$

$$\left\{ \begin{array}{l} \ddot{x}_2 = \frac{m_1 \left[x_1 - \left(1 - \frac{\kappa \rho_{12}^2}{2} \right) x_2 \right]}{\left(1 - \frac{\kappa \rho_{12}^2}{4} \right)^{3/2} \rho_{12}^3} + \frac{m_3 \left[x_3 - \left(1 - \frac{\kappa \rho_{32}^2}{2} \right) x_2 \right]}{\left(1 - \frac{\kappa \rho_{32}^2}{4} \right)^{3/2} \rho_{32}^3} - \kappa (\dot{x}_2^2 + \dot{y}_2^2 + \kappa B_2) x_2 \\ \ddot{y}_2 = \frac{m_1 \left[y_1 - \left(1 - \frac{\kappa \rho_{12}^2}{2} \right) y_2 \right]}{\left(1 - \frac{\kappa \rho_{12}^2}{4} \right)^{3/2} \rho_{12}^3} + \frac{m_3 \left[y_3 - \left(1 - \frac{\kappa \rho_{32}^2}{2} \right) y_2 \right]}{\left(1 - \frac{\kappa \rho_{32}^2}{4} \right)^{3/2} \rho_{32}^3} - \kappa (\dot{x}_2^2 + \dot{y}_2^2 + \kappa B_2) y_2 \end{array} \right. \quad (3.13b)$$

$$\left\{ \begin{array}{l} \ddot{x}_3 = \frac{m_1 \left[x_1 - \left(1 - \frac{\kappa \rho_{13}^2}{2} \right) x_3 \right]}{\left(1 - \frac{\kappa \rho_{13}^2}{4} \right)^{3/2} \rho_{13}^3} + \frac{m_2 \left[x_2 - \left(1 - \frac{\kappa \rho_{32}^2}{2} \right) x_3 \right]}{\left(1 - \frac{\kappa \rho_{32}^2}{4} \right)^{3/2} \rho_{32}^3} - \kappa (\dot{x}_3^2 + \dot{y}_3^2 + \kappa B_3) x_3 \\ \ddot{y}_3 = \frac{m_1 \left[y_1 - \left(1 - \frac{\kappa \rho_{13}^2}{2} \right) y_3 \right]}{\left(1 - \frac{\kappa \rho_{13}^2}{4} \right)^{3/2} \rho_{13}^3} + \frac{m_2 \left[y_2 - \left(1 - \frac{\kappa \rho_{32}^2}{2} \right) y_3 \right]}{\left(1 - \frac{\kappa \rho_{32}^2}{4} \right)^{3/2} \rho_{32}^3} - \kappa (\dot{x}_3^2 + \dot{y}_3^2 + \kappa B_3) y_3 \end{array} \right. \quad (3.13c)$$

$$\left\{ \begin{array}{l} \ddot{x}_4 = \frac{m_1 \left[x_1 - \left(1 - \frac{\kappa \rho_{14}^2}{2} \right) x_4 \right]}{\left(1 - \frac{\kappa \rho_{14}^2}{4} \right)^{3/2} \rho_{14}^3} + \frac{m_2 \left[x_2 - \left(1 - \frac{\kappa \rho_{42}^2}{2} \right) x_4 \right]}{\left(1 - \frac{\kappa \rho_{42}^2}{4} \right)^{3/2} \rho_{42}^3} + \frac{m_3 \left[x_3 - \left(1 - \frac{\kappa \rho_{43}^2}{2} \right) x_4 \right]}{\left(1 - \frac{\kappa \rho_{43}^2}{4} \right)^{3/2} \rho_{43}^3} \\ \quad - \kappa (\dot{x}_4^2 + \dot{y}_4^2 + \kappa B_4) x_4 \\ \ddot{y}_4 = \frac{m_1 \left[y_1 - \left(1 - \frac{\kappa \rho_{14}^2}{2} \right) y_4 \right]}{\left(1 - \frac{\kappa \rho_{14}^2}{4} \right)^{3/2} \rho_{14}^3} + \frac{m_2 \left[y_2 - \left(1 - \frac{\kappa \rho_{42}^2}{2} \right) y_4 \right]}{\left(1 - \frac{\kappa \rho_{42}^2}{4} \right)^{3/2} \rho_{42}^3} + \frac{m_3 \left[y_3 - \left(1 - \frac{\kappa \rho_{43}^2}{2} \right) y_4 \right]}{\left(1 - \frac{\kappa \rho_{43}^2}{4} \right)^{3/2} \rho_{43}^3} \\ \quad - \kappa (\dot{x}_4^2 + \dot{y}_4^2 + \kappa B_4) y_4. \end{array} \right. \quad (3.13d)$$

We will next show that if the non-negligible masses are equal, then there exist some kite-shaped relative equilibria.

Theorem 5. Consider the curved 4-body problem with masses $m_1 = m_2 = m_3 := m > 0$ and $m_4 = 0$. Then, in \mathbb{S}_κ^2 , there exists at least one kite-shaped relative equilibrium for which the equal masses lie at the vertices of an equilateral triangle, whereas the

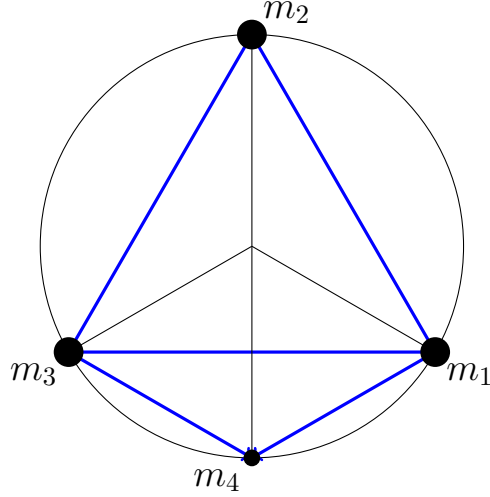


Figure 3.3: A kite configuration of 3 equal masses and one massless body.

negligible mass is at the intersection of the extension of one height of the triangle with the circle on which all the bodies move.

Proof. We will check a solution of the form

$$\begin{aligned}
 x_1 &= r \cos \alpha t, & y_1 &= r \sin \alpha t, \\
 x_2 &= r \cos \left(\alpha t + \frac{2\pi}{3} \right), & y_2 &= r \sin \left(\alpha t + \frac{2\pi}{3} \right) \\
 x_3 &= r \cos \left(\alpha t + \frac{4\pi}{3} \right), & y_3 &= r \sin \left(\alpha t + \frac{4\pi}{3} \right), \\
 x_4 &= r \cos \left(\alpha t - \frac{\pi}{3} \right), & y_4 &= r \sin \left(\alpha t - \frac{\pi}{3} \right),
 \end{aligned}$$

where

$$\rho_{12}^2 = \rho_{13}^2 = \rho_{23}^2 = 3r^2, \quad \rho_{43}^2 = \rho_{41}^2 = r^2, \quad \rho_{24}^2 = 4r^2.$$

Substituting these expressions into the above system, we are led to the conclusion that the following two equations must be satisfied,

$$\begin{aligned}
 \alpha^2 &= \frac{m}{\sqrt{3}r^3(1 - \frac{3\kappa r^2}{4})^{3/2}}, \\
 \alpha^2 &= \frac{m}{4r^3(1 - \kappa r^2)^{3/2}} + \frac{m}{r^3(1 - \frac{\kappa r^2}{4})^{3/2}}.
 \end{aligned}$$

Comparing these equations we obtain the condition for the existence of the kite-shaped

relative equilibria,

$$\frac{1}{\sqrt{3}(1 - \frac{3\kappa r^2}{4})^{3/2}} = \frac{1}{4(1 - \kappa r^2)^{3/2}} + \frac{1}{(1 - \frac{\kappa r^2}{4})^{3/2}}.$$

Straightforward computations show that r is a solution of this equation if it is a root of the polynomial

$$\begin{aligned} P(r) = & a_{24}r^{24} + a_{22}r^{22} + a_{20}r^{20} + a_{18}r^{18} + a_{16}r^{16} + a_{14}r^{14} + a_{12}r^{12} + \\ & a_{10}r^{10} + a_8r^8 + a_6r^6 + a_4r^4 + a_2r^2 + a_0, \\ a_{24} = & \frac{6697290145}{16777216}\kappa^{12}, \quad a_{22} = -\frac{2884257825}{524288}\kappa^{11}, \quad a_{20} = \frac{18063189465}{524288}\kappa^{10}, \\ a_{18} = & -\frac{4241985935}{32768}\kappa^9, \quad a_{16} = \frac{21267471735}{65536}\kappa^8, \\ a_{14} = & -\frac{584429805}{1024}\kappa^7, \quad a_{12} = \frac{737853351}{1024}\kappa^6, \quad a_{10} = -\frac{41995431}{64}\kappa^5, \\ a_8 = & \frac{109080063}{256}\kappa^4, \quad a_6 = -\frac{1530101}{8}\kappa^3, \\ a_4 = & \frac{446217}{8}\kappa^2, \quad a_2 = -9318\kappa, \quad a_0 = 649 \end{aligned}$$

that belongs to the interval $r \in (0, \kappa^{-1/2})$ for \mathbb{S}_κ^2 . To find out if we have such a root, we make the substitution $x = r^2$, and obtain the polynomial

$$\begin{aligned} Q(x) = & a_{24}x^{12} + a_{22}x^{11} + a_{20}x^{10} + a_{18}x^9 + a_{16}x^8 + a_{14}x^7 + a_{12}x^6 + \\ & a_{10}x^5 + a_8x^4 + a_6x^3 + a_4x^2 + a_2x + a_0. \end{aligned}$$

By Descartes's rule of signs the number of positive roots depends on the number of changes of sign of the coefficients, which in turn depends on the sign of κ . So let us discuss the two cases separately.

In \mathbb{S}_κ^2 , i.e. for $\kappa > 0$, there are twelve changes of sign, so Q can have twelve, ten, eight, six, four, two, or zero positive roots, so this does not guarantee the existence of a positive root. However, we can notice that $Q(\frac{\kappa^{-1}}{2}) = -2.4959 < 0$ and $Q(0) = 649 > 0$, so a root must exist for $x \in (0, \kappa^{-1/2})$, i.e. for $r \in (0, \kappa^{-1})$, a remark that proves the existence of at least one kite-shaped relative equilibrium. \square

3.3 The case of negative curvature for $N = 4$ on \mathbb{H}_κ^2

As we have mentioned in Section 3.1, the relative equilibria on \mathbb{H}_κ^2 can be of three different kinds, depending on the special group of isometry which is acting on that surface. In this way we can have elliptic, parabolic or hyperbolic relative relative. In [11] the authors prove the non-existence of parabolic relative equilibria. More recently, in [31], the authors prove the non-existence of polygonal hyperbolic relative equilibria. In this section, we have restricted our analysis to the case of elliptic relative equilibria on \mathbb{H}_κ^2 , also known as hyperbolic elliptic relative equilibria.

3.3.1 The case of two massless bodies on \mathbb{H}_κ^2

It is known (see for instance [6]), that for any $N \in \mathbb{N}$, $m > 0$ and $z > 1$, there are two values of ω , one positive and one negative such that, the isometry matrix $A(\omega t)$ defined by equation (3.3), generates relative equilibria where the masses are located at the vertices of a regular N -gon. For these reasons and the results proved in the previous section. Unexpectedly, this is the case.

