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Applications of Some Subclasses of Meromorphic Functions Associated with the q-Derivatives of the q-Binomials

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Abstract: In this article, we make use of the q-binomial theorem to introduce and study two new subclasses $\aleph(\alpha q, q)$ and $\aleph(\alpha, q)$ of meromorphic functions in the open unit disk $\mathbb{U}$, that is, analytic functions in the punctured unit disk $\mathbb{U}^* = \mathbb{U} \setminus \{0\} = \{z : z \in \mathbb{C} \quad \text{and} \quad 0 < |z| < 1\}$. We derive inclusion relations and investigate an integral operator that preserves functions which belong to these function classes. In addition, we establish a strict inequality involving a certain linear convolution operator which we introduce in this article. Several special cases and corollaries of our main results are also considered.

Keywords: q-derivatives of binomials; Jackson's q-integrals; analytic functions; starlike and convex functions; analytic and meromorphic functions; hadamard product (or convolution); q-Bernardi integral operator

MSC: 30C45; 30C80; 33D15

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1. Introduction, Definitions and Preliminaries

Basic (or quantum or q-) derivatives have various applications in different branches of mathematics that make the study of Geometric Function Theory interesting and pertinent (see [1,2]). Some applications of the q-operator are studied by Aral and Gupta [3] and Elhaddad et al. [4,5]. Many of the different q-derivative problems can also be found in [6–12].

In this article, we assume that $0 < q < 1$. We start with the definitions and various results from the q-analysis, including the q-factorial $[\nu]_q!$ for every non-negative integer $\nu \in \mathbb{N}_0$, which is characterized by

$$[\nu]_q! = \begin{cases} [\nu]_q[\nu-1]_q \cdots [2]_q[1]_q & (\nu \in \mathbb{N} \setminus \{1\}) \quad 1 \\ 1 & (\nu = 1) \qquad\quad 0 \\ 0 & (\nu = 0), \end{cases} \tag{1}$$

where, in general,

$$[\lambda]_q = \begin{cases} \dfrac{1-q^\lambda}{1-q} & (\lambda \in \mathbb{C} \setminus \{0\}) \\ \\ 0 & (\lambda = 0), \end{cases} \tag{2}$$

$\mathbb{N}$ being the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. In the limit case when $q \to 1-$, $[\lambda]_q$ tends to the complex number $\lambda \in \mathbb{C}$.

Here are some definitions and concepts from the q-calculus, including the q-derivative operator $\mathfrak{D}_q$ and the q-integral, as follows:

$$\mathfrak{D}_q[f(z)] = \begin{cases} \dfrac{f(z)-f(qz)}{(1-q)z} & (z \neq 0) \\ \\ f'(0) & (z = 0), \end{cases} \tag{3}$$

provided that the derivative $f'(0)$ exists. When $q \to 1-$, the q-derivative $\mathfrak{D}_q[f(z)]$ reduces to the ordinary derivative $f'(z)$.

The q-derivative operator $\mathfrak{D}_q$, defined by (3), satisfies each of the following properties:

$$\mathfrak{D}_q[f(z)h(z)] = \mathfrak{D}_q[f(z)]h(z) + f(qz)\mathfrak{D}_q[h(z)], \tag{4}$$

$$\mathfrak{D}_q\left[\frac{f(z)}{h(z)}\right] = \frac{\mathfrak{D}_q[f(z)]h(z) - f(z)\mathfrak{D}_q[h(z)]}{h(z)h(qz)} \tag{5}$$

and

$$\mathfrak{D}_q\big[\log\big(f(z)\big)\big] = \frac{\mathfrak{D}_q[f(z)]}{f(z)}. \tag{6}$$

As long ago as 1910, Jackson [2] introduced the q-integral defined by

$$\int_0^z f(\tau)d_q\tau = (1-q)z\sum_{k=0}^{\infty} q^k f(q^k z), \tag{7}$$

provided that the series involved in (7) is convergent. In particular, for a power function $f(z) = z^k$, we note that

$$\int_0^z f(\tau)d_q\tau = \int_0^z \tau^k d_q\tau = \frac{1}{[k+1]_q} z^{k+1} \qquad (k \neq -1), \tag{8}$$

and

$$\lim_{q\to 1-} \int_0^z f(\tau)d_q\tau = \lim_{q\to 1-} \frac{1}{[k+1]_q} z^{k+1} = \frac{1}{k+1} z^{k+1} = \int_0^z f(\tau)d\tau, \tag{9}$$

where $\int_0^z f(\tau)d\tau$ denotes the ordinary integral.

Several researchers have investigated many different subclasses of analytic functions by using the q-derivative operator $\mathfrak{D}_q$ defined by (3) (see, for example, [4,5,13,14]).

