THE THEOREM OF VON ZEIPEL FOR QUASIHOMOGENEOUS POTENTIALS

by

Cristina-Mona Popescu
B.Sc., University Babeş-Bolyai of Cluj-Napoca, 1992.

A thesis submitted in partial fulfillment
of the requirements for the degree of

Master of Science

in the Department
of Mathematics and Statistics.

We accept this thesis as conforming
to the required standard.

[redacted]

Dr. F. N. Diacu, Dept. of Math. & Stat., University of Victoria

[redacted]

Dr. R. Illner, Dept. of Math. & Stat., University of Victoria

[redacted]

Dr. C. Bose, Dept. of Math. & Stat., University of Victoria

[redacted]

Dr. J. Tatum, Dept. of Physics, University of Victoria

Supervisor: Dr. F. N. Diacu.

Abstract

In this thesis we consider the motion of n particles under the influence of a law of attraction given by a quasihomogeneous potential function depending on two parameters. Our main goal is to prove that if a solution of the n-body problem has a singularity at which the moment of inertia has a finite limit, then the singularity is due to a collision. In other words, a pseudocollision may occur only if the motion becomes unbounded in finite time.

Examiners [redacted]

Dr. F. N. Diacu, Dept. of Math. & Stat., University of Victoria

[redacted]

Dr. R. Illner, Dept. of Math. & Stat., University of Victoria

[redacted]

Dr. C. Bose, Dept. of Math. & Stat., University of Victoria

[redacted]

Dr. J. Tatum, Dept. of Physics, University of Victoria

Table of Contents

INTRODUCTION

This paper is concerned with the study of the n-body problem of particles moving in the 3-dimensional Euclidien space $\mathbb{R}^3$ under the influence of a law of attraction described by quasihomogeneous potentials (see [3]). For given initial conditions, the problem has regular or singular solutions. We consider the case of singular solutions.

A natural question one may ask is whether all singularities are due to collisions. This question was initially formulated in the Newtonian n-body problem, where there are examples of noncollision singularities. Our goal is to offer a necessary condition for the existence of noncollision singularities, analogous to a result of the Swedish mathematician Hugo Von Zeipel (see [7],[8]). In the context of the classical n-body problem he stated that: "If a solution has a singularity at t^* and if some of the particles do not tend to finite limiting positions as t tends to t^*, then one necessarily has $\lim_{t \to t^*} R(t) = \infty$, where R is the maximum of the mutual distances between particles."

This thesis has four chapters. The first chapter gives a brief description of the n-body problem with quasihomogeneous potentials and its ten first integrals.

In chapter two we show that the minimum distance between all pairs of particles must approach zero at a singularity. This result generalizes Painlevé's criterion for the encounter of a singularity in the classical n-body problem (see [6]) and allows us to formulate the definition which establishes the nature of singularities of the n-body problem with quasihomogeneous potentials.

In chapter three we decompose the moment of inertia of the system of particles into components corresponding to subsystems. For this, we define a new inner product on $\mathbb{R}^{3n}$ and regard $\mathbb{R}^{3n}$ as the direct sum of two linear orthogonal subspaces.

In chapter four we continue to study the behavior of singular solutions. We prove that at a singularity instant, the moment of inertia I always has a limit (which may be finite or infinite). We conclude with the proof of Von Zeipel's theorem for the n-body problem with quasihomogeneous potentials, which states that: "If t^* is a singularity for a solution $(\mathbf{q},\mathbf{p})$ of the n-body problem with quasihomogeneous potentials and $\lim_{t\to t^*} I(\mathbf{q}(t))$ is finite, then t^* is a collision singularity." This means that a necessary condition for having a noncollision singularity is that the motion of the particles becomes unbounded in finite time.

Chapter 1

THE N-BODY PROBLEM

In this section we formulate the equations of motion of the n-body problem and establish its first integrals.

Consider n particles (bodies, point masses) in the 3-dimensional Euclidien space $\mathbb{R}^3$. Denote by $m_i > 0$ the **mass** and by $\mathbf{q}_i = (\mathbf{q}_i^1, \mathbf{q}_i^2, \mathbf{q}_i^3)$ the **position** of the i-th particle, $i = 1, \ldots, n$, in a fixed Cartesian coordinate system, so that $\mathbf{q}_i$ is an element of $\mathbb{R}^3$ and call

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, ..., \mathbf{q}_n) \in \mathbb{R}^{3n}$$

the **configuration** of the n-particles. A configuration $(\mathbf{q}_1, \mathbf{q}_2, ..., \mathbf{q}_n)$ of the particles is called a **collision** if $\mathbf{q}_i = \mathbf{q}_j$ for some $i \neq j$.

Suppose the particles move in a potential field with potential energy $-W$, where W is the **quasihomogeneous potential function** defined on $\mathbb{R}^{3n} - \Delta$ by

$$W(\mathbf{q}) = U(\mathbf{q}) + V(\mathbf{q}). \tag{1.1}$$

The functions $U : \mathbb{R}^{3n} - \Delta \to \mathbb{R}_+$ and $V : \mathbb{R}^{3n} - \Delta \to \mathbb{R}_+$ are given by

$$U(\mathbf{q}) = \sum_{1 \le i < j \le n} \frac{\alpha(m_i, m_j)}{|\mathbf{q}_i - \mathbf{q}_j|^a}, \qquad a \in \mathbb{R}, a \ge 1,$$

$$V(\mathbf{q}) = \sum_{1 \le i < j \le n} \frac{\beta(m_i, m_j)}{|\mathbf{q}_i - \mathbf{q}_j|^b}, \qquad b \in \mathbb{R}, b > a,$$

where α and β are functions of the masses satisfying the following conditions

$$\alpha(m_i, m_j) > 0, \quad \beta(m_i, m_j) > 0, \quad \forall i, j = 1, ..., n,$$

and

$$\alpha(m_i, m_j) = \alpha(m_j, m_i), \quad \beta(m_i, m_j) = \beta(m_j, m_i), \quad \forall i, j = 1, \dots, n.$$

Note that $| \,.\, | : \mathbb{R}^3 \to [0, \infty)$ is the Euclidien norm and $\mathbb{R}_+ = (0, \infty)$. The set Δ is defined as

$$\Delta = \bigcup_{1 \leq i < j \leq n} \Delta_{ij}, \tag{1.2}$$

where Δ_{ij} is called the set of collisions of the i-th and j-th particles, i.e.,

$$\Delta_{ij} = \{\mathbf{q} \in \mathbb{R}^{3n} | \mathbf{q}_i = \mathbf{q}_j\}, \; i < j. \tag{1.3}$$

Thus Δ_{ij} is a linear subspace of $\mathbb{R}^{3n}$. Δ denotes the set of all collision configurations. Observe that Δ_{ij}, Δ are closed sets in $\mathbb{R}^{3n}$ and U, V, W are analytic functions on the open set $\mathbb{R}^{3n} - \Delta$.

The motion of the particles in Newtonian formulation is described by the nonlinear second order autonomous system of differential equations on $\mathbb{R}^{3n} - \Delta$

$$m_i \ddot{\mathbf{q}}_i = \nabla_i W(\mathbf{q}), \quad i = 1, \dots, n, \tag{1.4}$$

where the double dot denotes the second derivative with respect to time t and the symbol ∇_i denotes the **gradient** with respect to the i-th variable, i. e.,

$$\nabla_i W(\mathbf{q}) = \frac{\partial W}{\partial \mathbf{q}_i}(\mathbf{q}) = (\frac{\partial W}{\partial q_i^1}(\mathbf{q}), \frac{\partial W}{\partial q_i^2}(\mathbf{q}), \frac{\partial W}{\partial q_i^3}(\mathbf{q})).$$

Let $M = \text{diag}(m_1, m_1, m_1, \dots, m_n, m_n, m_n)$ be a $3n$-dimensional diagonal matrix, each index $i \in \{1, 2, \dots, n\}$ appearing three times and let ∇ be the gradient operator $\nabla = (\nabla_1, \dots, \nabla_n)$. Then, the equations of motion (1.4) take the abbreviated form

$$M\ddot{\mathbf{q}} = \nabla W(\mathbf{q})$$

or

$$\ddot{\mathbf{q}} = M^{-1}\nabla W(\mathbf{q}),$$

where $M^{-1} = \mathrm{diag}(m_1^{-1}, m_1^{-1}, m_1^{-1}, \ldots, m_n^{-1}, m_n^{-1}, m_n^{-1})$. Newton's equations can be rewritten in such way that they become a first order system of $6n$ differential equations with the vector field defined on $(\mathbb{R}^{3n} - \Delta) \times \mathbb{R}^{3n}$,

$$\begin{cases} \dot{\mathbf{q}}_i = m_i^{-1}\mathbf{p}_i \\ \\ \dot{\mathbf{p}}_i = \nabla_i W(\mathbf{q}) \quad i = 1, \ldots, n, \end{cases} \tag{1.5}$$

where $\mathbf{p}_i = m_i\dot{\mathbf{q}}_i$, $i = 1, \ldots, n$ denotes the **momentum** of the i-th particle. Let $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_n)$ be the momentum of the system of particles. Then the abbreviated form for (1.5) is

$$\begin{cases} \dot{\mathbf{q}} = M^{-1}\mathbf{p} \\ \\ \dot{\mathbf{p}} = \nabla W(\mathbf{q}). \end{cases} \tag{1.6}$$

The **n-body problem** consists of solving the system of equations (1.6) for arbitrary preassigned initial data in $(\mathbb{R}^{3n} - \Delta) \times \mathbb{R}^{3n}$.

Let us construct the ten known first integrals of the n-body problem. Recall that a continuously differentiable function $F : (\mathbb{R}^{3n} - \Delta) \times \mathbb{R}^{3n} \rightarrow \mathbb{R}$ of the $6n$ variables $\mathbf{q}, \mathbf{p}$ is said to be a **first integral** for the system of equations (1.6) if it is constant along each solution $(\mathbf{q},\mathbf{p})$ of it (i.e., $F(\mathbf{q}, \mathbf{p}) = c$ (constant) along a solution $(\mathbf{q},\mathbf{p})$). A relation like this between the components of a solution reduces the dimension of the system by one. Given l integrals $F_1, F_2, \ldots, F_l$ of (1.6), they are said to be **independent** if the Jacobian matrix formed with the $6n$ partial derivatives with respect to $\mathbf{q}_i, \mathbf{p}_i$, $i = 1, \ldots, n$ has rank l. It is known that systems of m first order differential equations have locally m independent first integrals. In general, a function f is said to be **algebraic** with respect to the variable x if there exists a polynomial $P(x, y)$ such that $P(x, f(x)) = 0$.

Observe that

$$\begin{aligned}\nabla_i W(\mathbf{q}) &= \nabla_i U(\mathbf{q}) + \nabla_i V(\mathbf{q}) \\ &= a\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{\alpha(m_i, m_j)}{|\mathbf{q}_j - \mathbf{q}_i|^{a+2}}(\mathbf{q}_j - \mathbf{q}_i) + b\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{\beta(m_i, m_j)}{|\mathbf{q}_j - \mathbf{q}_i|^{b+2}}(\mathbf{q}_j - \mathbf{q}_i), \qquad (1.7)\end{aligned}$$

for $\mathbf{q} \in \mathbb{R}^{3n} - \Delta$, for $i \in \{1, 2, \ldots, n\}$. By (1.4), (1.7), and the property of symmetry of the functions α and β we have

$$\sum_{i=1}^{n} \dot{\mathbf{p}}_i = \sum_{i=1}^{n} m_i \ddot{\mathbf{q}}_i = \sum_{i=1}^{n} \nabla_i W(\mathbf{q})$$

$$= \sum_{i=1}^{n} \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{\alpha(m_i, m_j)}{|\mathbf{q}_j - \mathbf{q}_i|^{a+2}}(\mathbf{q}_j - \mathbf{q}_i) + b\sum_{i=1}^{n} \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{\beta(m_i, m_j)}{|\mathbf{q}_j - \mathbf{q}_i|^{b+2}}(\mathbf{q}_j - \mathbf{q}_i) = \mathbf{0}.$$

By integrating this equation with respect to t, we obtain

$$\sum_{i=1}^{n} \mathbf{p}_i(t) = \mathbf{c}_1, \qquad (1.8)$$

where $\mathbf{c}_1 = (c_1^1, c_1^2, c_1^3) \in \mathbb{R}^3$ is a constant of integration. We obtained thus three scalar first integrals called the **momentum integrals.**

By integrating equation (1.8) we obtain

$$\sum_{i=1}^{n} m_i \mathbf{q}_i(t) - \mathbf{c}_1 t = \mathbf{c}_2, \qquad (1.9)$$

where $\mathbf{c}_2 = (c_2^1, c_2^2, c_2^3) \in \mathbb{R}^3$ is also a constant of integration. Thus we found another three scalar first integrals, called the **integrals of the center of mass.** The **center of mass** of the system of particles is defined at time t by

$$\mathbf{c}(t) = \frac{\sum_{i=1}^{n} m_i \mathbf{q}_i(t)}{\sum_{i=1}^{n} m_i}.$$

We define the **kinetic energy** of the system of particles as to be

$$T : \mathbb{R}^{3n} \to \mathbb{R}_+, \quad T(\mathbf{p}) = \frac{1}{2}\sum_{i=1}^{n} \frac{|\mathbf{p}_i|^2}{m_i}. \tag{1.10}$$

Using (1.4), we have

$$\begin{aligned} \frac{d}{dt}T(\mathbf{p}(t)) &= \frac{1}{2}\frac{d}{dt}\left(\sum_{i=1}^{n} m_i|\dot{\mathbf{q}}_i(t)|^2\right) = \sum_{i=1}^{n} m_i\dot{\mathbf{q}}_i(t)\ddot{\mathbf{q}}_i^T(t) \\ &= \sum_{i=1}^{n} \dot{\mathbf{q}}_i(t)\nabla_i^T W(\mathbf{q}(t)) = \frac{d}{dt}W(\mathbf{q}(t)), \end{aligned}$$

where the upper index "T" means transposition. By integrating this equation we obtain the **integral of energy**

$$T(\mathbf{p}(t)) - W(\mathbf{q}(t)) = h, \tag{1.11}$$

where $h \in \mathbb{R}$ is an integration constant called the **constant of energy**.

We define the **angular momentum** of the system of particles with respect to the origin of the Cartesian system to be the function

$$A : (\mathbb{R}^{3n} - \Delta) \times \mathbb{R}^{3n} \to \mathbb{R}^3, \quad A(\mathbf{q},\mathbf{p}) = \sum_{i=1}^{n}(\mathbf{q}_i \times \mathbf{p}_i),$$

where "$\times$" is the cross product of two vectors in $\mathbb{R}^3$, i.e.,

$$\mathbf{q}_i \times \mathbf{p}_i = (q_i^2 p_i^3 - q_i^3 p_i^2, q_i^3 p_i^1 - q_i^1 p_i^3, q_i^1 p_i^2 - q_i^2 p_i^1).$$

One can easily see that

$$\frac{d}{dt}(\mathbf{q}_i \times \mathbf{p}_i) = \frac{d\mathbf{q}_i}{dt} \times \mathbf{p}_i + \mathbf{q}_i \times \frac{d\mathbf{p}_i}{dt}.$$

Then, we have

$$\frac{d}{dt}\left(\sum_{i=1}^{n}(\mathbf{q}_i \times \mathbf{p}_i)\right) = \sum_{i=1}^{n} m_i(\dot{\mathbf{q}}_i \times \dot{\mathbf{q}}_i) + \sum_{i=1}^{n} m_i(\mathbf{q}_i \times \ddot{\mathbf{q}}_i) = \sum_{i=1}^{n} m_i(\mathbf{q}_i \times \ddot{\mathbf{q}}_i).$$

By (1.4),

$$\sum_{i=1}^{n} m_i(\mathbf{q}_i \times \ddot{\mathbf{q}}_i) = \sum_{i=1}^{n}(\mathbf{q}_i \times \nabla_i W(\mathbf{q}))$$

$$= a\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{\alpha(m_i, m_j)}{|\mathbf{q}_j - \mathbf{q}_i|^{a+2}}(\mathbf{q}_i \times \mathbf{q}_j) - a\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{\alpha(m_i, m_j)}{|\mathbf{q}_j - \mathbf{q}_i|^{a+2}}(\mathbf{q}_i \times \mathbf{q}_i)$$

$$+b\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{\beta(m_i, m_j)}{|\mathbf{q}_j - \mathbf{q}_i|^{b+2}}(\mathbf{q}_i \times \mathbf{q}_j) - b\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{\beta(m_i, m_j)}{|\mathbf{q}_j - \mathbf{q}_i|^{b+2}}(\mathbf{q}_i \times \mathbf{q}_i).$$

But $\mathbf{q}_i \times \mathbf{q}_i = \mathbf{0}$, $\mathbf{q}_i \times \mathbf{q}_j = -(\mathbf{q}_j \times \mathbf{q}_i)$ and thus

$$\frac{d}{dt}\left(\sum_{i=1}^{n}(\mathbf{q}_i \times \mathbf{p}_i)\right) = \mathbf{0}.$$

Integrating, we obtain

$$\sum_{i=1}^{n}(\mathbf{q}_i \times \mathbf{p}_i) = \mathbf{c}_3, \tag{1.12}$$

where $\mathbf{c}_3 = (c_3^1, c_3^2, c_3^3) \in \mathbb{R}^3$ is an integration constant. The three scalar first integrals obtained are called the **integrals of the angular momentum**.