In order to facilitate the notations, from here on we will assume, without loss of generality, that the negative curvature is equal to -1 . In this subsection we consider the case when $m_1 = m_2 = m > 0$ and $m_3 = m_4 = 0$. We must check the existence or non-existence of a solution of the form

$$\mathbf{q} = (q_1, q_2, q_3, q_4) \in \mathbb{H}_\kappa^2, \quad \mathbf{q}_i = (x_i, y_i), \quad i = 1, 2, 3, 4.$$

$$x_1 = r \cos \alpha t, \quad y_1 = r \sin \alpha t,$$

$$x_2 = -r \cos \alpha t, \quad y_2 = -r \sin \alpha t,$$

$$x_3 = r \cos(\alpha t + \pi/2) = -r \sin \alpha t, \quad y_3 = r \sin(\alpha t + \pi/2) = r \cos \alpha t,$$

$$x_4 = -r \cos(\alpha t + \pi/2) = r \sin \alpha t, \quad y_4 = -r \sin(\alpha t + \pi/2) = -r \cos \alpha t,$$

where

$$x_i^2 + y_i^2 = r^2, \quad \rho^2 = \rho_{13}^2 = \rho_{14}^2 = \rho_{23}^2 = \rho_{24}^2 = 2r^2, \quad \rho_{12}^2 = \rho_{34}^2 = 4r^2.$$

Substituting these expressions into the system (11) for $\kappa = -1 < 0$, the first four

equations lead us to

$$\alpha^2 = \frac{m}{4r^3(1+r^2)^{3/2}},$$

whereas the last four equations yield

$$\alpha^2 = \frac{m}{\sqrt{2}r^3(1+\frac{r^2}{2})^{3/2}}.$$

So, to have a solution, the equation

$$\frac{m}{4r^3(1+r^2)^{3/2}} = \frac{m}{\sqrt{2}r^3(1+\frac{r^2}{2})^{3/2}},$$

must be satisfied. This equation is equivalent to

$$4(1+r^2)^{3/2} = \sqrt{2}(1+\frac{r^2}{2})^{3/2},$$

which leads to

$$3r^2 = -2,$$

which is a contradiction. Hence, these orbits do not exist on \mathbb{H}_{-1}^2 .

3.3.2 The case of one massless body on \mathbb{H}_{-1}^2

Let $m_1, m_2, m_3 = m > 0$ and assume that $m_4 = 0$, without loss of generality, we can restrict our study to the unit hyperbolic sphere for negative curvature. Then we will check a solution of the form

$$\begin{aligned} x_1 &= r \cos \alpha t, & y_1 &= r \sin \alpha t, \\ x_2 &= r \cos \left(\alpha t + \frac{2\pi}{3} \right), & y_2 &= r \sin \left(\alpha t + \frac{2\pi}{3} \right), \\ x_3 &= r \cos \left(\alpha t + \frac{4\pi}{3} \right), & y_3 &= r \sin \left(\alpha t + \frac{4\pi}{3} \right), \\ x_4 &= r \cos \left(\alpha t - \frac{\pi}{3} \right), & y_4 &= r \sin \left(\alpha t - \frac{\pi}{3} \right), \end{aligned}$$

where

$$\rho_{12}^2 = \rho_{13}^2 = \rho_{23}^2 = 3r^2, \quad \rho_{43}^2 = \rho_{41}^2 = r^2, \quad \rho_{24}^2 = 4r^2.$$

Substituting these expressions into the system (12) for $\kappa = -1 < 0$, we are led to the conclusion that the following two equations must be satisfied,

$$\alpha^2 = \frac{m}{\sqrt{3}r^3(1 + \frac{3r^2}{4})^{3/2}},$$

$$\alpha^2 = \frac{m}{4r^3(1 + r^2)^{3/2}} + \frac{m}{r^3(1 + \frac{r^2}{4})^{3/2}}.$$

Comparing these equations we obtain the condition for the existence of the kite-shaped relative equilibria,

$$\frac{1}{\sqrt{3}(1 + \frac{3r^2}{4})^{3/2}} = \frac{1}{4(1 + r^2)^{3/2}} + \frac{1}{(1 + \frac{r^2}{4})^{3/2}}.$$

Straightforward computations show that r is a solution of this equation if it is a root of the polynomial

$$P(r) = a_{24}r^{24} + a_{22}r^{22} + a_{20}r^{20} + a_{18}r^{18} + a_{16}r^{16} + a_{14}r^{14} + a_{12}r^{12} +$$

$$a_{10}r^{10} + a_8r^8 + a_6r^6 + a_4r^4 + a_2r^2 + a_0,$$

$$a_{24} = \frac{6697290145}{16777216}, \quad a_{22} = \frac{2884257825}{524288}, \quad a_{20} = \frac{18063189465}{524288},$$

$$a_{18} = \frac{4241985935}{32768}, \quad a_{16} = \frac{21267471735}{65536},$$

$$a_{14} = \frac{584429805}{1024}, \quad a_{12} = \frac{737853351}{1024}, \quad a_{10} = \frac{41995431}{64},$$

$$a_8 = \frac{109080063}{256}, \quad a_6 = \frac{1530101}{8},$$

$$a_4 = \frac{446217}{8}, \quad a_2 = 9318, \quad a_0 = 649.$$

Since all coefficients of the polynomial $P(r)$ are positives, by Descartes rule of signs, this polynomial does not have positive roots. Therefore there are no kite solutions for this type of configuration in \mathbb{H}_{-1}^2 .

Chapter 4

Relative Equilibria in The Curved 5-Body Problem

4.1 Relative equilibria on the equator for 5-body problem

Returning to the equations of motion (3.6) on the equator of \mathbb{S}_κ^2 , we consider the 5-body problem and place the bodies at the vertices of a pentagon inscribed in the equator of \mathbb{S}_κ^2 that doesn't lie within any semicircle. (If the bodies are in some half of a hemisphere they cannot form relative equilibria, as shown in [8]). Then the expressions of $s_i, i = 1, 2, \dots, 10$, are

$$\begin{aligned}
 s_1 &:= \frac{\kappa^{3/2} \sin(\varphi_1 - \varphi_2)}{|\sin(\varphi_1 - \varphi_2)|^3}, & s_2 &:= \frac{\kappa^{3/2} \sin(\varphi_1 - \varphi_3)}{|\sin(\varphi_1 - \varphi_3)|^3}, & s_3 &:= \frac{\kappa^{3/2} \sin(\varphi_1 - \varphi_4)}{|\sin(\varphi_1 - \varphi_4)|^3}, \\
 s_4 &:= \frac{\kappa^{3/2} \sin(\varphi_1 - \varphi_5)}{|\sin(\varphi_1 - \varphi_5)|^3}, & s_5 &:= \frac{\kappa^{3/2} \sin(\varphi_2 - \varphi_3)}{|\sin(\varphi_2 - \varphi_3)|^3}, & s_6 &:= \frac{\kappa^{3/2} \sin(\varphi_2 - \varphi_4)}{|\sin(\varphi_2 - \varphi_4)|^3}, \\
 s_7 &:= \frac{\kappa^{3/2} \sin(\varphi_2 - \varphi_5)}{|\sin(\varphi_2 - \varphi_5)|^3}, & s_8 &:= \frac{\kappa^{3/2} \sin(\varphi_3 - \varphi_4)}{|\sin(\varphi_3 - \varphi_4)|^3}, & s_9 &:= \frac{\kappa^{3/2} \sin(\varphi_3 - \varphi_5)}{|\sin(\varphi_3 - \varphi_5)|^3}, \\
 s_{10} &:= \frac{\kappa^{3/2} \sin(\varphi_4 - \varphi_5)}{|\sin(\varphi_4 - \varphi_5)|^3}.
 \end{aligned}$$

From the equations of motion (3.6), we then obtain the system

$$\begin{cases} -m_2s_1 - m_3s_2 - m_4s_3 - s_4m_5 = 0 \\ m_1s_1 - m_3s_5 - m_4s_6 - m_5s_7 = 0 \\ m_1s_2 + m_2s_5 - m_4s_8 - m_5s_9 = 0 \\ m_1s_3 + m_2s_6 + m_3s_8 - m_5s_{10} = 0 \\ m_1s_4 + m_2s_7 + m_3s_9 - m_4s_{10} = 0, \end{cases}$$

which has infinitely many solutions,

$$\begin{aligned} m_1 &= \frac{s_{10}s_5 - s_6s_9 + s_7s_8}{s_1s_8 - s_2s_6 + s_3s_5} \gamma \\ m_2 &= \frac{s_{10}s_2 - s_3s_9 + s_4s_8}{s_1s_8 - s_2s_6 + s_3s_5} \gamma \\ m_3 &= \frac{s_{10}s_1 - s_3s_7 + s_4s_6}{s_1s_8 - s_2s_6 + s_3s_5} \gamma \\ m_4 &= \frac{s_9s_1 - s_2s_7 + s_4s_5}{s_1s_8 - s_2s_6 + s_3s_5} \gamma \\ m_5 &= \gamma, \end{aligned}$$

with $\gamma \neq 0$. So to form a relative equilibrium, five bodies lying at the vertices of a pentagon inscribed in the equator that doesn't stay in any semicircle, must have masses as given above.

4.1.1 Example of relative equilibria on the equator for equal masses

We will next construct an example in the 5-body problem in which 5 bodies of equal masses move on the equator.

Example 1. In the curved 5-body problem in \mathbb{S}_κ^2 , if the pentagon at whose vertices the masses lie is regular, then $m_1 = m_2 = m_3 = m_4 = m_5 > 0$.

Proof. For $m_i > 0, i = 1, 2, 3, 4, 5$, let the the angles between the masses and the x -axis viewed from the center of the Euclidean plane be:

$$\varphi_1 = 0, \quad \varphi_2 = \frac{2\pi}{5}, \quad \varphi_3 = \frac{4\pi}{5}, \quad \varphi_4 = \frac{6\pi}{5}, \quad \varphi_5 = \frac{8\pi}{5}.$$

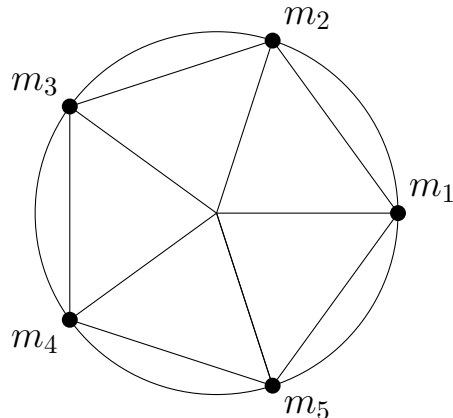


Figure 4.1: The configuration of a regular polygon of 5 equal masses.

Then

$$s_1 := \frac{-1}{\sin^2(\frac{2\pi}{5})}, s_2 := \frac{-1}{\sin^2(\frac{4\pi}{5})}, s_3 := \frac{-1}{\sin^2(\frac{6\pi}{5})}, s_4 := \frac{-1}{\sin^2(\frac{8\pi}{5})}, s_5 := \frac{-1}{\sin^2(\frac{2\pi}{5})},$$

$$s_6 := \frac{-1}{\sin^2(\frac{4\pi}{5})}, s_7 := \frac{-1}{\sin^2(\frac{6\pi}{5})}, s_8 := \frac{-1}{\sin^2(\frac{2\pi}{5})}, s_9 := \frac{-1}{\sin^2(\frac{4\pi}{5})}, s_{10} := \frac{-1}{\sin^2(\frac{2\pi}{5})}.$$

Using (3.6), we obtain

$$m_i = [\gamma, \gamma, \gamma, \gamma, \gamma]^T, \quad i = 1, 2, 3, 4, 5.$$

Any value of the angular velocity makes this configuration a relative equilibrium. \square

Remark. It is not difficult to show that this result can be extended to any odd number of bodies. If the number of bodies is even, singularities occur, so such regular polygons cannot form relative equilibria.