Let Σ be the class of meromorphic functions in $\mathbb{U}$, that is, the class of analytic functions in the punctured unit disk

$$\mathbb{U}^* = \mathbb{U} \setminus \{0\} = \{z : z \in \mathbb{C} \quad \text{and} \quad 0 < |z| < 1\}. \tag{10}$$

with the power-series expansion of the form:

$$f(z) = \frac{1}{z} + \sum_{\nu=0}^{\infty} a_\nu z^\nu. \tag{11}$$

When f is a function of the form (11), it is clear that

$$\mathfrak{D}_q[f(z)] = -\frac{1}{qz^2} + \sum_{\nu=0}^{\infty} [\nu]_q \, a_\nu \, z^{\nu-1}. \tag{12}$$

Definition 1. *Let the function f be as in (11) and suppose that*

$$h(z) = \frac{1}{z} + \sum_{\nu=0}^{\infty} b_\nu \, z^\nu. \tag{13}$$

Then, the Hadamard product (or convolution) of f and h is given by

$$(f * h)(z) := (h * f)(z) = \frac{1}{z} + \sum_{\nu=0}^{\infty} a_\nu \, b_\nu \, z^\nu =: (h * f)(z). \tag{14}$$

In the year 1989, Ganigi and Uralegaddi [15] introduced an operator $\mathcal{D}^n : \Sigma \to \Sigma$, which they defined by means of the Hadamard product (or convolution) as follows:

$$\begin{aligned} \mathcal{D}^n[f(z)] &= \frac{1}{z(1-z)^{n+1}} * f(z) \\ &= \frac{1}{z}\left(\frac{z^{n+1} f(z)}{n!}\right)^{(n)} \\ &= \frac{1}{z} + (n+1)a_0 + \frac{(n+1)(n+2)}{2!} a_1 z + \cdots, \end{aligned}$$

where $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. By using the convolution operator $\mathcal{D}^n$, Ganigi and Uralegaddi [15] considered the function class $\mathcal{M}_n$ given by

$$\mathcal{M}_n = \left\{ f : f \in \Sigma \quad \text{and} \quad \Re\left(\frac{\mathcal{D}^{n+1}[f(z)]}{\mathcal{D}^n[f(z)]} - 2\right) < -\frac{n}{n+1} \quad (z \in \mathbb{U}^*;\ n \in \mathbb{N}_0) \right\}. \tag{15}$$

They also showed that $\mathcal{M}_{n+1} \subset \mathcal{M}_n$. More recently, Ahmad and Arif [16] introduced and studied the q-operator $\mathfrak{R}_q^n : \Sigma \to \Sigma$ defined by

$$\mathfrak{R}_q^n[f(z)] = \frac{1}{z} + \sum_{\nu=1}^{\infty} \frac{[n+\nu+1]_q!}{[n]_q![\nu+1]_q!} \, a_\nu \, z^\nu \qquad (n \in \mathbb{N}). \tag{16}$$

The classical binomial expansion is known to possess the following q-extension in terms of the basic (or quantum or q-) hypergeometric function ${}_r\Phi_s$ with r numerator parameters and s denominator parameters. We have (see, for details [17] (pp. 346–348) and [18] (p. 8))

$$\begin{aligned} {}_1\Phi_0(\lambda; \text{—}; q, z) &:= \sum_{n=0}^{\infty} \frac{(\lambda; q)_n}{(q; q)_n} \, z^n \\ &= \frac{(\lambda z; q)_\infty}{(q; q)_\infty} \qquad (|z| < 1;\ |q| < 1), \end{aligned}$$

where $(\lambda; q)_n$ denotes the q-shifted factorial which is defined for $\lambda \in \mathbb{C}$ by

$$(\lambda; q)_n := \begin{cases} 1 & (n = 0) \\ \prod_{j=0}^{n-1} (1 - \lambda q^j) & (n \in \mathbb{N}). \end{cases} \tag{17}$$

It is easy to see from (17) that

$$\lim_{q\to 1-}\left\{\frac{(q^{\lambda},q)_n}{(1-q)^n}\right\}=(\lambda)_n \qquad (\lambda\in\mathbb{C};\ n\in\mathbb{N}_0), \tag{18}$$

where $(\lambda)_n$ is the well-known Pochhammer symbol defined by

$$(\lambda)_0=1 \qquad \text{and} \qquad (\lambda)_n=\lambda(\lambda+1)\cdots(\lambda+n-1) \quad (n\in\mathbb{N}). \tag{19}$$

We now introduce the function $\psi(\alpha,q,z)$ given by

$$\psi(\alpha,q,z)=z^{-1}\,{}_1\Phi_0(a;-\!\!-;q,z)=\frac{1}{z}+\sum_{\nu=0}^{\infty}\frac{(\alpha;q)_{\nu+1}}{(q;q)_{\nu+1}}z^{\nu} \qquad (z\in\mathbb{U}^*). \tag{20}$$

