ROTOPULSATING ORBITS OF THE CURVED N-BODY PROBLEM

by

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B.Sc., Isfahan University of Technology, 2011

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ABSTRACT

The 3-dimensional gravitational N-body problem, $N \geq 2$, in spaces of constant Gaussian curvature $\kappa \neq 0$, i.e. on spheres $\mathbb{S}^3_k$, for $\kappa > 0$, and on hyperbolic manifolds $\mathbb{H}^3_k$, for $\kappa < 0$, is considered. In the 3-dimensional curved N-body problem, the new concept of rotopulsating orbits is defined. This type of solution is used when the bodies rotate and change size during the motion. Considering the possibility of having these bodies in spaces of positive or negative curvature, it is feasible to use the following classification: positive elliptic, positive elliptic-elliptic, negative elliptic, negative hyperbolic, and negative elliptic-hyperbolic. The necessary and sufficient criteria for the existence of rotopulsators are provided. Results will be obtained that describe their qualitative behaviour, which will then be applied to find examples for each type of rotopulsating orbits.

Contents

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Chapter 1

Introduction

In this thesis the curved N-body problem will be studied, defined as the gravitational motion of N point particles of masses $m_1, m_2, \ldots, m_N > 0$ in a space of constant curvature, $\kappa \neq 0$. This problem was studied previously by several mathematicians, such as Lejeune Dirichlet, Ernest Schering, [6], [7], Wilhelm Killing, [8], [9], [10], and Heinrich Liebmann, [11], [12]. Initially this research direction was undertaken in the 1830s by Janos Bolyai and Nikolai Lobachevsky, who independently formulated a curved 2-body problem in the hyperbolic space $\mathbb{H}^3$.

Different results came from [1], wherein the equations of motion of the curved N-body problem for any $N \geq 2$ and $\kappa \neq 0$ were obtained. The existence of several classes of relative equilibria, including the Lagrangian orbits have been proved. Relative equilibria are orbits for which the configuration of the system remains congruent with itself for all time, i.e. the distances between any two bodies are constant while in motion. In [1], the criterion for the existence of relative equilibria for the N-body problem of celestial mechanics in spaces of constant curvature, i.e. the 3-sphere for positive curvature and the hyperbolic 3-sphere for constant negative curvature have been formulated. Also the qualitative behavior of these orbits and some particular examples have been described.

In Euclidean space, these are solutions more general than relative equilibria, namely orbits for which the configuration of the system remains similar with itself. In this class of solutions, the relative distances between particles change proportionally during the motion, i.e. the size of the system may vary, though its shape remains the same. These solutions have been called homographic, therefore homographic solutions are orbits whose configuration remains similar to itself throughout the motion.

Specifically, when there is a rotation without expansion or contraction, the homographic orbits are called relative equilibria. Homothetic solutions are homographic orbits that experience expansion and/or contraction, but no rotation.

On the sphere and on the hyperbolic sphere, the only similar figures are the congruent ones. Therefore the term homographic, which implies similarity, makes us no sense anymore. We will therefore replace it with the concept of rotopulsating orbits. These types of rotopulsating orbits were studied in the curved 3-body problem in [1].

The first orbit, called Lagrangian, forms equilateral triangle, every time. The Euclidean plane of this triangle is always orthogonal to the axis of rotation. This assumption appears to be true because Lagrangian relative equilibria, which are a particular type of rotopulsating Lagrangian orbits, have this property, and relative equilibria without this property do not exist. The existence of rotopulsating Lagrangian orbits has been proved and their complete classification in the case of equal masses, for $\kappa > 0$, and $\kappa < 0$ has been provided. Moreover, the property that Lagrangian solutions with non-equal masses cannot exist has been proved.

Another type of rotopulsating solution of the curved 3-body problem is called Eulerian. At any time, the bodies of an Eulerian rotopulsating orbit are on a (possibly) rotating geodesic. The existence of these orbits has been proved. Moreover, for equal masses, their complete classification for $\kappa > 0$ and $\kappa < 0$ have been provided.

The third type of solution is the hyperbolic rotopulsating orbit, which occurs only for negative curvature. When the bodies are on the same hyperbolically rotating geodesic, a class of solutions referred to as hyperbolic Eulerian have been found, where every orbit is a hyperbolic Eulerian relative equilibrium. It was proved in [1] that all hyperbolic Eulerian relative equilibria are unstable. In this thesis we provide the first investigation of the rotopulsating orbits in $\mathbb{S}^3$ and $\mathbb{H}^3$.

In chapter 2 the equations of motion in $\mathbb{S}^3$ and $\mathbb{H}^3$ are introduced. Then the integral of energy and six integrals of the total angular momentum are obtained. At the end of this chapter, we define rotopulsating orbits and classify them in five groups: positive elliptic, positive elliptic-elliptic, negative elliptic, negative hyperbolic and negative elliptic-hyperbolic. The main results are obtained in chapter 3. The solutions given in each definition are then analyzed. Then we provide a criteria for the existence of each type of rotopulsating orbits. Examples of each type of rotpulsating orbit are provided in chapter 4. For the Lagrangian type and the Eulerian types, the negative hyperbolic and negative elliptic-hyperbolic rotopulsators are provided.

Chapter 2

Extension of Homographic Orbits

2.1 Equations of motion

The main purpose of the first part of this chapter is to obtain the equations of motion of the curved N-body problem on the 3-dimensional spheres and hyperbolic spheres. We will start by introducing the basics of the geometry and defining the natural metric of the sphere and the hyperbolic sphere, and will unify circular and hyperbolic trigonometry. In the next step, a definition will be presented for the curved potential function, afterwards, Euler's formula for homogeneous functions will be used for the curved potential. The classical variational theory of constrained Lagrangian dynamics will be introduced in order to obtain the Euler-Lagrange equations with constraints. Having found the equations of motion for the curved N-body problem, after selecting suitable coordinate and time rescaling transformations the study of the problem can be reduced to $\mathbb{S}^3$ (the unit 3-sphere) and $\mathbb{H}^3$ (the unit hyperbolic 3-sphere). At the end of the study the possibility of putting the equations of motion into Hamiltonian form will be presented. From this form, the first integrals will be developed.

2.1.1 The Basics of the Geometry

Consider

$$\mathbb{S}^3_\kappa = \{(w,x,y,z)|w^2+x^2+y^2+z^2=\kappa^{-1}\} \tag{2.1}$$

to be the 3-dimensional sphere of curvature $\kappa > 0$. and

$$\mathbb{H}^3_\kappa = \{(w,x,y,z)|w^2+x^2+y^2-z^2=\kappa^{-1}, z>0\} \tag{2.2}$$

to be the 3-dimensional hyperbolic sphere of curvature $\kappa < 0$.

$\mathbb{S}^3_\kappa$ is embedded in $\mathbb{R}^4$ endowed with standard inner product, whereas $\mathbb{H}^3_\kappa$ is embedded in the Minkovski space $\mathbb{R}^{3,1}$, which is $\mathbb{R}^4$ with the Lorentz inner product, $\boxdot$, defined below.

As we will show later it is possible to reduce the equations of motions research to the unit sphere $\mathbb{S}^3$ and unit hyperbolic sphere $\mathbb{H}^3$ by applying suitable coordinate and time-rescaling transformations where,

$$\mathbb{S}^3 = \{(w,x,y,z)|w^2+x^2+y^2+z^2=1\}, \tag{2.3}$$

for positive curvature, and

$$\mathbb{H}^3 = \{(w,x,y,z)|w^2+x^2+y^2-z^2=-1\}, \tag{2.4}$$

for negative curvature.

$$\text{For} \quad a=(a_w,a_x,a_y,a_z) \quad \text{and} \quad b=(b_w,b_x,b_y,b_z) \quad \text{in} \quad \mathbb{R}^4 \tag{2.5}$$

we define the inner product $a \odot b$ as

$$a \cdot b := a_w b_w + a_x b_x + a_y b_y + a_z b_z \text{ for } \kappa \geqslant 0, \tag{2.6}$$

and

$$a \boxdot b := a_w b_w + a_x b_x + a_y b_y - a_z b_z \text{ for } \kappa < 0. \tag{2.7}$$

2.1.2 Definition of the Metric

The distance between a and b on the manifolds $\mathbb{S}_k^3$ and $\mathbb{H}_k^3$, which, according to the corresponding inner products, is defined as:

$$d_\kappa(\mathbf{a}, \mathbf{b}) := \begin{cases} \kappa^{-1/2} \cos^{-1}(\kappa \mathbf{a} \cdot \mathbf{b}) & \kappa > 0 \\ |\mathbf{a} - \mathbf{b}|, & \kappa = 0 \\ (-\kappa)^{-1/2} \cosh^{-1}(\kappa \mathbf{a} \boxdot \mathbf{b}), & \kappa < 0, \end{cases} \tag{2.8}$$

For $\kappa > 0$, $\kappa = \frac{1}{R^2}$, where R is the radius of $\mathbb{S}_k^3$. For $\kappa < 0$, $\kappa = \frac{1}{(iR)^2}$, where iR is the radius of the hyperbolic 3-sphere, $\mathbb{H}_k^3$. The standard Euclidean norm is specified by the vertical bars. When $\kappa \to 0$, then $R \to \infty$, whether $\kappa > 0$ or $\kappa < 0$. With R approaching infinity, $\mathbb{S}_k^3$ and $\mathbb{H}_k^3$ turn into $\mathbb{R}^3$, where a and b make parallel vectors, as a result the distance between the vectors is presented by the Euclidean distance, as defined in (2.8). In order to find the equations of motion by applying a variational principle, the distance from the 3-dimensional manifolds of constant curvature $\mathbb{S}_k^3$ and $\mathbb{H}_k^3$ will be extended to the 4-dimensional ambient space in which they are embedded. The distance between a and b is therefore defined as

$$\bar{d}_\kappa(\mathbf{a}, \mathbf{b}) := \begin{cases} \kappa^{-1/2} \cos^{-1} \frac{\kappa \mathbf{a} \cdot \mathbf{b}}{\sqrt{\kappa \mathbf{a} \cdot \mathbf{a}} \sqrt{\kappa \mathbf{b} \cdot \mathbf{b}}} & \kappa > 0 \\ |\mathbf{a} - \mathbf{b}|, & \kappa = 0 \\ (-\kappa)^{-1/2} \cosh^{-1/2} \frac{\kappa \mathbf{a} \boxdot \mathbf{b}}{\sqrt{\kappa \mathbf{a} \boxdot \mathbf{a}} \sqrt{\kappa \mathbf{b} \boxdot \mathbf{b}}} & \kappa < 0. \end{cases} \tag{2.9}$$

In $\mathbb{S}_k^3$, $\sqrt{\kappa \mathbf{a} \cdot \mathbf{a}} = \sqrt{\kappa \mathbf{b} \cdot \mathbf{b}} = 1$ and in $\mathbb{H}_k^3$,$\sqrt{\kappa \mathbf{a} \boxdot \mathbf{a}} = \sqrt{\kappa \mathbf{b} \boxdot \mathbf{b}} = 1$. This new definition reduces to the distance defined in (2.8), when $\bar{d}_\kappa$ is restricted to the corresponding 3-dimensional manifolds of constant curvature, i.e. $d_\kappa = \bar{d}_\kappa$ in $\mathbb{S}_k^3$ and $\mathbb{H}_k^3$.

2.1.3 Generalized trigonometry

To find the equations of motion for the curved N-body problem in constant positive and constant negative curvature spaces, the trigonometric κ-functions, which unify the circular and hyperbolic trigonometry, will be defined in this section. The definition of κ-sine, sn_κ, is:

$$\operatorname{sn}_\kappa(x) := \begin{cases} \kappa^{-1/2} \sin \kappa^{1/2} x & \text{if} \quad \kappa > 0 \\ x & \text{if} \quad \kappa = 0 \\ (-\kappa)^{-1/2} \sinh (-\kappa)^{1/2} x & \text{if} \quad \kappa < 0, \end{cases} \tag{2.10}$$

the definition of κ-cosine, $\operatorname{csn}_\kappa$, is:

$$\operatorname{csn}_\kappa(x) := \begin{cases} \cos \kappa^{1/2} x & \text{if} \quad \kappa > 0 \\ 1 & \text{if} \quad \kappa = 0 \\ \cosh (-\kappa)^{1/2} x & \text{if} \quad \kappa < 0. \end{cases} \tag{2.11}$$

We defined the κ-tangent, tn_κ, and κ-cotangent, $\operatorname{ctn}_\kappa$, as follows:

$$\operatorname{tn}_\kappa(x) := \frac{\operatorname{sn}_\kappa(x)}{\operatorname{csn}_\kappa(\mathrm{x})} \text{ and } \operatorname{ctn}_\kappa(x) := \frac{\operatorname{ctn}_\kappa(x)}{\operatorname{sn}_\kappa(x)}. \tag{2.12}$$

This generalized approach can be used to derive the whole trigonometry, though the fundamental formula below is all we further need,

$$\kappa \operatorname{sn}_\kappa^2(x) + \operatorname{csn}_\kappa^2(x) = 1. \tag{2.13}$$

It is important to know that all the defined trigonometric κ-functions are continuous with respect to κ. The real parameter κ did not receive any definition when formulating the unified trigonometric κ-functions, but it will represent the constant curvature of $\mathbb{S}^3_k$ and $\mathbb{H}^3_k$.

2.1.4 Definition of the Potential Function

The coordinates of masses for N-bodies $m_1, m_2, \ldots, m_N > 0$ in $\mathbb{R}^4$, for $\kappa > 0$, and in $\mathbb{R}^{3,1}$, for $\kappa < 0$ are introduced using $\mathbf{q}_i = (w_i, x_i, y_i, z_i)$, $i = 1, 2, \ldots, N$. Having defined $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N)$ as the configuration of the system and $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N)$, with $\mathbf{p_i} = \mathbf{m_i}\dot{\mathbf{q}}_\mathbf{i}$, $i = 1, 2, \ldots, N$ as the momentum of the system, we define the gradient operator relative to the vector $\mathbf{q}_i$ as

$$\widetilde{\nabla}_{\mathbf{q_i}} := (\partial_{w_i}, \partial_{x_i}, \partial_{y_i}, \sigma\partial_{z_i}). \tag{2.14}$$

where σ is the signum function

$$\sigma = \begin{cases} 1 & , \quad \text{for } \kappa > 0 \\ -1 & , \quad \text{for } \kappa < 0. \end{cases} \tag{2.15}$$

We define potential of the curved N-body problem as the function $-U_\kappa$, where

$$U_\kappa(\mathbf{q}) := \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1 j\neq i}^{N} m_i m_j \mathrm{ctn}_\kappa\left(d_\kappa\left(\mathbf{q}_i, \mathbf{q}_j\right)\right), \tag{2.16}$$

represents the curved force function. The gravitational constant G does not occur because we can rescale the units to make G = 1. For $\kappa = 0$,

$$\mathrm{ctn}_0\left(d_0\left(\mathbf{q}_i, \mathbf{q}_j\right)\right) = \left|\mathbf{q}_i - \mathbf{q}_j\right|^{-1}, \tag{2.17}$$

so we obtain the classical Newtonian potential in the Euclidean case. As a result, it is expected to have a continuously varying potential U_κ relative to the curvature κ. In the following steps, a case will be considered for evaluation where $\kappa \neq 0$. By using the fundamental trigonometric formula (2.18), one can calculate

$$U_\kappa(\mathbf{q}) = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N} \frac{m_i m_j (\sigma\kappa)^{\frac{1}{2}} \frac{\kappa \mathbf{q_i} \odot \mathbf{q_j}}{\sqrt{\kappa \mathbf{q_i} \odot \mathbf{q_i}}\sqrt{\kappa \mathbf{q_j} \odot \mathbf{q_j}}}}{\sqrt{\sigma - \sigma(\frac{\kappa \mathbf{q_i} \odot \mathbf{q_j}}{\sqrt{\kappa \mathbf{q_i} \odot \mathbf{q_i}}\sqrt{\kappa \mathbf{q_j} \odot \mathbf{q_j}}})^2}}, \kappa \neq 0, \tag{2.18}$$

which is equivalent to:

$$U_\kappa(\mathbf{q}) = \sum_{1\leq i<j\leq N} \frac{m_i m_j |\kappa|^{1/2} \kappa \mathbf{q_i} \odot \mathbf{q_j}}{[\sigma(\kappa \mathbf{q_i} \odot \mathbf{q_i})(\kappa \mathbf{q_j} \odot \mathbf{q_j}) - \sigma(\kappa \mathbf{q_i} \odot \mathbf{q_j})^{\mathbf{2}}]^{\mathbf{1/2}}}, \kappa \neq 0. \tag{2.19}$$

2.1.5 Euler's formula for homogeneous functions

In this section, Euler's formula for homogeneous functions and its application to the curved potential will be presented.

Definition 1. *$F : \mathbb{R}^m \to \mathbb{R}$ is considered as a homogeneous function of degree $\alpha \in \mathbb{R}$ if for all $\eta \neq 0$ and $q \in \mathbb{R}^m$,*

$$F(\eta \mathbf{q}) = \eta^\alpha F(\mathbf{q}). \tag{2.20}$$

Based on the Euler's formula, one can represent any homogeneous function of degree $\alpha \in \mathbb{R}$ as:

$$\mathbf{q} \cdot \nabla F(\mathbf{q}) = \alpha F(\mathbf{q}) \tag{2.21}$$

for all $\mathbf{q} \in \mathbb{R}^m$. It is important to note that for any $\eta \neq 0$, $U_\kappa(\eta\mathbf{q}) = U_\kappa(\mathbf{q}) = \eta^0 U_\kappa(\mathbf{q})$. This implies that the curved potential is a homogeneous function of degree zero. Using the same notations and defining $m = 3N$, one can write Euler's formula as:

$$\mathbf{q} \odot \widetilde{\nabla} F(\mathbf{q}) = \alpha F(\mathbf{q}). \tag{2.22}$$

Noting that $\alpha = 0$ for U_κ with $\kappa \neq 0$,

$$\mathbf{q} \odot \widetilde{\nabla} U_\kappa(\mathbf{q}) = 0, \tag{2.23}$$

The curved force function is derived as $U_\kappa(\mathbf{q}) = \frac{1}{2}\sum_{i=1}^{n} U_\kappa^i(\mathbf{q}_i)$ where

$$U_\kappa^i(\mathbf{q}_i) = \sum_{j=1, j\neq i}^{n} \frac{m_i m_j (\sigma\kappa)^{\frac{1}{2}} \frac{\kappa \mathbf{q_i} \odot \mathbf{q_j}}{\sqrt{\kappa \mathbf{q_i} \odot \mathbf{q_i}}\sqrt{\kappa \mathbf{q_j} \odot \mathbf{q_j}}}}{\sqrt{\sigma - \sigma(\frac{\kappa \mathbf{q_i} \odot \mathbf{q_j}}{\sqrt{\kappa \mathbf{q_i} \odot \mathbf{q_i}}\sqrt{\kappa \mathbf{q_j} \odot \mathbf{q_j}}})^2}}, \kappa \neq 0. \tag{2.24}$$

are considered as homogeneous functions of degree 0. Euler's formula can be applied to F:$R^3 \rightarrow R$ in order to conclude that $\mathbf{q}_i \odot \widetilde{\nabla}_{\mathbf{q}_i} U_\kappa^i(\mathbf{q}) = 0$. Then applying the identity $\widetilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q}) = \widetilde{\nabla}_{\mathbf{q}_i} U_\kappa^i(\mathbf{q})$ the following conclusion can be drawn.

$$\mathbf{q}_i \odot \widetilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q}) = 0, i = 1, 2, ..., N. \tag{2.25}$$

2.1.6 Constrained Lagrangian dynamics

In this section, the classical variational theory of constrained Lagrangian dynamics will be used in order to derive the equations of motion of the curved N-body problem. Following the instruction of this theory for a system of N particles moving on a manifold, let

$$L = T - V \tag{2.26}$$

be the Lagrangian, where T and V represent the kinetic energy and the potential energy of the system, respectively. With vectors $\mathbf{q}_i$,$\dot{\mathbf{q}}_i$, $i = 1, 2, ..., N$ being the positions and velocities of the particles and the equations $f^i = 0, i = 1, 2, ..., N$, characterizing the constraints, the description of the motion is presented by the Euler-Lagrange equations with constraints

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}_i}\right) - \frac{\partial f^i}{\partial \mathbf{q}_i} - \lambda^i(t)\frac{\partial L}{\partial \mathbf{q}_i} = 0,\ i = 1, 2, ..., N, \tag{2.27}$$

where λ^i; $i = 1, 2, ..., N$, are called Lagrange multipliers. In these equations, the distance is considered in the entire ambient space. This classical result makes the derivation of the equations of motion of the curved N-body problem possible.

