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Abstract

A threshold parameter $\mathcal{R}_0$ is identified for an SIRS epidemiological model which has nonlinear incidence and a distributed delay for transfer out of the removed class. For $\mathcal{R}_0 < 1$, the disease free equilibrium is proved to be the global attractor for all solutions.

Keywords: Distributed delay – Epidemiological model – Global attractor – Threshold

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1. Introduction

In many epidemiological models a threshold parameter is identified. For parameter values below the threshold value the disease dies out, whereas for values above the threshold value the disease level generally approaches an endemic equilibrium. This threshold parameter is usually called the (basic) *reproduction number* (or contact number) see, e.g. [2, 3, 4, 5, 11]. The classical SIRS model takes a closed population divided into three classes S, I and R (denoting respectively the fractions of individuals that are susceptible, infectious and temporarily removed or recovered), and assumes bilinear incidence βIS. Removed individuals are assumed to become susceptible again at a rate proportional to R. The reproduction number, with threshold value 1, is the ratio of constants $\beta/(\gamma+\mu)$, where β, γ, μ denote respectively the contact, recovery and birth rates [4]; it measures the average number of adequate contacts per infective during the infectious period.

If the incidence rate is assumed to have a more general nonlinear form, the model can exhibit a wider range of dynamical behaviour. Periodic solutions are found for some parameter values, while for others the threshold concept may disappear or be modified by initial conditions, see [8, 11, 12]. The reasons for considering nonlinear incidence are reviewed in [5].

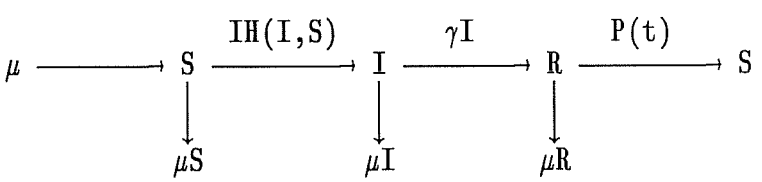
As analyzed in [7], periodic solutions may also occur in models with bilinear incidence but which assume a general probability for transfer out of the removed class. Models which include both nonlinear incidence and a *constant* period of temporary immunity are considered in [6, 8]. Results there leave unresolved the question of *global* stability of the disease free equilibrium in an interval below the threshold value. Our aim here is to

resolve this question; in fact we are able to prove the global result for an SIRS model with general nonlinear incidence and a distributed delay in R. We formulate our model in section 2, prove global attraction of the disease free quilibrium and a sharp threshold result in section 3, and conclude by giving examples in section 4.

Recently several authors have developed models for the spread of HIV/AIDS; see, for example, [2, 9, 10]. Because of the nature of this disease, which is usually fatal, these models have different assumptions from ours; thus we are unable to give direct comparisons of our results with the thresholds found by these authors.

2. The Model

Let S, I, R denote respectively the fractions of the constant population that are susceptible, infectious and removed with temporary immunity. Parameters $\mu \geq 0$, $\gamma > 0$ denote respectively the birth (and death) rate and recovery rate. The disease is assumed to have no vertical transmission and to be nonfatal. Nonlinear incidence takes the form IH(I,S) as in [12] and P(t) is the probability of remaining removed t units after becoming removed; see assumptions below. Our SIRS model thus has the following transfer diagram



and for $t \geq 0$ is described by the equations

$$I'(t) = I(t)\, H(I(t), S(t)) - (\gamma+\mu)I(t), \tag{2.1}$$

$$R(t) = R_0(t)e^{-\mu t} + \gamma \int_0^t I(x)e^{-\mu(t-x)}P(t-x)dx, \tag{2.2}$$

$$S(t) + I(t) + R(t) = 1. \tag{2.3}$$

Here $R_0(t)$ is the fraction initially removed and still removed at time t, and initially the infective fraction is $I_0 = I(0) \geq 0$. These equations can be combined into a single integro–differential equation for $I(t)$, namely,

$$I'(t) = I(t)H\left[I(t),\ 1 - I(t) - R_0(t)e^{-\mu t} - \gamma \int_0^t I(x)e^{-\mu(t-x)}P(t-x)dx\right] - (\gamma+\mu)I(t). \tag{2.4}$$

We now list the epidemilogically reasonable hypotheses imposed on the functions above.

