A collection of algorithmic and complexity results for variants of the Firefighter Problem

by

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B.Math., University of Waterloo, 2008

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ABSTRACT

The Firefighter Process models the spread and defence of a fire using a simple graph. We consider the following discrete-time process: at $t = 0$ some vertex of the graph begins burning. At each subsequent step we may defend a vertex from burning and the fire spreads from all burning vertices to all undefended neighbours. We consider the related problems of maximising the number of saved vertices, protecting a specified set from burning and maximising the weight of the saved vertices. We close three open problems concerning these decision problems and their related optimisation problems using the notion of a strategy, the sequence of defended vertices.

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Acknowledgments

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Chapter 1

Introduction

We consider the process first introduced by Hartnell [6] at the *25th Manitoba Conference on Combinatorial Mathematics and Computing* at the University of Manitoba in 1995: At $t = 0$ some set of vertices of a graph begin burning. At each subsequent time a set of firefighters can defend some set of vertices and the fire spreads from all burning vertices to each of their neighbours that have yet to be defended. Once a vertex burns or is defended it remains as such for the remainder of the process.

Since the introduction of this process a number of different combinatorial optimisation problems based on this process have been devised and studied. In 2009, Finbow and MacGillivray published a survey on the work that had been done to date on such problems [4]. In their survey they discuss a variety of algorithms and complexity results that have been obtained for these problems, report the work done by Moeller and Wang [12] on a version of the problem on finite grids and look at results obtained by Fogarty, as well as those by Devlin and Hartke on infinite grids [5] [2]. Since then a variety of different results by a number of authors have been obtained.

In 2007, Ng and Raff [13] examined a variant of the problem in which the number of available firefighters at each step is a periodic sequence. They looked at the two-dimensional infinite grid and showed that the fire can be contained if the average number of firefighters per period exceeded $\frac{3}{2}$.

In 2011, Iwaikawa, Kamiyama and Matsui [8] improved on approximation algorithms for a version of the problem on rooted trees. They were able to improve on an existing approximation algorithm to obtain a *0.7144 - approximation*.

In addition to grids and algorithms, the *surviving rate* of a graph has been well studied. The *surviving rate* measures the average number of vertices that can be saved if a fire starts at an arbitrary vertex. Cai et al. [9] showed, in 2010, that an n

vertex graph with tree-width at most k has *surviving rate* of at least $1 - O(\frac{k^2 \log n}{n})$.

This thesis will continue work examining approximation algorithms and complexity results.

In Section 2.1 a summary of terms and notation related to the process described above are formalised. In particular, the notion of a strategy is formalised in Section 2.1.1.

This thesis closes three open problems presented in [4] related to the following three objectives:

1. Determine whether a pre-specified quantity of vertices can be prevented from burning.

2. Determine whether a pre-specified set of vertices can be prevented from burning.

3. For a weight function from the vertices to the integers, determine whether a pre-specified weight can be prevented from burning.

We will review results obtained regarding this first objective in Section 2.2.1. In Chapter 3 we examine greedy strategies related to this objective. Here we close an open problem about how well a specific greedy algorithm approximates the optimal solution for an optimisation version of the first objective.

The second objective is presented in Chapter 4. Prior results concerning this objective are reviewed in Section 2.2.2. In Chapter 4 we also examine the restriction to graphs with maximum degree three, where the fire starts at a vertex of degree two. Here, we close an open problem concerning the complexity of the problem given such a restriction.

Finally, we formally introduce the third objective in Chapter 5 and consider the distinction between weight functions that assign arbitrary weights and those which assign {0,1} weights. Using the results from Chapter 4, we close an open problem concerning the complexity of the third objective.

This work concludes with a brief discussion of open problems related to each of the three objectives considered. This discussion is found in Chapter 6.

Chapter 2

Background

Let $G = (V, E)$ be a simple graph with vertex set V and edge set E. If G is a graph and $r \in V$, we call the ordered pair (G, r), $r \in V$, a *rooted graph with root r*. This will often be shortened to *a rooted graph* (G, r). We call a tree T a *binary tree* if it has maximum degree three and the root of the tree has degree two. For a binary tree T, we call T a *full binary tree* if T is a binary tree in which each vertex is either a leaf of has exactly two descendants [14].We call T a *complete binary tree* if T is a binary tree with leaves on at most two adjacent levels $l - 1$ and l in which leaves on the bottommost level l lie in the leftmost positions of l [14]. For all other graph theoretic terms not defined herein we refer to Bondy and Murty [1].

2.1 Firefighter Processes: Definitions and Terminology

Consider the rooted graph (G, r). The *firefighter process* proceeds as follows: At time $t = 0$ a fire breaks out at r. At each subsequent time step, one unburned vertex of G may be defended from burning and the fire spreads to each undefended vertex adjacent to a burning vertex. Once a vertex is defended it remains defended for the remainder of the process. Similarly, once a vertex is burning it remains burning for the remainder of the process. The process ends when every burning vertex has all of its neighbours either burning or defended. At the conclusion of the process any vertex that is neither burned nor defended is called *protected*. Together, the *defended* and the *protected* vertices are the *saved* vertices.

Figure 2.1 illustrates the process at work. At $t = 1$, v_1 is defended while v_2 and

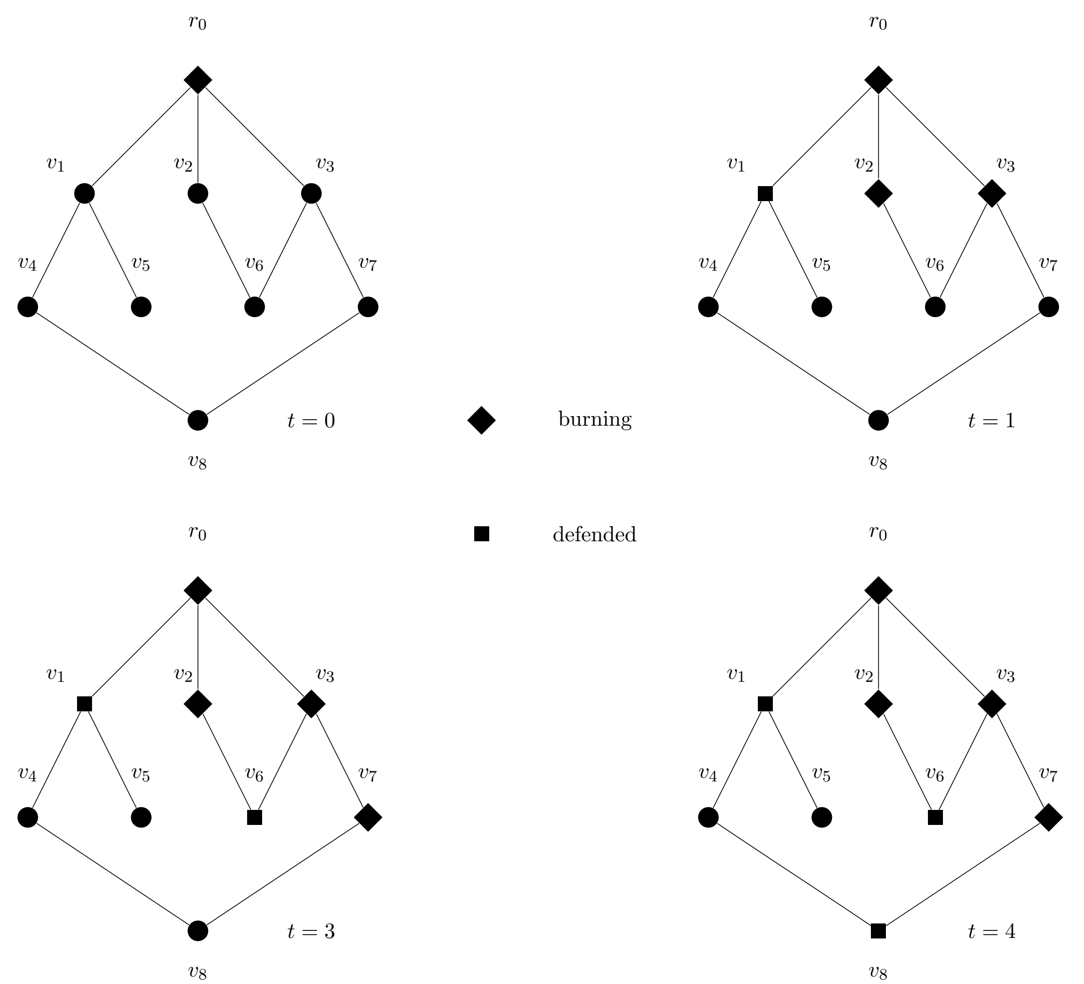

Figure 2.1: A sample process

v_3 burn. At $t = 2$, v_6 is defended and v_7 burns. Finally, at $t = 3$, v_8 is defended and no new vertex burns. The process ends because every neighbour of a burning vertex is either defended or burning. We notice that v_4 was neither defended nor burned, thus it was protected. In general, if at any time during the process every path from a burning vertex to a vertex v contains a defended vertex, then v is protected as it cannot burn.

Looking again at Figure 2.1, we notice that the state of the process after time $t = k$ depends solely upon the state of the process at time $t = k - 1$ and what was defended at $t = k$. Further we notice that once a burning vertex has spread the fire to an adjacent vertex it no longer affects the process. Thus, we can consider each time step $t = i$ as the beginning of a new instance (G_i, r_i) of the process by identifying

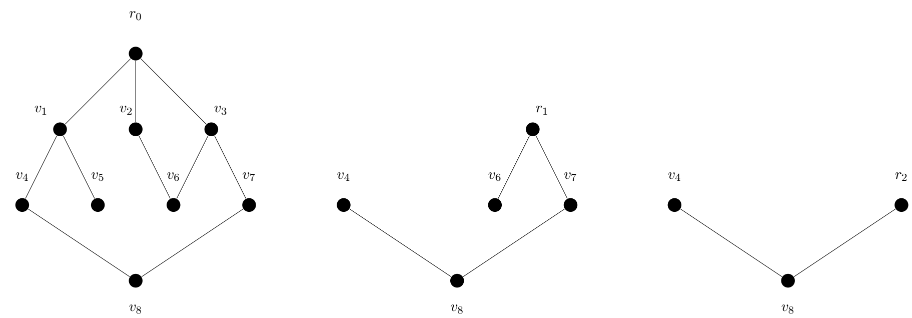

Figure 2.2: Each time-step can be considered a new instance of the process

all burning vertices into a single vertex, r_i, and deleting any vertex that has been defended or protected. Figure 2.2 shows the same process as Figure 2.1 using this method of reduction. Using this method we notice that v_5 is removed after $t = 1$ because it is in a component with no burning vertex (in fact it is an isolated vertex). All vertices in any such component may be deleted as they are protected.

2.1.1 Strategies

For a given pair (G, r) there are many different firefighter processes, each corresponding to a different way to defend the vertices. We consider the sequence $D = (d_1, d_2, \ldots d_t)$, $d_i \in V(G)$, of vertices, where d_i is defended at time i, and call a feasible (i.e., d_i is neither burning nor defended time $i-1$) such sequence a *strategy*. We denote the number, t, of vertices defended under D as $|D|$ and the number of vertices burned as b_D. For the *firefighter process* on rooted graph (G, r) with strategy D, we let H_D denote the subgraph of G formed by the vertices that burned and the edges over which the fire spread the strategy D.