Thus, the left sides of (1.8), (1.9), (1.11), (1.12) are ten first integrals algebraic with respect to $\mathbf{q}$, $\mathbf{p}$ and t for the n-body problem, which are easily seen to be independent. One may ask if there are any additional integrals of this kind. The German mathematician H.Bruns proved in 1887 [1] that there are no further first integrals of (1.6) algebraic with respect to $\mathbf{q}$, $\mathbf{p}$ and t, independent of these ten.

With the aid of the ten first integrals, by eliminating ten coordinates $\mathbf{q}, \mathbf{p}$ from the equations of motion (1.6), the system can be reduced to $6n - 10$ first order differential equations. The local existence and uniqueness of the analytic solution of (1.6), for given initial data in $(\mathbb{R}^{3n} - \Delta) \times \mathbb{R}^{3n}$, is assured by results of the theory of ordinary differential equations (for the existence theorem of Cauchy, see [7]).

We shall close this section by a remark which will be used from now on. The equations (1.8) and (1.9) show us that, from the physical point of view, the center of mass of the n-bodies system is moving uniformly (with constant speed $\mathbf{c}_1(\sum_{i=1}^n m_i)^{-1}$, if $\mathbf{c}_1 \neq \mathbf{0}$) or rests (at $\mathbf{c}_2$, if $\mathbf{c}_1 = \mathbf{0}$) for the time interval where the solution is defined.

We assume in the following that the center of mass is at rest (fixed) at the origin O. This restricts the configurations to the invariant set

$$Q = \{\mathbf{q} \in \mathbb{R}^{3n} | \sum_{i=1}^{n} m_i \mathbf{q}_i = \mathbf{0}\}$$

and the momentum to the invariant set

$$P = \{\mathbf{p} \in \mathbb{R}^{3n} \mid \sum_{i=1}^{n} \mathbf{p}_i = \mathbf{0}\}.$$

Since $T - W$ is constant along solutions of (1.6), the set defined by

$$M(h) = \{(\mathbf{q}, \mathbf{p}) \in (Q - \Delta) \times P \mid (T - W)(\mathbf{q}, \mathbf{p}) = h\}$$

is an invariant set for each real constant h (i.e., if $(\mathbf{q}, \mathbf{p})(0) \in M(h)$, then $(\mathbf{q}, \mathbf{p})(t) \in M(h)$ for all t where the solution is defined). The set $M(h)$ is called the **constant energy surface**. Thus, without loss of generality, we shall fix an $h \in \mathbb{R}$ and restrict the equations of motion to the invariant set $M(h)$.

Chapter 2

NATURE OF SINGULARITIES

Suppose we are given an initial position $\mathbf{q}(0) \in \mathbb{R}^{3n} - \Delta$ and an initial velocity $\dot{\mathbf{q}}(0) \in \mathbb{R}^{3n}$. Then the standard theorems of differential equations assure the existence and uniqueness of an analytic solution of equations (1.6), defined locally on some interval (t^-, t^+) with $0 \in (t^-, t^+)$. Because of the symmetry of mechanical laws with respect to the past and future, one can study the problem on $(t^-, 0]$ or on $[0, t^+)$ without loss of generality. We choose to work on the second interval. The solution can be extended analytically to a maximal interval $[0, t^*)$, with $0 < t^+ \leq t^* \leq \infty$.

Definition 2.1. *If $t^* = \infty$, the solution is called* **regular**. *If $t^* < \infty$, the solution is said to be* **singular**, *t^* is called a* **singularity of the solution** *and we say that the solution experiences a singularity at t^* (i.e., is defined and is analytic on $[0, t^*)$ but not at t^*).*

Our study will be concerned with the behavior of singular solutions of the n-body problem. We shall start by describing the physical behavior of the particles close to t^*. To this end, we define the function

$$\rho : \mathbb{R}^{3n} \to [0, \infty), \quad \rho(\mathbf{q}) = \min_{1 \leq i < j \leq n} q_{ij}, \tag{2.1}$$

where $q_{ij} = |\mathbf{q}_i - \mathbf{q}_j|$. Because $t^* > 0$, we shall use the notation $t \to t^*$ for

$t \to t^*$ with $t < t^*$.

A first result stated in the following proposition tells us that the minimum distance between all pairs of particles must approach zero at a singularity.

Proposition 2.2. *If* $(\mathbf{q},\mathbf{p})$ *is a solution of the equations* (1.6) *defined on the maximal interval* $[0, t^*)$, *then* t^* *is a singularity of the solution if and only if*

$$\lim_{t\to t^*} \inf \rho(\mathbf{q}(t)) = 0. \tag{2.2}$$

Proof: Recall that $l \in \mathbb{R}$ is the limit inferior of $\rho \circ \mathbf{q}$ at t^*, i.e., $l = \lim_{t\to t^*} \inf \rho(\mathbf{q}(t))$ if and only if the following conditions are satisfied:
i) for each $\epsilon > 0$ and for each $\delta > 0$ there exists a $t_\delta \in [0, t^*)$ such that $|t_\delta - t^*| < \delta$ and $\rho(\mathbf{q}(t_\delta)) < l + \epsilon$,
ii) for each $\epsilon > 0$ there exists a $\delta > 0$ such that for all $t \in [0, t^*)$ satisfying $|t - t^*| < \delta$, we have $\rho(\mathbf{q}(t)) > l - \epsilon$.
L is the limit superior of $\rho \circ \mathbf{q}$ at t^*, i.e., $L = \lim_{t\to t^*} \sup \rho(\mathbf{q}(t))$ if and only if
i′) for each $\epsilon > 0$ and for each $\delta > 0$ there exists a $t_\delta \in [0, t^*)$ such that $|t_\delta - t^*| < \delta$ and $\rho(\mathbf{q}(t_\delta)) > L - \epsilon$,
ii′) for each $\epsilon > 0$ there exists a $\delta > 0$ such that for all $t \in [0, t^*)$ satisfying $|t - t^*| < \delta$, we have $\rho(\mathbf{q}(t)) < L + \epsilon$.
Notice also that $\lim_{t\to t^*} \sup \rho(\mathbf{q}(t)) = \infty$ if and only if
iii′) there exists a sequence $(t_n)_{n\in\mathbf{N}}$, $t_n \in [0, t^*)$, $\forall n \in \mathbf{N}$ with $t_n \to t^*$ as $n \to \infty$, such that $\rho(\mathbf{q}(t_n)) \to \infty$ when $n \to \infty$.

Let us prove the necessity of the condition stated in Proposition 2.2. Let $l = \lim_{t\to t^*} \inf \rho(\mathbf{q}(t))$. $\rho : \mathbb{R}^{3n} \to [0, \infty)$, so $l \geq 0$. Assume $l > 0$. Then, there exists a $\theta > 0$ such that $l \geq \theta$. Take $\mu \in (0, \theta)$ and $\epsilon = l - \mu$. By ii), there exists a $\delta > 0$ such that if $t \in [0, t^*)$ with $|t - t^*| < \delta$ then $\rho(\mathbf{q}(t)) > l - \epsilon = \mu$. Let $t_0 = t^* - \delta/2$. Then $\rho(\mathbf{q}(t)) > \mu$ on $[t_0, t^*)$, which implies that

$$q_{ij}(t) > \mu, \quad \forall t \in [t_0, t^*), \quad \forall i, j = 1, 2, \ldots, n, \; i \neq j. \tag{2.3}$$

By (1.4) and (1.7) we have

$$m_i\ddot{\mathbf{q}}_i = a\sum_{\substack{j=1\\j\neq i}}^{n}\frac{\alpha(m_i,m_j)}{{q_{ij}}^{a+2}}(\mathbf{q}_j-\mathbf{q}_i)+b\sum_{\substack{j=1\\j\neq i}}^{n}\frac{\beta(m_i,m_j)}{{q_{ij}}^{b+2}}(\mathbf{q}_j-\mathbf{q}_i),\ \forall i=1,\ldots,n.$$

This implies

$$|\ddot{\mathbf{q}}_i| \leq a\sum_{\substack{j=1\\j\neq i}}^{n}\frac{\alpha(m_i,m_j)}{m_i{q_{ij}}^{a+1}}+b\sum_{\substack{j=1\\j\neq i}}^{n}\frac{\beta(m_i,m_j)}{m_i{q_{ij}}^{b+1}},\quad \forall i=1,\ldots,n,$$

on $[t_0,t^*)$. From here, by (2.3) we have

$$|\ddot{\mathbf{q}}_i| < \frac{a}{\mu^{a+1}}\sum_{\substack{j=1\\j\neq i}}^{n}\frac{\alpha(m_i,m_j)}{m_i}+\frac{b}{\mu^{b+1}}\sum_{\substack{j=1\\j\neq i}}^{n}\frac{\beta(m_i,m_j)}{m_i},\quad \forall i=1,\ldots,n, \tag{2.4}$$

on $[t_0,t^*)$, i.e., $\ddot{\mathbf{q}}_i$ is bounded by a positive constant on $[t_0,t^*)$. Denote by M this constant, i.e., the right side of (2.4).

The next step is to prove that for $t\to t^*$, $\lim\dot{\mathbf{q}}(t)$ and $\lim\mathbf{q}(t)$ exist, this means $\lim\dot{\mathbf{q}}_i(t)$ and $\lim\mathbf{q}_i(t)$ exist, for all $i=1,\ldots,n$.

For this we shall use Cauchy's criterion for the existence of the limit of a function at a point, criterion which, applied to our case, states: "The limit $\lim_{t\to t^*}\dot{\mathbf{q}}_i(t)$ (respectively $\lim_{t\to t^*}\mathbf{q}_i(t)$) exists if and only if for $\epsilon>0$ there exists $\delta>0$ such that whenever $t_1,t_2\in[t_0,t^*)$ with $|t_2-t^*|<\delta$, then $|\dot{\mathbf{q}}_i(t_1)-\dot{\mathbf{q}}_i(t_2)|<\epsilon$ (respectively $|\mathbf{q}_i(t_1)-\mathbf{q}_i(t_2)|<\epsilon$)."

First, we shall show the existence of $\lim_{t\to t^*}\dot{\mathbf{q}}_i(t)$, $\forall i=1,\ldots,n$. Fix $i\in\{1,2,\ldots,n\}$. Let $\epsilon>0$ and $\delta=\epsilon/2M$. Let $t_1,t_2\in[t_0,t^*)$, $t_1<t_2$, such that $|t_1-t^*|<\delta$ and $|t_2-t^*|<\delta$. The solution is analytic on $[0,t^*)$, thus differentiable on (t_1,t_2). The mean-value theorem assures the existence of a $\xi_i\in(t_1,t_2)$ such that

$$\dot{\mathbf{q}}_i(t_2)-\dot{\mathbf{q}}_i(t_1)=\ddot{\mathbf{q}}_i(\xi_i)(t_2-t_1).$$

Then, by (2.4) we have

$$|\dot{\mathbf{q}}_i(t_2)-\dot{\mathbf{q}}_i(t_1)|=|\ddot{\mathbf{q}}_i(\xi_i)|\cdot|t_2-t_1|<M|t_2-t_1|,$$

where

$$|t_2 - t_1| \leq |t_2 - t^*| + |t_1 - t^*| < 2\delta.$$

Thus,

$$|\dot{\mathbf{q}}_i(t_2) - \dot{\mathbf{q}}_i(t_1)| < 2M\delta = \frac{2M\epsilon}{2M} = \epsilon.$$

Therefore, by Cauchy's criterion, $\lim_{t\to t^*} \dot{\mathbf{q}}_i(t)$ exists.

Let $\mathbf{p}_i^* = \lim_{t\to t^*} \dot{\mathbf{q}}_i(t)$, $\forall i = 1, \ldots, n$ and let $\mathbf{p}^* = (\mathbf{p}_1^*, \mathbf{p}_2^*, \ldots, \mathbf{p}_n^*)$. Now fix $i \in \{1, 2, \ldots, n\}$. We shall show that $\lim_{t\to t^*} \mathbf{q}_i(t)$ exists. Let $\epsilon > 0$. By the definition of the limit of a function at a point, there exists a $\delta' > 0$ such that $\forall t \in [t_0, t^*)$ satisfying $|t - t^*| < \delta'$, we have

$$|\dot{\mathbf{q}}_i(t) - \mathbf{p}_i^*| < \epsilon.$$

Then

$$|\dot{\mathbf{q}}_i(t)| \leq |\dot{\mathbf{q}}_i(t) - \mathbf{p}_i^*| + |\mathbf{p}_i^*| < \epsilon + |\mathbf{p}_i^*|, \tag{2.5}$$

with t satisfying $|t - t^*| < \delta'$. Let $\delta = \min\{\delta', \epsilon/2(\epsilon + |\mathbf{p}_i^*|)\}$ and let $t_1, t_2 \in [t_0, t^*)$, $t_1 < t_2$, such that $|t_1 - t^*| < \delta$. The mean-value theorem assures the existence of a $\xi_i \in (t_1, t_2)$ such that

$$\mathbf{q}_i(t_2) - \mathbf{q}_i(t_1) = \dot{\mathbf{q}}_i(\xi_i)(t_2 - t_1). \tag{2.6}$$

Combining (2.5) and (2.6) we obtain

$$|\mathbf{q}_i(t_2) - \mathbf{q}_i(t_1)| < (\epsilon + |\mathbf{p}_i^*|) \cdot |t_2 - t_1| < (\epsilon + |\mathbf{p}_i^*|)2\delta \leq \epsilon.$$

Therefore, by Cauchy's criterion, $\lim_{t\to t^*} \mathbf{q}_i(t)$ exists. Let $\mathbf{q}_i^* = \lim_{t\to t^*} \mathbf{q}_i(t)$, $\mathbf{q}^* = (\mathbf{q}_1^*, \ldots, \mathbf{q}_n^*)$, and $q_{ij}^* = |\mathbf{q}_i^* - \mathbf{q}_j^*|$ for all $i, j = 1, \ldots, n$, $i \neq j$. Letting $t \to t^*$ in relation (2.3), we obtain

$$q_{ij}^* \geq \mu > 0, \quad \forall i, j = 1, \ldots, n, \quad i \neq j. \tag{2.7}$$

To the equations (1.6) we further apply the existence and uniqueness theorem for differential equations with initial conditions sufficiently close to t^*, $\mathbf{q}^*$ (relation (2.7) assures that the initial position is in $\mathbb{R}^{3n} - \Delta$) and $\mathbf{p}^*$. It follows that there is an interval $(t^* - \gamma, t^* + \gamma)$ with $\gamma > 0$ depending on θ and μ, but not on t^*, where the solution remains analytic, i.e., the solution is defined on t^*, a contradiction. The necessity is thus proved.