(2.5) $R_0(t) \geq 0$, is continuous, nonincreasing with $\lim_{t\to\infty} R_0(t) = 0$.

(2.6) $P(t) \geq 0$, is nonincreasing, piecewise continuous with $P(0) = 1$, $P(\infty) = 0$, and $\int_0^\infty P(t)dt = \omega$. The positive, finite parameter ω is

the mean period of immunity, and the death adjusted mean period of immunity is $\tilde{\omega} = \int_0^{\infty} e^{-\mu t} P(t)dt$.

(2.7) $H(I,S) \geq 0$, is continuous and Lipschitzian on $[0,1] \times [0,1]\backslash\{0,0\}$, and is nondecreasing with respect to S with $H(I,0) = 0$. (This is a little stronger than we need; we only require $H(I, 1-(1+\gamma\tilde{\omega})I)$ to be Lipschitzian.)

As in [6], the model described above is well-posed and the positively invariant region is $\{(I,R): I \geq 0, R \geq 0, I + R \leq 1\}$.

The system (2.1)–(2.3) with $R_0(t) = 0$, has the equilibrium $(S,I,R) = (1,0,0)$, but for $R_0(t) \not\equiv 0$ has no constant solution. However, by our hypotheses, we can expect the solutions to approach some constants (or asymptotic equilibria) under certain conditions; see, for example, [2]. To locate these asymptotic equilibria, we need to study the limiting equation of (2.4), namely:

$$I'(t) = I(t)H\left[I(t),\ 1 - I(t) - \gamma \int_{-\infty}^{t} I(x)e^{-\mu(t-x)} P(t-x)dx\right] - (\gamma+\mu)I(t). \tag{2.8}$$

Let $I(t)$ be a solution of (2.4) and $\{t_n\}_{n=1}^{\infty}$ be a sequence such that $t_n \to \infty$ as $n \to \infty$. If $\tilde{I}(t) = \lim_{n\to\infty} I(t+t_n)$, then $\tilde{I}(t)$ is a solution of (2.8); see [13, p. 172]. Hence every asymptotic equilibrium of (2.4) is an equilibrium of (2.8). If $R_0(t) = 0$ for $t \geq \omega$ and $P(t)$ is a step

function, then the limiting equation coincides with the original equation for $t \geq \omega$. In this case, the endemic equilibrium is actually a positive equilibrium of the corresponding limiting equation.

Equation (2.8) always has the trivial solution $I = 0$, the disease free equilibrium (DFE). Nontrivial endemic equilibria with $I = I_e \in (0,1]$ exist precisely where I_e satisfies the equation

$$(\gamma+\mu) - H(I,\ 1 - (1+\gamma\tilde{\omega})I) = 0. \tag{2.9}$$

The number of solutions I_e depends on $H(I,S)$. For the case in which $H(I,S) = \beta I^{p-1}S/(1+\alpha I^q)$ with constants $\beta > 0$, $p \geq 1$, $\alpha \geq 0$, $q \in \{p-1,p\}$ there may be 0, 1 or 2 nontrivial equilibria depending on the values of α and $\beta/(\gamma+\mu)$; see [6] for analysis with $\alpha = 0$, and [8] for $\alpha \geq 0$. This form of H generalizes the classical bilinear incidence and includes many incidence functions considered by other authors; see references in [8].

3. Global Attraction of the Disease Free Equilibrium

Local stability of the DFE is examined by linearizing equation (2.1) for I, about $(I,S) = (0,1)$. Thus the DFE is *locally* asymptotically stable (LAS) precisely when $(\gamma+\mu) > H(0,1)$. If all solutions which start in the feasible region approach the DFE as $t \to \infty$, then the DFE is the *global attractor*. We now state our main result.