2.1.7 Derivation of the equations of motion

The requirement for deriving the Lagrangian of the curved N body problem is to define the kinetic energy of the system of bodies as:

$$T_\kappa(\mathbf{q}, \dot{\mathbf{q}}) := \frac{1}{2}\sum_{i=1}^{N} m_i(\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i)(\kappa \mathbf{q}_i \odot \mathbf{q}_i). \tag{2.28}$$

The definition of the kinetic energy was constructed using the factors $\kappa \mathbf{q}_i \odot \mathbf{q}_i = 1$; $i = 1, 2, ..., N$, in order to enable the equations of motion using a Hamiltonian structure. V is the potential energy which is defined as $V = -U_\kappa$. As a result, the Lagrangian of the curved N-body system is presented as:

$$L_\kappa(\mathbf{q}, \dot{\mathbf{q}}) := T_\kappa(\mathbf{q}, \dot{\mathbf{q}}) + U_\kappa(\mathbf{q}). \tag{2.29}$$

The requirement of the theory of constrained Lagrangian dynamics is to use a distance defined in the ambient space. This requirement was met in deriving definition (2.12), therefore the equations of motion are:

$$\frac{d}{dt}\left(\frac{\partial L_\kappa}{\partial \dot{\mathbf{q}}_i}\right) - \frac{\partial L_\kappa}{\partial \mathbf{q}_i} - \lambda_\kappa^i(t)\frac{\partial f_\kappa^i}{\partial \mathbf{q}_i} = 0,\ i = 1, 2, ..., N, \tag{2.30}$$

where the constraints $f_\kappa^i = 0, i = 1, 2, ..., N$ are derived using the function $f_\kappa^i = \mathbf{q}_i \odot \mathbf{q}_i - \kappa^{-1}$. This enables the body of mass m_i to stay on the surface of constant curvature κ, and λ_k^i is considered as the Lagrangian multiplier that corresponds to the same body. Considering that $\mathbf{q}_i \odot \mathbf{q}_i = \kappa^{-1}$ presents $\dot{\mathbf{q}}_i \odot \mathbf{q}_i = 0$:

$$\frac{d}{dt}\left(\frac{\partial L_\kappa}{\partial \dot{\mathbf{q}}_i}\right) = m_i\ddot{\mathbf{q}}\left(\kappa \mathbf{q}_i \odot \mathbf{q}_i\right) + 2m_i\ddot{\mathbf{q}}\left(\kappa \dot{\mathbf{q}}_i \odot \mathbf{q}_i\right) = m_i\ddot{\mathbf{q}},\ i = 1, 2, ..., N. \tag{2.31}$$

This relation, together with

$$\frac{\partial L_\kappa}{\partial \mathbf{q}_i} = m_i\kappa\left(\dot{\mathbf{q}} \odot \dot{\mathbf{q}}\right)\mathbf{q}_i + \widetilde{\nabla}_{\mathbf{q}_i}U_\kappa\left(\mathbf{q}\right),\ i = 1, 2, ..., N, \tag{2.32}$$

implies that equations (2.38) are equivalent to

$$m_i\ddot{\mathbf{q}}_i - m_i\kappa\left(\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i\right)\dot{\mathbf{q}}_i - \widetilde{\nabla}_{\mathbf{q}_i}U_\kappa\left(\mathbf{q}\right) - 2\lambda_\kappa^i(t)\,\mathbf{q}_i = 0,\ i = 1, 2, ..., N, \tag{2.33}$$

To determine λ_κ^i, it is important to note that $0 = \ddot{f}_\kappa^i = 2\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i + 2\left(\mathbf{q}_i \odot \ddot{\mathbf{q}}_i\right)$, so

$$\mathbf{q}_i \odot \ddot{\mathbf{q}}_i = -\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i,\ i = 1, 2, ..., N. \tag{2.34}$$

Remarking that $\odot$-multiplying equations (2.41) by $\mathbf{q_i}$ and using Euler's formula (2.33), the equations of motion became:

$$m_i\left(\mathbf{q}_i \odot \ddot{\mathbf{q}}_i\right) - m_i\left(\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i\right) - \mathbf{q}_i \odot \widetilde{\nabla}_{\mathbf{q}_i}U_\kappa\left(\mathbf{q}\right) = 2\lambda_\kappa^i\mathbf{q}_i \odot \mathbf{q}_i = 2\kappa^{-1}\lambda_\kappa^i,\ i = 1, 2, ..., N, \tag{2.35}$$

which, by (2.42), means that $\lambda_\kappa^i = -\kappa m_i\left(\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i\right)$, i= 1, 2, ..., N. By putting the Lagrange multipliers, λ_κ^i, into equations (2.41), the equations of motion and their constraints, which can be inserted in Hamiltonian form, turn into

$$m_i\ddot{\mathbf{q}} = \widetilde{\nabla}_{\mathbf{q}_i}U_\kappa\left(\mathbf{q}\right) - m_i\kappa\left(\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i\right)\mathbf{q}_i,\ \mathbf{q}_i \odot \mathbf{q}_i = \kappa^{-1},\ \kappa \neq 0,\ i = 1, 2, ..., N, \quad (2.36)$$

where the $\mathbf{q}_i$ gradient of the curved force function has the form

$$\widetilde{\nabla}_{\mathbf{q}_i}U_\kappa(\mathbf{q}) = \sum_{j=1, j\neq i}^{n} \frac{\frac{m_i m_j(\sigma\kappa)^{1/2}(\sigma\kappa\mathbf{q}_j - \sigma\frac{\kappa^2\mathbf{q}_i\odot\mathbf{q}_j}{\kappa\mathbf{q}_i\odot\mathbf{q}_i}\mathbf{q}_i)}{\sqrt{\kappa\mathbf{q}_i\odot\mathbf{q}_i}\sqrt{\kappa\mathbf{q}_j\odot\mathbf{q}_j}}}{[\sigma - \sigma(\frac{\kappa\mathbf{q}_i\odot\mathbf{q}_j}{\sqrt{\kappa\mathbf{q}_i\odot\mathbf{q}_i}\sqrt{\kappa\mathbf{q}_j\odot\mathbf{q}_j}})^2]^{3/2}}, \kappa \neq 0, \quad (2.37)$$

which can be written as:

$$\widetilde{\nabla}_{\mathbf{q}_i}U_\kappa(\mathbf{q}) = \sum_{j=1, j\neq i}^{n} \frac{m_i m_j|\kappa|^{3/2}(\kappa\mathbf{q}_j \odot \mathbf{q}_j)[(\kappa\mathbf{q}_i \odot \mathbf{q}_i)\mathbf{q}_j - (\kappa\mathbf{q}_i \odot \mathbf{q}_j)\mathbf{q}_i]}{[\sigma(k\mathbf{q}_i \odot \mathbf{q}_i)(\kappa\mathbf{q}_j \odot \mathbf{q}_j) - \sigma(\kappa\mathbf{q}_i \odot \mathbf{q}_j)^2]^{3/2}}. \quad (2.38)$$

Considering $\kappa\mathbf{q}_i \odot \mathbf{q}_i = 1$, the gradient can be derived as:

$$\widetilde{\nabla}_{\mathbf{q}_i}U_\kappa(\mathbf{q}) = \sum_{j=1, j\neq i}^{n} \frac{m_i m_j|k|^{3/2}[\mathbf{q}_j - (\kappa\mathbf{q}_i \odot \mathbf{q}_j)\mathbf{q}_i]}{[\sigma - \sigma(\kappa\mathbf{q}_i \odot \mathbf{q}_j)^2]^{3/2}}, \kappa \neq 0. \quad (2.39)$$

When trying to show the homogeneity of the gradient, or when there is a need to differentiate it, the original form (2.39) has to be used. In other cases, one can use the simpler form (2.39) of the gradient of the force function. Equations (2.36) and (2.38) provide a description for the N-body problem on surfaces of constant curvature for $\kappa \neq 0$.

2.1.8 Hamiltonian formulation

In order to use a more general theory to describe any new problem, the theory of Hamiltonian systems is used as a framework in the classical N-body problem. Newtonian gravitation is extended by Hamiltonian equations to spaces of constant curvature and the motion of the N-body problem is described by

$$H_\kappa(\mathbf{q},\mathbf{p}) = T_k(\mathbf{q},\mathbf{p}) - U_k(\mathbf{q}), \tag{2.40}$$

where

$$T_\kappa(\mathbf{q},\mathbf{p}) = \frac{1}{2}\sum_{i=1}^{n} m_i{}^{-1}(\mathbf{p}_i \odot \mathbf{p}_i)(\kappa\mathbf{q}_i \odot \mathbf{q}_i), \tag{2.41}$$

is the kinetic energy of the system of particles and

$$U_\kappa(\mathbf{q}) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} \frac{m_i m_j (\sigma\kappa)^{\frac{1}{2}} \frac{\kappa\mathbf{q}_i\odot\mathbf{q}_j}{\sqrt{\kappa\mathbf{q}_i\odot\mathbf{q}_i}\sqrt{\kappa\mathbf{q}_j\odot\mathbf{q}_j}}}{\sqrt{\sigma - \sigma(\frac{\kappa\mathbf{q}_i\odot\mathbf{q}_j}{\sqrt{\kappa\mathbf{q}_i\odot\mathbf{q}_i}\sqrt{\kappa\mathbf{q}_j\odot\mathbf{q}_j}})^2}}, \kappa \neq 0, \tag{2.42}$$

equivalent to

$$U_\kappa(\mathbf{q}) = \sum_{1\leq i<j\leq n} \frac{m_i m_j |k|^{1/2} \kappa\mathbf{q}_i \odot \mathbf{q}_j}{[\sigma(k\mathbf{q}_i \odot \mathbf{q}_i)(k\mathbf{q}_j \odot \mathbf{q}_j) - \sigma(k\mathbf{q}_i \odot \mathbf{q}_j)^2]^{1/2}}, \kappa \neq 0. \tag{2.43}$$

represents the force function.

The equations of motion (2.44) are Hamiltonian. The Hamiltonian function H_κ can be written as

$$\begin{cases} H_\kappa(\mathbf{q},\mathbf{p}) := \frac{1}{2}\sum_{i=1}^{n} m_i^{-1}(\mathbf{p}_i \odot \mathbf{p}_i)(\kappa\mathbf{q}_i \odot \mathbf{q}_i) - U_\kappa(\mathbf{q}) \\ \mathbf{q}_i \odot \mathbf{q}_i = \kappa^{-1}, \kappa \neq 0, i = 1, 2, ..., N. \end{cases} \tag{2.44}$$

As a result, the equation (2.41) turns into a $6N$-dimensional first order system of differential equations with $2N$ constraints

$$\begin{cases} \dot{\mathbf{q}}_i = \widetilde{\nabla}_{\mathbf{p}_i} H_\kappa(\mathbf{q}, \mathbf{p}) = m_i{}^{-1}\mathbf{p}_i \\ \dot{\mathbf{p}}_i = -\widetilde{\nabla}_{\mathbf{q}_i} H_\kappa(\mathbf{q}, \mathbf{p}) = \widetilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{p}) - m_i{}^{-1}\kappa(\mathbf{p}_i \cdot \mathbf{p}_i)\mathbf{q}_i \\ \mathbf{q}_i \cdot \mathbf{q}_i = k^{-1}, \mathbf{q}_i \odot \mathbf{p}_i = 0, \quad k \neq 0, \quad i = 1, 2, ..., N, \end{cases} \tag{2.45}$$

where the $\mathbf{q}_i$ gradient of the curved force function is written as:

$$\widetilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q}) = \sum_{j=1, j\neq i}^{n} \frac{\frac{m_i m_j (\sigma\kappa)^{1/2}(\sigma\kappa\mathbf{q}_j - \sigma\frac{\kappa^2 \mathbf{q}_i \odot \mathbf{q}_j}{\kappa\mathbf{q}_i \odot \mathbf{q}_i}\mathbf{q}_i)}{\sqrt{\kappa\mathbf{q}_i \odot \mathbf{q}_i}\sqrt{\kappa\mathbf{q}_j \odot \mathbf{q}_j}}}{[\sigma - \sigma(\frac{\kappa\mathbf{q}_i \odot \mathbf{q}_j}{\sqrt{\kappa\mathbf{q}_i \odot \mathbf{q}_i}\sqrt{\kappa\mathbf{q}_j \odot \mathbf{q}_j}})^2]^{3/2}}, \kappa \neq 0 \tag{2.46}$$

which is equivalent to

$$\widetilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q}) = \sum_{j=1, j\neq i}^{n} \frac{m_i m_j |\kappa|^{3/2}(\kappa\mathbf{q}_j \odot \mathbf{q}_j)[(\kappa\mathbf{q}_i \odot \mathbf{q}_i)\mathbf{q}_j - (\kappa\mathbf{q}_i \odot \mathbf{q}_j)\mathbf{q}_i]}{[\sigma(\kappa\mathbf{q}_i \odot \mathbf{q}_i)(\kappa\mathbf{q}_j \odot \mathbf{q}_j) - \sigma(\kappa\mathbf{q}_i \odot \mathbf{q}_j)^2]^{3/2}} \tag{2.47}$$

Considering that $\kappa\mathbf{q}_i \odot \mathbf{q}_i = 1$, this gradient turns into:

$$\widetilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q}) = \sum_{j=1, j\neq i}^{n} \frac{m_i m_j |k|^{3/2}[\mathbf{q}_j - (\kappa\mathbf{q}_i \odot \mathbf{q}_j)\mathbf{q}_i]}{[\sigma - \sigma(\kappa\mathbf{q}_i \odot \mathbf{q}_j)^2]^{3/2}}, k \neq 0 \tag{2.48}$$

It is important to note that whether the kinetic energy is written as

$$T_\kappa(\mathbf{q}, \mathbf{p}) = \frac{1}{2}\sum_{i=1}^{n} m_i{}^{-1}(\mathbf{p}_i \odot \mathbf{p}_i) \tag{2.49}$$

or

$$T_\kappa(\mathbf{q}, \mathbf{p}) = \frac{1}{2}\sum_{i=1}^{n} m_i{}^{-1}(\mathbf{p}_i \odot \mathbf{p}_i)(\kappa\mathbf{q}_i \odot \mathbf{q}_i) \tag{2.50}$$

the equations of motion remain the same, but in the former case they can not be written in Hamiltonian forms.

2.1.9 The integral of energy

The integral of energy can be derived using the Hamiltonian function:

$$H_\kappa(\mathbf{q}, \mathbf{p}) = h \tag{2.51}$$

where h represents the energy constant, in the form of an integration constant. It can be re-written as:

$$T_\kappa(\mathbf{q}, \dot{\mathbf{q}}) - U_\kappa(\mathbf{q}) = h \tag{2.52}$$

2.1.10 Independence of the value of the curvature

To eliminate the parameter κ from the equation of motion in spheres and hyperbolic manifolds with any dimension, it is necessary to change the coordinates and rescale the time. A coordinate and time-rescaling transformations is presented by

$$\mathbf{q}_i = |\kappa|^{-1/2}\,\mathbf{r}_i, i = 1, 2, ..., N \text{ and } \tau = |\kappa|^{3/4}\,t \tag{2.53}$$

Assuming $\mathbf{r}_i''$ and $\mathbf{r}_i'$ to be the first and second derivative of $\mathbf{r}_i$ in rescaled time variable τ, the equation of motion (2.44) can be written as:

$$\mathbf{r}_i'' = \sum_{j=1 j\neq i}^{N} \frac{m_j\left[\mathbf{r}_j - \sigma\left(\mathbf{r}_i \odot \mathbf{r}_j\right)\mathbf{r}_i\right]}{\left[\sigma - \sigma\left(\mathbf{r}_i \odot \mathbf{r}_j\right)^2\right]^{3/2}} - \sigma\left(\mathbf{r}'_i \odot \mathbf{r}'_i\right)\mathbf{r}_i, i = 1, 2, ..., N. \tag{2.54}$$

This equation does not show an explicit dependence on κ. Notice that $\sigma = 1$ for $\kappa > 0$ and $\sigma = -1$ for $\kappa < 0$ it is obtained:

$$\mathbf{r}_i \odot \mathbf{r}_i = |\kappa|\,\mathbf{q}_i \odot \mathbf{q}_i = |\kappa|\,\kappa^{-1} = \sigma. \tag{2.55}$$

Therefore for positive curvature one can write $\mathbf{r}_i \in \mathbb{S}^3$, i=1,2,...,N, and for negative curvature $\mathbf{r}_i \in \mathbb{H}^3$, i=1,2,...,N. This illustrates an independent qualitative behavior for the orbits relative to the curvature's value. As a result, for a positive curvature, this study will be restricted to the unit sphere and for a negative curvature the study will be limited to the unit hyperbolic sphere

The equation is re-written by having the variable $\mathbf{r}_i$ is replaced by $\mathbf{q}_i$, and replacing the rescaled time τ by t. Upper dots are used instead of primes to present the derivatives.

The equations of motion become:

$$\ddot{\mathbf{q}}_i = \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{m_j[\mathbf{q}_j - \sigma(\mathbf{q}_i \odot \mathbf{q}_j)\mathbf{q}_i]}{[\sigma - \sigma(\mathbf{q}_i \odot \mathbf{q}_j)^2]^{3/2}} - \sigma(\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i)\mathbf{q}_i, \;\; \mathbf{q}_i \odot \mathbf{q_i} = \sigma, \quad i = 1, 2, \ldots, N. \quad (2.56)$$

In case of a positive curvature, the equations of motion are written as:

$$\ddot{\mathbf{q}}_i = \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{m_j[\mathbf{q}_j - (\mathbf{q}_i \cdot \mathbf{q}_j)\mathbf{q}_i]}{[1 - (\mathbf{q}_i \cdot \mathbf{q}_j)^2]^{3/2}} - (\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i)\mathbf{q}_i, \;\; \mathbf{q}_i \cdot \mathbf{q_i} = 1, \quad i = 1, 2, \ldots, N, \qquad (2.57)$$

where $\cdot$ represents the standard inner product. Based on the constraints, the motion takes place on the unit sphere $\mathbb{S}^3$. The following system is used for negative curvature

$$\ddot{\mathbf{q}}_i = \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{m_j[\mathbf{q}_j + (\mathbf{q}_i \boxdot \mathbf{q}_j)\mathbf{q}_i]}{[(\mathbf{q}_i \boxdot \mathbf{q}_j)^2 - 1]^{3/2}} + (\dot{\mathbf{q}}_i \boxdot \dot{\mathbf{q}}_i)\mathbf{q}_i, \;\; \mathbf{q}_i \boxdot \mathbf{q_i} = -1, \quad i = 1, 2, \ldots, N, \quad (2.58)$$

where $\boxdot$ represents the Lorentz inner product. Based on the constraints, the motion takes place on the unit hyperbolic sphere $\mathbb{H}^3$.

It is important to note that the final term of each equation that has the Lagrange multipliers is derived based on the constraints that keep the bodies moving on the manifold. These terms are not present in Euclidean space.

The force function and its gradient are written as

$$U(\mathbf{q}) = \sum_{1\le i<j\le n} \frac{\sigma m_i m_j \mathbf{q}_i \odot \mathbf{q}_j}{[\sigma(\mathbf{q}_i \odot \mathbf{q}_i)(\mathbf{q}_j \odot \mathbf{q}_j) - \sigma(\mathbf{q}_i \odot \mathbf{q}_j)^2]^{1/2}}, \tag{2.59}$$

$$\widetilde{\nabla}_{\mathbf{q}_i} U(\mathbf{q}) = \sum_{j=1, j\ne i}^{N} \frac{m_i m_j [\mathbf{q}_j - \sigma(\mathbf{q}_i \odot \mathbf{q}_j)\mathbf{q}_i]}{[\sigma - \sigma(\mathbf{q}_i \odot \mathbf{q}_j)^2]^{3/2}}, \tag{2.60}$$

respectively, and the kinetic energy is written as:

$$T(\mathbf{q}, \dot{\mathbf{q}}) := \frac{1}{2}\sum_{i=1}^{N} m_i(\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i)(\sigma \mathbf{q}_i \odot \mathbf{q}_i). \tag{2.61}$$

2.1.11 The integrals of the angular momentum

The existence of six angular momentum integrals for equations (2.13) will be proved in this section. To facilitate this, some new concepts will be introduced. The generalization of a vector is the bivector. It is important to note that a scalar has dimension 0, a vector has dimension 1, and a bivector has dimension 2. Bivectors are constructed with the help of the wedge product.

