MAXIMIZING THE SPECTRAL RADIUS OF FIXED TRACE DIAGONAL PERTURBATIONS OF NONNEGATIVE MATRICES

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DMS-684-IR October 1994

October 3, 1994

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[*] The work of this author was supported in part by National Science Foundation grant DMS 92-00899 and Office of Naval Research contract N00014-90-J-1739.

[**] This research was supported by the fund for the promotion of research at the Technion.

[†] The research of these authors was supported by grant no. 90-00471 from the United States - Israel Binational Science Foundation (BSF), Jerusalem, Israel.

[††] This research was partially supported by NSERC grant A-8214 and the University of Victoria Committee on Faculty Research and Travel.

[†††] This research was partially supported by NSERC grant A-8965 and the University of Victoria Committee on Faculty Research and Travel.

Abstract

Let A be an n-by-n irreducible, entry-wise nonnegative matrix. For a given $t > 0$, we consider the problem of maximizing the Perron root of a nonnegative, diagonal, trace t perturbation of A. Because of the convexity of the Perron root as a function of diagonal entries, the maximum occurs for some tE_{ii}. Such an index i, which is called a winner, may depend on t. We show how to determine the (nonempty) set of indices i that are winners for all sufficiently small t and the possibly different (nonempty) set of indices that are winners for all sufficiently large t. We also show how to determine if there are indices that are winners for all t.

1. INTRODUCTION

Let $A \equiv [a_{ij}] \geq 0$ be an n-by-n irreducible, entry-wise nonnegative matrix, and let $t > 0$ be given. We consider the problem, in contrast to [JOSvD], of determining

(1) $$\mu(A) \equiv \max_D \{\rho(A+D) \colon D \geq 0 \text{ is diagonal, trace } D = t\},$$

in which ρ denotes the spectral radius (Perron root). If $D \equiv \operatorname{diag}(d_{ii}) \geq 0$ has trace t, then

$$A + D = \sum_{i=1}^{n} \frac{d_{ii}}{t} (A + tE_{ii}),$$

in which E_{ii} denotes the n-by-n matrix whose (i,i) entry is 1 while all other entries are 0. Since the spectral radius of a nonnegative matrix is a convex function of its diagonal entries (see, e.g., [Co 81]),

$$\rho(A+D) = \rho\left(\sum_{i=1}^{n} \frac{d_{ii}}{t} (A + tE_{ii})\right) \leq \sum_{i=1}^{n} \frac{d_{ii}}{t} \rho(A + tE_{ii}),$$

and thus

$$\rho(A+D) \leq \max_{1 \leq i \leq n} \rho(A + tE_{ii}).$$

Consequently,

$$\mu(A) = \max_{1 \leq i \leq n} \rho(A + tE_{ii});$$

that is, the maximum in (1) is attained by some diagonal matrix tE_{ii} having precisely one nonzero diagonal entry.

Because of the above characterization of $\mu(A)$, it is of interest to investigate the function

$$\mu(A;t) \equiv \max_{1 \le i \le n} \rho\left(A + tE_{ii}\right) \qquad \forall t \ge 0.$$

We let

$$\rho_i(t) \equiv \rho\left(A + tE_{ii}\right), \text{ for } 1 \le i \le n \text{ and } \forall t \ge 0,$$

and note, by the Perron-Frobenius theory, that $\rho_i(t)$ is a strictly monotonically increasing function. Since $\rho(A + tI) = \rho(A) + t$, then by monotonicity of the Perron root $\rho_i(t) < \rho(A) + t$. It is important to emphasize the dependence of $\mu(A;t)$ on t, as the indices i at which it is attained may depend upon t, as we shall see.

In Sections 2 and 3, respectively, we consider the behavior of $\rho_i(t)$ for large values of t and for values of t near 0. Our analysis uses part of Theorem 3.12 of [HRR 92], which (for our purposes) can be summarized as follows. We denote by $A^{\#}$ the group inverse of A, which when it exists satisfies $AA^{\#}A = A$, $A^{\#}AA^{\#} = A^{\#}$ and $AA^{\#} = A^{\#}A$; see [CM, p. 124].