4.2 Relative Equilibria in the curved 5-body problem in \mathbb{S}^2 and \mathbb{H}^2

We consider a 5-body problem on 2-dimensional surfaces of constant curvature κ , with four of the masses arranged at the vertices of a square and the fifth mass at the north pole of the sphere. The five-body set up is discussed for $\kappa > 0$ and for $\kappa < 0$. When the curvature is positive, it is shown that relative equilibria exists when the four masses at the vertices of the square are either equal or two of them are infinitesimal such that

it doesn't affect the motion of the remaining three masses. However with two pairs of masses at the vertices of the square, no relative equilibria exists. In the hyperbolic case, $\kappa < 0$, there exist two values for the angular velocity which produce negative elliptic relative equilibria when the masses at the vertices of the square are equal. We also show that the solutions with non-equal masses do not exist in \mathbb{H}^2 .

Let $\mathbf{q}_i = (x_i, y_i, z_i)$ be the coordinates of the point mass m_i , satisfying the constraint

$$x_i^2 + y_i^2 + \sigma z_i^2 = \kappa^{-1},$$

where σ is the signum function

$$\sigma := \begin{cases} +1 & \text{for } \kappa > 0, \\ -1 & \text{for } \kappa < 0. \end{cases}$$

The inner product between the vectors $\mathbf{q}_i = (x_i, y_i, z_i)$ and $\mathbf{q}_j = (x_j, y_j, z_j)$ is given by

$$q^{ij} = \mathbf{q}_i \cdot \mathbf{q}_j := x_i x_j + y_i y_j + \sigma z_i z_j. \quad (4.1)$$

The distance between the bodies m_i and m_j in \mathbb{S}_κ^2 and \mathbb{H}_κ^2 is defined as

$$q_{ij} := [(x_i - x_j)^2 + (y_i - y_j)^2 + \sigma(z_i - z_j)^2]^{1/2} \quad (4.2)$$

After a change of coordinates and a re-parametrization of time introduced by R. Martínez and C. Simó [28], that we do not repeat here, we can assume without loss of generality that $\kappa = \pm 1$.

Consider five point masses, $m_i > 0$, $i = 1, 2, 3, 4$, whose position vectors, velocities, and accelerations are given by

$$\mathbf{q}_i = (x_i, y_i, z_i), \quad \dot{\mathbf{q}}_i = (\dot{x}_i, \dot{y}_i, \dot{z}_i), \quad \ddot{\mathbf{q}}_i = (\ddot{x}_i, \ddot{y}_i, \ddot{z}_i), \quad i = 1, 2, 3, 4,$$

the mass m_5 is fixed, located at $(0, 0, 1)$ for both signs of the curvature. Then, as shown in [11], the equations of motion take the form

$$\ddot{\mathbf{q}}_i = \sum_{j=1, j \neq i}^N \frac{m_j [\mathbf{q}_j - \sigma(\mathbf{q}_i \cdot \mathbf{q}_j) \mathbf{q}_i]}{[\sigma - \sigma(\mathbf{q}_i \cdot \mathbf{q}_j)]^{3/2}} - \sigma(\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i) \mathbf{q}_i, \quad i = 1, 2, 3, 4, 5. \quad (4.3)$$

On components, the equations of motion and the constraints can be written as

$$\left\{ \begin{array}{l} \ddot{x}_i = \sum_{j=1, j \neq i}^N \frac{m_j [x_j - \sigma(\mathbf{q}_i \cdot \mathbf{q}_j) x_i]}{[\sigma - \sigma(\mathbf{q}_i \cdot \mathbf{q}_j)^2]^{3/2}} - \sigma(\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i) x_i, \\ \ddot{y}_i = \sum_{j=1, j \neq i}^N \frac{m_j [y_j - \sigma(\mathbf{q}_i \cdot \mathbf{q}_j) y_i]}{[\sigma - \sigma(\mathbf{q}_i \cdot \mathbf{q}_j)^2]^{3/2}} - \sigma(\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i) y_i, \\ \ddot{z}_i = \sum_{j=1, j \neq i}^N \frac{m_j [z_j - \sigma(\mathbf{q}_i \cdot \mathbf{q}_j) z_i]}{[\sigma - \sigma(\mathbf{q}_i \cdot \mathbf{q}_j)^2]^{3/2}} - \sigma(\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i) z_i, \\ x_i^2 + y_i^2 + \sigma z_i^2 = \sigma, \quad x_i \dot{x}_i + y_i \dot{y}_i + \sigma z_i \dot{z}_i = 0, \quad i = 1, \dots, n. \end{array} \right. \quad (4.4)$$

4.2.1 The positively curved 5-body problem with one mass at the north pole and four masses form a square in \mathbb{S}^2

In the positively curved 5-body problem, we assume that one body of mass m_5 is fixed at the north pole and the other four bodies of masses $m_1, m_2, m_3,$ and m_4 are located at the vertices of a rotating square which is orthogonal to z -axis and parallel with the equator $z = 0$, we must exclude the equator, since here we have that the masses are located in antipodal positions, which correspond to antipodal singularities and are excluded from our configuration space [12]. The bodies are moving on the unit sphere \mathbb{S}^2 , which has constant curvature 1 (see Figure 4.2).

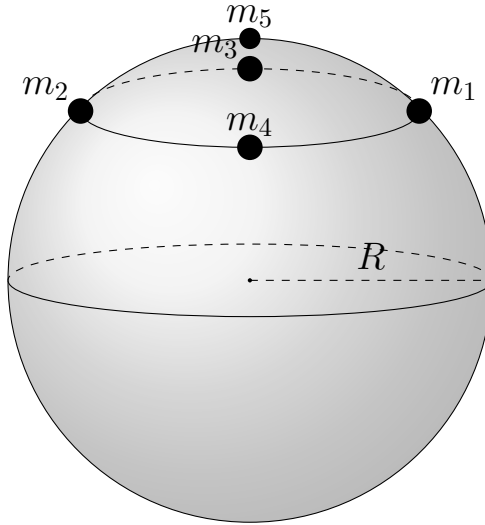


Figure 4.2: The case of 4 equal masses and one body at the north pole

Lemma 6. In the curved 5-body problem in \mathbb{S}^2 , if four bodies of masses $m_1 = m_2 = m_3 = m_4 = m$ are located at the vertices of a rotating square and the fifth body of mass m_5 is fixed at the north pole $(0, 0, 1)$, then we can compute the angular velocity of the system which depends on the masses of the five bodies and the distances between them.

Proof. When $\kappa = 1$, the equations of motion are given by

$$\ddot{\mathbf{q}}_i = \sum_{j=1, j \neq i}^N \frac{m_j [\mathbf{q}_j - (\mathbf{q}_i \cdot \mathbf{q}_j) \mathbf{q}_i]}{[1 - (\mathbf{q}_i \cdot \mathbf{q}_j)^2]^{3/2}} - (\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i) \mathbf{q}_i, \quad i = 1, 2, 3, 4, 5. \quad (4.5)$$

We are interested in solutions of the form

$$\mathbf{q} = (q_1, q_2, q_3, q_4, q_5) \in \mathbb{S}^2, \quad \mathbf{q}_i = (x_i, y_i, z_i), \quad i = 1, 2, 3, 4, 5.$$

$$\begin{cases} x_1 = r \cos \alpha t, & y_1 = r \sin \alpha t, & z_1 = \pm \sqrt{1 - r^2}, \\ x_2 = -r \cos \alpha t, & y_2 = -r \sin \alpha t, & z_2 = \pm \sqrt{1 - r^2}, \\ x_3 = -r \sin \alpha t, & y_3 = r \cos \alpha t, & z_3 = \pm \sqrt{1 - r^2}, \\ x_4 = r \sin \alpha t, & y_4 = -r \cos \alpha t, & z_4 = \pm \sqrt{1 - r^2}, \\ x_5 = 0, & y_5 = 0, & z_5 = 1, \end{cases} \quad (4.6)$$

where r denotes the radius of the circle in which the square configuration rotates, and α denotes the angular velocity of the rotation.

Due to the symmetry, without loss of generality, we will write the equations of motion for x_i and ignore y_i . Consider

$$\begin{cases} \mathbf{q}_1 \cdot \mathbf{q}_2 = \mathbf{q}_3 \cdot \mathbf{q}_4 = 1 - 2r^2 = A, \\ \mathbf{q}_1 \cdot \mathbf{q}_3 = \mathbf{q}_1 \cdot \mathbf{q}_4 = \mathbf{q}_2 \cdot \mathbf{q}_3 = \mathbf{q}_2 \cdot \mathbf{q}_4 = 1 - r^2 = B, \\ \mathbf{q}_i \cdot \mathbf{q}_5 = \pm \sqrt{1 - r^2} = C, \quad i = 1, 2, 3, 4. \end{cases} \quad (4.7)$$

Substituting these coordinates and expressions into the equations of motion (4.5) corresponding to the x_i coordinate, we get

$$\ddot{x}_i = R x_i, \quad i = 1, 2, 3, 4. \quad (4.8)$$

where

$$R = -\frac{m}{4r^3\sqrt{B}} - \frac{2mB}{[r^2(2-r^2)]^{3/2}} \pm \frac{m_5C}{r^3} - r^2\alpha^2.$$

We get the following condition for the existence of the square pyramid relative equilibria in \mathbb{S}^2

$$\alpha^2 = \frac{m}{4r^3(1-r^2)^{3/2}} + \frac{2m}{r^3(2-r^2)^{3/2}} \pm \frac{m_5}{r^3(1-r^2)^{1/2}}. \quad (4.9)$$

Equation (4.9) is the angular velocity of this system and depends on the four equal masses m , the mass of the body at the north pole, and the distances between them. This completes the proof of Lemma 6. \square

Lemma 7. For the pyramidal configuration given in Lemma 6, when $m_1 = m_2 = m > 0$ and $m_3 = m_4 = M > 0$, the square pyramidal relative equilibria exists only if $m = M$.

Proof. This is a consequence of the fact that a regular n -gon, on a circle parallel to the equator, all masses must be equal. Here we present the following proof for our case.

Let $m_1 = m_2 = m$ and $m_4 = m_3 = M$ be the two pairs of masses at vertices of a square, such that m_1, m_2 are at the opposite ends of one diagonal and m_3, m_4 are at the opposite ends of the other diagonal, where m_5 is at the north pole $(0, 0, 1)$. Using the configurations given by (4.6) and substituting these coordinates and expressions (4.7) into the equations of motion (4.5) corresponding to x_1 and x_2 , we get

$$\alpha^2 = \frac{m}{4r^3(1-r^2)^{3/2}} + \frac{2M}{r^3(2-r^2)^{3/2}} \pm \frac{m_5}{r^3(1-r^2)^{1/2}}$$

The equation corresponding to x_3 and x_4 yield

$$\alpha^2 = \frac{M}{4r^3(1-r^2)^{3/2}} + \frac{2m}{r^3(2-r^2)^{3/2}} \pm \frac{m_5}{r^3(1-r^2)^{1/2}}.$$

To have a relative equilibria, we must have

$$\frac{m}{4r^3(1-r^2)^{3/2}} + \frac{2M}{r^3(2-r^2)^{3/2}} = \frac{M}{4r^3(1-r^2)^{3/2}} + \frac{2m}{r^3(2-r^2)^{3/2}}.$$

This is equivalent to

$$\frac{m(\delta - 8\gamma)}{4\gamma\delta} = \frac{M(\delta - 8\gamma)}{4\gamma\delta},$$

where $\gamma = (1-r^2)^{3/2}$ and $\delta = (2-r^2)^{3/2}$. This equation can be satisfied only if $m = M$. Consequently, the solution for the square pyramidal problem in \mathbb{S}^2 must have equal masses. \square

Lemma 8. In the restricted case when $m_1 = m_2 = 0$, there is a circle of radius $r = \sqrt{2/3}$, parallel with the equator, such that a square configuration inscribed in this circle, with m_1, m_2 at the opposite ends of one diagonal and m_3, m_4 at the opposite ends of the other diagonal, forms a relative equilibrium and the mass at the north pole does not influence the existence of relative equilibria.