Let e_w, e_x, e_y, and e_z be a basis of $\mathbb{R}^4$. Then $e_w \wedge e_x, e_w \wedge e_y, e_w \wedge e_z, e_x \wedge e_y, e_x \wedge e_z$, and $e_y \wedge e_z$ form a basis of the Grassmann algebra (the space of bi-vectors together with the wedge product is called a Grassmann algebra) over $\mathbb{R}^4$.

$$For \quad \mathbf{u} = (u_w, u_x, u_y, u_z) \quad and \quad \mathbf{v} = (v_w, v_x, v_y, v_z) \quad of \quad \mathbb{R}^4, \tag{2.62}$$

the definition of the wedge product is written as:

$$\mathbf{u}\wedge\mathbf{v} := (u_w v_x - u_x v_w)e_w\wedge e_x + (u_w v_y - u_y v_w)e_w\wedge e_y +$$
$$(u_w v_z - u_z v_w)e_w\wedge e_z + (u_x v_y - u_y v_x)e_x\wedge e_y +$$
$$(u_x v_z - u_z v_x)e_x\wedge e_z + (u_y v_z - u_z v_y)e_y\wedge e_z \tag{2.63}$$

Assuming $\sum_{i=1}^N m_i\mathbf{q}_i\wedge\mathbf{p}_i$ as the definition of the total angular momentum of the particles of masses $m_1, m_2, \ldots, m_N > 0$ in $\mathbb{R}^4$, the total angular momentum is considered to be conserved for the equations of motion, i.e.

$$\sum_{i=1}^n m_i\mathbf{q}_i\wedge\mathbf{p}_i = c \tag{2.64}$$

where $c = c_{wx}e_w\wedge e_x + c_{wy}e_w\wedge e_y + c_{wz}e_w\wedge e_z + c_{xy}e_x\wedge e_y + c_{xz}e_x\wedge e_z + c_{yz}e_y\wedge e_z$, with the coefficients c_{wx}, c_{wy}, c_{wz}, c_{xy}, c_{xz}, $c_{yz}\in R$. Then

$$\sum_{i=1}^N m_i\ddot{\mathbf{q}}_i\wedge\mathbf{q}_i = \sum_{i=1}^N\sum_{j=1\ j\neq i}^N \frac{\sigma m_i m_j\mathbf{q}_i\wedge\mathbf{q}_j}{[\sigma-\sigma(\mathbf{q}_i\odot\mathbf{q}_j)^2]^{3/2}}$$
$$-\sum_{i=1}^N\left[\textstyle\sum_{j=1\ j\neq i}^N \frac{\sigma m_i m_j\mathbf{q}_i\odot\mathbf{q}_j}{[\sigma-\sigma(\mathbf{q}_i\odot\mathbf{q}_j)^2]^{3/2}} - \sigma m_i(\mathbf{q}_i\odot\mathbf{q}_i)\right]\mathbf{q}_i\wedge\mathbf{q}_i = 0, \tag{2.65}$$

so

$$\textstyle\sum_{i=1}^N m_i(w_i\dot{x}_i - \dot{w}_i x_i) = c_{wx}, \qquad \sum_{i=1}^N m_i(w_i\dot{y}_i - \dot{w}_i y_i) = c_{wy} \tag{2.66}$$

$$\textstyle\sum_{i=1}^N m_i(w_i\dot{z}_i - \dot{w}_i z_i) = c_{wz}, \qquad \sum_{i=1}^N m_i(x_i\dot{y}_i - \dot{x}_i y_i) = c_{xy} \tag{2.67}$$

$$\textstyle\sum_{i=1}^N m_i(x_i\dot{z}_i - \dot{x}_i z_i) = c_{xz}, \qquad \sum_{i=1}^N m_i(y_i\dot{z}_i - \dot{y}_i z_i) = c_{yz}, \tag{2.68}$$

represent the six integrals of the total angular momentum.

2.2 Definitions and Classifications

In this section some concepts will be introduced which will be explored in the next parts of the thesis. Five types of orbits, two in $\mathbb{S}^3$ and 3 in $\mathbb{H}^3$ are defined. For each type, the building expressions of the natural definition will be simplified. The definition for several types of rotopulsating orbits of the curved N-body problem will be presented. There will be an elaboration of the options and choices made in classification afterwards. The Euclidean homographic solution is capable of being extended to spaces of where the constant curvature is not zero using the rotopulsating orbit.

This concept was introduced in the 2-dimensional case in two previous papers [5], [9]. The concept presented in these works was to consider the configurations that have been studies before (mostly polygons) to remain homographic using a perspective in the ambient Euclidean space. Considering these configurations in intrinsic terms, especially in a case where a move from two to three dimensions is expected, is more natural. For example, the only similarity on $\mathbb{S}^2$ and $\mathbb{H}^2$ is the congruent triangles. With a triangle in hand, the angles are not the same in a larger or smaller version of it while keeping the length ratios (measured in the manifold's metric) the same on the sides. The next step is then to define the homographic orbits relative to the Euclidean geometric figures corresponding to them. This is not reasonable, because having a constant ratio of the chords of the Euclidean distances among bodies does not mean having constant arc-distance ratios measured with the help of the natural distances given by the manifold's metric. In order to elaborate this in $\mathbb{S}^2$, a square and a regular octagon inscribed in a circle of radius 1 is considered. If D is the length of the diameter, S the length of the side of the square, and O the length of the side of the octagon, while $\widehat{D}$, $\widehat{S}$, and $\widehat{O}$ represent the lengths of the corresponding arcs of the circle, one can write

$$\frac{\overset{\frown}{D}}{\overset{\frown}{S}} = \frac{\overset{\frown}{S}}{\overset{\frown}{O}} = 2 \ \text{ but } \ \frac{2}{\sqrt{2}} = \frac{D}{S} \neq \frac{S}{O} = \sqrt{2+\sqrt{2}},$$

so having constant arc ratios does not mean having constant chord ratios or vice versa. As a result, it is necessary to, in a way, capture the dilation/contraction and/or the rotational aspects of homographic solutions and to recover the original definition where the curvature approaches zero in order to extend the concept of homographic orbit to spaces of constant curvature. This is the reason for introducing a new adjective, rotopulsating, here that suggests both these features of the orbit without the need to prove the similarity of the configuration. Rotopulsating orbits will be named rotopulsators. The following definition is based on the concept of relative equilibrium of the curved N-body problem, presented in [6], [7]. Different types of relative equilibria regarding the isometric rotation groups of $\mathbb{S}^3$ and $\mathbb{H}^3$ are presented.

Definition 2 (Rotopulsating positive elliptic orbits).
Let $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N)$*, be a position of masses* $m_1, m_2, ...m_N > 0, N \geqslant 2$*, on the manifold* $\mathbb{S}^3$*, where* $\mathbf{q}_i = (w_i, x_i, y_i, z_i)$*,* $\ i = 1, 2, \ldots, N$*. Then a solution of system (2.21) of the form*

$$w_i = r_i(t)\cos[\alpha(t) + a_i], \quad x_i = r_i(t)\sin[\alpha(t) + a_i], \quad y_i = y_i(t), \quad z_i = z_i(t) \qquad (2.69)$$

where a_i*,* $\ i = 1, 2, \ldots, N$*, are constants, there is no limit for the function* α*, while the functions* r_i, y_i*, and* z_i *meet the requirements*

$$0 \leq r_i \leq 1; \quad -1 \leq y_i, z_i \leq 1; \quad \text{and} \quad r_i^2 + y_i^2 + z_i^2 = 1, \quad i = 1, 2, \ldots, N,$$

and $c_{yz} = 0$,

is called a rotopulsating positive elliptic orbit.

If all the mutual distances are constant, the solution is called a relative equilibrium.

Remark 1. With non constant conditions present and having $c_{yz} = 0$ it is expected to have a system with elliptic rotation relative to the wx-plane, while no elliptic rotation is expected in the yz-plane. One could expect to see rotations relative to other base planes.

Definition 3 (**Rotopulsating positive elliptic-elliptic orbits**).

Let $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N)$, *be a position of masses* $m_1, m_2, ...m_N > 0, N \geqslant 2$, *on the manifold* $\mathbb{S}^3$, *where* $\mathbf{q}_i = (w_i, x_i, y_i, z_i)$, $\quad i = 1, 2, \ldots, N$. *Then a solution of system (2.21) of the form*

$$
\begin{aligned}
w_i = r_i(t)\cos[\alpha(t) + a_i],\ x_i = r_i(t)\sin[\alpha(t) + a_i],\\
y_i = \rho_i(t)\cos[\beta(t) + b_i],\ z_i = \rho_i(t)\sin[\beta(t) + b_i],
\end{aligned}
\tag{2.70}
$$

where $a_i, b_i,\ i = 1, 2, \ldots, N$ *are constants and there is no limit for the functions* α *and* β, *while the functions* r_i *and* ρ_i *meet the requirements*

$$
0 \leq r_i, \rho_i \leq 1 \ \text{ and } \ r_i^2 + \rho_i^2 = 1,\ i = 1, 2, \ldots, N.
$$

is called a rotopulsating positive elliptic-elliptic orbit.

If all the mutual distances are constant, the solution is called a relative equilibrium.

Remark 2. Considering α and β as non constant parameters, it is expected to have a system with two elliptic rotations relative to wx-plane and yz-plane. One could expect to see rotations relative to other base planes.

Definition 4 (Rotopulsating negative elliptic orbits).

Let $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N)$*, be a position of masses* $m_1, m_2, ...m_n > 0, n \geqslant 2$*, on the manifold* $\mathbb{H}^3$*, where* $\mathbf{q}_i = (w_i, x_i, y_i, z_i), \quad i = 1, 2, \ldots, N$*. Then a solution of system (2.21) of the form*

$$w_i = r_i(t)\cos[\alpha(t) + a_i], \quad x_i = r_i(t)\sin[\alpha(t) + a_i], \quad y_i = y_i(t), \quad z_i = z_i(t), \qquad (2.71)$$

where $a_i,\ i = 1, 2, \ldots, N$*, are constants and there is no limit for the function* α*, while the functions* r_i, y_i*, and* z_i *meet the requirements*

$$z_i \geq 1 \quad \text{and} \quad r_i^2 + y_i^2 - z_i^2 = -1, \quad i = 1, 2, \ldots, N.$$

and $c_{yz} = 0$*,*

is called a rotopulsating negative elliptic orbit.

If all the mutual distances are constant, the solution is called a relative equilibrium.

Remark 3. Considering α as a non constant parameter, and $c_{yz} = 0$ it is expected to have a system with an elliptic rotation relative to the wx-plane, but no hyperbolic rotation relative to the yz-plane. One could expect to see rotations relative to other base planes.

Definition 5 (Rotopulsating negative hyperbolic orbits).

Let $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N)$*, be a position of masses* $m_1, m_2, ...m_N > 0, N \geqslant 2$*, on the manifold* $\mathbb{H}^3$*, where* $\mathbf{q}_i = (w_i, x_i, y_i, z_i), \quad i = 1, 2, \ldots, N$*. Then a solution of system (2.21) of the form*

$$w_i = w_i(t),\ x_i = x_i(t),\ y_i = \eta_i(t)\sinh[\beta(t) + b_i],\ z_i = \eta_i(t)\cosh[\beta(t) + b_i], \qquad (2.72)$$

where $b_i,\ i = 1, 2, \ldots, N$*, are constants, no restriction is imposed on the function* β*,* *whereas the functions* w_i, x_i, z_i*, and* η_i *satisfy the conditions*

$$z_i \geq 1 \quad \text{and} \quad w_i^2 + x_i^2 - \eta_i^2 = -1, \quad i = 1, 2, \ldots, N.$$

and $c_{wx} = 0$*,*

is called a rotopulsating negative hyperbolic orbit.

If all the mutual distances are constant, the solution is called a relative equilibrium.

Remark 4. Considering β as a non constant parameter and with $c_{wx} = 0$ it is expected to have a system with a hyperbolic rotation relative to the yz-plane, but no elliptic rotation relative to the wx-plane. One could expect to see rotations relative to other base planes

Definition 6 (Rotopulsating negative elliptic-hyperbolic orbits).
Let $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N)$*, be a position of masses* $m_1, m_2, ...m_N > 0, N \geqslant 2$*, on the manifold* $\mathbb{H}^3$*, where* $\mathbf{q}_i = (w_i, x_i, y_i, z_i), \quad i = 1, 2, \ldots, N$*. Then a solution of system (2.21) of the form*

$$\begin{aligned} w_i &= r_i(t)\cos[\alpha(t) + a_i], \quad x_i = r_i(t)\sin[\alpha(t) + a_i], \\ y_i &= \eta_i(t)\sinh[\beta(t) + b_i], \quad z_i = \eta_i(t)\cosh[\beta(t) + b_i], \end{aligned} \tag{2.73}$$

where $a_i, b_i,\ i = 1, 2, \ldots, N$ *are constants and there is no limit for the function* α, β*, while the functions* r_i, η_i*, and* z_i *meet the requirements*

$$z_i \geq 1 \quad \text{and} \quad r_i^2 - \eta_i^2 = -1, \quad i = 1, 2, \ldots, N.$$

is called a rotopulsating negative elliptic-hyperbolic orbit.

If all the mutual distances are constant, the solution is called a relative equilibrium.

Remark 5. Considering α and β it is expected to have a system with an elliptic rotation relative to the wx-plane and a hyperbolic rotation relative to the yz-plane. One could expect to see rotations relative to other base planes.

Chapter 3

Criteria for the Existence of Rotopulsators

In chapter 2, we defined rotopulsators of the curved N-body problem as a starting point for this research. We will next provide existence criteria for these orbits and later apply them to find particular rotations.

3.1 Rotopulsating positive elliptic orbits

In this section a criterion will be provided for the existence of positive elliptic rotopulsators. The solution introduced in Definition 2 can be explained in more detail considering this criterion. The integral of energy and the six integrals of the total angular momentum will be derived, which are specific for these orbits. These results provide necessary and sufficient conditions to prove the existence of positive elliptic rotopulsators.

Criterion 1. *A solution of the type (2.69) is a rotopulsating positive elliptic orbit for* *system (2.57) if and only if*

$$\dot{\alpha} = \frac{c}{\sum_{j=1}^{N} m_j(1 - y_j^2 - z_j^2)}, \tag{3.1}$$

where c is a constant value and the variables y_i, z_i, $i = 1, 2, \ldots, N$, meet the requirements of the system of $2N$ second-order differential equations

$$\begin{cases} \ddot{y}_i = \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{m_j(y_j - q_{ij} y_i)}{(1-q_{ij}^2)^{3/2}} - G_i y_i \\ \ddot{z}_i = \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{m_j(z_j - q_{ij} z_i)}{(1-q_{ij}^2)^{3/2}} - G_i z_i, \\ r_i \ddot{\alpha} + 2\dot{r}_i \dot{\alpha} = \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{m_j r_j \sin(a_j - a_i)}{(1-q_{ij}^2)^{\frac{3}{2}}} \end{cases} \tag{3.2}$$

where

$$G_i := \frac{\dot{y}_i^2 + \dot{z}_i^2 - (y_i \dot{z}_i - z_i \dot{y}_i)^2}{1 - y_i^2 - z_i^2} + \frac{c^2(1 - y_i^2 - z_i^2)}{\left[\sum_{j=1}^{N} m_j(1 - y_j^2 - z_j^2)\right]^2}, \tag{3.3}$$

$i = 1, 2, \ldots, N$, and, for any $i, j \in \{1, 2, \ldots, N\}$, q_{ij} is represented by

$$q_{ij} := \mathbf{q}_i \cdot \mathbf{q_j} = (1 - y_i^2 - z_i^2)^{\frac{1}{2}}(1 - y_j^2 - z_j^2)^{\frac{1}{2}} \cos(a_i - a_j) + y_i y_j + z_i z_j.$$

Proof. Considering a solution of the type (2.69) under the initial conditions in the theorem, one can conclude from the computation that for any $i, j \in \{1, 2, \ldots, N\}$

$$q_{ij} := \mathbf{q}_i \cdot \mathbf{q_j} = (1 - y_i^2 - z_i^2)^{\frac{1}{2}}(1 - y_j^2 - z_j^2)^{\frac{1}{2}} \cos(a_i - a_j) + y_i y_j + z_i z_j,$$

$$\dot{w}_i = \dot{r} \cos \alpha - r\dot{\alpha} \sin \alpha \tag{3.4}$$

$$\dot{x}_i = \dot{r} \sin \alpha + r\dot{\alpha} \cos \alpha \tag{3.5}$$

$$\dot{y}_i = \dot{y} \tag{3.6}$$

$$\dot{z}_i = \dot{z} \tag{3.7}$$

and for any $i = 1, 2, \ldots, N$ it is presented that

$$\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i = \frac{\dot{y}_i^2 + \dot{z}_i^2 - (y_i \dot{z}_i - z_i \dot{y}_i)^2 + (1 - y_i^2 - z_i^2)^2 \dot{\alpha}^2}{1 - y_i^2 - z_i^2}.$$

For all $i = 1, 2, \ldots, N$, each r_i can be expressed in terms of y_i and z_i to derive

$$r_i = (1 - y_i^2 - z_i^2)^{\frac{1}{2}}, \quad \dot{r}_i = -\frac{y_i \dot{y}_i + z_i \dot{z}_i}{(1 - y_i^2 - z_i^2)^{\frac{1}{2}}},$$

$$\ddot{r}_i = \frac{(y_i \dot{z}_i - z_i \dot{y}_i)^2 - \dot{y}_i^2 - \dot{z}_i^2 - (1 - y_i^2 - z_i^2)(y_i \ddot{y}_i + z_i \ddot{z}_i)}{(1 - y_i^2 - z_i^2)^{\frac{3}{2}}}.$$

$$\ddot{w}_i = (\ddot{r} - r\dot{\alpha}^2)\cos\alpha - (r\ddot{\alpha} + 2\dot{r}\dot{\alpha})\sin\alpha \tag{3.8}$$

$$\ddot{x}_i = (\ddot{r} - r\dot{\alpha}^2)\sin\alpha + (r\ddot{\alpha} + 2\dot{r}\dot{\alpha})\cos\alpha \tag{3.9}$$

$$\ddot{y}_i = \ddot{y} \tag{3.10}$$

$$\ddot{z}_i = \ddot{z} \tag{3.11}$$

With putting a solution of the type (2.69) into system (2.57) and using the formulas presented above, for the equations corresponding to $\ddot{y}_i$ and $\ddot{z}_i$ it is obtained that

$$\ddot{y}_i = \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{m_j (y_j - q_{ij} y_i)}{(1 - q_{ij}^2)^{\frac{3}{2}}} - \frac{[\dot{y}_i^2 + \dot{z}_i^2 - (y_i \dot{z}_i - z_i \dot{y}_i)^2] y_i}{1 - y_i^2 - z_i^2} - (1 - y_i^2 - z_i^2) y_i \dot{\alpha}^2, \tag{3.12}$$

$$\ddot{z}_i = \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{m_j(z_j - q_{ij}z_i)}{(1-q_{ij}^2)^{\frac{3}{2}}} - \frac{[\dot{y}_i^2 + \dot{z}_i^2 - (y_i\dot{z}_i - z_i\dot{y}_i)^2]z_i}{1-y_i^2-z_i^2} - (1-y_i^2-z_i^2)z_i\dot{\alpha}^2, \quad (3.13)$$

$$\ddot{w}_i = (\ddot{r}_i - r_i\dot{\alpha}^2)\cos(\alpha + a_i) - (r_i\ddot{\alpha} + 2\dot{r}_i\dot{\alpha})\sin(\alpha + a_i) \quad (3.14)$$

$$\ddot{w}_i = \sum_{j=1}^{N} \frac{m_j[r_j\cos(\alpha+a_j) + q_{ij}r_i\cos(\alpha+a_i)]}{(q_{ij}^2-1)^{3/2}} + (\dot{r}_i^2 + r_i^2\dot{\alpha}^2 - \frac{r_i^2\dot{r}_i^2}{(1+r_i^2)} - (1+r_i^2)\dot{\beta}^2)r_i\cos(\alpha+a_i) \quad (3.15)$$

But

$$\cos(\alpha + a_j) = \cos(\alpha + a_i)\cos(a_i - a_j) + \sin(\alpha + a_i)\sin(a_i - a_j) \quad (3.16)$$

$$\ddot{w}_i = \sum_{j=1}^{N} \frac{m_j[r_j\cos(\alpha+a_i)\cos(a_i-a_j) + r_j\sin(\alpha+a_i)\sin(a_i-a_j) + q_{ij}r_i\cos(\alpha+a_i)]}{(q_{ij}^2-1)^{3/2}} + (\dot{r}_i^2 + r_i^2\dot{\alpha}^2 - \frac{r_i^2\dot{r}_i^2}{(1+r_i^2)} - (1+r_i^2)\dot{\beta}^2)r_i\cos(\alpha+a_i) \quad (3.17)$$

$$\ddot{r}_i - r_i\dot{\alpha}^2 = \sum_{j=1}^{N} \frac{m_j[r_j\cos(a_i-a_j) + q_{ij}r_i]}{(q_{ij}^2-1)^{3/2}} + (\dot{r}_i^2 + r_i^2\dot{\alpha}^2 - \frac{r_i^2\dot{r}_i^2}{(1+r_i^2)} - (1+r_i^2)\dot{\beta}^2) \quad (3.18)$$

$$\ddot{\alpha} = -\frac{2\dot{r}_i\dot{\alpha}}{r_i} - \sum_{j=1}^{N} \frac{m_j r_j \sin(a_i - a_j)}{r_i(q_{ij}^2-1)^{3/2}} \quad (3.19)$$

whereas the equations related to $\ddot{w}_i$ and $\ddot{x}_i$, through extensive calculations using (3.12) and (3.13), lead to identities or equations

$$r_i\ddot{\alpha} + 2\dot{r}_i\dot{\alpha} = \sum_{\substack{j=1\\j\neq i}}^{N} \frac{m_j r_j \sin(a_j - a_i)}{(1-q_{ij}^2)^{\frac{3}{2}}}, \quad i = 1, 2, \ldots, N. \tag{3.20}$$

