Charge and Mass Multipole Moments in General Relativity

by

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Abstract

Multipole moments in the Weyl class of static, axially-symmetric electrovacuum spacetimes are studied from both the the Erez-Rosen and Geroch-Hansen-Hoenselaers formalisms. The Erez-Rosen formalism is generalized from their uncharged formalism to describe the electric and gravitational fields of a charged body. Both formalisms show the interdependency of mass and charge on both the mass and charge multipole moments. We suggest how one can utilize both formalisms to experimentally determine the multipole structure of a spacetime. Two methods for determining the Newtonian limit of a pure vacuum axially-symmetric spacetime are examined. One method is inadequate for determining such multipole moments. The other method is generalized to describe the Newtonian limit of charged spacetimes. The charged Curzon solution was studied as an example.

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Chapter 1

Introduction

Multipole moments have been useful in the description of electromagnetic and Newtonian gravitational fields. The use of a multipole potential as an approximation to the exact potential is sometimes the only feasible option for extremely complicated systems. Of course one must stay within the domain of validity of the approximation, but this is satisfactory for investigating many phenomena in nature. An example of the use of multipole expansions is in electromagnetism, where the dominant term in the expansion can be used to obtain an analytic expression for the radiation arising from an antenna. Exact solutions for such a complex problem cannot be obtained easily. With the advent of satellites in the 1960's, it has been possible to measure the earth's Newtonian gravitational potential as a multipole expansion. This technique of mapping satellite motion to a parameterized multipole potential has led to the discovery that the earth is actually pear-shaped [1].

This thesis is concerned with the extension of the concept of multipole moments into general relativity. In 1959, Erez and Rosen [2] proposed a method of constructing the gravitational field of a particle with mass multipole moments in general relativity. The procedure was based upon the Weyl [3] - Levi-Civita [4] formulation of the static axially-symmetric vacuum Einstein equations. The method closely resembled the usual procedure for identifying multipole moments in Newtonian mechanics.

It has since been pointed out that the definition of mass multipoles presented by Erez and Rosen is not a covariant definition since the exact form of the expansion depends explicitly on the coordinate system. Several authors have devised fairly complicated definitions of covariant multipole moments [5-8] and these definitions have subsequently been shown to be equivalent to each other [9,10]. The desire to create a covariant definition of multipole moments is a necessity if multipole moments are to have any physically significant meaning in curved spacetimes.

In this thesis we will examine the charge and mass multipole moments of a static axially-symmetric electrovacuum solution to Einstein's field equations from both the Geroch-Hansen-Hoenselaers (GHH) covariant formalism and the Erez-Rosen coordinate dependent formalism.

The GHH covariant formalism defines multipole moments as certain symmetric and traceless tensors at infinity [5]. The assumed symmetry of the spacetime allows one to express the multipole moments as scalars at infinity. These scalars are the basis for comparison between the two formalisms. From the moments obtained through this formalism, we find the true physical multipole structure of the spacetime. These moments explicitly illustrate the duality of mass and charge expected within the Weyl class of electrovacuum spacetimes.

The Erez-Rosen formalism for charge and mass multipole moments is a straightforward extension of their uncharged formalism [11]. The mass multipole moments are identified as the coefficients of the terms $r^{-(n+1)}P_n(\cos\theta)$ from the expansion of the g_{00} component of the metric tensor. Even though the Erez-Rosen expansion is coordinate dependent, it is still a viable method in that it gives a physically understandable form of the field behaviour with respect to its far-field limit. It is thus possible to investigate properties of the field at a finite distance away from the source. The GHH moments are limited in this respect, since they are only defined at spatial infinity. However these moments have the virtue of being true scalars, unlike the Erez-Rosen moments.

If one wishes to experimentally measure the physical moments of a spacetime, we propose how one would use both formalisms to make such a measurement. We discuss why each formalism on its own would not be sufficient for accomplishing such a task.

The outline of the thesis is as follows: Chapter 2 introduces the Weyl class of axially-symmetric spacetimes. We discuss how one exactly solves the uncharged vacuum Einstein Field Equations (EFEs), given the Weyl metric and the symmetry of the spacetime. Many distinct representations are possible. However we shall concentrate on the representation utilized by Erez-Rosen, who use ellipsoidal coordinates to solve the field equations. We will then examine how one solves the Einstein-Maxwell vacuum field equations. This is quite easily accomplished if one assumes that the g_{00} component of the metric tensor is a function of the electrostatic potential as Weyl did in 1917. The specific solution of the Einstein-Maxwell field equations we use will be referred to as the Erez-Rosen solution.

Chapter 3 presents multipole moments for a static electrovacuum through the Erez-Rosen formalism. Both charge and mass multipoles are identified in the same manner as the uncharged case of Erez and Rosen. At this stage we make no claims as to the validity of the presented definition of multipole moments nor do we enter into a detailed discussion of the moments. This is left until Chapter 5, after the covariant definition has been presented and used to calculate the moments for the spacetime which we are investigating.

Chapter 4 presents the GHH formalism for calculating covariant multipole moments. We first give a summary of how one defines multipole moments in this formalism. The methodology assumes a mathematical approach in an unphysical spacetime rather than remaining within the physical spacetime. In the second section of this chapter, we explicitly calculate both the charge and mass moments for the spacetime under investigation. The part of the formalism developed by Hansen and then Hoenselaers defines moments for stationary spacetimes. Since the spacetime of interest is static, the actual calculation of the moments is simplified. Again, we defer comments on the calculated moments until chapter 5.

In chapter 5 we discuss the results of the previous two chapters. We examine the physical basis of the definitions, highlighting the desirable and undesirable aspects of each. In this chapter we explain how it would be possible to experimentally measure the physical multipole moments. This entails using certain aspects of the Erez-Rosen formalism in conjunction with the GHH formalism. Neither formalism on its own could accomplish such a task.

In chapter 6 we study two definitions of the Newtonian limit for axially-symmetric uncharged spacetimes. One is a coordinate independent definition presented by Ehlers [12] and the other is a coordinate dependent one proposed by Cooperstock [11] obtained from the Erez-Rosen formalism. We shall discuss why Ehlers' definition is insufficient for determining Newtonian multipole moments for the Erez-Rosen solution. We show that Cooperstock's consideration of the properties of the source itself is necessary to correctly identify Newtonian moments if they exist. It is then shown how Cooperstock's definition can be extended to include charged sources. This leads to a general formula for both charge and mass multipole moments in the

Newtonian limit.

The final chapter examines Curzon's axially symmetric solution as another example of the application of the two formalisms studied. We show that the GHH moments for the Curzon solution or any other axially symmetric solution can be obtained without applying the GHH formalism for each case. Instead, one can expand the solution being studied in terms of orthogonal Legendre functions. From the coefficients of this expansion, the moments can be found by using the general formula for the GHH moments as given in chapter 4.

Chapter 2

The Weyl Class of Vacuum and Electrovacuum

Weyl [3], in 1917, showed that static, axially-symmetric gravitational fields in vacuum or electrovacuum could be represented by the line element

$$ds^2 = e^w dt^2 - e^{v-w}(d\rho^2 + dz^2) - \rho^2 e^{-w} d\phi^2 \tag{2.1}$$

in cylindrical polar coordinates, where w and v are functions of ρ and z. He further showed that in vacuum, the Einstein field equations reduce to

$$\nabla^2 \bar{w} \equiv \bar{w}_{,\rho\rho} + \bar{w}_{,zz} + \frac{\bar{w}_{,\rho}}{\rho} = 0 \tag{2.2}$$

$$\bar{v}_{,\rho} = \frac{\rho}{2}(\bar{w}^2_{,\rho} - \bar{w}^2_{,z}), \qquad \bar{v}_{,z} = \rho \bar{w}_{,\rho} \bar{w}_{,z} \tag{2.3}$$

where the bars indicate that the equations hold in vacuum and commas indicate partial differentiation with respect to the indicated variable.

One will notice that Eq. (2.2) is simply the Laplace equation. To obtain an exact solution to the EFEs, one would find a solution to Eq. (2.2) and then find v via quadratures.

This formulation induces a distortion of the coordinates, as evidenced by the fact that a solution which looks like a Newtonian monopole potential, mr^{-1}, $r = (\rho^2 + z^2)^{1/2}$, taken for $\bar{w}$ yields a particle with higher multipole moments in the physical spacetime [13] (see Chapter 7 for further study). To retrieve the spherically symmetric Schwarzschild metric, the potential which would normally represent a line mass in Newtonian theory,

$$\bar{w} = \frac{m}{l} \ln \left(\frac{R_1 + R_2 - 2l}{R_1 + R_2 + 2l} \right), \tag{2.4}$$

$$R_{1,2}^2 = (z \pm l)^2 + \rho^2, \tag{2.5}$$

must be used in conjunction with the linear mass density $\frac{m}{2l} = \frac{1}{2}$ [3].

Weyl [3], Bach and Weyl [14], Curzon [13], Erez and Rosen [2] and others have examined the various fields generated from some of the simpler solutions to the Laplace equation. We are interested in the solution found by Erez and Rosen. There is no requirement that the field equations (2.2) and (2.3) be solved in cylindrical polar coordinates. To simplify the representation of the Schwarzschild solution in the axially symetric spacetime, Erez and Rosen used ellipsoidal coordinates (λ, μ)

given by

$$\lambda = \frac{R_1 + R_2}{2l}, \quad \mu = \frac{R_1 - R_2}{2l} \tag{2.6}$$

with $l = m$, $\lambda > 1$, $-1 \leq \mu \leq 1$.

The Laplace equation (2.2) in terms of λ and μ is

$$\left[\left(\lambda^2 - 1\right) \bar{w}_{,\lambda}\right]_{,\lambda} + \left[\left(1 - \mu^2\right) \bar{w}_{,\mu}\right]_{,\mu} = 0 \tag{2.7}$$

which, upon separation of variables, via

$$\frac{\bar{w}}{2} = L(\lambda)M(\mu), \tag{2.8}$$

yields

$$\left[\left(\lambda^2 - 1\right) L_{,\lambda}\right]_{,\lambda} - n(n+1)L = 0,$$

$$\left[\left(1 - \mu^2\right) M_{,\mu}\right]_{,\mu} + n(n+1)M = 0, \quad n = 0, 1, \ldots \tag{2.9}$$

with the separation constant expressed in the integer form above for well-behaved solutions. The solution of these differential equations are linear combinations of Legendre polynomials and Legendre functions of the second kind. Moreover, for $\bar{w}$ to be well-behaved at infinity, the solution must take the form [2]

$$\frac{1}{2}\bar{w} = \sum_{n=0}^{\infty} -c_n P_n(\mu) Q_n(\lambda), \tag{2.10}$$

with $P_n(\mu)$, the Legendre polynomial, $Q_n(\lambda)$, the Legendre function of the second kind and c_n a constant. The Schwarzschild solution is found by taking

$$P_0(\mu) = 1, \quad Q_0(\lambda) = \frac{1}{2}\ln\left(\frac{\lambda+1}{\lambda-1}\right), \tag{2.11}$$

with $c_0 = 1$, $c_n = 0$ for $n \geq 1$ in Eq. (2.10). They noted that it is the transformation

$$\lambda = \frac{r}{m} - 1, \quad \mu = \cos\theta \tag{2.12}$$

which brings Eq. (2.11) (which is also the special Newtonian line-mass potential solution of Eq. (2.4),with $l = m$) into the standard Schwarzschild form

$$ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{2.13}$$

Clearly, if some of the constants c_n are not zero for $n > 0$, the solution $\bar{w}$ will no longer be spherically symmetric. Erez and Rosen explicitly calculated the field for a particle with $c_0 = 1$, $c_1 = 0$, $c_2 \neq 0$, $c_n = 0$, $n \geq 3$. Using the transformation Eqs. (2.12) which identify the spherical coordinates, they extracted the quadrupole moment from the $r^{-3}P_2(\cos\theta)$ coefficient in the same manner as in Newtonian theory.