Proof. Suppose that the masses m_1 and m_2 are very small and do not influence the motion of other particles, but are affected by them. More precisely, we consider the configuration for the square, such that the masses $m_1 = m_2 = 0$ are at the opposite ends of one diagonal and $m_3, m_4 = M$ are at the opposite ends of the other diagonal, where m_5 fixed at the north pole $(0, 0, 1)$. Substituting into the equations of motion (4.5) corresponding to x coordinates we obtain

$$\begin{aligned}\ddot{x}_1 &= -\frac{2MBr \cos \alpha t}{r^3(2-r^2)^{3/2}} \pm \frac{m_5 Cr \cos \alpha t}{r^3} - r^3 \alpha^2 \cos \alpha t, \\ \ddot{x}_2 &= \frac{2MBr \cos \alpha t}{r^3(2-r^2)^{3/2}} \pm \frac{m_5 Cr \cos \alpha t}{r^3} + r^3 \alpha^2 \cos \alpha t, \\ \ddot{x}_3 &= \frac{MBr \sin \alpha t}{4r^3 B^{3/2}} \pm \frac{m_5 Cr \sin \alpha t}{r^3} + r^3 \alpha^2 \sin \alpha t, \\ \ddot{x}_4 &= -\frac{MBr \sin \alpha t}{4r^3 B^{3/2}} \pm \frac{m_5 Cr \sin \alpha t}{r^3} - r^3 \alpha^2 \sin \alpha t, \\ \ddot{x}_5 &= 0.\end{aligned}$$

From equations corresponding to m_1 and m_2 , we obtain

$$\alpha^2 = \frac{2M}{r^3(2-r^2)^{3/2}} + \frac{m_5}{r^3(1-r^2)^{1/2}}, \quad (4.10)$$

similarly, equations of motion corresponding to m_3 and m_4 , lead us to

$$\alpha^2 = \frac{M}{4r^3(1-r^2)^{3/2}} + \frac{m_5}{r^3(1-r^2)^{1/2}}. \quad (4.11)$$

For relative equilibria to exist the right hand side of equations (4.10) and (4.11) must be equal; therefore

$$\frac{2M}{r^3(2-r^2)^{3/2}} = \frac{M}{4r^3(1-r^2)^{3/2}}.$$

Notice that we arrived at the condition for the existence of a relative equilibria for the

restricted 4-body problem with 2 equal masses and 2 negligible masses. Thus the square pyramidal body problem of 2 equal masses and 2 negligible masses reduced to 4-body problem with two equal masses and 2 negligible masses. This is due to the force on the body at the north pole being cancelled out. In fact, we treat the square pyramid of 2 negligible masses as the 4-body problem of 2 negligible masses. Hence, we recovered the same result studied in [2]; in \mathbb{S}_κ^2 , there is a circle of radius $r = \sqrt{2/3}\kappa^{-1/2}$, parallel with the equator, such that a square configuration inscribed in this circle, with opposite vertices having the same mass, forms a relative equilibrium. But here we restrict our study to the unit sphere of curvature 1. \square

Remark 1. Unlike the square pyramid problem with equal masses in \mathbb{S}^2 , the angular velocity α depends only on M not m_5 .

Now we are in a condition to state and prove the first main theorem concerning the case of positive curvature.

Theorem 9. Consider the positively curved square 5-body problem in \mathbb{S}^2 with four masses inscribed in a circle of radius $0 < r < 1$, parallel with the equator, while the fifth mass is fixed at the north pole. If $m_i = m, i = 1, 2, 3, 4$, then relative equilibria exists for all positive values of m and M when $z > 0$. However, when $z < 0$ relative equilibria exists only in

$$R_1(\Gamma, z) = \{(\Gamma, z) \mid -1 < z < 0 \wedge \Gamma > 2\sqrt{\frac{z^2}{(z^2 + 1)^3} - \frac{1}{4z^2}}\}.$$

Proof. Let $\Gamma = \frac{m_5}{m}$, then equation (4.9) can be written as:

$$\frac{\alpha^2}{m} = \pm \frac{1}{4z^3(1 - z^2)^{3/2}} \pm \frac{2}{(1 - z^4)^{3/2}} \pm \frac{\Gamma}{z(1 - z^2)^{3/2}} = f^\pm(\Gamma, z)$$

where $z^3 = (1 - r^2)^{3/2}$, $r^3 = \pm(1 - z^2)^{3/2}$, $(2 - r^2)^{3/2} = (z^2 + 1)^{3/2}$, and $(1 + z^2)^{3/2}(1 - z^2)^{3/2} = (1 - z^4)^{3/2}$. To show the existence of Relative Equilibria (RE) we need to show that $f^\pm(\Gamma, z) \geq 0$. Let $z > 0$, then

$$f^+(\Gamma, z) = \frac{1}{4z^3(1 - z^2)^{3/2}} + \frac{2}{(1 - z^4)^{3/2}} + \frac{\Gamma}{z(1 - z^2)^{3/2}}. \quad (4.12)$$

It is trivial to see that $f^+(\Gamma, z) > 0$ when $z \in (0, 1)$ which implies the existence of RE for all values of z and Γ .

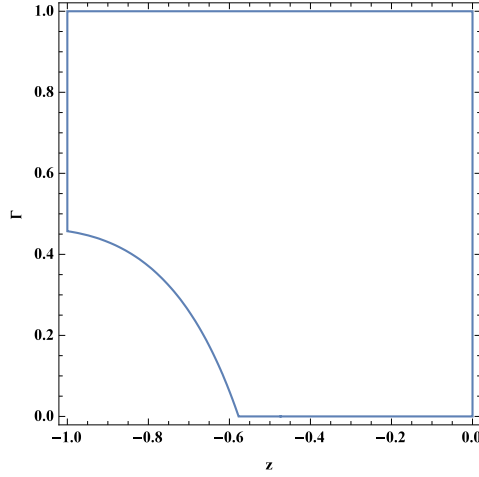


Figure 4.3: Region $R_1(\Gamma, z)$ (colored).

In the case when $z < 0$, we have

$$f^-(\Gamma, z) = -\frac{1}{4z^3(1-z^2)^{3/2}} - \frac{2}{(1-z^4)^{3/2}} - \frac{\Gamma}{z(1-z^2)^{3/2}}. \quad (4.13)$$

To find regions of RE when $z < 0$, we rewrite equation (4.13).

$$f^-(\Gamma, z) = \frac{\mu(\Gamma, z)}{\delta(z)},$$

where

$$\begin{aligned} \mu(\Gamma, z) &= \sqrt{1-z^2} \left(8z^3 + (z^2+1)^{3/2} (4\Gamma z^2 + 1) \right), \\ \delta(z) &= 4z^3 (1-z^2) (z^2-1) (z^2+1)^{3/2}. \end{aligned}$$

The function $\delta(z) > 0$ when $z \in (-1, 0)$ as z^3 and (z^2-1) are negative and two of the remaining components are positive when $z \in (-1, 0)$. Using algebraic manipulation, it can be shown that $\mu(\Gamma, z) > 0$ in

$$R_1(\Gamma, z) = \{(\Gamma, z) \mid -1 < z < 0 \wedge \Gamma > 2\sqrt{\frac{z^2}{(z^2+1)^3} - \frac{1}{4z^2}}\}. \quad (4.14)$$

This shows the existence of RE in the region $R_1(\Gamma, z)$, shown in Figure 4.3. \square

4.2.2 The negatively curved 5-body problem with one mass at the vertex and four masses form a square in \mathbb{H}^2

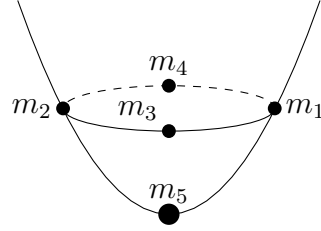


Figure 4.4: The case of 4 equal masses with one body at the vertex in \mathbb{H}^2

In this section we study the dynamics of the 5-bodies that interact in \mathbb{H}^2 and will investigate the existence of elliptic relative equilibria of the square pyramidal configuration which has four masses at the vertices of a square and one mass at the vertex of \mathbb{H}^2 (see Figure 4.4).

Lemma 10. The negatively curved square pyramidal 5-body problem with two pairs of equal mass in \mathbb{H}^2 has no elliptic relative equilibria; here we include the restricted case $m_1 = m_2 = 0$.

The proof of this lemma is analogous to Lemmas 7 and 8, and we omit it here.

Theorem 11. Consider the negatively curved square 5-body problem in \mathbb{H}^2 with four masses $m_i = m, i = 1, 2, 3, 4$ at the vertices of a square and the fifth mass fixed at the vertex. If $m_i = m, i = 1, 2, 3, 4$, relative equilibria exist for all positive values of m, m_5 and r (or $z > 1$).

Proof. When $\kappa = -1$, the equations of motion are given by

$$\ddot{\mathbf{q}}_i = \sum_{j=1, j \neq i}^N \frac{m_j [\mathbf{q}_j + (\mathbf{q}_i \cdot \mathbf{q}_j) \mathbf{q}_i]}{[1 + (\mathbf{q}_i \cdot \mathbf{q}_j)^2]^{3/2}} + (\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i) \mathbf{q}_i, \quad i = 1, 2, 3, 4, 5. \quad (4.15)$$

Then, we search for solutions of the form

$$\begin{cases} x_1 = r \cos at, & y_1 = r \sin at, & z_1 = \sqrt{1 + r^2}, \\ x_2 = -r \cos at, & y_2 = -r \sin at, & z_2 = \sqrt{1 + r^2}, \\ x_3 = -r \sin at, & y_3 = r \cos at, & z_3 = \sqrt{1 + r^2}, \\ x_4 = r \sin at, & y_4 = -r \cos at, & z_4 = \sqrt{1 + r^2}, \\ x_5 = 0, & y_5 = 0, & z_5 = 1, \end{cases} \quad (4.16)$$

Consider the Lorentz inner product between particles

$$\begin{cases} \mathbf{q}_1 \sqcap \mathbf{q}_2 = \mathbf{q}_3 \sqcap \mathbf{q}_4 = -(1 + 2r^2), & \mathbf{q}_i \sqcap \mathbf{q}_5 = -\sqrt{1 + r^2}, \quad i = 1, 2, 3, 4. \\ \mathbf{q}_1 \sqcap \mathbf{q}_3 = \mathbf{q}_1 \sqcap \mathbf{q}_4 = \mathbf{q}_2 \sqcap \mathbf{q}_3 = \mathbf{q}_2 \sqcap \mathbf{q}_4 = -(1 + r^2). \end{cases} \quad (4.17)$$

Substituting these coordinates and expressions into the equations of motion (4.15) corresponding to the x_i coordinate, we get

$$\ddot{x}_i = R x_i, \quad \ddot{x}_5 = 0, \quad i = 1, 2, 3, 4. \quad (4.18)$$

where

$$R = -\frac{m}{4r^3\sqrt{1+r^2}} - \frac{2m(1+r^2)}{r^3(r^2+2)^{3/2}} - \frac{m_5\sqrt{1+r^2}}{r^3} + r^2\alpha^2.$$

Comparing the above equations to obtain conditions for the existence of relative equilibria, we obtain

$$\alpha^2 = \frac{m}{4r^3(1+r^2)^{3/2}} + \frac{2m}{r^3(2+r^2)^{3/2}} + \frac{m_5}{r^3(1+r^2)^{1/2}} =: f(r). \quad (4.19)$$

This shows the dependence of angular velocity of the system on the four equal masses, m , the mass of the body at the north pole, and r . Thus for any m , m_5 , and r in \mathbb{H}^2 , there are always two values of α satisfying the above equation, one corresponding to each direction of rotation.