For every $i = 1, 2, \ldots, N$, the ith equation in (3.20) is further multiplied by $m_i r_i$, and summation of N equations results in the following equations, considering:

$$\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N} \frac{m_i m_j r_i r_j \sin(a_j - a_i)}{(1-q_{ij}^2)^{\frac{3}{2}}} = 0.$$

$$m_i r_r \ddot{\alpha} = -2m_i\dot{r}_i\dot{\alpha} - m_i \sum_{j=1}^{N} \frac{m_j r_j \sin(a_i - a_j)}{(q_{ij}^2 - 1)^{3/2}} \tag{3.21}$$

Thus we obtain the equation

$$\left(\sum_{i=1}^{N} m_i r_i^2\right)\ddot{\alpha} + 2\left(\sum_{i=1}^{N} m_i r_i \dot{r}_i\right)\dot{\alpha} = 0,$$

And the equation is derived as:

$$\dot{\gamma} = -\frac{2(\sum_{i=1}^{N} m_i\dot{\gamma}_i)\gamma}{\sum_{i=1}^{N} m_i r_i} \to \gamma = \frac{c}{\sum_{i=1}^{N} m_i r_i^2} \tag{3.22}$$

$$\dot{\alpha} = \frac{c}{\sum_{i=1}^{N} m_i r_i^2} = \frac{c}{\sum_{i=1}^{N} m_i(1 - y_i^2 - z_i^2)},$$

where c is an integration constant. As a result, equations (3.12) and (3.13) turn into

$$\ddot{y}_i = \sum_{\substack{j=1\\j\neq i}}^{N} \frac{m_j(y_j - q_{ij}y_i)}{(1-q_{ij}^2)^{\frac{3}{2}}} - \frac{[\dot{y}_i^2 + \dot{z}_i^2 - (y_i\dot{z}_i - z_i\dot{y}_i)^2]y_i}{1 - y_i^2 - z_i^2} - \frac{c^2(1 - y_i^2 - z_i^2)y_i}{\left[\sum_{j=1}^{N} m_j(1 - y_j^2 - z_j^2)\right]^2}, \tag{3.23}$$

$$\ddot{z}_i = \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{m_j(z_j - q_{ij}z_i)}{(1-q_{ij}^2)^{\frac{3}{2}}} - \frac{[\dot{y}_i^2 + \dot{z}_i^2 - (y_i\dot{z}_i - z_i\dot{y}_i)^2]z_i}{1-y_i^2-z_i^2} - \frac{c^2(1-y_i^2-z_i^2)z_i}{\left[\sum_{j=1}^N m_j(1-y_j^2-z_j^2)\right]^2}, \tag{3.24}$$

$i = 1, 2, \ldots, N$, where, recall, for any $i, j \in \{1, 2, \ldots, N\}$:

$$q_{ij} := \mathbf{q}_i \cdot \mathbf{q_j} = (1-y_i^2-z_i^2)^{\frac{1}{2}}(1-y_j^2-z_j^2)^{\frac{1}{2}} \cos(a_i - a_j) + y_i y_j + z_i z_j.$$

□

Remark 6. It is important to note that c_{wx}, the total angular momentum, is in the form

$$c_{wx} = \sum_{i=1}^{N} m_i(w_i\dot{x}_i - \dot{w}_i x_i) = \dot{\alpha} \sum_{i=1}^{N} m_i(1-y_i^2-z_i^2) = c,$$

therefore c (which is a non zero constant) describes the rotation of the particles relative to the wx-plane.

Remark 7. Criterion 1 is satisfactory in order to confirm the existence of any candidate solution, though it might be difficult to use if that candidate is not a solution. Therefore, there is a value in having simpler nonexistence methods, which can be derived using the integrals of motion. In order to prove nonexistence, enough to show to present that the functions involved in the integrals are not constant. It is important to note that the first expression in the result stated below stands for the energy integral. The others are derived from five out of six total angular momentum integrals.

Corollary 1. *If system (2.57) has a solution of the form (2.69), then the following expressions are constant:*

energy:

$$h = \sum_{i=1}^{N} \frac{m_i[\dot{y}_i^2 + \dot{z}_i^2 - (y_i\dot{z}_i - z_i\dot{y}_i)^2]}{2(1 - y_i^2 - z_i^2)} + \frac{c^2}{2\sum_{i=1}^{N} m_i(1 - y_i^2 - z_i^2)} - \sum_{1\le i<j\le N} \frac{m_i m_j q_{ij}}{(1 - q_{ij}^2)^{\frac{1}{2}}}, \tag{3.25}$$

total angular momentum relative to the wy-plane:

$$c_{wy} = \textstyle\sum_{i=1}^{N} m_i(w_i\dot{y}_i - \dot{w}_i y_i)$$

$$c_{wy} = \sum_{i=1}^{N} m_i\left[(1 - y_i^2 - z_i^2)^{\frac{1}{2}}\dot{y}_i + \frac{(y_i\dot{y}_i + z_i\dot{z}_i)y_i}{(1 - y_i^2 - z_i^2)^{\frac{1}{2}}}\right]\cos(\alpha + a_i) + \frac{c}{\sum_{i=1}^{N} m_i(1 - y_i^2 - z_i^2)} \sum_{i=1}^{N}(1 - y_i^2 - z_i^2)^{\frac{1}{2}} y_i \sin(\alpha + a_i), \tag{3.26}$$

total angular momentum relative to the wz-plane:

$$c_{wz} = \textstyle\sum_{i=1}^{N} m_i(w_i\dot{z}_i - \dot{w}_i z_i)$$

$$c_{wz} = \sum_{i=1}^{N} m_i\left[(1 - y_i^2 - z_i^2)^{\frac{1}{2}}\dot{z}_i + \frac{(y_i\dot{y}_i + z_i\dot{z}_i)z_i}{(1 - y_i^2 - z_i^2)^{\frac{1}{2}}}\right]\cos(\alpha + a_i) + \frac{c}{\sum_{i=1}^{N} m_i(1 - y_i^2 - z_i^2)} \sum_{i=1}^{N}(1 - y_i^2 - z_i^2)^{\frac{1}{2}} z_i \sin(\alpha + a_i), \tag{3.27}$$

total angular momentum relative to the xy-plane:

$$c_{xy} = \textstyle\sum_{i=1}^{N} m_i(x_i\dot{y}_i - \dot{x}_i y_i)$$

$$c_{xy} = \sum_{i=1}^{N} m_i \left[(1 - y_i^2 - z_i^2)^{\frac{1}{2}} \dot{y}_i + \frac{(y_i \dot{y}_i + z_i \dot{z}_i) y_i}{(1 - y_i^2 - z_i^2)^{\frac{1}{2}}} \right] \sin(\alpha + a_i)$$
$$- \frac{c}{\sum_{i=1}^{N} m_i (1 - y_i^2 - z_i^2)} \sum_{i=1}^{N} (1 - y_i^2 - z_i^2)^{\frac{1}{2}} y_i \cos(\alpha + a_i), \tag{3.28}$$

total angular momentum relative to the xz-plane:

$$c_{xz} = \sum_{i=1}^{N} m_i (x_i \dot{z}_i - \dot{x}_i z_i)$$

$$c_{xz} = \sum_{i=1}^{N} m_i \left[(1 - y_i^2 - z_i^2)^{\frac{1}{2}} \dot{z}_i + \frac{(y_i \dot{y}_i + z_i \dot{z}_i) z_i}{(1 - y_i^2 - z_i^2)^{\frac{1}{2}}} \right] \sin(\alpha + a_i)$$
$$- \frac{c}{\sum_{i=1}^{N} m_i (1 - y_i^2 - z_i^2)} \sum_{i=1}^{N} (1 - y_i^2 - z_i^2)^{\frac{1}{2}} z_i \cos(\alpha + a_i), \tag{3.29}$$

total angular momentum relative to the yz-plane:

$$c_{yz} = \sum_{i=1}^{N} m_i (y_i \dot{z}_i - \dot{y}_i z_i)$$

$$c_{yz} = \sum_{i=1}^{N} m_i (y_i \dot{z}_i - \dot{y}_i z_i) = 0. \tag{3.30}$$

3.2 Rotopulsating positive elliptic-elliptic orbits

A criterion for the existence of positive elliptic-elliptic to analyze the solution introduced in Definition 3 is provided in this section. Afterwards, the integral of energy and the six integrals of the total angular momentum is derived which are specific for the orbits. It is elaborated that the results provide necessary and sufficient conditions for the existence of positive elliptic-elliptic rotopulsators

Criterion 2. *A solution of the type (2.70) is a rotopulsating positive elliptic-elliptic orbit for system (2.57) if and only if*

$$\dot{\alpha} = \frac{c_1}{\sum_{i=1}^N m_i r_i^2}, \quad \dot{\beta} = \frac{c_2}{M - \sum_{i=1}^N m_i r_i^2}, \tag{3.31}$$

with c_1, c_2 constants, and the variables $r_1, r_2, \ldots, r_N$ satisfy the N second-order differential equations

$$\begin{aligned} \ddot{r}_i &= r_i(1-r_i^2)\left[\frac{c_1^2}{(\sum_{i=1}^N m_i r_i^2)^2} - \frac{c_2^2}{(M - \sum_{i=1}^N m_i r_i^2)^2}\right] - \frac{r_i \dot{r}_i^2}{1-r_i^2} \\ &+ \sum_{\substack{j=1\\ j\neq i}}^N \frac{m_j[r_j(1-r_i^2)\cos(a_i - a_j) - r_i(1-r_i^2)^{\frac{1}{2}}(1-r_j^2)^{\frac{1}{2}}\cos(b_i - b_j)]}{(1-\epsilon_{ij}^2)^{\frac{3}{2}}}, \end{aligned} \tag{3.32}$$

$$r_i\ddot{\alpha} + 2\dot{r}_i\dot{\alpha} = -\sum_{\substack{j=1\\ j\neq i}}^N \frac{m_j r_j \sin(a_i - a_j)}{(1-\epsilon_{ij}^2)^{\frac{3}{2}}}, \quad i = 1, 2, \ldots, N, \tag{3.33}$$

$$\ddot{\beta} = -\frac{2\dot{\rho}_i\dot{\beta}}{\rho_i} - \frac{1}{\rho_i}\sum_{j=1}^N \frac{m_j p_j \sin(b_i - b_j)}{(1-\epsilon_{ij}^2)^{3/2}} \tag{3.34}$$

$$r_i\ddot{\beta} + 2\dot{r}_i\dot{\beta} = -\sum_{\substack{j=1\\ j\neq i}}^N \frac{m_j(1-r_j^2)^{\frac{1}{2}}\sin(b_i - b_j)}{(1-\epsilon_{ij}^2)^{\frac{3}{2}}}, \quad i = 1, 2, \ldots, N. \tag{3.35}$$

where, for any $i, j \in \{1, 2, \ldots, N\}$ with $i \neq j$, as denoted

$$\epsilon_{ij} := \mathbf{q}_i \cdot \mathbf{q}_j = r_i r_j \cos(a_i - a_j) + (1-r_i^2)^{\frac{1}{2}}(1-r_j^2)^{\frac{1}{2}}\cos(b_i - b_j).$$

Proof. Considering a solution of the type (2.70) for system (2.57). with having ρ_i in the form of r_i, $i = 1, 2, \ldots, N$, it is obtained that

$$\rho_i = (1-r_i^2)^{\frac{1}{2}}, \quad \dot{\rho}_i = -\frac{r_i\dot{r}_i}{(1-r_i^2)^{\frac{1}{2}}}, \quad \ddot{\rho}_i = -\frac{\dot{r}_i^2 + r_i(1-r_i^2)\ddot{r}_i}{(1-r_i^2)^{\frac{3}{2}}},$$

$$\epsilon_{ij} := \mathbf{q}_i \cdot \mathbf{q}_j = r_i r_j \cos(a_i - a_j) + (1 - r_i^2)^{\frac{1}{2}} (1 - r_j^2)^{\frac{1}{2}} \cos(b_i - b_j),$$

$$\dot{w}_i = \dot{r} \cos\alpha - r\dot{\alpha} \sin\alpha \tag{3.36}$$

$$\dot{x}_i = \dot{r} \sin\alpha + r\dot{\alpha} \cos\alpha \tag{3.37}$$

$$\dot{y}_i = \dot{\rho} \cos\beta - \rho\dot{\beta} \sin\beta \tag{3.38}$$

$$\dot{z}_i = \dot{\rho} \sin\beta + \rho\dot{\beta} \cos\beta \tag{3.39}$$

$$\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i = \dot{r}_i^2 + r_i^2 \dot{\alpha}^2 + \frac{r_i^2 \dot{r}_i^2}{1 - r_i^2} + (1 - r_i^2)\dot{\beta}^2.$$

$$\ddot{W}_i = \sum_{j=1}^{N} \frac{m_j [r_j \cos(\alpha + a_j) - \epsilon_{ij} r_i \cos(\alpha + a_i)]}{(1 - \epsilon_{ij}^2)^{3/2}} - (\dot{r}_i^2 + \dot{r}_i^2 \dot{\alpha}^2 + \rho_i^2 + \rho_i^2 \dot{\beta}^2) r_i \cos(\alpha + a_i) \tag{3.40}$$

$$\cos(\alpha + a_j) = \cos(\alpha + a_i)\cos(a_i - a_j) + \sin(\alpha + a_i)\sin(a_i - a_j) \tag{3.41}$$

$$\ddot{w}_i = \sum_{j=1}^{N} \frac{m_j [\cos(\alpha + a_i)\cos(a_i - a_j) + r_j \sin(\alpha + a_i)\sin(a_i - a_j) - \epsilon_{ij} r_i \cos(\alpha + a_i)]}{(1 - \epsilon_{ij}^2)^{3/2}}$$
$$- (\dot{r}_i^2 + r_i^2 \dot{\alpha}^2 + \rho_i^2 + \rho_i^2 + \rho_i^2 \dot{\beta}^2) r_i \cos(\alpha + a_i) \tag{3.42}$$

$$\ddot{r}_i - r_i \dot{\alpha}^2 = \sum_{j=1}^{N} \frac{m_j [r_j \cos(a_i - a_j) - \epsilon_{ij} r_i]}{(1 - \epsilon_{ij}^2)^{3/2}} - (\dot{r}_i^2 + r_i^2 \dot{\alpha}^2 + \dot{\rho}_i^2 + \rho_i^2 \dot{\beta}^2) r_i \tag{3.43}$$

$$-r_i \ddot{\alpha} - 2\dot{r}_i \dot{\alpha}_i = \sum_{j=1}^{N} \frac{m_j r_j \sin(a_i - a_j)}{(1 - \epsilon_{ij}^2)^{3/2}} \tag{3.44}$$

$$\begin{aligned} \ddot{r}_i &= r_i \dot{\alpha}^2 - r_i^3 \dot{\alpha}^2 - r_i \dot{r}_i^2 - r_i \dot{\rho}_i^2 - r_i \rho_i^2 \dot{\beta}^2 \\ &+ \sum_{j=1}^{N} \frac{m_j [r_j \cos(a_i - a_j) - r_i^2 r_j \cos(a_i - a_j) - r_i \rho_i \rho_j \cos(b_i - b_j)]}{(1 - \epsilon_{ij}^2)^{3/2}} \end{aligned} \tag{3.45}$$

$$\ddot{\alpha} = -\frac{2\dot{r}_i\dot{\alpha}}{r_i} - \sum_{j=1}^{N} \frac{m_j r_j \sin(a_i - a_j)}{r_i(1-\epsilon_{ij}^2)^{3/2}} \tag{3.46}$$

$$r_i\dot{r}_i + \rho_i\dot{\rho}_i = 0 \tag{3.47}$$

$$\dot{\rho}_i = -\frac{r_i\dot{r}_i}{\sqrt{1-r_i^2}} \quad , \quad \dot{\rho}_i^2 = \frac{r_i^2\dot{r}_i^2}{1-r_i^2} \tag{3.48}$$

With putting a solution candidate of the form (2.70) into system (2.57), and using the formulas presented above, the equations representing $\ddot{w}_i$ and $\ddot{x}_i$ the following equations will be derived

$$\begin{aligned}\ddot{r}_i = r_i\dot{\alpha}^2 - r_i^3\dot{\alpha}^2 - r_i\dot{r}_i^2 - \frac{r_i^3\dot{r}_i^2}{1-r_i^2} - r_i(1-r_i^2)\dot{\beta}^2 \\ + \sum_{j=1}^{N} \frac{m_j[r_j(1-r_i^2)\cos(a_i-a_j) - r_i\sqrt{(1-r_i)(1-r_j)}\cos(b_i-b_j)]}{(1-\epsilon_{ij}^2)^{3/2}}\end{aligned} \tag{3.49}$$

$$\begin{aligned}\ddot{r}_i = r_i(1-r_i^2)(\dot{\alpha}^2-\dot{\beta}^2) - \frac{r_i\dot{r}_i^2}{1-r_i^2} \\ + \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{m_j[r_j(1-r_i^2)\cos(a_i-a_j) - r_i(1-r_i^2)^{\frac{1}{2}}(1-r_j^2)^{\frac{1}{2}}\cos(b_i-b_j)]}{(1-\epsilon_{ij}^2)^{\frac{3}{2}}},\end{aligned} \tag{3.50}$$

$$\ddot{w}_i = (\ddot{r} - r\dot{\alpha}^2)\cos\alpha - (r\ddot{\alpha} + 2\dot{r}\dot{\alpha})\sin\alpha \tag{3.51}$$

$$\ddot{x}_i = (\ddot{r} - r\dot{\alpha}^2)\sin\alpha + (r\ddot{\alpha} + 2\dot{r}\dot{\alpha})\cos\alpha \tag{3.52}$$

$$\ddot{y}_i = (\ddot{\rho} - \rho\dot{\beta}^2)\cos\beta - (\rho\ddot{\beta} + 2\dot{\rho}\dot{\beta})\sin\beta \tag{3.53}$$

$$\ddot{z}_i = (\ddot{\rho} - \rho\dot{\beta}^2)\sin\beta + (\rho\ddot{\beta} + 2\dot{\rho}\dot{\beta})\cos\beta \tag{3.54}$$

$$\ddot{\alpha} = -\frac{2\dot{r}_i\dot{\alpha}}{r_i} - \frac{1}{r_i}\sum_{j=1}^{N}\frac{m_j r_j \sin(a_i - a_j)}{(1-\epsilon_{ij}^2)^{3/2}} \tag{3.55}$$

$$r_i\ddot{\alpha} + 2\dot{r}_i\dot{\alpha} = -\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{m_j r_j \sin(a_i - a_j)}{(1-\epsilon_{ij}^2)^{\frac{3}{2}}}, \quad i = 1, 2, \ldots, N, \tag{3.56}$$

whereas for the equations corresponding to $\ddot{y}_i, \ddot{z}_i$, we find equations (3.50) again as well as the equations

$$\ddot{\beta} = -\frac{2\dot{\rho}_i\dot{\beta}}{\rho_i} - \frac{1}{\rho_i}\sum_{j=1}^{N}\frac{m_j p_j \sin(b_i - b_j)}{(1-\epsilon_{ij}^2)^{3/2}} \tag{3.57}$$

$$r_i\ddot{\beta} + 2\dot{r}_i\dot{\beta} = -\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{m_j(1-r_j^2)^{\frac{1}{2}}\sin(b_i - b_j)}{(1-\epsilon_{ij}^2)^{\frac{3}{2}}}, \quad i = 1, 2, \ldots, N. \tag{3.58}$$

Equations (3.56) can be solved the same way the equations (3.20) were solved,

$$m_i r_i\ddot{\alpha} = -2m_i\dot{r}_i\dot{\alpha} - m_i\sum_{j=1}^{N}\frac{m_j r_j \sin(a_i - a_j)}{(1-\epsilon_{ij}^2)^{3/2}} \quad i = 1, 2, \cdots, n \tag{3.59}$$

$$(\sum_{i=1}^{N} m_i r_i)\ddot{\alpha} + 2(\sum_{i=1}^{N} m_i\dot{r}_i)\dot{\alpha} = 0 \Longrightarrow \dot{\alpha} = \gamma \tag{3.60}$$

$$\dot{\gamma} = -\frac{2(\sum_{i=1}^{N} m_i\dot{r}_i)\gamma}{\sum_{i=1}^{N} m_i r_i}, \quad \gamma = \frac{c_1}{m_1 r_1^2 + \cdots + m_N r_N^2} \tag{3.61}$$

$$\dot{\alpha} = \frac{c_1}{\sum_{i=1}^{N} m_i r_i^2},$$

where c_1 is an integration constant. To solve equations (3.58), using the similar solution technique, with a difference in having multiplied by $m_i(1-r_i^2)^{\frac{1}{2}}$ instead of $m_i r_i$, for each $i = 1, 2, \ldots, N$ the corresponding equation will be derived. After addition

$$(\sum_{i=1}^{N} m_i \rho_i)\ddot{\beta} + 2(\sum_{i=1}^{N} m_i \dot{\rho}_i)\dot{\beta} = 0 \Longrightarrow \dot{\beta} = \delta \tag{3.62}$$

$$\begin{aligned}\dot{\delta} = -\frac{2\sum_{i=1}^{N} m_i \dot{\rho}_i \delta}{\sum_{i=1}^{N} m_i \rho_i} \quad , \delta &= \frac{c_2}{m\rho_1 + \cdots + m_N \rho_N^2} = \frac{c_2}{m_1 - m_1 r_1^2 + \cdots + m_N - m_N r_N^2} \\ &= \frac{c_2}{M - (m_1 r_1^2 + \cdots + m_N r_N^2)} = \delta\end{aligned} \tag{3.63}$$

$$M = m_1 + \cdots + m_N$$

$$\dot{\beta} = \frac{c_2}{M - \sum_{i=1}^{N} m_i r_i^2},$$

where $M = \sum_{i=1}^{N} m_i$ and c_2 is an integration constant. Then equations (3.50) turns into

$$\begin{aligned}\ddot{r}_i &= r_i(1 - r_i^2)\left[\frac{c_1^2}{(\sum_{i=1}^{N} m_i r_i^2)^2} - \frac{c_2^2}{(M - \sum_{i=1}^{N} m_i r_i^2)^2}\right] - \frac{r_i \dot{r}_i^2}{1 - r_i^2} \\ &+ \sum_{\substack{j=1\\ j\neq i}}^{N} \frac{m_j[r_j(1 - r_i^2)\cos(a_i - a_j) - r_i(1 - r_i^2)^{\frac{1}{2}}(1 - r_j^2)^{\frac{1}{2}}\cos(b_i - b_j)]}{(1 - \epsilon_{ij}^2)^{\frac{3}{2}}},\end{aligned} \tag{3.64}$$

where, recall, for any $i, j \in \{1, 2, \ldots, N\}$ with $i \neq j$

$$\epsilon_{ij} := \mathbf{q}_i \cdot \mathbf{q}_j = r_i r_j \cos(a_i - a_j) + (1 - r_i^2)^{\frac{1}{2}}(1 - r_j^2)^{\frac{1}{2}}\cos(b_i - b_j).$$

□

Remark 8. It is important to note that the constants c_{wx} and c_{yz} of the total angular momentum are

$$c_{wx} = \sum_{i=1}^{N} m_i(w_i \dot{x}_i - \dot{w}_i x_i) = \dot{\alpha} \sum_{i=1}^{N} m_i r_i^2 = c_1,$$

$$c_{yz} = \sum_{i=1}^{N} m_i(y_i \dot{z}_i - \dot{z}_i y_i) = \dot{\beta} \sum_{i=1}^{N} m_i(1 - r_i^2) = c_2,$$

respectively, as a result, c_1 and c_2 represent the rotation of the particle system relative to the wx-plane and yz-plane, respectively.