Theorem 3.1

Assume the model hypotheses, and in addition that for the limiting equation the disease free equilibrium is locally asymptotically stable, and that there is no endemic equilibrium. Then the disease free equilibrium is the global attractor for all solutions of (2.4).

The global result of this theorem is conjectured for a step function $P(t)$, that is $P(t) = 1$ on $[0, \omega]$ and 0 on (ω, ∞), in [6, below Th. 4.2] for nonlinear incidence $IH(I,S) = \beta I^p S$, $p > 1$; and in [8, below Th. 4.1] for a more general incidence function $\beta g(I)S$. These use the Lyapunov function I (see also [11]) to give a simple proof of global stability valid only for a *subset* of parameter values, so no sharp threshold result can be stated. Further discussion of these results is given in section 4.

By contrast, our proof uses analytical techniques to show that $I(t)$ is eventually nonincreasing and approaches the DFE. We proceed by a sequence of lemmas, for all of which we make the assumptions of theorem 3.1.

Lemma 3.2

Under the assumptions of theorem 3.1, *if* ϵ *is defined by*

$$\epsilon = \min_{0 \le I \le 1/(1+\gamma\tilde{\omega})} \left\{ (\gamma+\mu) - H(I,\ 1-(1+\gamma\tilde{\omega})I) \right\},$$

then $\epsilon > 0$.

Proof. Assume $\epsilon = 0$. Then either there exists an $I > 0$ such that $(\gamma+\mu) - H(I,\ 1-(1+\gamma\tilde{\omega})I) = 0$, or $(\gamma+\mu) - H(0,1) = 0$. The first equation is

(2.9) which yields an endemic equilibrium; the second equation means that the DFE is neutrally stable (the linearization of (3.1) gives a zero eigenvalue). Both these contradict our assumptions, thus $\epsilon \neq 0$. From hypothesis (2.7),

$$(\gamma+\mu) - H\left[\frac{1}{1 + \gamma\tilde{w}} , 0\right] = (\gamma+\mu) > 0,$$

so, by continuity, $\epsilon > 0$. ■

For the ϵ defined in lemma 3.2, since $H(I,S)$ is uniformly continuous on $[0,1] \times [0,1]\backslash\{0,0\}$, there exists $\rho > 0$ (independent of I), so that

$$\text{if } |S_1 - S_2| < \rho, \text{ then } |H(I,S_1) - H(I,S_2)| < \epsilon/2. \tag{3.1}$$

Also, by (2.6), for the given ρ, there exists $T > 0$ such that

$$\gamma \int_T^\infty e^{-\mu t} P(t)dt < \rho/3. \tag{3.2}$$

Lemma 3.3

Under the assumptions of theorem 3.1 *and for* T *above, for any given nontrivial solution* $I(t) > 0$ *define*

$$\delta_n = \min\left\{I(t): \ t \in [nT, (n+1)T]\right\}, \ n = 1, 2, \ldots, \tag{3.3}$$

and let t_n *be chosen such that* $I(t_n) = \delta_n$, $t_n \in [nT, (n+1)T]$. *Then either* $t_n \neq (n+1)T$ *for all* n; *or there exists* $t' > 0$ *such that* $I(t)$ *is*

nonincreasing on $[t', \infty)$, *that is* $I(t)$ *is an eventually nonincreasing function.*

Proof. Assume $t_n = (n+1)T$ for some n, thus $I(t_n) \leq I(t)$ for $t \in [nT, (n+1)T]$. Equation (2.4) gives

$$I'(t_n) = I(t_n)\Big[H(I(t_n),\ 1-I(t_n)-R_0(t_n)e^{-\mu t_n}$$

$$- \gamma\int_0^{t_n} I(x)e^{-\mu(t_n-x)}\ P(t_n-x)dx)-(\gamma+\mu)\Big]$$

$$\leq I(t_n)\Big[H(I(t_n),\ 1-I(t_n)-\gamma\int_0^{nT} I(x)e^{-\mu(t_n-x)}\ P(t_n-x)dx$$

$$- \gamma I(t_n)\int_{nT}^{t_n} e^{-\mu(t_n-x)}\ P(t_n-x)dx) - (\gamma+\mu)\Big],$$

as H is nonincreasing with respect to S and $R_0(t_n) \geq 0$.

