SOLUTION OF A CERTAIN MULTIPLE INTEGRAL EQUATION

by

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ABSTRACT

An explicit solution is derived formally for a certain multiple integral equation involving a multidimensional fractional integral of Riemann–Liouville type. The main inversion theorem proved here provides a generalization of a result due to W.L. Wainwright [3]. A simple illustration of the theorem, involving the classical Laguerre function, is also presented.

1. INTRODUCTION AND PRELIMINARIES

We begin by defining an n–dimensional analogue of the familiar Riemann–Liouville fractional integral by (cf. [1, p. 181 et seq.)

(1.1) $$\mathcal{R}_{\mu_1,\ldots,\mu_n} f(x_1,\ldots,x_n) = \frac{1}{\Gamma(\mu_1)\cdots\Gamma(\mu_n)}$$

$$\cdot \int_0^{x_1} \cdots \int_0^{x_n} \prod_{j=1}^{n} \left\{(x_j-t_j)^{\mu_j-1}\right\} f(t_1,\ldots,t_n)dt_1 \cdots dt_n,$$

provided that the multiple integral exists. As is customary in the theory of fractional calculus [op. cit., p. 181], the operator $\mathcal{R}_{\mu_1,\ldots,\mu_n}$ is defined (by its analytic continuation) for all (real or complex) values of $\mu_1, \ldots, \mu_n$.

If $f(t_1,\ldots,t_n)$ is piecewise continuous for each $t_j \in [0, \infty)$, $j = 1, \ldots, n$, and if

(1.2) $$|f(t_1,\ldots,t_n)| \leq M_0 \exp(\xi_1 t_1 + \cdots + \xi_n t_n)$$

for all $t_j \geqq T_j$ $(j = 1,\ldots,n)$, M_0 and T_j being positive constants, then the n–dimensional Laplace transform of $f(t_1,\ldots,t_n)$ is defined by

$$\mathscr{L}\{f(t_1,\ldots,t_n)\colon s_1,\ldots,s_n\} = F(s_1,\ldots,s_n) \tag{1.3}$$

$$= \int_0^\infty \cdots \int_0^\infty \exp(-s_1t_1-\cdots-s_nt_n)\, f(t_1,\ldots,t_n)dt_1 \cdots dt_n,$$

where, for convergence, $\mathrm{Re}(s_j-\xi_j) > 0$, $j = 1, \ldots, n$.

The n–dimensional Fourier transform of a function $f(x_1,\ldots,x_n)$ is defined, as usual, by (cf. [2, p. 1136])

$$\mathscr{F}(\omega_1,\ldots,\omega_n) = \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \exp(-i\omega_1x_1-\cdots-i\omega_nx_n) \tag{1.4}$$

$$\cdot\, f(x_1,\ldots,x_n)dx_1 \cdots dx_n,\ -\infty < \omega_j < \infty \quad (j = 1,\ldots,n),$$

together with its inversion formula:

$$f(x_1,\ldots,x_n) = (2\pi)^{-n} \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \exp(i\omega_1x_1+\cdots+i\omega_nx_n) \tag{1.5}$$

$$\cdot\, \mathscr{F}(\omega_1,\ldots,\omega_n)d\omega_1 \cdots d\omega_n.$$

With a view to generalizing an earlier result of Wainwright [3], we apply these multidimensional transforms in order to solve the multiple integral equation:

$$g(x_1,\ldots,x_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathcal{R}_{\mu_1,\ldots,\mu_n} f(x_1,\ldots,x_n) \cdot h(\mu_1,\ldots,\mu_n) d\mu_1 \cdots d\mu_n, \tag{1.6}$$

where the functions f and g are so prescribed that their n–dimensional Laplace transforms exist, and the unknown function h is such that the multiple integral satisfies the required convergence conditions.

The following result will also be needed in our present investigation:

LEMMA. <u>Let the multidimensional fractional integral of</u> $f(x_1,\ldots,x_n)$, <u>defined by</u> (1.1), <u>exist</u>.

<u>Then</u>

$$\mathcal{L}\left\{\mathcal{R}_{\mu_1,\ldots,\mu_n} f(x_1,\ldots,x_n) : s_1,\ldots,s_n\right\} = s_1^{-\mu_1} \cdots s_n^{-\mu_n} F(s_1,\ldots,s_n), \tag{1.7}$$

<u>provided that the</u> n–<u>dimensional Laplace transform of</u> $f(x_1,\ldots,x_n)$ <u>exists</u>.

The proof of the lemma is fairly straightforward; indeed, if we apply the definitions (1.1) and (1.3), invert the order of integrations, and evaluate the inner multiple integral by using an elementary result [1, p. 202, Equation (11)], we arrive at the assertion (1.7).