By means of the Hadamard product (or convolution) with the function $\psi(\alpha,q,z)$, we define a linear operator $\mathfrak{L}(\alpha,q)$ on Σ as follows:

$$\mathfrak{L}(\alpha,q)[f(z)]=\psi(\alpha,q,z)*f(z)=\frac{1}{z}+\sum_{\nu=0}^{\infty}\frac{(\alpha;q)_{\nu+1}}{(q;q)_{\nu+1}}\,a_{\nu}\,z^{\nu}. \tag{21}$$

Then, by applying Equation (12), it is easily seen that

$$z\mathfrak{D}_q\big[\mathfrak{L}(\alpha,q)[f(z)]\big]=-\frac{1}{qz}+\sum_{\nu=0}^{\infty}[\nu]_q\,\frac{(\alpha;q)_{\nu+1}}{(q;q)_{\nu+1}}\,a_{\nu}\,z^{\nu}. \tag{22}$$

Furthermore, for the operator $\mathfrak{L}(\alpha,q)$, we can readily show that

$$\left(\frac{1-\alpha}{1-q}\right)\mathfrak{L}(\alpha q,q)[f(z)]=\left(\frac{1-\alpha q}{1-q}\right)\mathfrak{L}(\alpha,q)[f(z)]+\alpha qz\mathfrak{D}_q\big[\mathfrak{L}(\alpha,q)[f(z)]\big]. \tag{23}$$

Remark 1. *For $f(z)\in\Sigma$, we have*

$$\mathfrak{L}(q^{\beta+1},q)[f(z)]=\frac{1}{z}+\sum_{\nu=0}^{\infty}\frac{(q^{\beta+1};q)_{\nu+1}}{(q;q)_{\nu+1}}\,a_{\nu}\,z^{\nu}=\mathfrak{R}_q^n[f(z)]. \tag{24}$$

Definition 2. *Let $\aleph(\alpha,q)\ \ (0\leqq\alpha<q)$ be in the subclass of Σ whose members are defined by the following condition*:

$$\Re\left(\frac{\mathfrak{L}(\alpha q,q)[f(z)]}{\mathfrak{L}(\alpha,q)[f(z)]}-(1+q)\right)<-\frac{q-\alpha}{1-\alpha} \qquad (z\in\mathbb{U}). \tag{25}$$

Remark 2. *If we set $\alpha=q^{\beta+1}$ in Definition 2 of the function class $\aleph(\alpha,q)$, a new function class $\widetilde{\aleph}_{\beta,q}\ \ (\beta\geqq 0)$ is obtained, which is given in Definition 3 below.*

Definition 3. *A function $f\in\Sigma$ of the form (11) is said to belong to the class $\widetilde{\aleph}_{\beta,q}\ \ (\beta\geqq 0)$ if it satisfies the following condition*:

$$\Re\left(\frac{\mathfrak{R}_q^{\beta+1}[f(z)]}{\mathfrak{R}_q^{\beta}[f(z)]}-(1+q)\right)<\frac{[\beta]_q}{[\beta+1]_q} \qquad (z\in\mathbb{U}). \tag{26}$$

We note the following limit relationship:

$$\lim_{q\to 1-}\{\widetilde{\aleph}_{\beta,q}\}=\mathcal{M}_n. \tag{27}$$

In recent years, the q-derivative operator $\mathfrak{D}_q$ was extensively used to create a number of analytic function classes (see, for examples [19–25]). In this paper, we construct inclusion relations for the families of meromorphic functions which we have introduced and described above. We also successfully apply some class-preserving integral operators upon the functions belonging to these classes of functions. Additionally, we derive a strict inequality involving a linear convolution operator which we introduce in this article.

We choose to remark in passing that the inclusion properties, the class-preserving q-integral, and the strict inequalities involving linear convolution operators, which we have presented below for functions in the subclasses of the normalized meromorphic function class Σ, not only yield various known (or new) results upon appropriate specializations, but are also potentially capable of further developments for other function classes that can be introduced by means the operators of q-derivatives and q-integrals.

2. The q-Jack Lemma and ITS Consequences

We need the following lemma to prove our main results in this section.

Lemma 1 (**q-Jack Lemma [26]**)**.** *Let the function $u(z)$ be analytic in $\mathbb{U}$ with $u(0) = 0$. If $|u(z)|$ attains its maximum value on the circle $|z| = r \quad (r < 1)$ at a point z_0, then*

$$z_0\mathfrak{D}_q[u(z_0)] = ku(z_0), \tag{28}$$

where k is real and $k \geqq 1$.

We now state our first main result as Theorem 1 below. This result provides an inclusion property of the subclass $\aleph(\alpha, q) \quad (0 \leqq \alpha < q)$ of the normalized meromorphic function class Σ, which is given by Definition 2.

