The Distance of Potentially Stable Sign Patterns To The Unstable Matrices

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ABSTRACT

The relative distance of a potentially stable sign pattern to the unstable matrices is introduced. The computation of the precise value of this distance is a nonlinear optimization problem. For order 2, an optimal stable matrix in the unique minimally potentially stable sign pattern is determined analytically. This analysis of the order 2 minimally potentially stable sign pattern shows how difficult the computation of this distance can be. Furthermore, it is shown that this distance can be arbitrarily small for potentially stable sign patterns of large order. Therefore, an adequate estimation of this distance becomes increasingly important.

A solution is introduced for such an estimation. Firstly, guided by the heuristic hierachy of the minimally potentially stable tree sign patterns, graph theory is used to obtain a good minimal subpattern. Secondly, a sequence of algorithms is introduced to estimate the distance to the unstable matrices of the minimal components of this subpattern that have a properly signed nest. The algorithms are applied to those order 3 and 4 minimally potentially stable tree sign patterns that have a properly signed nest, as well as to order 5 minimally potentially stable rooted tree sign patterns. An ad hoc procedure is used to find a good stable matrix in each of the two minimally potentially stable tree sign patterns of order 4 that do not have a properly signed nest.

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The Distance of Potentially Stable Sign Patterns

To The Unstable Matrices

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§1. Introduction

In science and technology, a problem is often formulated in terms of functional relationships between certain endogenous (internal) variables $(x_1, \cdots, x_n)$ and certain exogenous (external) variables $(\alpha_1, \cdots, \alpha_m)$:

(1.1) $f_i(x_1, \cdots, x_n, \alpha_1, \cdots, \alpha_m)$, for $i = 1, 2, \cdots, n$.

For example, in economic theory, $(x_1, \cdots, x_n)$ are economic variables and $(\alpha_1, \cdots, \alpha_m)$ are certain shift parameters (see [HLMQ] and [Q]). In a particular labor market problem in economics (see [Q, p. 301]),

$x_1 =$ price level, $x_2 =$ interest rate, and $x_3 =$ wage rate,

$m = 1$ and $\alpha_1 =$ parameter shifting the demand for labor,

$f_1(x_1, x_2, x_3, \alpha_1) =$ excess demand for product,

$f_2(x_1, x_2, x_3, \alpha_1) =$ excess demand for money balances,

$f_3(x_1, x_2, x_3, \alpha_1) =$ excess demand for labor.

An equilibrium position of the system (1.1) is a vector $(\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_n, \hat{\alpha}_1, \cdots, \hat{\alpha}_m)$ such that

$$f_i(\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_n, \hat{\alpha}_1, \cdots, \hat{\alpha}_m) = 0, \text{ for } i = 1, 2, \cdots, n.$$

A central problem in the theory of equilibria is the "(local) stability of an equilibrium question": does a system remain in an equilibrium position when slightly disturbed from it?

Consider an equilibrium position $(\hat{x}_1, \cdots, \hat{x}_n, \hat{\alpha})$ of (1.1) with $m = 1$, i.e.,

$$(1.2) \quad f_i(\hat{x}_1, \cdots, \hat{x}_n, \hat{\alpha}) = 0, \text{ for } i = 1, 2, \cdots, n.$$

In this context, an example of a dynamical adjustment system is

$$(1.3) \quad dx_i/dt = b_i f_i(x_1, \cdots, x_n, \hat{\alpha}), \text{ for } i = 1, 2, \cdots, n,$$

where $\hat{\alpha}$ is the constant from (1.2), b_i is a positive constant (referred to as the "speed of adjustment" of the i-th variable) and $x_i(0) = \hat{x}_i$, i.e., starting at the equilibrium position (see [Q]). To determine the stability of the equilibrium position $(\hat{x}_1, \cdots, \hat{x}_n, \hat{\alpha})$ of (1.2), on replacing the right side of (1.3) by its linear Taylor expansion at $(\hat{x}_1, \cdots, \hat{x}_n)$, we obtain

$$(1.4) \quad dx_i/dt = b_i(\textstyle\sum_{j=1}^{n} \partial f_i/\partial x_j(\hat{x}_1, \cdots, \hat{x}_n, \hat{\alpha})(x_j - \hat{x}_j)).$$

Let $A = [a_{ij}]$, where $a_{ij} = \partial f_i / \partial x_j(\hat{x}_1, \cdots, \hat{x}_n, \hat{\alpha})$, be the $n \times n$ Jacobian matrix of the f_i's at $(\hat{x}_1, \cdots, \hat{x}_n, \hat{\alpha})$, D be the diagonal matrix with $d_{ii} = b_i$, and z be the vector with j-th entry equal to $x_j - \hat{x}_j$. Then (1.4) can be rewritten as

(1.5) $dz/dt = DAz,$

a linear differential system. Given a fixed A, the asymptotic stability of the equilibrium position $(\hat{x}_1, \cdots, \hat{x}_n, \hat{\alpha})$ is equivalent to the existence of D such that (1.5) is asymptotically stable, i.e., $z(t) \to 0$ as $t \to \infty$.

For a diagonalizable matrix B for which $PBP^{-1} = \text{diag}(\lambda_1, \cdots, \lambda_n)$, the linear differential system

$$dz/dt = Bz$$

can be simplified by the change of variable $y = Pz$ to give

$$dy/dt = \text{diag}(\lambda_1, \cdots, \lambda_n)y.$$

So

$$\begin{aligned} & z(t) \to 0 \text{ as } t \to \infty \\ \iff & \quad y(t) \to 0 \text{ as } t \to \infty \\ \iff & \quad Re(\lambda_j) < 0, \text{ for } j = 1, 2, \cdots, n. \end{aligned}$$

For a nondiagonalizable matrix B, a similar result can be derived using the Jordan normal form (see [HS]). The above motivates the following definition.

Definition 1.1. A square matrix is (negative) **stable** *if all of its eigenvalues have negative real parts.*

Thus the "stability of an equilibrium question" can be restated as follows: *if A is the Jacobian matrix of the f_i's at an equilibrium position, is there a positive diagonal matrix D such that DA is stable?*

However, the situation may not be as simple as stated above. For example, in economics, chemistry, biology, and the social sciences, we are often concerned with the situation where the f_i's are not completely known. Instead, only the variables appearing in the relationships and the nature of their direct influence are known. That is, it is known if $\partial f_i/\partial x_j$ is zero, positive or negative; consequently, only the signs of the entries of A are known. These circumstances lead to consideration of the stablility of sign pattern classes of matrices, which is a central theme of this thesis.

Definition 1.2. The **sign pattern** *of a real matrix A is the matrix with entries +, - or 0 obtained by replacing each entry of A by its sign. Matrix A determines a* **sign pattern class** *consisting of all matrices with the same sign pattern as A.*

Note that, in general, we do not distinguish between a sign pattern and the sign pattern class that it determines.

Definition 1.3. A sign pattern is **potentially stable** *if there is a stable matrix* *in the sign pattern class. A sign pattern is* **sign stable** *if all matrices in the sign pattern class are stable.*

For example, $\begin{pmatrix} - & + \\ - & 0 \end{pmatrix}$ is a sign stable pattern. However, in practical applications, sign stable patterns do not occur very often (as pointed out by the US economist James Quirk [Q]). More frequently occurring are sign pattern classes in which some matrices are stable and some matrices are unstable. For example, the sign pattern class $\begin{pmatrix} - & + \\ - & + \end{pmatrix}$ is potentially stable but not sign stable, since the matrix $\begin{pmatrix} -1 & 1 \\ -1 & 1/2 \end{pmatrix}$ is stable but the matrix $\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$ is unstable.

Definition 1.4. A matrix A is **row scalable stable** *if there is a positive diagonal matrix D such that DA is stable.*

Notice that the matrices A and DA have the same sign pattern when D is a positive diagonal matrix. So determining if a sign pattern is potentially stable will be the main focus in answering the "stability of an equilibrium question". Reformulating the problem in this way, we face two fundamental questions:

(i) Which sign patterns are potentially stable?

(ii) In a potentially stable sign pattern, which matrices are row scalable stable?

Fisher and Fuller [FF], whose work was inspired by the famous British economist

John R. Hicks [H1], gave a sufficient condition leading to an affirmative answer for the second question. The first question is still far from being answered.

Definition 1.5. ([JMOV]) An $n \times n$ matrix has a **properly signed nest** *if up to a symmetric rearrangement of rows and columns, the leading principal minor of order k has sign $(-1)^k$ for $k = 1, \cdots, n$.*

Theorem ([FF]). If a matrix has a properly signed nest, then it is row scalable stable.

It follows from this theorem that if a sign pattern contains a matrix having a properly signed nest, then it is potentially stable. The converse of the theorem is not true (see $\mathcal{A}_{4,6}$ and $\mathcal{A}_{4,7}$ in Section 7). It is worthwhile pointing out that, in certain important economic applications, every matrix from a given sign pattern class has a properly signed nest. For example, every matrix in the sign pattern class

$$\begin{pmatrix} - & 0 & + \\ 0 & 0 & + \\ - & - & 0 \end{pmatrix}$$

has a properly signed nest. However, the constructions in the existing proofs of Fisher and Fuller's theorem (see [FF], [B1]) result in matrices that are very close to being unstable, and so are not of practical value. In the presence of round off errors, such matrices may even be unstable.

Unfortunately, it is not the case that Fisher and Fuller's theorem can be extended to conclude that there is a universal positive constant c such that, for any stable matrix A having a properly signed nest, there is a positive diagonal matrix D such that the distance of $DA/\| DA \|$ to the unstable matrices is at least c. The precise definition of the distance of a stable matrix to the unstable matrices is given in Definition 2.1. We prove that (see Example 2.3), for each $n \geq 2$, there is an $n \times n$ minimally potentially stable tridiagonal sign pattern (i.e., irreducible and replacing any nonzero entry by zero results in an unstable sign pattern) $\mathcal{A}_n$ such that the distance of $A/\| A \|$ to the unstable matrices is less than $1/n$ for any stable matrix A in $\mathcal{A}_n$.

The proof of the theorem of Fisher and Fuller determines a positive diagonal matrix D such that DA is stable with simple, real eigenvalues. However, it seems to be difficult to find a matrix D such that DA is stable with some complex eigenvalues so that the distance of $DA/\| DA \|$ from the unstable matrices is greater than that given by the Fisher and Fuller result. Thus we believe that it is unrealistic to directly improve their theorem to go from a matrix with a properly signed nest to a numerically stable matrix. Given a fixed potentially stable sign pattern and a matrix A in this pattern that has a properly signed nest, rather than attempting to find a positive diagonal matrix D so as to maximize the distance of $DA/\| DA \|$ from the unstable matrices, we aim to determine a matrix in the pattern that is not too close to any unstable matrix. Such a matrix we call a "good" stable matrix.

The details of the approach and an outline of the thesis are now described. In Section 2, relative measures of the distance to the unstable matrices for a fixed matrix and for a sign pattern are introduced. In Section 3, the concept of a minimally potentially stable sign pattern is introduced, and a conjecture regarding the relative distance of a general potentially stable sign pattern to the unstable matrices in terms of its minimal subpatterns is given. The structure of minimally potentially stable tree sign patterns is explored, especially those corresponding to a rooted tree. For order 2, there is (up to permutation and signature similarity) only one minimally potentially stable sign pattern and an optimal stable matrix (i.e., the relative distance from the unstable matrices is maximized) in this pattern is determined analytically in Section 4. This nontrivial analysis shows that it is unrealistic to find an optimal stable matrix in a pattern of higher order. Examples of some good stable matrices for particular tree sign patterns are given in Section 5. In Section 6, attention is restricted to those minimally potentially stable tree sign patterns that contain a matrix with a properly signed nest. Based on some theoretical results and numerical experience, some algorithms are given to find a good stable matrix in such a pattern. In Section 7, these heuristic algorithms are applied to find a good stable matrix for each minimally potentially stable tree sign pattern of order 3 and 4, and for rooted trees of order 5; an ad hoc procedure is used for those patterns without a properly signed nest. A hierarchy of minimally potentially stable tree sign patterns with respect to the relative distance from the unstables is given in Section 8. Here criteria are identified that contribute to this relative distance being large or small. In Section 9, some conclusions and directions for future research are given.

§2. The Relative Distance to the Unstable Matrices

We denote the spectral radius of an $n\times n$ matrix $B=[b_{ij}]$ by $\rho(B)$. The matrix 2-norm of B is $\| B \|_2 = \sqrt{\rho(B^*B)}$, and $\| B \|_F = \sqrt{tr(B^*B)} = \sqrt{\sum_i \sum_j |b_{ij}|^2}$ is the Frobenius norm of B.

In [V], Van Loan pointed out that the maximum real part of the eigenvalues is not an adequate measurement of the distance of a given stable matrix to the unstable matrices. For example, consider the $n \times n$ matrices

$$A = \begin{pmatrix} -1/2 & 1 & 0 & \cdots & 0 \\ 0 & -1/2 & 1 & & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1/2 & 1 \\ 0 & 0 & \cdots & 0 & -1/2 \end{pmatrix},$$

$$B = \begin{pmatrix} -1/2 & 1 & 0 & \cdots & 0 \\ 0 & -1/2 & 1 & & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1/2 & 1 \\ 1/2^n & 0 & \cdots & 0 & -1/2 \end{pmatrix}.$$

A is stable and B is unstable (singular), but

$$\| A - B \|_2 = 1/2^n,$$

although the eigenvalues of A are all a distance $1/2$ away from the imaginary axis.

Van Loan [V] proposed an alternate measurement of the distance of a (possibly complex) stable matrix A to the set of all unstable matrices, using the Frobenius norm. The notation $d_{us}(A)$ is from [HW].

Definition 2.1 If A is a stable matrix, then

$$d_{us}(A) = \min\{\| A - E \|_F : E \text{ is a (possibly complex) unstable matrix}\}.$$

In the literature, $d_{us}(A)$ is sometimes called the complex stability radius of A, and in the case that E is restricted to be a real matrix, the corresponding distance is called the real stability radius; see, for example [HW]. In [H2], it is shown that the Frobenius norm can be replaced by the matrix 2-norm in Definition 2.1. We include a proof in Theorem 2.2, which uses the following result of Van Loan, in which $\sigma_{\min}(B)$ denotes the minimum singular value of a matrix B.

Theorem 2.1 ([V]). For a stable matrix A,

$$d_{us}(A) = \min\{\sigma_{min}(A - \mu iI) : \mu \in \mathbf{R}\}.$$

Proof. The crucial fact used here is that, for a matrix A,

$$(2.1) \quad \sigma_{min}(A) = \min\{\| A - E \|_F : E \text{ is singular}\} \quad \text{(see [V, p. 246])}.$$

For any unstable matrix E, consider the linear section $\alpha A + (1-\alpha)E,\ 0 \le \alpha \le 1$ connecting the stable matrix A and the unstable matrix E. By continuity of eigenvalues, there must exist α_0, $0 \le \alpha_0 < 1$, such that $\alpha_0 A + (1-\alpha_0)E$ has an eigenvalue with real part zero. Thus $\alpha_0 A + (1-\alpha_0)E$ is unstable and

$$\| A - (\alpha_0 A + (1-\alpha_0)E) \|_F \le \| A - E \|_F.$$

Consequently,

$$(2.2) \quad d_{us}(A) = \min\{\| A - E \|_F : E \text{ has an eigenvalue with real part zero}\}.$$

On the other hand, for any real number μ, by (2.1)

$$\sigma_{\min}(A - \mu i I) = \min\{\| (A - \mu i I - E) \|_F : E \text{ is singular}\}$$

Thus, on letting $B = \mu i I + E$,

$$(2.3) \quad \min\{\sigma_{\min}(A - \mu i I) : \mu \in \mathbf{R}\}$$

$$= \min\{\| A - B \|_F : B \text{ has an eigenvalue } \mu i\}.$$

Now (2.2) and (2.3) together give the desired result. □

The Frobenius norm in Definition 2.1 can be replaced by the 2-norm, as shown in the following result.