The foregoing is for uncharged bodies, and it is natural to consider the case of a charged body in general relativity. Weyl showed that for charged bodies with axial symmetry, if one imposed the condition that w be functionally related to Φ, then the unique functional relationship for asymptotically Minkowskian boundary conditions is

$$e^{w} = 1 - \frac{2m}{q_c}\Phi + \Phi^2 \tag{2.14}$$

where q_c is the charge of the body. Weyl then defined the function

$$\chi \equiv \int \left(1 - \frac{2m}{q_c}\Phi + \Phi^2\right)^{-1} d\Phi \tag{2.15}$$

and showed that it satisfies the Laplace equation (2.2). Integrating this function yields

$$\chi = \frac{a}{a^2 - 1} \ln\left(\frac{\Phi - a}{\Phi - a^{-1}}\right), \tag{2.16}$$

where a is a constant defined by

$$\frac{1 + a^2}{a} = \frac{2m}{q_c}. \tag{2.17}$$

Since we know that χ satisfies the Laplace equation, its form with respect to a given coordinate system is easily deduced. Therefore, using Eqs. (2.16) and (2.14),

e^w and Φ are known. To simplify matters somewhat, we can eliminate Φ between Eqs. (2.16) and (2.14). This yields [15]

$$e^w = \frac{(1-a^2)^2 f}{(a^2 f - 1)^2}, \tag{2.18}$$

with

$$f \equiv a^{-2} \exp\left(\frac{a^2-1}{a}\chi\right). \tag{2.19}$$

Eliminating χ between Eq. (2.16) and (2.19) gives

$$\Phi = \frac{a(f-1)}{(a^2 f - 1)}. \tag{2.20}$$

It can be shown that the field equations for v take the same form as in the uncharged case, viz,

$$v_{,\rho} = \frac{\rho}{2}\left[(\ln f)^2_{,\rho} - (\ln f)^2_{,z}\right]$$

$$v_{,z} = \rho(\ln f)_{,\rho}(\ln f)_{,z}. \tag{2.21}$$

It is evident from Eq. (2.19) that if χ satisfies the Laplace equation, then $\ln f$ must also satisfy the Laplace equation. Therefore if $\ln f$ is chosen to have the same form as $\bar{w}$ in Eq. (2.10), we will have a solution to the electrovacuum field equations which is also asymptotically flat. One can also see that v can differ from the corresponding known $\bar{v}$ by at most an additive constant.

The spherically symmetric charged generalization of the Schwarzschild metric, the Reissner-Nordström metric [16,17], is found by taking

$$f = \left(\frac{x-2l}{x+2l}\right), \tag{2.22}$$

$$x \equiv R_1 + R_2 \tag{2.23}$$

where

$$l = m\sqrt{1 - \frac{q_c^2}{m^2}} \tag{2.24}$$

for the "undercharged" case $q_c^2 < m^2$, with a in the range $-1 < a < 1$. The "overcharged" ($q_c^2 > m^2$) and "critically charged" ($q_c^2 = m^2$) cases are considered in detail in Ref. [15]. The close connection between Eqs. (2.4) and (2.22) is to be noted, as well as the expanded role which is played by the l parameter in the case of charged bodies.

In the next chapter we will take a specific solution of the Einstein-Maxwell field equations and examine both the charge and mass moments as identified through the Erez-Rosen formalism.

Chapter 3

Multipoles of Charge and Mass: Erez-Rosen Formalism

In building charge multipoles through the Erez-Rosen formalism, one of the first steps is to identify the connection between spherical coordinates and the Weyl canonical coordinates (cylindrical coordinates). These will be the charged analogues of Eq. (2.12). To do so, we first note that the Reissner-Nordström metric in spherical polar coordinates

$$
\begin{aligned}
ds^2 &= \left(1 - \frac{2m}{r} + \frac{q_c^2}{r^2}\right) dt^2 - \left(1 - \frac{2m}{r} + \frac{q_c^2}{r^2}\right)^{-1} dr^2 + \\
&\quad - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)
\end{aligned}
\tag{3.1}
$$

takes the form

$$
\begin{aligned}
ds^2 &= \frac{(R_1+R_2)^2-4(m^2-q_c^2)}{(R_1+R_2+2m)^2}dt^2 - \frac{(R_1+R_2+2m)^2}{4R_1R_2}(d\rho^2+dz^2)+ \\
&\quad - \frac{(R_1+R_2+2m)^2}{(R_1+R_2)^2-4(m^2-q_c^2)}\rho^2 d\phi^2 \qquad (3.2)
\end{aligned}
$$

in Weyl coordinates (t,ρ,z,ϕ). It is straightforward to verify that the required transformation linking Eqs. (3.1) and (3.2) is

$$
\rho = \sqrt{r^2-2mr+q_c^2}\,\sin\theta,
$$

$$
z = (r-m)\cos\theta. \qquad (3.3)
$$

In what follows, we will restrict our attention to the "undercharged" case, Eqs. (2.22–2.24). With the choice $q_c \geq 0$, we must choose the negative root when solving for a in Eq. (2.17) so that in the limit $q_c \to 0\,,\ a\,(q_c) \to 0$, i.e.

$$
a = \frac{m}{q_c} - \sqrt{\frac{m^2}{q_c^2}-1}. \qquad (3.4)
$$

Thus, from Eqs. (3.3), (2.5), (2.6) and (2.23), we find

$$
x = 2l\lambda = 2(r-m),
$$

$$
2l\mu = 2l\cos\theta. \qquad (3.5)
$$

The role of charge in changing the λ coordinate above in comparison to the vacuum case of Eq. (2.12) is to be noted.

It was stated in the previous chapter that $\ln f$ in the charged case will play a role in the charged formalism corresponding to $\bar{w}$ in the uncharged formalism. The general asymptotically flat solution to the Einstein-Maxwell field equations separated in ellipsoidal coordinates within the framework of the Weyl class can be represented by

$$\frac{1}{2}\ln f = \sum_{n=0}^{\infty} -c_n P_n(\mu) Q_n(\lambda). \tag{3.6}$$

In the remainder of the thesis we will refer to this equation as the Erez-Rosen solution. To keep the calculations manageable we shall choose to keep only the first 5 terms. Therefore, in terms of the more convenient coordinates x and $\cos\theta$ of Eqs. (3.5),

$$\begin{aligned} \ln f \;=\; & \left[\ln\left(\frac{x-2l}{x+2l}\right)\right]\left[1+\frac{c_1P_1(\cos\theta)x}{2l}+\frac{c_2P_2(\cos\theta)}{2}\left(\frac{3x^2}{4l^2}-1\right)+\right. \\ & \left.+\frac{c_3P_3(\cos\theta)}{2}\left(\frac{5x^3}{8l^3}-\frac{3x}{2l}\right)+\frac{c_4P_4(\cos\theta)}{8}\left(\frac{35x^4}{16l^4}-\frac{15x^2}{2l^2}+3\right)\right] \\ & +2c_1P_1(\cos\theta)+\frac{3}{2}c_2P_2(\cos\theta)x+c_3P_3(\cos\theta)\left(\frac{5x^2}{4l^2}-\frac{4}{3}\right)+ \\ & +\frac{5c_4P_4(\cos\theta)}{8}\left(\frac{7x^3}{4l^3}-\frac{11x}{3l}\right) \end{aligned} \tag{3.7}$$

where $c_0 \equiv 1$, and the values of the Legendre functions of the second kind have been substituted in the form appropriate for the domains of the coordinates. Eqs. (3.7), (2.18) and (2.20) constitute the charged solution with v found by quadratures in Eq. (2.21). Even though we have not used the most general solution given by Eq. (3.6), the expression for $\ln f$ in Eq. (3.7) is still an exact solution to the field equations.

The procedure for calculating the multipole moments in the Erez-Rosen formalism is very straight-forward. Using our chosen solution Eq. (3.7) in Eqs. (2.18) and (2.20) and the new spherical coordinates determined by Eq. (3.5), we calculate the asymptotic expansion in r of Φ and e^w. The results are

$$\begin{aligned}
\Phi = \quad & \frac{q_c}{r} + \frac{1}{3r^2} q_c c_1 P_1 \sqrt{m^2 - q_c^2} + \frac{2}{15r^3} q_c c_2 P_2 \left(m^2 - q_c^2\right) + \\
+ \quad & \frac{1}{r^4} \left[\frac{2}{35} c_3 P_3 q_c \left(m^2 - q_c^2\right)^{3/2} + \frac{2}{15} c_2 P_2 q_c m \left(m^2 - q_c^2\right) + \right. \\
& \left. - \frac{1}{9} c_1^2 P_1^2 q_c m \left(m^2 - q_c^2\right) - \frac{2}{15} c_1 P_1 q_c \left(m^2 - q_c^2\right)^{3/2} \right] + \\
+ \quad & \frac{1}{r^5} \left[\frac{8}{315} q_c c_4 P_4 \left(m^2 - q_c^2\right)^2 + \frac{4}{35} c_3 P_3 q_c m \left(m^2 - q_c^2\right)^{3/2} + \right. \\
& - \frac{4}{45} c_1 P_1 c_2 P_2 q_c m \left(m^2 - q_c^2\right)^{3/2} + \\
& + \frac{2}{105} c_2 P_2 q_c \left(6m^2 + q_c^2\right) \left(m^2 - q_c^2\right) + \\
& - \frac{1}{9} c_1^2 P_1^2 q_c \left(2m^2 - q_c^2\right) \left(m^2 - q_c^2\right) + \\
& \left. - \frac{4}{15} c_1 P_1 q_c m \left(m^2 - q_c^2\right)^{3/2} \right] + \cdots,
\end{aligned} \tag{3.8}$$