In the search for strategies that produce firefighter processes that adhere to certain criteria, we examine minimal strategies. We call a strategy D *minimal* if there exists no strategy D' such that $H_{D'}$ is a proper subgraph of H_D. Consider the rooted graph in Figure 2.3 and the following pair of strategies: $D = (v_2, v_7, v_9, v_{10})$ and $D' = (v_2, v_4, v_7, v_{10})$. Each of these two strategies gives a different firefighter process on (G, r). Figure 2.3 also shows H_D and $H_{D'}$. Notice D is not a minimal strategy as H_D contains $H_{D'}$ as a proper subgraph.

We now exhibit a pair of lemmas which will be useful when attempting to construct firefighter processes with minimal strategies.

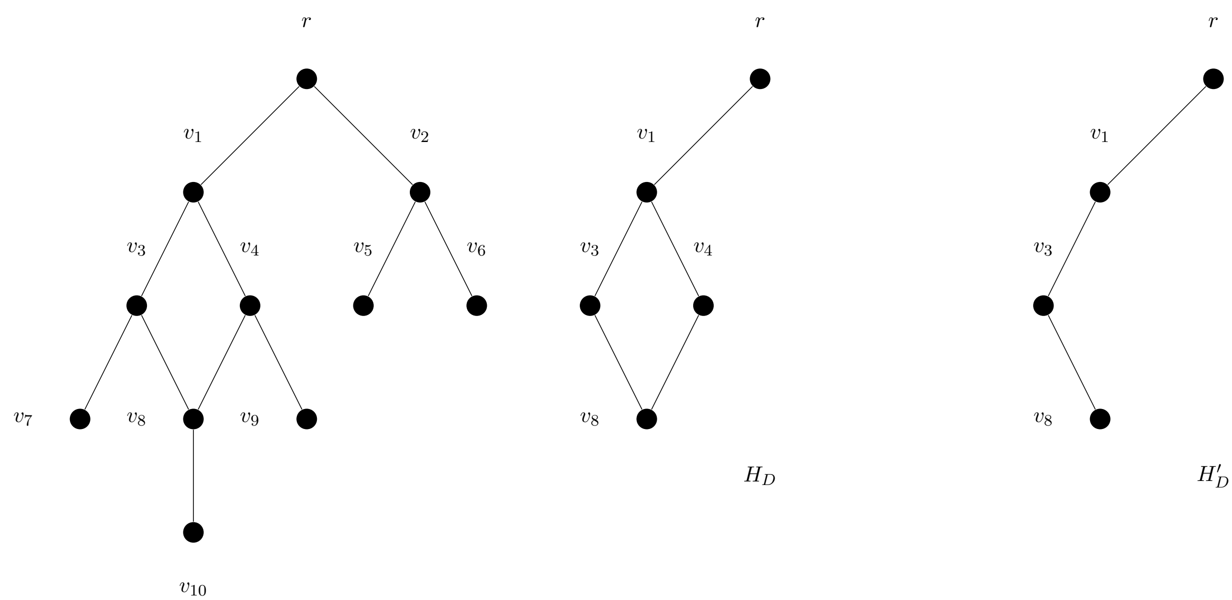

Figure 2.3: The Application of Strategies D and D'

Lemma 2.1.1 (The Ordering Lemma). *Let $D = (d_1, d_2, \ldots, d_t)$ be a strategy for a firefighter process on (G, r). If there exists i such that $d_{G_{i-1}}(d_{i+1}, r_{i-1}) < d_{G_{i-1}}(d_i, r_{i-1})$ then the strategy $D' = (d_1, d_2, \ldots, d_{i-1}, d_{i+1}, d_i \ldots d_t)$ has the property that $H_{D'}$ is a subgraph of H_D.*

Proof. We proceed by induction on the earliest time at which a pair of out of order vertices appear in a strategy $D = (d_1, d_2, \ldots, d_k)$. Consider first the case that $d_G(d_2, r) < d_G(d_1, r)$. Let $D' = (d_2, d_1, d_3, d_4, \ldots, d_k)$. Then $H_{D'}$ is a subgraph of H_D. Consider now the case that $1 < j < k$ is the earliest time such that $d_{G_{j-1}}(d_{j+1}, r_{j-1}) < d_{G_{j-1}}(d_j, r_{j-1})$. Since the first $j - 1$ vertices appear in order, we may consider the instance (G_{j-1}, r_{j-1}) and the strategies $F = (d_j, d_{j+1}, d_{j+2}, d_{j+3}, \ldots, d_k)$ and $F' = (d_{j+1}, d_j, d_{j+2}, d_{j+3}, \ldots, d_k)$. By induction, H_F is a subgraph of $H_{F'}$ and so $H_{D'}$ is a subgraph of H_D, where $D' = (d_1, d_2, \ldots, d_{j-1}, d_{j+1}, d_{j+2}, \ldots, d_k)$.

□

Lemma 2.1.2 (The Tree Lemma). *If D is a strategy for the Firefighter Process where the i^{th} entry d_i is not adjacent to r_{i-1} and there is a unique path P from r_{i-1} to d_i in G_{i-1}, then the strategy $D' = (d_1, d_2, \ldots, d_{i-1}, x \ldots d_t)$ formed by replacing d_i with the vertex x of P adjacent to r_{i-1} has the property that $H_{D'}$ is a proper subgraph of H_D.*

Proof. Let $D = (d_1, d_2, \ldots, d_t)$ be a strategy for the Firefighter Process where the *ith* entry d_i is not adjacent to r_{i-1} in G_{i-1} and there is a single path P from r_i to d_i. Let x be the neighbour of r_{i-1} on P. Consider the strategy $D' = (d_1, d_2, \ldots, d_{i-1}, x, d_{i+1}, \ldots, d_t)$ formed by replacing d_i with x. Under such a strategy, no vertex on P burns. Thus $H_{D'}$ is a proper subgraph of H_D . □

The Tree Lemma generalizes an observation by MacGillivray and Wang [11] about firefighting on trees: to save the most vertices in a tree, each defended vertex needs to be adjacent to a burning vertex. The Tree Lemma generalises this observation to show that a strategy that defends a vertex at time i which is not adjacent to the fire and has only a single path to r_{i-1} in G_i is not a minimal strategy.

2.1.2 Guiding the Fire - A special case when $\Delta(G) \leq 3$ and $deg(r) \leq 2$

Consider the *Firefighter Process* on a rooted graph (G, r) where $\Delta(G) \leq 3$ and $deg(r) \leq 2$ and a strategy D that always defends a vertex adjacent to a burning vertex. By first defending a vertex adjacent the the root, the fire spreads from the root to a single undefended neighbour. Thus, when we defend adjacent to the fire at $t = 1$, $deg(r_1) \leq 2$. Continuing in this fashion results in the fire spreading to at most one new vertex at each time step – the fire can be contained to a path. This property allows us to guide the fire along any path in the graph. However, if the root does not have degree two or if $\Delta(G) > 3$ then defending adjacent to a burning vertex can result in many new vertices burning. In our consideration of the three objectives outlined in Chapter 1, we will consider the division between those instances on rooted graphs (G, r) for which $\Delta(G) \leq 3$ and $deg(r) \leq 2$ and those for which one of these properties does not hold. We will call such a rooted graph (G, r) for which $\Delta(G) \leq 3$ and $deg(r) \leq 2$ a *2-subcubic graph (G, r)*.

2.2 Three Decision Problems

Based on Hartnell's [6] description of the discrete-time process that models the spread of a fire, we examine three related problems.

The first, the *Firefighter Problem* asks, given a rooted graph, whether or not a pre-specified number of vertices can be saved under some strategy. Even for trees there is a sharp dividing line based upon the maximum degree of the graph and the degree of the root between instances that can be shown to be solved in polynomial time and those for which there is a polynomial time reduction from an NP-complete problem [10]. Given the complexity of deciding whether a given number can be saved, we will also examine some greedy strategies for a related optimisation problem on rooted trees. We solve an open problem given in [4] concerning the relationship between the maximum number of vertices that can be saved and the number saved by a particular greedy strategy. This result is discussed further in Chapter 3.

The second, the *S-Fire Problem*, asks, given a rooted graph and a subset of the vertices, whether there is a strategy that ensures that no vertex in the specified subset burns. A polynomial time algorithm to solve whether the leaves of a tree can be saved exists for binary trees [10]. Further, there is a polynomial transformation to instances of graphs rooted at a vertex of degree three with maximum degree two from an NP-complete problem [10]. We examine the restriction where the root is a vertex of degree two and the maximum degree of the graph is three and find a polynomial transformation from an NP-complete problem. Further, we examine the problem in its original context by considering the case where the root is a set of vertices and we allow more than one vertex to be defended at a time. These results are discussed in Chapter 4

Finally, the *Weighted Firefighter Problem* asks, given a rooted graph and a weight function from the vertices to the integers, whether some pre-specified total weight can be prevented from burning. In fact, the prior two problems can be formulated in terms of the Weighted Firefighter Problem. From this observation we will use our results about the S-Fire Problem to describe restrictions that can be solved in polynomial time, and also determine collections for which a polynomial transformation from an NP-complete problem exists.

2.2.1 The Firefighter Problem

Following [11], the decision-version of the Firefighter Problem is formalised as follows

FIREFIGHTER
INSTANCE: A rooted graph (G, r) and an integer $k \geq 1$
QUESTION: Is there a finite sequence $(d_1, d_2 \dots d_t)$ of vertices of G such that if the fire breaks out at r then

- vertex d_i is neither defended nor burning at time i,
- at time t no undefended vertex is adjacent to a burning vertex, and
- at least k vertices are not burning at the end of time t?

In an instance of FIREFIGHTER on (G, r) with integer k we are seeking a strategy D such that b_D is no more than $n - k$.

Consider the following optimisation problem related to FIREFIGHTER.

OPT-FIREFIGHTER
INSTANCE: A rooted graph (G, r)
PROBLEM: Over all finite sequence $(d_1, d_2 \dots d_t)$ of vertices of G, such that if the fire breaks out at r where

- vertex d_i is neither defended nor burning at time i and
- at time t no undefended vertex is adjacent to a burning vertex,

MAXIMISE $n - b_D$.

For a rooted graph (G, r), we denote the maximum number of vertices that can be saved as MVS(G, r). We observe that MVS$(G, r) = max\{n - b_D|\ D\ is\ a\ strategy\}$.

In [4], the authors examine MVS(G, r) for some simple classes of graphs. Since finding an optimal solution for an instance OPT-FIREFIGHTER will allow us to answer the decision version of FIREFIGHTER we will examine FIREFIGHTER through the lens of determining MVS(G, r).

Consider OPT-FIREFIGHTER on (G, r) and the firefighter process on (G, r) with a strategy D that minimises b_D. Since b_D is optimised, D is a minimal strategy. Therefore, for a YES instance of FIREFIGHTER on a rooted graph (G, r) with integer k, there is a minimal strategy that ensures that at least k vertices are saved.

We begin our examination of FIREFIGHTER by first considering $\Delta(G) \leq 3$ and $deg(r) \leq 2$ for an instance of FIREFIGHTER on (G, r) with integer k. As we saw in Section 2.1.2, restricting to such instances allows for strategies that contain the fire to a single path. We first focus our attention on instances of FIREFIGHTER on (G, r) with integer k for which (G, r) is 2-subcubic and a tree.