Let us prove the sufficiency. Suppose $\lim_{t\to t^*} \inf \rho(\mathbf{q}(t)) = 0$. From the equations of motion $m_i\ddot{\mathbf{q}}_i = \nabla_i W(\mathbf{q})$, $i = 1, \ldots, n$, we observe that if $\ddot{\mathbf{q}}(t)$ is unbounded as $t \to t^*$, i.e., if there exists an $i \in \{1, \ldots, n\}$ and a sequence $(t_n)_{n\in\mathbb{N}}$, $t_n \in [0, t^*)$, $\forall n \in \mathbb{N}$, with $t_n \to t^*$ as $n \to \infty$ such that $\ddot{\mathbf{q}}_i(t_n) \to \infty$, then $\lim_{n\to\infty} \nabla_i W(\mathbf{q}(t_n)) = \infty$, i.e., $\lim_{t\to t^*} \sup \nabla_i W(\mathbf{q}(t)) = \infty$. This makes the equations of motion meaningless. The solution cannot be defined at t^*, and so t^* is a singularity of the solution. Assume that $\ddot{\mathbf{q}}$ is bounded, i.e., there exists a $c_1 > 0$ such that

$$||\ddot{\mathbf{q}}(t)|| \leq c_1, \quad \forall t \in [0, t^*),$$

where $||\ .\ ||$ is the Euclidien norm in $\mathbb{R}^{3n}$. But $\ddot{\mathbf{q}}(t) = M^{-1}\nabla W(\mathbf{q}(t))$, so it follows that $\nabla W(\mathbf{q}(t))$ is bounded on $[0, t^*)$. Notice that $|\ddot{\mathbf{q}}_i(t)| \leq ||\ddot{\mathbf{q}}(t)||$, $\forall i = 1, \ldots, n$, and apply Cauchy's criterion for the existence of $\lim_{t\to t^*} \dot{\mathbf{q}}(t)$ as we did previously. It follows that $\lim_{t\to t^*} \dot{\mathbf{q}}(t)$ exists. Denote it by $\mathbf{p}^*$. Let $\epsilon > 0$. Then there exists a $\delta > 0$ such that for $t \in [0, t^*)$ satisfying $|t - t^*| < \delta$ we have $||\dot{\mathbf{q}}(t) - \mathbf{p}^*|| < \epsilon$. Thus,

$$||\dot{\mathbf{q}}(t)|| \leq ||\dot{\mathbf{q}}(t) - \mathbf{p}^*|| + ||\mathbf{p}^*|| < \epsilon + ||\mathbf{p}^*||$$

on $(t^* - \delta, t^*)$. Recall that

$$\frac{d}{dt} W(\mathbf{q}(t)) = (\nabla W(\mathbf{q}(t)), \dot{\mathbf{q}}(t)), \quad \forall t \in [0, t^*),$$

where $(\ .\ ,.\ ) : \mathbb{R}^{3n} \times \mathbb{R}^{3n} \to \mathbb{R}$ is the standard inner product defined over $\mathbb{R}^{3n}$. The inner product satisfies the Schwarz inequality, so

$$|\frac{d}{dt} W(\mathbf{q}(t))| \leq ||\nabla W(\mathbf{q}(t))|| \cdot ||\dot{\mathbf{q}}(t))||, \quad \forall t \in [0, t^*).$$

Since $\dot{\mathbf{q}}(t)$ and $\nabla W(\mathbf{q}(t))$ are bounded on $[0, t^*)$, it follows that $|(d/dt)W(\mathbf{q}(t))| \leq c_2$, $\forall t \in (t^* - \delta, t^*)$, c_2 being a positive constant. Fix $t_3 \in (t^* - \delta, t^*)$. Let $t \in (t^* - \delta, t^*)$ be an arbitrary moment. By the mean-value theorem, there exists a ξ_t between t_3 and t such that

$$|W(\mathbf{q}(t_3)) - W(\mathbf{q}(t))| = |\frac{d}{dt}W(\mathbf{q}(\xi_t))| \cdot |t_3 - t| \leq c_2\delta.$$

This implies that

$$|W(\mathbf{q}(t))| < c_2\delta + |W(\mathbf{q}(t_3))|, \quad \forall t \in (t^* - \delta, t^*),$$

i.e., W is bounded on $(t^* - \delta, t^*)$. On the other hand, $\lim_{t \to t^*} \inf \rho(\mathbf{q}(t)) = 0$ implies that

$$\lim_{t \to t^*} \sup W(\mathbf{q}(t)) = \infty,$$

i.e., W is unbounded on $[0, t^*)$, a contradiction with the above conclusion. The proposition is thus proved.

Painlevé established a finer result than Proposition 2.2.(see [6]), stating his criterion as follows:

Theorem 2.3. (Painlevé). *If* $(\mathbf{q}, \mathbf{p})$ *is a solution of the equations* (1.6) *defined on the maximal interval* $[0, t^*)$, *then* t^* *is a singularity of the solution if and only if*

$$\lim_{t \to t^*} \rho(\mathbf{q}(t)) = 0. \tag{2.8}$$

Proof: Recall that $\lim_{t \to t^*} \rho(\mathbf{q}(t)) = L$ if and only if

$$\lim_{t \to t^*} \inf \rho(\mathbf{q}(t)) = \lim_{t \to t^*} \sup \rho(\mathbf{q}(t)) = L.$$

With this remark, the sufficiency follows obviously by Proposition 2.2. To prove the necessity we apply again Proposition 2.2. and obtain that

$$\lim_{t \to t^*} \inf \rho(\mathbf{q}(t)) = 0.$$

It remains to prove that $\lim_{t \to t^*} \sup \rho(\mathbf{q}(t)) = 0$. Let $L = \lim_{t \to t^*} \sup \rho(\mathbf{q}(t))$. Then $L \geq 0$. Assume $L > 0$. Then, there exists a $\theta > 0$ such that $L \geq \theta$. Let

$\epsilon = L - \theta$. By i$'$) page 11, for each $m \in \mathbb{N}$ there exists a $t_m \in [0, t^*)$ such that $|t_m - t^*| < 1/m$ and $\rho(\mathbf{q}(t_m)) > L - \epsilon = \theta$. This means there is a sequence $(t_m)_{m \in \mathbb{N}}$, $t_m \to t^*$ when $n \to \infty$, such that

$$q_{ij}(t_m) \geq \rho(\mathbf{q}(t_m)) > \theta, \quad \forall m \in \mathbb{N}, \quad \forall i, j = 1, \ldots, n, \ i \neq j. \tag{2.9}$$

From here and by the definition (1.1) of W, we have

$$W(\mathbf{q}(t_m)) < \sum_{1 \leq i < j \leq n} \frac{\alpha(m_i, m_j)}{\theta^a} + \sum_{1 \leq i < j \leq n} \frac{\beta(m_i, m_j)}{\theta^b}, \quad \forall m \in \mathbb{N}. \tag{2.10}$$

Denote by μ the constant which bounds the sequence $(W(\mathbf{q}(t_m)))_{m \in \mathbb{N}}$, $\mu > 0$. Then (2.10) becomes

$$W(\mathbf{q}(t_m)) < \mu, \quad \forall m \in \mathbb{N}. \tag{2.11}$$

By the integral of energy (1.11) it follows that

$$T(\mathbf{p}(t_m)) < \mu + h, \quad \forall m \in \mathbb{N}. \tag{2.12}$$

Let $M = \max\{m_i, \ i = 1, \ldots, n\}$. Then, by the definition (1.10) of T, we have

$$\sum_{i=1}^{n} |\mathbf{p}_i(t_m)|^2 < 2M(\mu + h), \quad \forall m \in \mathbb{N}.$$

This means

$$||\dot{\mathbf{q}}(t_m)||^2 = \sum_{i=1}^{n} |\dot{\mathbf{q}}_i(t_m)|^2 \leq \sum_{i=1}^{n} |\mathbf{p}_i(t_m)|^2 < 2M(\mu + h),$$

i.e.,

$$||\dot{\mathbf{q}}(t_m)|| < \gamma, \quad \forall m \in \mathbb{N}, \tag{2.13}$$

where $\gamma = \sqrt{2M(\mu + h)}$ is a positive constant.

Applying the existence theorem for some t_k, $\mathbf{q}(t_k)$, $\mathbf{p}(t_k)$ with t_k sufficiently close to t^* (relation (2.9) assures that the initial position is in $\mathbb{R}^{3n} - \Delta$), it follows that there is an interval $(t^* - \eta, t^* + \eta)$ with $\eta > 0$ depending on θ

and γ, but not on t^*, where the solution remains analytic, i.e., the solution is defined on t^*, a contradiction. The sufficiency is thus proved.

Define the function $d : \mathbb{R}^{3n} \times \mathbb{R}^{3n} \to [0, \infty)$ by

$$d(\mathbf{q}, \Delta) = \inf\{ \, ||\mathbf{q} - \mathbf{p}||, \ \mathbf{p} \in \Delta\}, \quad \mathbf{q} \in \mathbb{R}^{3n}, \ \Delta \subset \mathbb{R}^{3n}. \tag{2.14}$$

$d(\mathbf{q}, \Delta)$ denotes the Euclidien distance in $\mathbb{R}^{3n}$ from the point $\mathbf{q}$ to the set Δ defined at (1.2).

Observation 2.4. *With ρ defined at (2.1) we have*

$$\rho(\mathbf{q}) = \sqrt{2} d(\mathbf{q}, \Delta), \quad \forall \mathbf{q} \in \mathbb{R}^{3n}.$$

Using this remark, Theorem 2.3. can be stated as it follows:

Corollary 2.5. *If* $(\mathbf{q},\mathbf{p})$ *is a solution of the equations* (1.6) *defined on the maximal interval* $[0, t^*)$, *then* t^* *is a singularity of the solution if and only if*

$$\lim_{t \to t^*} d(\mathbf{q}(t), \Delta) = 0.$$

This statement gives us a slightly different interpretation of Theorem 2.3. It says that t^* is a singularity for the solution of (1.6) if and only if the distance from $\mathbf{q}(t)$ to the set Δ tends to zero as $t \to t^*$. This is the same as saying: "$\mathbf{q}(t)$ tends to Δ as $t \to t^*$". Based on this remark, we shall formulate the following definition which establishes the nature of singularities of the n-body problem.

Definition 2.6. *Suppose that* $(\mathbf{q},\mathbf{p})$ *has a singularity at* t^*. *This singularity is called a* **collision singularity** *if* $\mathbf{q}(t)$ *tends to a definite limit when* $t \to t^*$, *i.e., if there exists a* $\mathbf{q}^* \in \Delta$ *such that* $\lim_{t \to t^*} \mathbf{q}(t) = \mathbf{q}^*$. *The singularity is then due to a collision. Otherwise, the singularity is called a* **pseudocollision** *or a* **noncollision singularity**.

Thus, there might be two possibilities for $\mathbf{q}(t)$ to approach the set Δ:

1. By tending to some point $\mathbf{q}^* \in \Delta$ as $t \to t^*$. Then each of the particles has some limiting position at time t^*. Since $\mathbf{q}^* \in \Delta$, at least two of these

limiting positions must coincide, which means that these particles must collide as $t \to t^*$.

2. By approaching Δ without a limit. In this case at least one particle may oscillate between some others without colliding, but coming closer and closer to a collision. Then $d(\mathbf{q}, \Delta)$ tends to zero as $t \to t^*$, but $\mathbf{q}(t)$ does not have a definite limit.

The goal of this paper is to prove that in the n-body problem with quasihomogeneous potentials, if the motion remains bounded at the singularity, then the singularity is a collision.

Chapter 3

A DECOMPOSITION

To achieve our main goal, which consists of the proof of Von Zeipel's theorem for quasihomogeneous potentials, we consider first Von Zeipel's decomposition of the moment of inertia of the system of particles into components corresponding to subsystems. Today this decomposition is best understood in terms of the geometry of the space $\mathbb{R}^{3n}$ viewed as a Hilbert space with inner product defined in the following way:

$$< \ .\ ,\ .\ >: \mathbb{R}^{3n} \times \mathbb{R}^{3n} \to \mathbb{R}, \quad < \mathbf{q}, \mathbf{p} > = \sum_{i=1}^{n} m_i(\mathbf{q}_i, \mathbf{p}_i),$$

where $(\ .\ ,\ .\ ): \mathbb{R}^{3n} \times \mathbb{R}^{3n} \to \mathbb{R}$ denotes the standard inner product on $\mathbb{R}^3$, i.e.,

$$(\mathbf{q}_i, \mathbf{p}_i) = \mathbf{q}_i \mathbf{p}_i^T = q_i^1 p_i^1 + q_i^2 p_i^2 + q_i^3 p_i^3.$$

In the rest of this paper we denote by $||\ .\ ||: \mathbb{R}^{3n} \to [0, \infty)$ the norm induced by this inner product (i.e., $||\mathbf{q}|| = \sqrt{< \mathbf{q}, \mathbf{q} >}$), while $|\ .\ |: \mathbb{R}^3 \to [0, \infty)$ further denotes the Euclidien norm in $\mathbb{R}^3$. Also, we consider the gradient of W with respect to this inner product and denote it by ∇W. That is $\nabla W(\mathbf{q})$ is the vector in $\mathbb{R}^{3n}$ defined by

$$< \nabla W(\mathbf{q}), \mathbf{p} > = DW(\mathbf{q})\mathbf{p}, \quad \forall \mathbf{q} \in \mathbb{R}^{3n} - \Delta, \quad \forall \mathbf{p} \in \mathbb{R}^{3n}, \qquad (3.1)$$

where $DW(\mathbf{q})\mathbf{p}$ is the derivative of W at $\mathbf{q}$ (which is the linear map $DW(\mathbf{q}):\mathbb{R}^{3n}\to\mathbb{R}$) evaluated at $\mathbf{p}$. Let $\nabla=(\nabla_1,\ldots,\nabla_n)$, where ∇_i is the gradient with respect to the i-th particle, $\forall i=1,\ldots,n$. On the other hand, we have the following fundamental equality

$$DW(\mathbf{q})\mathbf{p}=\sum_{i=1}^{n}(\frac{\partial W}{\partial \mathbf{q}_i}(\mathbf{q}),\mathbf{p}_i). \tag{3.2}$$

By (3.1), (3.2) we obtain

$$\sum_{i=1}^{n} m_i(\nabla_i W(\mathbf{q}),\mathbf{p}_i)=\sum_{i=1}^{n}(\frac{\partial W}{\partial \mathbf{q}_i}(\mathbf{q}),\mathbf{p}_i),\quad \forall \mathbf{q}\in\mathbb{R}^{3n}-\Delta,\quad \forall \mathbf{p}\in\mathbb{R}^{3n},$$

and it follows that

$$m_i\nabla_i W(\mathbf{q})=\frac{\partial W}{\partial \mathbf{q}_i}(\mathbf{q}),\quad \forall \mathbf{q}\in\mathbb{R}^{3n}-\Delta.$$

Then, the equations of motion can be written as

$$\ddot{\mathbf{q}}_i=\nabla_i W(\mathbf{q}),\quad \forall i=1,\ldots,n$$

or in the abbreviated form

$$\ddot{\mathbf{q}}=\nabla W(\mathbf{q}). \tag{3.3}$$

Definition 3.1. *The* **moment of inertia** *of the system of particles is defined as*

$$I:\mathbb{R}^{3n}\to[0,\infty),\quad I(\mathbf{q})=||\mathbf{q}||^2=\sum_{i=1}^{n} m_i|\mathbf{q}_i|^2. \tag{3.4}$$

I is a physical measure of the distribution of particles in space.

Denote by N the set of the n integers which label the n particles,

$$N=\{1,2,\ldots,n\}.$$

Let μ be a subset of N, and call the set of particles having indices in μ, a subsystem μ of the system of n particles. The set of points in the set Δ

corresponding to a collision between all the particles in the subsystem μ is denoted by

$$\Delta_\mu = \{\mathbf{q} \in \mathbb{R}^{3n} | \ \mathbf{q}_i = \mathbf{q}_j, \quad \forall i, j \in \mu\}.$$

Thus, points in Δ_μ can be regarded as points of "total collapse" of the subsystem μ. Observe that if $\mu = \emptyset$, Δ_μ is undefined. If μ contains a single point, then $\Delta_\mu = \mathbb{R}^{3n}$. If $\mu = \{i, j\}$, $i, j \in N$, $i \neq j$, then $\Delta_\mu = \Delta_{ij}$, as defined in formula (1.3).

Let ω be a **partition** of N, i.e., a set of mutually disjoint subsets of N whose union is all of N. A partition of N corresponds to a decomposition of the total system into subsystems, each corresponding to one of the elements of the partition. Let Δ_ω be the set of points in $\mathbb{R}^{3n}$ corresponding to total collapse simultaneously in each subsystem, i.e.,

$$\Delta_\omega = \bigcap_{\mu\in\omega} \Delta_\mu = \bigcap_{\mu\in\omega} \{\mathbf{q} \in \mathbb{R}^{3n} | \ \mathbf{q}_i = \mathbf{q}_j, \quad \forall i, j \in \mu\}. \tag{3.5}$$

Notice that since Δ_μ is a hyperplane for every $\mu \in \omega$, it follows that Δ_ω is a hyperplane in $\mathbb{R}^{3n}$. If $\mu \subset N$, we define the **center of mass** of the corresponding subsystem as

$$c_\mu : \mathbb{R}^{3n} \to \mathbb{R}^3, \quad c_\mu(\mathbf{q}) = \frac{\sum\limits_{i\in\mu} m_i \mathbf{q}_i}{\sum\limits_{i\in\mu} m_i}. \tag{3.6}$$

Define a map

$$\pi_\omega : \mathbb{R}^{3n} \to \mathbb{R}^{3n}, \quad (\pi_\omega \mathbf{q})_i = c_\mu(\mathbf{q}) \quad \text{if } i \in \mu \in \omega, \tag{3.7}$$

i.e., the components $(\pi_\omega)_i$ corresponding to all i of an element μ of ω are equal to the center of mass of the corresponding subsystem μ. One can easily see that c_μ and therefore π_ω are linear functions (i.e., if $\mathbf{q}, \mathbf{p} \in \mathbb{R}^{3n}, \alpha, \beta \in \mathbb{R}$, then $\pi_\omega(\alpha\mathbf{q} + \beta\mathbf{p}) = \alpha\pi_\omega(\mathbf{q}) + \beta\pi_\omega(\mathbf{p})$). As a linear function from a $3n$ dimensional normed linear space to the same linear space (actually $\mathbb{R}^{3n}$ is a Banach space), π_ω is continuous.