Let $\Gamma = \frac{m_5}{m}$, where $r > 0$ and $z^2 = r^2 + 1$, then equation (4.19) can be written as

$$\frac{\alpha^2}{m} = \frac{1}{4z^3(z^2-1)^{3/2}} + \frac{2}{(z^4-1)^{3/2}} + \frac{\Gamma}{z(z^2-1)^{3/2}} =: f(z).$$

The function $f(z)$ is positive for $z > 1$. Thus, for $m, \Gamma > 0$ and $z > 1$, there are positive and negative angular velocity α that lead to an elliptic relative equilibrium. This completes the proof of the Theorem. \square

Chapter 5

Relative Equilibria in The Curved 6-Body Problem

5.1 Relative equilibria on the equator for 6-body problem

Returning to the equations of motion on the equator (3.6), we consider the 6-body problem and place the bodies at the vertices of a hexagon inscribed in the equator of \mathbb{S}_κ^2 that does not lie within any semicircle. (If the bodies are in some half of a hemisphere they cannot form relative equilibria, as shown in [8]). Then the expressions of $s_i, i = 1, 2, \dots, 15$, are

$$\begin{aligned}
 s_1 &:= \frac{\kappa^{3/2} \sin(\varphi_1 - \varphi_2)}{|\sin(\varphi_1 - \varphi_2)|^3}, & s_2 &:= \frac{\kappa^{3/2} \sin(\varphi_1 - \varphi_3)}{|\sin(\varphi_1 - \varphi_3)|^3}, & s_3 &:= \frac{\kappa^{3/2} \sin(\varphi_1 - \varphi_4)}{|\sin(\varphi_1 - \varphi_4)|^3}, \\
 s_4 &:= \frac{\kappa^{3/2} \sin(\varphi_1 - \varphi_5)}{|\sin(\varphi_1 - \varphi_5)|^3}, & s_5 &:= \frac{\kappa^{3/2} \sin(\varphi_1 - \varphi_6)}{|\sin(\varphi_1 - \varphi_6)|^3}, & s_6 &:= \frac{\kappa^{3/2} \sin(\varphi_2 - \varphi_3)}{|\sin(\varphi_2 - \varphi_3)|^3}, \\
 s_7 &:= \frac{\kappa^{3/2} \sin(\varphi_2 - \varphi_4)}{|\sin(\varphi_2 - \varphi_4)|^3}, & s_8 &:= \frac{\kappa^{3/2} \sin(\varphi_2 - \varphi_5)}{|\sin(\varphi_2 - \varphi_5)|^3}, & s_9 &:= \frac{\kappa^{3/2} \sin(\varphi_2 - \varphi_6)}{|\sin(\varphi_2 - \varphi_6)|^3}, \\
 s_{10} &:= \frac{\kappa^{3/2} \sin(\varphi_3 - \varphi_4)}{|\sin(\varphi_3 - \varphi_4)|^3}, & s_{11} &:= \frac{\kappa^{3/2} \sin(\varphi_3 - \varphi_5)}{|\sin(\varphi_3 - \varphi_5)|^3}, & s_{12} &:= \frac{\kappa^{3/2} \sin(\varphi_3 - \varphi_6)}{|\sin(\varphi_3 - \varphi_6)|^3}, \\
 s_{13} &:= \frac{\kappa^{3/2} \sin(\varphi_4 - \varphi_5)}{|\sin(\varphi_4 - \varphi_5)|^3}, & s_{14} &:= \frac{\kappa^{3/2} \sin(\varphi_4 - \varphi_6)}{|\sin(\varphi_4 - \varphi_6)|^3}, & s_{15} &:= \frac{\kappa^{3/2} \sin(\varphi_5 - \varphi_6)}{|\sin(\varphi_5 - \varphi_6)|^3}.
 \end{aligned}$$

Theorem 12. A necessary condition that the hexagons inscribed in the equator of \mathbb{S}_κ^2 , with the six masses $m_1, m_2, m_3, m_4, m_5, m_6 > 0$ at its vertices and avoiding antipodal

positions, forms a relative equilibrium is that

$$\begin{aligned}
& s_3 s_9 s_{11} - s_3 s_8 s_{12} + s_4 s_7 s_{12} - s_4 s_9 s_{10} - s_5 s_7 s_{11} + s_5 s_8 s_{10} - s_2 s_7 s_{15} + s_2 s_8 s_{14} \\
& - s_2 s_9 s_{13} + s_3 s_6 s_{15} - s_4 s_6 s_{14} + s_5 s_6 s_{13} + s_1 s_{10} s_{15} - s_1 s_{11} s_{14} + s_1 s_{12} s_{13} = 0.
\end{aligned} \tag{5.1}$$

Proof. From the equations of motion (3.6), we then obtain the system

$$\begin{cases}
-m_2 s_1 - m_3 s_2 - m_4 s_3 - m_5 s_4 - m_6 s_5 = 0 \\
m_1 s_1 - m_3 s_6 - m_4 s_7 - m_5 s_8 - m_6 s_9 = 0 \\
m_1 s_2 + m_2 s_6 - m_4 s_{10} - m_5 s_{11} - m_6 s_{12} = 0 \\
m_1 s_3 + m_2 s_7 + m_3 s_{10} - m_5 s_{13} - m_6 s_{14} = 0 \\
m_1 s_4 + m_2 s_8 + m_3 s_{11} + m_4 s_{13} - m_6 s_{15} = 0, \\
m_1 s_5 + m_2 s_9 + m_3 s_{12} + m_4 s_{14} + m_5 s_{15} = 0.
\end{cases}$$

To have a non-trivial solution of the masses, the determinant of the above system must vanish, which is equivalent to equation (5.1). \square

5.2 The 6-body problem with 3 massless bodies and 3 equal masses in \mathbb{S}_κ^2

In this section we study the existence of relative equilibria of the restricted (3+3)-body problem that lies on a circle parallel with the xy -plane in \mathbb{S}_κ^2 . Such that $m_4 = m_5 = m_6 = m > 0$ in an equilateral triangle inscribed in the circle and $m_1 = m_2 = m_3 = 0$ in an equilateral triangle inscribed in the same circle. Therefore, we are seeking the existence of a solution of the form

$$\mathbf{q} = (q_1, q_2, q_3, q_4, q_5, q_6) \in \mathbb{S}_\kappa^2, \quad \mathbf{q}_i = (x_i, y_i), \quad i = 1, 2, 3, 4, 5, 6.$$

where

$$x_1 = r \cos(\alpha t), \quad y_1 = r \sin(\alpha t)$$

$$x_2 = r \cos(\alpha t + 2\pi/3), \quad y_2 = r \sin(\alpha t + 2\pi/3)$$

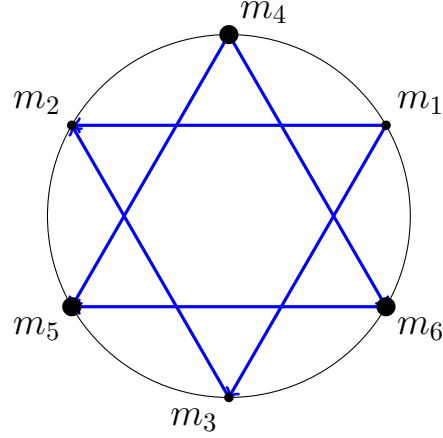


Figure 5.1: The case of 3 equal masses and 3 massless bodies.

$$x_3 = r \cos(\alpha t + 4\pi/3), \quad y_3 = r \sin(\alpha t + 4\pi/3)$$

$$x_4 = r \cos(\alpha t + \pi/3), \quad y_4 = r \sin(\alpha t + \pi/3)$$

$$x_5 = r \cos(\alpha t + \pi), \quad y_5 = r \sin(\alpha t + \pi),$$

$$x_6 = r \cos(\alpha t + 5\pi/3), \quad y_6 = r \sin(\alpha t + 5\pi/3).$$

Then the equivalent equations of motion take the form

$$\left\{ \begin{array}{l} \ddot{x}_1 = \frac{m_4 \left[x_4 - \left(1 - \frac{\kappa \rho_{14}^2}{2} \right) x_1 \right]}{\left(1 - \frac{\kappa \rho_{14}^2}{4} \right)^{3/2} \rho_{14}^3} + \frac{m_5 \left[x_5 - \left(1 - \frac{\kappa \rho_{51}^2}{2} \right) x_1 \right]}{\left(1 - \frac{\kappa \rho_{51}^2}{4} \right)^{3/2} \rho_{51}^3} + \frac{m_6 \left[x_6 - \left(1 - \frac{\kappa \rho_{61}^2}{2} \right) x_1 \right]}{\left(1 - \frac{\kappa \rho_{61}^2}{4} \right)^{3/2} \rho_{61}^3} - \kappa (\dot{x}_1^2 + \dot{y}_1^2 + \kappa B_1) x_1 \\ \ddot{y}_1 = \frac{m_4 \left[y_4 - \left(1 - \frac{\kappa \rho_{14}^2}{2} \right) y_1 \right]}{\left(1 - \frac{\kappa \rho_{14}^2}{4} \right)^{3/2} \rho_{14}^3} + \frac{m_5 \left[y_5 - \left(1 - \frac{\kappa \rho_{51}^2}{2} \right) y_1 \right]}{\left(1 - \frac{\kappa \rho_{51}^2}{4} \right)^{3/2} \rho_{51}^3} + \frac{m_6 \left[y_6 - \left(1 - \frac{\kappa \rho_{61}^2}{2} \right) y_1 \right]}{\left(1 - \frac{\kappa \rho_{61}^2}{4} \right)^{3/2} \rho_{61}^3} - \kappa (\dot{x}_1^2 + \dot{y}_1^2 + \kappa B_1) y_1 \end{array} \right.$$

$$\left\{ \begin{array}{l} \ddot{x}_2 = \frac{m_4 \left[x_4 - \left(1 - \frac{\kappa \rho_{24}^2}{2} \right) x_2 \right]}{\left(1 - \frac{\kappa \rho_{24}^2}{4} \right)^{3/2} \rho_{24}^3} + \frac{m_5 \left[x_5 - \left(1 - \frac{\kappa \rho_{52}^2}{2} \right) x_2 \right]}{\left(1 - \frac{\kappa \rho_{52}^2}{4} \right)^{3/2} \rho_{52}^3} + \frac{m_6 \left[x_6 - \left(1 - \frac{\kappa \rho_{62}^2}{2} \right) x_2 \right]}{\left(1 - \frac{\kappa \rho_{62}^2}{4} \right)^{3/2} \rho_{62}^3} - \kappa (\dot{x}_2^2 + \dot{y}_2^2 + \kappa B_2) x_2 \\ \ddot{y}_2 = \frac{m_4 \left[y_4 - \left(1 - \frac{\kappa \rho_{24}^2}{2} \right) y_2 \right]}{\left(1 - \frac{\kappa \rho_{24}^2}{4} \right)^{3/2} \rho_{24}^3} + \frac{m_5 \left[y_5 - \left(1 - \frac{\kappa \rho_{52}^2}{2} \right) y_2 \right]}{\left(1 - \frac{\kappa \rho_{52}^2}{4} \right)^{3/2} \rho_{52}^3} + \frac{m_6 \left[y_6 - \left(1 - \frac{\kappa \rho_{62}^2}{2} \right) y_2 \right]}{\left(1 - \frac{\kappa \rho_{62}^2}{4} \right)^{3/2} \rho_{62}^3} - \kappa (\dot{x}_2^2 + \dot{y}_2^2 + \kappa B_2) y_2 \end{array} \right.$$