Theorem. 2.2. For a stable matrix A,

$$d_{us}(A) = \min\{\| A - E \|_2 : E \text{ is unstable}\}.$$

Proof. Using the fact that

$$\sigma_{\min}(A) = \min\{\| A - E \|_F : E \text{ is singular}\}$$
$$= \min\{\| A - E \|_2 : E \text{ is singular}\}$$

(see [W, Cor. 7.3.10]) and the proof of Theorem 2.1,

$$d_{us}(A) = \min\{\sigma_{\min}(A - \mu i I) : \mu \in \mathbf{R}\}$$
$$= \min\{\| A - \mu i I - E \|_2 : E \text{ is singular}, \mu \in \mathbf{R}\}$$
$$= \min\{\| A - B \|_2 : B \text{ has an eigenvalue with real part zero}\}$$
$$= \min\{\| A - B \|_2 : B \text{ is unstable}\} \quad \square$$

Corollary 2.3. If A is a stable matrix then

$$d_{us}(A) \leq \min\{|Re(\lambda)| : \lambda \text{ is an eigenvalue of } A\}.$$

Proof. Let $\lambda = p + iq$ be an eigenvalue of A. Then iq is an eigenvalue of $A - pI$. Thus, by Theorem 2.2, $d_{us}(A) \leq \| A - (A - pI) \|_2 = |p|$. □

Corollary 2.4. If A is a real stable $n \times n$ matrix, then $d_{us}(A) \leq |tr(A)|/n$.

Proof. Since A is real and stable,

$$|tr(A)| = |\sum_j Re(\lambda_j)| = \sum_j |Re(\lambda_j)| \geq n \min_j |Re(\lambda_j)|.$$

The result now follows from Corollary 2.3. □

Observe that, if A is stable, so is cA for a positive constant c, but $d_{us}(cA) = c\, d_{us}(A)$ can be arbitrarily large. Notice that $\| cA \|_2$ can be very large as well.

Therefore, we define a normalized measure of closeness to the unstable matrices as follows.

Definition 2.2. For a stable matrix A, the **relative distance of A to the unstable matrices** *is*

$$\delta_{us}(A) = d_{us}(A)/\| A \|_2.$$

Here the 2-norm is used rather than the Frobenius norm, so that $\delta_{us}(-I_n)$ (see Example 2.2 below) is independent of n; recall that for the $n \times n$ identity matrix I_n, $\| I_n \|_2 = 1$ and $\| I_n \|_F = \sqrt{n}$. Note that $\delta_{us}(cA) = \delta_{us}(A)$ for $c > 0$. Notice also that, if U is a unitary matrix, then $\delta_{us}(A) = \delta_{us}(UAU^*)$, i.e., δ_{us} is invariant under unitary similarity. The following result shows that 1 is the upper bound for $\delta_{us}(A)$.

Lemma 2.5. For any stable matrix A, $0 < \delta_{us}(A) \leq 1$.

Proof. This follows from Theorem 2.1 with $\mu = 0$:

$$d_{us}(A) \leq \sigma_{min}(A) = (\text{min eigenvalue of } A^*A)^{1/2} \leq \sqrt{\rho(A^*A)} = \| A \|_2. \quad \square$$

The following two examples illustrate the range of $\delta_{us}(A)$.

Example 2.1. If $A = \begin{pmatrix} -1 & 2^k \\ -2^k & 0 \end{pmatrix}$, then

$$\| \frac{A}{2^k} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \|_2 = \frac{1}{2^k}.$$

Since the 2-norm of $A/2^k$ is greater than 1, consequently,

$$\delta_{us}(A) = \delta_{us}(\frac{A}{2^k}) < \frac{1}{2^k},$$

which can be made as small as possible.

Example 2.2. Let $A = -I_n$. Since

$$\sigma_{min}(-I_n - i\mu I_n) = \sqrt{1+\mu^2}$$

for $\mu \in \mathbf{R}$,

$$\delta_{us}(-I_n) = 1$$

by Theorem 2.1. Consequently,

$$\delta_{us}(-I_n) = \delta_{us}(-cI_n) = 1, \text{ for } c > 0,$$

which is the maximum possible value for $\delta_{us}(A)$ by Lemma 2.5.

It is well known that the relative distance of a nonsingular matrix A to the nearest singular matrix is equal to the reciprocal of the condition number, i.e.,

$$\frac{1}{\kappa_2(A)} = \frac{1}{\| A \|_2 \| A^{-1} \|_2};$$

see, e.g., [W, Cor. 7.3.10].

Lemma 2.6. For any stable matrix A, $\delta_{us}(A) \leq \frac{1}{\kappa_2(A)}$.

Proof. From Theorem 2.1, with $\mu = 0$,

$$d_{us}(A) \leq \sigma_{\min}(A) = \frac{1}{\| A^{-1} \|_2};$$

see, e.g., [W, p. 416]. Thus,

$$\delta_{us}(A) = \frac{d_{us}(A)}{\| A \|_2} \leq \frac{1}{\| A \|_2 \| A^{-1} \|_2}. \square$$

We now extend the concept of distance to the unstable matrices to sign patterns.

Definition 2.3. For a potentially stable sign pattern class $\mathcal{A}$, **the distance of the sign pattern class $\mathcal{A}$ to the unstable matrices** *is defined as*

$$\delta_{us}(\mathcal{A}) = \sup_A \{\delta_{us}(A) : A \in \mathcal{A} \text{ is stable}\}.$$

Notice that, by compactness of the set

$$\{A : \| A \|_2 = 1, \ A \text{ is } n \times n \text{ real valued matrix}\},$$

and continuity of $\delta_{us}(A)$, there is a stable matrix $A_0 \in \mathcal{A}$ with $\| A_0 \|_2 = 1$ such that

$$\delta_{us}(\mathcal{A}) = d_{us}(A_0) = \max\{\delta_{us}(A) : A \in \mathcal{A} \text{ is stable and } \| A \|_2 = 1\}.$$

For such a matrix A_0, any matrix cA_0 with $c > 0$ is called an **optimal stable matrix** in the sign pattern class $\mathcal{A}$. For example, $-cI_n$ is an optimal stable matrix in the $n \times n$ sign pattern class diag$(-, \cdots, -)$. It is well known that the computation of $d_{us}(A)$ is a difficult nonlinear optimization problem. Therefore, finding an optimal stable matrix in a potentially stable sign pattern class is often extremely difficult. However, a lower bound on $\delta_{us}(\mathcal{A})$ may be possible.

Example 2.3. It is well known that the $n \times n$ tridiagonal matrix sign pattern

$$\mathcal{A}_n = \begin{pmatrix} - & + & & & & 0 \\ - & 0 & + & & & \\ & - & 0 & + & & \\ & & - & 0 & \ddots & \\ & & & \ddots & \ddots & \\ & & & & \ddots & \ddots & + \\ 0 & & & & & - & 0 \end{pmatrix}$$

is a sign stable pattern. Note that any matrix $A \in \mathcal{A}_n$ can be normalized such that its unique nonzero diagonal entry is $a_{11} = -1$, and, if e_1 denotes the column n-vector $(1, 0, \cdots, 0)^t$

$$\| A \|_2 \geq \| Ae_1 \|_2 = \sqrt{1 + a_{21}^2} > 1.$$

So by Corollary 2.4,

$$\delta_{us}(A) < 1/n,$$

for all $A \in \mathcal{A}_n$ and thus

$$\delta_{us}(\mathcal{A}_n) < 1/n.$$

Although the computation of $\delta_{us}(A)$ is difficult, one of the most feasible methods is based on the following result.

Theorem 2.7 ([B2]). For any real stable matrix A and positive number α, let

$$H(\alpha) = \begin{pmatrix} A & -\alpha I \\ \alpha I & -A^T \end{pmatrix}.$$

Then $H(\alpha)$ has an eigenvalue with real part zero if and only if $\alpha \geq d_{us}(A)$.

Corollary 2.8. For any real stable matrix A and positive number α, let

$$HH(\alpha) = \begin{pmatrix} A & -\alpha \| A \|_2 I \\ \alpha \| A \|_2 I & -A^T \end{pmatrix}.$$

Then $HH(\alpha)$ has an eigenvalue with real part zero if and only if $\alpha \geq \delta_{us}(A)$.

For a sequence of real stable $n \times n$ matrices $\{A_k\}_{k=1}^N$ and a number $t > 0$, these results suggest the following procedure for attempting to show that $\delta_{us}(A_k) > t$ for $k = 1, 2, \cdots, N$.

Algorithm 2.1.

For $k = 1$ to N

$$HH = \begin{pmatrix} A_k & -t\| A_k \|_2 I \\ t\| A_k \|_2 I & -A_k^T \end{pmatrix};$$

For $j = 1$ to $2n$

If $(|Re(\lambda_j(HH))| < 10^{-10})$

print "Break at k";

break;

End if

End for // for j

End for // for k

If, for any of the matrices A_k, there is an eigenvalue λ_j of the associated matrix HH such that $|Re(\lambda_j(HH))| < 10^{-10}$, then HH likely has an eigenvalue with real part zero, and thus $\delta_{us}(A_k) \leq t$. Otherwise, it is likely that $\delta_{us}(A_k) > t$ for $k = 1, 2, \cdots, N$.

For a fixed potentially stable sign pattern $\mathcal{A}$, this procedure further suggests a method for attempting to show that $\delta_{us}(\mathcal{A}) > t$. Let $\mathcal{S}$ be a subset of the stable matrices A in $\mathcal{A}$ such that $0 < \beta \leq \| A \|_2 \leq \gamma$. If a finite sequence of real stable matrices $\{A_k\}_{k=1}^N$ in $\mathcal{S}$ can be selected that is sufficiently dense in $\mathcal{S}$, then the above procedure can be applied to this sequence to try to determine if $\delta_{us}(\mathcal{A}) > t$.

Notice that such a selection is highly nontrivial. The validity of obtaining a lower bound for $\delta_{us}(\mathcal{A})$ in this way is supported by Theorem 2.10 below, the proof of which depends on the following lemma that is also of independent interest.

Lemma 2.9. For any two stable matrices A_1 and A_2 of the same order,

$$|d_{us}(A_1) - d_{us}(A_2)| \leq \| A_1 - A_2 \|_2.$$

Proof. For any unstable matrix E, by Theorem 2.2,

$$d_{us}(A_1) \leq \| A_1 - E \|_2 \leq \| A_1 - A_2 \|_2 + \| A_2 - E \|_2.$$

Taking the minimum of the right hand side over all unstable matrices E, and using Theorem 2.2 again, we obtain

$$d_{us}(A_1) \leq \| A_1 - A_2 \|_2 + d_{us}(A_2).$$

Interchanging the roles of A_1 and A_2 and repeating the above process,

$$d_{us}(A_2) \leq \| A_1 - A_2 \|_2 + d_{us}(A_1),$$

which gives the result. □

Theorem 2.10. For any two stable matrices A_1 and A_2 of the same order,

$$|\delta_{us}(A_1) - \delta_{us}(A_2)| \leq \frac{2\| A_1 - A_2 \|_2}{\max(\| A_1 \|_2, \| A_2 \|_2)}.$$

Proof. Without loss of generality, assume $\| A_1 \|_2 \geq \| A_2 \|_2$. Then

$$|\delta_{us}(A_1) - \delta_{us}(A_2)|$$

$$\leq \frac{|d_{us}(A_1) - d_{us}(A_2)|}{\| A_1 \|_2} + \frac{d_{us}(A_2)}{\| A_2 \|_2} \frac{(\| A_1 \|_2 - \| A_2 \|_2)}{\| A_1 \|_2}$$

$$\leq \frac{2\| A_1 - A_2 \|_2}{\| A_1 \|_2},$$

by Lemmas 2.5 and 2.9. □

Corollary 2.11. If $A_1, A_2 \in \mathcal{A}_n$, defined as in Example 2.3, and

$$A_1 = \begin{pmatrix} -1 & a_1 & & & & 0 \\ -a_1 & 0 & a_2 & & & \\ & -a_2 & 0 & a_3 & & \\ & & -a_3 & 0 & \ddots & \\ & & & \ddots & \ddots & \ddots \\ & & & & \ddots & \ddots & a_{n-1} \\ 0 & & & & & -a_{n-1} & 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} -1 & b_1 & & & & & 0 \\ -b_1 & 0 & b_2 & & & & \\ & -b_2 & 0 & b_3 & & & \\ & & -b_3 & 0 & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \ddots & \ddots & b_{n-1} \\ 0 & & & & & -b_{n-1} & 0 \end{pmatrix}$$

where $a_j, b_j > 0$, then

$$|\delta_{us}(A_1) - \delta_{us}(A_2)| \le \frac{4 \max_j(|a_j - b_j|)}{\max(\| A_1 \|_2, \| A_2 \|_2)}.$$

Proof. Let $c_j = a_j - b_j$, $1 \le j \le n-1$. By Theorem 2.10, it suffices to show that

$$\| A_1 - A_2 \|_2 \le 2 \max_j(|c_j|).$$

Let $M = \max_j |c_j|$. For any unit vector $x = (x_1, x_2, \cdots, x_n)^T$,

$$\| (A_1 - A_2)x \|_2^2 = c_1^2 x_2^2 + \textstyle\sum_{j=1}^{n-2} (c_{j+1}x_{j+2} - c_j x_j)^2 + c_{n-1}^2 x_{n-1}^2$$

$$\le 4M^2 \textstyle\sum_{j=1}^{n} x_j^2 \le 4M^2,$$

from which the result follows. □

We close this section by determining (for use in Section 4) the characteristic polynomial of the matrix $H(\alpha)$, as defined in Theorem 2.7.

Theorem 2.12. The characteristic polynomial of $H(\alpha)$ is

$$\det(H(\alpha) - \lambda I_{2n}) = \det((\alpha^2 + \lambda^2)I_n - AA^T + \lambda(A^T - A)),$$

which is an even polynomial in λ.

Proof. For any $n \times n$ matrix F,

$$\begin{pmatrix} F & I_n \\ I_n & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & I_n \\ I_n & -F \end{pmatrix}.$$

Consider

$$\begin{pmatrix} 0 & I_n \\ I_n & -F \end{pmatrix} \begin{pmatrix} A & -\alpha I_n \\ \alpha I_n & -A^T \end{pmatrix} \begin{pmatrix} F & I_n \\ I_n & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha F - A^T & \alpha I_n \\ AF + FA^T - \alpha(I_n + F^2) & A - \alpha F \end{pmatrix}.$$

If $F = (A - \lambda I_n)/\alpha$, then $A - \alpha F - \lambda I_n = 0$ and so

$$\det(H(\alpha) - \lambda I_n) = (-1)^n \alpha^n \det(AF + FA^T - \alpha(I_n + F^2))$$

$$= \det((\alpha^2 + \lambda^2)I_n - AA^T + \lambda(A^T - A)).$$

Observe that, if B is symmetric and C is skew symmetric, then

$$\det(B + \lambda C) = \det(B - \lambda C), \text{ for all } \lambda,$$

i.e., $\det(B + \lambda C)$ is an even polynomial in λ. Applying this result with $B = (\alpha^2+\lambda^2)I_n - AA^T$ and $C = A^T - A$, we conclude that the characteristic polynomial of $H(\alpha)$ must be an even polynomial in λ. $\square$

Corollary 2.13. $det(H(\alpha)) = det(\alpha^2 I_n - AA^T)$.

Replacing λ^2 in $\det((\alpha^2+\lambda^2)I_n - AA^T + \lambda(A^T - A))$ by $-\mu$ gives a polynomial of degree n in μ. Theorem 2.12 shows that the $2n \times 2n$ matrix $H(\alpha)$ has an eigenvalue with real part zero if and only if this degree n polynomial in μ has a nonnegative real root. This observation is used in Section 4 to simplify our computations.