$$
\begin{aligned}
e^w = \quad & 1+2\Big\{-\frac{m}{r}-\frac{1}{r^2}\left(\frac{1}{3}c_1P_1m\sqrt{m^2-q_c^2}-\frac{q_c^2}{2}\right)+ \\
- \quad & \frac{1}{r^3}\left[\frac{2}{15}c_2P_2m\left(m^2-q_c^2\right)-\frac{1}{3}c_1P_1q_c^2\sqrt{m^2-q_c^2}\right]+ \\
- \quad & \frac{1}{r^4}\left[\frac{2}{35}c_3P_3m\left(m^2-q_c^2\right)^{3/2}+\frac{2}{15}c_2P_2\left(m^2-q_c^2\right)^2+\right. \\
& \left.-\frac{1}{9}c_1^2P_1^2\left(2m^2+q_c^2\right)\left(m^2-q_c^2\right)\right]+ \\
- \quad & \frac{1}{r^5}\left[\frac{8}{315}c_4P_4m\left(m^2-q_c^2\right)^2+\right. \\
& +\frac{2}{35}c_3P_3\left(2m^2-q_c^2\right)\left(m^2-q_c^2\right)^{3/2}+ \\
& -\frac{1}{630}c_2P_2\left(m^2-q_c^2\right)\left[28\left(2m^2+q_c^2\right)\sqrt{m^2-q_c^2}+\right. \\
& \left.-57c_1P_1m\left(m^2-q_c^2\right)\right]-\frac{2}{9}c_1^2P_1^2m\left(m^2-q_c^2\right)^2+ \\
& \left.-\frac{2}{15}c_1P_1\left(2m^2-q_c^2\right)\left(m^2-q_c^2\right)^{3/2}\right]+\cdots\Big\} \qquad (3.9)
\end{aligned}
$$

where the P_k's are understood to be $P_k(\cos\theta)$.

While the form of Eq. (3.9) is convenient to illustrate the fall-off properties in powers of r^{-1}, the Erez-Rosen formalism requires us to order them in terms of $r^{-(n+1)}P_n(\cos\theta)$, i.e.

$$
\begin{aligned}
e^w = \quad & 1+2\Big\{-\frac{1}{r}\left(m-\frac{q_c^2}{2r}\right)-\frac{c_1}{3r^2}P_1\sqrt{m^2-q_c^2}\times \\
& \times\left(m-\frac{q_c^2}{r}-\frac{2\left(2m^2-q_c^2\right)\left(m^2-q_c^2\right)}{5r^3}+\cdots\right)+
\end{aligned}
$$

$$
\begin{aligned}
- & \frac{2}{15r^3} c_2 P_2 \left(m^2 - q_c^2\right) \times \\
& \times \left(m + \frac{m^2 - q_c^2}{r} + \frac{\left(2m^2 + q_c^2\right)\sqrt{m^2 - q_c^2}}{3r^2} + \cdots\right) + \\
+ & \frac{1}{r^4}\left[-\frac{2}{35} c_3 P_3 \left(m^2 - q_c^2\right)^{3/2} \left(m + \frac{\left(2m^2 - q_c^2\right)}{r} + \cdots\right) + \right. \\
& \left. -\frac{1}{18} c_1^2 P_1^2 \left(2m^2 - q_c^2\right)\left(m^2 - q_c^2\right)\right] + \\
& + \frac{1}{r^5}\left[-\frac{8}{315} c_4 P_4 \left(m^2 - q_c^2\right)^2 \left(m - O\left(\frac{1}{r}\right) + \cdots\right) + \right. \\
& -\frac{19}{210} c_2 P_2 c_1 P_1 m \left(m^2 - q_c^2\right)^2 + \\
& \left. + \frac{2}{9} c_1^2 P_1^2 m \left(m^2 - q_c^2\right)^2\right] + \cdots \Big\}.
\end{aligned} \tag{3.10}
$$

Also it is more useful from a physical standpoint to express the series in this manner. Note that the coefficient of r^{-1} in Eq. (3.10) can be interpreted as the effective mass [18]

$$
m(r) \equiv m - \frac{q_c^2}{2r} \tag{3.11}
$$

of the system within a sphere up to r. The quantity is less than m by the amount of electromagnetic field energy exterior to this sphere. It is only in the limit as $r \to \infty$ that the entire mass m, which includes the electromagnetic field contribution, is seen. Similarly, from the coefficient of $r^{-2} P_1(\cos\theta)$, we note that the mass dipole moment is

$$
D_m = \frac{1}{3} c_1 \sqrt{m^2 - q_c^2} \left(m - \frac{q_c^2}{r} - \frac{2\left(2m^2 - q_c^2\right)\left(m^2 - q_c^2\right)}{5r^3} + \cdots\right) \tag{3.12}
$$

which, again, has a charge-dependent part which falls off as r^{-1}. The quadrupole moment

$$\text{Quad.}_m = \frac{2}{15}c_2\left(m^2 - q_c^2\right)\left(m + \frac{m^2 - q_c^2}{r} + \frac{\left(2m^2 + q_c^2\right)\sqrt{m^2 - q_c^2}}{3r^2} + \cdots\right) \quad (3.13)$$

also shows this behaviour. However, the procedure is not so clear for the higher moments. Eqs. (3.14) and (3.15) are a possible way of reordering the octupole, and 16-pole *potentials* (shown below):

$$\text{Oct.}_m = \frac{1}{r^4}\left[\frac{2}{35}\,c_3P_3\left(m^2 - q_c^2\right)^{3/2}\left(m + \frac{2m^2 - q_c^2}{r} + \cdots\right)\right.$$
$$\left. + \frac{1}{18}c_1^2P_1^2\left(2m^2 + q_c^2\right)\left(m^2 - q_c^2\right)\right] \quad (3.14)$$

$$\text{16-pole}_m = \frac{1}{r^5}\left[\frac{8}{315}\,c_4P_4\left(m^2 - q_c^2\right)^2\left(m + O\left(\frac{1}{r}\right) + \cdots\right)\right.$$
$$\left. + \left[\frac{19}{210}c_1P_1c_2P_2 - \frac{2}{9}c_1^2P_1^2\right]m\left(m^2 - q_c^2\right)^2\right]. \quad (3.15)$$

The criteria of this reordering of terms is not well defined for field contributions like

$$\frac{1}{18}c_1^2P_1^2\left(2m^2 - q_c^2\right)\left(m^2 - q_c^2\right) \quad (3.16)$$

in Eq. (3.14). For example, this particular term may have been written as a term in the dipole moment that falls off as r^{-2}. Our placement of this term reflects our *a priori* knowledge of the covariant moments. This, however, is still not a perfect correspondence. We shall see in chapter 6 that these terms are negligible if we are dealing with a Newtonian source. Thus, they are referred to as "relativistic correction" terms. Erez and Rosen [2] refer to these same terms as "cross terms". For any r-dependent multipole moment, we refer to the $r^{-(n+1)}P_n(\cos\theta)$ term as the Newtonian part of that multipole. For example, the Newtonian part of D_m is $\frac{1}{3}c_1 m\sqrt{m^2-q_c^2}$.

In the same manner as the mass moments, the charge moments are identified as the coefficients of the $r^{-(n+1)}P_n(\cos\theta)$ terms in the expansion of Φ. Reordering Eq. (3.8) yields for the electric dipole moment

$$D_{q_c} = \frac{1}{3}c_1 q_c\sqrt{m^2-q_c^2}\left(1 - \frac{2\left(m^2-q_c^2\right)}{5r^2} - \frac{4m\left(m^2-q_c^2\right)}{5r^3} + \cdots\right) \tag{3.17}$$

and the electric quadrupole moment

$$\text{Quad.}_{q_c} = \frac{2}{15}c_2 q_c\left(m^2-q_c^2\right)\left(1 + \frac{m}{r} + \frac{6m^2+q_c^2}{7r^2} + \cdots\right). \tag{3.18}$$

The electric octupole and 16-pole potentials are respectively

$$\text{Oct.}_{q_c} = \frac{1}{r^4}\left[\frac{2}{35}c_3 P_3 q_c\left(m^2-q_c^2\right)^{3/2}\left(1+\frac{2m}{r}+\cdots\right) - \frac{1}{9}c_1^2 P_1^2 q_c m\left(m^2-q_c^2\right)\right], \tag{3.19}$$

$$\begin{aligned} \text{16-pole}_{q_c} = \frac{1}{r^5}\Bigg[&\frac{8}{315}\,q_c c_4 P_4\left(m^2-q_c^2\right)^2\left(1+O\left(\frac{1}{r}\right)+\cdots\right) \\ &- \frac{4}{45}c_1P_1c_2P_2q_c m\left(m^2-q_c^2\right)^{3/2} - \frac{1}{9}c_1^2P_1^2q_c\left(2m^2-q_c^2\right)\left(m^2-q_c^2\right)\Bigg]. \end{aligned} \tag{3.20}$$

Again, we have "relativistic correction" terms at the octupole and 16-pole levels. We shall now turn our attention to a covariant definition of multipole moments.

Chapter 4

Covariant Multipole Moments

4.1 Summary of the Geroch-Hansen-Hoenselaers Formalism

The development of a covariant definition of multipole moments has taken place over the past 20 years. Recently, Hoenselaers and Perjés [19] have established the procedure for calculating the electromagnetic and gravitational multipole moments of fields of stationary axially-symmetric electrovacuum spacetimes. This generalizes the earlier work of Hoenselaers [20], Hansen [6], Geroch [5] and others. We shall refer to calculations of moments in a charged spacetime as the Geroch-Hansen-Hoenselaers (GHH) formalism.

In the GHH formalism, multipole moments are defined as certain symmetric and traceless tensors at infinity on the background 3-manifold of the timelike Killing vector trajectories of the spacetime. It was Geroch [5,21] who first examined the possibility of defining multipole moments this way. He started by examining the different ways one could define multipole moments in flat-space. He identified three equivalent interpretations; coefficients in a multipole expansion, moments of the source distribution and objects associated with the conformal group. It became apparent to him that the first two interpretations would be very difficult to generalize to curved spacetimes. Therefore, he developed multipole moments from the standpoint of the conformal group (see Ref. [21] for a discussion of Newtonian multipole moments from this point of view).

A requirement of the GHH formalism is that the spacetime be asymptotically flat. The definition, established by Geroch [5], depends on the existence of a 3-manifold $\mathcal{S}$ with metric $\tilde{h}_{ab}$ such that

i) $\mathcal{S} = S \cup \Lambda$ where Λ is a single point at infinity,

ii) $\tilde{h}_{ab} \equiv \Omega^2 h_{ab}$ is a smooth metric on $\mathcal{S}$,

iii) $\Omega\,|_\Lambda = 0,\ \tilde{D}_a\Omega\,|_\Lambda = 0,\ \tilde{D}_a\tilde{D}_b\Omega\,|_\Lambda = 2\tilde{h}_{ab}$,

where S is the 3-manifold with metric h_{ab}, $\tilde{D}_a$ is the covariant derivative associated with $\tilde{h}_{ab}$ and Ω is called a conformal factor. Item *ii)* is the definition of a conformal transformation and $\tilde{h}_{ab}$ is called the conformally transformed metric of h_{ab} (note: the 4-manifold of the physical spacetime is identified with the metric g_{ab}).