By the Tree Lemma, any minimal strategy for an instance of FIREFIGHTER on (T, r) with integer k, where T is a binary tree, will always defend adjacent to the newest burning vertex. This strategy ensures that the fire is restricted to a path. The process will terminate under such a strategy when the newest burning vertex, r_i, has degree less than three in T. The predecessor of r_i will have burned immediately prior to r_i, and defending adjacent to r_i (if possible) will defend r_i's only unburned neighbour, terminating the process. From this we can find all minimal defensive strategies D and determine MVS(G, r) [3].

Continuing our examination of the restriction FIREFIGHTER to 2-subcubic graphs (G, r), we move to the case where G is not an an acyclic graph. In [3] Finbow et al. take up this problem and produce the following result.

Theorem 2.2.1. *[3] There is a polynomial time algorithm for FIREFIGHTER restricted to (G, r) with integer k when (G, r) is a 2-subcubic graph.*

We now turn our attention to those instances for which either $\Delta(G) > 3$ or $deg(r) > 2$.

In [10] King and MacGillivray find a polynomial transformation from an NP-complete problem to instances FIREFIGHTER on (T, r) with integer k where T is a tree rooted at a vertex of degree three.

The authors expand upon this reduction to show that FIREFIGHTER is NP-complete even when restricted to cubic graphs [10].

Theorem 2.2.2. *[10] FIREFIGHTER is NP-complete even if restricted to graphs with maximum degree three. The problem is solvable in polynomial time for graphs of maximum degree three, provided the fire starts at a vertex of degree two.*

2.2.2 The S-Fire Problem

Consider now a variant of the Firefighter Problem where it must be determined whether a pre-specified $S \subseteq V(G)$ can be saved. Following [4] the decision problem for SFIRE is formalised as follows:

$SFIRE$
INSTANCE: A rooted graph (G, r) and a subset $S \subseteq V(G) - \{r\}$.
QUESTION: If the fire breaks out at r, is there a strategy under which no vertices in S burn? That is, does there exist a finite sequence $(d_1, d_2, \ldots, d_t)$ of vertices of G such that if the fire breaks out at r, then

- vertex d_i is neither burning nor defended at time i.
- at time t no undefended vertex is adjacent to a burning vertex, and
- no vertex in S is burned at the end of time t?

For every YES instance of SFIRE on (G, r) with set S there is some minimal strategy D such that for all $v \in V(H_D)$, $v \notin S$. We call such a strategy a *successful strategy*. We call a *successful strategy* an *away successful strategy* if D is a successful strategy and for some $i \leq t$, r_{i-1} is not adjacent to d_i in G_{i-1} – we defend away from the fire. Finally, if a strategy D is both *a successful strategy* and also has the property that for all i, r_{i-1} is adjacent to d_i in G_{i-1}, it is called an *adjacent successful strategy*.

For a rooted graph (G, r) and a set $S \subseteq V(G)$, we call a vertex $v \in V(G) - (S \cup \{r\})$ *reachable* if there is a path from r to v containing no vertex in S. Thus $v \notin S \cup \{r\}$ is reachable if it is in the component containing r in the graph $G[V - S]$. If a vertex $v \in V(G) - (S \cup \{r\})$ is not reachable we call it *unreachable*. The *reachable* vertices are those that lie in same component as r after the vertices of S are removed. The *unreachable* vertices are those that lie in another component.

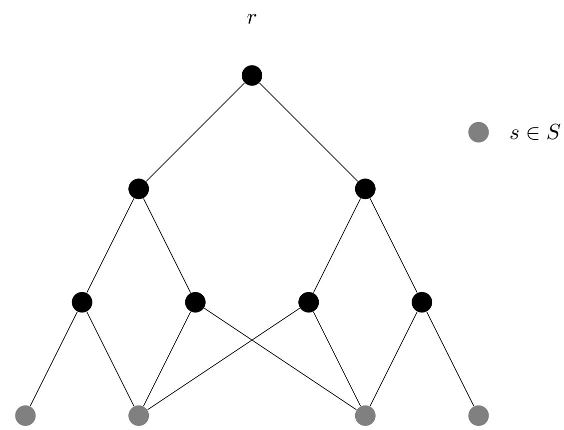

Figure 2.4: A NO instance of SFIRE

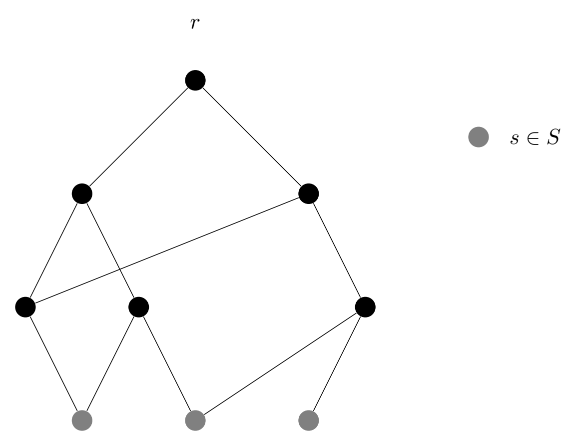

Figure 2.5: A YES instance of SFIRE

We make the following observation about instances of SFIRE on 2-subcubic graphs with set S:

Observation 1. *If, under some strategy D, some unreachable vertex of G burns, then some some vertex of S must have burned.*

Observation 2. *Consider $u, v \in S$ and $uv \in E$. Since no strategy that that allows the fire to spread between a pair of vertices in S is a strategy that prevents vertices in S from burning, every vertex of S can be saved if and only if every vertex of S can be saved on the rooted graph (G', r) where $G' = G - uv$.*

These observations allow for the following reductions to an instance of SFIRE on (G, r) with set S:

Reduction 1. *If x is an unreachable vertex, then x may be removed from the graph. If $s \in S$ has no neighbour that is a reachable vertex, s may be removed from the graph.*

Reduction 2. *Any edge between a pair of vertices in S may be removed.*

We call an instance of SFIRE on (G', r) with set S formed by the application of the above reductions to an instance of SFIRE on (G, r) with set S the *reduced instance*. A *reduced instance* of SFIRE has the properties that:

1. S' is an independent set.

2. Every vertex in $V - S'$ is reachable, i.e., $G[V - S']$ is connected.

Since solving an instance SFIRE amounts to solving the related reduced instance, we need only consider reduced instances.

As with the FIREFIGHTER, we consider separately those reduced instances of SFIRE on 2-subcubic graphs (G, r) with set S, and those on non-2-subcubic graphs.

In [10] King and MacGillivray examine instances of SFIRE on (T, r) with set S, where T is a tree and S is set containing the leaves of the tree with the following decision problem:

3FLFIRE
INSTANCE: A rooted tree (T, r) with $\Delta(T) \leq 3$.
QUESTION: If the fire breaks out at r, is there a strategy under which no leaf of T burns? That is, does there exist a finite sequence $(d_1, d_2, \ldots, d_t)$ of vertices of T such that if the fire breaks out at r, then

- vertex d_i is neither burning nor defended at time i,

- at time t no undefended vertex is adjacent to a burning vertex, and

- no leaf in T is burned at the end of time t.

From this related problem they derive the following result by modifying the reduction used in the proof of Theorem 2.2.2.

Theorem 2.2.3. *[10] SFIRE is NP-complete, even if S is the set of leaves of a tree of maximum degree three.*

King and MacGillivray also consider those instances of SFIRE where T is a binary tree.

Proposition 2.2.4. *[10] Let (T, r) be a binary tree. If the fire breaks out at r, then all of the leaves can be saved if and only if T is not full.*

They go on to note that for an instance of SFIRE on a full binary tree where S is the set of leaves, there is a strategy to have at most one vertex of S burn – direct the fire toward any leaf.

These results leave open the complexity of SFIRE restricted to 2-subcubic graphs (G, r). Given the similarity between FIREFIGHTER and SFIRE, the results for FIREFIGHTER in Section 2.2.1 suggest the possibility of the existence of a polynomial time algorithm for such instances. In Chapter 4 we will show a reduction to an instance of SFIRE a 2-subcubic graph (G, r) with set S from an instance of 3FLFIRE. This reduction shows that SFIRE is an NP-complete problem even when restricted to 2-subcubic graphs (G, r). In Chapter 4 we will also examine SFIRE in the context first described by Hartnell – the possibility for fires to start at multiple vertices and for a number of firefighters to be utilised at a single time.

2.2.3 The Weighted Firefighter Problem

Consider the generalisation of FIREFIGHTER in which each vertex is assigned a weight and the objective becomes to determine whether a subset of vertices having at least a given total weight can be saved. For a graph G rooted at r, and a weight function $w : V(G) \to \mathbb{Z}$ we define $MVS_w(G, r)$ to be the maximum weight that can be saved over all strategies. Formally, we define the Weighted Firefighter Decision Problem (WFIRE) as follows [4].

WFIRE
INSTANCE: A rooted graph (G, r) an integer k and a weight function $w : V(G) \to \mathbb{Z}$.
QUESTION: Is $MVS_w(G, r) \geq k$? Or, is there a finite sequence $(d_1, d_2 \ldots d_t)$ of vertices of G such that if the fire breaks out at r, then

- vertex d_i is neither defended nor burning at time i,
- at time t no undefended vertex is adjacent to a burning vertex, and
- the total weight of the non-burning vertices at time t is at least k?

As with FIREFIGHTER we can define a related optimisation problem:

OPT-WFIRE
INSTANCE: A rooted graph (G, r), and a weight function $w : V(G) \rightarrow \mathbb{Z}$.
PROBLEM: Over all finite sequences $(d_1, d_2 \ldots d_t)$ of vertices of G, such that if the fire breaks out at r, where

- vertex d_i is neither defended nor burning at time i, and
- at time t no undefended vertex is adjacent to a burning vertex,

MAXIMISE: $\sum_{v \in V} w(v) - b_w$.

For a rooted graph (G, r) and a weight function $w : V(G) \rightarrow \mathbb{Z}$ we denote the maximum weight of vertices that can be saved as $MVS_w(G, r)$. We observe that

$$MVS_w(G, r) = max \left\{ \sum_{v \in V} w(v) - w_D | D \; is \; a \; strategy \right\}$$

where w_D is the total weight of the saved vertices for the Firefighter Process on (G, r) with strategy D.

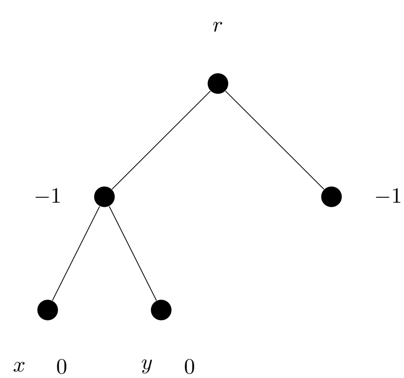

Figure 2.6: A minimal strategy will not satisfy $\text{MVS}_w(G, r) = 0$

For the analysis of FIREFIGHTER and SFIRE we have relied upon the fact that when an instance of such a problem is a YES instance there is a minimal strategy that satisfies the objective. However, as we see in Figure 2.6 this will not be the case for graphs with arbitrary weights due the introduction of negative weights. The strategy in Figure 2.6 that maximises the weight of the saved vertices is to defend vertices x and y. This strategy, however, is not a minimal strategy. With non-negative weights, for a pair of strategies D and D' where H_D is a proper subgraph of $H_{D'}$, the sum of the weight of the vertices in H_D will be at most the sum of the weight of the burned vertices in $H_{D'}$. With this in mind we consider restricting the problem to weights that are either 0 or 1.