§3. Minimally Potentially Stable Sign Patterns

Definition 3.1. A sign pattern $\mathcal{P}$ is **reducible** *if there exists a permutation matrix Q such that*

$$QPQ^T = \begin{pmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ 0 & \mathcal{P}_{22} \end{pmatrix},$$

where $\mathcal{P}_{11}$ and $\mathcal{P}_{22}$ are square, nonempty sign patterns. A sign pattern that is not reducible is called **irreducible**.

The stability of a reducible sign pattern $\mathcal{P}$ is determined completely by that of the patterns $\mathcal{P}_{11}$ and $\mathcal{P}_{22}$. Therefore, without loss of generality, it suffices to consider irreducible potentially stable sign patterns; see [JS, p. 7].

Definition 3.2. A sign pattern is **minimally potentially stable** *if it is irreducible, potentially stable and replacing any nonzero entry by zero results in a sign pattern such that every matrix in this sign pattern is unstable (i.e., it is an unstable sign pattern).*

For example, the sign pattern

$$\mathcal{P} = \begin{pmatrix} - & + \\ - & 0 \end{pmatrix}$$

is irreducible and potentially stable (in fact, it is sign stable). To verify that $\mathcal{P}$ is minimally potentially stable, let

$$A = \begin{pmatrix} -a & b \\ -c & 0 \end{pmatrix},$$

where $a, b, c \geq 0$. Then $\det(A) = bc$ and $\operatorname{tr}(A) = -a$. If A is stable, then it has a positive determinant and a negative trace, hence $a, b, c > 0$. Thus $\mathcal{P}$ is minimally potentially stable.

We now show that minimally potentially stable sign patterns are the atoms among potentially stable sign patterns, in the sense that every potentially stable sign pattern contains a subpattern that is a direct sum of minimally potentially stable sign patterns.

Definition 3.3. A **subpattern** $\mathcal{P}'$ *of a sign pattern* $\mathcal{P}$ *is a sign pattern obtained by replacing one or more nonzero entries of* $\mathcal{P}$ *by zero.*

If an irreducible potentially stable sign pattern is not minimal, then it must have a potentially stable subpattern. If this subpattern is reducible, it can be decomposed into irreducible blocks. If any of these irreducible blocks is not minimally potentially stable, the above process can be repeated. After a finite number of steps, a direct sum of minimally potentially stable sign patterns is obtained that is a subpattern of the given potentially stable sign pattern.

Definition 3.4. A **minimal subpattern** *of a potentially stable sign pattern is a* *subpattern that is a direct sum of minimally potentially stable sign patterns.*

For example, $\mathcal{P}_1 = \begin{pmatrix} - & + \\ - & 0 \end{pmatrix}$ and $\mathcal{P}_2 = \begin{pmatrix} - & 0 \\ 0 & - \end{pmatrix}$ are both minimal subpatterns of $\mathcal{P} = \begin{pmatrix} - & + \\ - & - \end{pmatrix}$. However, it follows by Lemma 2.5, Example 2.2 and a continuity argument that

$$\delta_{us}(\mathcal{P}) = \delta_{us}(\mathcal{P}_2) = 1,$$

whereas (see Section 4) $\delta_{us}(\mathcal{P}_1) < 0.273$. Thus, not all minimal subpatterns of a potentially stable sign pattern $\mathcal{P}$ can be used to determine $\delta_{us}(\mathcal{P})$. However, experience with several examples suggests the following conjecture.

Conjecture. If $\mathcal{P}$ is a potentially stable sign pattern, then

$$\delta_{us}(\mathcal{P}) = \max\{\delta_{us}(\mathcal{P}') : \mathcal{P}' \text{ is a minimal subpattern of } \mathcal{P}\}.$$

In Section 8, we describe how to determine a minimal subpattern $\mathcal{P}'$ of a given sign pattern $\mathcal{P}$ so that $\delta_{us}(\mathcal{P}')$ is a good approximation to $\delta_{us}(\mathcal{P})$.

To understand the structure of minimally potentially stable sign patterns and the choice of a good minimal subpattern of a potentially stable sign pattern, we consider a graph representation of a sign pattern.

Definition 3.5. The (undirected) **graph** *of an $n \times n$ sign pattern is a graph on* *vertices* $1, 2, \cdots, n$ *with an undirected edge between* i *and* $j \neq i$ *if and only if either the* (i, j) *entry or the* (j, i) *entry is nonzero.*

It is well known that if a sign pattern $\mathcal{P}$ is irreducible and its graph is a tree, then its (i, j) and (j, i) entries are either both zero or both nonzero. Moreover, if $A = [a_{ij}] \in \mathcal{P}$, then the eigenvalues of A depend on the product $a_{ij}a_{ji}$ $(i \neq j)$, and not on the individual values a_{ij} or a_{ji}. Therefore, the stability classification of such a sign pattern depends only on the sign of $a_{ij}a_{ji}$ and is independent of the sign of a_{ij} or a_{ji}. See [JS] for related concepts on trees, including the following definition.

Definition 3.6. A sign pattern is called a **tree sign pattern (t.s.p)** *if it is irreducible and the graph of the pattern is a tree.*

For a t.s.p., the following signed graph representation is useful because it contains all the information pertinent to classifying the stability of a pattern.

Definition 3.7. Let G *be the undirected graph of a tree sign pattern* $\mathcal{P} = [p_{ij}]$*. The i-th vertex is signed* $+$ *or* $-$ *in agreement with the sign of* p_{ii} *(if* $p_{ii} = 0$*, then the vertex* i *is unsigned and is called a zero vertex), and the edge between* i *and* j *is similarly signed* $+$ *or* $-$ *according to the sign of* $p_{ij}p_{ji}$*. This graph is called the* **signed tree** *of a tree sign pattern.*

For example, the signed tree of $\begin{pmatrix} - & + & 0 \\ - & + & + \\ 0 & - & 0 \end{pmatrix}$ is

A **signature matrix** is a nonsingular diagonal matrix with diagonal entries ± 1. A signature similarity is a similarity transformation given by a signature matrix. A similarity transformation given by a permutation matrix is called a permutation similarity. Up to signature similarity and permutation similarity (which do not affect the stability of a t.s.p.), there is a one-to-one correspondence between tree sign patterns and signed trees.

Example 3.1.

$$\begin{pmatrix} - & + & 0 \\ - & + & + \\ 0 & - & 0 \end{pmatrix}, \begin{pmatrix} - & - & 0 \\ + & + & + \\ 0 & - & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & - & 0 \\ + & + & - \\ 0 & + & - \end{pmatrix}$$

correspond to the (unordered) signed tree

From now on, we identify a t.s.p. (up to signature and permutation similarity) with its signed tree.

We now state a theorem that uses the following concept and indicates the power of the signed tree representation.

Definition 3.8. A **complete matching** *in a signed tree is a set of disjoint edges and nonzero vertices that covers the vertex set of the tree.*

Note that a signed tree of a tree sign pattern has a complete matching if and only if there exists a matrix in this sign pattern that has a nonzero determinant.

Now, we begin to explore the structure of minimally potentially stable sign patterns. If a signed tree has a unique nonzero vertex, then we call it a **rooted tree**, with the nonzero vertex as the root.

Theorem 3.1. A potentially stable tree sign pattern with exactly one nonzero diagonal entry (i.e., corresponding to a rooted tree) is minimally potentially stable.

Proof. Since the trace of a stable matrix must be negative, a potentially stable sign pattern cannot have all diagonal entries zero. Let $\mathcal{A} = [a_{ij}]$ be a sign pattern satisfying the conditions in the theorem. If $a_{ij} \neq 0$, $i \neq j$, then replacing a_{ij} or a_{ji} by zero breaks the signed tree into two (connected) subtrees, one of which must have all diagonal entries zero and thus is not potentially stable. This proves that $\mathcal{A}$ is minimal. □

Note that the converse of Theorem 3.1 is not true in general; see Example 3.3 in this section. However, more can be said on sign patterns corresponding to a rooted tree. A signed tree with all edges negative is called **skew-symmetric.**

Definition 3.9. A tree sign pattern with exactly one nonzero diagonal entry (which is negative) and having a skew-symmetric signed tree is called a **canonical t.s.p.**

Example 3.2. The following is a canonical t.s.p. that is potentially stable but not sign stable:

For example, the matrix

$$\begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix}.$$

has this sign pattern, but is unstable since it has eigenvalues $\pm i$.

Theorem 3.2. An $n \times n$ tree sign pattern corresponding to a rooted tree is minimally potentially stable if and only if it is a canonical tree sign pattern and has a complete matching.

Proof. Assume that the t.s.p. is canonical and has a complete matching. As in the proof of [JMOV, Corollary 3.7], the t.s.p. allows a properly signed nest. By the theorem of Fisher and Fuller [FF] (see Section 1), the t.s.p. is potentially stable, and is minimal by Theorem 3.1.

For the converse, assume that the t.s.p. is minimally potentially stable. By the proof of Theorem 4.2 in [JMOV], each edge of the t.s.p. is negative. Thus it is a canonical t.s.p., and the potentially stability implies that it has a complete matching. □

Theorem 3.3. Let $\mathcal{P}$ be an $n \times n$ t.s.p whose signed tree is a rooted tree. If $A, B \in \mathcal{P}$, then there exist positive diagonal matrices D_1 and D_2 such that $B = D_1 A D_2$.

Proof. We use induction on n. For $n = 2$, without loss of generality

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{pmatrix},$$

and the assertion follows since

$$\begin{pmatrix} \frac{b_{11}}{a_{11}} & 0 \\ 0 & \frac{b_{21}}{a_{21}} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{b_{12}a_{11}}{b_{11}a_{12}} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & 0 \end{pmatrix}.$$

Assume now that the result is true for any rooted tree sign pattern of order $n-1$, and let $\mathcal{P}$ be an $n \times n$ tree sign pattern whose signed tree is a rooted tree. Let $A, B \in \mathcal{P} = [p_{ij}]$ and assume without loss of generality that $p_{11} \neq 0$ and vertex n of the signed tree corresponding to $\mathcal{P}$ is a leaf. Specifically, suppose that p_{nk} and p_{kn} are the only nonzero entries in the n-th row and column of $\mathcal{P}$. Let $\hat{\mathcal{P}}$ denote the t.s.p. of order $n-1$ obtained by deleting row and column n from

$\mathcal{P}$. Clearly, the signed tree of $\hat{\mathcal{P}}$ is also rooted. By the induction hypothesis, if $\hat{A}, \hat{B} \in \hat{\mathcal{P}}$, then there exist positive diagonal matrices $\hat{D}_1 = \operatorname{diag}(d_1, \cdots, d_{n-1})$ and $\hat{D}_2 = \operatorname{diag}(c_1, \cdots, c_{n-1})$ such that $\hat{B} = \hat{D}_1 \hat{A} \hat{D}_2$. Then

$$\begin{pmatrix} \hat{D}_1 & 0 \\ 0 & \frac{b_{nk}}{a_{nk}c_k} \end{pmatrix} A \begin{pmatrix} \hat{D}_2 & 0 \\ 0 & \frac{b_{kn}}{a_{kn}d_k} \end{pmatrix} = B,$$

which completes the proof. □

Note that if $A, B \in \mathcal{P}$ as in Theorem 3.3, then they are either both singular or both nonsingular. However, the result of Theorem 3.3 is not in general true for a non-rooted tree, even if the sign pattern is minimally potentially stable. For example, the pattern may contain matrices with determinant $+, 0$, and $-$; see, for example, the t.s.p. $\mathcal{A}_{4,5}$ in Section 7. In the following example, all matrices in the given non-rooted t.s.p. $\mathcal{P}$ are nonsingular, but the conclusion of Theorem 3.3 does not hold.

Example 3.3. The pattern $\begin{pmatrix} - & + & + \\ - & + & 0 \\ + & 0 & 0 \end{pmatrix}$ is a minimally potentially stable t.s.p. (see $\mathcal{A}_{3,2}$ in Section 7) whose underlying graph is the non-rooted tree

$$\circ \overset{+}{\text{———}} \ominus \overset{-}{\text{———}} \oplus$$

The matrices

$$A = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 1/2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 1/4 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

are both in the pattern, but B is not a row and column scaling of A.

The following result gives an upper bound for $\delta_{us}(D_1AD_2)$ in terms of entries of A and A^{-1}.

Theorem 3.4. If $B = D_1AD_2$ is a stable $n \times n$ matrix, where D_1 and D_2 are positive diagonal matrices, then

$$\delta_{us}(B) \leq \frac{n|\det(A)|}{\sum_{i,j}|a_{ij}||A_{ji}|},$$

where $A = (a_{ij})$, $A^{-1} = (A_{ji})/det(A)$ and A_{ji} denotes the cofactor of a_{ij}.

Proof. It follows from Lemma 2.6 that

$$(3.1) \quad \delta_{us}(B) \leq \frac{1}{\| B \|_2 \| B^{-1} \|_2} \leq \frac{1}{(\frac{1}{n}\mathrm{tr}(B^TB))^{1/2}(\frac{1}{n}\mathrm{tr}(B^{-T}B^{-1}))^{1/2}},$$

since $(\mathrm{tr}(B^TB))^{1/2} = \| B \|_F \leq \sqrt{n}\| B \|_2$ (see, e.g., [HJ, p. 314-315]). Let $B = (b_{ij})$ and $B^{-1} = (B_{ji})/\det(B)$. Then

$$\left(\frac{1}{n}\mathrm{tr}(B^TB)\right)^{1/2}\left(\frac{1}{n}\mathrm{tr}(B^{-T}B^{-1})\right)^{1/2} = \frac{(\sum_{i,j} b_{ij}^2)^{1/2}(\sum_{i,j} B_{ji}^2)^{1/2}}{n|\det(B)|}.$$

Let $D_1 = \mathrm{diag}(d_1, \cdots, d_n)$ and $D_2 = \mathrm{diag}(c_1, \cdots, c_n)$. Then

$$b_{ij} = d_i a_{ij} c_j, \quad B_{ji} = \frac{d_1 \cdots d_n}{d_i} A_{ji} \frac{c_1 \cdots c_n}{c_j}.$$

In particular, $b_{ij} B_{ji} = a_{ij} A_{ji} (d_1 \cdots d_n)(c_1 \cdots c_n)$. Also,

$$\det(B) = d_1 \cdots d_n c_1 \cdots c_n \det(A).$$

So by the Cauchy-Schwarz Inequality,

$$\left(\frac{1}{n}\mathrm{tr}(B^T B)\right)^{1/2} \left(\frac{1}{n}\mathrm{tr}(B^{-T} B^{-1})\right)^{1/2}$$

$$\geq \frac{\sum_{i,j} |b_{ij} B_{ji}|}{n|\det(B)|}$$

$$= \frac{\sum_{i,j} |a_{ij} A_{ji}|}{n|\det(A)|},$$

and the result follows from (3.1). □

Finally, the following result is used to restrict the magnitudes of entries in finding an approximation to an optimal stable matrix in a given sign pattern; see Section 6.

Theorem 3.5. For a stable matrix A in a minimally potentially stable sign pattern,

$$\delta_{us}(A) \leq \frac{\min\{|a_{ij}| : a_{ij} \neq 0\}}{\max_{i,j}\{|a_{ij}|\}}.$$

Proof. If $a_{ij} \neq 0$, let $B = A - a_{ij}E_{ij}$, where E_{ij} is the matrix with the (i, j) entry 1 and all other entries 0. Since the sign pattern of A is minimally potentially stable, the matrix B is unstable. Thus by Theorem 2.2, $d_{us}(A) \leq \| A - B \|_2 = |a_{ij}|$. On the other hand, $\| A \|_2 \geq \max_{i,j}\{|a_{ij}|\}$. The conclusion follows from these two inequalities. □

§4. An Optimal Stable Matrix of Order Two

We first consider 2×2 minimally potentially stable sign patterns. Recall from Section 3 that the sign pattern $\mathcal{A} = \begin{pmatrix} - & + \\ - & 0 \end{pmatrix}$ is a minimally potentially stable sign pattern (that is also sign stable).