The metric h_{ab} is defined as

$$h_{ab} = \lambda g_{ab} - \xi_a \xi_b, \tag{4.1}$$

where λ is the norm of the timelike Killing vector ξ^a. The norm is defined as $\lambda \equiv \xi^a \xi_a$. However, for asymptotically flat, axially-symmetric *stationary* spacetimes, the metric g_{ab} can be written as [20]

$$ds^2 = \lambda \left(dt - \omega d\phi\right)^2 - \lambda^{-1} \left[e^v \left(d\rho^2 + dz^2\right) + \rho^2 d\phi^2\right]. \tag{4.2}$$

We see that the g_{00} component of the metric tensor can be identified with λ. (ω is called the twist of the timelike Killing vector. See Hansen [6] for its definition.)

We now present the tensor definition of the multipole moments. By assigning the potential $\tilde{\phi}$ the conformal weight of $-1/2$, i.e. $\tilde{\phi} = \Omega^{-1/2}\phi$, tensors are recursively defined on $\mathcal{S}$ as [5]

$$\mathcal{P} = \tilde{\phi}, \quad \mathcal{P}_{a_1} = \tilde{D}_{a_1}\tilde{\phi}$$

$$\mathcal{P}_{a_1 \cdots a_{n+1}} = \mathcal{C}\left[\tilde{D}_{a_{n+1}}\mathcal{P}_{a_2 \cdots a_n} - \frac{n}{2}(2n-1)\tilde{R}_{a_1 a_2}\mathcal{P}_{a_3 \cdots a_{n+1}}\right], \tag{4.3}$$

where $\mathcal{C}$ is an operator that takes the symmetric trace free part of the tensor in the square brackets. If one examines the first term of Eq. (4.3) inside the square brackets, we see that it takes the form of multipole moments in a flat (Newtonian) spacetime, i.e.

$$\mathcal{P}_{a_1 \cdots a_n} = \phi_{,a_1 \cdots a_n}. \tag{4.4}$$

Eq. (4.4) has the properties that it is symmetric and trace free, and most importantly that a given moment is altered by an amount dependent only on lower moments under shifts in the origin in a flat spacetime. For example, it is well known that the values of the quadrupole moment about two nearby origins differ by an amount which is dependent only on the dipole moment.

Geroch [5] has shown that in order to retain the latter property in a curved spacetime, it is necessary to alter the definition of the flat-space multipole moments by the second term shown in Eq. (4.3). Geroch states that the property of the moments of different rank being properly related to each other under changes in origin is a necessary condition on the moments in order for them to be interpreted as multipole moments.

For a stationary, electrically charged gravitational field, the potential ϕ can be any one of the potentials ϕ_M, ϕ_J, ϕ_E, ϕ_H, which in turn are defined from two other potentials ξ and q by

$$\xi = \phi_M + i\phi_J, \tag{4.5}$$

$$q = \phi_E + i\phi_H. \tag{4.6}$$

(note: the potential ξ is not to be confused with the Killing vector ξ^a). Hoenselaers [19] identifies ϕ_M and ϕ_J as the mass and rotation potentials, and ϕ_E and ϕ_H as the electric and magnetic scalar potentials. *The 2^n-pole moment of ϕ is defined as the value of $\mathcal{P}_{a_1 \cdots a_{n+1}}$ at Λ.* It is thus a question of finding ξ and q in a suitable coordinate system to determine $\mathcal{P}_{a_1 \cdots a_n}|_\Lambda$.

The functions ξ and q are gravitational and electromagnetic potentials related to the Ernst potentials $\mathcal{E}$ and Ψ [22]. The relationship is

$$\mathcal{E} = \frac{1-\xi}{1+\xi}, \tag{4.7}$$

$$\Psi = \frac{q}{1+\xi}. \tag{4.8}$$

The Ernst potentials themselves are defined as the complex functions [22]

$$\mathcal{E} = \lambda + i\omega - \Psi\Psi^*, \tag{4.9}$$

$$\Psi = \Phi + i\gamma \tag{4.10}$$

where Φ is the electrostatic potential of the physical spacetime and γ is related to the electromagnetic vector potential (see Ref. [22]). The rationale for introducing the Ernst potentials are firstly, that they enable one to express the Einstein field equations as a very compactly written set of elliptic differential equations and secondly, that they enable one to use the properties of the resultant elliptical differential equations to facilitate the calculations of the multipole moments [23].

To summarize: the ingredients necessary to calculate multipole moments are a solution g_{ab} to the Einstein-Maxwell field equations and an electromagnetic 4-vector $A^i = (\Phi, \mathbf{A})$. With this knowledge we find $\mathcal{E}$ and Ψ through Eqs. (4.9) and (4.10), solve for ξ and q through Eqs. (4.12) and (4.8), determine the potential ϕ through Eqs. (4.5) and (4.6), and finally evaluate Eq. (4.3) at Λ to obtain the moments.

4.2 Calculation of the Covariant Multipole Moments for the Erez-Rosen Solution

For electrostatic (non-stationary) solutions, comparing Eqs. (2.1) and (4.2), we see that

$$\lambda = e^w = g_{00} \tag{4.11}$$

and

$$\omega = 0. \tag{4.12}$$

Furthermore, the 4-vector potential $A^i = (\Phi, \mathbf{0})$. It can easily be shown that as a consequence of this choice of gauge, the condition $\gamma = 0$ must hold in Eq. (4.10). Hence, the Ernst potentials and all subsequent variables are real quantities. Since we are in the Weyl class,

$$e^w = \lambda = 1 - \frac{2m}{q_c}\Phi + \Phi^2. \tag{4.13}$$

Therefore, we can solve for Φ in terms of λ (i.e. g_{00}). Substituting $\gamma = 0$ and Eqs. (4.12) and (4.10) into Eq. (4.9), we obtain

$$\mathcal{E} = \lambda - \Phi^2. \tag{4.14}$$

Inverting Eqs. (4.7) and (4.8) yields

$$\begin{aligned} \xi &= \frac{1-\mathcal{E}}{1+\mathcal{E}} = \frac{1-\lambda+\Phi^2}{1+\lambda-\Phi^2}, \\ q &= (1+\xi)\Phi = \frac{2\Phi}{1+\lambda-\Phi^2} \end{aligned} \tag{4.15}$$

Solving Eq. (4.13) for Φ gives

$$\Phi(\lambda) = \frac{m}{q_c} \pm \sqrt{\frac{m^2}{q_c^2} - 1 + \lambda}, \tag{4.16}$$

(assuming positive charge, we must choose the negative sign). From Eqs. (4.15) and (4.16), we have ξ and q as functions of λ.

At this point it is convenient to explicitly express ξ and q in terms of the coordinates (ρ, z, ϕ). This requires the specific solution which we have chosen in chapter 2. From Eqs. (2.18) and (3.4), we have

$$\lambda = \frac{(1-a^2)^2 f}{(a^2 f - 1)^2}, \quad a \equiv \frac{m}{q_c} - \frac{1}{q_c}\sqrt{m^2 - q_c^2} \tag{4.17}$$

and f is found through Eq. (3.7). However, a more convenient coordinate system is one in which infinity is mapped into the origin. This choice is made because the multipole tensors which are evaluated at infinity in the old coordinate system will now be evaluated at the origin. The required transformation is [20]

$$\tilde{\rho} = \frac{\rho}{\rho^2 + z^2}, \quad \tilde{z} = \frac{z}{\rho^2 + z^2}, \quad \tilde{\phi} = \phi. \tag{4.18}$$

Choosing the conformal factor as $\Omega = \tilde{\rho}^2 + \tilde{z}^2$ allows us to express $\tilde{\xi}$ and $\tilde{q}$ as functions of $\tilde{z}$ and $\tilde{\rho}$. It can be verified that this choice of conformal factor satisfies the conditions outlined in item *iii)* of this section.

Hoenselaers and Perjés [19] have expressed the Einstein-Maxwell field equations in terms of ξ and q. Transforming to the tilded coordinate system leaves the field equations in the same form. It follows from the standard theory of elliptic differential equations that when $\tilde{\xi}$ and $\tilde{q}$ are analytic in a neighbourhood of the origin, they are determined uniquely by their respective values on the axis ($\tilde{\rho} = 0$) [19]. Thus, we can express $\tilde{\xi}$ and $\tilde{q}$ as Taylor series about the point $\tilde{z} = 0$:

$$\tilde{\xi}(\tilde{\rho} = 0) = \sum_{n=0}^{\infty} m_n \tilde{z}^n, \tag{4.19}$$

$$\tilde{q}(\tilde{\rho} = 0) = \sum_{n=0}^{\infty} q_n \tilde{z}^n. \tag{4.20}$$

The final step is to substitute Eqs. (4.19) and (4.20) into (4.3) via the conformally transformed versions of (4.5) and (4.6). Then, by setting $\tilde{z} = 0$, $\mathcal{P}_{a_1 \cdots a_n}$ is evaluated at Λ.