2. SOLUTION OF THE MULTIPLE INTEGRAL EQUATION (1.6)

When the n–dimensional Laplace transform operator $\mathscr{L}$ is applied to the multiple integral equation (1.6), using the operational relation (1.7), we find that

(2.1) $$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} s_1^{-\mu_1} \cdots s_n^{-\mu_n} h(\mu_1,\dots,\mu_n) d\mu_1 \cdots d\mu_n = \frac{G(s_1,\dots,s_n)}{F(s_1,\dots,s_n)},$$

where $F(s_1,\dots,s_n) \neq 0$ and $G(s_1,\dots,s_n)$ denote the n–dimensional Laplace transforms of $f(x_1,\dots,x_n)$ and $g(x_1,\dots,x_n)$, respectively.

Setting $s_j = e^{i\omega_j}$ $(j = 1,\dots,n)$, (2.1) becomes

(2.2) $$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp(-i\omega_1\mu_1 - \cdots - i\omega_n\mu_n)\, h(\mu_1,\dots,\mu_n) d\mu_1 \cdots d\mu_n = \frac{G(e^{i\omega_1},\dots,e^{i\omega_n})}{F(e^{i\omega_1},\dots,e^{i\omega_n})}.$$

Now apply the multidimensional Fourier inversion formula (1.5), and we formally obtain

(2.3) $$h(\mu_1,\dots,\mu_n) = (2\pi)^{-n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp(i\omega_1\mu_1 + \cdots + i\omega_n\mu_n) \cdot \frac{G(e^{i\omega_1},\dots,e^{i\omega_n})}{F(e^{i\omega_1},\dots,e^{i\omega_n})} d\omega_1 \cdots d\omega_n,$$

which provides an explicit solution of the multiple integral equation (1.6).

3. AN ILLUSTRATIVE EXAMPLE

For $n = 1$, the solution (2.3) corresponds to the solution given in the one–dimensional case of the integral equation (1.6) by Wainwright [3, p. 300, Equation (13)].

For a simple application of our results, we let

$$g(x_1,\ldots,x_n) = x_1^{\lambda_1} \cdots x_n^{\lambda_n}, \tag{3.1}$$

so that

$$G(s_1,\ldots,s_n) = \prod_{j=1}^{n} \left\{\frac{\Gamma(\lambda_j+1)}{s_j^{\lambda_j+1}}\right\}, \quad \mathrm{Re}(\lambda_j) > -1 \quad (j = 1,\ldots,n). \tag{3.2}$$

Furthermore, in terms of the classical Laguerre function $L_\nu^{(\alpha)}(x)$ with arbitrary ν, we set

$$f(x_1,\ldots,x_n) = \prod_{j=1}^{n} \left\{x_j^{\alpha_j} L_{\nu_j}^{(\alpha_j)}(x_j)\right\}, \tag{3.3}$$

so that

$$F(s_1,\ldots,s_n) = \prod_{j=1}^{n} \left\{\frac{\Gamma(\alpha_j+\nu_j+1)}{\Gamma(\nu_j+1)} \frac{(s_j-1)^{\nu_j}}{s_j^{\alpha_j+\nu_j+1}}\right\}, \tag{3.4}$$

$$\mathrm{Re}(\alpha_j) > -1 \quad (j = 1,\ldots,n).$$

Upon substituting from (3.1) to (3.4) into the pair of equations (1.6) and (2.3), we arrive at once at the following result:

If

(3.5) $$x_1^{\lambda_1} \cdots x_n^{\lambda_n} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h(\mu_1,\ldots,\mu_n)$$

$$\cdot\ \mathscr{R}_{\mu_1,\ldots,\mu_n} \prod_{j=1}^{n} \left\{ x_j^{\alpha_j} L_{\nu_j}^{(\alpha_j)}(x_j) \right\} d\mu_1 \cdots d\mu_n,$$

then

(3.6) $$h(\mu_1,\ldots,\mu_n) = (2\pi)^{-n} \prod_{j=1}^{n} \left\{ \frac{\Gamma(\lambda_j+1)\,\Gamma(\nu_j+1)}{\Gamma(\alpha_j+\nu_j+1)} \right\}$$

$$\cdot \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{\exp(i\omega_1\sigma_1+\cdots+i\omega_n\sigma_n)}{(e^{i\omega_1}-1)^{\nu_1} \cdots (e^{i\omega_n}-1)^{\nu_n}}\, d\omega_1 \cdots d\omega_n,$$

where, for convenience,

$$\sigma_j = \alpha_j + \mu_j + \nu_j - \lambda_j \quad (j = 1, \ldots, n).$$

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REFERENCES

[1] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, Tables of Integral Transforms, Vol. II, McGraw–Hill, New York, London and Toronto, 1954.

[2] K. Rektorys (Ed.), Survey of Applicable Mathematics (Translated from the Czech by R. Výborný et al.), M.I.T. Press, Cambridge, Massachusetts, 1969.

[3] W.L. Wainwright, A functional relation, in Fractional Calculus and Its Applications (B. Ross, Ed.), pp. 298–305, Springer–Verlag, Berlin, Heidelberg and New York, 1975.

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