Lemma 4.1. The sign pattern $\mathcal{A} = \begin{pmatrix} - & + \\ - & 0 \end{pmatrix}$ *is the only* 2×2 *minimally potentially stable sign pattern, up to signature and permutation similarity.*

Proof. Let $\mathcal{B}$ be an arbitrary irreducible 2×2 potentially stable sign pattern. Because of irreducibility, $\mathcal{B}$ is neither upper nor lower triangular. Furthermore, $\mathcal{B}$ must have at least one negative diagonal entry, that without loss of generality is in the (1,1) position.

Thus, let $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in \mathcal{B}$ be stable with $b_{11} < 0$. Then $\det B = b_{11}b_{22} - b_{12}b_{21} > 0$. If $b_{22} = 0$, then $b_{12}b_{21} < 0$ and $\mathcal{B} = \mathcal{A}$ (up to signature similarity) is minimally potentially stable. If $b_{22} > 0$, then $\mathcal{A}$ is a potentially stable subpattern of $\mathcal{B}$, making the pattern $\mathcal{B}$ not minimally potentially stable. If $b_{22} < 0$, then setting either b_{12} or b_{21} to zero gives a sign stable subpattern of $\mathcal{B}$, making the pattern $\mathcal{B}$ not minimally potentially stable. □

As $\delta_{us}(A) = \delta_{us}(cA)$ for $c > 0$, without loss of generality, assume that

$$A = \begin{pmatrix} -1 & a \\ -b & 0 \end{pmatrix} \in \mathcal{A},$$

with $a, b > 0$. Under this normalization, we only need to determine a, b so that $\delta_{us}(A) = \delta_{us}(\mathcal{A})$ gives an optimal stable matrix of order two.

From Theorem 2.1, recall that

$$d_{us}(A) = \min\{\sigma_{\min}(A - \mu iI) : \mu \in \mathbf{R}\}.$$

We now determine an analytic formula for

$$f(\mu) = \sigma_{\min}(A - \mu iI),$$

where A is the above 2×2 matrix.

Lemma 4.2. $f^2(\mu) = \frac{1}{2}(c + 2\mu^2 - \sqrt{d + 4\mu^2(a+b)^2})$, *where*

$$c = \operatorname{tr}(A^T A) = 1 + a^2 + b^2, \quad d = c^2 - 4a^2b^2 = (1 + b^2 - a^2)^2 + 4a^2.$$

Proof. Since

$$f^2(\mu) = \text{min eigenvalue of } [(A - \mu iI)^*(A - \mu iI)],$$

and it is easily verified that

$$\operatorname{tr}((A - \mu iI)^*(A - \mu iI)) = c + 2\mu^2,$$

and

$$\det((A-\mu iI)^*(A-\mu iI)) = (ab-\mu^2)^2+\mu^2,$$

it follows that

$f^2(\mu)$

$= \frac{1}{2}(\mathrm{tr}(A-\mu iI)^*(A-\mu iI) - \sqrt{(\mathrm{tr}((A-\mu iI)^*(A-\mu iI))^2 - 4\det((A-\mu iI)^*(A-\mu iI))}$

$= \frac{1}{2}(c+2\mu^2-\sqrt{d+4\mu^2(a+b)^2})$. □

In order to determine $d_{us}(A)$, note that

$$\frac{d(f^2(\mu))}{d\mu} = \frac{1}{2}\left(4\mu - \frac{4\mu(a+b)^2}{\sqrt{d+4\mu^2(a+b)^2}}\right) = 0$$

if and only if either $\mu = 0$ or $(a+b)^4 = d + 4\mu^2(a+b)^2$. We consider two cases.

Case 1. $d < (a+b)^4$.

In this case, $\mu = 0$ and $\mu^2 = \frac{(a+b)^4-d}{4(a+b)^2}$ specify three critical points. Since $f^2(\mu) = f^2(-\mu)$ and $f^2(\mu) \to +\infty$ as $\mu \to \pm\infty$, $\mu = 0$ gives a local maximum and $\mu^2 = \frac{(a+b)^4-d}{4(a+b)^2}$ gives local minima. So

$$d_{us}^2(A) = \frac{1}{2}\left(c + \frac{(a+b)^4-d}{2(a+b)^2} - (a+b)^2\right) = \frac{4ab-1}{4(a+b)^2}.$$

Case 2. $d \geq (a+b)^4$.

In this case, $\mu = 0$ is the only critical point of $f^2(\mu)$. Since $f^2(\mu) \to +\infty$ as $\mu \to \pm\infty$,

$$d_{us}^2(A) = \sigma_{\min}^2(A) = \frac{1}{2}(c - \sqrt{d}).$$

Note that $d_{us}^2(A)$ is continuous at $d = (a+b)^4$.

We summarize these two cases in the following theorem.

Theorem 4.3. Let $A = \begin{pmatrix} -1 & a \\ -b & 0 \end{pmatrix}$, $c = 1 + a^2 + b^2$ *and* $d = c^2 - 4a^2b^2$. *Then*

$$d_{us}^2(A) = \begin{cases} \frac{4ab-1}{4(a+b)^2}, & \text{if } d < (a+b)^4, \\ \frac{1}{2}(c - \sqrt{d}), & \text{if } d \geq (a+b)^4. \end{cases}$$

Corollary 4.4. Let $B = \begin{pmatrix} -1 & t \\ -t & 0 \end{pmatrix}$. *Then*

$$d_{us}^2(B) = \begin{cases} \frac{4t^2-1}{16t^2}, & \text{if } 1 + 4t^2 < 16t^4, \\ \frac{1}{2}(1 + 2t^2 - \sqrt{1+4t^2}), & \text{if } 1 + 4t^2 \geq 16t^4. \end{cases}$$

In the next theorem, it is shown that the value of $\delta_{us}(\mathcal{A})$ occurs at a matrix B as in Corollary 4.4. Note that any matrix A as in Theorem 4.3 is diagonally similar to such a matrix B, but that in general $\delta_{us}(A) \neq \delta_{us}(B)$.

Theorem 4.5. If $A = \begin{pmatrix} -1 & a \\ -b & 0 \end{pmatrix}$ *and* $B = \begin{pmatrix} -1 & t \\ -t & 0 \end{pmatrix}$ *with* $t = \sqrt{ab}$, *then* $d_{us}(A) \leq d_{us}(B)$. *Equality holds if and only if* $A = B$.

Proof. With $c = 1 + a^2 + b^2$ and $d = c^2 - 4a^2b^2$ as before, consider Case 1 above:

$$d < (a+b)^4 \iff c^2 < (a+b)^4 + 4a^2b^2$$

$$\iff 1 + (2 - 4ab)(a+b)^2 < 4ab.$$

Thus the arithmetic geometric mean inequality $(a+b)^2 \geq (2\sqrt{ab})^2 = 4t^2$ gives $d < (a+b)^4$ implies $1 + (2 - 4t^2)4t^2 < 4t^2$, which is equivalent to $1 + 4t^2 < 16t^4$. By Corollary 4.4, if $1 + 4t^2 < 16t^4$, then

$$d_{us}^2(B) = \frac{4t^2 - 1}{16t^2} \geq \frac{4ab - 1}{4(a+b)^2} = d_{us}^2(A),$$

using the arithmetic geometric mean inequality. Clearly, in this case, $d_{us}(A) = d_{us}(B)$ implies that $a = b = t$, i.e., $A = B$.

For Case 2 above, if $d \geq (a+b)^4$ and $1 + 4t^2 \geq 16t^4$, then

$d_{us}^2(A) = \frac{2a^2b^2}{c+\sqrt{d}}$

$\leq \frac{2t^4}{1+2t^2+\sqrt{1+4t^2}}$ since $a^2 + b^2 \geq 2t^2$

$= \frac{1}{2}(1 + t^2 - \sqrt{1+4t^2})$

$= d_{us}^2(B)$, by Corollary 4.4.

Again, in this case, $d_{us}(A) = d_{us}(B)$ implies that $A = B$.

The only case left is when $d \geq (a+b)^4$ and $1 + 4t^2 < 16t^4$, i.e.,

$$d_{us}^2(A) = \frac{1}{2}(c - \sqrt{d}) \text{ and } d_{us}^2(B) = \frac{4t^2 - 1}{16t^2}.$$

The desired inequality $d_{us}^2(A) \le d_{us}^2(B)$ in this case follows from Lemma 4.8 below.

□

We prove Lemma 4.8 indirectly using a result of Byers (see Theorem 2.7) and the notation $c = 1 + a^2 + b^2, d = c^2 - 4a^2b^2$ as before.

Lemma 4.6. If $A = \begin{pmatrix} -1 & a \\ -b & 0 \end{pmatrix}$ *and* $H(\alpha) = \begin{pmatrix} A & -\alpha I \\ \alpha I & -A^T \end{pmatrix}$, *where* $a, b, \alpha > 0$, *then the characteristic polynomial of* $H(\alpha)$ *is* $\lambda^4 + E_2\lambda^2 + E_4 = 0$, *where*

$$E_2 = 2ab + 2\alpha^2 - 1,\ E_4 = \det(H(\alpha)) = \alpha^4 - \alpha^2 c + a^2b^2.$$

Proof. From Theorem 2.12, the characteristic polynomial of $H(\alpha)$ is $\det((\alpha^2 + \lambda^2)I_2 + \lambda(A^T - A) - AA^T)$. Expanding this 2×2 determinant gives the result.

□

Setting $x = -\lambda^2$, $H(\alpha)$ has an eigenvalue with real part zero if and only if $x^2 - E_2x + E_4 = 0$ has a nonnegative real solution (in which case, $\lambda = \pm i\sqrt{x}$ is an eigenvalue of $H(\alpha)$), and this occurs if and only if

$$E_4 \le 0 \ \text{ or } (E_4 > 0, E_2 > 0, \text{ and } E_2^2 \ge 4E_4).$$

However,

$$E_4 > 0 \Longleftrightarrow \alpha^2 > \frac{c+\sqrt{d}}{2}$$

or

$$\alpha^2 < \frac{c-\sqrt{d}}{2}.$$

Also

$$E_2 > 0 \Longleftrightarrow \alpha^2 > \frac{1-2ab}{2}$$

and

$$E_2^2 \geq 4E_4 \Longleftrightarrow \alpha^2 \geq \frac{4ab-1}{4(a+b)^2}.$$

This leads to the following result.

Lemma 4.7. With the notation as in Theorem 4.3,

$$d_{us}^2(A) \leq \frac{c-\sqrt{d}}{2}.$$

Moreover, the strict inequality occurs here if and only if $\frac{1-2ab}{2} < \frac{c-\sqrt{d}}{2}$ *and* $\frac{4ab-1}{4(a+b)^2} < \frac{c-\sqrt{d}}{2}$.

Proof. Since

$$\sigma_{\min}^2(A) = \frac{c-\sqrt{d}}{2},$$

the inequality in this lemma follows from Theorem 2.1.

Assume $d_{us}^2(A) < \sigma_{\min}^2(A)$. Let α be a positive number such that

$$d_{us}^2(A) < \alpha^2 < \sigma_{\min}^2(A).$$

By Theorem 2.7 and the above discussion on the characteristic polynomial of $H(\alpha)$, $H(\alpha)$ has an eigenvalue with real part zero but $E_4 > 0$. Thus we must have $E_2 > 0$ and $E_2^2 \geq 4E_4$, which implies that

$$\max\left(\frac{1-2ab}{2}, \frac{4ab-1}{4(a+b)^2}\right) \leq \alpha^2 < \sigma_{\min}^2(A).$$

For the converse, there must be some positive number α such that

$$\max\left(\frac{1-2ab}{2}, \frac{4ab-1}{4(a+b)^2}\right) < \alpha^2 < \sigma_{\min}^2(A).$$

Following the above discussion on the characteristic polynomial of $H(\alpha)$, we see that $E_4 > 0, E_2 > 0$ and $E_2^2 \geq 4E_4$, which implies that $H(\alpha)$ has an eigenvalue with real part zero. By Theorem 2.7,

$$d_{us}^2(A) \leq \alpha^2 < \sigma_{\min}^2(A).$$

This completes our proof. □

Lemma 4.8. With the notation as in Theorem 4.3 and Corollary 4.4, if $d \geq (a+b)^4$ *and* $1 + 4t^2 < 16t^4$, *then* $d_{us}(A) < d_{us}(B)$.

Proof. Lemma 4.7 can be restated as: $d^2_{us}(A) = \frac{c-\sqrt{d}}{2}$ if and only if either $\frac{1-2ab}{2} \geq d^2_{us}(A)$ or $\frac{4ab-1}{4(a+b)^2} \geq d^2_{us}(A)$.

If $d^2_{us}(A) \leq \frac{4ab-1}{4(a+b)^2}$, then

$$d^2_{us}(B) = \frac{4t^2-1}{4(2t)^2} \geq \frac{4ab-1}{4(a+b)^2} \geq d^2_{us}(A).$$

If $d^2_{us}(B) = d^2_{us}(A)$ in this case, then $A = B$, which implies that $1 + 4t^2 \geq 16t^4$, contradicting the assumption that $1+4t^2 < 16t^4$. Therefore, in this case $d^2_{us}(B) > d^2_{us}(A)$.

If $\frac{4ab-1}{4(a+b)^2} < d^2_{us}(A)$, we must have $\frac{1-2ab}{2} \geq d^2_{us}(A)$. On the other hand, $1+4t^2 < 16t^4$ if and only if

$$\frac{1-2t^2}{2} < \frac{4t^2-1}{16t^2},$$

and $d^2_{us}(B) = \frac{4t^2-1}{16t^2}$. Thus

$$d^2_{us}(A) \leq \frac{1-2ab}{2} < \frac{4t^2-1}{16t^4} = d^2_{us}(B). \quad \square$$

By direct computation,

$\| A \|^2_2 = \max$ eigenvalue of $(A^TA) = \frac{1}{2}(c+\sqrt{d})$

$\geq \frac{1}{2}(1+2t^2+\sqrt{1+4t^2}) = \| B \|^2_2$.

Thus, by Theorem 4.5,

$$\delta_{us}^2(A) \le \delta_{us}^2(B),$$

and consequently it suffices to look for an optimal stable matrix in the sign pattern class $\mathcal{A}$ in the form

$$B = \begin{pmatrix} -1 & t \\ -t & 0 \end{pmatrix}, \text{ with } t > 0.$$

We can restate Corollary 4.4 for the matrix B as

$$(4.1) \quad \delta_{us}^2(B_t) = \begin{cases} \frac{4t^2-1}{8t^2(1+2t^2+\sqrt{1+4t^2})}, & \text{if } t^2 > \frac{1+\sqrt{5}}{8}, \\ \frac{4t^4}{(1+2t^2+\sqrt{1+4t^2})^2}, & \text{if } t^2 \le \frac{1+\sqrt{5}}{8}. \end{cases}$$

Using this formula, we can determine the absolute maximum of $\delta_{us}^2(B_t)$, which leads to the following result.

Theorem 4.9. For the sign pattern $\mathcal{A} = \begin{pmatrix} - & + \\ - & 0 \end{pmatrix}$,

$$\delta_{us}(\mathcal{A}) = \frac{1}{3}\sqrt{\frac{2}{3}} = 0.2721655...$$

Moreover, this optimal value is achieved only at pB *where* $B = \begin{pmatrix} -1 & t_1 \\ -t_1 & 0 \end{pmatrix} \in \mathcal{A}$, *with* $t_1 = \sqrt{3}/2$ *and* $p > 0$.