For stationary metrics with axial symmetry, one can further simplify the formula for the multipole moments. For axially-symmetric metrics there also exists an angular Killing vector. The multipole moments must be invariant under the action of this additional Killing vector. However, the only tensors which are invariant with respect to the axial Killing vector at Λ are the metric h_{ab} and the axis vector itself denoted by η^a [6]. It can be deduced that the multipole moments must be multiples of $\mathcal{C}[\tilde{\eta}_{a_1} \cdots \tilde{\eta}_{a_n}]|_\Lambda$, i.e. the symmetric trace free outer products of the axis vector with itself. Thus the 2^n moments are completely determined by the *numbers*

$$\mathcal{P}_n = \frac{1}{n!} \mathcal{P}_{a_1 \cdots a_n} \tilde{\eta}^{a_1} \cdots \tilde{\eta}^{a_n}|_\Lambda . \tag{4.21}$$

Since $\eta^a = (0, 1, 0)$,

$$\mathcal{P}_n = \frac{1}{n!} \mathcal{P}_{\underbrace{z \ldots z}_{n}}|_\Lambda \ . \tag{4.22}$$

Hoenselaers et al. have derived for both uncharged [20, 24] and charged [19] spacetimes, the moments of Eq. (4.22) as functions of the m_n and q_n, which are defined by Eq. (4.19) and (4.20). Denoting $\mathcal{M}_n$ as the mass moments and $\mathcal{Q}_n$ as the charge moments, the result of applying Eqs. (4.19) and (4.20) to Eq. (4.3) up to the 16-pole moment is [19]

$$\begin{aligned}
\mathcal{M}_n &= m_n \text{ for } n = 0, 1, 2, 3 \\
\mathcal{M}_4 &= m_4 - \frac{1}{7} M_{20} m_0^* + \frac{8}{105} S_{10} q_1^* m_0^* m_0 - \frac{3}{70} S_{10} q_1^* + \\
&\quad - \frac{8}{105} S_{10} q_0^* m_0 m_1^* + \frac{1}{7} S_{20} q_0^* \\
\mathcal{Q}_n &= q_n \text{ for } n = 0, 1, 2, 3 \\
\mathcal{Q}_4 &= q_4 - \frac{1}{7} Q_{20} q_0^* - \frac{8}{105} S_{10} m_1^* q_0^* q_0 - \frac{3}{70} S_{10} q_1^* + \\
&\quad + \frac{8}{105} S_{10} m_0^* q_0 q_1^* - \frac{1}{7} S_{20} m_0^* \qquad (4.23)
\end{aligned}$$

where

$$\begin{aligned}
M_{ij} &= m_i m_j - m_{i-1} m_{j+1}, \\
Q_{ij} &= q_i q_j - q_{i-1} q_{j+1},
\end{aligned}$$

for $i > j + 1$ and

$$S_{ij} = m_i q_j - m_{i-1} q_{j+1}.$$

We have left $\mathcal{M}_n$ and $\mathcal{Q}_n$ as complex quantities for generality, since the Hoenselaers-Perjés derivation is valid for stationary spacetimes.

We can summarize the procedure for calculating mass and charge covariant multipoles as follows: For a given solution of the Einstein-Maxwell electrovac field equations (Eqs. (4.17) and (3.7)), express the solutions in the new coordinate system $\tilde{\rho}, \tilde{z}$. Then determine the gravitational and electromagnetic potentials $\tilde{\xi}(\tilde{\rho})$ and $\tilde{q}(\tilde{z})$ along the symmetry axis $\tilde{\rho} = 0$ (through Eqs. (4.15) and (4.16)). In practice, it is best to take the Taylor series of ξ and q about $\tilde{z} = 0$ first and then make the conformal transformation. This eases the calculations since $\Omega^{-1/2} = \tilde{z}^{-1}$ along $\tilde{\rho} = 0$. Therefore

$$\tilde{\xi} = \frac{\xi}{\tilde{z}} = \frac{1}{\tilde{z}} \sum_{k=1}^{\infty} b_k \tilde{z}^k = \sum_{k=1}^{\infty} b_k \tilde{z}^{k-1} = \sum_{n=0}^{\infty} m_n \tilde{z}^n. \tag{4.24}$$

Note that the Taylor expansion for ξ starts at $k = 1$. This is expected because the conformally transformed potential $\tilde{\xi}$ cannot have any singularities at Λ. A similar result applies to $\tilde{q}$.

Therefore the first four mass and charge covariant multipole moments calculated from Eq. (4.23) for the Erez-Rosen solution are

$$\begin{aligned}
\mathcal{M}_0 &= m \\
\mathcal{M}_1 &= \frac{1}{3} c_1 m \sqrt{m^2 - q_c^2} \\
\mathcal{M}_2 &= \frac{2}{15} c_2 m \left(m^2 - q_c^2\right) \\
\mathcal{M}_3 &= \frac{2}{105} (3c_3 - 7c_1) m \left(m^2 - q_c^2\right)^{3/2} \\
\mathcal{M}_4 &= \frac{2}{315} m \left(m^2 - q_c^2\right)^2 \left[4c_4 - 6c_2 - 15c_1^2\right],
\end{aligned}$$

$$\begin{aligned}
\mathcal{Q}_0 &= q_c \\
\mathcal{Q}_1 &= \frac{1}{3} c_1 q_c \sqrt{m^2 - q_c^2} \\
\mathcal{Q}_2 &= \frac{2}{15} c_2 q_c \left(m^2 - q_c^2 \right) \\
\mathcal{Q}_3 &= \frac{2}{105} (3c_3 - 7c_1) q_c \left(m^2 - q_c^2 \right)^{3/2} \\
\mathcal{Q}_4 &= \frac{2}{315} q_c \left(m^2 - q_c^2 \right)^2 \left[4c_4 - 6c_2 - 15c_1^2 \right] .
\end{aligned} \tag{4.25}$$

Chapter 5

Discussion

We will not directly compare the coordinate dependent Erez-Rosen multipole moments with the invariant GHH moments since they are not referring to multipole moments in the same manner. The GHH moments are considered the physically significant entities since they are derived from tensor fields. Erez-Rosen moments look very similar to the covariant ones, but they lack the tensorial definition which is necessary to establish them as covariant physical constructs. In this sense the Erez-Rosen moments must be labelled coordinate dependent. However, within a spherical coordinate system, it is reasonable to refer to the expansion of g_{00} with respect to r as a "multipole expansion". Although the Erez-Rosen moments must be labelled as coordinate dependent moments, the field represented by this expansion is a physical field and the Erez-Rosen moments contain interesting information about the gravitational field of the source which the GHH formalism does not contain.

We shall first examine the GHH moments. From Eq. (4.25) we see the increasing complexity of the moments in the series. Eq. (4.25) reveals that if we wish to construct a source with a certain multipole structure, such as a monopole plus a quadrupole only, then it is necessary to set c_1 and c_3 equal to zero to make $\mathcal{M}_1$ and $\mathcal{M}_3$ zero. However the increased complexity of the poles higher than octupole places constraints on all the c_n's for $n \geq 4$. This is a result of setting $\mathcal{M}_n = 0, n \geq 4$ to eliminate all higher poles. In effect what is happening is the relativistic parts of the higher moments are cancelling the Newtonian parts so that physically there is only a monopole and quadrupole moment to the source. If one were to take the set of c_n this example produces and then calculate the Erez-Rosen moments, one would come to the erroneous conclusion that the source includes many physical multipoles of order higher than quadrupole.

This procedure shows that unlike Newtonian theory, it is possible to construct "pure multipoles" for objects of finite extent. By "pure multipoles" we mean an object with a monopole moment and only one other multipole moment as described above. In Newtonian theory, one cannot mathematically eliminate the higher order terms. The higher order terms are usually eliminated through physical arguments (they are shown to be negligible in comparison with the lower order terms). In Maxwell's electromagnetic theory, it is possible to obtain, for example, a pure dipole source, but one must shrink the multipole to a mathematical point in such a way as to keep the potential finite.

The above gives the criterion for constructing a source with a certain multipole structure. Suppose instead one wishes to experimentally determine the multipole structure of a field. We will briefly outline how the Erez-Rosen and GHH formalisms can be used together to make such a determination. The objective is to make a measurement of the parameters c_n. This can be done by mapping the orbit of a satellite onto the geodesics calculable from the metric expressed in the "multipole expansion" form of the Erez-Rosen formalism. Note that one must choose a coordinate system in order to calculate the geodesics. It is also necessary to explicitly determine the metric function v of Eq. (2.21) in a multipole expansion form. The mapping of the satellite orbit to the geodesics can be achieved by the common practice of numerically adjusting the c_n's to make a "best fit". Once the c_n's are known, one simply calculates the GHH moments to determine the physical multipole structure. Another approach one could take to determine the c_n's is to map the motion of a test charge onto a trajectory calculable from the electric field potential Φ of Eq. (3.8) and Maxwell's covariant equations. Either prescription is sufficient to characterize the c_n's for the field generated by a charged massive body.

Additional observations regarding the GHH moments are as follows: Firstly, if the m which is a common factor in the mass moments is exchanged with q_c and all occurrences of $m^2 - q_c^2$ are left unchanged, then we immediately obtain the charge moments. This illustrates a symmetry between the mass and charge moments, which we expect when working within the Weyl class of electrovacuum solution (specifically Eq. (2.14)). Secondly, all the mass multipoles except the monopole are a function of the charge. This shows that the charge affects the mass moments. What is more unusual is that the mass affects the charge moments. This is not experienced in flat-space electromagnetic theory. Therefore, choosing to work in the Weyl class of electrovacuum spacetime inextricably links the electric and gravitational fields. This link manifests itself in the multipole moments of each field. The Erez-Rosen moments, which we now turn to, also illuminate this charge-mass mixing but in a coordinate dependent way.

It is important to examine the g_{00} component in this coordinate dependent manner because it provides a more intuitive picture of a test particle's motion in the field as is seen with the Reissner-Nordström solution. As was indicated in chapter 3, one can interpret Eq. (3.11) as the effective mass of a purely monopole source which is dependent on the electromagnetic field energy interior to the sphere of radius r in spherical coordinates. One would expect that the higher moments would also show this r-dependent behaviour. Eqs. (3.11–3.15) show that this does indeed occur in the Erez-Rosen moments. This information is not extractable from the GHH moments since they are defined at spatial infinity only. It is obvious that the r-dependence of the moments is subject to the particular type of spherical coordinates chosen.

However, this is not a cause for concern since we must be in some coordinate system in order to make a measurement. In an analogous fashion to the mass moments, Eqs. (3.17–3.20) show how the electric field can be interpreted as being dependent on the mass-energy inside a sphere of radius r. Again, we emphasize that this phenomenon has no analogue in flat-space electromagnetic theory.

It is interesting to note that the first three Erez-Rosen moments of both mass and charge, if evaluated in the limit $r \to \infty$, are identical to the GHH moments. It is understandable for the first two moments, since the second term of Eq. (4.3) is identically zero. It is unclear as to why the quadrupole is identical.

Chapter 6

The Newtonian Limit

The preceding chapters have dealt with the problem of calculating relativistic covariant moments for axially-symmetric charged spacetimes. We will now examine the problem of determining the Newtonian limit of such spacetimes. We will first discuss two definitions for finding the Newtonian mass multipoles of a pure vacuum spacetime (i.e. uncharged). One method is shown to be inadequate for determining such multipoles. The other method, which is adequate, can easily be extended to encompass finding the classical mass and charge moments for charged axially-symmetric spacetimes.

As we have mentioned, many papers have been written on how to calculate relativistic covariant multipole moments for axially-symmetric spacetimes [5–8,25]. A procedure for extracting the Newtonian limit for axially-symmetric spacetimes has been suggested by Ehlers [12].

However, it was shown by Perry and Bohun [26] that it is not possible to extract the Newtonian multipole moments solely based on Ehlers' definition. Rather, one must consider a nearly classical source in order to see the classical multipole structure of the field as proposed by Cooperstock [11]. We shall outline the arguments presented by Perry and Bohun and then discuss the Newtonian limit's extension to charged spacetimes.