A Restriction of WFIRE to $\{0, 1\}$ weights

Consider the decision problem for the restriction of the Weighted Firefighter Problem to a version where $w(v) \in \{0, 1\}$ and a related optimisation problem.

W01FIRE
INSTANCE: A rooted graph (G, r), an integer k and a weight function $w : V(G) \to \{0, 1\}$.
QUESTION: Is $MVS_w(G, r) \geq k$? That is, is there a finite sequence $d_1, d_2 \ldots d_t$ of vertices of G such that if the fire breaks out at r, then

- vertex d_i is neither defended nor burning at time i,
- at time t no undefended vertex is adjacent to a burning vertex, and
- the total weight of the non-burning vertices at time t is at least k?

OPT-W01FIRE
INSTANCE: A rooted graph (G, r), and a weight function $w : V(G) \to \{0, 1\}$.
PROBLEM: Over all finite sequence $(d_1, d_2 \ldots d_t)$ of vertices of G such that if the fire breaks out at r where,

- vertex d_i is neither defended nor burning at time i and,
- at time t no undefended vertex is adjacent to a burning vertex.

MAXIMISE: $\sum_{v \in V} w(v) - b_w$.

As mentioned in [4] we can use W01FIRE to bring FIREFIGHTER and SFIRE into a common framework. An instance of FIREFIGHTER on (G, r) with integer k can be formed as an instance of W01FIRE simply by taking the weight function

$w(v) = 1$ for all $v \in V$. Similarly we may express an instance of SFIRE on (G, r) with set S as an instance of W01FIRE as follows:

$$w(v) = \begin{cases} 0 & : v \notin S \\ 1 & : v \in S \end{cases}$$

and by setting $k = n - |S|$ [4]. In Chapter 5 examine the complexity of W01FIRE and WFIRE.

Chapter 3

Greedy Algorithms for Firefighter

Given the complexity of determining MVS(T, r), even for a tree rooted at a vertex of degree three and having maximum degree three, it is worth examining some simple approximation algorithms for MVS(T, r). Strategies that greedily determine which vertex to defend at each time step are a natural choice. In this section we examine a pair of greedy algorithms for determining MVS(T, r): the Weighted Greedy Algorithm and the Degree Greedy Algorithm.

For any algorithm that approximates an optimal solution to optimisation problem, it is of interest to know whether the computed solution is guaranteed to be "close" to optimal in either absolute or relative terms. We call an algorithm an *ϵ-approximation algorithm*, $\epsilon \in [0, 1]$, if the ratio of the computed solution to the optimal solution is always at least ϵ.

3.1 The Weighted Greedy Algorithm

The Weighted Greedy Algorithm makes a decision on which neighbour of r_{i-1} to defend at time i based upon how many vertices will be saved by defending each possible neighbour. Since each vertex in T has a unique path to r, defending a vertex u will prevent any of u's descendants from burning. Thus we may assign to each vertex a weight – the number of vertices that will be saved if the vertex is defended. This value works out to be the number of children of the vertex plus one. The Weighted Greedy Algorithm defends, at each step, the vertex adjacent to r_i with the greatest weight. For the rooted tree (T, r) in Figure 3.1 the Weighted Greedy Algorithm is optimal; however, for the rooted tree in Figure 3.2 it is not. Examples can be

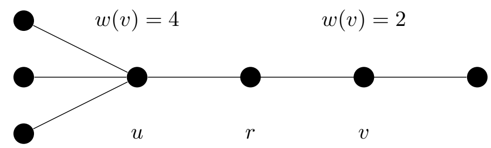

Figure 3.1: The weighted greedy algorithm is optimal

constructed to have the number of saved vertices be arbitrarily far from MVS(T, r) in absolute terms. However, in relative terms the difference between the number of vertices saved by the Weighted Greedy Algorithm and MVS(T, r) is bounded [7].

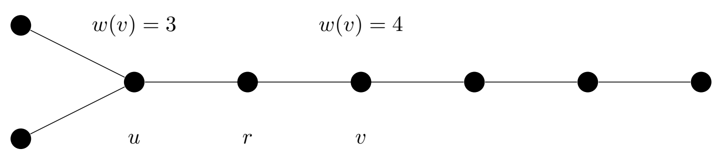

Figure 3.2: The weighted greedy algorithm is not optimal

Theorem 3.1.1. *[7] If $Greedy_w(T, r)$ denotes the number of vertices saved using the Weighted Greedy Algorithm, then*

$$\frac{1}{2} \leq \frac{Greedy_w(T, r)}{MVS(T, r)}.$$

That is, the Weighted Greedy Algorithm is a $\frac{1}{2}$-approximation algorithm for OPT-FIREFIGHTER on trees.

Hartnell and Li [7] show this bound to be sharp by examining the family of rooted graphs (G_k, r) obtained by subdividing one edge of a star on $k \geq 3$ vertices k times and setting r to be the vertex of degree two adjacent to the vertex of degee k. (See Figure 3.3.) Applying the Weighted Greedy Algorithm saves all of the vertices on the path except r and allows all but one vertex of the original star to burn. This strategy saves a total of $k + 1$ vertices. An optimal strategy first defends the centre vertex of the star and then defends the vertex adjacent to r_2. This strategy will save $2(k - 1)$ vertices. The ratio between these two values tends to two as k is increased. Thus the Weighted Greedy Algorithm saves $\frac{1}{2}\cdot$ MVS(T, r) vertices [7].

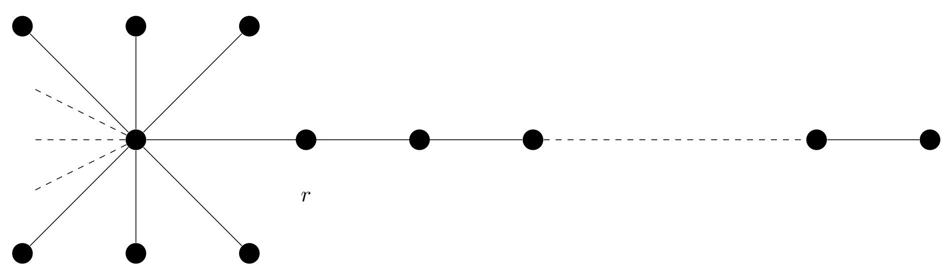

Figure 3.3: A tree satisfying $Greedy_w(T, r) = 1/2 \cdot MVS(T, r)$.

3.2 The Degree Greedy Algorithm

In [11] MacGillivray and Wang examine the Degree Greedy Algorithm. This algorithm defends, at each step, the vertex of highest degree adjacent to r_i. They show that this strategy finds an optimum solution for caterpillars, but not for arbitrary trees. They leave open the existence of a constant $c \in (0, 1]$ such that the algorithm saves at least $c \cdot MVS(T, r)$ vertices [11]. However, by construction it can be seen that no such constant exists.

Theorem 3.2.1. *If $Greedy_d(T, r)$ denotes the number of vertices saved using the Degree Greedy Algorithm, then there is no $c \in (0, 1]$ such that is always the case that*

$$Greedy_d(T, r) \geq c \cdot MVS(T, r)$$

.

Proof. Let J be a full and complete binary tree of height k rooted at v. Let P_k be the path of length k with vertex sequence $x_0, x_1, \ldots, x_k$. For $0 \leq i \leq k$, let S_i be a copy of the star $K_{1,3}$ with centre vertex w_i. Construct a tree T with $2^{k+1} + 5k - 1$ vertices by joining x_0 to v, and x_i to w_i for all $i \leq k$. (See Figure 3.4)

Consider OPT-FIREFIGHTER on (T, x_0). With strategy $D = (v, x_2, z_1)$ nine vertices burn. This strategy is optimal. Thus $MVS_w(T, r) = 2^{k+1} + 5k - 10$. However, with the Degree Greedy Algorithm, every vertex of J will burn along with every vertex on the path of length k. Thus $Greedy_d(T, r) = 4k$.

$$\lim_{k \to \infty} \frac{Greedy_d(T, r)}{MVS(T, r)} = \lim_{k \to \infty} \frac{4k}{2^{k+1} + 5k - 10} = 0.$$

There is no $c \in (0, 1]$ such that the Degree Greedy Algorithm saves at least $c \cdot MVS(T, r)$ vertices. □

Corollary 3.2.2. *For any $\epsilon > 0$, the Degree Greedy Algorithm is not an ϵ - approximation algorithm for OPT-FIREFIGHTER.*

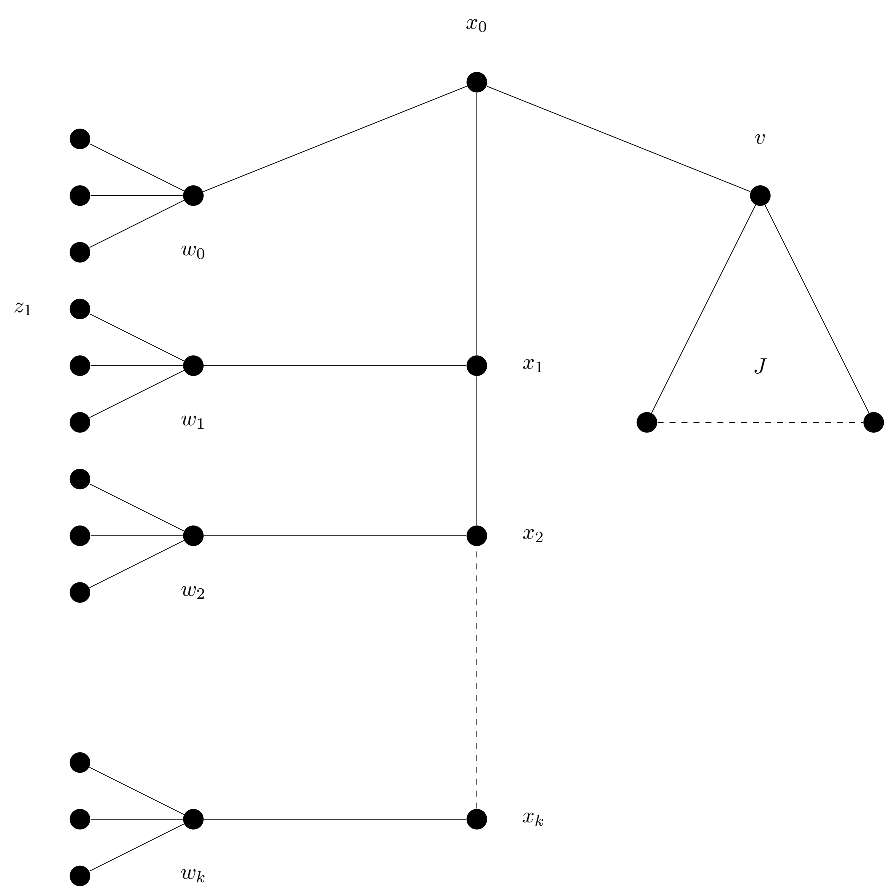

Figure 3.4: A construction where $\lim_{k\to\infty} \frac{Greedy_d(T,r)}{MVS(T,r)} = 0$

Chapter 4

The SFIRE Problem

As with the complexity results for FIREFIGHTER, we consider the restriction of our problem to graphs with maximum degree three and rooted at a vertex of degree two. We consider these graphs because when FIREFIGHTER is restricted to such graphs it can be solved in polynomial time.