Remark 9. One can conclude from (3.31), that $\dot{\alpha}$ and $\dot{\beta}$ are not independent of each other, the relationship governing this dependence is

$$\frac{c_1}{\dot{\alpha}} + \frac{c_2}{\dot{\beta}} = M, \tag{3.65}$$

written assuming that the α and β are not constant. In particular, if α and β are only different in one additive constant, then they are linear functions of time, i.e

$$\dot{\alpha} = \dot{\beta} = \frac{c_1 + c_2}{M}. \tag{3.66}$$

Then system (3.32) turns into

$$\ddot{r}_i = -\frac{r_i \dot{r}_i^2}{1 - r_i^2} + \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{m_j [r_j(1 - r_i^2)\cos(a_i - a_j) - r_i(1 - r_i^2)^{\frac{1}{2}}(1 - r_j^2)^{\frac{1}{2}}\cos(b_i - b_j)]}{(1 - \epsilon_{ij}^2)^{\frac{3}{2}}}, \tag{3.67}$$

$i = 1, 2, \ldots, N$.

Criterion 2 can confirm the existence of any solution candidate for a rotopulsating positive elliptic-elliptic orbit, but it may be difficult to use if that candidate is not a solution. It is preferred to have a simpler nonexistence methods. There is a way to obtain such a result using the integrals of motion. Proving the nonexistence is as straightforward as showing that the functions involved in the integrals are not constant. It is important to note that the first expression in the following result represents the energy integral, while the other equations are deducted from five of the six total angular momentum integrals.

Corollary 2. *If system* (2.57) *has a solution of the form (2.70), then the following expressions are constant*

energy:

$$h = \frac{1}{2}\sum_{i=1}^{N}\frac{m_i\dot{r}_i^2}{1-r_i^2} + \frac{\dot{\alpha}^2+\dot{\beta}^2}{2} - \sum_{1\le i<j\le N}\frac{m_im_j\epsilon_{ij}}{(1-\epsilon_{ij}^2)^{\frac{3}{2}}}, \tag{3.68}$$

total angular momentum relative to the wy-plane:

$$c_{wy} = \textstyle\sum_{i=1}^{N} m_i(w_i\dot{y}_i - \dot{w}_iy_i)$$

$$\begin{aligned} c_{wy} = \frac{1}{2}\sum_{i=1}^{N} m_i\Big[& r_i(1-r_i^2)^{\frac{1}{2}}(\dot{\alpha}+\dot{\beta})\sin(\alpha-\beta+a_i-b_i) \\ &+r_i(1-r_i^2)^{\frac{1}{2}}(\dot{\alpha}-\dot{\beta})\sin(\alpha+\beta+a_i+b_i) - \frac{\dot{r}_i}{(1-r_i^2)^{\frac{1}{2}}}\cos(\alpha-\beta+a_i-b_i) \\ &-\frac{\dot{r}_i}{(1-r_i^2)^{\frac{1}{2}}}\cos(\alpha+\beta+a_i+b_i)\Big], \end{aligned} \tag{3.69}$$

total angular momentum relative to the wz-plane:

$$c_{wz} = \textstyle\sum_{i=1}^{N} m_i(w_i\dot{z}_i - \dot{w}_iz_i)$$

$$\begin{aligned} c_{wz} = \frac{1}{2}\sum_{i=1}^{N} m_i\Big[& r_i(1-r_i^2)^{\frac{1}{2}}(\dot{\alpha}+\dot{\beta})\cos(\alpha+\beta+a_i+b_i) \\ &+r_i(1-r_i^2)^{\frac{1}{2}}(\dot{\alpha}-\dot{\beta})\cos(\alpha-\beta+a_i-b_i) + \frac{\dot{r}_i}{(1-r_i^2)^{\frac{1}{2}}}\sin(\alpha-\beta+a_i-b_i) \\ &-\frac{\dot{r}_i}{(1-r_i^2)^{\frac{1}{2}}}\sin(\alpha+\beta+a_i+b_i)\Big], \end{aligned} \tag{3.70}$$

total angular momentum relative to the xy-plane:

$$c_{xy} = \sum_{i=1}^{N} m_i(x_i\dot{y}_i - \dot{x}_i y_i)$$

$$c_{xy} = -\frac{1}{2}\sum_{i=1}^{N} m_i\Big[r_i(1-r_i^2)^{\frac{1}{2}}(\dot{\alpha}+\dot{\beta})\cos(\alpha-\beta+a_i-b_i)$$
$$+r_i(1-r_i^2)^{\frac{1}{2}}(\dot{\alpha}-\dot{\beta})\cos(\alpha+\beta+a_i+b_i)+\frac{\dot{r}_i}{(1-r_i^2)^{\frac{1}{2}}}\sin(\alpha-\beta+a_i-b_i) \quad (3.71)$$
$$+\frac{\dot{r}_i}{(1-r_i^2)^{\frac{1}{2}}}\sin(\alpha+\beta+a_i+b_i)\Big],$$

$$c_{xz} = \sum_{i=1}^{N} m_i(x_i\dot{z}_i - \dot{x}_i z_i)$$

total angular momentum relative to the xz-plane:

$$c_{xz} = \frac{1}{2}\sum_{i=1}^{N} m_i\Big[r_i(1-r_i^2)^{\frac{1}{2}}(\dot{\alpha}+\dot{\beta})\sin(\alpha-\beta+a_i-b_i)$$
$$-r_i(1-r_i^2)^{\frac{1}{2}}(\dot{\alpha}-\dot{\beta})\sin(\alpha+\beta+a_i+b_i)-\frac{\dot{r}_i}{(1-r_i^2)^{\frac{1}{2}}}\cos(\alpha-\beta+a_i-b_i) \quad (3.72)$$
$$-\frac{\dot{r}_i}{(1-r_i^2)^{\frac{1}{2}}}\cos(\alpha+\beta+a_i+b_i)\Big],$$

3.3 Rotopulsating negative elliptic orbits

A criterion will be presented in this section for the existence of negative elliptic rotopulsators to be used to analyze the solution introduced in Definition 3. In the next step, the integral of energy and the six integrals of the total angular momentum will be derived which are specific for these orbits. These results provide necessary and sufficient conditions for the existence of negative elliptic rotopulsators.

Criterion 3. *A solution of the type (2.71) is a rotopulsating positive elliptic orbit for* *system* (2.58) *if and only if*

$$\dot{\alpha} = \frac{b}{\sum_{j=1}^{N} m_j(z_j^2 - y_j^2 - 1)}, \tag{3.73}$$

where b *is a constant, and the variables* $y_i, z_i,\ i = 1, 2, \ldots, N$, *meet the requirements of the system of* $2N$ *second-order differential equations*

$$\begin{cases} \ddot{y}_i = \sum_{\substack{j=1\\ j\neq i}}^{N} \frac{m_j(y_j+\mu_{ij}y_i)}{(\mu_{ij}^2-1)^{3/2}} + F_i y_i \\ \ddot{z}_i = \sum_{\substack{j=1\\ j\neq i}}^{N} \frac{m_j(z_j+\mu_{ij}z_i)}{(\mu_{ij}^2-1)^{3/2}} + F_i z_i \\ r_i\ddot{\alpha} + 2\dot{r}_i\dot{\alpha} = \sum_{\substack{j=1\\ j\neq i}}^{N} \frac{m_j r_j \sin(a_j-a_i)}{(\mu_{ij}^2-1)^{3/2}}, \end{cases} \tag{3.74}$$

where

$$F_i := \frac{[(y_i\dot{z}_i - z_i\dot{y}_i)^2 + \dot{z}_i^2 - \dot{y}_i^2]}{z_i^2 - y_i^2 - 1} + \frac{b^2(z_i^2 - y_i^2 - 1)}{[\sum_{j=1}^{N} m_j(z_j^2 - y_j^2 - 1)]^2}, \tag{3.75}$$

$i = 1, 2, \ldots, N$, *and, for any* $i, j \in \{1, 2, \ldots, N\}$, μ_{ij} *is given by*

$$\mu_{ij} := \mathbf{q}_i \cdot \mathbf{q_j} = (z_i^2 - y_i^2 - 1)^{\frac{1}{2}}(z_j^2 - y_j^2 - 1)^{\frac{1}{2}} \cos(a_i - a_j) + y_i y_j - z_i z_j.$$

Proof. Considering a solution of the type (2.71) which has the initial conditions presented above, and for any $i, j \in \{1, 2, \ldots, N\}$ there is:

$$\mu_{ij} := \mathbf{q}_i \boxdot \mathbf{q_j} = (z_i^2 - y_i^2 - 1)^{\frac{1}{2}}(z_j^2 - y_j^2 - 1)^{\frac{1}{2}} \cos(a_i - a_j) + y_i y_j - z_i z_j,$$

and for any $i = 1, 2, \ldots, N$ it is found that

$$\dot{\mathbf{q}}_i \boxdot \dot{\mathbf{q}}_i = \frac{(y_i\dot{z}_i - z_i\dot{y}_i)^2 + \dot{z}_i^2 - \dot{y}_i^2 + (z_i^2 - y_i^2 - 1)^2\dot{\alpha}^2}{z_i^2 - y_i^2 - 1}.$$

For all $i = 1, 2, \ldots, N$, each r_i can be presented in the form of y_i and z_i to derive

$$r_i = (z_i^2 - y_i^2 - 1)^{\frac{1}{2}}, \quad \dot{r}_i = \frac{z_i\dot{z}_i - y_i\dot{y}_i}{(z_i^2 - y_i^2 - 1)^{\frac{1}{2}}},$$

$$\ddot{r}_i = \frac{(z_i^2 - y_i^2 - 1)(z_i\ddot{z}_i - y_i\ddot{y}_i) + \dot{y}_i^2 - \dot{z}_i^2 - (y_i\dot{z}_i - z_i\dot{y}_i)^2}{(z_i^2 - y_i^2 - 1)^{\frac{3}{2}}}.$$

Putting a solution with (2.71) type into system (2.58) and using the above formulas, the equations representing $\ddot{y}_i$ and $\ddot{z}_i$ are derived that

$$\ddot{y}_i = \sum_{\substack{j=1\\ j\neq i}}^{N} \frac{m_j(y_j + \mu_{ij}y_i)}{(\mu_{ij}^2 - 1)^{\frac{3}{2}}} + \frac{[(y_i\dot{z}_i - z_i\dot{y}_i)^2 + \dot{z}_i^2 - \dot{y}_i^2]y_i}{z_i^2 - y_i^2 - 1} + (z_i^2 - y_i^2 - 1)y_i\dot{\alpha}^2, \tag{3.76}$$

$$\ddot{z}_i = \sum_{\substack{j=1\\ j\neq i}}^{N} \frac{m_j(z_j + \mu_{ij}z_i)}{(\mu_{ij}^2 - 1)^{\frac{3}{2}}} + \frac{[(y_i\dot{z}_i - z_i\dot{y}_i)^2 + \dot{z}_i^2 - \dot{y}_i^2]z_i}{z_i^2 - y_i^2 - 1} + (z_i^2 - y_i^2 - 1)z_i\dot{\alpha}^2, \tag{3.77}$$

whereas for the equations representing $\ddot{w}_i$ and $\ddot{x}_i$, leads to following identities or equations detailed computations using (3.76) and (3.77):

$$\begin{aligned}\ddot{w}_i = &\sum_{j=1, j\neq i}^{N} \frac{m_j[r_j\cos(\alpha + a_j) + \mu_{ij}r_i\cos(\alpha + a_i)]}{(\mu_{ij}^2 - 1)} \\ &+ [\frac{\dot{z}_i^2 - \dot{y}_i^2 + (y_i\dot{z}_i - z_i\dot{y}_i)^2}{z_i^2 - y_i^2 - 1} + (z_i^2 - y_i^2 - 1)\dot{\alpha}^2]r_i\cos(\alpha + a_i)\end{aligned} \tag{3.78}$$

$$\cos(\alpha + a_j) = \cos(\alpha + a_i)\cos(a_i - a_j) + \sin(\alpha + a_i)\sin(a_i - a_j) \tag{3.79}$$

$$\begin{aligned}\ddot{w}_i = &\sum_{i=1, j\neq i}^{N} \frac{m_j[r_j\cos(\alpha + a_j)\cos(a_i - a_j) + r_j\sin(\alpha + a_i)\sin(a_i - a_j) + \mu_{ij}r_i\cos(\alpha + a_i)]}{(m_{ij}^2 - 1)^{3/2}} \\ &+ [\frac{\dot{z}_i^2 - \dot{y}_i^2 + (y_i\dot{z}_i - z_i\dot{y}_i)^2}{z_i^2 - y_i^2 - 1} + (z_i^2 - y_i^2 - 1)\dot{\alpha}^2]r_i\cos(\alpha + a_i)\end{aligned} \tag{3.80}$$

$$-r_i\ddot{\alpha} - 2\dot{r}_i\dot{\alpha} = \sum_{j=1}^{N} \frac{m_j r_j \sin(a_i - a_j)}{(\mu_{ij} - 1)^{3/2}} \tag{3.81}$$

$$r_i\ddot{\alpha} + 2\dot{r}_i\dot{\alpha} = -\sum_{j=1}^{N} \frac{m_j r_j \sin(a_i - a_j)}{(\mu_{ij}^2 - 1)^{3/2}} \tag{3.82}$$

$$r_i\ddot{\alpha} + 2\dot{r}_i\dot{\alpha} = \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{m_j r_j \sin(a_j - a_i)}{(\mu_{ij}^2 - 1)^{\frac{3}{2}}}, \quad i = 1, 2, \dots, N. \tag{3.83}$$

It is clear that for every $i = 1, 2, \dots, N$, after multiplying the ith equation in (3.83) by $m_i r_i$, calculating the summation of the resulting N equations:

$$\sum_{i=1}^{N}\sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{m_i m_j r_i r_j \sin(a_j - a_i)}{(\mu_{ij}^2 - 1)^{\frac{3}{2}}} = 0.$$

Thus the following equation is derived

$$\left(\sum_{i=1}^{N} m_i r_i^2\right)\ddot{\alpha} + 2\left(\sum_{i=1}^{N} m_i r_i \dot{r}_i\right)\dot{\alpha} = 0,$$

which has the solution

$$\dot{\alpha} = \gamma, \quad \dot{\gamma} = -\frac{2(\sum_{i=1}^{N} m_i r_i \dot{r}_i)\gamma}{\sum_{i=1}^{N} m_i r_i^2}, \quad \gamma = \frac{b}{\sum_{i=1}^{N} m_i r_i^2} \tag{3.84}$$

$$\dot{\alpha} = \frac{b}{\sum_{i=1}^{N} m_i r_i^2} = \frac{b}{\sum_{i=1}^{N} m_i (z_i^2 - y_i^2 - 1)},$$

where b is an integration constant. As a result, equations (3.76) and (3.77) turn into

$$\ddot{y}_i = \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{m_j(y_j + \mu_{ij} y_i)}{(\mu_{ij}^2 - 1)^{\frac{3}{2}}} + \frac{[(y_i\dot{z}_i - z_i\dot{y}_i)^2 + \dot{z}_i^2 - \dot{y}_i^2]y_i}{z_i^2 - y_i^2 - 1} + \frac{b^2(z_i^2 - y_i^2 - 1)y_i}{[\sum_{j=1}^{N} m_j(z_j^2 - y_j^2 - 1)]^2}, \tag{3.85}$$

$$\ddot{z}_i = \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{m_j(z_j + \mu_{ij}z_i)}{(\mu_{ij}^2 - 1)^{\frac{3}{2}}} + \frac{[(y_i\dot{z}_i - z_i\dot{y}_i)^2 + \dot{z}_i^2 - \dot{y}_i^2]z_i}{z_i^2 - y_i^2 - 1} + \frac{b^2(z_i^2 - y_i^2 - 1)z_i}{[\sum_{j=1}^{N} m_j(z_j^2 - y_j^2 - 1)]^2}, \tag{3.86}$$

$i = 1, 2, \ldots, N$, where, recall, for any $i, j \in \{1, 2, \ldots, N\}$

$$\mu_{ij} := \mathbf{q}_i \cdot \mathbf{q_j} = (z_i^2 - y_i^2 - 1)^{\frac{1}{2}}(z_j^2 - y_j^2 - 1)^{\frac{1}{2}}\cos(a_i - a_j) + y_i y_j - z_i z_j.$$

□

Remark 10. It is important to note that the constant c_{wx} of the total angular momentum is

$$c_{wx} = \sum_{i=1}^{N} m_i(w_i\dot{x}_i - \dot{w}_i x_i) = \dot{\alpha}\sum_{i=1}^{N} m_i(z_i^2 - y_i^2 - 1) = b,$$

so b represents the rotation of the particles relative to the wx-plane.