Ehlers' definition of the Newtonian limit is [12]

$$\Phi_N = \lim_{\lambda \to 0} \frac{1}{2\lambda} \bar{w}(\rho, z, \lambda)\,, \tag{6.1}$$

where $\lambda = c^{-2}$ (c = speed of light) and $\bar{w}(\rho, z, \lambda)$ is the metric function of Eq. (2.1) containing the parameter λ.

The asymptotically flat solution $\bar{w}$ of the vacuum EFEs is

$$\frac{\bar{w}}{2} = -\sum_{n=0}^{\infty} c_n P_n(y) Q_n(x)\,, \tag{6.2}$$

where

$$x \equiv \frac{r_+ + r_-}{2GM\lambda}\,, y \equiv \frac{r_+ - r_-}{2GM\lambda}\,,$$

and

$$r_\pm = \sqrt{\rho^2 + (z \pm GM\lambda)^2}\,.$$

Expressing $Q_n(x)$ as a descending power series in x,

$$Q_n(x) = 2^n \sum_{s=0}^{\infty} b(n,s) x^{-(n+1+2s)} \tag{6.3}$$

with

$$b(n,s) \equiv \frac{(n+s)!\,(n+2s)!}{s!\,(2n+2s+1)!} \tag{6.4}$$

and using the transformation between x, y and spherical coordinates [2]

$$x = \frac{r}{GM\lambda} - 1,$$

$$y = \cos\theta, \tag{6.5}$$

one can rewrite Eq. (6.2) as

$$\frac{1}{2}\bar{w}(\lambda) = \frac{1}{2}\sum_{n=0}^{\infty}\bar{w}_n = -\sum_{n=0}^{\infty} c_n P_n(\cos\theta) 2^n \sum_{s=0}^{\infty} b(n,s)\left(\frac{GM\lambda}{r}\right)^{n+2s+1} \times$$

$$\times \left[1 + \sum_{j=1}^{\infty} \frac{(n+2s+1)\cdots(n+2s+j)}{j!}\left(\frac{GM\lambda}{r}\right)^j\right]. \tag{6.6}$$

For the reason of avoiding lengthy complicated formulas, we shall choose $\bar{w}$ of Eq. (6.6) to have the form

$$\bar{w} = \bar{w}_0 + \bar{w}_1 + \bar{w}_2 + \bar{w}_3\,. \tag{6.7}$$

With $n = 0, 1, 2, 3$ put into Eq. (6.6), all terms of equal power in λ up to order $O(\lambda^5)$ are collected together to yield

$$\frac{1}{2\lambda}\;\bar{w}(\lambda) = -c_0\frac{GM}{r} - \left[c_0 + \frac{1}{3}c_1 P_1(\cos\theta)\right]\left(\frac{GM}{r}\right)^2\lambda +$$

$$
\begin{aligned}
&- \left[\frac{4}{3}c_0 + \frac{2}{3}c_1 P_1(\cos\theta) + \frac{2}{15}c_2 P_2(\cos\theta)\right]\left(\frac{GM}{r}\right)^3 \lambda^2 + \\
&- \left[2c_0 + \frac{6}{5}c_1 P_1(\cos\theta)\frac{2}{5}c_2 P_2(\cos\theta) + \frac{2}{35}c_3 P_3(\cos\theta)\right]\left(\frac{GM}{r}\right)^4 \lambda^3 + \\
&- \left[\frac{16}{5}c_0 + \frac{32}{15}c_1 P_1(\cos\theta) + \frac{32}{35}c_2 P_2(\cos\theta) + \frac{8}{35}c_3 P_3(\cos\theta)\right]\left(\frac{GM}{r}\right)^5 \lambda^4 + \\
&- \left[\frac{16}{3}c_0 + \frac{80}{21}c_1 P_1(\cos\theta) + 2c_2 P_2(\cos\theta) + \frac{40}{63}c_3 P_3(\cos\theta)\right]\left(\frac{GM}{r}\right)^6 \lambda^5. \qquad (6.8)
\end{aligned}
$$

Now suppose that the $\{c_n\}$ have no λ-dependence. It is obvious that setting $\lambda = 0$ in Eq. (6.8) would result in the Newtonian limit being just a monopole. This approach implies that even the general solution given by Eq. (6.2) always reduces to a monopole potential in the Newtonian limit. This is an incorrect conclusion and therefore one cannot apply Ehlers' definition in this manner.

In a recent paper [27], Ehlers' definition was employed to find the Newtonian moments for the Erez-Rosen solution. Perry and Bohun [26] showed that the approach used in [27] was not sufficient for extracting the Newtonian multipole structure of the field for the reasons stated above.

So the question remains how can one justifiably expect to attain a Newtonian limit in the form

$$
\Phi_N = GM \sum_{n=0}^{\infty} a_n \frac{P_n(\cos\theta)}{r^{n+1}}. \qquad (6.9)
$$

Cooperstock [11] shows that one must take into consideration the weak field limit of the source itself in order to "see" a multipole structure in the Newtonian limit of the field. This gives the physical justification for eliminating these "cross terms" appearing in Eq. (6.8). His method does not rely on parameterizing the speed of light constant c. Therefore any suitable units of measurement will do. As a consequence this method is quite easily understood when geometerized units are used ($G = c = 1$) where the explicit dependence of G and λ disappear. This method is sufficient to extract the Newtonian multipole structure given by Eq. (6.9) provided $\bar{w}$ is of the form of Eq. (6.2).

Perry and Bohun proceeded to take Cooperstock's definition of a nearly Newtonian (classical) source in traditional units so that it could be applied to Ehlers' definition of the Newtonian limit. Once this connection is made, one obtains Eq. (6.9) in the limit $\lambda \to 0$.

Cooperstock defines a "nearly Newtonian source" as being a source which satisfies the condition

$$\frac{GM\lambda}{L} \ll 1. \tag{6.10}$$

where L is the characteristic size of the source. He also states that the characteristic size of Newtonian 2^{l-1}-pole multipoles are GML^{l-1}, $l = 1, 2, \ldots$. Therefore each expansion coefficient a_n of Eq. (6.9) must be bounded above by

$$a_n \sim L^n . \tag{6.11}$$

If we examine the Newtonian-like terms of Eq. (6.8), we must have

$$c_n \sim \left(\frac{L}{GM\lambda}\right)^n \tag{6.12}$$

as the upper bound on $\{c_n\}$. If we make the substitution

$$c_n = d_n \left(\frac{L}{GM\lambda}\right)^n \tag{6.13}$$

where the d_n's are dimensionless and of order 1, into Eq. (6.8), then the resulting equation is

$$\frac{1}{2}\bar{w}(\lambda) = -\sum_{n=0}^{\infty} P_n(\cos\theta)2^n \sum_{s=0}^{\infty} b(n,s)d_n \left(\frac{L}{r}\right)^n \left(\frac{GM}{r}\right)^{2s+1} \lambda^{2s} \times$$
$$\times \left[1 + \sum_{j=1}^{\infty} \frac{(n+2s+1)(n+2s+2)\cdots(n+2s+j)}{j!} \left(\frac{GM\lambda}{r}\right)^j\right] . \tag{6.14}$$

Taking Ehlers' limit one obtains

$$\Phi_N = -GM \sum_{n=0}^{\infty} d_n \frac{n!L^n}{(2n+1)!!} \frac{P_n(\cos\theta)}{r^{n+1}} . \tag{6.15}$$

Therefore it is concluded that Eq. (6.15) represents the Newtonian limit of the EFE solution $\bar{w}$ only when all the expansion coefficients are of the order of the upper limit of a nearly Newtonian source given in Eq. (6.12).

It has been shown that it is not possible to extract the multipole moments solely based on Ehlers' definition of the Newtonian limit. One must consider a nearly classical source in order to see the classical multipole structure of a field whose solution is representable by Eq. (6.2) in the Weyl class. This brings us to the step of determining the Newtonian limit for the case of a charged spacetime. The Newtonian limit is a set of conditions placed on the general relativistic scenario. Applying the conditions does not guarantee that the limit will agree with that of the "true" Newtonian theory. One may find situations where there is no Newtonian analogue.

For instance, Newtonian theory does not equate energy to mass. For a charged spherical mass, the mass monopole moment would be just the inertial mass m and similarly the charge monopole would be the classical charge q_c. However, we know from Eq. (3.11) that it is possible to interpret the Reissner-Nordström solution as having a mass monopole which increases in size with respect to the radius r via the $-q_c^2/2r$ contribution. The total mass m is interpreted as being the equivalent mass energy of both the inertial mass of the object and its electric field energy. Since, in Newtonian theory, inertial mass is all that contributes to the gravitational field, the Newtonian observer must conclude that the energy density found in Poisson's equation for a charged object is different from that for an uncharged (otherwise identical) object. This is one possible way for the Newtonian observer to salvage the theory. This special treatment of charged objects is not *a priori* knowledge; it is only observation which could bring the Newtonian observer to this conclusion. (A Newtonian observer could also take this phenomenon as a hint that his/her theory may have problems.)

If we adopt the argument that the radially dependent contributions to a mass moment can be "absorbed" by requiring a different mass density than the corresponding uncharged case, then Eq. (3.12) yields a Newtonian dipole moment of

$$\mathcal{N}_1 = \frac{1}{3} c_1 m \sqrt{m^2 - q_c^2} \tag{6.16}$$

in geometrized units ($G = c = 1$). For the octupole moment (Eq. (3.14)), we employ a modified version of Cooperstock's criterion for a nearly Newtonian source, i.e.

$$\frac{\sqrt{m^2 - q_c^2}}{L} \ll 1, \tag{6.17}$$

to eliminate the "cross term" and thus deduce the value

$$\mathcal{N}_3 = \frac{2}{35} c_3 m \left(m^2 - q_c^2\right)^{3/2}. \tag{6.18}$$

Proceeding in this fashion, the general expression for the Newtonian limit of the mass multipoles is

$$\mathcal{N}_k = \frac{k!}{(2k+1)!!} c_k m \left(m^2 - q_c^2\right)^{k/2}. \tag{6.19}$$

Since we expect the Newtonian multipoles to be of the order mL^k, we see that the $\{c_k\}$ must be of the order

$$c_k \sim \left(\frac{L}{\sqrt{m^2 - q_c^2}}\right)^k \tag{6.20}$$

for a charged body. Letting

$$c_k = d_k \left(\frac{L}{\sqrt{m^2 - q_c^2}}\right)^k, \tag{6.21}$$

we find

$$\mathcal{N}_k = \frac{k!}{(2k+1)!!} d_k m L^k. \tag{6.22}$$

The determination of the charge moments is completely analogous to that of the mass moments. The upper bound of Eq. (6.20) on the c_k's also applies to the charge moments. The coefficients are bounded above in this manner for the charged case because in the classical limit we expect $D_{q_c} \sim q_c L$, $Q_{q_c} \sim q_c L^2$, etc. Eq. (3.17–3.20) shows that the charge multipoles have radial dependence and thus the Newtonian observer would be required to modify the charge density in Maxwell's equations as well. Hence the general expression for the charge moments in the Newtonian limit is

$$\begin{aligned} \mathcal{N}_k(\text{charge}) &= \frac{k!}{(2k+1)!!} c_k q_c \left(m^2 - q_c^2\right)^{k/2}. \\ &= \frac{k!}{(2k+1)!!} d_k q_c L^k. \end{aligned} \tag{6.23}$$

It is of interest to consider the question of choice of spherical coordinates. While Schwarzschild (and in our charged case Reissner-Nordström) coordinates are preferred in various ways, they are by no means the unique spherical coordinates. Isotropic coordinates come to mind as being useful in certain contexts. However, the transformation from one system to the other brings in higher powers of m, which are ignored in the Newtonian limit. It is to be emphasized that the Poisson equation, and hence the formal mathematical basis for the Newtonian limit of general relativity, is extracted from the Einstein field equations by retaining only the first order in m (with the added stipulation that source velocities are $\ll c$). Thus, the particular choice of spherical coordinates is irrelevant in this regard. However, from the stand-point of the Erez-Rosen moments, there is an evident sensitivity to choice of coordinates. The Erez-Rosen moments extracted from Eqs. (3.8) and (3.10) may then be said to be based upon Schwarzschild (for $q_c = 0$) or Reissner-Nordström (for $q_c \neq 0$) coordinate expansions. They are physically well-founded because the radial coordinate r is related to proper spherical surface area at r by $4\pi r^2$.