We consider first the structure of H_D for strategies on reduced instances of SFIRE on a 2-subcubic graph (G, r). Figure 4.1 shows an instance, a strategy and its burned subgraph H_D. Under D no vertex of S burns, thus this is a YES instance of SFIRE. However, we notice that H_D contains a cycle. Therefore, D is not successful strategy as it is not minimal. Using this cycle we may construct a successful strategy that guides the fire around the cycle.

Lemma 4.0.3. *If D is a minimal strategy for a reduced YES instance of SFIRE on a 2-subcubic graph (G, r) with set S, then H_D is a tree.*

Proof. We proceed by contradiction. For an instance of SFIRE on (G, r) with set S, let D be a minimal strategy such that H_D contains a cycle C. Let $x \in V(C)$ be the vertex with the earliest burning time, i. There is a path P in H_D from r to x that intersects C only in x. If x has neighbours u and v in C, then consider the strategy D' that guides the fire down P to x and then through u and around C. The process will terminate when v's other neighbour in C is burning. Under such a strategy, the only vertices to burn are vertices on P and $C - \{v\}$. Since every vertex in P and C burned under D, $H_{D'}$ is a proper subgraph of H_D, a contradiction.

□

We consider first those YES reduced instances of SFIRE on (G, r) with set S for which there is an adjacent successful strategy.

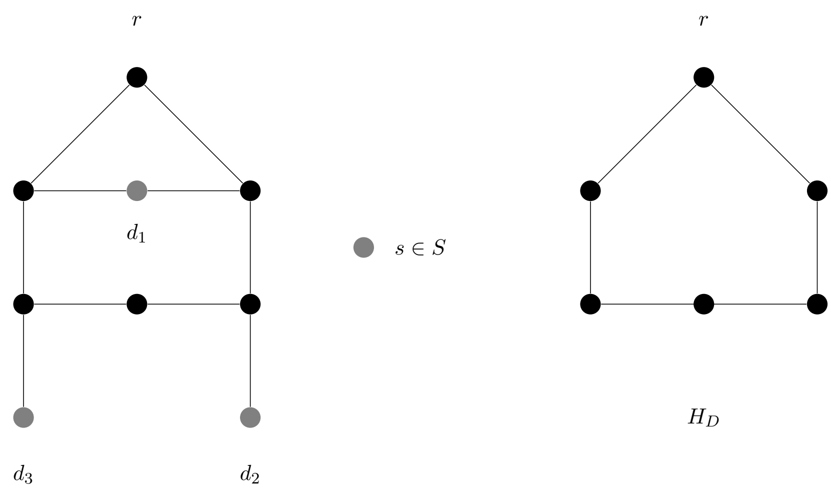

Figure 4.1: A instance of SFIRE and a strategy that saves all vertices in S

Proposition 4.0.4. *Let (G, r) be a 2-subcubic graph with set S. For any adjacent successful strategy D, the graph H_D is a path.*

Proof. Consider an instance of SFIRE on a 2-subcubic graph (G, r) with set S with adjacent successful strategy D. Since (G, r) is 2-subcubic and D is an adjacent successful strategy, at most one new vertex burns at each time $i \leq t$. Thus, there is exactly one vertex at distance $i \leq t$ from r in H_D. Therefore H_D is a path.

□

Proposition 4.0.5. *For a YES reduced instance of $SFIRE$ on a 2-subcubic graph (G, r) with set S, if $D = (d_1, d_2, \ldots d_t)$ is a successful strategy where H_D is a path having one end at r, then D is an adjacent successful strategy.*

Proof. We proceed via contradiction. Assume that for some successful strategy D where H_D is a path having one end as r has an entry d_i where d_i is not adjacent to r_i. Since H_D is a path, at time $i + 1$ exactly one new vertex is burning. This means r_i has degree one. Defending r_i's neighbour instead of d_i produces a strategy D' such that $H_{D'}$ is a proper subgraph of H_D. This is a contradiction since D was assumed to be minimal. □

We are able to classify those YES instances of SFIRE on 2-subcubic graphs with adjacent successful strategies D by looking at the the vertex at greatest distance from r in the path H_D. We will consider the cases where (i) this vertex has fewer than three neighbours in G and (ii) this vertex has exactly three neighbours in G.

Proposition 4.0.6. *Consider a YES reduced instance of SFIRE on a 2-subcubic graph (G, r) with set S where (G, r) has a reachable vertex v of degree less than three. The instance of SFIRE has an adjacent successful strategy.*

Proof. Let x be the nearest reachable vertex to r with $deg(r) \leq 2$. Let $d_G(r, x) = k$. Consider the path P from r to x that has length k and contains no vertex of S. The strategy D that guides the fire down this path will terminate after k time-steps. Since every vertex of P excluding r and x has degree three, this strategy is minimal. Thus D is an adjacent successful strategy. □

We consider now those YES reduced instances of SFIRE on (G, r) with set S with adjacent successful strategy D and where every reachable vertex has degree three. By Proposition 4.0.4, H_D is a path. If this path has length k, then the vertex at distance

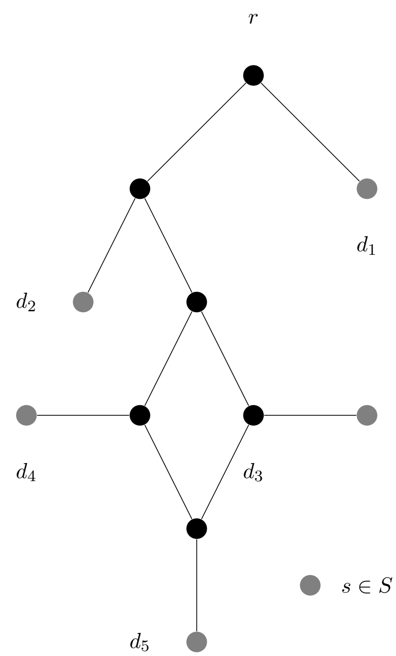

Figure 4.2: $G[V - S]$ contains a cycle

k from r has a neighbour that was defended at time $i < k$. We consider the two cases where $d_i \in S$ and $d_i \notin S$. In the first case (see Figure 4.2) we notice that $G[V - S]$ contains a cycle. This cycle can be used to construct an adjacent successful strategy by simply guiding the fire around this cycle. In the second case (see Figure 4.3) we see that there is some vertex of S that is adjacent to a pair of vertices in H_D. In this case we construct our adjacent successful strategy by guiding the fire down the path from r containing this pair of vertices. We formalise these observation as follows:

Proposition 4.0.7. *Consider a reduced instance of SFIRE on the 2-subcubic graph (G, r) with set S. If $G[V - S]$ contains a cycle, then there is an adjacent successful strategy.*

Proof. Suppose $G[V - S]$ contains the cycle C. Let x a the nearest vertex to r on the cycle, and let P be a shortest path from r to x. Let x have neighbours u and v. Consider the strategy that guides the fire down P and around C through x and u. In such a strategy, when x was the newest burning vertex, v was defended. Therefore when v's other neighbour in C is the newest burning vertex, it has at most one neighbour that is neither defended nor burning. Defending this neighbour will terminate the process. Since all of the burned vertices were either on P or C, no

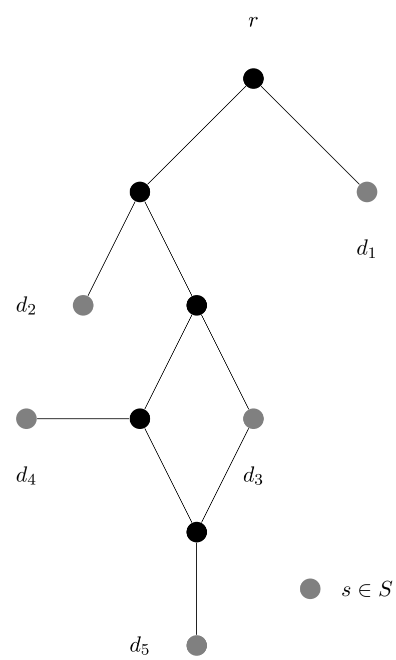

Figure 4.3: $G[V - S]$ is a tree and a vertex of S has a pair of neighbours on the same path from r

vertex of S burns. Thus this strategy is a successful adjacent strategy. □

Proposition 4.0.8. *Consider a reduced instance of SFIRE on the 2-subcubic graph* (G, r) *with set* S*, where G[V-S] is a tree. If there exists* $s \in S$ *having a pair of neighbours* u *and* v *such that* u *is a predecessor of* v *in* $G[V-S]$*, then the instance is a YES instance with an adjacent successful strategy.*

Proof. Let $s \in S$ such that s has a pair of neighbours u and v such that u is a predecessor of v in $G[V-S]$. Let P be the path in $G[V-S]$ having ends r and v. Consider the strategy that guides the fire down P. Since s is a neighbour of u, s is defended when u is the newest burning vertex. Thus when v is the newest burning vertex it has at most one neighbour that is neither defended nor burning. Defending this neighbour will terminate the process. Since P is a path in $G[V-S]$, under such a strategy no vertex of S burns. Therefore the instance is a YES instance with an adjacent successful strategy.

□

Combining these facts together gives the following theorem:

Theorem 4.0.9. *A YES reduced instance of SFIRE on a 2-subcubic graph* (G, r) *with set* S *has an adjacent successful strategy if and only if one of the following properties hold:*

1. *G[V-S] contains a cycle;*
2. *G contains a reachable vertex of degree two;*
3. *G[V-S] is a tree and there is a vertex* $s \in S$ *such that* s *has a pair of neighbours,* u *and* v *such that* u *is an ancestor of* v *in G[V-S].*

Proof. Assume that for a YES reduced instance of SFIRE on (G, r) with set S, an adjacent successful strategy D exists where $|D| = k$. Consider the vertex that burns at $t = k - 1$, i.e., the last vertex to burn. If this vertex has degree two in G, then property one holds directly. Assume then that $deg(x) = 3$. Since the process terminated at time k, some a neighbour of x was defended at time i and the other at time k. If $d_i \notin S$ then $G[V-S]$ contains a cycle and property two holds. Otherwise, $d_i \in S$ and property three holds.

The converse holds directly by Proposition 4.0.6, Proposition 4.0.7 and Proposition 4.0.8.