$$\begin{cases}
\ddot{x}_3 = \frac{m_4 \left[x_4 - \left(1 - \frac{\kappa \rho_{34}^2}{2} \right) x_3 \right]}{\left(1 - \frac{\kappa \rho_{34}^2}{4} \right)^{3/2} \rho_{34}^3} + \frac{m_5 \left[x_5 - \left(1 - \frac{\kappa \rho_{53}^2}{2} \right) x_3 \right]}{\left(1 - \frac{\kappa \rho_{53}^2}{4} \right)^{3/2} \rho_{53}^3} + \frac{m_6 \left[x_6 - \left(1 - \frac{\kappa \rho_{63}^2}{2} \right) x_3 \right]}{\left(1 - \frac{\kappa \rho_{63}^2}{4} \right)^{3/2} \rho_{63}^3} - \kappa (\dot{x}_3^2 + \dot{y}_3^2 + \kappa B_3) x_3 \\
\ddot{y}_3 = \frac{m_4 \left[y_4 - \left(1 - \frac{\kappa \rho_{34}^2}{2} \right) y_3 \right]}{\left(1 - \frac{\kappa \rho_{34}^2}{4} \right)^{3/2} \rho_{34}^3} + \frac{m_5 \left[y_5 - \left(1 - \frac{\kappa \rho_{53}^2}{2} \right) y_3 \right]}{\left(1 - \frac{\kappa \rho_{53}^2}{4} \right)^{3/2} \rho_{53}^3} + \frac{m_6 \left[y_6 - \left(1 - \frac{\kappa \rho_{63}^2}{2} \right) y_3 \right]}{\left(1 - \frac{\kappa \rho_{63}^2}{4} \right)^{3/2} \rho_{63}^3} - \kappa (\dot{x}_3^2 + \dot{y}_3^2 + \kappa B_3) y_3
\end{cases}$$

$$\begin{cases}
\ddot{x}_4 = \frac{m_5 \left[x_5 - \left(1 - \frac{\kappa \rho_{54}^2}{2} \right) x_4 \right]}{\left(1 - \frac{\kappa \rho_{54}^2}{4} \right)^{3/2} \rho_{54}^3} + \frac{m_6 \left[x_6 - \left(1 - \frac{\kappa \rho_{64}^2}{2} \right) x_4 \right]}{\left(1 - \frac{\kappa \rho_{64}^2}{4} \right)^{3/2} \rho_{64}^3} - \kappa (\dot{x}_4^2 + \dot{y}_4^2 + \kappa B_4) x_4 \\
\ddot{y}_4 = \frac{m_5 \left[y_5 - \left(1 - \frac{\kappa \rho_{54}^2}{2} \right) y_4 \right]}{\left(1 - \frac{\kappa \rho_{54}^2}{4} \right)^{3/2} \rho_{54}^3} + \frac{m_6 \left[y_6 - \left(1 - \frac{\kappa \rho_{64}^2}{2} \right) y_4 \right]}{\left(1 - \frac{\kappa \rho_{64}^2}{4} \right)^{3/2} \rho_{64}^3} - \kappa (\dot{x}_4^2 + \dot{y}_4^2 + \kappa B_4) y_4
\end{cases}$$

$$\begin{cases}
\ddot{x}_5 = \frac{m_4 \left[x_4 - \left(1 - \frac{\kappa \rho_{54}^2}{2} \right) x_5 \right]}{\left(1 - \frac{\kappa \rho_{54}^2}{4} \right)^{3/2} \rho_{54}^3} + \frac{m_6 \left[x_6 - \left(1 - \frac{\kappa \rho_{65}^2}{2} \right) x_5 \right]}{\left(1 - \frac{\kappa \rho_{65}^2}{4} \right)^{3/2} \rho_{65}^3} - \kappa (\dot{x}_5^2 + \dot{y}_5^2 + \kappa B_5) x_5 \\
\ddot{y}_5 = \frac{m_4 \left[y_4 - \left(1 - \frac{\kappa \rho_{54}^2}{2} \right) y_5 \right]}{\left(1 - \frac{\kappa \rho_{54}^2}{4} \right)^{3/2} \rho_{54}^3} + \frac{m_6 \left[y_6 - \left(1 - \frac{\kappa \rho_{65}^2}{2} \right) y_5 \right]}{\left(1 - \frac{\kappa \rho_{65}^2}{4} \right)^{3/2} \rho_{65}^3} - \kappa (\dot{x}_5^2 + \dot{y}_5^2 + \kappa B_5) y_5
\end{cases}$$

$$\begin{cases}
\ddot{x}_6 = \frac{m_4 \left[x_4 - \left(1 - \frac{\kappa \rho_{64}^2}{2} \right) x_6 \right]}{\left(1 - \frac{\kappa \rho_{64}^2}{4} \right)^{3/2} \rho_{64}^3} + \frac{m_5 \left[x_5 - \left(1 - \frac{\kappa \rho_{65}^2}{2} \right) x_6 \right]}{\left(1 - \frac{\kappa \rho_{65}^2}{4} \right)^{3/2} \rho_{65}^3} - \kappa (\dot{x}_6^2 + \dot{y}_6^2 + \kappa B_6) x_6 \\
\ddot{y}_6 = \frac{m_4 \left[y_4 - \left(1 - \frac{\kappa \rho_{64}^2}{2} \right) y_6 \right]}{\left(1 - \frac{\kappa \rho_{64}^2}{4} \right)^{3/2} \rho_{64}^3} + \frac{m_5 \left[y_5 - \left(1 - \frac{\kappa \rho_{65}^2}{2} \right) y_6 \right]}{\left(1 - \frac{\kappa \rho_{65}^2}{4} \right)^{3/2} \rho_{65}^3} - \kappa (\dot{x}_6^2 + \dot{y}_6^2 + \kappa B_6) y_6
\end{cases}$$

where

$$\begin{aligned}
\rho_{12}^2 &= \rho_{46}^2 = \rho_{56}^2 = \rho_{13}^2 = 3r^2, \\
\rho_{16}^2 &= \rho_{36}^2 = \rho_{41}^2 = \rho_{42}^2 = \rho_{52}^2 = \rho_{53}^2 = r^2, \\
\rho_{62}^2 &= \rho_{15}^2 = \rho_{43}^2 = 4r^2.
\end{aligned}$$

Substituting these expressions into the first six equations of the system, we obtain the angular velocity

$$\alpha^2 = \frac{m}{4r^3(1 - \kappa r^2)^{3/2}} + \frac{m}{r^3(1 - \frac{\kappa r^2}{4})^{3/2}},$$

whereas the last six equations yield

$$\alpha^2 = \frac{m}{\sqrt{3}r^3(1 - \frac{3\kappa r^2}{4})^{3/2}}.$$

So, to have a solution, the equations for the angular velocity must be satisfied. This is equivalent to

$$\frac{1}{4(1 - \kappa r^2)^{3/2}} + \frac{1}{(1 - \frac{\kappa r^2}{4})^{3/2}} = \frac{1}{\sqrt{3}(1 - \frac{3\kappa r^2}{4})^{3/2}}.$$

Let us notice that we obtain the same result for the 4-body problem with one massless body and three equal masses, namely, the kite-shaped relative equilibria in \mathbb{S}_κ^2 . With all the above we have proved the following result.

Theorem 13. Given equal masses $m_4 = m_5 = m_6 = m > 0$ lie at the vertices of an equilateral triangle inscribed in a circle parallel with the plane of the equator, whereas the massless bodies $m_1 = m_2 = m_3 = 0$ form another an equilateral triangle inscribed in the same circle. Then in \mathbb{S}_κ^2 , there exists at least one kite-shaped relative equilibrium.

Chapter 6

The 2-and 3-Centre Problems on \mathbb{H}^2

This problem concerns the motion of a particle of mass M subject to the attraction of two fixed particles of mass m located symmetrically on a surface of constant negative curvature κ , which without loss of generality we assume $\kappa = -1$.

It is well known that this surface of negative curvature \mathbb{H}_κ^2 corresponds to a model of the hyperbolic geometry. In this chapter we are using two of these models for \mathbb{H}_κ^2 :

- The Pseudosphere or Weierstrass model \mathbb{L}^2 , which corresponds to the upper part of the hyperboloid $x^2 + y^2 - z^2 = -1$ embedded in $\mathbb{R}^{2,1}$, that is \mathbb{R}^3 endowed with the Lorenz inner product $(+, +, -)$.
- The Poincaré upper semiplane model $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 | y > 0\}$, endowed with the metric $ds^2 = (dx^2 + dy^2)/y^2$ [12].

6.1 The equations of motion on the Poincaré upper semi-plane model

The N -body problem in the Poincaré upper semiplane model \mathbb{H}^2 can be defined as the simple mechanical system on the configuration space $Q = (\mathbb{H}^2)^N \setminus \Delta$, where Δ is the set of collision configurations, and whose Lagrangian $L : TQ \rightarrow \mathbb{R}$ is given by

$$L = \frac{1}{2} \left(\sum_i m_i v_i^2 \right) - \sum_{i < j} m_i m_j U(d_{ij}).$$

In the above equation m_i and v_i^2 are, respectively, the mass and the square of the hyperbolic norm of the velocity of the i^{th} particle. The positive number d_{ij} is the

hyperbolic distance between the i^{th} and the j^{th} particle and the potential $U : \mathbb{R}^+ \rightarrow \mathbb{R}$ is given by

$$U(d) = -\coth(d), \quad d > 0. \quad (6.1)$$

This choice of potential is a generalization of the classical N -body problem in the following way: The Newtonian potential $U_N(d) = \frac{1}{d}$ is (proportional to) the fundamental solution of the Laplacian operator on \mathbb{R}^3 , while the proposed potential $U(d)$ is (proportional to) the fundamental solution of the Laplace-Beltrami operator on \mathbb{H}^2 . For more information see [1].

We now proceed to write explicit formulae for the equations of motion for the N -body problem in the \mathbb{H}^2 model. The equations are written in terms of the global cartesian coordinates.

Using that the Riemannian distance, d , between two points $(x_1, y_1), (x_2, y_2) \in \mathbb{H}^2$ satisfies

$$\cosh(d) = 1 + \frac{(x_1 - x_2)^2 + (y_1 - y_2)^2}{2y_1y_2}. \quad (6.2)$$

We obtain that

$$d_{ij} = \cosh^{-1} \left(1 + \frac{(x_i - x_j)^2 + (y_i - y_j)^2}{2y_iy_j} \right) \quad (6.3)$$

and the relationship

$$\coth(\cosh^{-1}(z)) = \frac{z}{\sqrt{z^2 - 1}}$$

that is valid for all $z > 1$ we can write

$$L = \frac{1}{2} \left(\sum_i m_i \frac{\dot{x}_i^2 + \dot{y}_i^2}{y_i^2} \right) + \sum_{i < j} m_i m_j \frac{(x_i - x_j)^2 + y_i^2 + y_j^2}{\sqrt{((x_i - x_j)^2 + (y_i - y_j)^2)((x_i - x_j)^2 + (y_i + y_j)^2)}}. \quad (6.4)$$

The Euler-Lagrange equations for the above Lagrangian can be cast in the form:

$$\begin{aligned} \ddot{x}_i &= \frac{2\dot{x}_i\dot{y}_i}{y_i} + \sum_{j \neq i} \frac{m_j 8(x_j - x_i)y_i^4 y_j^2}{[((x_i - x_j)^2 + (y_i - y_j)^2)((x_i - x_j)^2 + (y_i + y_j)^2)]^{3/2}}, \\ \ddot{y}_i &= \frac{\dot{y}_i^2 - \dot{x}_i^2}{y_i} + \sum_{j \neq i} \frac{m_j 4y_i^3 y_j^2 ((x_i - x_j)^2 + y_j^2 - y_i^2)}{[((x_i - x_j)^2 + (y_i - y_j)^2)((x_i - x_j)^2 + (y_i + y_j)^2)]^{3/2}}, \end{aligned} \quad (6.5)$$

where i runs from 1 to N . The first term appearing in the right of the above equations is the inertia of the point mass m_i manifesting its tendency to move along a geodesic of \mathbb{H}^2 . The second term is the force due to the gravitational interaction with the other

masses.