Theorem 1. *It is asserted that $\aleph(\alpha q, q) \subset \aleph(\alpha, q)$ for each $q \quad (0 < q < 1)$ and for each $\alpha \quad (0 \leqq \alpha < q)$.*

Proof. Let $f \in \aleph(\alpha q, q)$. Then

$$\Re\left(\frac{\mathfrak{L}(\alpha q^2, q)[f(z)]}{\mathfrak{L}(\alpha q, q)[f(z)]} - (1+q)\right) < -\frac{q(1-\alpha)}{1-\alpha q}. \tag{29}$$

In order to show that

$$\Re\left(\frac{\mathfrak{L}(\alpha q, q)[f(z)]}{\mathfrak{L}(\alpha, q)[f(z)]} - (1+q)\right) < -\frac{q-\alpha}{1-\alpha}, \tag{30}$$

we define $u(z)$ in $\mathbb{U}$ by

$$\frac{\mathfrak{L}(\alpha q, q)[f(z)]}{\mathfrak{L}(\alpha, q)[f(z)]} - (1+q) = -\left[\frac{q-\alpha}{1-\alpha} + \frac{\alpha(1-q)}{1-\alpha}\left(\frac{1-u(z)}{1+u(z)}\right)\right]. \tag{31}$$

Clearly, $u(0) = 0$. Moreover, Equation (31) can be expressed as follows:

$$\frac{\mathfrak{L}(\alpha q, q)[f(z)]}{\mathfrak{L}(\alpha, q)[f(z)]} = \frac{(1-\alpha) + (1+\alpha-2\alpha q)u(z)}{(1-\alpha)[1+u(z)]}. \tag{32}$$

Now, by using some basic computations and the properties of the q-derivative, we find from (32) that

$$\begin{aligned} &\frac{z\mathfrak{D}_q\mathfrak{L}(\alpha q,q)[f(z)]}{\mathfrak{L}(\alpha q,q)[f(z)]} - \frac{z\mathfrak{D}_q\mathfrak{L}(\alpha,q)[f(z)]}{\mathfrak{L}(\alpha,q)[f(z)]} \\ &\quad = \frac{2\alpha(1-q)z\mathfrak{D}_q[u(z)]}{[(1-\alpha)+(1+\alpha-2\alpha q)u(z)][1+u(z)]}. \end{aligned} \tag{33}$$

From (23) and (31), we obtain

$$\begin{aligned}\frac{\mathfrak{L}(\alpha q^2,q)[f(z)]}{\mathfrak{L}(\alpha q,q)[f(z)]} - (1+q) + \frac{q(1-\alpha)}{1-\alpha q} \\ = \frac{\alpha q(1-q)}{(1-\alpha q)}\left(- \frac{1-u(z)}{1+u(z)} \right. \\ \left. + \frac{2\alpha q(1-q)z\mathfrak{D}_q[u(z)]}{[(1-\alpha)+(1+\alpha-2\alpha q)u(z)][1+u(z)]} \right).\end{aligned} \tag{34}$$

We assume that $|u(z)| < 1$ in $z \in \mathbb{U}$. Therefore, by Lemma 1, there exists a point $z_0 \in \mathbb{U}$ such that

$$z_0\mathfrak{D}_q[u(z_0)] = ku(z_0), \tag{35}$$

where $|u(z_0)| < 1$ and $k \geqq 1$. From (34) and (35), we obtain

$$\begin{aligned}\frac{\mathfrak{L}(\alpha q^2,q)[f(z_0)]}{\mathfrak{L}(\alpha q,q)[f(z_0)]} - (1+q) + \frac{q(1-\alpha)}{1-\alpha q} \\ = \frac{\alpha q(1-q)}{(1-\alpha q)}\left(- \frac{1-u(z_0)}{1+u(z_0)} \right. \\ \left. + \frac{2\alpha q(1-q)ku(z_0)}{[(1-\alpha)+(1+\alpha-2\alpha q)u(z_0)][1+u(z_0)]} \right).\end{aligned}$$

We thus obtain

$$\Re\left(\frac{\mathfrak{L}(\alpha q^2,q)[f(z)]}{\mathfrak{L}(\alpha q,q)[f(z)]} - (1+q) + \frac{q(1-\alpha)}{1-\alpha q} \right) > 0, \tag{36}$$

which contradicts (29). Hence, $|u(z)| < 1$, and from (31), it follows that $f(z) \in \aleph(\alpha q, q)$. □

If we put $\alpha = q^{\beta+1}$ in Theorem 1, we obtain the following result.