Proof. Consider first the second part of (4.1). With $x = t^2$ and the function $g(x) = \frac{2x}{1+2x+\sqrt{1+4x}}$, we claim that $g(x)$ is increasing when $x > 0$.

To prove this claim, consider $h(x) = \frac{1}{g(x)}$. Then

$$h'(x) = -\frac{1 + 2x + \sqrt{1+4x}}{2x^2\sqrt{1+4x}} < 0, \ \text{if } x > 0,$$

which proves the claim. Thus, $t_0^2 = \frac{1+\sqrt{5}}{8}$ gives the maximum of $\delta_{us}(B_t)$ for $t^2 \in (0, \frac{1+\sqrt{5}}{8}]$. In order to determine this maximum, notice that $\frac{1+\sqrt{5}}{2}$ is a solution of the quadratic equation $y^2 - y - 1 = 0$. Thus $y = \sqrt{y+1}$, i.e., $\frac{1+\sqrt{5}}{2} = \sqrt{1 + 4t_0^2}$. Substituting this into the denominator of the second part of (4.1) gives

$$\delta_{us}(B_t) \leq \delta_{us}(B_{t_0}) = \frac{1+\sqrt{5}}{7+3\sqrt{5}} = 0.2360679...,$$

if $t^2 \leq \frac{1+\sqrt{5}}{8}$.

Now consider the first part of (4.1). It is clear that $\delta_{us}(B_t) \to 0$ as $t \to \infty$. So to determine the maximum of $\delta_{us}^2(B_t)$ for $t^2 \geq \frac{1+\sqrt{5}}{8}$, let $x = t^2$ and consider the minimum of the function

$$p(x) = \frac{8x(1 + 2x + \sqrt{1+4x})}{4x - 1},$$

for $x > \frac{1+\sqrt{5}}{8}$. Taking the derivative, we obtain

$$p'(x) = 0$$

$$\Longleftrightarrow 4(8x(1 + 2x + \sqrt{1+4x})) = 8(4x-1)(1 + 2x + \sqrt{1+4x} + x(2 + \tfrac{2}{\sqrt{1+4x}}))$$

$$\Longleftrightarrow 6x + 1 - 8x^2 = \sqrt{1+4x}(8x^2 - 4x - 1)$$

$$\Longrightarrow 64x^3 - 64x^2 + 8x + 3 = 0, \ \text{by squaring the above.}$$

The only solution of the last equation that is also greater than $\frac{1+\sqrt{5}}{8}$ is $3/4$. Check that, indeed, $p'(3/4) = 0$.

Thus with $t_1 = \sqrt{3}/2$, the first part of (4.1) gives

$$\delta_{us}(B_{t_1}) = \frac{1}{3}\sqrt{\frac{2}{3}} = 0.2721655...$$

Since $\delta_{us}(B_{t_0}) < \delta_{us}(B_{t_1})$ and the fact that the relative distance is invariant under multiplication by a positive constant, our proof is completed. □

Comment: The fundamental computation in the derivation of Theorem 4.9 for $n = 2$ is the explicit analytic solution of a quadratic equation. Thus, in principle, it is possible to compute an optimal stable matrix for $n = 3$ or 4 in any minimally potentially stable sign pattern, although the computation would be more difficult. In order to use Theorem 2.7 to prove a result analogous to Theorem 4.5, we have to obtain necessary and sufficient conditions for a cubic or a quartic polynomial (with real coefficients) to have a nonnegative solution. Clearly, for $n \geq 5$, it is not possible to use this approach. Therefore, for $n \geq 3$, we turn our attention to "good" stable matrices rather than optimal stable matrices. We first give some examples in the next section.

§5. Examples of Good Stable Matrices

Let $\mathcal{A}$ be a potentially stable sign pattern with $\delta_{us}(\mathcal{A}) < M$. A stable matrix $A \in \mathcal{A}$ is called a **good** stable matrix if there is a positive number m such that $m < \delta_{us}(A)$ and the ratio m/M is not too small (e.g., in the following, we use $m/M > 0.01$).

Example 5.1. Consider the $n \times n$ tridiagonal sign stable pattern

$$\mathcal{A}_n = \begin{pmatrix} - & + & & & & \\ - & 0 & + & & 0 & \\ & - & 0 & + & & \\ & & - & 0 & \ddots & \\ & & & \ddots & \ddots & \ddots \\ & 0 & & & \ddots & + \\ & & & & - & 0 \end{pmatrix}.$$

In *Example 2.3*, it was shown that

$$\delta_{us}(\mathcal{A}_n) < \frac{1}{n}.$$

For $n \geq 3$, let $A_n \in \mathcal{A}_n$ be

$$A_n = \begin{pmatrix} -1 & a_1 & & & & \\ -a_1 & 0 & a_2 & & 0 & \\ & -a_2 & 0 & a_3 & & \\ & & -a_3 & 0 & \ddots & \\ & & & \ddots & \ddots & \ddots & \\ & 0 & & & \ddots & & a_{n-1} \\ & & & & & -a_{n-1} & 0 \end{pmatrix}$$

with $a_j = 0.8 - (j-1)s_n$ for $j = 1, 2, \cdots, n-1$, where $s_n = 0.6/(n-2)$. An implementation of Algorithm 2.1 in MATLAB (see the Appendix) shows that, for $3 \le n \le 550$,

$$\delta_{us}(A_n) > \frac{1}{8n}.$$

We believe that this bound holds for all n, but we have not been able to prove it. Based on these computations, A_n is a good stable matrix in the sign pattern $\mathcal{A}_n$. Furthermore, this result shows that $\delta_{us}(\mathcal{A}_n) = O(\frac{1}{n})$. Note: $\frac{1}{8n}$ is not a tight bound for $\delta_{us}(\mathcal{A}_n)$. For example,

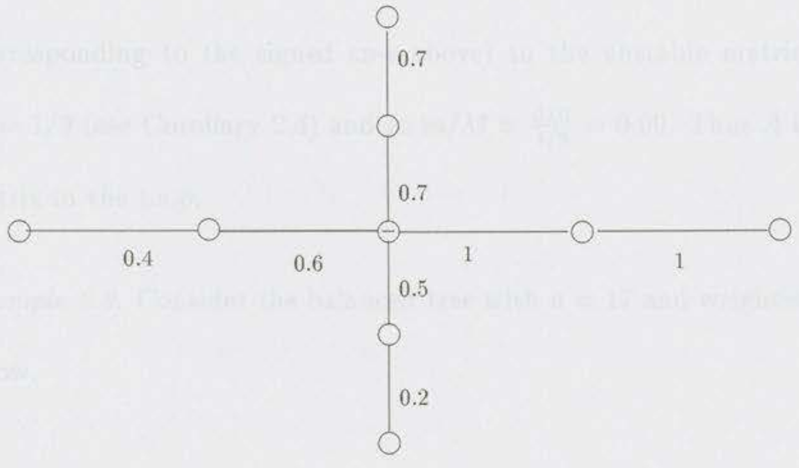
$$\delta_{us}(\mathcal{A}_5) > 0.08305 > \frac{1}{8 \times 5} = 0.025,$$

see Example 6.3.

Note that, in Example 5.1, the (i,j) and (j,i) entries of $\mathcal{A}_n$ have the same magnitude for all $i \neq j$ and the parameters a_i are evenly decreasing from 0.8 to 0.2. If $a_{ij} = -a_{ji} = w > 0$, we say that w is the **weight** of the matrix on the edge (i,j) of the underlying graph.

In the remainder of this section, various star shaped rooted trees are considered. Such a tree is said to be **balanced** if the number of edges on each branch from the root is equal. We describe a heuristic way for assigning weights to the edges of such a tree, or equivalently, assigning values to the entries of a matrix with its graph.

Example 5.2. Consider the balanced tree with $n = 9$ and the weighting as illustrated below.



Taking the root as vertex 1, this weighting is equivalent to assigning the entries of the matrix $A = [a_{ij}]$ as

$$a_{11} = -1, a_{12} = -a_{21} = 1, a_{13} = -a_{31} = 0.7, a_{14} = -a_{41} = 0.6,$$

$$a_{15} = -a_{51} = 0.5, a_{26} = -a_{62} = 1, a_{37} = -a_{73} = 0.7, a_{48} = -a_{84} = 0.4,$$

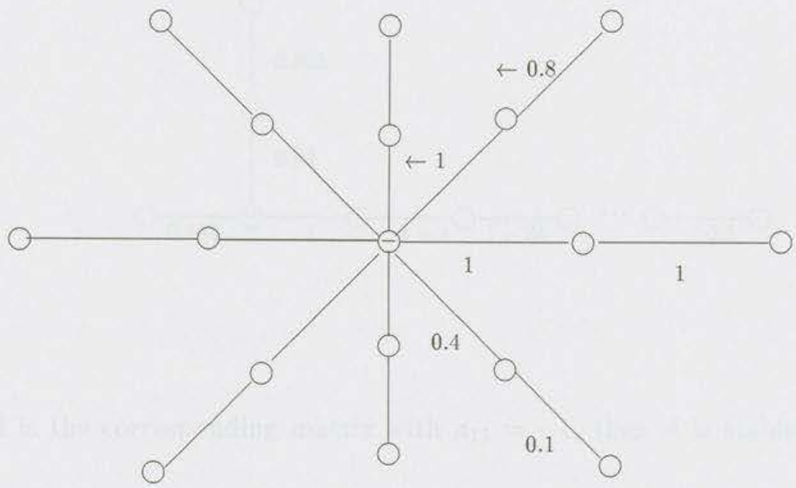
$$a_{59} = -a_{95} = 0.2,$$

with all other entries zero. In this case, numerical computation using Algorithm 2.1 gives

$$0.01 < \delta_{us}(A) < 0.015.$$

Since $\| A \|_2 > 1$ (as in Example 2.1) the relative distance from the sign pattern (corresponding to the signed tree above) to the unstable matrices is less than $M = 1/9$ (see Corollary 2.4) and so $m/M = \frac{0.01}{1/9} = 0.09$. Thus A is a good stable matrix in the t.s.p.

Example 5.3. Consider the balanced tree with $n = 17$ and weighting as illustrated below.

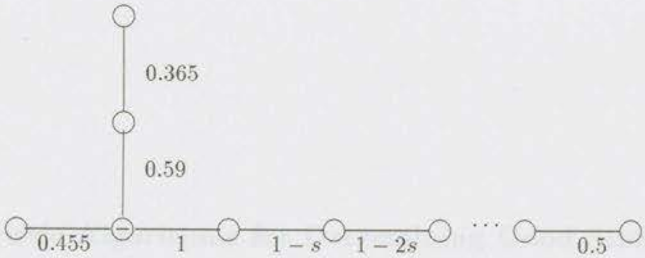


First assign -1 to the root vertex, and weight 1 to the edges on one branch, as illustrated above. The heuristic used for the remaining edges is that weights are assigned to the edges adjacent to the root (counterclockwise), then those a distance 2 away from the root. More precisely, the weights on the inner circle centered at the root are $\{1, 1, 1, 1, 0.9, 0.6, 0.4, 0.4\}$ counterclockwise; the weights on the outer circle centered at the root are $\{1, 0.8, 0.7, 0.5, 0.4, 0.3, 0.2, 0.1\}$ counterclockwise. If A is the corresponding matrix, then it is stable and numerical computation gives

$$0.0007501 < \delta_{us}(A) \leq 0.0007502.$$

Since $\| A \|_2 > 1$, the relative distance from the sign pattern (corresponding to the signed tree above) to the unstable matrices is less than $M = 1/17$ and so $m/M = \frac{0.0007501}{1/17} = 0.0127517$. Thus A is a good stable matrix in the t.s.p.

Example 5.4. Consider the following very unbalanced rooted tree with $n = 14$ and the weighting below, where $s = 0.5/9$.



If A is the corresponding matrix with $a_{11} = -1$, then A is stable and numerical computation gives

$$0.01 < \delta_{us}(A) < 0.015.$$

Since $\| A \|_2 > 1$, the relative distance from the sign pattern (corresponding to the signed tree above) to the unstable matrices is less than $M = 1/14$, and so $m/M = \frac{0.01}{1/14} = 0.14$. Thus, A is a good stable matrix in the t.s.p.

For most potentially stable sign patterns $\mathcal{A}$, we do not have adequate bounds m and M (for $\delta_{us}(\mathcal{A})$) so that $m/M > 0.01$. In such a case, the definition of a good stable matrix given above is not applicable. Instead, we use the term "good stable matrix" somewhat loosely; see Section 6.

§6. Heuristic Algorithms for Determining Good Stable Matrices

In this section, we consider the determination of a good stable matrix in a minimally potentially stable t.s.p. that contains a matrix with a properly signed nest. Since the construction of a good stable matrix with a given sign pattern is dependent upon the particular structure of the sign pattern, we cannot expect that a general algorithm exists for constructing a good stable matrix. However, there are still some general approaches for a minimally potentially stable t.s.p. containing a matrix with a properly signed nest. We begin by introducing some new terminology, which extends that of Section 5.

Definition 6.1. A matrix A is **weight symmetric** *if* $|a_{ij}| = |a_{ji}|$ *for all* $i \neq j$. *Furthermore,* $|a_{ij}| = |a_{ji}| = w > 0$ *if and only if the edge* (i, j) *in the graph of A has* **weight** w *and is denoted as follows:*

Similarly, $|a_{ii}| = w > 0$ *if and only if the vertex i of the graph of A has* **weight** w. *A matrix A is* **normalized** *if* $a_{11} = -1$. *Similarly, a labeled weighted tree is* **normalized** *if vertex 1 has negative sign and weight 1.*

Note that all the example matrices in Section 5 have $a_{ij} = -a_{ji}$ and $a_{11} = -1$ and thus are weight symmetric and normalized.

We adopt the following conventions throughout the rest of the thesis.

(1) For a given signed tree, we always choose a corresponding t.s.p. with all of its strictly upper triangular entries nonnegative.

(2) If a signed tree is labeled with vertex 1 negative, we always consider a normalized matrix in the corresponding sign pattern (i.e., $a_{11} = -1$). Consequently the weight 1 may be omitted on vertex 1 in the weighted tree.

If a matrix in a t.s.p. is given, there is a unique way to label the signed tree compatible with the matrix. If the matrix has a leading properly signed nest, then vertex 1 must be a negative vertex. Moreover, if the matrix is weight symmetric, there is a unique way to assign the weights on the labeled signed tree such that the matrix has this weighting. Conversely, given a normalized labeling and weighting of a signed tree, under the above conventions, it corresponds to a unique normalized and weight symmetric matrix in the sign pattern. Therefore, under our conventions, weight symmetric and normalized matrices in a t.s.p. are in one-to-one correspondence with normalized, labeled, and weighted signed trees.

Example 6.1. The singular matrix

$$\begin{pmatrix} -1 & 0.5 & 1 \\ -0.5 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

is weight symmetric and normalized, and corresponds to the following normalized, labeled, weighted signed tree:

The heuristics used in Section 5 to construct good stable matrices in a given minimally potentially stable t.s.p. (containing a matrix with a properly signed nest) result in matrices with the following attributes:

(1) They are weight symmetric and normalized.

(2) Each nonzero entry has magnitude in the range $[0.1, 1]$.

In a fixed t.s.p., a matrix satisfying (1) and (2) above is said to be a **candidate matrix**.

We now focus attention on the construction of good stable matrices that are candidate matrices in a fixed minimally potentially stable t.s.p. $\mathcal{A}$. Note that, in general, the existence of a stable candidate matrix has not been proved. However, in practical numerical computation, we have never had difficulty finding stable candidate matrices. The following procedure specifies the main ideas of the heuristics developed in this section. First fix a labeling of a minimally potentially stable t.s.p. $\mathcal{A}$ with vertex 1 having negative sign, and then proceed as follows to determine a sequence of stable candidate matrices $A^{(k)} \in \mathcal{A}$.