Let us establish the null space (kernel) and the image (range) of π_ω. The null space of π_ω is the set

$$X_\omega = \{\mathbf{q} \in \mathbb{R}^{3n} |\ \pi_\omega \mathbf{q} = \mathbf{0}\},$$

where $\mathbf{0} = (\mathbf{0}, \ldots, \mathbf{0}) \in \mathbb{R}^{3n}$ is the null element of $\mathbb{R}^{3n}$.
We have

$$\begin{aligned} \pi_\omega \mathbf{q} = \mathbf{0} \quad &\Longleftrightarrow \quad (\pi_\omega \mathbf{q})_i = \mathbf{0}, \ \ \forall i \in \mu, \ \ \forall \mu \in \omega \quad \Longleftrightarrow \quad c_\mu(\mathbf{q}) = \mathbf{0}, \ \ \forall \mu \in \omega \\ &\Longleftrightarrow \quad \frac{\sum_{i\in\mu} m_i \mathbf{q}_i}{\sum_{i\in\mu} m_i} = \mathbf{0}, \ \ \forall \mu \in \omega \quad \Longleftrightarrow \quad \sum_{i\in\mu} m_i \mathbf{q}_i = \mathbf{0}, \ \ \forall \mu \in \omega \ . \end{aligned}$$

Thus,

$$X_\omega = \{\mathbf{q} \in \mathbb{R}^{3n} |\sum_{i\in\mu} m_i \mathbf{q}_i = \mathbf{0}, \ \ \forall \mu \in \omega\ \}. \tag{3.8}$$

The image of π_ω is the set

$$\pi_\omega(\mathbb{R}^{3n}) = \{\pi_\omega \mathbf{q} |\ \mathbf{q} \in \mathbb{R}^{3n}\}.$$

Proposition 3.2. *The following equality applies*

$$\pi_\omega(\mathbb{R}^{3n}) = \Delta_\omega.$$

Proof: Let us prove the inclusion $\pi_\omega(\mathbb{R}^{3n}) \subseteq \Delta_\omega$. Take $\mathbf{p} \in \mathbb{R}^{3n}$. By (3.7), $(\pi_\omega \mathbf{p})_i = c_\mu(\mathbf{p})$ for $i \in \mu \in \omega$, i.e., for all $\mu \in \omega$, $(\pi_\omega \mathbf{p})_i = (\pi_\omega \mathbf{p})_j = c_\mu(\mathbf{p})$ for all $i, j \in \mu$. This means $\pi_\omega \mathbf{p} \in \cap_{\mu\in\omega}\{\mathbf{q} \in \mathbb{R}^{3n} |\ \mathbf{q}_i = \mathbf{q}_j \ \ \forall i, j \in \mu\} = \Delta_\omega$. To prove that $\Delta_\omega \subseteq \pi_\omega(\mathbb{R}^{3n})$, take $\mathbf{q} \in \Delta_\omega$. Then, by (3.5), for all $\mu \in \omega$, $\mathbf{q}_i = \mathbf{q}_j, \ \ \forall i, j \in \mu$. Computing

$$(\pi_\omega \mathbf{q})_i = \frac{\sum_{i\in\mu} m_i \mathbf{q}_i}{\sum_{i\in\mu} m_i} = \frac{(\sum_{i\in\mu} m_i)\mathbf{q}_i}{\sum_{i\in\mu} m_i} = \mathbf{q}_i, \ \ \forall i \in \mu, \ \forall \mu \in \omega,$$

we observe that $\mathbf{q} \in \pi_\omega(\mathbb{R}^{3n})$. We notice also that $\pi_\omega(\mathbf{q}) = \mathbf{q}$ for all $\mathbf{q} \in \Delta_\omega$.

One can easily see that X_ω and Δ_ω are closed subsets of $\mathbb{R}^{3n}$. We shall show that $\mathbb{R}^{3n}$ can be written as the direct sum of Δ_ω and X_ω.

Proposition 3.3. *The sets X_ω and Δ_ω are orthogonal subspaces of $(\mathbb{R}^{3n}, < . , . >)$ and*

$$\mathbb{R}^{3n} = X_\omega \oplus \Delta_\omega,$$

where the symbol "$\oplus$" denotes the direct sum of X_ω and Δ_ω.

Proof: Due to the linearity of π_ω, its nullspace and range are linear subspaces of $\mathbb{R}^{3n}$. To prove their orthogonality, let $q \in X_\omega$, $p \in \Delta_\omega$. We shall show that $< \mathbf{q}, \mathbf{p} > = 0$. By (3.5) and (3.8) we have

$$< \mathbf{q}, \mathbf{p} > = \sum_{i=1}^{n} m_i \mathbf{q}_i \mathbf{p}_i^T = \sum_{\mu \in \omega} \sum_{i \in \mu} m_i \mathbf{q}_i \mathbf{p}_i^T = \sum_{\mu \in \omega} (\sum_{i \in \mu} m_i \mathbf{q}_i) \mathbf{p}_\mu^T = 0,$$

where, for $\mu \in \omega$ we have denoted those $\mathbf{p}_i$ with $i \in \mu$ by $\mathbf{p}_\mu$.

To prove that $\mathbb{R}^{3n}$ is the direct sum of the two subspaces, we have to show that

$$X_\omega \cap \Delta_\omega = \{\mathbf{0}\} \tag{3.9}$$

and that

$$\mathbb{R}^{3n} = X_\omega + \Delta_\omega, \tag{3.10}$$

where $X_\omega + \Delta_\omega = \{ \mathbf{x} + \mathbf{y} | \ \mathbf{x} \in X_\omega, \ \mathbf{y} \in \Delta_\omega\}$. From (3.9) and (3.10) it follows then easily that any element in $\mathbb{R}^{3n}$ can be uniquely represented as the sum of elements of X_ω and Δ_ω.

To prove that $X_\omega \cap \Delta_\omega = \{\mathbf{0}\}$, observe that $\mathbf{0} \in X_\omega$ and $\mathbf{0} \in \Delta_\omega$. Let $\mathbf{q} \in X_\omega \cap \Delta_\omega$. Then $\forall \mu \in \omega$, $\sum_{i \in \mu} m_i \mathbf{q}_i = \mathbf{0}$ and $\mathbf{q}_i = \mathbf{q}_j$, $\forall i, j \in \mu$. Denote by $\mathbf{q}_\mu$ those $\mathbf{q}_i$ with $i \in \mu$. Then $\forall \mu \in \omega$, $\sum_{i \in \mu} m_i \mathbf{q}_i = (\sum_{i \in \mu} m_i) \mathbf{q}_\mu = \mathbf{0}$, i.e., $\mathbf{q}_\mu = \mathbf{0}$ $\forall \mu \in \omega$, i.e., $\mathbf{q} = \mathbf{0}$. Thus, (3.9) is proved. To prove that $\mathbb{R}^{3n} = X_\omega + \Delta_\omega$, we have to show that given a $\mathbf{q} \in \mathbb{R}^{3n}$, there exist $\mathbf{x} \in X_\omega$ and $\mathbf{y} \in \Delta_\omega$ such that $\mathbf{q} = \mathbf{x} + \mathbf{y}$. Take $\mathbf{y} = \pi_\omega \mathbf{q}$ and thus $\mathbf{y} \in \Delta_\omega$. Let $\mathbf{x} = \mathbf{q} - \mathbf{y}$. It remains to prove that $\mathbf{x} \in X_\omega$. Let $\mu \in \omega$. We have

$$\sum_{i \in \mu} m_i \mathbf{x}_i = \sum_{i \in \mu} m_i [\mathbf{q}_i - (\pi_\omega \mathbf{q})_i] = \sum_{i \in \mu} m_i \mathbf{q}_i - \sum_{i \in \mu} m_i (\pi_\omega \mathbf{q})_i$$

$$= \sum_{i\in\mu} m_i\mathbf{q}_i - \sum_{i\in\mu} m_i c_\mu \mathbf{q} = \sum_{i\in\mu} m_i\mathbf{q}_i - \sum_{i\in\mu} m_i \left(\frac{\sum_{i\in\mu} m_i\mathbf{q}_i}{\sum_{i\in\mu} m_i}\right)$$
$$= \sum_{i\in\mu} m_i\mathbf{q}_i - \sum_{i\in\mu} m_i\mathbf{q}_i \left(\frac{\sum_{i\in\mu} m_i}{\sum_{i\in\mu} m_i}\right) = \mathbf{0},$$

and thus $\mathbf{x} \in X_\omega$. The proposition is proved.

Some important properties of π_ω are stated in the following:

Proposition 3.4. *π_ω is an orthogonal projection of $\mathbb{R}^{3n}$ onto Δ_ω, is self-adjoint and idempotent.*

Proof: To prove that π_ω is an orthogonal projection of $\mathbb{R}^{3n}$ onto Δ_ω, let $\mathbf{q} \in \mathbb{R}^{3n}$ and consider the unique $\mathbf{x} \in X_\omega$ and the unique $\mathbf{y} \in \Delta_\omega$ such that $\mathbf{q} = \mathbf{x} + \mathbf{y}$. We have to show that $\pi_\omega\mathbf{q} = \mathbf{y}$. Let $\mu \in \omega$. Then, for all $i \in \mu$ we have

$$(\pi_\omega\mathbf{q})_i = \frac{\sum_{i\in\mu} m_i\mathbf{q}_i}{\sum_{i\in\mu} m_i} = \frac{\sum_{i\in\mu} m_i(\mathbf{x}_i + \mathbf{y}_i)}{\sum_{i\in\mu} m_i} = \frac{\sum_{i\in\mu} m_i\mathbf{y}_i}{\sum_{i\in\mu} m_i} = \frac{(\sum_{i\in\mu} m_i)\mathbf{y}_\mu}{\sum_{i\in\mu} m_i} = \mathbf{y}_i,$$

i.e., $\pi_\omega\mathbf{q} = \mathbf{y}$.

Recall that π_ω is a **self-adjoint** operator if it satisfies the following condition

$$< \pi_\omega\mathbf{q}, \mathbf{p} > = < \mathbf{q}, \pi_\omega\mathbf{p} >, \quad \forall \mathbf{q}, \mathbf{p} \in \mathbb{R}^{3n}. \tag{3.11}$$

Let $\mathbf{q} = \mathbf{x} + \mathbf{y}$ and $\mathbf{p} = \mathbf{x}' + \mathbf{y}'$, with $\mathbf{x}, \mathbf{x}' \in X_\omega$ and $\mathbf{y}, \mathbf{y}' \in \Delta_\omega$. Observing that $< \mathbf{y}, \mathbf{x}' > = 0$ and $< \mathbf{x}, \mathbf{y}' > = 0$ by the orthogonality of X_ω and Δ_ω, we have

$$< \pi_\omega\mathbf{q}, \mathbf{p} > = < \pi_\omega(\mathbf{x} + \mathbf{y}), \mathbf{x}' + \mathbf{y}' > = < \mathbf{y}, \mathbf{x}' > + < \mathbf{y}, \mathbf{y}' > = < \mathbf{y}, \mathbf{y}' >$$

and

$$< \mathbf{q}, \pi_\omega\mathbf{p} > = < \mathbf{x} + \mathbf{y}, \pi_\omega(\mathbf{x}' + \mathbf{y}') > = < \mathbf{x}, \mathbf{y}' > + < \mathbf{y}, \mathbf{y}' > = < \mathbf{y}, \mathbf{y}' >,$$

thus (3.11) is proved.

Recall that π_ω is an **idempotent** operator if it satisfies the condition

$$\pi_\omega{}^2 = \pi_\omega,$$

where $\pi_\omega{}^2 = \pi_\omega \circ \pi_\omega$.
Let $\mathbf{x} \in \mathbb{R}^{3n}$. Then $\pi_\omega \mathbf{x} \in \Delta_\omega$ and so $\pi_\omega(\pi_\omega \mathbf{x}) = \pi_\omega \mathbf{x}$, using the remark we made at the end of the proof of Proposition 3.2. Thus π_ω is idempotent, and the proof is complete.

Define $\Pi_\omega : \mathbb{R}^{3n} \to \mathbb{R}^{3n}$ by

$$\Pi_\omega = id - \pi_\omega,$$

where id is the identity function $id : \mathbb{R}^{3n} \longrightarrow \mathbb{R}^{3n}$, $id(\mathbf{x}) = \mathbf{x}$, $\forall \mathbf{x} \in \mathbb{R}^{3n}$. Obviously, Π_ω is linear.

Proposition 3.5. *The following properties hold*
a) Π_ω *is an orthogonal projection of* $\mathbb{R}^{3n}$ *onto* X_ω.
b) $\forall \mathbf{q} \in \mathbb{R}^{3n}$,

$$||\mathbf{q}||^2 = ||\pi_\omega \mathbf{q}||^2 + ||\Pi_\omega \mathbf{q}||^2. \tag{3.12}$$

c) $||\pi_\omega|| \leq 1$.

Proof: Let $\mathbf{q} \in \mathbb{R}^{3n}$ and let $\mathbf{q} = \mathbf{x} + \mathbf{y}$ be its unique representation with $\mathbf{x} \in X_\omega$, $\mathbf{y} \in \Delta_\omega$.
a) We have

$$\Pi_\omega \mathbf{q} = (id - \pi_\omega)\mathbf{q} = id(\mathbf{x} + \mathbf{y}) - \pi_\omega(\mathbf{x} + \mathbf{y}) = \mathbf{x} + \mathbf{y} - \mathbf{y} = \mathbf{x},$$

i.e., Π_ω is an orthogonal projection of $\mathbb{R}^{3n}$ onto X_ω.
b) Because $< \mathbf{x}, \mathbf{y} > = < \mathbf{y}, \mathbf{x} >= 0$, we have

$$\begin{aligned} ||\mathbf{q}||^2 &= < \mathbf{q}, \mathbf{q} > = < \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y} > = < \mathbf{x}, \mathbf{x} > +2 < \mathbf{x}, \mathbf{y} > + < \mathbf{y}, \mathbf{y} > \\ &= ||\mathbf{x}||^2 + ||\mathbf{y}||^2 = ||\Pi_\omega \mathbf{q}||^2 + ||\pi_\omega \mathbf{q}||^2, \end{aligned}$$

and (3.12) is proved.
c) Since π_ω is linear and continuous, its norm is defined by

$$||\pi_\omega|| = \sup\{ \ ||\pi_\omega \mathbf{q}||, \ \ ||\mathbf{q}|| \leq 1\}.$$

Relation (3.12) implies that $||\pi_\omega \mathbf{q}|| \leq ||\mathbf{q}||$, $\forall \mathbf{q} \in \mathbb{R}^{3n}$ and thus $||\pi_\omega \mathbf{q}|| \leq 1$, $\forall \mathbf{q} \in \mathbb{R}^{3n}$ with $||\mathbf{q}|| \leq 1$. It follows that $\sup\{\ ||\pi_\omega \mathbf{q}||,\ \ ||\mathbf{q}|| \leq 1\} \leq 1$, i.e., $||\pi_\omega|| \leq 1$, and the proposition is proved.