Corollary 3. *If system* (2.58) *has a solution of the form (2.71), then the following expressions are constant:*

energy:

$$\begin{aligned} h = &\sum_{i=1}^{N} \frac{m_i[(y_i\dot{z}_i - z_i\dot{y}_i)^2 + \dot{z}_i^2 - \dot{y}_i^2]}{2(z_i^2 - y_i^2 - 1)} \\ &+ \frac{b^2}{2\sum_{i=1}^{N} m_i(z_i^2 - y_i^2 - 1)} + \sum_{1\leq i<j\leq N} \frac{m_i m_j \mu_{ij}}{(\mu_{ij}^2 - 1)^{\frac{1}{2}}}, \end{aligned} \tag{3.87}$$

total angular momentum relative to the wy-plane:

$$c_{wy} = \sum_{i=1}^{N} m_i \left[(z_i^2 - y_i^2 - 1)^{\frac{1}{2}} \dot{y}_i + \frac{(y_i \dot{y}_i - z_i \dot{z}_i) y_i}{(z_i^2 - y_i^2 - 1)^{\frac{1}{2}}} \right] \cos(\alpha + a_i)$$
$$+ \frac{b}{\sum_{i=1}^{N} m_i (z_i^2 - y_i^2 - 1)} \sum_{i=1}^{N} m_i (z_i^2 - y_i^2 - 1)^{\frac{1}{2}} y_i \sin(\alpha + a_i), \tag{3.88}$$

total angular momentum relative to the wz-plane:

$$c_{wz} = \sum_{i=1}^{N} m_i \left[(z_i^2 - y_i^2 - 1)^{\frac{1}{2}} \dot{z}_i + \frac{(y_i \dot{y}_i - z_i \dot{z}_i) z_i}{(z_i^2 - y_i^2 - 1)^{\frac{1}{2}}} \right] \cos(\alpha + a_i)$$
$$+ \frac{b}{\sum_{i=1}^{N} m_i (z_i^2 - y_i^2 - 1)} \sum_{i=1}^{N} m_i (z_i^2 - y_i^2 - 1)^{\frac{1}{2}} z_i \sin(\alpha + a_i), \tag{3.89}$$

total angular momentum relative to the xy-plane:

$$c_{xy} = \sum_{i=1}^{N} m_i \left[(z_i^2 - y_i^2 - 1)^{\frac{1}{2}} \dot{y}_i + \frac{(y_i \dot{y}_i - z_i \dot{z}_i) y_i}{(z_i^2 - y_i^2 - 1)^{\frac{1}{2}}} \right] \sin(\alpha + a_i)$$
$$- \frac{b}{\sum_{i=1}^{N} m_i (z_i^2 - y_i^2 - 1)} \sum_{i=1}^{N} m_i (z_i^2 - y_i^2 - 1)^{\frac{1}{2}} y_i \cos(\alpha + a_i), \tag{3.90}$$

total angular momentum relative to the xz-plane:

$$c_{xz} = \sum_{i=1}^{N} m_i \left[(z_i^2 - y_i^2 - 1)^{\frac{1}{2}} \dot{z}_i + \frac{(y_i \dot{y}_i - z_i \dot{z}_i) z_i}{(z_i^2 - y_i^2 - 1)^{\frac{1}{2}}} \right] \sin(\alpha + a_i)$$
$$- \frac{b}{\sum_{i=1}^{N} m_i (z_i^2 - y_i^2 - 1)} \sum_{i=1}^{N} m_i (z_i^2 - y_i^2 - 1)^{\frac{1}{2}} z_i \cos(\alpha + a_i), \tag{3.91}$$

total angular momentum relative to the yz-plane:

$$c_{yz} = \sum_{i=1}^{N} m_i(y_i\dot{z}_i - \dot{y}_i z_i). \tag{3.92}$$

3.4 Rotopulsating negative hyperbolic orbits

A criterion will be presented in this section for the existence of negative hyperbolic rotopulsators to be used to analyze the solution introduced in Definition 5. In the next step, the integral of energy and the six integrals of the total angular momentum will be derived which are specific for these orbits. This result provides necessary and sufficient conditions for the existence of negative hyperbolic rotopulsators.

Criterion 4. *A solution of the type (2.72) is a rotopulsating positive elliptic orbit for system* (2.58) *if and only if*

$$\dot{\beta} = \frac{a}{\sum_{j=1}^{N} m_j(w_j^2 + x_j^2 + 1)}, \tag{3.93}$$

where a is a constant, and the variables $w_i, x_i,\ i = 1, 2, \ldots, N,$ *that meet the requirements of the system of* $2N$ *second-order differential equations*

$$\begin{cases} \ddot{w}_i = \sum_{\substack{j=1\\ j\neq i}}^{N} \frac{m_j(w_j+\nu_{ij}w_i)}{(\nu_{ij}^2-1)^{3/2}} + H_i w_i \\ \ddot{x}_i = \sum_{\substack{j=1\\ j\neq i}}^{N} \frac{m_j(x_j+\nu_{ij}x_i)}{(\nu_{ij}^2-1)^{3/2}} + H_i x_i \\ \eta_i\ddot{\beta} + 2\dot{\eta}_i\dot{\beta} = \sum_{\substack{j=1\\ j\neq i}}^{N} \frac{m_j\eta_j \sinh(b_j-b_i)}{(\nu_{ij}^2-1)^{3/2}}, \end{cases} \tag{3.94}$$

where

$$H_i := \frac{[(w_i\dot{x}_i - x_i\dot{w}_i)^2 + \dot{w}_i^2 + \dot{x}_i^2]}{w_i^2 + x_i^2 + 1} + \frac{a^2(w_i^2 + x_i^2 + 1)}{[\sum_{j=1}^{N} m_j(w_j^2 + x_j^2 + 1)]^2}, \tag{3.95}$$

$i = 1, 2, \ldots, N,$ *and, for any* $i, j \in \{1, 2, \ldots, N\}$, ν_{ij} *is given by*

$$\nu_{ij} := \mathbf{q}_i \boxdot \mathbf{q_j} = w_i w_j + x_i x_j - (w_i^2 + x_i^2 + 1)^{\frac{1}{2}} (w_j^2 + x_j^2 + 1)^{\frac{1}{2}} \cosh(b_i - b_j).$$

Proof. Considering a solution of the type (2.72) with having the initial conditions presented earlier. Then, for any $i, j \in \{1, 2, \ldots, N\}$ there is

$$\nu_{ij} := \mathbf{q}_i \boxdot \mathbf{q_j} = w_i w_j + x_i x_j - (w_i^2 + x_i^2 + 1)^{\frac{1}{2}} (w_j^2 + x_j^2 + 1)^{\frac{1}{2}} \cosh(b_i - b_j),$$

for any $i = 1, 2, \ldots, N$ it is found that

$$\dot{\mathbf{q}}_i \boxdot \dot{\mathbf{q}}_i = \frac{(w_i \dot{x}_i - x_i \dot{w}_i)^2 + \dot{w}_i^2 + \dot{x}_i^2 + (w_i^2 + x_i^2 + 1)^2 \dot{\beta}^2}{w_i^2 + x_i^2 + 1}.$$

For all $i = 1, 2, \ldots, N$, each r_i can be presented in the type of y_i and z_i to derive

$$\eta_i = (w_i^2 + x_i^2 + 1)^{\frac{1}{2}}, \quad \dot{\eta}_i = \frac{w_i \dot{w}_i + x_i \dot{x}_i}{(w_i^2 + x_i^2 + 1)^{\frac{1}{2}}},$$

$$\ddot{\eta}_i = \frac{(w_i^2 + x_i^2 + 1)(w_i \ddot{w}_i + x_i \ddot{x}_i) + \dot{w}_i^2 + \dot{x}_i^2 + (w_i \dot{x}_i - x_i \dot{w}_i)^2}{(w_i^2 + x_i^2 + 1)^{\frac{3}{2}}}.$$

Putting a solution with (2.72) into system (2.58) and using the above formulas, the equations representing $\ddot{w}_i$ and $\ddot{x}_i$ are derived that

$$\ddot{w}_i = \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{m_j (w_j + \nu_{ij} w_i)}{(\nu_{ij}^2 - 1)^{\frac{3}{2}}} + \frac{[(w_i \dot{x}_i - x_i \dot{w}_i)^2 + \dot{w}_i^2 + \dot{x}_i^2] w_i}{w_i^2 + x_i^2 + 1} + (w_i^2 + x_i^2 + 1) w_i \dot{\beta}^2, \quad (3.96)$$

$$\ddot{x}_i = \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{m_j (x_j + \nu_{ij} x_i)}{(\mu_{ij}^2 - 1)^{\frac{3}{2}}} + \frac{[(w_i \dot{x}_i - x_i \dot{w}_i)^2 + \dot{w}_i^2 + \dot{x}_i^2] x_i}{w_i^2 + x_i^2 + 1} + (w_i^2 + x_i^2 + 1) x_i \dot{\beta}^2, \quad (3.97)$$

whereas for the equations representing $\ddot{y}_i$ and $\ddot{z}_i$, leads to following identities or equations detailed computations using (3.96) and (3.97),

$$\eta_i\ddot{\beta} + 2\dot{\eta}_i\dot{\beta} = \sum_{\substack{j=1\\j\neq i}}^{N} \frac{m_j\eta_j \sinh(b_j - b_i)}{(\nu_{ij}^2 - 1)^{\frac{3}{2}}}, \quad i = 1, 2, \ldots, N. \tag{3.98}$$

For every $i = 1, 2, \ldots, N$, and further multiplying the ith equation in (3.83) by $m_i r_i$, add the resulting N equations, it is noticed that

$$\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N} \frac{m_i m_j \eta_i\eta_j \sinh(b_j - b_i)}{(\nu_{ij}^2 - 1)^{\frac{3}{2}}} = 0.$$

Thus the equation is obtained

$$\left(\sum_{i=1}^{N} m_i\eta_i^2\right)\ddot{\beta} + 2\left(\sum_{i=1}^{N} m_i\eta_i\dot{\eta}_i\right)\dot{\beta} = 0,$$

which has the solution

$$\dot{\beta} = \frac{a}{\sum_{i=1}^{N} m_i\eta_i^2} = \frac{a}{\sum_{i=1}^{N} m_i(w_i^2 + x_i^2 + 1)},$$

where a is an integration constant. As a result, equations (3.96) and (3.97) turn into:

$$\ddot{w}_i = \sum_{\substack{j=1\\j\neq i}}^{N} \frac{m_j(w_j + \nu_{ij}w_i)}{(\nu_{ij}^2 - 1)^{\frac{3}{2}}} + \frac{[(w_i\dot{x}_i - x_i\dot{w}_i)^2 + \dot{w}_i^2 + \dot{x}_i^2]w_i}{w_i^2 + x_i^2 + 1} + \frac{a^2(w_i^2 + x_i^2 + 1)w_i}{[\sum_{j=1}^{N} m_j(w_j^2 + x_j^2 + 1)]^2}, \tag{3.99}$$

$$\ddot{x}_i = \sum_{\substack{j=1\\j\neq i}}^{N} \frac{m_j(x_j + \nu_{ij}x_i)}{(\nu_{ij}^2 - 1)^{\frac{3}{2}}} + \frac{[(w_i\dot{x}_i - x_i\dot{w}_i)^2 + \dot{w}_i^2 + \dot{x}_i^2]x_i}{w_i^2 + x_i^2 + 1} + \frac{a^2(w_i^2 + x_i^2 + 1)x_i}{[\sum_{j=1}^{N} m_j(w_j^2 + x_j^2 + 1)]^2}, \tag{3.100}$$

$i = 1, 2, \ldots, N$, where, recall, for any $i, j \in \{1, 2, \ldots, N\}$

$$\nu_{ij} := \mathbf{q}_i \boxdot \mathbf{q_j} = w_iw_j + x_ix_j - (w_i^2 + x_i^2 + 1)^{\frac{1}{2}}(w_j^2 + x_j^2 + 1)^{\frac{1}{2}} \cosh(b_i - b_j).$$

□

Remark 11. It is important to note that the constant c_{yz} of the angular momentum is

$$c_{yz} = \sum_{i=1}^{N} m_i(y_i\dot{z}_i - \dot{y}_i z_i) = -\dot{\beta}\sum_{i=1}^{N} m_i(w_i^2 + x_i^2 + 1) = -a.$$

Corollary 4. *If system* (2.58) *has a solution of the form (2.72), then the following expressions are constant:*

energy:

$$\begin{aligned} h = &\sum_{i=1}^{N} \frac{m_i[(w_i\dot{x}_i - x_i\dot{w}_i)^2 + \dot{w}_i^2 + \dot{x}_i^2]}{2(w_i^2 + x_i^2 + 1)} \\ &+ \frac{a^2}{2\sum_{j=1}^{N} m_j(w_j^2 + x_j^2 + 1)} + \sum_{1 \le i < j \le N} \frac{m_i m_j \nu_{ij}}{(\nu_{ij}^2 - 1)^{1/2}}, \end{aligned} \tag{3.101}$$

total angular momentum relative to the wx-plane:

$$c_{wx} = \sum_{i=1}^{N} m_i(w_i\dot{x}_i - \dot{w}_i x_i), \tag{3.102}$$

total angular momentum relative to the wy-plane:

$$\begin{aligned} c_{wy} = &\sum_{i=1}^{N} \frac{m_i[x_i(w_i\dot{x}_i - x_i\dot{w}_i) - \dot{w}_i]}{(w_i^2 + x_i^2 + 1)^{\frac{1}{2}}} \sinh(\beta + b_i) \\ &+ \frac{a}{\sum_{j=1}^{N} m_j(w_j^2 + x_j^2 + 1)} \sum_{i=1}^{N} m_i w_i (w_i^2 + x_i^2 + 1)^{\frac{1}{2}} \cosh(\beta + b_i) \end{aligned} \tag{3.103}$$

total angular momentum relative to the wz-plane:

$$
\begin{aligned}
c_{wz} &= \sum_{i=1}^{N} \frac{m_i[x_i(w_i\dot{x}_i - x_i\dot{w}_i) - \dot{w}_i]}{(w_i^2 + x_i^2 + 1)^{\frac{1}{2}}} \cosh(\beta + b_i) \\
&+ \frac{a}{\sum_{j=1}^{N} m_j(w_j^2 + x_j^2 + 1)} \sum_{i=1}^{N} m_i w_i (w_i^2 + x_i^2 + 1)^{\frac{1}{2}} \sinh(\beta + b_i)
\end{aligned} \tag{3.104}
$$

total angular momentum relative to the xy-plane:

$$
\begin{aligned}
c_{xy} &= \sum_{i=1}^{N} \frac{m_i[w_i(\dot{w}_i x_i - w_i\dot{x}_i) - \dot{x}_i]}{(w_i^2 + x_i^2 + 1)^{\frac{1}{2}}} \sinh(\beta + b_i) \\
&+ \frac{a}{\sum_{j=1}^{N} m_j(w_j^2 + x_j^2 + 1)} \sum_{i=1}^{N} m_i x_i (w_i^2 + x_i^2 + 1)^{\frac{1}{2}} \cosh(\beta + b_i)
\end{aligned} \tag{3.105}
$$

total angular momentum relative to the xz-plane:

$$
\begin{aligned}
c_{xz} &= \sum_{i=1}^{N} \frac{m_i[w_i(\dot{w}_i x_i - w_i\dot{x}_i) - \dot{x}_i]}{(w_i^2 + x_i^2 + 1)^{\frac{1}{2}}} \cosh(\beta + b_i) \\
&+ \frac{a}{\sum_{j=1}^{N} m_j(w_j^2 + x_j^2 + 1)} \sum_{i=1}^{N} m_i x_i (w_i^2 + x_i^2 + 1)^{\frac{1}{2}} \sinh(\beta + b_i)
\end{aligned} \tag{3.106}
$$

3.5 Rotopulsating negative elliptic-hyperbolic orbits

A criterion will be presented in this section for the existence of negative elliptic-hyperbolic rotopulsators to be used to analyze the solution introduced in Definition 6. In the next step, the integral of energy and the six integrals of the total angular momentum will be derived which are specific for these orbits. These results provide necessary and sufficient conditions for the existence of negative elliptic-hyperbolic rotopulsators.

Criterion 5. *A solution of the type (2.73) is a rotopulsating negative elliptic-hyperbolic orbit for system* (2.58) *if and only if*

$$\dot{\alpha} = \frac{d_1}{\sum_{i=1}^N m_i r_i^2}, \quad \dot{\beta} = \frac{d_2}{M + \sum_{i=1}^N m_i r_i^2}, \tag{3.107}$$

with d_1, d_2 *constants, and the variables* r_i*,* $i = 1, \ldots, N$*, satisfy the* N *second-order differential equations*

$$\begin{aligned} \ddot{r}_i &= r_i(1+r_i^2)\left[\frac{d_1^2}{(\sum_{i=1}^N m_i r_i^2)^2} - \frac{d_2^2}{(M + \sum_{i=1}^N m_i r_i^2)^2}\right] + \frac{r_i \dot{r}_i^2}{1+r_i^2} \\ &+ \sum_{\substack{j=1\\ j\neq i}}^N \frac{m_j[r_j(1+r_i^2)\cos(a_i - a_j) - r_i(1+r_i^2)^{\frac{1}{2}}(1+r_j^2)^{\frac{1}{2}}\cosh(b_i - b_j)]}{(\delta_{ij}^2 - 1)^{\frac{3}{2}}}, \end{aligned} \tag{3.108}$$

$$r_i\ddot{\alpha} + 2\dot{r}_i\dot{\alpha} = \sum_{\substack{j=1\\ j\neq i}}^N \frac{m_j r_j \sin(a_j - a_i)}{(\delta_{ij}^2 - 1)^{\frac{3}{2}}}, \quad i = 1, 2, \ldots, N, \tag{3.109}$$

$$r_i\ddot{\beta} + 2\dot{r}_i\dot{\beta} = \sum_{\substack{j=1\\ j\neq i}}^N \frac{m_j (1+r_j^2)^{\frac{1}{2}} \sin(b_j - b_i)}{(\delta_{ij}^2 - 1)^{\frac{3}{2}}}, \quad i = 1, 2, \ldots, N. \tag{3.110}$$

where, for any $i, j \in \{1, 2, \ldots, N\}$ *with* $i \neq j$*, it is denoted*

$$\delta_{ij} := \mathbf{q}_i \boxdot \mathbf{q}_j = r_i r_j \cos(a_i - a_j) - (1+r_i^2)^{\frac{1}{2}}(1+r_j^2)^{\frac{1}{2}}\cosh(b_i - b_j).$$

Proof. Considering a solution of the type (2.73) for system (2.58). with having ρ_i the form of r_i, $i = 1, 2, \ldots, N$, it is presented that

$$\eta_i = (1+r_i^2)^{\frac{1}{2}}, \quad \dot{\eta}_i = \frac{r_i \dot{r}_i}{(1+r_i^2)^{\frac{1}{2}}}, \quad \ddot{\eta}_i = \frac{\dot{r}_i^2 + r_i(1+r_i^2)\ddot{r}_i}{(1+r_i^2)^{\frac{3}{2}}},$$

$$\delta_{ij} := \mathbf{q}_i \boxdot \mathbf{q}_j = r_i r_j \cos(a_i - a_j) - (1 + r_i^2)^{\frac{1}{2}}(1 + r_j^2)^{\frac{1}{2}} \cosh(b_i - b_j),$$

$$\dot{\mathbf{q}}_i \boxdot \dot{\mathbf{q}}_i = \dot{r}_i^2 + r_i^2\dot{\alpha}^2 - \frac{r_i^2\dot{r}_i^2}{1 + r_i^2} - (1 + r_i^2)\dot{\beta}^2.$$

$$\ddot{w}_i = (\ddot{r}_i - r_i\dot{\alpha}^2)\cos(\alpha + a_i) - (r_i\ddot{\alpha} + 2\dot{r}_i\dot{\alpha})\sin(\alpha + a_i) \tag{3.111}$$

$$\ddot{w}_i = \sum_{j=1}^{N} \frac{m_j[r_j \cos(\alpha + a_j) + q_{ij} r_i \cos(\alpha + a_i)]}{(q_{ij}^2 - 1)^{3/2}} + (\dot{r}_i^2 + r_i^2\dot{\alpha}^2 - \frac{r_i^2\dot{r_i}^2}{(1 + r_i^2)} - (1 + r_i^2)\dot{\beta}^2) r_i \cos(\alpha + a_i) \tag{3.112}$$

But

$$\cos(\alpha + a_j) = \cos(\alpha + a_i)\cos(a_i - a_j) + \sin(\alpha + a_i)\sin(a_i - a_j) \tag{3.113}$$

$$\ddot{w}_i = \sum_{j=1}^{N} \frac{m_j[r_j \cos(\alpha + a_i)\cos(a_i - a_j) + r_j \sin(\alpha + a_i)\sin(a_i - a_j) + q_{ij} r_i \cos(\alpha + a_i)]}{(q_{ij}^2 - 1)^{3/2}} + (\dot{r}_i^2 + r_i^2\dot{\alpha}^2 - \frac{r_i^2\dot{r}_i^2}{(1 + r_i^2)} - (1 + r_i^2)\dot{\beta}^2) r_i \cos(\alpha + a_i) \tag{3.114}$$

$$\ddot{r}_i - r_i\dot{\alpha}^2 = \sum_{j=1}^{N} \frac{m_j[r_j \cos(a_i - a_j) + q_{ij} r_i]}{(q_{ij}^2 - 1)^{3/2}} + (\dot{r}_i^2 + r_i^2\dot{\alpha}^2 - \frac{r_i^2\dot{r}_i^2}{(1 + r_i^2)} - (1 + r_i^2)\dot{\beta}^2) \tag{3.115}$$

$$\ddot{r}_i = r_i\dot{\alpha}^2 + r_i^3\dot{\alpha}^2 + r_i\dot{r}_i^2 - r_i\dot{\eta}_i^2 - r_i\eta_i^2\dot{\beta}^2 + \sum_{j=1}^{N} \frac{m_j[r_j\cos(a_i - a_j) + r_i^2 r_j\cos(a_i - a_j) - r_i\eta_i\eta_j\cos h(b_i - b_j)]}{(q_{ij}^2 - 1)^{3/2}} \tag{3.116}$$

$$\ddot{\alpha} = -\frac{2\dot{r}_i\dot{\alpha}}{r_i} - \sum_{j=1}^{N} \frac{m_j r_j \sin(a_i - a_j)}{r_i(q_{ij}^2 - 1)^{3/2}} \tag{3.117}$$