Thus $I'(t_n) \leq I(t_n)\Big[H(I(t_n),\ 1 - (1+\gamma\tilde{w})I(t_n)$

$$- \gamma\int_0^{nT} (I(x)-I(t_n))e^{-\mu(t_n-x)}P(t_n-x)dx+\gamma I(t_n)\int_{-\infty}^{0} e^{-\mu(t_n-x)}P(t_n-x)dx)$$

$$- (\gamma+\mu)\Big], \text{ by the definition of } \tilde{w}.$$

By (3.1), $\left| H(I(t_n),\ 1-(1+\gamma\tilde{w})I(t_n)-\gamma\int_0^{nT}(I(x)-I(t_n))e^{-\mu(t_n-x)}P(t_n-x)dx \right.$

$$\left. + \gamma I(t_n)\int_{-\infty}^{0} e^{-\mu(t_n-x)}P(t_n-x)dx) - H(I(t_n),\ 1-(1+\gamma\tilde{w})I(t_n)) \right| < \epsilon/2$$

provided that

$$\left| \gamma\int_0^{nT}(I(x)-I(t_n))e^{-\mu(t_n-x)}P(t_n-x)dx + \gamma I(t_n)\int_{-\infty}^{0} e^{-\mu(t_n-x)}P(t_n-x)dx \right| < \rho.$$

But $\left| \gamma\int_0^{nT}(I(x)-I(t_n))e^{-\mu(t_n-x)}P(t_n-x)dx + \gamma I(t_n)\int_{-\infty}^{0} e^{-\mu(t_n-x)}P(t_n-x)dx \right|$

$$\leq 2\gamma \int_{-\infty}^{nT} e^{-\mu(t_n-x)}P(t_n-x)dx = 2\gamma \int_{t_n-nT}^{\infty} e^{-\mu s}P(s)ds$$

$$= 2\gamma \int_{T}^{\infty} e^{-\mu s}P(s)ds \leq 2\rho/3 \quad \text{by (3.2).}$$

So $I'(t_n) \leq I(t_n)\left[H(I(t_n),\ 1-(1+\gamma\tilde{w})I(t_n)) + \epsilon/2 - (\gamma+\mu)\right]$

$$\leq -\epsilon I(t_n)/2 < 0, \text{ by the definition of } \epsilon.$$

Hence, there is a maximal interval $[(n+1)T,\ t'')$ such that $I(t)$ is nonincreasing on it. If $t'' < \infty$, then $I'(t'') = 0$. Then $t'' - T > nT$ and $I(t'') \leq I(t)$ for $t \in [t''-T,\ t'']$. Replacing t_n by t'' and nT by

$t''-T$ in the above procedure, we have $I'(t'') < 0$; a contradiction. Hence $t'' = \infty$, that is $I(t)$ is a nonincreasing function after $t' = (n+1)T$. ■

Lemma 3.4

Under the assumptions of theorem 3.1 *and* T, t_n, δ_n *as in lemma* 3.3, *if* $t_n \neq (n+1)T$ *for all* $n \geq 1$, *then* $\delta_{n+1} > \delta_n$ *for all* $n \geq 1$.