As a concluding remark we would like to note that Cooperstock's method for calculating the Newtonian limit does not require knowledge of the covariant moments. It is based solely on the Erez-Rosen formalism. Thus we see that the Erez-Rosen expansions are perfectly reasonable in this limit.

Chapter 7

The Curzon Particle

In chapter 2 we introduced the Weyl class of electrovacuum solutions of the coupled Einstein-Maxwell field equations. We found that solutions to Laplace's equation generates solutions to the EFEs for axially-symmetric spacetimes. In this chapter we will compute the multipole moments for a charged Curzon particle using the methods developed in chapters 3 and 4. Both the Curzon and charged Curzon gravitational fields have been well studied (see Ref. [28] and references contained therein). However, the recent development of the covariant formalisms for charged multipole moments suggests that a re-examination of the multipole structure is in order. It will also serve as a simpler example in order to comprehend both the Erez-Rosen and GHH formalisms.

7.1 Erez-Rosen formalism

The Curzon particle is deemed to be a body which is representable by the exterior solution

$$\frac{1}{2}\bar{w} = -\frac{m}{R}, \quad R \equiv \sqrt{\rho^2 + z^2} \tag{7.1}$$

to Laplace's equation (2.2). Recall that R is not the true spherically symmetric radial coordinate in the Weyl spacetime.

We know that in the charged Weyl spacetime it is the function $\ln f$ which must satisfy the Laplace equation. Therefore the corresponding charged Curzon solution must be

$$\frac{1}{2}\ln f = -\frac{b}{R}, \tag{7.2}$$

where b is some constant to be evaluated. From Eq. (2.18) the metric component e^w is [29,15]

$$e^w = \frac{\left(1 - a^2\right)^2 e^{-2b/R}}{\left(a^2 e^{-2b/R} - 1\right)^2}. \tag{7.3}$$

The electric potential is

$$\Phi = \frac{a\left(e^{-2b/R} - 1\right)}{a^2 e^{-2b/R} - 1} \tag{7.4}$$

from Eq. (2.20).

For the Erez-Rosen formalism, we need to transform to spherical polar coordinates. Using Eq. (3.3) we find

$$\begin{aligned} z^2+\rho^2 &= (r-m)^2\cos^2\theta+\left(r^2-2mr+q_c^2\right)\sin^2\theta \\ &= r^2-2mr+m^2\cos^2\theta+q_c^2\sin^2\theta. \end{aligned} \tag{7.5}$$

Hence

$$\frac{1}{R}=\frac{1}{\sqrt{z^2+\rho^2}}=\frac{1}{\sqrt{r^2-2mr+m^2\cos^2\theta+q_c^2\sin^2\theta}}. \tag{7.6}$$

Substituting Eq. (7.6) into e^w and then expanding about $r\to\infty$ gives

$$\begin{aligned} e^w &= 1-\frac{2bm}{r\sqrt{m^2-q_c^2}}+\frac{b\left(-2m^2\sqrt{m^2-q_c^2}+2bm^2+bq_c^2\right)}{r^2\left(m^2-q_c^2\right)}+ \\ &\quad +\frac{1}{r^3}\mathcal{F}(b,m,q_c)+\frac{1}{r^4}\mathcal{G}(b,m,q_c)+\cdots, \end{aligned} \tag{7.7}$$

where $\mathcal{F}$ and $\mathcal{G}$ represent the next two terms. In the limit $r\to\infty$, one would expect that the monopole term would be the total mass m (including the electromagnetic field energy). This condition enables us to determine the constant b. By inspection of Eq. (7.7)

$$b=\sqrt{m^2-q_c^2}. \tag{7.8}$$

This is in agreement with Cooperstock and de la Cruz [15], who determine the value of b by an alternate method.

With the substitution $b=\sqrt{m^2-q_c^2}$, e^w reduces to

$$e^w=1+2\left\{-\frac{m}{r}+\frac{q_c^2}{2r^2}+\frac{m\left(m^2-q_c^2\right)\left(3\cos^2\theta-1\right)}{6r^3}+\right.$$

$$+\frac{\left(m^2-q_c^2\right)^2\left(3\cos^2\theta-1\right)}{6r^4}+\cdots\Big\}. \tag{7.9}$$

Similarly, the electrostatic potential reduces to

$$\Phi=\frac{q_c}{r}-\frac{q_c}{6r^3}\left(m^2-q_c^2\right)\left(3\cos^2\theta-1\right)-\frac{q_c m}{6r^4}\left(m^2-q_c^2\right)^2\left(3\cos^2\theta-1\right)+\cdots. \tag{7.10}$$

The final step is to order the potentials in terms of $r^{n+1}P_n(\cos\theta)$ so that the Erez-Rosen moments may be determined. The result is

$$e^w=1+2\left\{-\frac{1}{r}\left(m-\frac{q_c^2}{2r}\right)+\frac{\left(m^2-q_c^2\right)P_2(\cos\theta)}{3r^3}\left(m+\frac{m^2-q_c^2}{r}+\cdots\right)+\cdots\right\} \tag{7.11}$$

$$\Phi=\frac{q_c}{r}-\frac{q_c}{3r^3}P_2(\cos\theta)\left(m^2-q_c^2\right)\left(1+\frac{m}{r}+\cdots\right)+\cdots. \tag{7.12}$$

From Eq. (7.11) we see that the monopole term can be interpreted as the effective mass

$$m(r)=m-\frac{q_c^2}{2r} \tag{7.13}$$

of the gravitational field in exactly the same manner as in chapter 3. Since there is no r^{-2} term, the dipole moment is zero. The mass quadrupole moment is

$$-\frac{1}{3}\left(m^2-q_c^2\right)\left(m+\frac{m^2-q_c^2}{r}\cdots\right). \tag{7.14}$$

It exhibits the r-dependence discussed in the Erez-Rosen solution. Similarly, the charge monopole, dipole and quadrupole moments are respectively

$$M_{q_c}=q_c, \tag{7.15}$$

$$D_{q_c} = 0, \tag{7.16}$$

and

$$\text{Quad.}_{q_c} = -\frac{1}{3} q_c \left(m^2 - q_c^2\right) \left[1 + \frac{m}{r} + \cdots\right]. \tag{7.17}$$

The moments presented above are of course the coordinate dependent moments. The physical multipole structure must be determined through the GHH formalism.

7.2 The GHH Formalism

The first step in calculating the GHH moments is to transform e^w and Φ into the $\tilde{z}, \tilde{\rho}$ coordinate system. Inverting Eq. (4.18) yields the relation

$$\sqrt{z^2 + \rho^2} = \frac{1}{\sqrt{\tilde{z}^2 + \tilde{\rho}^2}} \equiv \frac{1}{\tilde{R}}. \tag{7.18}$$

Therefore

$$e^w = \frac{\left(1 - a^2\right)^2 e^{-2b\tilde{R}}}{\left(a^2 e^{-2b\tilde{R}} - 1\right)^2}. \tag{7.19}$$

With e^w written in the new coordinate system, we can substitute it into Eq. (4.15) and Eq. (4.16) to obtain the Ernst potentials $\xi(\tilde{z}, \tilde{\rho})$ and $q(\tilde{z}, \tilde{\rho})$. At this stage we do not conformally transform the Ernst potentials, but instead we take the Taylor series of ξ and q along the symmetry axis (i.e. set $\tilde{\rho} = 0$) about the point $\tilde{z} = 0$. The results are

$$\xi(\tilde{z}, \tilde{\rho} = 0) = m\tilde{z} - \frac{1}{3} m \left(m^2 - q_c^2\right) \tilde{z}^3 + \frac{2}{15} m \left(m^2 - q_c^2\right)^2 \tilde{z}^5 + \cdots, \tag{7.20}$$

$$q(\tilde{z},\tilde{\rho}=0) = q_c\tilde{z} - \frac{1}{3}q_c\left(m^2 - q_c^2\right)\tilde{z}^3 + \frac{2}{15}q_c\left(m^2 - q_c^2\right)^2\tilde{z}^5 + \cdots. \qquad (7.21)$$

To find the conformally transformed Ernst potentials, we simply divide Eqs. (7.20) and (7.21) by $\tilde{z}$. The final step is to identify the coefficients m_n and q_n by comparing Eq. (7.20) and Eq. (7.21) to Eq. (4.19) and Eq. (4.20) and substitute the coefficients into Eq. (4.23) to obtain the moments. The result of this procedure is

$$\begin{aligned}
\mathcal{M}_0 &= m \\
\mathcal{M}_1 &= 0 \\
\mathcal{M}_2 &= -\frac{1}{3}m\left(m^2 - q_c^2\right) \\
\mathcal{M}_3 &= 0 \\
\mathcal{M}_4 &= \frac{19}{105}m\left(m^2 - q_c^2\right)^2,
\end{aligned}$$

$$\begin{aligned}
\mathcal{Q}_0 &= q_c \\
\mathcal{Q}_1 &= 0 \\
\mathcal{Q}_2 &= -\frac{1}{3}q_c\left(m^2 - q_c^2\right) \\
\mathcal{Q}_3 &= 0 \\
\mathcal{Q}_4 &= \frac{19}{105}q_c\left(m^2 - q_c^2\right)^2.
\end{aligned} \qquad (7.22)$$

If we compare Eqs. (7.22) with the Erez-Rosen moments of Eqs. (7.13–7.17), we see that at infinity the first three moments are in agreement. We expect the first three moments to agree at infinity, while discrepancies should first appear at the octupole level. This is a reflection of the properties discussed in chapter 5 for the general Erez-Rosen solution of Eq. (3.6). More expansion terms in Eqs. (7.11–7.12) are required to verify this observation for the Curzon particle.