(i) Consider the set of all candidate matrices with weights from

$$\{0.1, 0.2, \cdots, 0.9, 1\}$$

that are stable. Find a matrix in this set with maximal $\delta_{us}(A)$ and denote it by $A^{(1)} = [a_{ij}^{(1)}]$. In the case that this set is empty, we broaden the set of weights, for example, to $\{0.10, 0.11, 0.12, \cdots, 0.99, 1\}$, until a stable candidate matrix is found; the sequential steps (ii), (iii), ... have to be modified accordingly. We continue to the following steps only when this set is nonempty.

(ii) Consider the set of all stable candidate matrices with the (i, j) entry equal to $a_{ij}^{(1)} \pm e_{ij}^{(1)}$, where $e_{ij}^{(1)} \in \{0, 0.01, 0.02, \cdots, 0.09\}$ and $a_{ij}^{(1)} \neq 0$. Find a matrix in this new set with maximal $\delta_{us}(A)$, and denote it by $A^{(2)} = [a_{ij}^{(2)}]$. Clearly,

$$\delta_{us}(A^{(1)}) \leq \delta_{us}(A^{(2)}).$$

(iii) Consider the set of all stable candidate matrices with (i, j) entry equal to $a_{ij}^{(2)} \pm e_{ij}^{(2)}$, where $e_{ij}^{(2)} \in \{0, 0.001, 0.002, \cdots, 0.009\}$ and $a_{ij}^{(2)} \neq 0$. Find a matrix in this new set with maximal $\delta_{us}(A)$, and denote it by $A^{(3)} = [a_{ij}^{(3)}]$. Clearly,

$$\delta_{us}(A^{(2)}) \leq \delta_{us}(A^{(3)}).$$

Continue this procedure until no significant increase in $\delta_{us}(A^{(k)})$ is achieved.

The theoretical basis of this systematic search is Theorem 2.10. In addition, it is implicitly assumed that $\delta_{us}(A)$ is smooth at an optimal stable matrix with respect to A in a minimally potentially stable t.s.p. (while this has not been proved in general, it is indeed supported by the 2×2 minimally potentially stable sign pattern in Section 4, and our numerical experience). However, there are two difficulties with the above procedure. Firstly, when is the procedure terminated? Secondly, it is too difficult to find the maximal $\delta_{us}(A)$ in a finite set (when it is large) and it is even too costly to determine the exact value of $\delta_{us}(A)$ for a stable matrix A. In order to be more practical, we lower our expectations and concentrate on finding a good stable matrix in each finite set. The following algorithm implements the above procedure and addresses these two difficulties. While the following algorithm is not necessarily optimal, it is a practical working algorithm.

For a finite set $\mathcal{S}$ of stable matrices from the fixed t.s.p. $\mathcal{A}$, denote $\max\{\delta_{us}(B) : B \in \mathcal{S}\}$ by $\delta_{us}(\mathcal{S})$. When the set $\mathcal{S}$ is large, the computation of the exact value of $\delta_{us}(\mathcal{S})$ remains difficult. As an alternative to computing $\delta_{us}(\mathcal{S})$, we find a small interval containing this value. A matrix $A \in \mathcal{S}$ is said to be a **good stable matrix in** $\mathcal{S}$ if we can find two intervals of length 10^{-r} and 10^{-r-1} that contain both $\delta_{us}(A)$ and $\delta_{us}(\mathcal{S})$. This leads to the following definition.

Definition 6.2. Let $\mathcal{S}$ be a finite set of $n \times n$ stable matrices. Let $b = 0.d_1 d_2 \cdots d_r$ be a positive number, where the d_j's are integers between 0 and 9 and $d_r > 0$. Let

$A \in \mathcal{S}$ be such that $b < \delta_{us}(A)$ and $\delta_{us}(\mathcal{S}) \le b + 10^{-r}$. If also $b + p10^{-r-1} < \delta_{us}(A)$ for some integer p between 0 and 9, and $\delta_{us}(\mathcal{S}) \le b + (p+1)10^{-r-1}$, then A is a **good stable matrix in $\mathcal{S}$**.

Note that a good stable matrix A in $\mathcal{S}$ may not be equal to $\delta_{us}(\mathcal{S})$. However,

$$b < \delta_{us}(A) \le \delta_{us}(\mathcal{S}) \le b + 10^{-r},$$

and

$$b + p10^{-r-1} < \delta_{us}(A) \le \delta_{us}(\mathcal{S}) \le b + (p+1)10^{-r-1}.$$

In the following algorithm, it is assumed that a fixed $n \times n$ minimally potentially stable t.s.p. $\mathcal{A}$ with a properly signed nest is given. Also, it is assumed that an arbitrary labeling of the signed tree with negative sign and weight 1 on vertex 1 is given. We then assign weights to the labelled signed tree to find a good stable candidate matrix.

Step 1. Let $\mathcal{S}_1$ be the set of all stable candidate matrices with weights from $\{0.1, 0.2, \cdots, 0.9, 1\}$, and proceed as follows to find a sequence of matrices leading to a good stable matrix in $\mathcal{S}_1$.

Step 1.1. (Initialization) Find $1 \le r \le 10$ such that there exists $A \in \mathcal{S}_1$ with $10^{-r} < \delta_{us}(A)$, but $\delta_{us}(\mathcal{S}_1) \le 10^{-r+1}$. Note that it is not necessarily the case that

$\delta_{us}(A) = \delta_{us}(\mathcal{S}_1)$, since the first matrix $A \in \mathcal{S}_1$ satisfying the above conditions is selected. However,

$$10^{-r} < \delta_{us}(A) \leq \delta_{us}(\mathcal{S}_1) \leq 10^{-r+1}.$$

Algorithm 6.1.1 implements Step 1.1. If no such r exists, then the search for a good stable matrix in the sign pattern $\mathcal{A}$ terminates.

Step 1.2. If Step 1.1 finds a value 10^{-r} with $1 \leq r \leq 10$, we refine the bound on $\delta_{us}(\mathcal{S}_1)$ within $(10^{-r}, 10^{-r+1}]$. Algorithm 6.1.2 determines a matrix $A \in \mathcal{S}_1$ (possibly different from the A determined in Step 1.1) and a value j_1 $(1 \leq j_1 \leq 9)$ such that $j_1 10^{-r} < \delta_{us}(A)$ and $\delta_{us}(\mathcal{S}_1) \leq (j_1+1)10^{-r}$. Thus,

$$j_1 10^{-r} < \delta_{us}(A) \leq \delta_{us}(\mathcal{S}_1) \leq (j_1+1)10^{-r}.$$

Step 1.3. Given that Step 1.2 has found a value $j_1 10^{-r}$, we further refine the bound on $\delta_{us}(\mathcal{S}_1)$ within $(j_1 10^{-r}, (j_1+1)10^{-r}]$. Algorithm 6.1.3 finds a matrix $A \in \mathcal{S}_1$ and a value j_2 $(0 \leq j_2 \leq 9)$ such that $j_1 10^{-r} + j_2 10^{-r-1} < \delta_{us}(A)$ and $\delta_{us}(\mathcal{S}_1) \leq j_1 10^{-r} + (j_2+1)10^{-r-1}$. Thus

$$j_1 10^{-r} + j_2 10^{-r-1} < \delta_{us}(A) \leq \delta_{us}(\mathcal{S}_1) \leq j_1 10^{-r} + (j_2+1)10^{-r-1}.$$

If this matrix A is the same as that determined in Step 1.2, then Step 1 terminates and, by Definition 6.2, A is a good stable matrix in $\mathcal{S}_1$.

Step 1.4. If Step 1.3 does not terminate, we continue to further refine the bound on $\delta_{us}(\mathcal{S}_1)$ within $(j_1 10^{-r} + j_2 10^{-r-1}, j_1 10^{-r} + (j_2+1)10^{-r-1}]$; see Algorithm 6.1.4.

The procedure in Step 1 continues until the same matrix A is determined in two consecutive steps, say Step 1.k and 1.(k+1), for some $k \geq 2$. Then, by Definition 6.2, A is a good stable matrix in $\mathcal{S}_1$ with

$$\begin{aligned} & j_1 10^{-r} + j_2 10^{-r-1} + \cdots + j_k 10^{-r-k+1} \\ & < \delta_{us}(A) \leq \delta_{us}(\mathcal{S}_1) \\ & \leq j_1 10^{-r} + j_2 10^{-r-1} + \cdots + (j_k + 1) 10^{-r-k+1}. \end{aligned}$$

We now state the algorithms for Step 1. Given a value t $(0 < t \leq 1)$ and a finite set $\mathcal{S}$ of stable matrices, the following function uses Corollary 2.8 to attempt to find a matrix $A \in \mathcal{S}$ for which $\delta_{us}(A) > t$ (see Algorithm 2.1 in Section 2). This function returns either such a stable matrix A or, if none is found, the zero matrix. Notice that, in a fixed t.s.p., this function is implemented by a special indexing of the set $\mathcal{S}$ in the sign pattern (see Example 6.2).

function **Search**$(t, \mathcal{S})$

$count = 0$;

For each $A \in \mathcal{S}$

$$HH = \begin{pmatrix} A & -t\| A \|_2 I_n \\ t\| A \|_2 I_n & -A^T \end{pmatrix};$$

eiv = eigenvalues of HH;

For $p = 1$ to $2n$

If (abs(real part(eiv(p))) $< 10^{-10}$)

count = 1;

break;

End if;

End for;

If (*count* = 0) // $\delta_{us}(A) > t$

return A;

End if;

End for;

return zeros(n, n);

Algorithm 6.1.1.

count[11] = 0;

For $r = 1$ to 10

$A^{(1)} = \text{Search}(10^{-r}, \mathcal{S}_1)$;

If ($A^{(1)} \neq \text{zeros}(n, n)$)

```
        // there does exist an A ∈ S_1 with δ_us(A) > 10^{-r}
        count[11] = 10^{-r};
        break;
    End if;
End for;
If ( count[11] = 0 )
    print ("It is likely that δ_us(A) ≤ 10^{-9}.");
    return; // The entire search algorithm terminates!
Else
    // Continue the search for a good stable matrix in S_1
    // by refining the interval (count[11], count[11] * 10]
    Start Algorithm 6.1.2;
End if;
```

Algorithm 6.1.2. Given the result of Algorithm 6.1.1.

```
For j = 9 to 1
    B = Search(count[11] * j, S_1);
    If ( B ≠ zeros(n, n) )
        A^(1) = B;
        count[12] = count[11] * j;
```

break;

End if;

End for;

// Continue the search for a good stable matrix in $\mathcal{S}_1$

// by refining the interval $(count[12], count[12] + count[11]]$

Start Algorithm 6.1.3;

Algorithm 6.1.3. Given the result of Algorithm 6.1.2.

$C_1 = A^{(1)}$;

For $j = 9$ to 0

$C_2 = \text{Search}(count[12] + count[11] * j/10, \mathcal{S}_1)$;

If ($C_2 \neq \text{zeros}(n, n)$)

$A^{(1)} = C_2$;

$count[13] = count[12] + count[11] * j/10$;

break;

End if;

End for

If ($C_1 = A^{(1)}$)

// C_1 is not changed and so $A^{(1)}$ is a good stable matrix in

// the set $\mathcal{S}_1$, stop!

return;

Else

// Continue the search for a good stable matrix in $\mathcal{S}_1$,

// by refining the smaller interval $(count[13], count[13] + count[11]/10]$

Start Algorithm 6.1.4;

End if;

Algorithm 6.1.4. Given the result of Algorithm 6.1.3, define $count[14] = count[13] + count[11] * j/100$ and proceed as above.

Example 6.2. Consider the tridiagonal 5×5 minimally potentially stable t.s.p. $\mathcal{A}_5$ given in Example 2.3. We use the above algorithms to find a good stable matrix in the set $\mathcal{S}_1$ of all candidate matrices with weights from $\{0.1, 0.2, \cdots, 0.9, 1\}$ in $\mathcal{A}_5$. Note that, since this t.s.p. is sign stable, all candidate matrices are stable. The MATLAB function **Search1** below is an implementation of the general function **Search** by a special indexing of $\mathcal{S}_1$ for this particular t.s.p.

function $[A] =$ **Search1**(t);

$A = \text{zeros}(5, 5)$;

$A(1, 1) = -1$;

for $j = 1 : 10$

$A(1, 2) = 0.1 * j$;

for $k = 1:10$

$A(2,3) = 0.1 * k;$

for $h = 1:10$

$A(3,4) = 0.1 * h;$

for $g = 1:10$

$count = 0;$

$A(4,5) = 0.1 * g;$

$A(2,1) = -A(1,2);$

$A(3,2) = -A(2,3);$

$A(4,3) = -A(3,4);$

$A(5,4) = -A(4,5);$

$HH = [A, -\text{norm}(A) * t * \text{eye}(5); \text{norm}(A) * t * \text{eye}(5), -A'];$

$eiv = \text{eig}(HH);$

for $p = 1:10$

if ($\text{abs}(\text{real}(eiv(p))) < 10^{-10}$)

$count = 1;$

break;

end;

end;

if ($count = 0$)

```
                disp('Succeeded in first search!');
                return;
            end;
        end;
    end;
  end;
end;
```

Using Algorithm 6.1.1 on this example, the first nonzero matrix $A^{(1)}$ that is returned by the function **Search1** occurs when $r = 2$; that is, $count[11] = 0.01$. Continuing to refine the search in $(0.01, 0.1]$ using Algorithm 6.1.2 leads to $count[12] = 0.07$. Continuing to refine the search in $(0.07, 0.08]$ using Algorithm 6.1.3 leads to $count[13] = 0.077$. Further refinement of the search in $(0.077, 0.078]$ using Algorithm 6.1.4 leads to $count[14] = 0.0779$. In these last two refinements, the matrix

$$A^{(1)} = \begin{pmatrix} -1 & 0.9 & 0 & 0 & 0 \\ -0.9 & 0 & 0.7 & 0 & 0 \\ 0 & -0.7 & 0 & 0.6 & 0 \\ 0 & 0 & -0.6 & 0 & 0.4 \\ 0 & 0 & 0 & -0.4 & 0 \end{pmatrix}$$

is unchanged, and thus $k = 3$ and $A^{(1)}$ is a good stable matrix in $\mathcal{S}_1$, with $0.0779 < \delta_{us}(A^{(1)}) \leq \delta_{us}(\mathcal{S}_1) \leq 0.0780$.

Step 2. This step is parallel to Step 1. Assume $A^{(1)} = [a_{ij}^{(1)}]$ is obtained from Step 1. Let $\mathcal{S}_2$ be the set of all stable candidate matrices in $\mathcal{A}$ with (i, j) entry $a_{ij}^{(1)} \pm e_{ij}^{(1)}$ if $a_{ij}^{(1)} \neq 0$, in which $e_{ij}^{(1)} \in \{0, 0.01, 0.02, \cdots, 0.09\}$. Proceed as follows to find a good stable matrix in $\mathcal{S}_2$. Note that, since we cannot exclude the existence of a stable matrix A in $\mathcal{S}_2$ with $\delta_{us}(A) > 10^{-r+1}$, where the integer r is taken from Step 1.1, we have to do an initialization step for the new set $\mathcal{S}_2$.

Step 2.1. (Initialization) With $k \geq 2$ and $r \in \{1, 2, \cdots, 10\}$ from the result of Step 1, find p such that $1 \leq p \leq r$ and there exists $A^{(2)} \in \mathcal{S}_2$ with

$$10^{-p} < \delta_{us}(A^{(2)}) \leq \delta_{us}(\mathcal{S}_2) \leq 10^{-p+1}.$$

Note that it is not necessarily the case that $\delta_{us}(A^{(2)}) = \delta_{us}(\mathcal{S}_2)$, since the first matrix $A \in \mathcal{S}_2$ satisfying the above condition is selected. Algorithm 6.2.1 implements Step 2.1.