THEOREM 1.1 [HRR 92]. *Let* F *and* G *be* $n \times n$ *real matrices, let* $H = F - \lambda I_n$ *where* λ *is a simple eigenvalue of* F, *let* u *and* v *be left and right eigenvectors, respectively, of* F *corresponding to* λ *and normalized so that* $u^T v = 1$. *Then for* $\delta > 0$

sufficiently small, an eigenvalue of $F + \delta G$ *is given by* $\lambda + \sum_{k=1}^{\infty} \lambda_k \delta^k$, *with corresponding right eigenvector given by* $v + \sum_{k=1}^{\infty} w_k \delta^k$. *Here* $\lambda_1 = u^T Gv$, $w_1 = -H^{\#} Gv$, *and for* $k = 2, 3, \ldots$

$$\lambda_k = u^T G w_{k-1}, \quad \text{and} \quad w_k = \sum_{j=1}^{k-2} \left(u^T G w_{k-j-1}\right) H^{\#} w_j + H^{\#}\left(\lambda_1 I_n - G\right) w_{k-1}. \quad \square$$

In Section 4, we compare arbitrary distinct pairs of functions $\rho_i(t)$ and $\rho_j(t)$, considering, for example, the case that $\rho_i(t) \equiv \rho_j(t)$ for all $t \geq 0$, and, when $\rho_i(t) \not\equiv \rho_j(t)$, the number of possible values t_0 for which $\rho_i(t_0) = \rho_j(t_0)$. Finally, in Section 5 we discuss the case in which $\rho_i(t) > \rho_j(t)$ for all $j \neq i$ and for all $t > 0$.

We first illustrate these concepts with the following simple example.

Example 1.2. Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{bmatrix} \geq 0.$$

Then

$$\rho_1(t) = \frac{a_{11} + t + \sqrt{(a_{11} + t)^2 + 4a_{12}a_{21}}}{2}$$

and

$$\rho_2(t) = \frac{a_{11} + t + \sqrt{(a_{11} + t)^2 + 4a_{12}a_{21} - 4a_{11}t}}{2}.$$

Thus, if $a_{11} = 0$, then $\mu(A;t) = \rho_1(t) = \rho_2(t)$ for all $t \geq 0$. However, if $a_{11} > 0$, then $\mu(A;t) = \rho_1(t)$ for all $t > 0$. □

We introduce the following terminology to describe the situations illustrated in the above example.

Definitions 1.3

If $\rho_i(t) \equiv \rho_j(t)$ for all $t \geq 0$, then i and j *tie*. Let $0 \leq t_0 < t_1$. If, for some i and j, $\rho_i(t) > \rho_j(t)$ for all $t \in (t_0, t_1)$, then i *dominates* j (for $t_0 < t < t_1$). If there is a $t_1 > 0$ such that i dominates each index (for $0 < t < t_1$) with which it is not tied, then i is called an *initial winner*. Similarly, if i dominates each index with which it is not tied (for all $t >$ some t_1), then i is called a *terminal winner*. Finally, if i dominates all indices with which it is not tied for all $t > 0$, then i is called a *universal winner*. There may be no universal winner, but there is always at least one initial winner and at least one terminal winner (see Theorem 4.2).

Let $0 < t_0 < t_1 < t_2$. If $\rho_i(t) < \rho_j(t)$ for all $t \in (t_0, t_1)$, $\rho_i(t_1) = \rho_j(t_1)$ and

$\rho_i(t) > \rho_j(t)$ for all $t \in (t_1, t_2)$, then the functions $\rho_i(t)$ and $\rho_j(t)$ are said to *switch* (or to have a *switch*) at t_1. □

With regard to Example 1.2, if $a_{11} = 0$, then 1 and 2 tie, whereas if $a_{11} > 0$, then 1 dominates 2 for all $t > 0$ and so index 1 is the universal winner. The functions $\rho_1(t)$ and $\rho_2(t)$ have no switches (regardless of the value of $a_{11} \geq 0$).

Finally, we note that the above concepts are invariant under multiplication of A by a positive scalar, under translation of A by a scalar matrix (so that they may be extended to the right-most eigenvalue of an essentially nonnegative matrix), and under diagonal similarity of A (so that they may be extended to matrices with nonnegative cycle products). Also, if A is reducible, then $\mu(A;t) = \max_{j=1,\ldots,k} \mu(A_{jj};t)$ where A_{jj} are the irreducible diagonal blocks of the Frobenius normal form of A, and our results can be applied to A_{jj}.