□

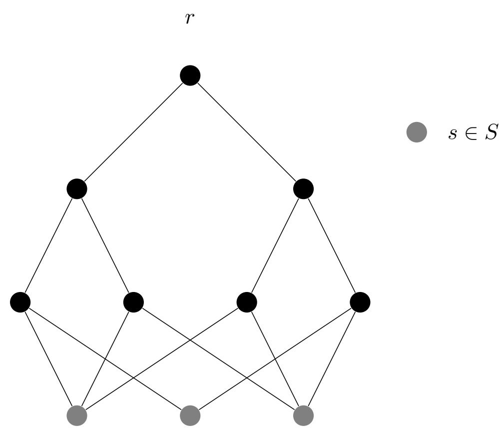

Figure 4.4: A YES instance where the vertices of S can be defended one by one

Having dispatched the case for which H_D is a path rooted at r, we turn our attention to YES instances for which the only successful strategies are away successful strategies. Figure 4.4 and Figure 4.5 provide two such examples. In Figure 4.4 we observe that we merely defend vertices in S until the process terminates. Since we may initially route the fire to any vertex in G, we can also consider this sort strategy after first routing the fire to another vertex.

Lemma 4.0.10. *Suppose that the 2-subcubic graph (G, r) with set S is a reduced instance of SFIRE. If there exists a reachable vertex x joined to r by a path P containing no vertices in S, and such that, for all i, $|\{s : d_{G_x}(x, s) < i\}| < i$, where G_x is the component of $G[G - (P - \{x\})]$ containing x then all vertices in S can be saved.*

Proof. Consider the strategy formed by routing the fire to x and then defending all vertices reachable in the instance of SFIRE on (G_i, r_i) with set S where $d_{G_i}(r, x) = i$. Under such a strategy every reachable vertex of S can be defended. Therefore D saves all of the vertices of S □

Adding these properties to the ones in Theorem 4.0.9 gives the following list of properties that imply a reduced instance SFIRE is a YES instance .

Proposition 4.0.11. *An reduced instance of SFIRE on (G, r) with set S where $\Delta(G) \leq 3$ and $deg(r) \leq 2$ is a YES instance if at least one of the following properties holds.*

1. *$G[V - S]$ contains a cycle;*

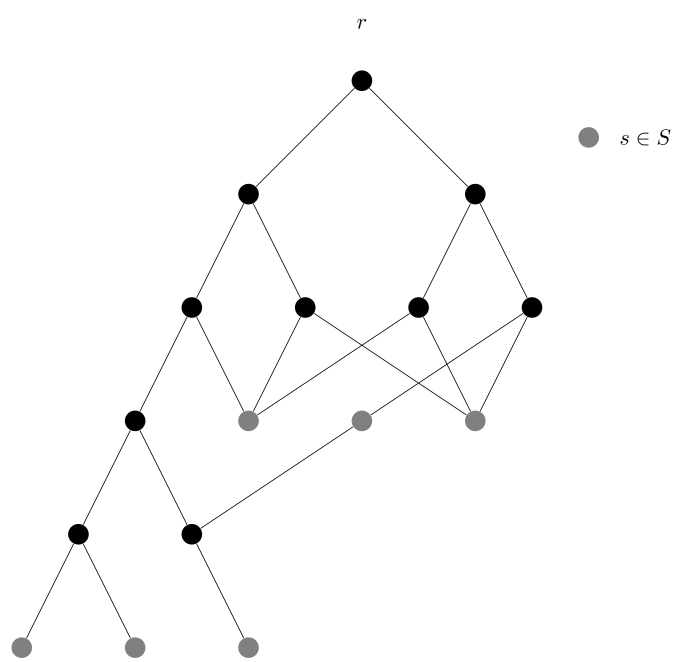

Figure 4.5: A YES instance where vertices away from the fire are defended

2. *G contains a reachable vertex of degree two;*

3. *$G[V-S]$ is a tree and there is a vertex $s \in S$ such that s has a pair of neighbours, u and v such that u is an ancestor of v in $G[V-S]$;*

4. *there exists x, such that for all i, $|\{s | d_{G_x}(x,s) \leq i \quad s \in S\}| \leq i$.*

Though these conditions are certainly sufficient, they are not necessary. Figure 4.5 is a YES instance of the problem that does not satisfy any of the properties above. The only successful strategy for Figure 4.5 starts by defending vertices of S and then defends a vertex not in S at $t = 4$. A successful strategy, D, that defends a vertex that is not adjacent to the fire forms a graph H_D which is a tree with vertex of degree three – it is an away successful strategy. What follows is an examination instances on such graphs.

Proposition 4.0.12. *For a reduced instance of SFIRE on a 2-subcubic graph (G, r) with set S where every successful strategy D has the property that H_D is not a path rooted at r, every successful strategy for the instance is an away successful strategy.*

Proof. Consider the a reduced instance of SFIRE on a 2-subcubic graph (G, r) with set S such that every successful strategy D has the property that H_D is not a path rooted at r. By Theorem 4.0.9, there is no adjacent successful strategy. Therefore every successful strategy is an away successful strategy.

□

Looking now at instances with only away successful strategies, we consider which vertices in G might be candidates to be defended even though they are not adjacent to the fire. As we saw in Figure 4.4 the vertices in S are viable candidates. In fact, every vertex in an away successful strategy that is defended away from the fire is a vertex of S. Furthermore, by the Ordering Lemma, it is the closest vertex of S to the fire.

Proposition 4.0.13. *Consider a YES reduced instance of SFIRE on a 2-subsubic graph (G, r) with set S where every successful strategy is an away successful strategy. For all successful strategies D, if $d_i \in D$ is not adjacent to r_{i-1} then $d_i \in S$.*

Proof. Let D be a successful strategy where $d_i \notin S$ not be adjacent to r_{i-1}. At time i consider the path P from r_{i-1} to d_i. Since d_i is not adjacent to r_{i-1}, there is a vertex u adjacent to r_{i-1} on this path. Consider the strategy D' formed by replacing d_i with u.

The vertices on the path from u to d_i are protected under such a strategy. Therefore H_D is a proper subgraph of $H_{D'}$, and so D is not minimal, a contradiction. □

We now determine the complexity of SFIRE restricted to 2-subcubic graphs.

Let G_{k+1} be the 2-subcubic graph shown in Figure 4.6. It consists of $k+1$ copies of the graph formed by the 11 vertices in the topmost four levels in the figure, with vertex q_i joined to vertex x_{i+1} for $i = 1, 2, \ldots, k$, plus the three vertices u, v and w, which are adjacent to p_{k+1}, p_{k+1} and q_{k+1}, respectively. Let $X = \{x_1, x_2, \ldots, x_{k+1}\}, Y = \{y_1, y_2, \ldots, y_{k+1}\}$, and $Z = \{z_1, z_2, \ldots, z_{k+1}\}$. The following lemmas describe a pair of important properties of (G_{k+1}, p_0).

Lemma 4.0.14. *There are two adjacent successful strategies for the SFIRE on* (G_{k+1}, p_0) *with set* $X \cup Y \cup Z$*. In both of these, no vertex of* Z *is defended.*

Proof. Let D be a successful adjacent strategy. By Theorem 4.0.9, D guides the fire to a reachable vertex of degree ≤ 3. The only such vertices are p_{k+1} and q_{k+1}. Since no vertex on the path from p_0 to q_{k+1} has a neighbour in Z, no vertex of Z is defended. Similarly, since no vertex on the path from p_0 to p_{k+1} has a neighbour in Z, no vertex of Z is defended. □

Lemma 4.0.15. *Any firefighter process on* (G_{k+1}, p_0) *that saves all vertices in* $X \cup Y \cup Z$ *but at some time defends a vertex not adjacent to a burning vertex, will have both* p_{k+1} *and* q_{k+1} *burning when the process terminates.*

Proof. Consider the earliest time j that a vertex is defended that is not adjacent to a burning vertex in a firefighter process on (G_{k+1}, p_0) that saves all vertices in $X \cup Y \cup Z$. Let d_j be the vertex defended at this time. By Proposition 4.0.13, $d_j \in X \cup Y \cup Z$. We consider three cases.

Case i. $j \equiv 0 \pmod 3$: At time j there is a single vertex burning at distance $j-1$. Each vertex at distance $j-1 \equiv 2 \pmod 3$ has an undefended neighbour in $X \cup Y \cup Z$. Thus d_j is adjacent to r_{j-1}, a contradiction.

Case ii. $j \equiv 2 \pmod 3$: At time j there is a single vertex burning at distance $j-1$. Without loss of generality, let $d_j = x_c$, where x_c is the vertex in X at distance 2 from r_{j-1}. In such a case, r_j has a pair of neighbours in $X \cup Y \cup Z$ (y_c and z_c).

Thus this cannot be a strategy that protects all vertices in $X \cup Y \cup Z$, a contradiction.

Case iii. $j \equiv 1 \pmod 3$: If a vertex is being defended at time $j \equiv 1 \pmod 3$, then some vertex at distance $d \equiv 0 \pmod 3$ is burning. The vertices at distance $d \equiv 0 \pmod 3$ are $\{p_0, p_1, \ldots p_k\}$ and $\{q_1, q_2, \ldots q_k\}$. We note that each of $\{q_1, q_2, \ldots q_k\}$ has a pair of undefended neighbours in $X \cup Y$. As such if one of these vertices is burning at time $j-1$ then d_j is adjacent to the fire. This fact discounts these vertices from consideration. Thus, we need only consider the case where the fire is routed to one of $\{p_0, p_1, \ldots p_k\}$ before defending a vertex away from the fire.

Let p_c be the vertex burning at time $j \equiv 0 \pmod 3$. Without loss of generality let $d_j = x_{c+1}$, $d_{j+1} = z_{c+1}$, and $d_{j+2} = y_{c+1}$. At time $j+2$ both p_{c+1} and q_{c+1} are burning. Since q_{c+1} is adjacent to x_{c+2}, $d_{j+3} = x_{c+2}$. Defending adjacent to the fire at time at this point will lead to at least one of z_{c+2} or y_{c+2} burning. Thus, without loss of generality, $d_{j+4} = z_{c+2}$ and $d_{j+5} = y_{c+2}$. Continuing in this fashion will result in p_{k+1} and q_{k+1} burning.

□

Theorem 4.0.16. *The restriction of SFIRE to 2-subcubic graphs is NP-complete.*

Proof. The problem is known to be in NP [11]. The transformation is from 3FLFIRE.

Consider an instance of 3FLFIRE on (T', a'). Let L' be the set of leaves of T', and $S' = \{s'_1, s'_2, \ldots, s'_k,\}$ be the set of degree two vertices of T'. Note that k is determined by S'. Construct an instance of SFIRE restricted to 2-subcubic graphs from T' and G_{k+1} by forming the graph H as follows.

- Add the edge $z_i s'_i$ for all $1 \leq i \leq k$
- Identify the neighbours of a' in T' with u, v and w, respectively;
- Remove a'

We note that H is a 2-subcubic graph, and that the construction can be carried out in polynomial time. We claim that all leaves of T' can be saved if and only if all vertices in $L' \cup X \cup Y \cup Z$ can be saved.