We digress briefly to give a physical interpretation of the relation (3.12). With the aid of the linear operators π_ω and Π_ω we define the functions

$$I_\omega : \mathbb{R}^{3n} \to [0, \infty), \quad I_\omega(\mathbf{q}) = ||\pi_\omega \mathbf{q}||^2, \quad \mathbf{q} \in \mathbb{R}^{3n} \tag{3.13}$$

and

$$J_\omega : \mathbb{R}^{3n} \to [0, \infty), \quad J_\omega(\mathbf{q}) = ||\Pi_\omega \mathbf{q}||^2, \quad \mathbf{q} \in \mathbb{R}^{3n}. \tag{3.14}$$

By simple computations, we have

$$\begin{aligned} I_\omega(\mathbf{q}) &= <\pi_\omega \mathbf{q}, \pi_\omega \mathbf{q}> = \sum_{i=1}^{n} m_i((\pi_\omega \mathbf{q})_i, (\pi_\omega \mathbf{q})_i) \\ &= \sum_{\mu\in\omega}(\sum_{i\in\mu} m_i)\cdot(c_\mu \mathbf{q}, c_\mu \mathbf{q}) = \sum_{\mu\in\omega}(\sum_{i\in\mu} m_i)\cdot|c_\mu \mathbf{q}|^2, \quad \forall \mathbf{q} \in \mathbb{R}^{3n}. \end{aligned}$$

Thus, $I_\omega(\mathbf{q})$ is the moment of inertia of a system of particles consisting, for each $\mu \in \omega$, of a fictitious particle of mass $\sum_{i\in\mu} m_i$ located at the center of mass of the subsystem corresponding to μ.

We also compute

$$\begin{aligned} J_\omega(\mathbf{q}) &= <\Pi_\omega \mathbf{q}, \Pi_\omega \mathbf{q}> = <\mathbf{q} - \pi_\omega \mathbf{q}, \mathbf{q} - \pi_\omega \mathbf{q}> \\ &= \sum_{i=1}^{n} m_i(\mathbf{q}_i - (\pi_\omega \mathbf{q})_i, \mathbf{q}_i - (\pi_\omega \mathbf{q})_i) \\ &= \sum_{\mu\in\omega}\sum_{i\in\mu} m_i(\mathbf{q}_i - c_\mu \mathbf{q}, \mathbf{q}_i - c_\mu \mathbf{q}) \\ &= \sum_{\mu\in\omega}\sum_{i\in\mu} m_i|\mathbf{q}_i - c_\mu \mathbf{q}|^2 = \sum_{\mu\in\omega} J_\mu(\mathbf{q}), \quad \forall \mathbf{q} \in \mathbb{R}^{3n}, \end{aligned}$$

where

$$J_\mu : \mathbb{R}^{3n} \to [0, \infty), \quad J_\mu(\mathbf{q}) = \sum_{i\in\mu} m_i|\mathbf{q}_i - c_\mu \mathbf{q}|^2, \quad \mathbf{q} \in \mathbb{R}^{3n}.$$

Thus, J_μ is the moment of inertia with respect to its center of mass of the subsystem corresponding to μ.

Therefore, relation (3.12) states that the moment of inertia of the system of particles can be decomposed into the sum of the moments of inertia of each of the subsystems plus the moment of inertia of a system composed of a fictitious particle at the center of mass of each subsystem.

The potential energy can be decomposed in an analogous way.

Define the functions $W_{ij} : \mathbb{R}^{3n} - \Delta \to \mathbb{R}_+,\ i,j \in N$ by

$$\forall \mathbf{q} \in \mathbb{R}^{3n} - \Delta, \quad W_{ij}(\mathbf{q}) = \begin{cases} \frac{1}{2}\frac{\alpha(m_i,m_j)}{|\mathbf{q}_i - \mathbf{q}_j|^a} + \frac{1}{2}\frac{\beta(m_i,m_j)}{|\mathbf{q}_i - \mathbf{q}_j|^b} & \text{if } i \neq j \\ 0 & \text{if } i = j. \end{cases}$$

Then we can write

$$\begin{aligned} W(\mathbf{q}) &= \sum_{1 \le i < j \le n} \left[\frac{\alpha(m_i,m_j)}{|\mathbf{q}_i - \mathbf{q}_j|^a} + \frac{\beta(m_i,m_j)}{|\mathbf{q}_i - \mathbf{q}_j|^b}\right] \\ &= \frac{1}{2}\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \left[\frac{\alpha(m_i,m_j)}{|\mathbf{q}_i - \mathbf{q}_j|^a} + \frac{\beta(m_i,m_j)}{|\mathbf{q}_i - \mathbf{q}_j|^b}\right] \\ &= \sum_{i,j=1}^{n} W_{ij}(\mathbf{q}) = \sum_{i \in N}\sum_{j \in N} W_{ij}(\mathbf{q}), \quad \forall \mathbf{q} \in \mathbb{R}^{3n} - \Delta. \qquad (3.15) \end{aligned}$$

Regard the subsystem of particles corresponding to μ as isolated from the rest of the system. Its quasihomogeneous potential function is then defined by $Z_\mu : \mathbb{R}^{3n} - \Delta \to \mathbb{R}_+$,

$$\begin{aligned} Z_\mu(\mathbf{q}) &= \sum_{\substack{i,j \in \mu \\ i < j}} \left[\frac{\alpha(m_i,m_j)}{|\mathbf{q}_i - \mathbf{q}_j|^a} + \frac{\beta(m_i,m_j)}{|\mathbf{q}_i - \mathbf{q}_j|^b}\right] = \frac{1}{2}\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \left[\frac{\alpha(m_i,m_j)}{|\mathbf{q}_i - \mathbf{q}_j|^a} + \frac{\beta(m_i,m_j)}{|\mathbf{q}_i - \mathbf{q}_j|^b}\right] \\ &= \sum_{i \in \mu}\sum_{j \in \mu} W_{ij}(\mathbf{q}), \quad \forall \mathbf{q} \in \mathbb{R}^{3n} - \Delta. \qquad (3.16) \end{aligned}$$

Define

$$Z_\omega : \mathbb{R}^{3n} - \Delta \to \mathbb{R}_+, \quad Z_\omega(\mathbf{q}) = \sum_{\mu \in \omega} Z_\mu(\mathbf{q}), \quad \forall \mathbf{q} \in \mathbb{R}^{3n} - \Delta. \qquad (3.17)$$

Then Z_ω is the total potential of the isolated subsystems.

Denote by $W_\omega : \mathbb{R}^{3n} - \Delta \to \mathbb{R}_+$ the remaining potential, i.e.,

$$W_\omega = W - Z_\omega. \tag{3.18}$$

This potential is due to the interactions between the subsystems. To be more precise, for any μ, ν elements of the partition ω, define the functions

$$W_{\mu\nu} : \mathbb{R}^{3n} - \Delta \to \mathbb{R}_+,$$

$$\forall \mathbf{q} \in \mathbb{R}^{3n} - \Delta, \quad W_{\mu\nu}(\mathbf{q}) = \begin{cases} \sum_{i\in\mu}\sum_{j\in\nu} W_{ij}(\mathbf{q}) & \text{if } \mu \cap \nu = \emptyset \\ 0 & \text{if } \mu = \nu. \end{cases} \tag{3.19}$$

Observe that using (3.15), (3.16), (3.17), (3.19) the following equalities hold

$$\begin{aligned} W(\mathbf{q}) &= \sum_{i\in N}\sum_{j\in N} W_{ij}(\mathbf{q}) = \sum_{\mu\in\omega}\left[\sum_{i\in\mu}\sum_{j\in\mu} W_{ij}(\mathbf{q})\right] + \\ &+ \sum_{\mu\in\omega}\sum_{\substack{\nu\in\omega \\ \nu\neq\mu}}\left[\sum_{i\in\mu}\sum_{j\in\nu} W_{ij}(\mathbf{q})\right] = \sum_{\mu\in\omega} Z_\mu(\mathbf{q}) + \\ &+ \sum_{\mu\in\omega}\sum_{\nu\in\omega} W_{\mu\nu}(\mathbf{q}) = Z_\omega(\mathbf{q}) + \sum_{\mu\in\omega}\sum_{\nu\in\omega} W_{\mu\nu}(\mathbf{q}), \quad \forall \mathbf{q} \in \mathbb{R}^{3n} - \Delta. \end{aligned}$$

From here, by (3.18) it follows that

$$W_\omega(\mathbf{q}) = \sum_{\mu\in\omega}\sum_{\nu\in\omega} W_{\mu\nu}(\mathbf{q}), \quad \forall \mathbf{q} \in \mathbb{R}^{3n} - \Delta, \tag{3.20}$$

an expression which characterizes the potential energy due to the interactions between the subsystems.

We shall state and prove some properties which are useful in our further study.

Proposition 3.6. *For all* $\mathbf{q} \in \mathbb{R}^{3n} - \Delta$ *we have*

$$\pi_\omega \nabla Z_\omega(\mathbf{q}) = \mathbf{0}. \tag{3.21}$$

Proof: Let $\mathbf{q} \in \mathbb{R}^{3n} - \Delta$. Let $\mathbf{z} \in \Delta_\omega$ such that $(\mathbf{q}+\mathbf{z}) \in \mathbb{R}^{3n} - \Delta$. Then $\mathbf{z}_i = \mathbf{z}_j$, $\forall i, j \in \mu$ and $\forall \mu \in \omega$. By (3.16) and (3.17) we have

$$\begin{aligned} Z_\omega(\mathbf{q}+\mathbf{z}) &= \sum_{\mu\in\omega} \sum_{i,j\in\mu} W_{ij}(\mathbf{q}+\mathbf{z}) \\ &= \sum_{\mu\in\omega} \sum_{i,j\in\mu} \frac{1}{2}\left[\frac{\alpha(m_i,m_j)}{|\mathbf{q}_i+\mathbf{z}_i-\mathbf{q}_j-\mathbf{z}_j|^a} + \frac{\beta(m_i,m_j)}{|\mathbf{q}_i+\mathbf{z}_i-\mathbf{q}_j-\mathbf{z}_j|^b}\right] \\ &= \sum_{\mu\in\omega} \sum_{i,j\in\mu} \frac{1}{2}\left[\frac{\alpha(m_i,m_j)}{|\mathbf{q}_i-\mathbf{q}_j|^a} + \frac{\beta(m_i,m_j)}{|\mathbf{q}_i-\mathbf{q}_j|^b}\right] \\ &= \sum_{\mu\in\omega} \sum_{i,j\in\mu} W_{ij}(\mathbf{q}) = Z_\omega(\mathbf{q}). \end{aligned}$$

Recalling that $\Delta_\omega = \pi_\omega(\mathbb{R}^{3n})$, we have $\mathbf{z} = \pi_\omega \mathbf{p}$ for some $\mathbf{p} \in \mathbb{R}^{3n}$. Let $D = \{\mathbf{p} \in \mathbb{R}^{3n} | \, (\pi_\omega \mathbf{p} + \mathbf{q}) \in \mathbb{R}^{3n} - \Delta\}$. The last equality can be then written as

$$Z_\omega(\mathbf{q} + \pi_\omega \mathbf{p}) = Z_\omega(\mathbf{q}), \quad \text{with } \mathbf{p} \in \mathbb{R}^{3n}, \; \mathbf{z} = \pi_\omega \mathbf{p}. \tag{3.22}$$

Differentiating (3.22) with respect to $\mathbf{p}$ on the open set D, we have

$$DZ_\omega(\mathbf{q} + \pi_\omega \mathbf{p}) = 0, \quad \forall \mathbf{p} \in D. \tag{3.23}$$

To compute the left hand side of (3.23), define $A : \mathbb{R}^{3n} \to \mathbb{R}^{3n}$, $A(\mathbf{p}) = \mathbf{q}$, $\mathbf{p} \in \mathbb{R}^{3n}$, to be a constant function with respect to $\mathbf{p}$, and $B : \mathbb{R}^{3n} \to \mathbb{R}^{3n}$, by $B(\mathbf{p}) = A(\mathbf{p}) + \pi_\omega \mathbf{p}$, $\mathbf{p} \in \mathbb{R}^{3n}$. We have thus

$$Z_\omega(\mathbf{q} + \pi_\omega \mathbf{p}) = Z_\omega(A(\mathbf{p}) + \pi_\omega \mathbf{p}) = Z_\omega(B(\mathbf{p})) = (Z_\omega \circ B)(\mathbf{p}), \quad \forall \mathbf{p} \in D.$$

By using the differentiation rules for composed functions it follows that

$$DZ_\omega(\mathbf{q} + \pi_\omega \mathbf{p}) = D(Z_\omega \circ B)(\mathbf{p}) = DZ_\omega(B(\mathbf{p})) \circ DB(\mathbf{p}), \quad \forall \mathbf{p} \in D. \tag{3.24}$$

But

$$DB(\mathbf{p}) = DA(\mathbf{p}) + D\pi_\omega(\mathbf{p}) = D\pi_\omega(\mathbf{p}) = \pi_\omega, \quad \forall \mathbf{p} \in D, \tag{3.25}$$

where we took into account that, by the linearity of π_ω, $D\pi_\omega(\mathbf{p}) = \pi_\omega$, $\forall \mathbf{p} \in \mathbb{R}^{3n}$. By (3.23), (3.24), (3.25) we have

$$DZ_\omega(\mathbf{q} + \pi_\omega \mathbf{p}) \circ \pi_\omega = 0, \quad \forall \mathbf{p} \in D. \tag{3.26}$$

Let $\mathbf{p} = \mathbf{0}$ ($\mathbf{0} \in D$) in (3.26). Then $\pi_\omega \mathbf{p} = \mathbf{0}$ and we obtain

$$DZ_\omega(\mathbf{q}) \circ \pi_\omega = 0,$$

i.e.,

$$(DZ_\omega(\mathbf{q}) \circ \pi_\omega)(\boldsymbol{\alpha}) = 0, \quad \forall \boldsymbol{\alpha} \in \mathbb{R}^{3n}$$

or

$$(DZ_\omega(\mathbf{q}))(\pi_\omega \boldsymbol{\alpha}) = 0, \quad \forall \boldsymbol{\alpha} \in \mathbb{R}^{3n}.$$

On the other hand, by (3.1) we have

$$(DZ_\omega(\mathbf{q}))(\pi_\omega \boldsymbol{\alpha}) = \, < \nabla Z_\omega(\mathbf{q}), \pi_\omega \boldsymbol{\alpha} >, \quad \forall \boldsymbol{\alpha} \in \mathbb{R}^{3n}.$$

The last two relations imply that

$$< \nabla Z_\omega(\mathbf{q}), \pi_\omega \boldsymbol{\alpha} > \, = 0, \quad \forall \boldsymbol{\alpha} \in \mathbb{R}^{3n}.$$

From here, using the fact that π_ω is self-adjoint, the property emphasized in Proposition 3.4., we have

$$< \pi_\omega(\nabla Z_\omega(\mathbf{q})), \boldsymbol{\alpha} > \, = 0, \quad \forall \boldsymbol{\alpha} \in \mathbb{R}^{3n},$$

which leads us to the conclusion: $(\pi_\omega \circ \nabla Z_\omega)(\mathbf{q}) = \mathbf{0}$.