Corollary 1. *Suppose that $\beta \geqq 0$. Then*

$$\widetilde{\aleph}_{\beta+1,q} \subset \widetilde{\aleph}_{\beta,q}. \tag{37}$$

Remark 3. *In the limit case when $q \to 1-$, Corollary 1 yields the corresponding result proved by Ganigi and Uralegaddi [15].*

3. The q-Bernardi Integral Operator

For a function $f \in \Sigma$, we denote by $\mathfrak{I}_{\rho,q}$ the q-Bernardi integral operator $\mathfrak{I}_{\rho,q}$ defined by (see [27–30])

$$F_q(z) := \mathfrak{I}_{\rho,q}[f(z)] = \frac{[\rho]_q}{z^{\rho+1}} \int_0^z \tau^\rho \, f(\tau)\, d_q\tau \qquad (\rho \in \mathbb{N}_0). \tag{38}$$

The q-Bernardi integral operator $\mathfrak{I}_{\rho,q} : \Sigma \to \Sigma$, defined in (38), satisfies the following relationship:

$$q^{\rho+1}z\mathfrak{D}_q\mathfrak{L}(\alpha,q)[F_q(z)] = [\rho]_q\mathfrak{L}(\alpha,q)[f(z)] - [\rho+1]_q\mathfrak{L}(\alpha,q)[F_q(z)]. \tag{39}$$

We now state and prove the following result. More precisely, Theorem 2 establishes the fact that the q-Bernardi integrals of functions in the subclass $\aleph(\alpha,q)$ $(0 \leqq \alpha < q)$ of the normalized meromorphic function class Σ, which are defined by (38), are also in the same class $\aleph(\alpha,q)$ $(0 \leqq \alpha < q)$. This essentially means that the q-Bernardi integrals preserve the class $\aleph(\alpha,q)$ $(0 \leqq \alpha < q)$.

Theorem 2. *If $f \in \Sigma$ defined by (11) is in the function class $\aleph(\alpha, q)$, then $F_q(z)$ defined by (38) also belongs to the class $\aleph(\alpha, q)$.*

Proof. Let
$$\Re\left(\frac{\mathfrak{L}(\alpha q, q)[f(z)]}{\mathfrak{L}(\alpha, q)[f(z)]} - (1+q)\right) < -\frac{q-\alpha}{1-\alpha}, \tag{40}$$
which, by using Equations (39) and (23), can be written as follows:
$$\Re\left(\frac{q^\rho(1-\alpha q)\frac{\mathfrak{L}(\alpha q^2,q)[F_q(z)]}{\mathfrak{L}(\alpha q,q)F_q(z)} + (\alpha q - q^\rho)}{(\alpha q - q^{\rho+1})\frac{\mathfrak{L}(\alpha,q)[F_q(z)]}{\mathfrak{L}(\alpha q,q)[F_q(z)]} + q^{\rho+1}(1-\alpha)} - (1+q)\right) < -\frac{q-\alpha}{1-\alpha}. \tag{41}$$

We need to prove that (41) implies the inequality given by
$$\Re\left(\frac{\mathfrak{L}(\alpha q, q)[F_q(z)]}{\mathfrak{L}(\alpha, q)[F_q(z)]} - (1+q)\right) < -\frac{q-\alpha}{1-\alpha}. \tag{42}$$

For this purpose, we define the function $u(z)$ in $\mathbb{U}$ by
$$\frac{\mathfrak{L}(\alpha q, q)[F_q(z)]}{\mathfrak{L}(\alpha, q)[F_q(z)]} - (1+q) = -\left[\frac{q-\alpha}{1-\alpha} + \frac{\alpha(1-q)}{1-\alpha}\left(\frac{1-u(z)}{1+u(z)}\right)\right]. \tag{43}$$

Clearly, we have $u(0) = 0$. Moreover, Equation (43) can be written as follows:
$$\frac{\mathfrak{L}(\alpha q, q)[F_q(z)]}{\mathfrak{L}(\alpha, q)[F_q(z)]} = \frac{(1-\alpha) + (1+\alpha-2\alpha q)u(z)}{(1-\alpha)[1+u(z)]}. \tag{44}$$

Now, by using some simple computations and the properties of the q-derivative, we find from (44) that
$$\begin{aligned} &\frac{z\mathfrak{D}_q\mathfrak{L}(\alpha q,q)[F_q(z)]}{\mathfrak{L}(\alpha q,q)[F_q(z)]} - \frac{z\mathfrak{D}_q\mathfrak{L}(\alpha,q)F_q(z)}{\mathfrak{L}(\alpha,q)[F_q(z)]} \\ &= \frac{2\alpha(1-q)z\mathfrak{D}_q[u(z)]}{[(1-\alpha)+(1+\alpha-2\alpha q)u(z)][1+u(z)]}. \end{aligned} \tag{45}$$