Proof. Since $t_n \neq (n+1)T$, we have $I'(t_n) \geq 0$. Suppose $\delta_{n+1} \leq \delta_n$. Then δ_{n+1} is a minimal value of $I(t)$ on $[nT, (n+2)T]$. Hence, by repeating the argument in lemma 3.3, we have

$$I'(t_{n+1}) = I(t_{n+1})\Big[H(I(t_{n+1}),1-I(t_{n+1})-R_0(t_{n+1})e^{-\mu t_{n+1}}$$

$$- \gamma\int_0^{nT} I(x)e^{-\mu(t_{n+1}-x)}P(t_{n+1}-x)dx$$

$$- \gamma\int_{nT}^{t_{n+1}} I(x)e^{-\mu(t_{n+1}-x)}P(t_{n+1}-x)dx)-(\gamma+\mu)\Big]$$

$$\leq -\epsilon I(t_{n+1})/2 < 0.$$

This contradicts $I'(t_{n+1}) \geq 0$, thus $\delta_{n+1} > \delta_n$ for all n. ■

Lemma 3.5

Under the assumptions of lemma 3.4, *there is a* $\delta > 0$ *such that* $\delta_{n+1} - \delta_n \geq \delta$ *for all* $n \geq 1$.

Proof. Assume there is no such δ, then for the given ρ in (3.1) there is an integer n such that $\gamma\tilde{\omega}(\delta_{n+1}-\delta_n) < \rho/3$.

Since $I'(t_{n+1}) \geq 0$,

$$0 \leq H(I(t_{n+1}), 1-I(t_{n+1})-R_0(t_{n+1})e^{-\mu t_{n+1}} - \gamma\int_0^{t_{n+1}} I(x)e^{-\mu(t_{n+1}-x)}P(t_{n+1}-x)dx) - (\gamma+\mu)$$

$$\leq H(\delta_{n+1}, 1-(1+\gamma\tilde{\omega})\delta_{n+1}+\gamma\tilde{\omega}(\delta_{n+1}-\delta_n) - \gamma\int_0^{nT} I(x)e^{-\mu(t_{n+1}-x)}P(t_{n+1}-x)dx$$

$$+ \gamma\delta_n\int_{-\infty}^{nT} e^{-\mu(t_{n+1}-x)}P(t_{n+1}-x)dx) - (\gamma+\mu),$$

as by lemma 3.4, δ_n is the minimum value of $I(t)$ on $[nT, (n+2)T]$. Therefore, by (3.1), (3.2) and lemma 3.2,
$0 \leq H(\delta_{n+1}, 1-(1+\gamma\tilde{\omega})\delta_{n+1}) + \epsilon/2 - (\gamma+\mu) \leq -\epsilon/2 < 0$, a contradiction. ■

We now use the above lemma to prove our main result.

Proof of Theorem 3.1. Consider any solution $I(t)$. If $I(t)$ is not eventually nonincreasing, then, by lemma 3.5, $\delta_{n+1} - \delta_n \geq \delta > 0$ for some $\delta > 0$ and all n. Therefore $\lim_{n\to\infty} I(t_n) \geq \lim_{n\to\infty} (n-2)\delta = +\infty$, which is impossible as $I(t) \leq 1$. Thus $I(t)$ is eventually nonincreasing. Let $I^* = \lim_{t\to\infty} I(t)$. Then from the discussion in section 2, I^* is an equilibrium of the limiting equation (2.8). Thus $I^* = 0$, which implies that the DFE is the global attractor to all solutions of (2.4). ■

We are now able to state our result as a sharp threshold theorem in terms of the threshold parameter $\mathcal{R}_0$ defined as

$$\mathcal{R}_0 = \max_{0\le I\le 1/(1+\gamma\tilde{w})} \frac{H(I,\ 1-(1+\gamma\tilde{w})I)}{\gamma + \mu} . \tag{3.4}$$

Theorem 3.6

Consider the model with the assumptions (2.5)–(2.7). *If* $\mathcal{R}_0 < 1$, *then the disease free equilibrium is the unique equilibrium for the limiting equation and it is the global attractor for all solutions of* (2.4). *If* $\mathcal{R}_0 > 1$, *there exists at least one endemic equilibrium for the limiting equation.*

Proof. Assume $\mathcal{R}_0 < 1$, then $H(0,1)/(\gamma+\mu) < 1$, thus the DFE is LAS. Also (2.9) is never satisfied, so there is no endemic equilibrium. Theorem 3.1 gives the global result. If $\mathcal{R}_0 > 1$, then by the continuity of H, there is an I_e satisfying (2.9). Hence I_e is a positive equilibrium for the limiting equation. ∎

The above result identifies the threshold parameter $\mathcal{R}_0$, and shows that the disease dies out below the threshold value 1.