Proposition 3.7. *Let* $(\mathbf{q},\mathbf{p})$ *be a solution of the equations* (3.3) *defined on the maximal interval* $[0, t^*)$. *The following relation takes place*

$$\frac{d^2}{dt^2} I_\omega(\mathbf{q}(t)) = 2||\pi_\omega \dot{\mathbf{q}}(t)||^2 + 2 < \pi_\omega \mathbf{q}(t), \nabla W_\omega(\mathbf{q}(t)) >, \quad \forall t \in [0, t^*). \tag{3.27}$$

Proof: From the definition (3.13) of I_ω, it follows that

$$\frac{d^2}{dt^2} I_\omega(\mathbf{q}(t)) = 2\left(||\frac{d}{dt}\pi_\omega \mathbf{q}(t)||^2 + < \frac{d^2}{dt^2}\pi_\omega \mathbf{q}(t), \pi_\omega \mathbf{q}(t) >\right), \quad \forall t \in [0, t^*). \tag{3.28}$$

Let $\mu \in \omega$, $i \in \mu$. Using (3.6) and (3.7), compute

$$\begin{aligned} \left(\frac{d}{dt}\pi_\omega \mathbf{q}(t)\right)_i &= \frac{d}{dt}(\pi_\omega \mathbf{q}(t))_i = \frac{d}{dt}\left(\frac{\sum_{i\in\mu} m_i \mathbf{q}_i(t)}{\sum_{i\in\mu} m_i}\right) = \\ &= \frac{\sum_{i\in\mu} m_i \dot{\mathbf{q}}_i(t)}{\sum_{i\in\mu} m_i} = (\pi_\omega \dot{\mathbf{q}}(t))_i, \quad \forall t \in [0, t^*), \end{aligned}$$

and further,

$$\begin{aligned} \left(\frac{d^2}{dt^2}\pi_\omega \mathbf{q}(t)\right)_i &= \left(\frac{d}{dt}\pi_\omega \dot{\mathbf{q}}(t)\right)_i = \frac{d}{dt}\left(\frac{\sum_{i\in\mu} m_i \dot{\mathbf{q}}_i(t)}{\sum_{i\in\mu} m_i}\right) = \\ &= \frac{\sum_{i\in\mu} m_i \ddot{\mathbf{q}}_i(t)}{\sum_{i\in\mu} m_i} = (\pi_\omega \ddot{\mathbf{q}}(t))_i, \quad \forall t \in [0, t^*). \end{aligned}$$

Thus,

$$\frac{d}{dt}\pi_\omega \mathbf{q}(t) = \pi_\omega \dot{\mathbf{q}}(t), \quad \frac{d^2}{dt^2}\pi_\omega \mathbf{q}(t) = \pi_\omega \ddot{\mathbf{q}}(t), \quad \forall t \in [0, t^*). \tag{3.29}$$

We have defined in (3.18) the quasihomogeneous potential W_ω. It follows that

$$\pi_\omega \nabla W_\omega(\mathbf{q}(t)) = \pi_\omega \nabla W(\mathbf{q}(t)) - \pi_\omega \nabla Z_\omega(\mathbf{q}(t)), \quad \forall t \in [0, t^*).$$

By the fact that $\mathbf{q}(t)$ satisfies the equations (3.3) on $[0, t^*)$ and by equation (3.21) proved in Proposition 3.6., we have next that

$$\pi_\omega \nabla W_\omega(\mathbf{q}(t)) = \pi_\omega \ddot{\mathbf{q}}(t), \quad \forall t \in [0, t^*). \tag{3.30}$$

By (3.28), (3.29), and taking into account that π_ω is self-adjoint and idempotent, properties stated in Proposition 3.4., (3.28) becomes

$$\begin{aligned} \frac{d^2}{dt^2} I_\omega(\mathbf{q}(t)) &= 2\left(||\pi_\omega \dot{\mathbf{q}}(t)||^2 + < \pi_\omega \mathbf{q}(t), \pi_\omega \nabla W_\omega(\mathbf{q}(t)) >\right) \\ &= 2||\pi_\omega \dot{\mathbf{q}}(t)||^2 + 2 < {\pi_\omega}^2 \mathbf{q}(t), \nabla W_\omega(\mathbf{q}(t)) > \\ &= 2||\pi_\omega \dot{\mathbf{q}}(t)||^2 + 2 < \pi_\omega \mathbf{q}(t), \nabla W_\omega(\mathbf{q}(t)) >, \quad \forall t \in [0, t^*). \end{aligned}$$

The proposition is proved.

Chapter 4

THE THEOREM OF VON ZEIPEL

One of the standard results of the theory of differential equations applied to the equations (1.6) is the following

Theorem 4.1. *Let* $(\mathbf{q},\mathbf{p})$ *be a solution of* (1.6) *on a maximal interval* $[0, t^*)$, $t^* < \infty$. *Then given any compact set* $K \subset \mathbb{R}^{3n} - \Delta \times \mathbb{R}^{3n}$, *there is some* $t \in [0, t^*)$ *with* $(\mathbf{q}(t), \mathbf{p}(t)) \notin K$.

This means, $||(\mathbf{q}, \mathbf{p})(t)||$ becomes unbounded when $t \to t^*$. Obviously, this always happens at a collision instant, because the velocities of the particles are infinite. (At a collision, when $W(t)$ tends to infinity, to satisfy the integral of energy (1.11), the kinetic energy has also to become infinite).

In 1908 Von Zeipel established an important condition for the occurrence of noncollision singularities of the classical n-body problem. He showed that a necessary condition for having a noncollision singularity at t^* is that the motion becomes unbounded in finite time. This theorem, as it appeared in his paper in May 1908 (see [8]), states:

Theorem 4.2. (Von Zeipel). *In the context of the classical* n*-body problem, if some of the particles do not tend to finite limiting positions as* $t \to t^*$, *then one necessarily has*

$$\lim_{t \to t^*} R(t) = \infty,$$

where R is the maximum of the mutual distances, i.e., $R(t) = \max_{1 \le i < j \le n} |\mathbf{q}_i(t) - \mathbf{q}_j(t)|$.

Remark 4.3. We could reformulate this theorem in the terms of Theorem 4.1. and obtain the following statement:
"If $(\mathbf{q},\mathbf{p})$ *is a solution of* (1.6) *on a maximal interval* $[0, t^*)$ *with* t^* *a noncollision singularity, then, given any compact set* $K \subset \mathbb{R}^{3n} - \Delta$, *there is some* $t \in [0, t^*)$ *with* $\mathbf{q}(t) \notin K$*".*
This means $||\mathbf{q}(t)||$ becomes unbounded when $t \to t^*$.

An article published in 1986 [5] by R.Mc Gehee provided a translation in modern mathematical language of Von Zeipel's proof, considering the equations (1.6) with the potential function for Newton's law of attraction

$$W : \mathbb{R}^{3n} - \Delta \to \mathbb{R}_+, \quad W(\mathbf{q}) = \sum_{1 \le i < j \le n} \frac{Gm_i m_j}{|\mathbf{q}_i - \mathbf{q}_j|},$$

where G is the gravitational constant (i.e., $a = b = 1$ and $\alpha(m_i, m_j) = \beta(m_i, m_j) = (G/2)m_i m_j$ in the definition (1.1) of W).

We shall show that Von Zeipel's theorem is true also for the quasihomogeneous potential defined at (1.1). First, however, we shall prove another important result stated in Proposition 4.5., by using

Lemma 4.4. (Lagrange-Jacobi). *Let* $(\mathbf{q},\mathbf{p})$ *be a solution of the equations* (3.3) *defined on a maximal interval* $[0, t^*)$. *On* $[0, t^*)$ *the following identity applies*

$$\frac{d^2}{dt^2} I(\mathbf{q}(t)) = 2(2-a)U(\mathbf{q}(t)) + 2(2-b)V(\mathbf{q}(t)) + 4h, \tag{4.1}$$

where h is the energy constant.

Proof: Differentiating relation (3.4) twice with respect to t, we obtain

$$\frac{d}{dt} I(\mathbf{q}(t)) \;=\; 2 < \dot{\mathbf{q}}(t), \mathbf{q}(t) >, \quad \forall t \in [0, t^*),$$

$$\frac{d^2}{dt^2}I(\mathbf{q}(t)) \;=\; 2\left(<\ddot{\mathbf{q}}(t),\mathbf{q}(t)> + <\dot{\mathbf{q}}(t),\dot{\mathbf{q}}(t)>\right), \quad \forall t\in[0,t^*). \quad (4.2)$$

By the definition (1.10) of the kinetic energy T, we have

$$<\dot{\mathbf{q}}(t),\dot{\mathbf{q}}(t)>= ||\dot{\mathbf{q}}(t)||^2 = \sum_{i=1}^{n} m_i|\dot{\mathbf{q}}_i(t)|^2 = 2T, \quad \forall t\in[0,t^*). \quad (4.3)$$

Using the equations of motion (1.4), we have

$$<\ddot{\mathbf{q}}(t),\mathbf{q}(t)>= \sum_{i=1}^{n} m_i\ddot{\mathbf{q}}_i\mathbf{q}_i^T(t) = \sum_{i=1}^{n}\frac{\partial W}{\partial \mathbf{q}_i}(\mathbf{q}(t))\cdot\mathbf{q}_i^T(t), \quad \forall t\in[0,t^*). \quad (4.4)$$

We shall show that W is a sum of two homogeneous functions, one of degree $-a$, the other of degree $-b$, in the coordinates $\mathbf{q}$. Let $r>0$, $\mathbf{q}\in\mathbb{R}^{3n}-\Delta$. Then $r\mathbf{q}\in\mathbb{R}^{3n}-\Delta$ and we have

$$U(r\mathbf{q}) \;=\; \sum_{1\le i<j\le n}\frac{\alpha(m_i,m_j)}{|r\mathbf{q}_i-r\mathbf{q}_j|^a} = \frac{1}{r^a}\sum_{1\le i<j\le n}\frac{\alpha(m_i,m_j)}{|\mathbf{q}_i-\mathbf{q}_j|^a} = r^{-a}U(\mathbf{q}),$$

$$V(r\mathbf{q}) \;=\; \sum_{1\le i<j\le n}\frac{\beta(m_i,m_j)}{|r\mathbf{q}_i-r\mathbf{q}_j|^b} = r^{-b}V(\mathbf{q}).$$

Thus, U is a homogeneous function of degree $-a$ in $\mathbf{q}$ and V is a homogeneous function of degree $-b$ in $\mathbf{q}$. Being differentiable on $\mathbb{R}^{3n}-\Delta$ (and $\mathbf{0}\notin\mathbb{R}^{3n}-\Delta$), U and V satisfy the hypothesis of Euler's theorem for homogeneous functions and thus we have

$$\sum_{i=1}^{n}\frac{\partial U}{\partial \mathbf{q}_i}(\mathbf{q})\cdot\mathbf{q}_i^T = -aU(\mathbf{q}), \quad \forall \mathbf{q}\in\mathbb{R}^{3n}-\Delta,$$

$$\sum_{i=1}^{n}\frac{\partial V}{\partial \mathbf{q}_i}(\mathbf{q})\cdot\mathbf{q}_i^T = -bV(\mathbf{q}), \quad \forall \mathbf{q}\in\mathbb{R}^{3n}-\Delta.$$

With the aid of these two equations, (4.4) becomes

$$<\ddot{\mathbf{q}}(t),\mathbf{q}(t)>= -aU(\mathbf{q}(t)) - bV(\mathbf{q}(t)), \quad \forall \mathbf{q}\in\mathbb{R}^{3n}-\Delta. \quad (4.5)$$

Using (4.3) and (4.5), (4.2) becomes

$$\frac{d^2}{dt^2}I(\mathbf{q}(t)) = -2aU(\mathbf{q}(t)) - 2bV(\mathbf{q}(t)) + 4T(t), \quad \forall t \in [0, t^*).$$

Using the integral of energy (1.11) in the above equation, we have

$$\frac{d^2}{dt^2}I(\mathbf{q}(t)) = 2(2-a)U(\mathbf{q}(t)) + 2(2-b)V(\mathbf{q}(t)) + 4h, \quad \forall t \in [0, t^*),$$

which is the conclusion of the Lemma.

The next result states that at a singularity instant the moment of inertia always has a limit (which may be finite or infinite).

Proposition 4.5. *Let* $(\mathbf{q},\mathbf{p})$ *be a solution of the equation* (3.3) *defined on a maximal interval* $[0, t^*)$. *If* $\mathbf{q}(t)$ *experiences a singularity at* t^*, *then there exists an* $I^* \in [0, \infty]$ *such that*

$$\lim_{t \to t^*} I(\mathbf{q}(t)) = I^*.$$

Proof: By Theorem 2.3. it follows that

$$\lim_{t \to t^*} U(\mathbf{q}(t)) = \infty, \quad \lim_{t \to t^*} V(\mathbf{q}(t)) = \infty.$$

We consider the following cases:

1) $a < b < 2$. Using the Lagrange-Jacobi relation (4.1) one can see that $(d^2/dt^2)I(\mathbf{q}(t)) > 0$ for any t sufficiently close to t^*, say $t \in [\tau, t^*)$. Hence, $(d/dt)I(\mathbf{q}(t))$ is an increasing function in this neighborhood of t^* and we may assume that $(d/dt)I(\mathbf{q}(t))$ is always positive or always negative in $[\tau, t^*)$. Thus, $(d/dt)I(\mathbf{q}(t)) > 0$ or $(d/dt)I(\mathbf{q}(t)) < 0$ on $[\tau, t^*)$ (if it should change sign there, say at τ', we need only to replace τ by a number between τ' and t^*), which implies the fact that the positive function I is monotonic increasing or decreasing near t^*, and therefore has a limit I^* when $t \to t^*$.

2) $2 < a < b$. Using again Lagrange-Jacobi relation (4.1), we obtain that $(d^2/dt^2)I(\mathbf{q}(t)) < 0$ for any t sufficiently close to t^* and thus $(d/dt)I(\mathbf{q}(t))$ is

a decreasing function in a neighborhood of t^*. The proof continues as in case 1).

3) $a < 2 < b$. By (4.1) we have

$$\begin{aligned}\frac{d^2}{dt^2}I(\mathbf{q}(t)) &= 4[U(\mathbf{q}(t)) + V(\mathbf{q}(t))] - 2aU(\mathbf{q}(t)) - 2bV(\mathbf{q}(t)) + 4h \\ &= 4W(\mathbf{q}(t)) - 2aU(\mathbf{q}(t)) - 2bV(\mathbf{q}(t)) + 4h \\ &> 4W(\mathbf{q}(t)) - 2bU(\mathbf{q}(t)) - 2bV(\mathbf{q}(t)) + 4h \\ &= 4W(\mathbf{q}(t)) - 2bW(\mathbf{q}(t)) + 4h = 2(2-b)W(\mathbf{q}(t)) + 4h.\end{aligned}$$

The proof is analogous with that of case 2), due to the fact that $2 < b$ and $\lim_{t\to t^*} W(\mathbf{q}(t)) = \infty$.

4) $a = 2$ or $b = 2$. Suppose $a = 2$. Then $2 - b < 0$ and by (4.1), $(d^2/dt^2)I(\mathbf{q}(t)) = 2(2-b)V(\mathbf{q}(t)) + 4h < 0$ in a neighborhood of t^*. The proof continues as in case 2). If $b = 2$, then $2 - a > 0$ and by (4.1), $(d^2/dt^2)I(\mathbf{q}(t)) = 2(2-a)U(\mathbf{q}(t)) + 4h > 0$ in a neighborhood of t^*. The rest of the proof is the same as in case 1). The proposition is thus proved.

We now use the above facts to state and prove an equivalent version of Von Zeipel's Theorem 4.2.

Theorem 4.6. (Von Zeipel). *If* $(\mathbf{q},\mathbf{p})$ *is a solution of the equations* (3.3) *defined on* $[0, t^*)$ *with a singularity at* t^* *and such that* $\lim_{t\to t^*} I(\mathbf{q}(t))$ *is finite, then* t^* *is a collision singularity.*

Proof: Denote by

$$I^* = \lim_{t\to t^*} I(\mathbf{q}(t)) \tag{4.6}$$

and let

$$\Delta^* = \bigcap_{0\le t<t^*} \overline{\mathbf{q}((t,t^*))}, \tag{4.7}$$

where $\overline{\mathbf{q}((t,t^*))}$ denotes the closure of the set

$$\mathbf{q}((t,t^*)) = \left\{\mathbf{q}(\tau) \in \mathbb{R}^{3n} - \Delta \mid \tau \in (t,t^*)\right\}, \quad t \in [0,t^*).$$

We prove now that Δ^* is non-empty and compact. The solution is defined on $[0, t^*)$, so $\forall t \in [0, t^*)$, $\mathbf{q}((t, t^*)) \neq \emptyset$. We shall prove that $\forall t \in [0, t^*)$, $\overline{\mathbf{q}((t, t^*))}$ is bounded. Because $\overline{\mathbf{q}((t, t^*))} \subseteq \overline{\mathbf{q}([0, t^*))}$, $\forall t \in [0, t^*)$, it suffices to show that $\overline{\mathbf{q}([0, t^*))}$ is bounded, i.e., there exists an $M > 0$ such that $||\mathbf{q}|| \leq M$, $\forall \mathbf{q} \in \overline{\mathbf{q}([0, t^*))}$. Let $\epsilon > 0$. By (4.6) and $I^* < \infty$, there exists a $\delta > 0$ such that for all t satisfying $t^* - \delta < t < t^*$ we have

$$| \; ||\mathbf{q}||^2 - I^* | < \epsilon. \tag{4.8}$$

Let $\mathbf{q} \in \overline{\mathbf{q}([0, t^*))}$ and $(\tau_n)_{n \in \mathbb{N}}$ be a sequence in $[0, t^*)$ such that $\mathbf{q}(\tau_n) \to \mathbf{q}$ when $n \to \infty$. This means there exist an $n_0 \in \mathbb{N}$ such that $||\mathbf{q}(\tau_n) - \mathbf{q}|| < \epsilon$, $\forall n \in \mathbb{N}$, $n \geq n_0$. By (4.8) it follows that for those τ_n with $n \geq n_0$ and $\tau_n \in (t^* - \delta, t^*)$ we have $||\mathbf{q}(\tau_n)|| < \sqrt{I^* + \epsilon}$, and further

$$||\mathbf{q}|| \leq ||\mathbf{q}(\tau_n) - \mathbf{q}|| + ||\mathbf{q}(\tau_n)|| < \sqrt{\epsilon + I^*} + \epsilon.$$

For $\tau_n \in [0, t^* - \delta]$, $n \in \mathbb{N}$, we have $\mathbf{q}(\tau_n) \in \mathbf{q}([0, t^* - \delta])$. On the compact interval $[0, t^* - \delta]$, $\mathbf{q}$ is continuous, so $\mathbf{q}([0, t^* - \delta])$ is a compact set in $\mathbb{R}^{3n}$ and therefore closed and bounded by, say, a constant $L > 0$. Taking $M = \max\{\sqrt{I^* + \epsilon} + \epsilon, L\}$, we can conclude that $\overline{\mathbf{q}([0, t^*))}$ is bounded by M. Being closed and bounded in $\mathbb{R}^{3n}$, $\overline{\mathbf{q}((t, t^*))}$ is compact for all $t \in [0, t^*)$. Take a finite subcollection of the collection $\mathcal{C} = \{\overline{\mathbf{q}((t, t^*))},\ 0 \leq t < t^*\}$, e.g., $\{C_i := \overline{\mathbf{q}((t_i, t^*))},\ 0 \leq t_1 < t_2 < \ldots < t_n < t^*\}$. Observe that $\bigcap_{i=1}^n C_i = \overline{\mathbf{q}((t_n, t^*))} \neq \emptyset$ and therefore $\mathcal{C}$ is a collection of subsets of the compact $\overline{\mathbf{q}([0, t^*))}$, which satisfies the finite intersection condition. It follows that $\bigcap_{C \in \mathcal{C}} \overline{C} \neq \emptyset$, i.e., $\bigcap_{0 \leq t < t^*} \overline{\overline{\mathbf{q}((t, t^*))}} \neq \emptyset$. This is equivalent to $\bigcap_{0 \leq t < t^*} \overline{\mathbf{q}((t, t^*))} \neq \emptyset$. By the definition of Δ^* it follows that $\Delta^* \neq \emptyset$. For any $t \in [0, t^*)$, $\overline{\mathbf{q}((t, t^*))}$ is closed, so $\bigcap_{0 \leq t < t^*} \overline{\mathbf{q}((t, t^*))}$ is closed, i.e., Δ^* is closed in $\mathbb{R}^{3n}$. Obviously, $\Delta^* \subset \overline{\mathbf{q}([0, t^*))}$. Being a closed subset of a compact set of $\mathbb{R}^{3n}$, Δ^* is compact.