Putting a solution with (2.73) into system (2.58) and using the above formulas, the equations representing $\ddot{w}_i$ and $\ddot{x}_i$ are derived that

$$\ddot{r}_i = r_i(1 + r_i^2)(\dot{\alpha}^2 - \dot{\beta}^2) + \frac{r_i\dot{r}_i^2}{1 - r_i^2} + \sum_{\substack{j=1\\ j\neq i}}^{N} \frac{m_j[r_j(1 + r_i^2)\cos(a_i - a_j) - r_i(1 + r_i^2)^{\frac{1}{2}}(1 + r_j^2)^{\frac{1}{2}}\cosh(b_i - b_j)]}{(\delta_{ij}^2 - 1)^{\frac{3}{2}}}, \tag{3.118}$$

$$r_i\ddot{\alpha} + 2\dot{r}_i\dot{\alpha} = \sum_{\substack{j=1\\ j\neq i}}^{N} \frac{m_j r_j \sin(a_j - a_i)}{(\delta_{ij}^2 - 1)^{\frac{3}{2}}}, \quad i = 1, 2, \ldots, N, \tag{3.119}$$

$$\ddot{r}_i = r_i\dot{\alpha}^2 + r_i^3\dot{\alpha}^2 + r_i\dot{r}_i^2 - \frac{r_i^3\dot{r}_i^2}{(1 + r_i^2)} - r_i(1 + r_i^2)\dot{\beta}^2 + \sum_{j=1}^{N} \frac{m_j[r_j(1 + r_i^2)\cos(a_i + a_j) - r_i\sqrt{(1 + r_i^2)(1 + r_j^2)}\cos h(b_i - b_j)]}{(q_{ij}^2 - 1)^{3/2}} \tag{3.120}$$

$$\ddot{\alpha} = -\frac{2\dot{r}_i^2\dot{\alpha}}{r_i} - \frac{1}{r_i}\sum_{j=1}^{N} \frac{m_j r_j \sin(a_i - a_j)}{(q_{ij}^2 - 1)^{3/2}} \tag{3.121}$$

$$\ddot{\beta} = -\frac{2\dot{\eta}_i\dot{\beta}}{\eta_i} - \frac{1}{\eta_i}\sum_{j=1}^{N} \frac{m_j \eta_j \sin h(b_i - b_j)}{(q_{ij}^2 - 1)^{3/2}} \tag{3.122}$$

$$\ddot{\beta} = -\frac{2\dot{\eta}_i\dot{\beta}}{\eta_i} - \frac{1}{\eta_i}\sum_{j=1}^{N}\frac{m_j(1+r_j^2)^{1/2}\sin h(b_i-b_j)}{(q_{ij}^2-1)^{3/2}} \tag{3.123}$$

$$m_i r_r \ddot{\alpha} = -2m_i\dot{r}_i\dot{\alpha} - m_i\sum_{j=1}^{N}\frac{m_j r_j \sin(a_i-a_j)}{(q_{ij}^2-1)^{3/2}} \tag{3.124}$$

$$(\sum_{i=1}^{N} m_i r_i)\ddot{\alpha} + 2(\sum_{i=1}^{N} m_i\dot{r}_i)\dot{\alpha} = 0, \quad \dot{\alpha} = \gamma \tag{3.125}$$

$$\dot{\gamma} = -\frac{2(\sum_{i=1}^{N} m_i\dot{\gamma}_i)\gamma}{\sum_{i=1}^{N} m_i r_i} \to \gamma = \frac{c}{\sum_{i=1}^{N} m_i r_i^2} \tag{3.126}$$

whereas for the equations corresponding to $\ddot{y}_i, \ddot{z}_i$, we find equations (3.118) again as well as the equations

$$r_i\ddot{\beta} + 2\dot{r}_i\dot{\beta} = \sum_{\substack{j=1\\ j\neq i}}^{N}\frac{m_j(1+r_j^2)^{\frac{1}{2}}\sin(b_j-b_i)}{(\delta_{ij}^2-1)^{\frac{3}{2}}}, \quad i = 1,2,\ldots,N. \tag{3.127}$$

$$(\sum_{i=1}^{N} m_i r_i)\ddot{\beta} + 2(\sum_{i=1}^{N} m_i\dot{r}_i)\dot{\beta} = 0, \quad \dot{\beta} = \gamma \tag{3.128}$$

$$\dot{\gamma} = -\frac{2(\sum_{i=1}^{N} m_i\dot{\gamma}_i)\gamma}{\sum_{i=1}^{N} m_i r_i} \to \gamma = \frac{c}{\sum_{i=1}^{N} m_i r_i^2} \tag{3.129}$$

These equations (3.119) can be solved in the same way that equations (3.20) were solved to obtain

$$\dot{\alpha} = \frac{d_1}{\sum_{i=1}^{N} m_i r_i^2},$$

where d_1 is an integration constant. To solve equations (3.127), and proceeding similarly, with the only change that for each $i = 1,2,\ldots,N$, the corresponding equation gets multiplied by $m_i(1+r_i^2)^{\frac{1}{2}}$ instead of $m_i r_i$, to obtain after addition that

$$\dot{\beta} = \frac{d_2}{M + \sum_{i=1}^{N} m_i r_i^2},$$

where $M = \sum_{i=1}^{N} m_i$ and d_2 is an integration constant. Then equations (3.118) turns into

$$\begin{aligned}\ddot{r}_i = r_i(1+r_i^2)\left[\frac{d_1^2}{(\sum_{i=1}^{N} m_i r_i^2)^2} - \frac{d_2^2}{(M+\sum_{i=1}^{N} m_i r_i^2)^2}\right] + \frac{r_i \dot{r}_i^2}{1+r_i^2} \\ + \sum_{\substack{j=1\\ j\neq i}}^{N} \frac{m_j[r_j(1+r_i^2)\cos(a_i-a_j) - r_i(1+r_i^2)^{\frac{1}{2}}(1+r_j^2)^{\frac{1}{2}}\cosh(b_i-b_j)]}{(\delta_{ij}^2-1)^{\frac{3}{2}}},\end{aligned} \tag{3.130}$$

where, recall, for any $i, j \in \{1, 2, \ldots, N\}$ with $i \neq j$

$$\delta_{ij} := \mathbf{q}_i \boxdot \mathbf{q}_j = r_i r_j \cos(a_i - a_j) - (1+r_i^2)^{\frac{1}{2}}(1+r_j^2)^{\frac{1}{2}}\cosh(b_i - b_j).$$

□

Remark 12. It is important to note that the constants c_{wx} and c_{yz} of the total angular momentum are

$$c_{wx} = \sum_{i=1}^{N} m_i(w_i \dot{x}_i - \dot{w}_i x_i) = \dot{\alpha} \sum_{i=1}^{N} m_i r_i^2 = d_1,$$

$$c_{yz} = \sum_{i=1}^{N} m_i(y_i \dot{z}_i - \dot{z}_i y_i) = -\dot{\beta} \sum_{i=1}^{N} m_i(1+r_i^2) = -d_2,$$

respectively, as a result d_1 and d_2 represent the rotation of the particle system relative to the wx-plane and yz-plane, respectively.

Remark 13. Equation (3.107),leads to a conclusion that $\dot{\alpha}$ and $\dot{\beta}$ are connected by the relationship

$$\frac{d_2}{\dot{\beta}} - \frac{d_1}{\dot{\alpha}} = M. \tag{3.131}$$

In the following steps, the proven criteria will be used.

Corollary 5. *If system* (2.58) *has a solution of the form (2.73), then the following expressions are constant:*

energy:

$$h = \sum_{i=1}^{N} \frac{m_i \dot{r}_i^2}{2(1+r_i^2)} + \frac{d_1^2}{2\sum_{j=1}^{N} m_j r_j^2} + \frac{d_2^2}{2(M + \sum_{j=1}^{N} m_j r_j^2)} + \sum_{1\leq i<j\leq N} \frac{m_i m_j \delta_{ij}}{(\delta_{ij}^2 - 1)^{\frac{1}{2}}}, \quad (3.132)$$

total angular momentum relative to the wy-plane:

$$\begin{aligned} c_{wy} = &- \sum_{i=1}^{N} \frac{m_i \dot{r}_i}{(1+r_i^2)^{\frac{1}{2}}} \cos(\alpha + a_i)\sinh(\beta + b_i) \\ &+ \sum_{i=1}^{N} m_i r_i (1+r_i^2)^{\frac{1}{2}} [\dot{\alpha}\sin(\alpha + a_i)\sinh(\beta + b_i) + \dot{\beta}\cos(\alpha + a_i)\cosh(\beta + b_i)], \end{aligned} \quad (3.133)$$

total angular momentum relative to the wz-plane:

$$\begin{aligned} c_{wz} = &- \sum_{i=1}^{N} \frac{m_i \dot{r}_i}{(1+r_i^2)^{\frac{1}{2}}} \cos(\alpha + a_i)\cosh(\beta + b_i) \\ &+ \sum_{i=1}^{N} m_i r_i (1+r_i^2)^{\frac{1}{2}} [\dot{\alpha}\sin(\alpha + a_i)\cosh(\beta + b_i) + \dot{\beta}\cos(\alpha + a_i)\sinh(\beta + b_i)], \end{aligned} \quad (3.134)$$

total angular momentum relative to the xy-plane:

$$\begin{aligned} c_{xy} = &- \sum_{i=1}^{N} \frac{m_i \dot{r}_i}{(1+r_i^2)^{\frac{1}{2}}} \sin(\alpha + a_i)\sinh(\beta + b_i) \\ &+ \sum_{i=1}^{N} m_i r_i (1+r_i^2)^{\frac{1}{2}} [\dot{\beta}\sin(\alpha + a_i)\cosh(\beta + b_i) - \dot{\alpha}\cos(\alpha + a_i)\sinh(\beta + b_i)], \end{aligned} \quad (3.135)$$

total angular momentum relative to the xz-plane:

$$c_{xz} = -\sum_{i=1}^{N} \frac{m_i \dot{r}_i}{(1+r_i^2)^{\frac{1}{2}}} \sin(\alpha + a_i)\cosh(\beta + b_i)$$
$$+ \sum_{i=1}^{N} m_i r_i (1+r_i^2)^{\frac{1}{2}} [\dot{\beta}\sin(\alpha + a_i)\sinh(\beta + b_i) - \dot{\alpha}\cos(\alpha + a_i)\cosh(\beta + b_i)]. \tag{3.136}$$

Chapter 4

Examples

4.1 Rotopulsating positive elliptic Lagrangian orbits

The Lagrangian orbits we present here are examples of positive elliptic rotopulsators of the curved 3-body problem. These bodies stay at the vertices of a rotating equilateral triangle in $\mathbb{S}^3$ that changes size. These systems rotate relative to the wx-plane, without a rotation relative to other planes.

Consider three equal masses, $m_1 = m_2 = m_3 =: m$, and a candidate solution of the form

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3), \quad \mathbf{q}_i = (w_i, x_i, y_i, z_i), \quad i = 1, 2, 3, \tag{4.1}$$

$$w_1 = r(t)\cos\alpha(t),\ x_1(t) = r(t)\sin\alpha(t),\ y_1 = y(t),\ z_1(t) = z(t),$$

$$w_2 = r(t)\cos[\alpha(t) + 2\pi/3],\ x_2(t) = r(t)\sin[\alpha(t) + 2\pi/3], y_2 = y(t),\ z_2(t) = z(t),$$

$$w_3 = r(t)\cos[\alpha(t) + 4\pi/3],\ x_3(t) = r(t)\sin[\alpha(t) + 4\pi/3], y_3 = y(t),\ z_3(t) = z(t).$$

As a result

$$q_{12} = q_{13} = q_{23} = \frac{3y^2 + 3z^2 - 1}{2}, \quad \dot{\alpha} = \frac{c}{3m(1 - y^2 - z^2)},$$

and that the variables y and z satisfy the equations

$$\begin{cases} \ddot{y} = F(y, z, \dot{y}, \dot{z})y \\ \ddot{z} = F(y, z, \dot{y}, \dot{z})z, \end{cases} \tag{4.2}$$

where

$$F(y, z, \dot{y}, \dot{z}) = \frac{8m}{\sqrt{3}(1 - y^2 - z^2)^{\frac{1}{2}}(1 + 3y^2 + 3z^2)^{\frac{3}{2}}} - \frac{c^2}{9m^2(1 - y^2 - z^2)}$$
$$- \frac{\dot{y}^2 + \dot{z}^2 - (y\dot{z} - \dot{y}z)^2}{1 - y^2 - z^2}.$$

The other equations are identically satisfied. As a result of (4.2), $\ddot{y}z = y\ddot{z}$, which means

$$y\dot{z} - z\dot{y} = k \text{ (constant)}.$$

But, considering (3.30), one can say that $3m(y\dot{z} - z\dot{y}) = c_{yz}$, and following this, $k = c_{yz}/3m$. It is important to note that the energy relation (3.25) turns into

$$\frac{3m[\dot{y}^2 + \dot{z}^2 - (y\dot{z} - \dot{y}z)^2]}{2(1 - y^2 - z^2)} + \frac{c^2}{6m(1 - y^2 - z^2)} - \frac{\sqrt{3}m^2(3y^2 + 3z^2 - 1)}{(1 - y^2 - z^2)^{\frac{1}{2}}(1 + 3y^2 + 3z^2)^{\frac{1}{2}}} = h,$$

which means that F can be written as

$$F(y, z) = \frac{2m[5 - 9(y^2 + z^2)^2]}{\sqrt{3}(1 - y^2 - z^2)^{\frac{1}{2}}(1 + 3y^2 + 3z^2)^{\frac{3}{2}}} - \frac{2h}{3m}.$$

Since

$$\sin\alpha + \sin(\alpha + 2\pi/3) + \sin(\alpha + 4\pi/3) = \cos\alpha + \cos(\alpha + 2\pi/3) + \cos(\alpha + 4\pi/3) = 0,$$

because of (3.26), (3.27), (3.28), and (3.29), it follows that $c_{wy} = c_{wz} = c_{xy} = c_{xz} = 0$, as the triangle does not rotate relative to the planes wy, wz, xy, and xz. Having no rotation relative to the plane yz, i.e. if $c_{yz} = 0$, then $k = 0$, so $y\dot{z} - z\dot{y} = 0$. With the assumption that z does not become zero, then it is concluded that $\frac{d}{dt}\frac{y}{z} = 0$. This means that $y(t) = \gamma z(t)$, where γ is a constant. If one assumes that $\delta = \gamma^2 + 1 \geq 1$, system (4.2) turns into the equation

$$\ddot{z} = \left[\frac{2m(5 - 9\delta^2 z^4)}{\sqrt{3}(1 - \delta z^2)^{\frac{1}{2}}(1 + 3\delta z^2)^{\frac{3}{2}}} - \frac{2h}{3m}\right] z. \tag{4.3}$$

It is necessary to study this system qualitatively and draw the phase plane for some values of h and $\delta \geq 1$, etc. In order to derive the roots of the polynomial, the first step is to calculate the fixed points of the vector field which are the zeros of the polynomial

$$P(z) = 27(9m^4 + h^2)\delta^4 z^8 - 18(15m^4 + h^2)\delta^2 z^4 - 8h^2\delta z^2 + 75m^4 - h^2.$$

Considering the Descartes's rule of signs, we have two cases: (i)$|h| < 5\sqrt{3}m^2$, where P has either two positive or no positive root; (ii) $|h| \geq 5\sqrt{3}m^2$, where P has exactly one positive root.

Remark 14. In a case where y or z is a nonzero constant, the motion takes place on a 2-dimensional non-great sphere. In a case where $y \equiv 0$ or $z \equiv 0$, the motion takes place on a 2-dimensional great sphere, i.e. on $\mathbb{S}^2$. A complete classification has been given for this, kind of Lagrangian orbits in [1].

4.2 Rotopulsating positive elliptic-elliptic Lagrangian orbits

The Lagrangian orbits we present here are examples of positive elliptic-elliptic relative equilibria of the curved 3-body problem. These bodies stay at the vertices of a rotating equilateral triangle in $\mathbb{S}^3$ that does not change size. These systems rotate relative to wx-plane and yz-plane without rotation relative to the other planes

Consider three equal masses, $m_1 = m_2 = m_3 =: m$, and a candidate solution of the form

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3), \quad \mathbf{q}_i = (w_i, x_i, y_i, z_i), \quad i = 1, 2, 3, \tag{4.4}$$

$$w_1 = r(t)\cos\alpha(t),\ x_1(t) = (t)\sin\alpha(t),\ y_1 = \rho(t)\cos\beta(t),\ z_1(t) = \rho(t)\sin\beta(t),$$

$$w_2 = r(t)\cos[\alpha(t) + 2\pi/3],\ x_2(t) = (t)\sin[\alpha(t) + 2\pi/3],$$

$$y_2 = \rho(t)\cos[\beta(t) + 2\pi/3],\ z_2(t) = \rho(t)\sin[\beta(t) + 2\pi/3],$$

$$w_3 = r(t)\cos[\alpha(t) + 4\pi/3],\ x_3(t) = (t)\sin[\alpha(t) + 4\pi/3],$$

$$y_3 = \rho(t)\cos[\beta(t) + 4\pi/3],\ z_3(t) = \rho(t)\sin[\beta(t) + 4\pi/3].$$

It is important to note that $\epsilon_{12} = \epsilon_{21} = \epsilon_{13} = \epsilon_{31} = \epsilon_{23} = \epsilon_{32} = -1/2$, which means that this particular elliptic-elliptic rotopulsating orbit is a relative equilibrium. Although r and ρ are not constant. In order to describe the motion of the bodies one has to analyze the behaviour of r.

Remark 15. In a case where there are separate masses assumed and the same function r is applied to all of them, it means that the masses must be equal. If in addition to this they are assumed to be distinct functions r_1, r_2, r_3, then

$$\epsilon_{ij} = -\frac{r_i r_j + (1-r_i^2)^{1/2}(1-r_j^2)^{1/2}}{2} = \frac{(r_i - r_j)^2 + [(1-r_i^2)^{1/2} - (1-r_j^2)^{1/2}]^2 - 2}{4},$$

meaning that the triangle does not have to be equilateral, therefore it is possible to have an orbit which is not Lagrangian at all.

It is important to note that, the bodies of a rotopulsating elliptic-elliptic relative equilibrium have a continuous movement among the Clifford tori of a given foliation of $\mathbb{S}^3$ which is different from the standard elliptic-elliptic relative equilibria, in which each body rotates on a fixed Clifford torus. In order to understand how this transition happens for the rotopulsating elliptic-elliptic Lagrangian orbit presented in (4.4), it is important to note that

$$r^2 + \rho^2 = 1 \tag{4.5}$$

$$\begin{cases} r\ddot{\alpha} + 2\dot{r}\dot{\alpha} = 0 \\ \rho\ddot{\beta} + 2\dot{\rho}\dot{\beta} = 0 \\ \ddot{r} = r(1-r^2)(\dot{\alpha}^2 - \dot{\beta}^2) - \frac{r\dot{r}^2}{1-r^2} \\ \ddot{\rho} = \rho(1-\rho^2)(\dot{\alpha}^2 - \dot{\beta}^2) - \frac{\rho\dot{\rho}^2}{1-\rho^2} \end{cases} \tag{4.6}$$

$$\dot{\rho} = -\frac{r\dot{r}}{\sqrt{1-r^2}} \tag{4.7}$$

$$\begin{cases} \dot{\alpha} = \gamma \\ \dot{\beta} = \delta \\ \dot{r} = u \\ \dot{\gamma} = -\frac{2u\gamma}{r} \\ \dot{\delta} = \frac{2ru\delta}{1-r^2} \end{cases} \tag{4.8}$$

$$m := m_1 = m_2 = m_3 \tag{4.9}$$

$$\gamma = \frac{c_1}{r^2}, \quad \delta = \frac{c_2}{1-r^2} \tag{4.10}$$

$$\begin{cases} \dot{r} = u \\ \dot{u} = \frac{c_1^2(1-r^2)^2-(c_2^2+u^2)r^4}{r^3(1-r^2)} \end{cases} \tag{4.11}$$

$$\begin{cases} \dot{r} = u \\ \dot{u} = \frac{(c_1^2-c_2^2+u^2)r^4-2c_1^2r^2+c_1^2}{r^3(1-r^2)} \end{cases} \tag{4.12}$$

$$\dot{\alpha} = \frac{c_1}{3mr^2}, \quad \dot{\beta} = \frac{c_2}{3m(1-r^2)}, \tag{4.13}$$

therefore the equation representing r turns into

$$\ddot{r} = \frac{1}{9m^2}\left[\frac{c_1^2(1-r^2)}{r^3} - \frac{r(9m^2\dot{r}^2+c_2^2)}{1-r^2}\right]. \tag{4.14}$$

The change of the angular velocities $\dot{\alpha}$ and $\dot{\beta}$ against r is presented using the equations in (4.13): when r approaches 0, $|\dot{\alpha}|$ is large, while $|\dot{\beta}|$ is small, and the other way around when r is close to 1. The statement $\dot{\alpha}$ and $\dot{\beta}$ are in line with equation (3.65). Here a qualitative description equivalent with the first-order system will be presented for equation (4.14)

$$\begin{cases} \dot{r} = u, \\ \dot{u} = \frac{c_1^2(1-r^2)}{9m^2r^3} - \frac{r(9m^2u^2+c_2^2)}{9m^2(1-r^2)}. \end{cases} \tag{4.15}$$

The point of initiation is to search for fixed points, which means that $\dot{r} = \dot{u} = 0$, which leads to:

$$(c_1^2 - c_2^2)r^4 - 2c_1^2r^2 + c_1^2 = 0.$$

There are a maximum of two fixed points $r_1 = \sqrt{\frac{c_1}{c_1-c_2}}$ and $r_2 = \sqrt{\frac{c_1}{c_1+c_2}}$, for $c_1 \neq c_2$, with only one point between 0 and 1, while for $c_1 = c_2$ there is a single fixed point, which is named $r_0 = 1/\sqrt{2}$

Remark 16. The energy relation (3.68) turns into

$$h = \frac{3m\dot{r}^2}{2(1-r^2)} + \frac{1}{18m^2}\left[\frac{c_1^2}{r^4} + \frac{c_2^2}{(1-r^2)^2}\right] + \frac{4m^2}{\sqrt{3}},$$

Remark 17. It is important to note that, when $\dot{\alpha} = \dot{\beta}$, then $r^2 = \frac{c_1}{c_1+c_2}$, which means that if r exists, it must be constant, so

$$\dot{\alpha} = \dot{\beta} = \frac{c_1 + c_2}{3m}.$$

As a result of these facts, the orbit turns into a standard (not a rotopulsating) relative equilibrium. It is also important to note that the third equation in (4.14) meets the requirements at the same time and the integrals (3.68), (3.69), (3.70), (3.71), and (3.72) are constant, as expected.