There is a distinct similarity between the Curzon GHH moments and those calculated in chapter 4. This is not a surprising fact. All possible axially symmetric fields in the Weyl class are representable by the general solution of Eq. (3.6). The charged Curzon particle is simply a specific case of the general solution. By correctly choosing the $\{c_n\}$ of Eq. (4.25) the Curzon multipole structure is realized. A simple comparison of the two sets of moments would indicate

$$c_0 = 1, \quad c_1 = 0, \quad c_2 = -\frac{5}{2}, \quad c_3 = 0, \quad c_4 = \frac{27}{8}. \tag{7.23}$$

There is an alternate method in which one could determine $\{c_n\}$ without directly calculating the GHH moments and then comparing them with the general set given in Eq. (4.25). It is well known that an arbitrary function can be expanded as a series in orthogonal functions. In this case, the problem is to solve

$$g(x_1, x_2) = \sum_{n=0}^{\infty} -c_n P_n(x_1) Q_n(x_2). \tag{7.24}$$

Both the Legendre function and the Legendre function of the second kind are orthogonal in their respective domains of $-1 \leq x_1 \leq 1$ and $x_2 \geq 1$. From Abramowitz and Stegun [30] the orthogonality relations are

$$\int_{-1}^{1} P_n(x_1)P_m(x_1)dx_1 = \frac{2\delta_m^n}{2n+1}, \tag{7.25}$$

and

$$\begin{aligned} \int_1^\infty Q_n(x_2)Q_m(x_2)dx_2 &= \frac{\delta_m^n}{2n+1}\sum_{k=0}^{\infty}(m+1+k)^{-2} \\ &= \frac{\delta_m^n}{2n+1}\left(\frac{\pi^2}{6} - N_n\right) \end{aligned} \tag{7.26}$$

where

$$N_n = \begin{cases} 0 & n = 0 \\ \sum_{k=1}^{n} k^{-2} & n \geq 1 \end{cases}$$

To determine $\{c_n\}$, we multiply both sides of Eq. (7.24) by $P_j(x_1)Q_k(x_2)$. Integrating over the domains of x_1 and x_2 yields

$$\begin{aligned} &\int_1^\infty \int_{-1}^{1} g(x_1,x_2)P_j(x_1)Q_k(x_2)dx_1\,dx_2 \\ &= \sum_{n=0}^{\infty} -c_n \int_{-1}^{1} P_n(x_1)P_j(x_1)dx_1 \int_1^\infty Q_n(x_2)Q_k(x_2)dx_2. \end{aligned} \tag{7.27}$$

Using the orthogonality relations, we find

$$\int_1^\infty \int_{-1}^{1} g(x_1,x_2)P_j(x_1)Q_k(x_2)dx_1\,dx_2 = -c_n \frac{2\delta_n^j}{(2j+1)}\frac{2\delta_n^k}{(2k+1)}\left(\frac{\pi^2}{6} - N_k\right). \tag{7.28}$$

Thus the coefficient c_k can be found through the formula

$$c_k = -\frac{(2k+1)^2}{2\left(\frac{\pi^2}{6} - N_k\right)} \int_1^\infty \int_{-1}^{1} g(x_1,x_2)P_k(x_1)Q_k(x_2)dx_1\,dx_2. \tag{7.29}$$

The Curzon particle is represented in the (λ, μ) coordinate system by

$$\frac{1}{2}\ln f = -\frac{\sqrt{m^2 - q_c^2}}{\sqrt{\lambda^2 + \mu^2 - 1}}. \tag{7.30}$$

The required transformation between z, ρ and λ, μ to obtain Eq. (7.30) is

$$z = l\lambda\mu,$$

$$\rho^2 = l^2\left(\lambda^2 - 1\right)\left(1 - \mu^2\right) \tag{7.31}$$

where $l = \sqrt{m^2 - q_c^2}$. To obtain $\{c_n\}$, we set $g(\lambda, \mu)$ equal to the right hand side of Eq. (7.30) and evaluate Eq. (7.29). Numerical integration of the first 5 coefficients revealed exactly the same results as shown in Eq. (7.23), as expected.

7.3 The Newtonian Limit

It is most convenient to use the physically intuitive method of Cooperstock [11] in order to determine the Newtonian limit of the Curzon particle, instead of Ehlers' mathematical definition. To begin with, we will examine the uncharged Curzon solution and then extend it to the charged case. Cooperstock uses the well known fact that in the weak field limit

$$g_{00} = 1 + 2\,f(r, \theta) \tag{7.32}$$

where $f(r,\theta)$ is an asymptotic expansion in r. The Newtonian potential is realized when the condition of a nearly Newtonian source is applied to $f(r,\theta)$; i.e.

$$\frac{m}{L} \ll 1 \tag{7.33}$$

in geometerized units.

Taking Eq. (7.11) and setting $q_c = 0$, we find

$$g_{00} = e^{w} = 1+2\left\{-\frac{m}{r}+\frac{m^3 P_2(\cos\theta)}{3r^3}\left(1+\frac{m}{r}+\cdots\right)-\frac{3m^5 P_4(\cos\theta)}{35r^5}+\cdots\right\} \tag{7.34}$$

However, we know from chapter 6 that the Newtonian potential has the general form

$$\Phi_N = -m\sum_{n=0}^{\infty} d_n \frac{n!L^n}{(2n+1)!!}\frac{P_n(\cos\theta)}{r^{n+1}}. \tag{7.35}$$

We know that the "cross terms" of Eq. (7.34) do not satisfy Eq. (7.33) and are thus negligible in the Newtonian limit. However, if we compare the $n = 2$ terms of Eq. (7.34) and Eq. (7.35), we find

$$\frac{m^3}{3} = d_2\frac{2}{15}mL^2 \tag{7.36}$$

Since $d_n \sim$ O(1) by definition, it implies

$$L \approx \sqrt{\frac{5}{2}}\, m = 1.58\, m. \tag{7.37}$$

The $n = 4$ term reveals

$$L \approx \sqrt[4]{\frac{27}{8}}\, m = 1.35\, m. \tag{7.38}$$

The comparison of the first few terms of the asymptotic expansion of Eqs. (7.34) and (7.35) shows that the characteristic size L of the Curzon particle is of the order m. These terms clearly do not satisfy Eq. (7.33). Hence one must conclude that they are negligible in the Newtonian limit. This pattern must continue for the higher moments, since the the series representation for g_{00} must converge. Hence the gravitational field for the uncharged Curzon particle reduces to a monopole in the Newtonian limit.

When considering the Newtonian limit of the gravitational field in a charged spacetime, it was suggested that any radial dependence involving charge be absorbed into the mass density. This would leave Eq. (7.11) in the form

$$e^w = 1+2\left\{-\frac{m}{r}+\frac{m\,(m^2-q_c^2)\,P_2(\cos\theta)}{3r^3}-\frac{3m\,(m^2-q_c^2)^2\,P_4(\cos\theta)}{35r^5}+\cdots\right\} \quad (7.39)$$

The characteristic size of the charged Curzon particle is

$$L \sim \sqrt{m^2-q_c^2}. \quad (7.40)$$

The analogy of the "nearly Newtonian" source for a charged spacetime implied in chapter 6 is

$$\frac{\sqrt{m^2-q_c^2}}{L} \ll 1. \quad (7.41)$$

Hence the charged Curzon particle also reduces to a monopole solution in the Newtonian limit, but with different mass density than the uncharged case.

The electrostatic potential of Eq. (7.12) has a completely analogous argument to the mass potential. Absorbing any radially dependent mass terms into the charge density, Eq. (7.12) becomes

$$\Phi = \frac{q_c}{r} - \frac{q_c}{3r^3}\left(m^2 - q_c^2\right) P_2(\cos\theta) + \cdots . \tag{7.42}$$

Recall the characteristic size of the electric dipole moment is $q_c L^2$. Thus, from Eq. (7.42)

$$L \sim \sqrt{m^2 - q_c^2}. \tag{7.43}$$

This does not satisfy the condition of Eq. (7.41). Hence the electrostatic potential of the charged Curzon particle also reduces to a monopole in the classical limit, as expected.

Chapter 8

Summary and Conclusions

This thesis studied multipole moments for the Weyl class of axially-symmetric, static electrovacuum. Two methods were employed in the study. First, the Erez-Rosen formalism for constructing gravitational fields from mass-only multipoles was generalized to describe both the gravitational and electric fields of a charged object. Multipole moments for the same solution to the EFEs were also calculated through the GHH covariant formalism.

It was established that the GHH method provided the physical basis for defining multipole moments. This stemmed from the fact that the defining equation is a tensor equation and that the moments exhibit desired properties shifts in the origin. It was seen that the GHH moments displayed the expected symmetry between the charge and mass moments. One could also see the interdependence of charge on the mass moments and vice versa. This phenomenon is not realized in classical electrostatics within the framework of Newtonian theory, where the gravitational field and electromagnetic field are completely independent of each other. The major drawback to the GHH moments is that the moments are defined "at infinity". As a consequence, one runs into problems with measuring the moments since measurements must be made at a finite distance away from the source.

It was proposed that the Erez-Rosen formalism could be used in conjunction with the GHH formalism in order to experimentally determine the multipole structure of a system. It is the coordinate dependent aspect of the Erez-Rosen formalism which allows one to determine $\{c_n\}$ through geodesic calculations. The method would entail mapping the motion of a satellite to geodesics with $\{c_n\}$ as parameters. Once the $\{c_n\}$'s are known, the GHH formalism would determine the physical multipole structure.

When considering the Newtonian limit of the Erez-Rosen uncharged solution, it was shown that the employment of Ehlers' definition of a Newtonian limit is insufficient. Cooperstock's consideration of the source itself is required in order to correctly identify the Newtonian moments. We then extend this procedure to include the charged Erez-Rosen solution.

The charged Curzon particle was studied as an application of the formalisms presented in this thesis. It was shown that the covariant moments of the charged Curzon particle is simply a specific case of the general solution given in chapter 4.

By expanding any axially-symmetric solution to Laplace's equation in terms of orthogonal Legendre functions, the coefficients $\{c_n\}$ can be determined. Once these are known, the multipole structure is found from Eq. (4.25). It was found that both the Curzon electrostatic and gravitational multipole structure reduced to monopole fields in the Newtonian limit.

In future work, it is hoped that the Weyl-class restriction can be eliminated so that one can explore systems where gravitational and electromagnetic fields are not functionally related. This could possibly break the symmetry one sees in the GHH moments and therefore one could model spacetimes with independent mass and charge multipole structure.

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