We shall show next that Δ^* is a subset of Δ. Let $\mathbf{q}^0 \in \Delta^*$. This means that $\mathbf{q}^0 \in \overline{\mathbf{q}((t, t^*))}$, $\forall t \in [0, t^*)$, i.e., $\forall t \in [0, t^*)$ there exists a sequence

$(\tau_n^t)_{n\in\mathbb{N}}$, $\tau_n^t \in (t, t^*)$, $\forall n \in \mathbb{N}$, such that $\mathbf{q}(\tau_n^t) \to \mathbf{q}^0$ as $n \to \infty$. From all these sequences choose a sequence $(\tau_n)_{n\in\mathbb{N}}$ which satisfies $0 \le \tau_1 < \tau_2 < \ldots < t^*$ and $\tau_n \to t^*$ when $n \to \infty$. The fact that t^* is a singularity implies, by Corollary 2.5., that $d(\mathbf{q}(t), \Delta) \to 0$ when $t \to t^*$. It follows that $d(\mathbf{q}(\tau_n), \Delta) \to 0$ when $n \to \infty$. Let $\epsilon > 0$. Then, there exists $n_1 \in \mathbb{N}$ such that

$$\inf\{\ ||\mathbf{q}(\tau_n) - \mathbf{p}||,\ \ \mathbf{p} \in \Delta\} < \epsilon/2,\ \ \forall n \ge n_1.$$

But $\mathbf{q}(\tau_n) \to \mathbf{q}^0$, so there exists $n_2 \in \mathbb{N}$ such that

$$||\mathbf{q}^0 - \mathbf{q}(\tau_n)|| < \epsilon/2,\ \ \forall n \ge n_2.$$

Take $n_0 = \max\{n_1, n_2\}$. Then for any $\mathbf{p} \in \Delta$ we have

$$||\mathbf{q}^0 - \mathbf{p}|| \le ||\mathbf{q}^0 - \mathbf{q}(\tau_n)|| + ||\mathbf{q}(\tau_n) - \mathbf{p}||,\ \ \forall n \ge n_0.$$

By the last three inequalities it follows that, for $n \ge n_0$, $n \in \mathbb{N}$,

$$\inf\{\ ||\mathbf{q}^0 - \mathbf{p}||,\ \ \mathbf{p} \in \Delta\} \le ||\mathbf{q}^0 - \mathbf{q}(\tau_n)|| + \inf\{\ ||\mathbf{q}(\tau_n) - \mathbf{p}||,\ \ \mathbf{p} \in \Delta\} < \epsilon.$$

Therefore, $d(\mathbf{q}^0, \Delta) < \epsilon$, $\forall \epsilon > 0$, i.e., $d(\mathbf{q}^0, \Delta) = 0$. Since the set Δ is closed, $\mathbf{q}^0 \in \Delta$. Thus we have proved that

$$\Delta^* \subset \Delta.$$

As a remark, note that

$$I(\mathbf{q}) = I^*,\ \ \forall \mathbf{q} \in \Delta^*.$$

For this, let $\mathbf{q}^0 \in \Delta^*$. Let $(\tau_n)_{n\in\mathbb{N}} \subset [0, t^*)$, $\tau_n \to t^*$ as $n \to \infty$, such that $\mathbf{q}(\tau_n) \to \mathbf{q}^0$ as $n \to \infty$. Then $||\mathbf{q}(\tau_n)|| \to ||\mathbf{q}^0||$ as $n \to \infty$. On the other hand, by (4.6) we have $||\mathbf{q}(\tau_n)||^2 \to I^*$ as $n \to \infty$. The last two relations imply that $I^* = ||\mathbf{q}^0||^2$, i.e., $I(\mathbf{q}^0) = I^*$.

For each partition ω of $N = \{1, 2, \ldots, n\}$ define

$$\Delta_\omega^* = \Delta^* \cap \Delta_\omega,$$

where Δ_ω is defined at (3.5). Denote by $\mathcal{S}$ the set of all partitions ω of N such that $\Delta^*_\omega \neq \emptyset$. We claim that $\mathcal{S}$ is non-empty, i.e., there exists at least one ω such that $\Delta^*_\omega \neq \emptyset$. Observe that

$$\Delta^*_\omega \subset \Delta^* \subset \Delta.$$

Take $\mathbf{q}^0 \in \Delta^*$ ($\Delta^* \neq \emptyset$). Then $\mathbf{q}^0 \in \Delta$, i.e., there exist $i_0, j_0 \in N$, $i_0 < j_0$ such that $\mathbf{q}^0_{i_0} = \mathbf{q}^0_{j_0}$. Let

$$\omega^0 = \{ \ \{i_0, j_0\}, \{k\}, \ k \in N - \{i_0, j_0\}\} .$$

ω^0 is a partition of N. Let $\mu_0 = \{i_0, j_0\}$, $\mu_k = \{k\}$, $k \in N - \{i_0, j_0\}$. Then $\mathbf{q}^0 \in \Delta_{\mu_0}(= \{\mathbf{q} \in \mathbb{R}^{3n} | \mathbf{q}_{i_0} = \mathbf{q}_{j_0}\} = \Delta_{i_0 j_0})$ and $\mathbf{q}^0 \in \Delta_{\mu_k}$ $(= \mathbb{R}^{3n})$, $\forall k \in N - \{i_0, j_0\}$. So $\mathbf{q}^0 \in \Delta_{\mu_0} \cap (\bigcap_{k \in N - \{i_0, j_0\}} \Delta_{\mu_k})$, i.e., $\mathbf{q}^0 \in \Delta_{\omega^0}$. The fact that $\Delta^*_{\omega^0} \neq \emptyset$ implies $\omega^0 \in \mathcal{S}$, i.e., $\mathcal{S} \neq \emptyset$.

By Corollary 2.5., we know that $\mathbf{q}(t)$ tends to Δ when $t \to t^*$. This means either $\mathbf{q}(t)$ tends to a definite limit when $t \to t^*$, i.e., there exists a $\mathbf{q}^* \in \Delta$ such that $\mathbf{q}(t) \to \mathbf{q}^*$ as $t \to t^*$, or there is at least one particle which does not have a limiting position at t^*, oscillating between some of the others (the case of a pseudocollision).

If the first case happens, then the proof is done (this situation includes also the case of total collapse, when $\mathbf{q}^* = \mathbf{0}$, i.e., all the particles collide at their center of mass).

Assuming the second case happens, we consider any partition $\omega \in \mathcal{S}$ of minimal cardinality. This choice assures that, for all $\mathbf{q} \in \Delta^*_\omega$, all the denominators of W_ω are nonzero for any $t \in [0, t^*)$ and hence that W_ω is well defined on Δ^*_ω (W_ω is defined at (3.18) and characterized at (3.20)). Our goal is to prove that for these partitions ω we have $\Delta^*_\omega = \Delta^*$.

We claim that Δ^*_ω is a compact set. Indeed, for any $\mu \in \omega$, Δ_μ is closed, so Δ_ω is closed, and taking into account that Δ^* is also closed, $\Delta^*_\omega = \Delta^* \cap \Delta_\omega$ is a closed set in $\mathbb{R}^{3n}$. Being a closed subset of the compact set Δ^*, Δ^*_ω is compact.

Since Δ_ω^* is compact, by the continuity of W_ω with respect to $\mathbf{q}$, there exists an open neighborhood G of Δ_ω^* and a finite constant M, depending on G, such that

$$||\nabla W_\omega(\mathbf{q})|| \leq M \quad \text{and} \quad |< \pi_\omega \mathbf{q}, \nabla W_\omega(\mathbf{q}) >| \leq M, \quad \forall \mathbf{q} \in G. \tag{4.9}$$

We identify the direct sum of X_ω and Δ_ω with their Cartesian product, i.e., we establish the isomorphism

$$X_\omega \oplus \Delta_\omega \cong X_\omega \times \Delta_\omega.$$

Every $\mathbf{q} \in \mathbb{R}^{3n}$ has the unique representation $\mathbf{q} = \mathbf{x} + \mathbf{y}$, with $\mathbf{x} \in X_\omega$ and $\mathbf{y} \in \Delta_\omega$. By Proposition 3.4. and Proposition 3.5., we know that $\mathbf{x} = \Pi_\omega \mathbf{q}$ and $\mathbf{y} = \pi_\omega \mathbf{q}$. Then the function $\Phi : X_\omega \oplus \Delta_\omega \to X_\omega \times \Delta_\omega$ given by $\Phi(\mathbf{q}) = (\Pi_\omega \mathbf{q}, \pi_\omega \mathbf{q})$ is well defined and one can easily check that Φ is an isomorphism of Hilbert spaces.

Consider Δ_ω as a space endowed with the relative topology (induced from $\mathbb{R}^{3n}$). As a subset of $\mathbb{R}^{3n}$, Δ_ω is a locally compact Hausdorff space in which $G \cap \Delta_\omega$ is open and Δ_ω^* is closed. Then, a fundamental result in topology assures the existence of a subset B of Δ_ω, relatively open in Δ_ω, such that $\overline{B}$ is compact and

$$\Delta_\omega^* \subset B \subset \overline{B} \subset \Delta_\omega \cap G \subset G.$$

Then $\overline{B}$ is compact in $\mathbb{R}^{3n}$. Denote by $\partial_\omega B = \overline{B} - B$ the boundary of B with respect to the topology relative to Δ_ω. $\partial_\omega B = \overline{B} \cap (\Delta_\omega - B)$, so $\partial_\omega B$ is a closed subset of the compact set $\overline{B}$ and therefore compact.

For each constant $\sigma > 0$ define

$$D_\sigma = \{\ \mathbf{x} \in X_\omega |\ ||\mathbf{x}|| < \sigma\}.$$

D_σ is an open subset of X_ω. Let $\overline{D_\sigma}$ denote its closure, i.e., $\overline{D_\sigma} = \{\ \mathbf{x} \in X_\omega |\ ||\mathbf{x}|| \leq \sigma\}$, and let ∂D_σ denote the boundary of D_σ. Define by

$$K_\sigma = \overline{D_\sigma} \times \overline{B} \subset X_\omega \times \Delta_\omega = \mathbb{R}^{3n}.$$

Being bounded and closed, $\overline{D_\sigma}$ is compact, therefore K_σ is compact.

We shall prove further that

$$\partial_\omega B \cap \Delta^* = \emptyset. \tag{4.10}$$

Denote by $C(B) = \mathbb{R}^{3n} - B$. Since $\overline{B} \subset \Delta_\omega$ and $\Delta^*_\omega \subset B$, the following sequence of relations takes place

$$\begin{aligned}
\partial_\omega B \cap \Delta^* &= (\overline{B} - B) \cap \Delta^* = \overline{B} \cap C(B) \cap \Delta_\omega \cap \Delta^* \subset \Delta_\omega \cap C(\Delta^*_\omega) \cap \Delta^* \\
&= \Delta_\omega \cap C(\Delta^* \cap \Delta_\omega) \cap \Delta^* = \Delta_\omega \cap (C(\Delta^*) \cup C(\Delta_\omega)) \cap \Delta^* \\
&= [(\Delta_\omega \cap C(\Delta^*)) \cup ((\Delta_\omega \cap C(\Delta_\omega))] \cap \Delta^* \\
&= \Delta_\omega \cap C(\Delta^*) \cap \Delta^* = \emptyset,
\end{aligned}$$

so (4.10) is proved. This implies that the distance between Δ^* and $\partial_\omega B$ is positive. Using the fact that Δ_ω is a hyperplane and that Δ is a finite union of hyperplanes, one can prove that the distance between Δ^* and $\partial_\Delta B$ is positive, where $\partial_\Delta B$ is the boundary of B in the topology relative to Δ. Therefore, there exists a $\sigma_0 > 0$ such that

$$\Delta^* \cap (\overline{D_{\sigma_0}} \times \partial_\Delta B) = \emptyset. \tag{4.11}$$

By a "reductio ad absurdum" argument we show now that there exists a $t_0 \in [0, t^*)$ such that

$$\mathbf{q}([t_0, t^*)) \cap (\overline{D_{\sigma_0}} \times \partial_\Delta B) = \emptyset. \tag{4.12}$$

For this suppose that $\forall t \in [0, t^*)$, $\mathbf{q}((t, t^*)) \cap (\overline{D_{\sigma_0}} \times \partial_\Delta B) \neq \emptyset$, i.e., $\exists (t_n)_{n \in \mathbb{N}}$, $t_n \to t^*$ as $n \to \infty$, such that $\mathbf{q}(t_n) \in \overline{D_{\sigma_0}} \times \partial_\Delta B$, $\forall n \in \mathbb{N}$. Since $\overline{D_{\sigma_0}} \times \partial_\Delta B$ is compact, $(\mathbf{q}(t_n))_{n \in \mathbb{N}}$ has a convergent subsequent $(\mathbf{q}(t_{n_k}))_{k \in \mathbb{N}}$ with $\lim_{k \to \infty} \mathbf{q}(t_{n_k}) = \mathbf{q}' \in \overline{D_{\sigma_0}} \times \partial_\Delta B$. By the definition of Δ^*, $\mathbf{q}' \in \Delta^*$. Hence $\mathbf{q}' \in \Delta^* \cap (\overline{D_{\sigma_0}} \times \partial_\Delta B)$, contradiction with (4.11).

We have $G \subset \mathbb{R}^{3n} = X_\omega \times \Delta_\omega$, thus, by $\overline{B} \subset G \cap \Delta_\omega$ and by the definition of D_σ, we may assume that σ_0 is chosen small enough so that

$$K_{\sigma_0} = \overline{D_{\sigma_0}} \times \overline{B} \subset G. \tag{4.13}$$

The most important step in the proof is to show that

$$\Delta^* \subseteq \Delta_\omega.$$

Assume the opposite, i.e., Δ^* is not a subset of Δ_ω. Let

$$\mathbf{q}(t) = (\mathbf{x}(t), \mathbf{y}(t)), \quad \forall t \in [0, t^*),$$

where $\mathbf{x}(t) = \Pi_\omega \mathbf{q}(t) \in X_\omega$, $\mathbf{y}(t) = \pi_\omega \mathbf{q}(t) \in \Delta_\omega$, $\forall t \in [0, t^*)$.