4.3 Rotopulsating negative elliptic Lagrangian orbits

In this section, a class of specific examples will be provided for negative elliptic rotopulsators of the curved 3-body problem, which are called Lagrangian orbits in $\mathbb{H}^3$. These systems have a rotation against the wx-plane, with no rotations against the other base planes. $m_1 = m_2 = m_3 =: m$, and a solution of type

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3), \quad \mathbf{q}_i = (w_i, x_i, y_i, z_i), \quad i = 1, 2, 3, \tag{4.16}$$

$$w_1 = r(t)\cos\alpha(t),\ x_1(t) = r(t)\sin\alpha(t),\ y_1 = y(t),\ z_1(t) = z(t),$$

$$w_2 = r(t)\cos[\alpha(t) + 2\pi/3],\ x_2(t) = r(t)\sin[\alpha(t) + 2\pi/3], y_2 = y(t),\ z_2(t) = z(t),$$

$$w_3 = r(t)\cos[\alpha(t) + 4\pi/3],\ x_3(t) = r(t)\sin[\alpha(t) + 4\pi/3], y_3 = y(t),\ z_3(t) = z(t).$$

Then

$$\mu_{12} = \mu_{13} = \mu_{23} = \frac{3y^2 - 3z^2 + 1}{2}, \quad \dot{\alpha} = \frac{b}{3m(z^2 - y^2 - 1)}$$

and that the variables y and z meet the requirements of the equations

$$\begin{cases} \ddot{y} = G(y, z, \dot{y}, \dot{z})y \\ \ddot{z} = G(y, z, \dot{y}, \dot{z})z, \end{cases} \tag{4.17}$$

where

$$G(y, z, \dot{y}, \dot{z}) = \frac{(y\dot{z} - \dot{y}z)^2 + \dot{z}^2 - \dot{y}^2}{z^2 - y^2 - 1} + \frac{b^2}{9m^2(z^2 - y^2 - 1)}$$
$$- \frac{8m}{\sqrt{3}(z^2 - y^2 - 1)^{\frac{1}{2}}(3z^2 - 3y^2 + 1)^{\frac{3}{2}}}.$$

The other equations are identically satisfied. From (4.17), it can be concluded that $\ddot{y}z = y\ddot{z}$, which means that

$$y\dot{z} - z\dot{y} = k \text{ (constant)}.$$

But, from (3.92) it is clear that $3m(y\dot{z} - z\dot{y}) = c_{yz}$, furthermore $k = c_{yz}/3m$. It is important to note that the energy relation (3.87) turns into

$$\frac{3m[(y\dot{z} - \dot{y}z)^2 + \dot{z}^2 - \dot{y}^2]}{2(z^2 - y^2 - 1)} + \frac{b^2}{6m(z^2 - y^2 - 1)} + \frac{\sqrt{3}m^2(3y^2 - 3z^2 + 1)}{(z^2 - y^2 - 1)^{\frac{1}{2}}(3z^2 - 3y^2 + 1)^{\frac{1}{2}}} = h,$$

meaning that G can be written as

$$G(y, z) = \frac{2h}{3m} - \frac{2m[5 - 9(y^2 - z^2)^2]}{\sqrt{3}(z^2 - y^2 - 1)^{\frac{1}{2}}(3z^2 - 3y^2 + 1)^{\frac{3}{2}}}.$$

Since

$$\sin\alpha + \sin(\alpha + 2\pi/3) + \sin(\alpha + 4\pi/3) = \cos\alpha + \cos(\alpha + 2\pi/3) + \cos(\alpha + 4\pi/3) = 0,$$

according to (3.88), (3.89), (3.90), and (3.91) is it clear that $c_{wy} = c_{wz} = c_{xy} = c_{xz} = 0$, implying that the triangle does not have any rotation relative to the planes wy, wz, xy, and xz. Having no rotation relative to the plane yz either, i.e. if $c_{yz} = 0$, means $k = 0$. As a result $y\dot{z} - z\dot{y} = 0$, and the assumption that z never turns to zero, leads to a conclusion that $\frac{d}{dt}\frac{y}{z} = 0$. It is $y(t) = \gamma z(t)$, where γ is a constant. If $\epsilon = 1 - \gamma^2$, is considered, the system (4.17) turns into

$$\ddot{z} = \left[\frac{2h}{3m} - \frac{2m(5 - 9\epsilon^2 z^4)}{\sqrt{3}(\epsilon z^2 - 1)^{\frac{1}{2}}(3\epsilon z^2 + 1)^{\frac{3}{2}}}\right] z. \tag{4.18}$$

It is important to note that since a point (w, x, y, z) on $\mathbb{H}^3$ meets the requirements of the equation $w^2 + x^2 + y^2 - z^2 = -1$ and $z \geq 1$, it can be necessarily shown that $\epsilon \geq 1$. It is clear that there is a fixed point on the vector field in (4.3) as $(z, \dot{z}) = (0, 0)$, but it is outside the domain $z \geq 1$. In case of having any other points, they need to be of the form $(z, 0)$, where the positive values of z are given by the roots of the polynomial

$$Q(z) = 27(h^2 - 9m^4)\epsilon^4 z^8 - 18(h^2 - 15m^4)\epsilon^2 z^4 - 8h^2\epsilon z^2 - h^2 - 75m^4.$$

Using Descartes's rule of signs and taking in mind that the two cases: (i) $|h| < \sqrt{15}m^2$, when Q has no positive roots at all; (ii) $|h| \geq \sqrt{15}m^2$, when Q has exactly one positive root, are not the same. Using case (i) results in having no fixed points for equation (4.3). Using case (ii) turn into having one fixed point because the unique positive root, $z_0 := z_0(m, h, \epsilon)$, is also larger than 1. In order to present this fact, let $\xi = \epsilon z^2$.

An association can be made in this way to Q the polynomial

$$\bar{Q}(\xi) = 27(h^2 - 9m^4)\xi^4 - 18(h^2 - 15m^4)\xi^2 - 8h^2\xi - h^2 - 75m^4.$$

Because $\epsilon \geq 1$, in order to prove that $z_0 \geq 1$ it is satisfactory to present that $\xi_0 := \epsilon z_0 \geq 1$. While moving in this direction it is important to note that $\bar{Q}(1) = -48m^4 < 0$. Since $\lim_{\xi \to +\infty} \bar{Q}(\xi) = +\infty$, $\bar{Q}$ has a single positive root that is also larger than 1. It is now possible to prove the following result and prove that having suitable conditions, there exist rotopulsating negative elliptic Lagrangian orbits that are not relative equilibria

Proposition 1. *In the curved* 3*-body problem in* $\mathbb{H}^3$*, with having masses* $m_1 = m_2 = m_3 =: m > 0$*, for admissible initial conditions the bodies form a negative elliptic rotopulsating Lagrangian orbit. In addition to this, on energy levels there are negative elliptic rotopulsating Lagrangian orbits that meet the requirements of the inequality* $|h| \geq \sqrt{15}m^2$*, these tend to (eject from) a negative elliptic Lagrangian relative equilibrium that is a fixed point for equation* (4.18)*.*

Proof. The above remarks and standard existence and uniqueness result for the theory of ordinary differential equations are in line with the general existence of negative elliptic rotopulsating Lagrangian orbits. If $z_0 > 1$ represents the fixed point of (4.3), then the related eigenvalues $\lambda_{1,2}$ are written using

$$\lambda - \frac{2h}{3m} + S(z_0, m, \epsilon) = 0,$$

where $S(z_0, m, \epsilon)$ is a finite number. Therefore, considering that h is large enough, one eigenvalue is positive and the other is negative, so there is a stable and an unstable manifold for the fixed point. This remark completes the proof. □

4.4 Rotopulsating negative hyperbolic Eulerian orbits

To check the existence of solutions of the form

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3), \quad \mathbf{q}_i = (w_i, x_i, y_i, z_i), \quad i = 1, 2, 3, \tag{4.19}$$

$$w_1 = 0, \quad x_1 = 0, \quad y_1 = \sinh\beta(t), \quad z_1 = \cosh\beta(t),$$

$$w_1 = w(t), \quad x_1 = x(t), \quad y_1 = \eta(t)\sinh\beta(t), \quad z_1 = \eta(t)\cosh\beta(t),$$

$$w_1 = -w(t), \quad x_1 = -x(t), \quad y_1 = \eta(t)\sinh\beta(t), \quad z_1 = \eta(t)\cosh\beta(t),$$

with $w^2 + x^2 - \eta^2 = -1$, we compute that:

$$\nu_{12} = \nu_{13} = (w^2 + x^2)^{\frac{1}{2}}, \quad \nu_{23} = -2(w^2 + x^2), \quad \dot\beta = \frac{a}{m(2w^2 + 2x^2 + 3)},$$

and the equations of motion turns the system into

$$\begin{cases} \ddot w = J(w, x, \dot w, \dot x)w \\ \ddot x = J(w, x, \dot w, \dot x)x, \end{cases} \tag{4.20}$$

where

$$J(w, x, \dot w, \dot x) = \frac{m(w^2 + x^2)^{\frac{1}{2}}}{(w^2 + x^2 - 1)^{\frac{3}{2}}} - \frac{m}{(2w^2 + 2x^2 + 1)^{\frac{1}{2}}(2w^2 + 2x^2 - 1)^{\frac{3}{2}}}$$

$$+ \frac{(w\dot x - x\dot w)^2 + \dot w^2 + \dot x^2}{w^2 + x^2 + 1} + \frac{a^2(w^2 + x^2 + 1)}{m^2(2w^2 + 2x^2 + 3)^2}.$$

The other equations are identically satisfied. It is important to note that the energy integral (3.101) turns into

$$\frac{m[(w\dot{x}-x\dot{w})^2+\dot{w}^2+\dot{x}^2]}{w^2+x^2+1}+\frac{a^2}{2m(2w^2+2x^2+3)}$$

$$+\frac{2m^2(w^2+x^2)^{\frac{1}{2}}}{(w^2+x^2-1)^{\frac{1}{2}}}-\frac{2m^2(w^2+x^2)}{(2w^2+2x^2+1)^{\frac{1}{2}}(2w^2+2x^2-1)^{\frac{1}{2}}}=h,$$

meaning that J can be presented as

$$J(w,x)=\frac{h}{m}+\frac{m[2(w^2+x^2)(2w^2+2x^2-1)+1]}{(2w^2+2x^2+1)^{\frac{1}{2}}(2w^2+2x^2-1)^{\frac{3}{2}}}$$

$$-\frac{m(w^2+x^2)^{\frac{1}{2}}(2w^2+2x^2-3)}{(w^2+x^2-1)^{\frac{3}{2}}}-\frac{a^2}{2m^2(2w^2+2x^2+3)^2}.$$

The conclusion from (4.20) is that $\ddot{w}x=w\ddot{x}$, which means

$$w\dot{x}-\dot{w}x=k \text{ (constant)}.$$

From (3.102) it can be shown that $2m(w\dot{x}-\dot{w}x)=c_{wx}$, it follows that $k=\frac{c_{wx}}{2m}$. From (3.103), (3.104), (3.105), and (3.106) it follows that $c_{wy}=c_{wz}=c_{xy}=c_{xz}=0$, therefore the orbit does not have any rotation against the planes wy, wz, xy, and xz. When there is no rotation against the plane wx either, i.e. if $c_{wx}=0$, then $k=0$, so $w\dot{x}-\dot{w}x=0$, and it is assumed that x never turns to zero, it can be concluded that $\frac{d}{dt}\frac{w}{x}=0$, so $w(t)=\xi x(t)$, where ξ is a constant. Assuming $\zeta=\xi^2+1\geq 1$, system (4.20) turns into

$$\ddot{x}=\left[\frac{h}{m}+\frac{m[4\zeta^2x^4-2\zeta x^2+1]}{(2\zeta x^2+1)^{\frac{1}{2}}(2\zeta x^2-1)^{\frac{3}{2}}}-\frac{m(\zeta x^2)^{\frac{1}{2}}(2\zeta x^2-3)}{(\zeta x^2-1)^{\frac{3}{2}}}-\frac{a^2}{2m^2(2\zeta x^2+3)^2}\right]x. \quad (4.21)$$

Standard existence and uniqueness results prove the existence of the rotopulsating orbits.

4.5 Rotopulsating negative elliptic-hyperbolic Eulerian orbits

Consider candidates for negative elliptic-hyperbolic rotopulsating Eulerian solutions with equal masses, presented by

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3), \quad \mathbf{q}_i = (w_i, x_i, y_i, z_i), \quad i = 1, 2, 3, \tag{4.22}$$

$$w_1 = 0,\ x_1 = 0,\ y_1 = \sinh\beta(t),\ z_1(t) = \cosh\beta(t),$$

$$w_2 = r(t)\cos\alpha(t),\ x_2 = r(t)\sin\alpha(t),\ y_2 = \eta(t)\sinh\beta(t),\ z_2(t) = \eta(t)\cosh\beta(t),$$

$$w_3 = -r(t)\cos\alpha(t),\ x_3 = -r(t)\sin\alpha(t),\ y_3 = \eta(t)\sinh\beta(t),\ z_3 = \eta(t)\cosh\beta(t),$$

Then the variables relevant to Criterion 5 turn into

$$m_1 = m_2 = m_3 =: m, \quad M = 3m, \quad a_1 = a_2 = a_3 = b_1 = b_2 = b_3 = 0,$$

$$r_1 = 0, \quad r_2 = r, \quad r_3 = -r, \quad \eta_1 = 1, \quad \eta_2 = \eta_3 = \eta = (1 + r^2)^{1/2},$$

$$\delta_{12} = \delta_{21} = \delta_{13} = \delta_{31} = -(1 + r^2)^{1/2}, \quad \delta_{23} = \delta_{32} = -2r^2 - 1,$$

$$\dot\alpha = \frac{d_1}{2mr^2}, \quad \dot\beta = \frac{d_2}{m(3 + 2r^2)}, \ \text{ with } d_1, d_2 \text{ constants.} \tag{4.23}$$

In system (3.108), the equation representing $\ddot r_1$ is an identity, while the equations representing $\ddot r_2$ and $\ddot r_3$ are identical, as written in

$$\ddot{r} = r(1+r^2)(\dot{\alpha}^2 - \dot{\beta}^2) + \frac{r\dot{r}^2}{1+r^2} - \frac{m(5+4r^2)}{4r^2(1+r^2)^{1/2}}.$$

As a result, the problem becomes as simple as having and understandable system

$$\begin{cases} \dot{r} = \rho \\ \dot{\rho} = r(1+r^2)\left[\frac{d_1^2}{4m^2r^4} - \frac{d_2^2}{m^2(3+2r^2)^2}\right] + \frac{r\rho^2}{1+r^2} - \frac{m(5+4r^2)}{4r^2(1+r^2)^{1/2}}. \end{cases} \tag{4.24}$$

It is important to note that if $d_1 = 0$, i.e. when the orbit has no rotation against the wx-plane, then $\dot{\alpha} = 0$, which is in line with (3.131) meaning that $\dot{\beta}$ is constant. Then (4.23) means that r is constant, in agreement with Theorem 8 in [1], which works on the problem in $\mathbb{H}^2$. In other words, the only rotopulsating solutions similar to this are the standard elliptic hyperbolic Eulerian relative equilibria. Now it is possible to prove the following result. Notice that for the sake of simplicity and by preserving the generality, m is taken equal to 1.

Proposition 2. *There is at least one elliptic-hyperbolic Eulerian relative equilibrium for every* $m = 1$*,* $d_1, d_2 \neq 0$*, with* $|d_1| < |d_2|$ *in the curved* 3*-body problem in* $\mathbb{H}^3$*, that eject the rotopulsating elliptic-hyperbolic Eulerian orbits eject.*

Proof. In order to prove the result presented above, one needs to show that system (4.24) has at least one fixed point with at least one real positive eigenvalue or one with positive real part. In order to illustrate this fact, it is important to note that the vector field of system (4.24) disappears if $\rho = 0$ and for those values of r for which

$$(1+r^2)^{3/2}[4(d_1^2 - d_2^2)r^4 + 12d_1^2r^2 + 9d_1^2] = r(5+4r^2)(3+2r^2)^2.$$

For any $d_1, d_2 \neq 0$, with $|d_1| < |d_2|$, and $r > 0$, the left hand side function will finally decrease, while the right hand side is always increasing. As a result, the equation has at least one positive solution, r_0, which depends on d_1 and d_2, therefore system (4.24) has at least one fixed point of the form $(\rho, r) = (0, r_0)$. The two eigenvalues of the fixed point are

$$\frac{r_0}{2(1+r_0^2)} \pm \frac{1}{2}\sqrt{\frac{r_0^2}{(1+r_0^2)^2} + 4g(r_0)},$$

where g is the derivative with respect to r of the right hand side in the second equation of (4.24). Independent of the value of $g(r_0)$, at least one eigenvalue is real positive or has positive real part. This remark completes the proof.

□

Bibliography

[1] F. Diacu and E. Pérez-Chavela, Homographic solutions of the curved 3-body problem, *J. Differential Equations* **250** (2011), 340-366.

[2] F. Diacu, On the singularities of the curved N-body problem, *Trans. Amer.Math. Soc.* **363**, 4 (2011), 2249-2264.

[3] F. Diacu, Polygonal homographic orbits of the curved 3-body problem, *Trans.Amer. Math. Soc.* **364**, 5 (2012), 2783-2802.

[4] F. Diacu and E. Pérez-Chavela, and M. Santoprete, The N-body problem in spaces of constant curvature. Part I: Relative equilibria, *Journal of Nonlinear Science*, **22**, 2 (2012), 247-266.

[5] F. Diacu and E. Pérez-Chavela, and M. Santoprete, The N-body problsem in spaces of constant curvature. Part II: Singularities, *J. Nonlinear Sci.* **22**, 2 (2012), 267-275.

[6] E. Schering, Die Schwerkraft im Gaussischen Raume, Nachr. Königl. Gesell. Wiss. Göttingen **15** (1870) 311-321.

[7] E. Schering, Die Schwerkraft in mehrfach ausgedehnten Gaussischen und Riemanschen Räumen, Nachr. Königl. Gesell.Wiss. Göttingen **6** (1873) 149-159.

[8] W. Killing, Die Rechnung in den nichteuklidischen Raumformen, *J. Reine Angew. Math.* **89** (1880) 265-287.

[9] W. Killing, Die Mechanik in den nichteuklidischen Raumformen, *J. Reine Angew. Math.* **98** (1885) 1-48.

[10] W. Killing, Die Nicht-Eukildischen Raumformen in Analytischer Behandlung, Teubner, Leipzig, 1885.

[11] H. Liebmann, Über die Zentralbewegung in der nichteuklidische Geometrie, Berichte Königl. Sächsischen Gesell. Wiss. *Math. Phys. Kl.* **55** (1903) 146-153.

[12] H. Liebmann, Nichteuklidische Geometrie, G.J. Göschen, Leipzig, 1905; second ed., G.J. Göschen, 1912; third ed., Walter de Gruyter, Berlin, Leipzig, 1923.