Step 2.2. Given the integer p $(1 \leq p \leq r)$, we refine the bound on $\delta_{us}(\mathcal{S}_2)$ within $(10^{-p}, 10^{-p+1}]$. Algorithm 6.2.2 determines a matrix $A \in \mathcal{S}_2$ and a value i_1 $(1 \leq i_1 \leq 9)$ such that $i_1 10^{-p} < \delta_{us}(A)$ and $\delta_{us}(\mathcal{S}_2) \leq (i_1 + 1)10^{-p}$. Thus,

$$i_1 10^{-p} < \delta_{us}(A) \leq \delta_{us}(\mathcal{S}_2) \leq (i_1 + 1)10^{-p}.$$

Step 2.3. Given that Step 2.2 has found a value $i_1 10^{-p}$, we further refine the bound on $\delta_{us}(\mathcal{S}_2)$ within $(i_1 10^{-p}, (i_1+1)10^{-p}]$. Algorithm 6.2.3 finds a matrix $A \in \mathcal{S}_2$ and a value i_2 $(0 \le i_2 \le 9)$ such that $i_1 10^{-p} + i_2 10^{-p-1} < \delta_{us}(A)$ and $\delta_{us}(\mathcal{S}_2) \le i_1 10^{-p} + (i_2+1)10^{-p-1}$. Thus

$$i_1 10^{-p} + i_2 10^{-p-1} < \delta_{us}(A) \le \delta_{us}(\mathcal{S}_2) \le i_1 10^{-p} + (i_2+1)10^{-p-1}.$$

If this matrix A is the same as that determined in Step 2.2, then Step 2 terminates and, by Definition 6.2, A is a good stable matrix in $\mathcal{S}_2$.

Step 2.4. If Step 2.3 does not terminate, we continue to further refine the bound on $\delta_{us}(\mathcal{S}_2)$ within the interval $(i_1 10^{-p} + i_2 10^{-p-1}, i_1 10^{-p} + (i_2+1)10^{-p-1}]$; see Algorithm 6.2.4.

The procedure in Step 2 continues until the same matrix A is determined in two consecutive steps, say Step 2.q and 2.(q+1), for some $q \ge 2$. Then, by Definition 6.2, A is a good stable matrix in $\mathcal{S}_2$ with

$$\begin{aligned}
&i_1 10^{-p} + i_2 10^{-p-1} + \cdots + i_q 10^{-p-q+1} \\
&< \delta_{us}(A) \le \delta_{us}(\mathcal{S}_2) \\
&\le i_1 10^{-p} + i_2 10^{-p-1} + \cdots + (i_q+1)10^{-p-q+1}.
\end{aligned}$$

We now state the algorithms for Step 2.

Algorithm 6.2.1.

$count[21] = 0;$

For $p = 1$ to r

$A^{(2)} = \text{Search}(10^{-p}, \mathcal{S}_2);$

If ($A^{(2)} \neq \text{zeros}(n, n)$)

// The first time encountering a p such that there exists a $A \in \mathcal{S}_2$

// so that $\delta_{us}(A) > 10^{-p}$

$count[21] = 10^{-p};$

break;

End if;

End for;

Start Algorithm 6.2.2;

Algorithm 6.2.2. Given the result of Algorithm 6.2.1.

For $j = 9$ to 1

$B = \text{Search}(count[21] * j, \mathcal{S}_2);$

If ($B \neq \text{zeros}(n, n)$)

$A^{(2)} = B;$

$count[22] = count[21] * j;$

break;

```
    End if;
End for;
Start Algorithm 6.2.3;
```

Algorithm 6.2.3. Given the result of Algorithm 6.2.2.

```
C_1 = A^(2);
For j = 9 to 0
    C_2 = Search(count[22] + count[21] * j/10, S_2);
    If ( C_2 ≠ zeros(n, n) )
        A^(2) = C_2;
        count[23] = count[22] + count[21] * j/10;
        break;
    End if;
End for;
If ( C_1 = A^(2) )
    // C_1 is not changed and so A^(2) is a good stable matrix in
    // the set S_2, stop!
    return;
Else
    // Continue the search for a good stable matrix in S_2
```

// by refining the interval $[count[23], count[23] + count[21]/10)$

Start Algorithm 6.2.4;

End if;

Algorithm 6.2.4. Given the result of Algorithm 6.2.3, define $count[24] = count[23] + count[21] * j/100$ and proceed as above.

If $count[11] = 10^{-r}$ and $r \geq 3$, we continue with Step 3 (similar to Step 2), but with $e_{ij}^{(2)} \in \{0, 0.001, 0.002, \cdots, 0.009\}$, and continue until the completion of Step r.

Example 6.3. Continue the computations of Example 6.2 for the tridiagonal 5×5 minimally potentially stable t.s.p. $\mathcal{A}_5$. Let $A^{(1)} = [a_{ij}^{(1)}]$ be the good stable matrix of $\mathcal{S}_1$ obtained from Example 6.2. Let $\mathcal{S}_2$ be the set of all stable candidate matrices with the (i, j) entry $a_{ij}^{(1)} \pm e_{ij}^{(1)}$, where $e_{ij}^{(1)} \in \{0, 0.01, 0.02, \cdots, 0.09\}$ and $a_{ij}^{(1)} \neq 0$. The MATLAB function **Search2** below is an implementation of the general function **Search** by a special indexing of $\mathcal{S}_2$.

function $[B] =$ **Search2**(t, A)

$B = \text{zeros}(5, 5);$

$B(1, 1) = -1;$

for $j = 0 : 9$

$B(1,2) = A(1,2) \pm 0.01 * j;$

for $k = 0:9$

$B(2,3) = A(2,3) \pm 0.01 * k;$

for $h = 0:9$

$B(3,4) = A(3,4) \pm 0.01 * h;$

for $g = 0:9$

$B(4,5) = A(4,5) \pm 0.01 * g;$

$count = 0;$

$B(2,1) = -B(1,2);$

$B(3,2) = -B(2,3);$

$B(4,3) = -B(3,4);$

$B(5,4) = -B(4,5);$

$HH = [B, -\text{norm}(B) * t * \text{eye}(5); \text{norm}(B) * t * \text{eye}(5), -B'];$

$eiv = \text{eig}(HH);$

for $p = 1:10$

if ($\text{abs}(\text{real}(eiv(p))) < 10^{-10}$)

$count = 1;$

break;

end;

end;

```
        if ( count = 0 )
          disp('Succeeded in second search!');
          return;
        end;
      end;
    end;
  end;
end;
```

Recall that the stopping point in Example 6.2 is $count[14] = 0.0779$. Algorithm 6.2.1 returns $count[21] = 0.01 = count[11]$, Algorithm 6.2.2 returns $count[22] = 0.08$, and Algorithm 6.2.3 returns $count[23] = 0.083$, Algorithm 6.2.4 returns $count[24] = 0.0830 = count[23]$, Algorithm 6.2.5 returns $count[25] = 0.08305$. In these last two refinements, the matrix

$$A^{(2)} = \begin{pmatrix} -1 & 0.95 & 0 & 0 & 0 \\ -0.95 & 0 & 0.72 & 0 & 0 \\ 0 & -0.72 & 0 & 0.60 & 0 \\ 0 & 0 & -0.60 & 0 & 0.44 \\ 0 & 0 & 0 & -0.44 & 0 \end{pmatrix}$$

is unchanged, and thus $A^{(2)}$ is a good stable matrix in $\mathcal{S}_2$ with $0.08305 < \delta_{us}(A^{(2)}) \leq 0.08306$.

When do we terminate the entire search algorithm? We terminate it when the increase of $\delta_{us}(A)$ is negligible. In our practical experience, this often occurs after Step r when $count[11] = 10^{-r}$ from Step 1.1, and the search algorithm is then stopped after Step r. Notice that, when $r = 1$, we terminate the entire search algorithm after Step 2. In Example 6.3, since $count[11] = 0.01$ from Example 6.2, we terminate the search algorithm after Step 2.

We have tested the entire algorithm on the 2×2 minimally potentially stable sign pattern given in Section 4. In our test, Step 1 finds a good stable matrix $A^{(1)}$ in the set $\mathcal{S}_1$, namely

$$A^{(1)} = \begin{pmatrix} -1 & 0.8 \\ -0.8 & 0 \end{pmatrix};$$

Step 2 further finds a good stable matrix $A^{(2)}$ in the set $\mathcal{S}_2$, namely

$$A^{(2)} = \begin{pmatrix} -1 & 0.87 \\ -0.87 & 0 \end{pmatrix},$$

with $0.2721602 < \delta_{us}(A^{(2)}) \leq 0.2721603$, which is very close to the optimal stable matrix determined analytically in Section 4.

When can we theoretically guarantee that an increase in $\delta_{us}(A)$ is negligible? Denote the stable matrix found in Step k by $A^{(k)}$. Note that $\| A^{(k)} \|_2 > 1$ (because it is normalized). With r from Step 1.1, let K denote the integer part of $r +$

$\log_{10}(20n\sqrt{n})$, so that $20n\sqrt{n}10^{-K-1} < 10^{-r}$. By Theorem 2.10 and using norm results in [HJ],

$$|\delta_{us}(A^{(k)}) - \delta_{us}(A^{(k-1)})|$$
$$\leq 2\| A^{(k)} - A^{(k-1)} \|_2 \leq 2\sqrt{n}\| A^{(k)} - A^{(k-1)} \|_\infty$$
$$\leq 2n\sqrt{n} \max |a_{ij}^{(k)} - a_{ij}^{(k-1)}|$$
$$\leq 18n\sqrt{n}10^{-k}.$$

Consequently, for $p \geq 0$,

$$|\delta_{us}(A^{(K+p+1)}) - \delta_{us}(A^{(K)})|$$
$$\leq \sum_{j=0}^{\infty} |\delta_{us}(A^{(K+j+1)}) - \delta_{us}(A^{(K+j)})|$$
$$\leq \sum_{j=0}^{\infty} 18n\sqrt{n}10^{-K-j-1}$$
$$= 20n\sqrt{n}10^{-K-1} < 10^{-r}.$$

Thus in the search for a good stable matrix in the sign pattern $\mathcal{A}$, the algorithm can definitely be stopped after Step K. Recall that $r = 2$ for Example 6.3, and thus $K = 4$, whereas in practice the search algorithm is terminated after Step 2.

§7. A Good Stable Matrix in each Minimally Potentially Stable Tree Sign Pattern of Orders 3 and 4.

Having formulated a heuristic algorithm in Section 6 to find a good stable matrix in a t.s.p., we apply this to all minimally potentially stable tree sign patterns of orders 3 and 4. We take advantage of the known list of all irreducible potentially stable tree sign patterns of orders 3 and 4 ([JS], [JMOV], [P]). From this list, the set of all (up to permutation and signature similarity) minimally potentially stable tree sign patterns of orders 3 and 4 is extracted. These are listed below.

$\mathcal{A}_{3,1}$:

$\mathcal{A}_{3,2}$:

$\mathcal{A}_{4,1}$:

$\mathcal{A}_{4,2}$:

$\mathcal{A}_{4,3}$:

$\mathcal{A}_{4,4}$:

$\mathcal{A}_{4,5}$:

$\mathcal{A}_{4,6}$:

$\mathcal{A}_{4,7}$:

$\mathcal{A}_{4,8}$:

$\mathcal{A}_{4,9}$:

$\mathcal{A}_{4,10}$:

$\mathcal{A}_{4,11}$:

We now prove that each of the above patterns is minimal. In order to show the minimality of an irreducible potentially stable sign pattern, it is sufficient to show that replacing any nonzero entry by zero results in an unstable sign pattern. The well-known Routh-Hurwitz theorem (Theorem A 2.7 in [HLMQ, p. 46]) gives necessary and sufficient conditions for a matrix to be stable. This may be used to prove the minimality of an irreducible potentially stable sign pattern. However, this is very costly and complicated. We use a much easier way that relies on our methods and results developed in Section 3.

It is obvious that, if A is a real stable $n \times n$ matrix, then

$$\text{tr}(A) < 0 \text{ and } (-1)^n \det(A) > 0.$$

Also, a reducible sign pattern is potentially stable if and only if its irreducible components are all potentially stable. Thus, as was shown in Theorem 3.1, all rooted trees (i.e., $\mathcal{A}_{3,1}$, $\mathcal{A}_{4,1}$ and $\mathcal{A}_{4,2}$) in the list are minimally potentially stable. Examining the remaining trees and removing any one edge results in at least one sign unstable irreducible component. Thus, if $i \neq j$ and $a_{ij} \neq 0$, then the sign pattern obtained by setting $a_{ij} = 0$ is not potentially stable. Furthermore, replacing any nonzero diagonal entry by zero in any of the patterns except $\mathcal{A}_{4,9}$ and $\mathcal{A}_{4,11}$ results in a rooted subtree and Theorem 3.2 shows that the corresponding subpattern is sign unstable. This leaves only the patterns $\mathcal{A}_{4,9}$ and $\mathcal{A}_{4,11}$ to show to be minimally potentially stable.

To show that pattern $\mathcal{A}_{4,9}$ is minimally potentially stable, label the signed tree as follows:

Take a generic stable matrix A_0 in the sign pattern

$$A_0 = \begin{pmatrix} -c_1 & a_1 & a_2 & 0 \\ b_1 & -c_2 & 0 & a_3 \\ -b_2 & 0 & c_3 & 0 \\ 0 & b_3 & 0 & 0 \end{pmatrix},$$

where $a_j, b_j > 0$ and $c_j \geq 0$. Since A_0 is stable,

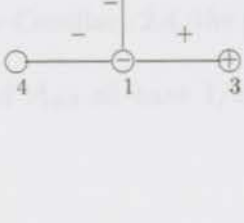
$$\det(A_0) = a_3 b_3 (c_1 c_3 - a_2 b_2) > 0,$$

and thus $c_1 c_3 \neq 0$. To show that $c_2 \neq 0$, note that as A_0 is stable, the sum of its 2×2 principal minors is positive (see [HJ, p. 42]). Thus

$$(c_1 c_2 - a_1 b_1) + (a_2 b_2 - c_1 c_3) - a_3 b_3 - c_2 c_3 > 0.$$

Since $\det(A_0) > 0$ requires $a_2 b_2 - c_1 c_3 < 0$, we must have $c_1 c_2 - a_1 b_1 > 0$, which implies $c_1 c_2 \neq 0$. Combined with the previous discussion on the nonzero off-diagonal entries, this shows that pattern $\mathcal{A}_{4,9}$ is minimal.

To show that pattern $\mathcal{A}_{4,11}$ is minimally potentially stable, label the signed tree as follows:



Take a generic stable matrix A_0 in the sign pattern

$$A_0 = \begin{pmatrix} -c_1 & a_1 & a_2 & a_3 \\ -b_1 & c_2 & 0 & 0 \\ b_2 & 0 & c_3 & 0 \\ -b_3 & 0 & 0 & 0 \end{pmatrix},$$

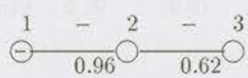
where $a_j, b_j > 0$ and $c_j \geq 0$. Since $\mathrm{tr}(A_0) < 0$, we must have $c_1 \neq 0$. Since

$$\det(A_0) = b_3 a_3 (c_2 c_3) > 0,$$

we must have $c_2 c_3 \neq 0$. Combined with the previous discussion on the nonzero off-diagonal entries, this shows that pattern $\mathcal{A}_{4,11}$ is minimal.