Suppose all of the leaves of T' can be saved using strategy D'. We use D' to construct a strategy for the instance of SFIRE on (H, p_0) with set $L' \cup X \cup Y \cup Z$. Consider the strategy that begins $(x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_{k+1}, y_{k+1}, z_{k+1})$. After

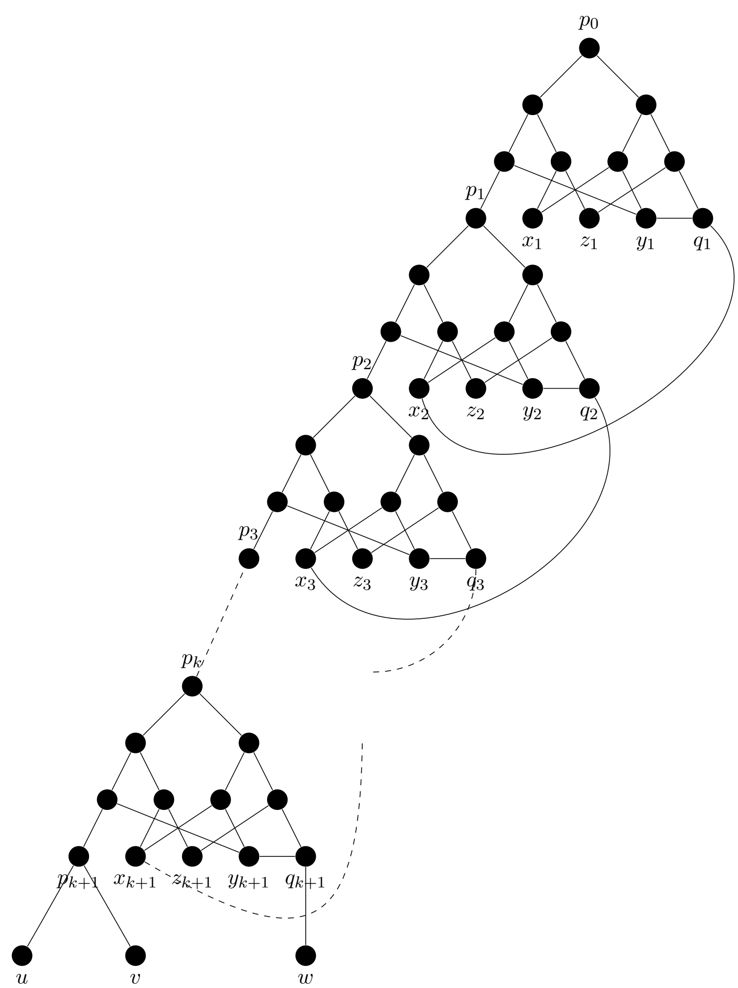

Figure 4.6: G_{k+1}

defending these vertices both q_{k+1} and p_{k+1} are burning. At this point combining the burning vertices into a single vertex yields 3FLFIRE on (T', a'). Applying D' will save all of the vertices in L'. This strategy saves all of the vertices in $L' \cup X \cup Y \cup Z$.

Suppose all vertices in $L' \cup X \cup Y \cup Z$ can be saved using strategy D. Then, by Lemma 4.0.14 and 4.0.15, we know that by the end of time $3(k+1)$ either no vertex of Z has been defended and p_{k+1} is burning, or some subset of the vertices of Z have been defended and both p_{k+1} and q_{k+1} are burning. In the first case, p_{k+1} is the root of a complete binary tree in $H_{3(k+1)}$ so some vertex of $L' \cup Z$ must burn, a contradiction. Thus, some subset of the vertices of Z have been defended and both p_{k+1} and q_{k+1} are burning. Therefore the strategy $D' = (d_{3(k+1)+1}, (d_{3(k+1)+2}, \ldots, (d_{3(k+1)+t})$, where t is the height of T' is a strategy that saves all of the leaves of T' in the instance of 3FLFIRE on (T', a') □

4.1 A Generalisation of SFIRE for multiple fires and firefighters

When Hartnell originally introduced the Firefighter Process, he did so considering the possibility that a set of vertices burn at the start of the process, and the possibility that multiple vertices may be defended at one time. Until now, we have considered the restricted version of this process that has the fire start at a single vertex and has a single vertex defended at each time step. Hartnell's more general of the process has been considered in many different situations, including containing fires on infinite grids. More about containing fires on grids can be found in Chapter 6

To properly consider SFIRE for Hartnell's original process, we must generalise some of the definitions found in Section 2.1. In particular, we must generalise our definitions to allow for the fire to start at multiple roots, and for a variable number of firefighters to be available at each time step.

For a positive integer f an *f-rooted graph* is a pair (G, F) where G is a simple graph and $F \subseteq V$ such that $|F| = f$.

For the *generalised firefighter process* we require an f-rooted graph, a sequence $\mathbf{t}$ of nonnegative integers: $\mathbf{t} = (t_1, t_2, \ldots t_t)$ called the *defence sequence* and a sequence of sets $D = (D_1, D_2, \ldots D_t)$ called *generalized strategy* where $D_i = \{d_i^1, d_i^2, \ldots d_i^{t_i}\}$, $d_i^j \in V(G)$ is the set of vertices defended at time i.

Consider the following decision problem:

S-**t**-FIRE
INSTANCE: A rooted graph (G, F), a subset $S \subseteq V(G)$ and a sequence $\mathbf{t} = (t_1, t_2, \ldots t_t)$.
QUESTION: If the fire breaks out on all vertices of F, is there a strategy using no more than t_i firefighters at time t under which no vertices in S burn? That is, does there exist a finite sequence $(D_1, D_2, \ldots, D_t)$ of sets vertices of G where for all $i \leq t$, $|D_i| \leq t_i$ such that if the fire breaks out at r, then

- vertex $d_i^j \in D_i$, $j \leq t_i$ is neither burning nor defended at time i
- at time t no undefended vertex is adjacent to a burning vertex, and
- no vertex in S is burned at the end of time t?

Theorem 4.1.1. *S-**t**-FIRE is NP-complete.*

Proof. Since S-**t**-FIRE contains SFIRE and SFIRE is NP-complete by Corollary **??**, the claim holds by restriction. □

Proposition 4.1.2. *For every instance of S-**t**-FIRE on the f-rooted graph (G, F) with set S and defence sequence $\mathbf{t}$ we can construct a related instance of S-**t**-FIRE on the $1-$ rooted graph (G', r) with set S and defence sequence $\mathbf{t}'$ such that the original instance is a YES instance if and only if the related instance is a YES instance.*

Proof. Consider an instance of S-**t**-FIRE on the f-rooted graph (G, F) with set S and defence sequence $\mathbf{t}$. We form an instance of S-**t**-FIRE on the $1-$*rooted* graph (G', r) with set S and defence sequence $\mathbf{t}'$ by creating a new vertex r, making it adjacent to all vertices in F, and letting $\mathbf{t}' = (0, t_1, t_2, \ldots, t_t)$.

At $t = 1$ in the instance of S-**t**-FIRE on the $1-$*rooted* graph (G', r) with set S and defence sequence $\mathbf{t}'$, all of the vertices of F are burning and no vertex of G has been defended – it is the instance of S-**t**-FIRE on the $1-$*rooted* graph (G', r) with set S and defence sequence $\mathbf{t}'$. Therefore solving the instance of S-**t**-FIRE on the $1-$*rooted* graph (G', r) with set S and defence sequence $\mathbf{t}'$ will also solve the instance of S-**t**-FIRE on the f-rooted graph (G, F) with set S and defence sequence $\mathbf{t}$. □

From this proposition we see that we need not consider instances for which $f > 1$, i.e, we may always assume that the fire starts at a single vertex.

We turn now to the generalisation that allows for a variable number of firefighters to be used at each time-step. Since SFIRE is NP-complete even for trees rooted at a vertex of degree three, we will restrict our consideration to instances on trees. We will first examine those instances with defence sequences that consist of 0's and 1's and show that solving such an instance is akin to solving a related instance for which $t_i = 1$ for all $i \leq t$. We will then use a similar technique to show that an instance with an arbitrary defence sequence can be solved using an algorithm that solves an instance for which $t_i = c$ for all $i \leq t$, where $c = max\{t_i | t_i \in \mathbf{t}\}$.

Proposition 4.1.3. *S-**t**-FIRE on f-rooted graphs polynomially transforms to S-**t**-FIRE on 1-rooted graphs.*

Proof. We consider the instance SFIRE on rooted tree (T', r) with set S' formed from an instance S-**t**-FIRE on the $1 - rooted$ graph (G, r) with set S and defence sequence $\mathbf{t}$ as follows:

- For each i such that $t_i = 0$, "append" a path P_i to r with leaf l_i and length i.
- Let $L = \bigcup l_i$
- Let $S' = S \cup L$.

The construction is shown in Figure 4.7.

We assume first that the instance SFIRE on the rooted tree (T', r) with set S' is a NO instance. This means that there is no strategy that can save all of the leaves of T'. Assume, for the sake of contradiction, that S-**t**-FIRE on the $1 - rooted$ graph (G, r) with set S and defence sequence $\mathbf{t}$ is a YES instance. Let D be a minimal strategy that saves all of the leaves of T. We can form a strategy D' for the instance SFIRE on rooted tree (T', r) with set S' by modifying D to let $d_i = l_i$ whenever $t_i = 0$. This strategy will save all of the leaves of T', a contradiction. Thus, if the instance SFIRE the rooted tree (T', r) with set S' is a NO instance then S-**t**-FIRE on the $1 - rooted$ graph (G, r) with set S and defence sequence $\mathbf{t}$ is also a NO instance.

We now assume that the instance SFIRE on rooted tree (T', r) with set S' is a YES instance with successful strategy D. We show that we can construct a strategy D' from D that shows S-**t**-FIRE on the $1 - rooted$ graph (G, r) with set S and defence sequence $\mathbf{t}$ to be a YES instance. We proceed by induction on

$$k = |\{d_i | d_i \notin V(T), t_i = 1\}|,$$

– the size of the set of vertices defended that are not in the original tree that are defended at a time i where $t_i = 1$. If this set is empty then for all i such that $t_i = 0$

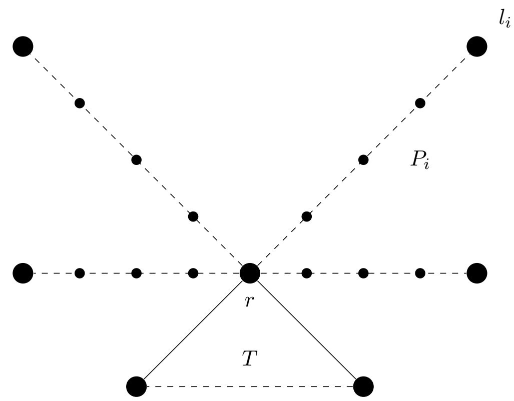

Figure 4.7: Forming T' from T in the proof of Proposition 4.1.3

we have $d_i = l_i$. In such a case, by removing these vertices we get a strategy that can be directly applied to S-**t**-FIRE on the $1 - \textit{rooted}$ graph (G, r) with set S and defence sequence **t** to defend all of the leaves in T.

Assume now that $k = q + 1$. Let j be the earliest time such that $t_j = 1$ and $d_j = x_i$, $x_i \in P_i$ Since l_i did not burn, we have $j < i$. Consider the strategy D' formed by modifying D such that $d'_j = d_i$ and $d'_i = l_i$. This strategy will still ensure that no leaf of T' burns and has the property that

$$|\{d'_i | d'_i \notin V(T), t_i = 1\}| < |\{d_i | d_i \notin V(T), t_i = 1\}|.$$

Thus our result holds by induction.

□

Having shown that we may use a single firefighter algorithm to solve a problem that uses a $\{1, 0\}$ sequence of firefighters on a tree we now generalize this result and show that an algorithm that uses a constant number of firefighters can solve a problem with an arbitrary number of firefighters at each step.