Thus, from (23) and (43), we obtain
$$\begin{aligned} &\frac{q^\rho(1-\alpha q)\frac{\mathfrak{L}(\alpha q^2,q)[F_q(z)]}{\mathfrak{L}(\alpha q,q)[F_q(z)]} + (\alpha q - q^\rho)}{(\alpha q - q^{\rho+1})\frac{\mathfrak{L}(\alpha,q)[F_q(z)]}{\mathfrak{L}(\alpha q,q)[F_q(z)]} + q^{\rho+1}(1-\alpha)} - (1+q) + \frac{q-\alpha}{1-\alpha} \\ &= \frac{\alpha(1-q)}{(1-\alpha)}\Bigg(-\frac{1-u(z)}{1+u(z)} \\ &+ \frac{2q^{\rho+2}(1-q)z\mathfrak{D}_q[u(z)]}{q\big[(1-q^\rho)+(1+q^\rho-2q^{\rho+1})u(z)\big][1+u(z)]}\Bigg). \end{aligned} \tag{46}$$

The final part of the proof of Theorem 2 is analogous to that of Theorem 1. We, therefore, choose to omit the details involved. □

If we put $\alpha = q^{\beta+1}$ $(\beta \geqq 0)$ in Theorem 2, we obtain the following corollary.

Corollary 2. *If $f \in \widetilde{\aleph}$, then the function $F_q(z)$ defined by (38) belongs to the class $\widetilde{\aleph}_{\beta,q}$.*

Remark 4. *If we let $q \to 1-$ in Corollary 2, we are led to the corresponding result given by Ganigi and Uralegaddi [15].*

4. Strict Inequalities Involving the Linear Convolution Operator $\mathfrak{L}(\alpha, q)$

In this section, we derive some strict inequalities which are based upon the linear convolution operators $\mathfrak{L}(\alpha q, q)$ and $\mathfrak{R}_q^{\beta}$, which are defined by (21) and (16), respectively.

Theorem 3. *If $f \in \Sigma$ satisfies the following strict inequality:*

$$\left|z\mathfrak{L}(\alpha q, q)[f(z)] - 1\right| < 1, \tag{47}$$

for $0 \leqq \alpha < q$, then

$$\left|z\mathfrak{L}(\alpha, q)[f(z)] - 1\right| < 1. \tag{48}$$

Proof. Defining the function $u(z)$ by

$$z\mathfrak{L}(\alpha, q)[f(z)] - 1 = u(z) \tag{49}$$

for $f \in \Sigma$, we see that

$$u(z) = u_1 z + u_2 z^2 + u_3(z) + \cdots$$

is analytic in $\mathbb{U}^*$, and $u(z) \neq 0$. It follows from (49) that

$$\alpha\mathfrak{L}(\alpha, q)[f(z)] + \alpha q z\mathfrak{D}_q(\alpha, q)[f(z)] = \alpha\mathfrak{D}_q[u(z)]. \tag{50}$$

Thus, by using (23), we find that

$$\mathfrak{L}(\alpha q, q)[f(z)] = \mathfrak{L}(\alpha, q)[f(z)] + \frac{\alpha(1-q)}{1-\alpha}\,\mathfrak{D}_q[u(z)]. \tag{51}$$

Therefore, we have

$$\left|z\mathfrak{L}(\alpha q, q)[f(z)] - 1\right| = u(z) + \frac{\alpha(1-q)}{1-\alpha}\,\mathfrak{D}_q[u(z)]. \tag{52}$$

Suppose that there exists a point $z_0 \in \mathbb{U}^*$ such that

$$\max_{|z|<|z_0|} |u(z)| = |u(z_0)| = 1. \tag{53}$$

We now apply Lemma 1 and let

$$z_0\mathfrak{D}_q[u(z_0)] = \kappa u(z_0),$$

so that $\kappa \geqq 1$. We then find from (52) that

$$\left|z\mathfrak{L}(\alpha q, q)[f(z)] - 1\right| = \left|1 + \frac{\alpha(1-q)}{1-\alpha}\,\kappa\right| \geqq 1, \tag{54}$$

which contradicts (47). Therefore, $|u(z)| < 1$ in $\mathbb{U}^*$, and thus,

$$\left|z\mathfrak{L}(\alpha, q)[f(z)] - 1\right| < 1 \qquad (z \in \mathbb{U}^*), \tag{55}$$

which evidently completes the proof of Theorem 3. □

Upon setting $\alpha = q^{\beta}$ $(\beta \geqq 0)$ in Theorem 3, we are led to the following corollary.