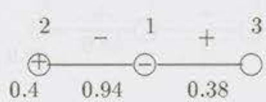
A good stable matrix in each of the minimally potentially stable sign patterns given in the list is depicted in the labeled and weighted graphs below. They are found by an application of the algorithm described in Section 6, except for the patterns $\mathcal{A}_{4,6}$ and $\mathcal{A}_{4,7}$ that contain no matrix having a properly signed nest. A good stable matrix in these cases is found through a large number of random examples and certain ad hoc adjustments. We list a stable matrix A with tight upper and lower bounds for $\delta_{us}(A)$ in each minimally potentially stable t.s.p. The fact that such a matrix is a good stable matrix is justified by the following argument. By Corollary 2.4, the patterns $\mathcal{A}_{3,1}$ and $\mathcal{A}_{3,2}$ have $1/3$, and the patterns $\mathcal{A}_{4,3}$, $\mathcal{A}_{4,4}$ and $\mathcal{A}_{4,9}$ all have $1/2$, as an upper bound for the relative distance to unstable matrices (by considering all stable matrices in a pattern with maximal magnitude 1 among negative diagonals). All other order 4 minimally potentially stable tree sign patterns have an upper bound of 1/4 for the relative distance to unstables. Thus, by the definition at the beginning of Section 5, the stable matrices listed below are all good stable matrices in their respective sign patterns.

$A_{3,1}$:



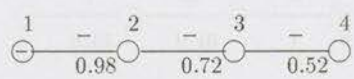
$$0.15545 < \delta_{us}(A_{3,1}) \leq 0.15546$$

$A_{3,2}$:



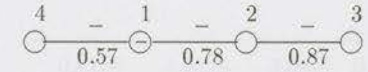
$$0.044969 < \delta_{us}(A_{3,2}) \leq 0.044970$$

$A_{4,1}$:



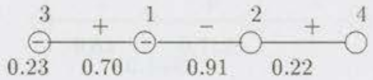
$$0.10859 < \delta_{us}(A_{4,1}) \leq 0.10860$$

$A_{4,2}$:



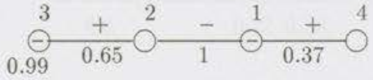
$$0.10864 < \delta_{us}(A_{4,2}) \leq 0.10865$$

$A_{4,3}$:



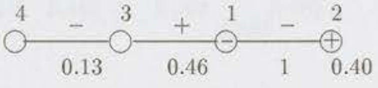
$$0.023805 < \delta_{us}(A_{4,3}) \leq 0.023806$$

$A_{4,4}$:



$$0.03828 < \delta_{us}(A_{4,4}) \leq 0.03829$$

$A_{4,5}$:

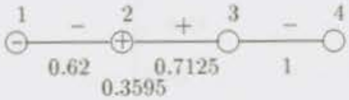


$$0.03174 < \delta_{us}(A_{4,5}) \leq 0.03175$$

$A_{4,6}$:

$$0.005 < \delta_{us}(A_{4,6}) \leq 0.006$$

$A_{4,7}$:

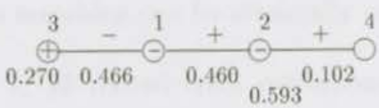


$$0.007 < \delta_{us}(A_{4,7}) \leq 0.008$$

$A_{4,8}$:

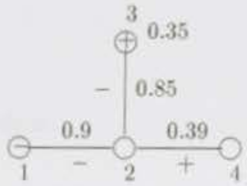
$$0.0211 < \delta_{us}(A_{4,8}) \leq 0.0212$$

$A_{4,9}$:



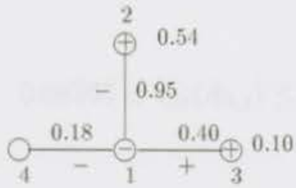
$$0.00835 < \delta_{us}(A_{4,9}) \leq 0.00836$$

$A_{4,10}$:



$$0.04432 < \delta_{us}(A_{4,10}) \leq 0.04433$$

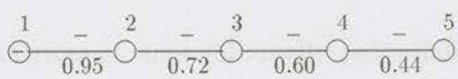
$A_{4,11}$:



$$0.0117 < \delta_{us}(A_{4,11}) \leq 0.0118$$

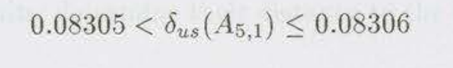
We conclude this section by listing all minimally potentially stable tree sign patterns corresponding to rooted trees with 5 vertices. All mutually nonisomorphic rooted trees with less than 50 vertices have been listed ([R]). By Theorem 3.2, only those with a complete matching can be minimally potentially stable. For example, there are exactly 3, 7, 10 rooted trees corresponding to minimally potentially stable sign patterns with 5, 6, 7 vertices, respectively. Numerical computation (with an older version of the algorithms) shows that all of these patterns have distance greater than 0.01 from the unstable matrices. Good stable matrices for the minimally potentially stable rooted trees with 5 vertices found using the algorithms of Section 6 are listed below.

$A_{5,1}$:



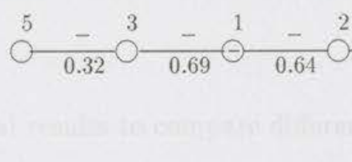
$$0.08305 < \delta_{us}(A_{5,1}) \leq 0.08306$$

$A_{5,2}$:



$$0.08308 < \delta_{us}(A_{5,2}) \leq 0.08309$$

$A_{5,3}$:



$$0.083083 < \delta_{us}(A_{5,3}) \leq 0.083084$$

§8. Hierarchy of Minimally Potentially Stable Tree Sign Patterns with respect to Distance to the Unstables.

The computation in the previous sections clearly shows that the relative distance to the unstable matrices of some minimally potentially stable tree sign patterns is greater than that of others. In this section, we specify the attributes that, based on numerical results, determine their distance to the unstable matrices relative to other such patterns. This suggests a hierarchy of these patterns, which we then use to find a good minimal subpattern (see Section 3) of a potentially stable sign pattern that has a spanning forest consisting of minimally potentially stable trees.

The use of numerical results to compare different minimally potentially stable tree sign patterns is limited by the fact that there is no complete listing of all such patterns beyond order 4. However, our numerical computations point to several observations that contribute to a hierarchical structure of minimally potentially stable tree sign patterns. We itemize these below in points 1-6.

We divide the minimally potentially stable tree sign patterns into three groups: those corresponding to rooted trees, those not corresponding to a rooted tree but containing a matrix with a properly signed nest, and the remaining patterns. The second group will be further subdivided by Definitions 8.1 and 8.2.

1. Within the first group above, the larger the order of a minimally potentially stable t.s.p., the smaller is its distance to the unstable matrices.

2. All minimally potentially stable rooted tree sign patterns of the same order have approximately the same distance to the unstable matrices.

3. Among minimally potentially stable tree sign patterns of the same order, rooted tree sign patterns are generally at a greater distance to the unstable matrices than nonrooted tree patterns.

4. Among the minimally potentially stable nonrooted tree sign patterns of the same order, those containing a matrix with a properly signed nest are generally further from the unstable matrices than those that do not contain such a matrix.

To classify nonrooted tree sign patterns that contain a matrix with a properly signed nest, we introduce the following two definitions. We say that a matrix has a **leading properly signed nest** if all its even order leading principal minors are positive and its odd order leading principal minors are negative.

Definition 8.1. Given a fixed labeling of a nonrooted tree corresponding to a minimally potentially stable t.s.p., if the pattern contains two matrices compatible with this labeling such that one has a leading properly signed nest and the other does not, then this labeling is called an **oscillating labeling***.*

Example 8.1. Consider the minimally potentially stable t.s.p. $\mathcal{A}_{4,9}$. Consider the labeling of the signed tree determined by the following matrix

$$A = \begin{pmatrix} -1 & a & b & 0 \\ a & -d_1 & 0 & c \\ -b & 0 & d_2 & 0 \\ 0 & c & 0 & 0 \end{pmatrix}$$

where $a, b, c, d_1, d_2 > 0$. The matrix A has a leading properly signed nest if and only if

$$\frac{a^2}{d_1} < 1, \ \frac{a^2}{d_1} + \frac{b^2}{d_2} > 1 \text{ and } \frac{b^2}{d_2} < 1.$$

For example,

$$\begin{pmatrix} -1 & 1 & 1 & 0 \\ 1 & -\sqrt{3/2} & 0 & 1 \\ -1 & 0 & \sqrt{3/2} & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

has a leading properly signed nest, but

$$\begin{pmatrix} -1 & 1 & 1 & 0 \\ 1 & -2 & 0 & 1 \\ -1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

does not since the leading 3×3 submatrix is singular. So this labeling is an oscillating labeling.

Definition 8.2. Consider a fixed oscillating labeling of a nonrooted tree corresponding to a minimally potentially stable t.s.p. An edge or nonzero vertex in the signed tree is called an **oscillating edge** *if by increasing or decreasing the weight on the edge or nonzero vertex (while fixing the other weights), it is possible to obtain both a normalized matrix in the sign pattern that has a leading properly signed nest as well as a normalized matrix in the pattern that does not. The total number of oscillating edges in a labeling is called the* **oscillating length** *of the labeling.*

In Example 8.1 above, a, b, d_1 and d_2 correspond to oscillating edges of the labeling but c does not. Thus the oscillating length of this labeling is 4. All of our numerical experience suggests that the oscillating length of an oscillating labeling is the same for all oscillating labelings of a fixed nonrooted minimally potentially stable t.s.p. that contains a matrix with a properly signed nest.

We now state the remaining two observations based on our numerical results.

5. Among nonrooted minimally potentially stable tree sign patterns that contain a matrix with a properly signed nest, a pattern that has an oscillating labeling is closer to the unstable matrices than one without an oscillating labeling.

6. Among nonrooted minimally potentially stable tree sign patterns that contain

a matrix with a properly signed nest and have an oscillating labeling, the larger the oscillating length, the closer is the pattern to the unstable matrices.

Example 8.2. Consider the minimally potentially stable t.s.p. $\mathcal{A}_{4,5}$. The labeling with a, b, c, d and $e > 0$ corresponding to the matrix

$$\begin{pmatrix} -1 & c & d & 0 \\ -c & b & 0 & 0 \\ d & 0 & 0 & e \\ 0 & 0 & -e & 0 \end{pmatrix}$$

is oscillating. It has a leading properly signed nest if and only if $c^2 > b$. Thus, this labeling has oscillating length 2. Using the algorithms in Section 6, there is a stable matrix A_0 in this sign pattern such that

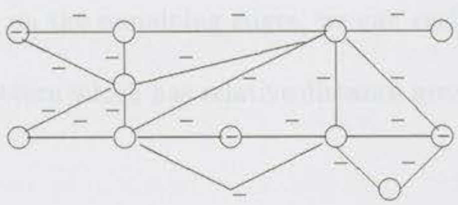
$$\delta_{us}(A_0) > 0.031.$$

However, the pattern $\mathcal{A}_{4,9}$ discussed in Example 8.1 has oscillating length 4 and the best stable matrix A_1 that we can find in $\mathcal{A}_{4,9}$ has

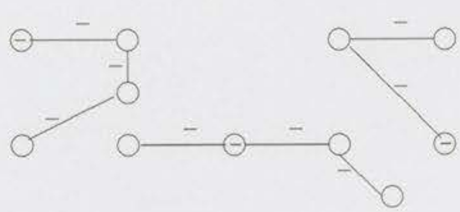
$$\delta_{us}(A_1) < 0.0084.$$

The above six observations are the basis for a heirarchy of minimally potentially stable tree sign patterns with respect to distance from the unstable matrices. We conclude this section with the following example, in which we show how to use some of the above observations to obtain a good minimal subpattern for a non-minimally potentially stable sign pattern with a spanning forest.

Example 8.3. The following 11×11 potentially stable sign pattern is from [JMOV].



Guided by observations 1, 2 and 3 above, the following spanning forest consisting of three minimally potentially stable rooted trees is a good minimal subpattern.



There is a stable matrix A in this minimal subpattern such that

$$\delta_{us}(A) > 0.10859,$$

which is approximately the minimum of $\delta_{us}(A_{3,1}), \delta_{us}(A_{4,1})$ and $\delta_{us}(A_{4,2})$ by using the weights given for these matrices in Section 7. Therefore, by filling in sufficiently small weights on the remaining edges, we can easily derive from A a stable matrix in the sign pattern which has relative distance greater than 0.10859 to the unstable matrices.

§9. Conclusions and Future Research

The distance of a potentially stable sign pattern to the unstable matrices is defined (Definition 2.3), based on the relative distance of a stable matrix to the unstable matrices (see Definitions 2.1 and 2.2). The computation of the precise value of this distance is a nonlinear optimization problem. Our analysis of the 2×2 sign stable pattern (see Section 4) shows how difficult the computation of this distance can be. Furthermore, Example 2.3 shows that this distance can be arbitrarily small for a pattern of large order. Also, the constructive proof of the theorem of Fisher and Fuller (see Section 1) does not provide stable matrices that are at an adequate relative distance from the unstable matrices. Therefore, an estimation of this distance for a potentially stable sign pattern is an interesting and practical problem.

The solution provided in this thesis uses the following strategy to find a good estimation. Firstly, guided by the heuristic hierachy of the minimally potentially stable tree sign patterns (see Section 8), graph theory is used to obtain a good minimal subpattern. Example 8.3 shows how such a surgery may be conducted. Secondly, the algorithms in Section 6, which are based on a result of Byers [B2], are used to estimate the distance of the minimal components that have a properly signed nest to the unstable matrices (some of these are listed in Section 7).

This solution brings in both opportunity and new challenge. This solution separates the problem into two stages: how to do the graph theoretical surgery and how to estimate the distance to the unstables from minimally potentially stable sign patterns. The second stage is solved analytically for order 2 (see Section 4) and numerically for orders 3 and 4 (see Section 7). Based on the rooted tree enumeration (see [R]), numerical estimates for minimally potentially stable rooted tree sign patterns for order 5 are given in Section 7, and the algorithms provided in Section 6 could be used for larger orders. Lists of minimally potentially stable nonrooted tree sign patterns for orders 5 and larger remain to be constructed.

Finally, the results in this thesis suggest the following new problems for future research.

1. Enumerate all nontree minimally potentially stable sign patterns of small order. Note that Example 4.1 of [JMOV] gives such a pattern of order 3.

2. Make the algorithms of Section 6 more computationally efficient for minimally potentially stable tree sign patterns, and extend them to tree patterns without a properly signed nest and to nontree patterns.

3. Prove that the heuristics on which the algorithms of Section 6 are based give good stable matrices in general.

4. Develop a graph-theoretic algorithm for the reduction of a (general) potentially stable sign pattern to a good minimal subpattern (see Sections 3 and 8).

Appendix

Let $\mathcal{A}_n$ denote the tridiagonal sign stable pattern defined in Example 2.3; see also Example 5.1. The input to the following MATLAB procedure is a pair of positive integers $p < q$. The procedure checks, for each n such that $p \leq n \leq q$, if $\frac{1}{8n} < \delta_{us}(\mathcal{A}_n)$. The procedure can be executed in the background and the returned result can be stored in a file named "vv". If the check is successful, it returns 0; otherwise, it returns 1.

```
function [count] = Tridiagonal(p, q);
count = 0;
for n = p : q
    A = zeros(n, n);
    A(1, 1) = -1;
    s = 0.8/(n - 2);
    d = 1/(8 * n);
    for k = 1 : (n - 1)
        A(k, k + 1) = 1 - s * (k - 1);
        A(k + 1, k) = -A(k, k + 1);
    end;
```

```
nn = norm(A);

HH = [A,-nn * d*eye(n);nn * d*eye(n),-A'];
eiv = eig(HH);
for p = 1 : 2 * n

  if ( abs(real(eiv(p))) < 10^-10 )

      count = 1;

      break;

  end;

end;

if (count = 1)

    break;

end;

end;

save vv count -ascii -double;
```

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