Proposition 4.1.4. *S-***t***-FIRE on 1-rooted trees with defence sequence* **t** *where* **t** *is an arbitrary sequence of nonnegative integers with maximum entry c, polynomially transforms to S-***t**′*-FIRE on 1-rooted , where* **t**′ *is the sequence with each entry as c.*

Proof. Let **t** be an arbitrary sequence of non-negative integers with largest element c. Suppose an instance (T, r) of S-**t**-FIRE is given. We construct an instance of S-**t**′-FIRE, where **t**′ is a sequence of the same length as **t** with every entry equal to c as follows:

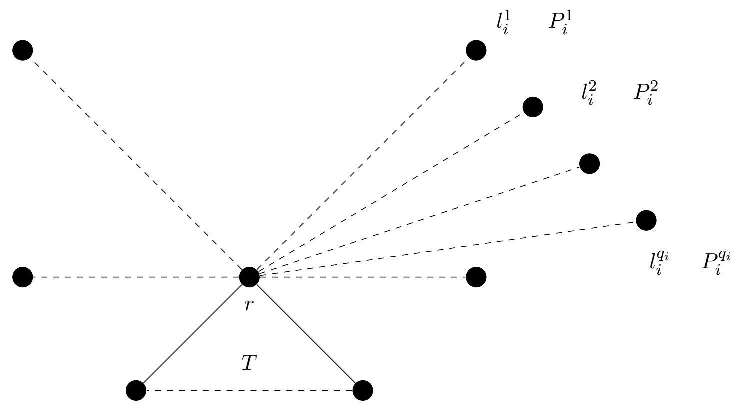

Figure 4.8: Forming T' from T in the proof of Proposition 4.1.4

- For each i such that $t_i < c$ "append" $q_i = c - t_i$ paths $P_i^1, P_i^2, \ldots P_i^{q_i}$ to r of length i with end vertices $l_i^1, \ldots l_i^{q_i}$. Let L_i be the set of end vertices on these paths of length i and let L be the set containing the elements in the union of the L_i's

- Let $S' = S \cup L$

- Let $r' = r$

The construction is illustrated in Figure 4.8.

Consider first the outcome where S-**t**-FIRE on the $1-$*rooted* tree (T', r') with set S' and defence sequence $\mathbf{t}'$ is a NO instance and strategy D that ensures no vertex of S burns in(T, r) with set S and defence sequence $\mathbf{t}$. Consider modifying D to form D' as follows:

- For each i such that $t_i < c$ let $D'_i = D_i \cup \{l_i^1, \ldots l_i^{q_i}\}$ where $q_i = c - t_i$.

- For each i such that $t_i = c$ let $D'_i = D_i$.

Since under D no vertex of S burns, no vertex of S will burn under D'. Further, since each vertex in L is defended, no vertex in S' burns. We have constructed a strategy for the instance S-**t**-FIRE on the $1-$*rooted* tree (T', r') with set S' and defence sequence $\mathbf{t}'$ that ensures no vertex of S' burns, a contradiction.

Consider now the outcome where S-**t**-FIRE on the $1-$*rooted* tree (T', r') with set S' and defence sequence $\mathbf{t}'$ is a YES instance and a strategy D' that ensures no vertex of S' burns. We proceed by induction on the size of

$$\mathcal{D}' = \left\{ D'_i : | \left\{ d_i^j \in D'_i : d_i^j \in V(T), j \leq c \right\} | > t_i \right\},$$

the set of entries in the sequence $(D'_1, D'_2, \ldots, D'_t)$ that defend only vertices on the original tree and defend more than the amount allotted by the defence sequence.

We notice that if this set is empty, i.e., for each D'_i there are no more than t_i vertices defended at time i that lie in T, we may remove all vertices that are not in T and directly apply this strategy to the instance S-**t**-FIRE on the $1 - \textit{rooted}$ graph (G, r) with set S and defence sequence $\mathbf{t}$.

Assume now that $|\mathcal{D}| = k + 1$. Let D'_i be the earliest set of moves such that $|\{d_i^j \in D_i : d_i^j \in V(T), j \leq c\}| > t_i$. Since no leaf in L_i burned, for each P_i^j there was some vertex $x_i^j \in P_i^j$ defended at some time no later than i. In fact there is at least one x_i^j defended strictly before time i. We may modify D' to form D as follows:

- For each $x_i^j \in P_i^j$ defended at time $\delta < i$ select some vertex $d_i^j \in V(T)$. Replace x_i^j in D'_δ with d_i^j to form D_δ and add l_i^j to D'_i to form D_i.

- Otherwise, let $D_a = D'_a$

We note that since $| \left\{ d_i^j \in D'_i : d_i^j \in V(T), j \leq c \right\} | > t_i$ there are enough vertices available in D'_i to make all the required swaps.

The result of this construction is a new strategy D that ensures no vertex of S' in the instance S-**t**-FIRE on the $1 - \textit{rooted}$ tree (T', r') with set S' and defence sequence $\mathbf{t}'$ will burn. Further, by replacing D'_i with D_i in D we reduce the size of $|\mathcal{D}'|$. Therefore, our result holds by induction.

□

Chapter 5

Weighted Firefighter Problems

Given the relationship between FIREFIGHTER and SFIRE, and W01FIRE outlined in Section 2.2.3, we can directly obtain the following results:

Corollary 5.0.5. *W01FIRE is NP-complete even for graphs with maximum degree three rooted at a vertex of degree two.*

Proof. Since an instance of SFIRE on (G, r) with set S can be expressed as an instance of W01FIRE and SFIRE is NP complete even for graphs with maximum degree three rooted at a vertex of degree two, W01FIRE is NP-complete by restriction. □

Corollary 5.0.6. *WFIRE is NP-complete.*

Proof. Since WFIRE contains W01FIRE, it is NP-complete by restriction. □

Proposition 5.0.7. *An instance of W01FIRE on a binary tree with integer k can be solved in polynomial time.*

Proof. Consider an instance of W01FIRE on a binary tree with integer k. By the Tree Lemma, a minimal strategy for such an instance always defends a vertex adjacent to the fire. Thus, under a minimal strategy, the process will end when the fire reaches either an internal vertex of degree two, or a leaf. Since there is a single path to each such vertex, with each leaf and vertex of degree two we can find the weight of the path from the root to the vertex. By the taking the smallest weight we can determine $MVS_w(T, r)$ and solve the instance of W01FIRE. □

We turn now to the full version of the Weighted Firefighter Problem by considering arbitrary weights.

5.1 The Weighted Firefighter Problem with Arbitrary Weights

By restriction we see that WFIRE is NP-complete. However, since FIREFIGHTER, SFIRE and W01FIRE can all be determined in polynomial time for binary trees, it is worth examining WFIRE on binary trees.

As noted in Section 5 and displayed in Figure 2.6, it is possible to have an instance of WFIRE where every minimal strategy allows less weight to burn than the strategy that produces $MVS_w(G, r)$. We will rely on this fact to obtain the following result.

Theorem 5.1.1. *The restriction of WFIRE to binary trees is NP-complete.*

Proof. The problem is clearly in NP. The transformation is from 3FLFIRE. Suppose an instance of 3FLFIRE, a rooted tree (T', r') with $deg(r) = 3$, is given. We construct an instance (T, r) of WFIRE, in which (T, r) is a binary tree, as follows.

Let G be the binary tree shown in Figure 5.1. Suppose that the three neighbours of r' in T' are u', v' and w'. Delete r', and identify u', v' and w' with the vertices u, v and w of G. This completes the construction of (T, r), which is clearly a binary tree. We complete the transformation by defining the weight function $w : V(T) \to \{-1, 0, 1\}$ and integer k. Let L be the set of leaves of T. Set $w(y) = w(x) = -1$, $w(a) = 1$ for all $a \in L$, and $w(a) = 0$ otherwise. Finally set $k = |L| + 1$. The transformation can be accomplished in polynomial time.

Note that in order to save a set of vertices of total weight at least k, the total weight of the burning vertices can not exceed -2. For this to be possible, both y and z must burn.

Suppose there is a strategy, D, for WFIRE on (T, r) under which the total weight of the burning vertices is at most -2. Since both y and z must burn, we must have $d_1 = x$, or $d_1 \in \{u, v, w\}$ and $d_2 = x$. In the first case, the instance (T_1, r_1) is equivalent to 3FLFIRE in (T', r'). In the second case, the instance (T_2, r_2) is equivalent to 3FLFIRE on (T', r') after one of u', v' and w' have been defended. Since the total weight of the burned vertices is -2, D must save all leaves of T'. Hence there is a strategy for 3FLFIRE on (T', r') under which all leaves are saved.

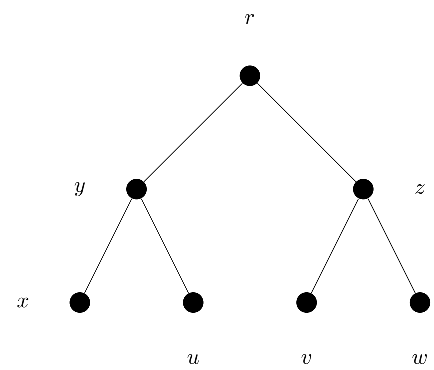

Figure 5.1: G in the construction in the proof of Theorem 5.1.1

Now suppose there is a strategy, D', for 3FLFIRE on (T', r') under which all leaves are saved. To defend (T, r) in such a way that the total weight of the burned vertices is at most -2, defend x at time 1 and then use the D' the instance (T_1, r_1).

Therefore, the restriction of WFIRE to binary trees is NP-complete. □

Chapter 6

Conclusions and Future Directions

Following this analysis we observe the trend that instances of decision problems using firefighter process on graphs of maximum degree three that are rooted at a vertex of degree two that can be solved by always defending a vertex adjacent to a burning vertex seem to be solvable in polynomial time. However, it is unknown whether this trend is merely a coincidence or indicative of some deeper structure in these sorts of problems. It would be worthwhile examining other instances of related problems on such graphs, where for a related optimisation problem, the strategy which yields the optimal value is one that always defends adjacent to the fire.

An optimisation version of SFIRE could be defined that would determine, given a rooted graph (G, r) and a set S, the maximum number of vertices in S that could be saved. However, the methods in Section 2.2.2 would no longer be valid – if we are looking to minimize the number of vertices of S that burn, it is possible that an unreachable vertex may burn, and that the fire may spread between a pair of vertices in S. Equivalently, W01FIRE could be used to model this problem.

The algorithms presented in Chapter 3 were for a version of the problem in which we wished to maximise the number of vertices saved by defending a single vertex at each time-step. Given the relationship between WFIRE and FIREFIGHTER, it would be worth examining greedy algorithms for the weighted version of the problem. Further, given the general version of the problem with multiple roots and possibility of defending multiple vertices at each time step, it would be worth examining greedy strategies for this more general version of the problem.

The problem of containing the fire on different sorts of infinite grids is examined in [5] and [2]. This problem relates closely to SFIRE, as to contain a fire on an infinite grid, all of the vertices at some distance from the root must be saved. By iteratively examining SFIRE for each possible distance from the root, we may find a set of vertices consisting of all vertices at some distance k such that every vertex in the set can be saved. This would imply that the fire can be contained on the grid. However, given the complexity of SFIRE for degree three rooted graphs, this does not seem to be a feasible option.

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