6.2 The negative curved two center problem

We consider two point particles with positive mass m_1 (called primaries), located symmetrically on the same geodesic with respect to the middle point between them; without loss of generality we assume that the geodesic is the unitary half circle centered at the origin, so the particles are fixed at positions $q_1 = (x_1, y_1)$ and $q_2 = (-x_1, y_1)$ with $|q_1| = |q_2| = 1$, $y_1 > 0$. We consider a particle of positive mass m with position $q = (x, y)$ moving under the influence of the primaries. According to (6.5), the equations of motion for the last particle are

$$\begin{aligned} \ddot{x} &= \frac{2\dot{x}\dot{y}}{y} + \frac{8(x_1 - x)y^4y_1^2}{\left[((x - x_1)^2 + (y - y_1)^2)((x - x_1)^2 + (y + y_1)^2) \right]^{3/2}} \\ &+ \frac{8(-x_1 - x)y^4y_1^2}{\left[((x + x_1)^2 + (y - y_1)^2)((x + x_1)^2 + (y + y_1)^2) \right]^{3/2}} \\ \ddot{y} &= \frac{\dot{y}^2 - \dot{x}^2}{y} + \frac{4y^3y_1^2((x - x_1)^2 + y_1^2 - y^2)}{\left[((x - x_1)^2 + (y - y_1)^2)((x - x_1)^2 + (y + y_1)^2) \right]^{3/2}} \\ &+ \frac{4y^3y_1^2((x + x_1)^2 + y_1^2 - y^2)}{\left[((x + x_1)^2 + (y - y_1)^2)((x + x_1)^2 + (y + y_1)^2) \right]^{3/2}} \end{aligned} \quad (6.6)$$

6.2.1 On the Weierstrass model \mathbb{L}^2

Here we give an alternative way to get the equations of motion for the two center problem on the Weierstrass model. Without loss of generality, we fix the points r_1 and r_2 at $(x_1, 0, \sqrt{x_1 + 1})$ and $(-x_1, 0, \sqrt{x_1 + 1})$, respectively. The particle $q \in \mathbb{L}^2$ of mass 1 has coordinates $q = (x, y, \sqrt{1 + x^2 + y^2})$.

The potential is defined as

$$U = -\cot(d(q, r_1)) - \cot(d(q, r_2)), \quad (6.7)$$

where d is the hyperbolic distance on \mathbb{L}^2 . Given $a, b \in \mathbb{H}^2$, it satisfies $\cosh(d(a, b)) = -a_1b_1 - a_2b_2 + a_3b_3 =: -a \odot b$. The kinetic energy is defined by

$$T = \frac{1}{2}(\dot{q} \odot \dot{q}). \quad (6.8)$$

Considering the following expression we obtain the equation of motion for the particle

q ,

$$\frac{d}{dt} \left(\frac{dL_k}{dq} \right) - \frac{dL_k}{dq} - \lambda \frac{df}{dq} = 0, \quad (6.9)$$

where λ is the Lagrange multiplier corresponding to the restriction f , which maintains the particle on \mathbb{L}^2 .

The equation of motion is

$$\ddot{q} = \sum_{i=1}^2 \frac{r_i + (r_i \odot q)q}{(-1 + (r_i \odot q)^2)^{3/2}} + (\dot{q} \odot \dot{q})q. \quad (6.10)$$

It is easy to verify that the Poincaré upper half plane model described above, can be written in complex variables as $\mathbb{H}^2 = \{w \in \mathbb{C} | \text{Im}(w) > 0\}$ with the Riemannian metric

$$-ds^2 = \frac{4}{(w - \bar{w})^2} dw d\bar{w}.$$

The particle q has coordinates (x, y) and, without loss of generality, we consider the points r_1 and r_2 at (x_1, y_1) and $(-x_1, y_1)$, respectively, with $|r_1| = |r_2| = 1$.

In this space the cotangent potential takes the form

$$U(q, \bar{q}) = - \sum_{i=1}^2 \frac{(\bar{q} - q)(\bar{r}_i - r_i) - 2(|q|^2 + 1)}{T_i},$$

where

$$T_i = (4(\text{Re}(q) - \text{Re}(r_i))^2 [(\text{Re}(q) - \text{Re}(r_i))^2 + 2(\text{Im}(q)^2 + \text{Im}(r_i)^2)] + 4(\text{Im}(q)^2 - \text{Im}(r_i)^2)^2)^{1/2},$$

and the kinetic energy is

$$T = \frac{2|\dot{q}|^2}{(q - \bar{q})^2}.$$

The equation of motion take the form

$$\ddot{q} = -2(q - \bar{q})^2 \sum_{j=1}^2 \frac{(\bar{q} - q)(\bar{r}_j - r_j)^2 (q - r_j)(\bar{r}_j - q)}{T_j^3} + \frac{2\dot{q}^2}{q - \bar{q}}. \quad (6.11)$$

Taking $q = x + iy$, after straightforward computations we recover the equations of motion (6.6).

6.2.2 Dynamics around the equilibrium point

It is easy to verify that the system defined in equation (6.6) has exactly one equilibrium point given by $(x, y, \dot{x}, \dot{y}) = (0, 1, 0, 0)$; and that the geodesic corresponding to the y -axis is invariant if $\dot{x}(0) = 0$. So we start by analyzing the motion on this invariant set.

By taking $x(t) \equiv 0$, the equation corresponding to y in (6.6) takes the form

$$\ddot{y} = \frac{\dot{y}^2}{y} + \frac{8y^3y_1^2(1-y^2)}{((x_1^2 + (y-y_1)^2)(x_1^2 + (y+y_1)^2))^{3/2}}. \quad (6.12)$$

We have the following result.

Proposition 2. The equilibrium point $(x(t), y(t)) = (0, 1)$ restricted to the geodesic identify with the y -axis is a center, that is all orbits around this point are periodic.

Proof. The system given by equation (6.12) can be written as

$$\begin{aligned} \dot{y} &= \nu, \\ \dot{\nu} &= \frac{\nu^2}{y} + \frac{8y^3y_1^2(1-y^2)}{((x_1^2 + (y-y_1)^2)(x_1^2 + (y+y_1)^2))^{3/2}}. \end{aligned} \quad (6.13)$$

The slope of the vector field is given by $g(y, \nu) = \frac{\dot{\nu}}{\dot{y}}$. Since $g(y, -\nu) = -g(y, \nu)$, the flow is symmetric with respect to the y -axis.

The direction of the solution curves of system (6.13) is horizontal (the slope of the flow is zero) along the curves

$$\nu = \pm \sqrt{-\frac{8y^4y_1^2(1-y^2)}{((x_1^2 + (y-y_1)^2)(x_1^2 + (y+y_1)^2))^{3/2}}} =: h(y).$$

We observe that the above function is not defined for $y < 1$, which means that for $y < 1$, the slope of the solution curves are never zero. The behavior of the slopes given by the function g can be seen by checking that (see Figure 6.1)

$$\lim_{\nu \rightarrow 0} g(y, \nu) = \infty \quad \text{if } y < 1, \quad \text{or} \quad -\infty \quad \text{if } y > 1$$

If we focus on $\nu > 0$, the function $g(y, \nu)$ is positive for any $y < h(y)$ and negative for any $y > h(y)$, from this observation and the symmetries of the flow we get the result. \square

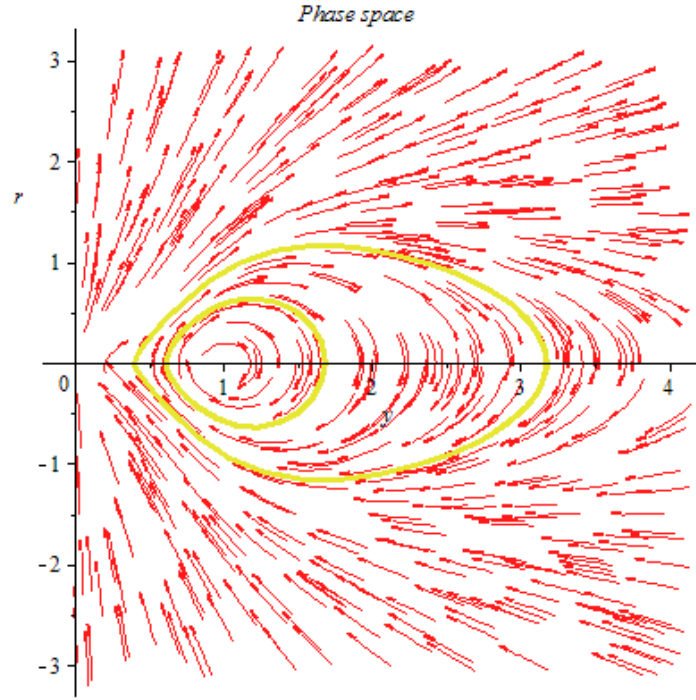


Figure 6.1: Phase space for $y(t), \dot{y}(t)$ ($x(t) = 0$).

6.2.3 Behavior of the global flow near $(0, 1, 0, 0)$.

Taking $(x_1, y_1) = (\cos \theta, \sin \theta)$ for some fix $0 < \theta < \pi/2$, and splitting the two second-order differential equations of equation (6.6) into 4 first-order differential equations, we get

$$\begin{cases} \dot{x} = u, \\ \dot{y} = v, \\ \dot{u} = \frac{2uv}{y} + \frac{E}{(AB)^{3/2}} - \frac{F}{(CD)^{3/2}}, \\ \dot{v} = \frac{v^2 - u^2}{y} + \frac{G}{(AB)^{3/2}} + \frac{H}{(CD)^{3/2}} \end{cases} \quad (6.14)$$

where

$$\begin{cases} A = (x - \cos \theta)^2 + (y - \sin \theta)^2, \\ B = (x - \cos \theta)^2 + (y + \sin \theta)^2, \\ C = (x + \cos \theta)^2 + (y - \sin \theta)^2, \\ D = (x + \cos \theta)^2 + (y + \sin \theta)^2, \\ E = 8(\cos \theta - x)y^4 \sin^2 \theta, \\ F = 8(\cos \theta + x)y^4 \sin^2 \theta, \\ G = 4y^3 \sin^2 \theta[(x - \cos \theta)^2 + \sin^2 \theta - y^2], \\ H = 4y^3 \sin^2 \theta[(x + \cos \theta)^2 + \sin^2 \theta - y^2]. \end{cases}$$

Straightforward computations, we obtain that the linear part of the flow given by (6.6) around the equilibrium point $(0, 1, 0, 0)$ is given by

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2a & 0 & 0 & 0 \\ 0 & -a & 0 & 0 \end{pmatrix}, \quad \text{where } a = \frac{2 \sin^2 \theta}{\cos^3 \theta} > 0,$$

Whose eigenvalues are the solutions of

$$(\lambda^2 - 2a)(\lambda^2 + a) = 0.$$

Since this equation has one positive root, the equilibrium point $(0, 1, 0, 0)$ is unstable. We have thus proved the following result.

Proposition 3. The equilibrium point $(0, 1, 0, 0)$ of the general 2-center problem is unstable.

In [37], the author proves that the two center problem in spaces of constant curvature, positive or negative, is integrable.

6.3 The three center problem

In this section, we work on a surface of constant negative curvature κ . Without loss of generality we assume that $\kappa = -1$. We take as a model for this surface the Poincaré's upper half semiplane \mathbb{H}^2 studied in the previous section.

The problem concerns to the motion of a particle of mass M on \mathbb{H}^2 , subject to the attraction of three fixed masses on the unit circle, two of them of mass m located symmetrically with respect to the y -axis, and the third one of mass μ located at the point $(0, 1)$.