We shall prove further that there exists a $\sigma \in (0, \sigma_0)$ such that $\mathbf{q}(t) \notin K_\sigma$ for infinitely many values of t tending to t^*. We assumed that $\Delta^* \not\subset \Delta_\omega$, so let $\mathbf{q}^0 \in \Delta^*$ such that $\mathbf{q}^0 \notin \Delta_\omega$. As we have seen before, there exists a sequence $(t_n)_{n\in\mathbb{N}} \subset [0, t^*)$, $t_n \to t^*$ as $n \to \infty$, such that $\mathbf{q}(t_n) \to \mathbf{q}^0$ as $n \to \infty$. This means $(\mathbf{x}(t_n))_{n\in\mathbb{N}}$ and $(\mathbf{y}(t_n))_{n\in\mathbb{N}}$ have limits as $n \to \infty$. Let $\mathbf{x}^0 = \lim_{n\to\infty} \mathbf{x}(t_n)$. Then $\mathbf{x}^0 \neq \mathbf{0}$. The set X_ω is closed, so $\mathbf{x}^0 \in X_\omega$. Let $\epsilon > 0$ such that $0 < ||\mathbf{x}^0|| - \epsilon < \sigma_0$. Then there exists a $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$ we have $|\ ||\mathbf{x}(t_n)|| - ||\mathbf{x}^0||\ | < \epsilon$, so $||\mathbf{x}(t_n)|| > ||\mathbf{x}^0|| - \epsilon$.
Let $\sigma = ||\mathbf{x}^0|| - \epsilon$. Thus, $\forall n \geq n_0$, $\mathbf{x}(t_n) \notin \overline{D_\sigma}$ and therefore

$$\forall n \geq n_0, \quad \mathbf{q}(t_n) \notin \overline{D_\sigma} \times \overline{B} = K_\sigma, \quad \text{with} \quad 0 < \sigma < \sigma_0.$$

Henceforth we fix σ at this value.

By (4.6), there exists a $\delta > 0$ such that

$$|I(\mathbf{q}(t)) - I^*| < \sigma^2/12, \quad \forall t \in [t_1, t^*), \tag{4.14}$$

where $t_1 \in (t^* - \delta, t^*)$ is chosen such that $t_1 \geq t_0$.

Observe that, due to the orthogonality of X_ω and Δ_ω and the fact that $\Delta^*_\omega \subset \Delta_\omega$, we have

$$d(\mathbf{q}(t), \Delta^*_\omega) \geq ||\mathbf{x}(t)|| = ||\Pi_\omega \mathbf{q}(t)|| = [J_\omega(\mathbf{q}(t))]^{1/2}, \quad \forall t \in [t_1, t^*), \tag{4.15}$$

where J_ω was defined at (3.14).

From the way the partition ω was defined and by the definition of Δ^*_ω we can state that $\mathbf{q}(t)$ comes infinitely often arbitrarily close to Δ^*_ω as $t \to t^*$. Since $\Delta^*_\omega \subset \overline{B} \subset \Delta_\omega$ and due to the orthogonality of X_ω and Δ_ω, we have

$$\Delta^*_\omega \subset \overline{D_\sigma} \times \overline{B} = K_\sigma.$$

We have shown that $\mathbf{q}(t)$ must leave K_σ infinitely often as $t \to t^*$. Thus, $\mathbf{q}(t)$ must enter and leave K_σ infinitely often as $t \to t^*$. Observe that $\sigma \in (0, \sigma_0)$ implies $\overline{D_\sigma} \subset \overline{D_{\sigma_0}}$. Thus, $\overline{D_\sigma} \times \partial_\Delta B \subset \overline{D_{\sigma_0}} \times \partial_\Delta B$. Using (4.12) and that $t_1 \geq t_0$, we conclude

$$\mathbf{q}([t_1, t^*)) \cap (\overline{D_\sigma} \times \partial_\Delta B) = \emptyset.$$

Therefore, $\mathbf{q}(t)$ must enter and leave K_σ infinitely often via $\partial D_\sigma \times B$ so long as $t_1 < t < t^*$, while $d(\mathbf{q}(t), \Delta^*_\omega) \to 0$ when $t \to t^*$. This implies the existence of a time interval T close to t^*, $T \subset (t_1, t^*)$, such that $\min_{t\in T} d^2(\mathbf{q}(t), \Delta^*_\omega) < \sigma^2/2$ or, using (4.15),

$$\min_{t\in T} J_\omega(\mathbf{q}(t)) < \sigma^2/2. \tag{4.16}$$

Consider as endpoints of T the moment instants τ_1 and τ_2, $t_1 < \tau_1 < \tau_2 < t^*$, when $\mathbf{q}(t)$ enters K_σ, i.e., $\mathbf{q}(\tau_1) \in \partial D_\sigma \times B$ and when $\mathbf{q}(t)$ leaves K_σ, i.e., $\mathbf{q}(\tau_2) \in \partial D_\sigma \times B$ with

$$\mathbf{q}(t) \in K_\sigma, \quad \forall t \in [\tau_1, \tau_2]. \tag{4.17}$$

Then $||\mathbf{x}(\tau_1)|| = ||\mathbf{x}(\tau_2)|| = \sigma$, or

$$J_\omega(\mathbf{q}(\tau_1)) = J_\omega(\mathbf{q}(\tau_2)) = \sigma^2. \tag{4.18}$$

Being continuous on the compact interval $[\tau_1, \tau_2]$, $J_\omega \circ \mathbf{q}$ attains its minimum on $[\tau_1, \tau_2]$, i.e., there exists a $\tau' \in [\tau_1, \tau_2]$ such that $J_\omega(\mathbf{q}(\tau')) = \min_{t\in[\tau_1,\tau_2]} J_\omega(\mathbf{q}(t))$. Therefore, by (4.16),

$$J_\omega(\mathbf{q}(\tau')) < \sigma^2/2. \tag{4.19}$$

Since such an interval T occurs arbitrarily close to t^*, we can consider

$$\tau_2 - \tau_1 < \sigma/\sqrt{3M}. \tag{4.20}$$

$I_\omega \circ \mathbf{q}$ attains its maximum on $[\tau_1, \tau_2]$, so there exists a $\overline{\tau} \in [\tau_1, \tau_2]$ such that

$$I_\omega(\mathbf{q}(\overline{\tau})) = \max_{t\in[\tau_1,\tau_2]} I_\omega(\mathbf{q}(t)). \tag{4.21}$$

Assume $\overline{\tau} \notin \{\tau_1, \tau_2\}$.
Equation (3.12) implies that

$$I(\mathbf{q}) = J_\omega(\mathbf{q}) + I_\omega(\mathbf{q}), \quad \forall \mathbf{q} \in \mathbb{R}^{3n}. \tag{4.22}$$

Relations (4.14), (4.18), (4.22) imply for $t = \tau_2$ that

$$I_\omega(\mathbf{q}(\tau_2)) < I^* - 11\sigma^2/12. \tag{4.23}$$

Relations (4.14), (4.19), (4.21), (4.22) imply for $t = \tau'$ that

$$I^* - \sigma^2/12 < J_\omega(\mathbf{q}(\tau')) + I_\omega(\mathbf{q}(\tau')) < \sigma^2/2 + I_\omega(\mathbf{q}(\overline{\tau})).$$

i.e.,

$$I_\omega(\mathbf{q}(\overline{\tau})) > I^* - 7\sigma^2/12. \tag{4.24}$$

Adding up (4.23) and (4.24) term by term, it follows that

$$I_\omega(\mathbf{q}(\overline{\tau})) - I_\omega(\mathbf{q}(\tau_2)) > \sigma^2/3. \tag{4.25}$$

Observe that $\sigma < \sigma_0$ implies $K_\sigma \subset K_{\sigma_0}$ and by (4.12), $K_\sigma \subset G$. From here and (4.17) it follows that $\mathbf{q}(t) \in G$, $\forall t \in [\tau_1, \tau_2]$ and therefore, the inequalities (4.9) are true on $[\tau_1, \tau_2]$, i.e.,

$$||\nabla W_\omega \mathbf{q}(t)|| \leq M \quad \text{and} \quad | < \pi_\omega \mathbf{q}(t), \nabla W_\omega \mathbf{q}(t) > | \leq M, \quad \forall t \in [\tau_1, \tau_2].$$

Using the second inequality and the relation established in Proposition 3.7., we have

$$\frac{d^2}{dt^2} I_\omega(\mathbf{q}(t)) + 2M \geq 0, \quad \forall t \in [\tau_1, \tau_2].$$

By integrating this continuous positive function from $\overline{\tau}$ to $t \in (\overline{\tau}, \tau_2]$, we obtain

$$\frac{d}{dt}I_\omega(\mathbf{q}(t)) - \frac{d}{dt}I_\omega(\mathbf{q}(\overline{\tau})) + 2M(t-\overline{\tau}) \geq 0, \quad \forall t \in (\overline{\tau}, \tau_2].$$

Since $\overline{\tau}$ is a local maximum, $(d/dt)I_\omega(\mathbf{q}(\overline{\tau})) = 0$. Integrating the last relation from $\overline{\tau}$ to τ_2, we obtain

$$I_\omega(\mathbf{q}(\tau_2)) - I_\omega(\mathbf{q}(\overline{\tau})) \geq -M(\tau_2 - \overline{\tau})^2.$$

Condition (4.20) now implies that

$$I_\omega(\mathbf{q}(\overline{\tau})) - I_\omega(\mathbf{q}(\tau_2)) < M\sigma^2/3M = \sigma^2/3,$$

which contradicts (4.25).

If $\overline{\tau} = \tau_1$, we obtain the same contradiction following the same proof.

If $\overline{\tau} = \tau_2$, relations (4.23), (4.24) are still true. Combining them, we obtain $I^* - 7\sigma^2/12 < I_\omega(\mathbf{q}(\tau_2)) < I^* - 11\sigma^2/12$, a contradiction.

Thus, we have shown that Δ^* must be a subset of Δ_ω. It follows that

$$\Delta^*_\omega = \Delta^*,$$

and furthermore, G is a neighborhood of Δ^*.

Let us now see that $\mathbf{x}(t) \to \mathbf{0}$ as $t \to t^*$. From the definition of Δ^*, if $t_n \to t^*$ when $n \to \infty$, then $\mathbf{q}(t_n) \to \mathbf{q}' \in \Delta^*$, or $(\mathbf{x}(t_n), \mathbf{y}(t_n)) \to (\mathbf{x}', \mathbf{y}') \in \Delta^* = \Delta^*_\omega \subset \Delta_\omega = \{\mathbf{0}\} \times \Delta_\omega$ when $n \to \infty$. Therefore $\mathbf{x}' = \mathbf{0}$. Since this is true for every sequence $t_n \to t^*$ as $n \to \infty$, it follows that

$$\lim_{t \to t^*} \mathbf{x}(t) = \mathbf{0}. \tag{4.26}$$

We show that there exists a $t_2 \in [0, t^*)$ such that $\mathbf{q}(t) \in G$, $\forall t \in (t_2, t^*)$. Assume the opposite, i.e., $\exists (t_n)_{n \in \mathbb{N}}$, $t_n \to t^*$ when $n \to \infty$, such that $\mathbf{q}(t_n) \in C(G) = \mathbb{R}^{3n} - G$, $\forall n \in \mathbb{N}$. I^* is finite, so there exists a compact K such that $\mathbf{q}([0, t^*)) \subset K$. Then $\mathbf{q}(t_n) \in K \cap C(G)$. The set $K \cap C(G)$ is compact, hence there exists a subsequence $(\mathbf{q}(t_{nk}))_{k\in\mathbb{N}}$ of the sequence $(\mathbf{q}(t_n))_{n\in\mathbb{N}}$ such that $\mathbf{q}(t_{nk}) \to \mathbf{q}'' \in K \cap C(G) \subset C(G)$ when $k \to \infty$. On the other hand, $\mathbf{q}'' \in \Delta^* \subset G$, a contradiction. Taking into account that $\pi_\omega \mathbf{q}(t) = \mathbf{y}(t)$, by the relations (3.29) we have $\ddot{\mathbf{y}}(t) = \pi_\omega \ddot{\mathbf{q}}(t)$ on $[0, t^*)$ which, combined with (3.30), gives

$$\ddot{\mathbf{y}}(t) = \pi_\omega \nabla W_\omega \mathbf{q}(t), \quad \forall t \in [0, t^*).$$

By the linearity and continuity of π_ω and ∇W_ω it follows that

$$||\ddot{\mathbf{y}}(t)|| \leq ||\pi_\omega|| \cdot ||\nabla W_\omega \mathbf{q}(t)||, \quad \forall t \in [0, t^*).$$

In Proposition 3.5.c) we have shown that $||\pi_\omega|| \leq 1$. By (4.9) we have $||\nabla W_\omega \mathbf{q}(t)|| \leq M$ for $t \in (t_2, t^*)$, so it follows that

$$||\ddot{\mathbf{y}}(t)|| \leq M, \quad \forall t \in (t_2, t^*),$$

which implies that

$$|\ddot{\mathbf{y}}_i(t)| \leq M, \quad \forall t \in (t_2, t^*), \quad \forall i = 1, \ldots, n.$$

In the proof of Proposition 2.2. we showed how a condition of this form implies the existence of $\lim_{t\to t^*} \dot{\mathbf{y}}(t)$ and of $\lim_{t\to t^*} \mathbf{y}(t)$ by using Cauchy's criterion. So we draw the conclusion that there exists a $\mathbf{q}^* \in \mathbb{R}^{3n}$ such that

$$\lim_{t\to t^*} \mathbf{y}(t) = \mathbf{q}^*. \tag{4.27}$$

Since $\mathbf{y}(t) \in \Delta_\omega$, $\forall t \in [0, t^*)$ and Δ_ω is a closed set,

$$\mathbf{q}^* \in \Delta_\omega.$$

By (4.26), (4.27) and since $\mathbf{q}(t) = \mathbf{x}(t) + \mathbf{y}(t)$ on $[0, t^*)$, it follows that

$$\lim_{t\to t^*} \mathbf{q}(t) = \mathbf{q}^*.$$

Finally, observe that $\Delta_\omega \in \Delta$ and conclude that $\mathbf{q}^* \in \Delta$. Von Zeipel's theorem for quasihomogeneous potentials is thus proved.

CONCLUSION

In this thesis we pointed out some problems which arise studying the qualitative behavior of solutions with singularities of the n-body problem. We showed that in the n-body problem with quasihomogeneous potentials the results of Painlevé and Von Zeipel concerning the classical (Newtonian) n-body problem remain true. The n-body problem with quasihomogeneous potentials has opened a new direction of investigation in celestial mechanics. Our study is a modest contribution to it.

Bibliography

[1] Bruns, H. (1887). *Über die Integrale des Vielkörper-Problems.* Acta Math. **30**, 49-91.

[2] Diacu, F. N. (1992). *Singularities of the N-Body Problem.-An Introduction to Celestial Mechanics-.* Centre de Recherches Mathématiques, Montréal.

[3] Diacu, F. N. & Illner, R. (1993). *Collision/ejection dynamics for particle systems with quasihomogeneous potentials.* Preprint University of Victoria, DMS-624-IR.

[4] Hirsch, M. W. & Smale, S. (1974). *Differential Equations, Dynamical Systems, and Linear Algebra.* Academic Press, Inc.

[5] Mc Gehee, R. (1986). *Von Zeipel's theorem on singularities in celestial mechanics.* Expositiones Mathematicae. **4**, 335-345.

[6] Painlevé, P. (1897). *Leçons sur la théorie analytique des équations différentielles.* Hermann, Paris.

[7] Siegel, C. L. & Moser, J. K. (1971). *Lectures on Celestial Mechanics.* Springer Verlag, Berlin-Heidelberg-New York.

[8] Von Zeipel, H. (1908). *Sur les singularités du problème des n corps.* Arkiv för Matematik, Astronomi, Fysik. **4**, 32, 1-4.

Vita

Surname:	Popescu
Given Names:	Cristina-Mona
Place of Birth:	Sibiu, Romania
Date of Birth:	February 8th, 1969

Educational Institutions Attended:

University Babeş-Bolyai of Cluj-Napoca, Cluj-Napoca	1987-1992
University of Victoria, Victoria	1992-1994

Degrees Awarded:

B. Sc. (Mathematics) University Babeş-Bolyai of Cluj-Napoca	1992

Honors and Awards:

University of Victoria Fellowship	1992-1994
Advanced Systems Institute Scholarship	1992
Petch Research Scholarship	1993
Martlet Chapter IODE Graduate Scholarship for Women	1993

Title of Thesis:

The Theorem of Von Zeipel for Quasihomogeneous Potentials

Author: [redacted]

CRISTINA-MONA POPESCU

18. May 1994

(Date)