Corollary 3. *If the function $f \in \Sigma$ satisfies the following strict inequality:*

$$\left|z\mathfrak{R}_q^{\beta+1}[f(z)] - 1\right| < 1 \qquad (z \in \mathbb{U}^*;\ \beta \geqq 0), \tag{56}$$

then

$$\left|z\mathfrak{R}_q^{\beta}[f(z)]-1\right|<1 \qquad (z\in\mathbb{U}^*;\ \beta\geqq 0). \tag{57}$$

5. Conclusions

There has been a resurgence of interest in the study of q-series and q-polynomials and related topics, which has a history dating back to the 19th century, as a result of the creation of quantum groups and their applications in mathematics and physics beginning in 1980 (see, for details, [31]). The works of Srivastava and Karlsson [17], Gasper and Rahman [18], Kac and Cheung [32], Annaby and Mansour [33], and Ismail et al. [34], among others, emerged primarily from that of Jackson [1,2]. In this study, a novel subclass of meromorphic functions in the punctured unit disk $\mathbb{U}^*$ has been introduced by using the q-binomial theorem. For functions belonging under this class, inclusion relations and a class-preserving q-integral operator have been investigated. We have also established a strict inequality involving the linear convolution operator $\mathfrak{L}(\alpha,q)$, which is defined by (21).

The inclusion properties and the class-preserving q-integral operators, as well as strict inequalities involving linear convolution operators, which we have presented for functions in the subclass $\aleph(\alpha,q)\ \ (0\leqq\alpha<q)$ of the normalized meromorphic function class Σ, not only yield various known (or new) results upon appropriate specializations, but are also potentially capable of further developments for other function classes that can be introduced by means the operators of q-derivatives and q-integrals. Remarkably, however, it is clearly demonstrated in the recently published survey cum expository review articles (see [35] (p. 340) and [36] (section 5, pp. 1511–1512); see also [31]) that any attempts to trivially translate the q-results in this paper and in other publications into the corresponding (p,q)-results by forcing in an obviously superfluous (or redundant) parameter p will surely not lead to any non-trivial consequences.

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References

1. Jackson, F.H. On q-functions and a certain difference Operator. *Trans. R. Soc.* **1908**, *46*, 253–281. [CrossRef]
2. Jackson, F.H. On q-definite integrals. *Quart. J. Pure Appl. Math.* **1910**, *41*, 193–203.
3. Aral, A.; Gupta, V. Generalized q-Baskakov operators. *Math. Slov.* **2011**, *61*, 619–634. [CrossRef]
4. Elhaddad, S.; Aldweby, H.; Darus, M. Some properties on a class of harmonic univalent functions defined by q-analogue of Ruscheweyh operator. *J. Math. Anal.* **2018**, *9*, 28–35.
5. Elhaddad, S.; Darus, M. On meromorphic functions defined by a new operator containing the Mittag-Leffter function. *Symmetry* **2019**, *11*, 210. [CrossRef]
6. Aldweby, H.; Darus, M. A note on q-integral operators. *Electron. Notes Discrete Math.* **2018**, *67*, 25–30. [CrossRef]
7. Aldweby, H.; Darus, M. Some subordination results on q-analogue of Ruscheweyh differential operator. *Abstr. Appl. Anal.* **2014**, *2014*, 958563. [CrossRef]
8. Kanas, S.; Altinkaya, Ş.; Yalçın, S. Subclass of k-uniformly starlike functions defined by the symmetric q-derivative Operator. *Ukr. Math. J.* **2019**, *70*, 727–1740. [CrossRef]
9. Kanas, S.; Răducanu, D. Some class of analytic functions related to conic domains. *Math. Slov.* **2014**, *64*, 1183–1196. [CrossRef]
10. Srivastava, H.M. Univalent functions, fractional calculus, and associated generalized hypergeometric functions. In *Univalent Functions, Fractional Calculus, and Their Applications*; Srivastava, H.M., Owa, S., Eds.; Halsted Press (Ellis Horwood Limited): Chichester, UK; John Wiley and Sons: New York, NY, USA; Chichester, UK; Brisbane, Australia; Toronto, ON, Canada, 1989; pp. 329–354.