4. Examples

We give some examples of threshold parameters $\mathcal{R}_0$ for incidence functions that are used in epidemiological modeling. In particular, we show

that our result covers the conjectures in [6, 8] as stated in the introduction.

(4.1) $H = \beta S$ gives the classical bilinear incidence βSI. Equation (3.4) gives $\mathcal{R}_0 = \beta/(\gamma+\mu)$, agreeing with the classical threshold result stated in the introduction. This threshold thus remains valid for a general probability for transfer out of the removed class. For the case in which the probability is a step function and vital dynamics are ignored $(\mu = 0)$, this is the model analyzed in [7]. For certain values of $\beta/\gamma > 1$, stable periodic solutions are found.

(4.2) $H = \beta I^{p-1} S^q$, $p \geq 1$, $q > 0$, gives a nonlinear incidence function which has been widely studied, see [6, 11, 12]. For this example

$$\mathcal{R}_0 = \frac{\beta(p-1)^{p-1} q^q}{(\gamma+\mu)(p+q-1)^{p+q-1}(1+\gamma\tilde{\omega})^{p-1}},$$

and our results confirm the global stability conjectures in [11], for the SIRS model with $P(t) = e^{-t/\omega}$; and in [6] for $P(t)$ a step function, $\mu = 0$ and $q = 1$.

(4.3) $H = \dfrac{\beta I^{p-1} S}{1 + \alpha I^q}$, $p \geq 1$, $q > 0$, gives a nonlinear incidence function incorporating saturation, see [8]. Here the parameter $\mathcal{R}_0$ cannot be found explicitly, since the maximum in (3.4) depends on a polynomial in I. Our implicit result agrees with that in [8] and confirms the global

stability conjecture made there [8, section 10] for this incidence function.

(4.4) $H = \beta g(I)S/I$ with suitable conditions on $g(I)$ is the form used in [8]. This gives

$$\mathcal{R}_0 = \frac{\beta}{\gamma + \mu} \max\left\{\lim_{I\to 0} \frac{g(I)}{I} \ , \ \frac{g^2(I_m)}{I_m^2\, g'(I_m)}\right\} ,$$

where I_m satisfies

$$I_m g'(I_m)(1 - (1+\gamma\tilde{\omega})I_m) - g(I_m) = 0.$$

This result confirms the conjectures on global stability in [8] for the SIRS model with $P(t) = e^{-t/\omega}$ or $P(t)$ a step function.

(4.5) $H = \beta S/(I+S)$, proportionate mixing, has been used in some general models for HIV/AIDS transmission, see, for example [9, 10]. However, as noted in the introduction, our model assumptions do not apply to a fatal disease. For this function, (3.4) gives $\mathcal{R}_0 = \beta/(\gamma+\mu)$, as in the classical case (4.1). (Note that although this incidence function is not Lipschitzian, the function $\beta(1-(1+\gamma\tilde{\omega})I)/[I+(1-(1+\gamma\tilde{\omega})I)]$ is a Lipschitzian function on $[0, 1]$; see (2.7). Thus our results apply to this case.)

(4.6) $H = \beta(1-e^{-aI})S/I$ for constant a is an incidence function taken in some models based on probabilistic considerations, see, for example, [1] and references therein. For this function, the maximum in (3.4) occurs at $I = 0$, giving $\mathcal{R}_0 = \beta a/(\gamma+\mu)$.

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