2. LARGE t: DETERMINATION OF THE TERMINAL WINNER(S).

In order to analyze $\rho_i(t) = \rho(A + tE_{ii})$ for large values of t and any fixed index i, we let $\varepsilon = 1/t$ and consider $\rho(E_{ii} + \varepsilon A)$, since $A + tE_{ii} = t(E_{ii} + \varepsilon A)$. We denote the (i, i) entry of A^ℓ by $a_{ii}^{(\ell)}$, where ℓ is any fixed integer, but usually write a_{ii} for $a_{ii}^{(1)}$.

For a positive integer m, a *partition* ϕ of m is a sequence of integers $\phi = (r_1, r_2, \ldots, r_s)$ such that $r_1 \geq r_2 \geq \ldots \geq r_s > 0$ and $\sum_{j=1}^{s} r_j = m$. The *principal partition* of m (i.e. the one with $s = 1$) is the partition (m). If π_m denotes the set of all partitions of m, then the *i-th diagonal product of A* corresponding to a partition $\phi = (r_1, r_2, \ldots, r_s) \in \pi_m$ is defined to be

$$m_{\phi, i} = a_{ii}^{(r_1)} \, a_{ii}^{(r_2)} \ldots a_{ii}^{(r_s)}.$$

For $m = 0$, the unique partition of m is defined to be $\phi = (0)$, and we let $\pi_0 = \{(0)\}$.

For sufficiently small $\varepsilon > 0$, we are interested in $\rho(E_{ii} + \varepsilon A)$, which is a perturbation of the simple eigenvalue $\lambda = 1$ of E_{ii}. The standard i-th unit vector is written $e^{(i)}$, so that $E_{ii} = e^{(i)} e^{(i)T}$.

THEOREM 2.1. *Let $A \geq 0$ be an n-by-n irreducible matrix. Then for sufficiently small $\varepsilon > 0$,*

$$\rho(E_{ii} + \varepsilon A) = 1 + \sum_{k=1}^{\infty} \lambda_{k,i} \, \varepsilon^k,$$

where $\lambda_{1,i} = a_{ii}$, $w_{1,i} = -(E_{ii} - I_n)Ae^{(i)}$, and for $k \geq 2$, $\lambda_{k,i} = e^{(i)T} A w_{k-1,i}$ and

$$w_{k,i} = \sum_{j=1}^{k-2} \left(e^{(i)T} A w_{k-j-1,i}\right)\left(E_{ii} - I_n\right) w_{j,i} + \left(E_{ii} - I_n\right)\left(a_{ii} I_n - A\right) w_{k-1,i}.$$

Moreover, $\lambda_{k,i}$ *is a linear combination of the form* $\sum_{\phi \in \pi_k} c_{\phi,i}\, m_{\phi,i}$, *the coefficient* $c_{\phi_0,i}$ *corresponding to the principal partition* $\phi_0 = (k)$ *is equal to* 1, *and for any* $\phi \in \pi_k$, *the real number* $c_{\phi,i}$ *depends only on* ϕ, *and not on* A *or* i.

Proof. Let $1 \le i \le n$. We use Theorem 1.1 with $F = E_{ii}$, $\lambda = 1$, $G = A$, $u = v = e^{(i)}$, $\delta = \varepsilon$ and we denote λ_k, w_k by $\lambda_{k,i}$, $w_{k,i}$ to show the dependence on i. With this notation, clearly $\lambda_{1,i} = a_{ii}$ and $w_{1,i} = -\left(E_{ii} - I_n\right) A e^{(i)}$ since $\left(E_{ii} - I_n\right)^{\#} = E_{ii} - I_n$. Similarly $\lambda_{k,i}$ and $w_{k,i}$ for $k \ge 2$ are as specified.