11. Srivastava, H.M.; Khan, B.; Khan, N.; Ahmad, Q.Z. Coefficient inequalities for q-starlike functions associated with the Janowski functions. *Hokkaido Math. J.* **2019**, *48*, 407–425. [CrossRef]
12. Srivastava, H.M.; Tahir, M.; Khan, B.; Ahmad, Q.Z.; Khan, N. Some general families of q-starlike functions associated with the Janowski functions. *Filomat* **2019**, *33*, 2613–2626. [CrossRef]
13. Alsoboh, A.; Darus, M. New subclass of analytic functions defined by q-differential operator with respect to k-symmetric points. *Int. J. Math. Comput. Sci.* **2019**, *14*, 761–773.
14. Alsoboh, A.; Darus, M. On Fekete-Szegö problem associatedwith q-derivative operator. *J. Phys. Conf. Ser.* **2019**, *1212*, 012003. [CrossRef]
15. Ganigi, M.R.; Uralegaddi, B.A. New criteria for meromorphic univalent functions. *Bull. Math. Soc. Sci. Math. Roum.* **1989**, *33*, 9–13.
16. Ahmad, B.; Arif, M. New subfamily of meromorphic convex functions in circular domain involving q-operator. *Int. J. Anal. Appl.* **2018**, *16*, 75–82.
17. Srivastava, H.M.; Karlsson, P.W. *Multiple Gaussian Hypergeometric Series*; Halsted Press (Ellis Horwood Limited): Chichester, UK; John Wiley and Sons: New York, NY, USA; Chichester, UK; Brisbane, Australia; Toronto, ON, Canada, 1985.
18. Gasper, G.; Rahman, M. Encyclopedia of Mathematics and Its Applications. In *Basic Hypergeometric Series*, 2nd ed.; Cambridge University Press: Cambridge, UK; London, UK; New York, NY, USA, 2004; Volume 96.
19. Aldawish, I.; Aouf, M.K.; Frasin, B.A.; Al-Hawary, T. New subclass of analytic functions defined by q-analogue of p-valent Noor integral operator. *AIMS Math.* **2021**, *6*, 10466–10484. [CrossRef]
20. Ali, E.E.; Bulboacă, T. Subclasses of multivalent analytic functions associated with a q-difference operator. *Mathematics* **2020**, *8*, 2184. [CrossRef]
21. Govindaraj, M.; Sivasubramanian, S. On a class of analytic functions related to conic domains involving q-calculus. *Anal. Math.* **2017**, *43*, 475–487. [CrossRef]
22. Khan, M.G.; Ahmad, B.; Khan, R.; Zubair, M.; Salleh, Z. On q-analogue of Janowski-type starlike functions with respect to symmetric points. *Demonstr. Math.* **2021**, *54*, 37–46. [CrossRef]
23. Piejko, K.; Sokół, J. On convolution and q-calculus. *Bol. Soc. Mat. Mex.* **2020**, *26*, 349–359. [CrossRef]
24. Piejko, K.; Sokół, J.; Trabka-Wiecław, K. On q-calculus and starlike Functions. *Iran. J. Sci. Technol. Trans. A Sci.* **2019**, *43*, 2879–2883. [CrossRef]
25. Srivastava, H.M.; Arif, M.; Raza, M. Convolution properties of meromorphically harmonic functions defined by a generalized convolution q-derivative operator. *AIMS Math.* **2021**, *6*, 5869–5885. [CrossRef]
26. Ademogullari, K.; Kahramaner, Y.; Polatoglu, Y. q-Harmonic mappings for which analytic part is q-convex functions. *Nonlinear Anal. Differ. Equ.* **2016**, *4*, 283–293. [CrossRef]
27. Long, P.-H.; Tang, H.; Wang, W.-S. Functional inequalities for several classes of q-starlike and q-convex type analytic and multivalent functions using a generalized Bernardi integral operator. *AIMS Math.* **2021**, *6*, 1191–1208. [CrossRef]
28. Noor, K.I. On analytic functions involving the q-Ruscheweyh derivative. *Fractal Fract.* **2019**, *3*, 10. [CrossRef]
29. Noor, K.I.; Riaz, S.; Noor, M.A. On q-Bernardi integral operator. *TWMS J. Pure Appl. Math.* **2017**, *8*, 3–11.
30. Srivastava, H.M.; Hadi, S.H.; Darus, M. Some subclasses of p-valent γ-uniformly type q-starlike and q-convex functions defined by using a certain generalized q-Bernardi integral operator. Rev. Real Acad. *Cienc. Exactas Fís. Nat. Ser. A Mat.* **2023**, *117*, 50. [CrossRef]
31. Srivastava, H.M. An introductory overview of the Bessel polynomials, the generalized Bessel polynomials and the q-Bessel polynomials. *Symmetry* **2023**, *15*, 822. [CrossRef]
32. Kac, V.; Cheung, P. *Quantum Calculus*; Springer Science and Business Media: Berlin/Heidelberg, Germany; New York, NY, USA, 2001.
33. Annaby, M.H.; Mansour, Z.S. *q-Fractional Calculus and Equations*; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 2012.
34. Ismail, M.E.-H.; Masson, D.R.; Rahman, M. (Eds.) Fields Institute Communications. In *Special Functions, q-Series and Related Topics*; American Mathematical Society: Providence, RI, USA, 1997.
35. Srivastava, H.M. Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis. *Iran. J. Sci. Technol. Trans. A Sci.* **2020**, *44*, 327–344. [CrossRef]
36. Srivastava, H.M. Some parametric and argument variations of the operators of fractional calculus and related special functions and integral transformations. *J. Nonlinear Convex Anal.* **2021**, *22*, 1501–1520.