6.3.1 The negative curved three center problem

We consider two point particles with positive mass m , located symmetrically on the same geodesic with respect to the third point with positive mass μ between them; without loss of generality we assume that the geodesic is the unitary half circle centered at the origin, so the three particles are fixed at positions $q_1 = (x_1, y_1)$, $q_2 = (-x_1, y_1)$ and $q_3 = (0, 1)$ with $|q_1| = |q_2| = |q_3| = 1$. Under the gravitational force of three fixed masses a fourth particle of positive mass M with position $q = (x, y)$ moving under the influence of the three fixed masses. The phase space is $\mathbf{T}^*(\mathbb{H}^2 \setminus \Delta)$ where $\Delta = \{(\pm x_1, y_1), (0, 1)\}$ are the collisions set. We see that the potential is analytic function in $\mathbb{H}^2 \setminus \Delta$.

According to (6.5), we express the equations of motion for the last particle in the following form

$$\begin{aligned}
\ddot{x} &= \frac{2\dot{x}\dot{y}}{y} + \frac{8m(x_1 - x)y^4y_1^2}{[(x - x_1)^2 + (y - y_1)^2][(x - x_1)^2 + (y + y_1)^2]^{3/2}} \\
&+ \frac{8m(-x_1 - x)y^4y_1^2}{[(x + x_1)^2 + (y - y_1)^2][(x + x_1)^2 + (y + y_1)^2]^{3/2}} \\
&- \frac{8\mu xy^4}{[(x^2 + (y - 1)^2)(x^2 + (y + 1)^2)]^{3/2}}, \\
\ddot{y} &= \frac{\dot{y}^2 - \dot{x}^2}{y} + \frac{4my^3y_1^2((x - x_1)^2 + y_1^2 - y^2)}{[(x - x_1)^2 + (y - y_1)^2][(x - x_1)^2 + (y + y_1)^2]^{3/2}} \\
&+ \frac{4my^3y_1^2((x + x_1)^2 + y_1^2 - y^2)}{[(x + x_1)^2 + (y - y_1)^2][(x + x_1)^2 + (y + y_1)^2]^{3/2}} \\
&+ \frac{4\mu y^3(x^2 + 1 - y^2)}{[(x^2 + (y - 1)^2)(x^2 + (y + 1)^2)]^{3/2}}
\end{aligned} \tag{6.15}$$

Then we reduced this problem to the vertical case and the geodesic case.

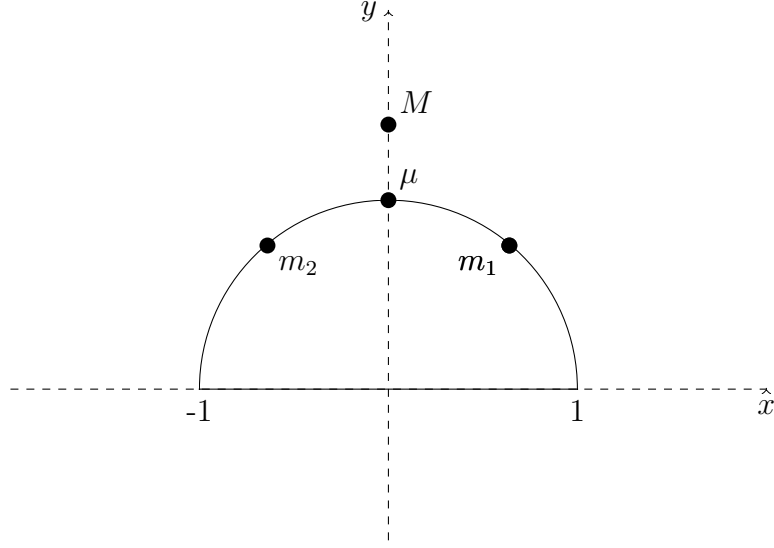


Figure 6.2: The vertical case

6.3.2 The vertical case

We restrict M to move on the y -axis. So let $q_4 = (0, y_4)$, $y > 0$, then (6.15) becomes

$$\ddot{y} = \frac{\dot{y}^2}{y} + \frac{8my^3y_1^2(1-y^2)}{[(x_1^2 + (y-y_1)^2)(x_1^2 + (y+y_1)^2)]^{3/2}} + \frac{4\mu y^3}{(y^2-1)^2} \quad (6.16)$$

On this invariant axis the equation of motion becomes singular at the collision $q_3 = (0, 1)$. So, the motion of the free particle M will form a binary collision with the fixed particle μ .

6.3.3 The geodesic case

In this case we have the collinear four-body problem: three fixed masses m and μ and the fourth body M is moving on the same geodesic. Let

$$q_1 = (\cos \theta_1(t), \sin \theta_1(t)), \quad q_2 = (-\cos \theta_1(t), \sin \theta_1(t)),$$

$$q_3 = (0, 1), \quad q_4 = (\cos \theta_2(t), \sin \theta_2(t)).$$

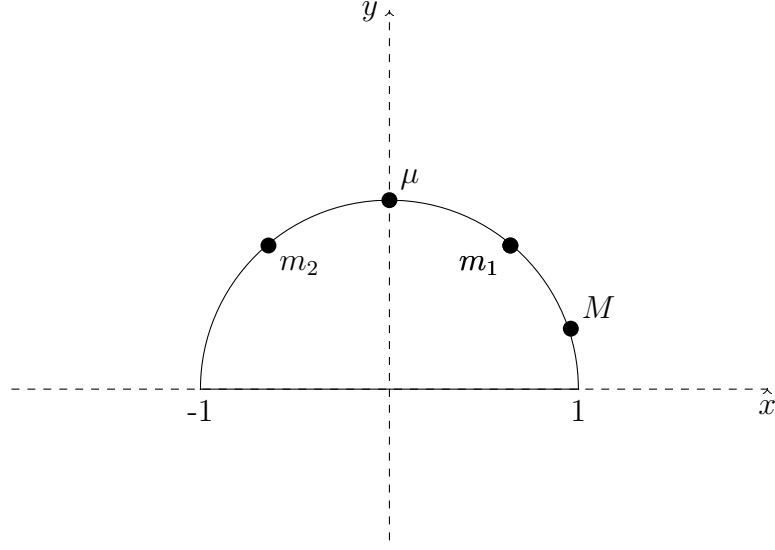


Figure 6.3: The geodesic case

According with (6.5), we express the equations of motion for the last particle in the following form

$$\begin{aligned}
\ddot{x} &= -2 \sin \theta_2(t) \\
&+ \frac{8m(\cos \theta_1 - \cos \theta_2) \sin^4 \theta_2 \sin^2 \theta_1}{[(2 - 2(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2))((2 - 2(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2)))]^{3/2}} \\
&+ \frac{8m(-\cos \theta_1 - \cos \theta_2) \sin^4 \theta_2 \sin^2 \theta_1}{[(2 + 2(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2))((2 - 2(\sin \theta_1 \sin \theta_2 - \cos \theta_1 \cos \theta_2)))]^{3/2}} \\
&- \frac{8\mu \cos \theta_2 \sin^4 \theta_2}{[(2 - 2 \sin \theta_2)(2 + 2 \sin \theta_2)]^{3/2}} \\
\ddot{y} &= \frac{1}{\sin \theta_2} \\
&+ \frac{4m \sin^3 \theta_2 \sin^2 \theta_1 ((\cos \theta_2 - \cos \theta_1)^2 + \sin^2 \theta_1 - \sin^2 \theta_2)}{[(2 - 2(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2))((2 - 2(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2)))]^{3/2}} \\
&+ \frac{4m \sin^3 \theta_2 \sin^2 \theta_1 ((\cos \theta_2 + \cos \theta_1)^2 + \sin^2 \theta_1 - \sin^2 \theta_2)}{[(2 + 2(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2))((2 + 2(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2)))]^{3/2}} \\
&+ \frac{8\mu \sin^3 \theta_2 \cos^2 \theta_2}{[(2 - 2 \sin \theta_2)(2 + 2 \sin \theta_2)]^{3/2}}
\end{aligned}$$

Notice that the denominator in the equation of motion

$$[(2 - 2(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2))((2 - 2(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2)))]^{3/2}$$

vanishes when $\theta_1 = \theta_2$. This means the free particle M and m form a collision.

Further, The denominator in the equation of motion

$$[(2 - 2 \sin^2 \theta_2)(2 + 2 \sin^2 \theta_2)]^{3/2}$$

vanishes when $\theta_2 = \pi/2$. This means the free particle M and μ form a collision.

6.3.4 Collisions

Singularity

Due to the fact that the potential function $U(q, \bar{q})$ depends on the inverse of the square of the hyperbolic distance, a collision implies a singularity in the equations of motion. A collision is caused when the distance between two masses becomes zero in finite time. During the collision, potential function is infinite and consequently, the velocity becomes infinite. Thus, the potential function $U(q, \bar{q})$ is singular on Δ .

Classifications of the binary collisions

In the three center problem, we consider three particular cases:

- The first case. The easiest and most accessible case is that the fourth particle moves on the invariant vertical line with $q_4 = (0, y_4)$, $y_4 > 0$. Note that the binary collision happens when $y_4 = 1$.
- The second case. The fourth particle moves on the same geodesic in which all the three fixed primaries lie. Let $q_4 = (x_4, y_4)$, $y_4 < \pm y_1$ with $|q_4| = 1$. Here we will have the 4 collinear problem. Note that the binary collision happens when $q_4 = q_1$.
- The third case. The fourth particle moves on the same geodesic between m and μ . Note that in this case we have two binary collisions which happen when $q_4 = q_1$ or $q_4 = q_3$.

Chapter 7

Conclusions and Future Work

The results and conclusions obtained in this thesis are summarized in this chapter. Some recommendations for future work are also discussed, in section (7.2).

7.1 Conclusions

In summary, we find new relative equilibria in the curved 4-body and curved 5-body problems, respectively, which move on the equator of the sphere.

We consider the case of two negligible masses and two equal masses, and we prove that square relative equilibria always exist on \mathbb{S}_κ^2 , but not on \mathbb{H}_κ^2 . Nevertheless, we put into the evidence a certain type of kite-shaped relative equilibria in the case of the one negligible mass of the 4-body problem. Then, we discuss the existence of solutions for the 5-body problem in which equilibrium solutions appear for particular geometrical configurations. Further, we show that for the 6-body problem in \mathbb{S}_κ^2 , there exists at least one kite-shaped relative equilibrium.

We derive the equations of motion for the 2-and 3-centre problems corresponding to the upper half-plane model. In particular, in the 2-center problem, we study the dynamics around the equilibrium point corresponding to the invariant axis. However, for the 3-center problem, we show the existence of three possible double collisions.

7.2 Future Work

While the relative equilibria of the curved N -body problem have been intensely studied in the past few years for positive masses, there is very little research done so far when

some of the masses are negligible. Therefore, my future research will mostly focus on this aspect of the problem.

In Chapter 3, we derive the necessary condition for the quadrilateral inscribed in the equator of \mathbb{S}_κ^2 that form a relative equilibrium. This could be continued to study the stability of this quadrilateral configurations.

In Chapter 4, we study the relative equilibria for the curved restricted 5 body problem. I will have to seek other, less symmetric configurations for when: one, two, or three of the masses are negligible. In all three possible cases: $\kappa < 0$, $\kappa = 0$, and $\kappa > 0$. Future investigations are necessary to explore the stability of the curved restricted 5-body problem in the 2-dimensional case on the unit sphere that can be drawn from this study.

Additionally, in the context of the singularity of the 3-centre problem on the Poincaré upper semi-plane model, the main question of this problem, is the integrability of the 3-center problem. Here the problem is more complicated because we have a particle at the point (0,1), which means that we must first regularize first the binary collision, then we will use the philosophy in [18]. We will attempt to apply the regularization processes to this case of the 3-centre problem to eliminate the singularity caused by the binary collision between μ and M . We will use the Levi-Civita transformation to study the motion near to this binary collision.

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