We now claim that for integers $k \ge 1$ and $p \ge 1$,

$$e^{(i)T} A^p\, w_{k,i} = \sum_{\phi \in \pi_{p+k}} c_{\phi,i}\, m_{\phi,i}. \tag{2}$$

with the coefficients $c_{\phi,i}$ having the properties stated in the theorem. We use proof by induction on k. For $k = 1$,

$$\begin{aligned} e^{(i)T} A^p w_{1,i} &= -e^{(i)T} A^p \left(E_{ii} - I_n\right) A e^{(i)} \\ &= a_{ii}^{(p+1)} - a_{ii} a_{ii}^{(p)}, \end{aligned}$$

which is in the form (2) with the coefficient corresponding to the principal partition equal to 1. Thus, the claim is true for $k = 1$.

We assume the claim to be true up to $k-1$, and prove it for k. We use the fact that

$$e^{(i)T} w_{\ell, i} = 0, \qquad \ell = 1, 2, \ldots$$

which follows easily from $e^{(i)T}(E_{ii} - I_n) = 0$. Thus from the expression for $w_{k,i}$, $k \geq 2$,

$$\begin{aligned}
e^{(i)T} A^p w_{k,i} &= \sum_{j=1}^{k-2} \left(e^{(i)T} A w_{k-j-1,i}\right)\left[e^{(i)T} A^p e^{(i)} e^{(i)T} w_{j,i} - e^{(i)T} A^p w_{j,i}\right] \\
&\quad + a_{ii}\, e^{(i)T} A^p e^{(i)} e^{(i)T} w_{k-1,i} - e^{(i)T} A^p e^{(i)} e^{(i)T} A w_{k-1,i} \\
&\quad - a_{ii}\left(e^{(i)T} A^p w_{k-1,i}\right) + e^{(i)T} A^{p+1} w_{k-1,i} \\
&= -\sum_{j=1}^{k-2} \left[\sum_{\phi \in \pi_{k-j}} c_{\phi,i}\, m_{\phi,i}\right]\left[\sum_{\phi \in \pi_{p-j}} c_{\phi,i}\, m_{\phi,i}\right] \\
&\quad - a_{ii}^{(p)} \sum_{\phi \in \pi_k} c_{\phi,i}\, m_{\phi,i} - a_{ii} \sum_{\phi \in \pi_{p+k-1}} c_{\phi,i}\, m_{\phi,i} \\
&\quad + \sum_{\phi \in \pi_{p+k}} c_{\phi,i}\, m_{\phi,i}.
\end{aligned}$$

Each term in this summation is in the form (2). Moreover, the contribution to the principal partition $(p+k)$ is obtained only from the last summand and the induction hypothesis shows that the coefficient of $a_{ii}^{(p+k)}$ is exactly 1. The induction hypothesis also shows that each coefficient $c_{\phi,i}$ is independent of A and i. Thus, the claim is proved.

Taking $p = 1$ and using $\lambda_{k,i} = e^{(i)T} A w_{k-1,i}$, then (2) gives the result that for $k \geq 2$

$$\lambda_{k,i} = a_{ii}^{(k)} + \sum_{\substack{\phi \in \pi_k \\ \phi \neq (k)}} c_{\phi,i}\, m_{\phi,i}. \quad \square$$

Using this theorem recursively, we record the first few terms in the expansion of $\rho(E_{ii} + \varepsilon A)$.

COROLLARY 2.2. *Let* $A \geq 0$ *be an* n-by-n *irreducible matrix. Then for sufficiently small* $\varepsilon > 0$,

$$\rho(E_{ii} + \varepsilon A) = 1 + a_{ii}\varepsilon + \left(a_{ii}^{(2)} - a_{ii}^{2}\right)\varepsilon^2 + \left(a_{ii}^{(3)} - 3a_{ii}a_{ii}^{(2)} + 2a_{ii}^{3}\right)\varepsilon^3$$

$$+ \left(a_{ii}^{(4)} - 4a_{ii}a_{ii}^{(3)} - 2\left(a_{ii}^{(2)}\right)^2 + 10a_{ii}^{2}a_{ii}^{(2)} - 5a_{ii}^{4}\right)\varepsilon^4 + \ldots \quad \square$$

We now use the expression for $\lambda_{k,i}$ in Theorem 2.1 to give an algorithm to find the indices that are terminal winners (S_{n-1} below) and thus to find $\mu(A;t)$ for sufficiently large values of t.

THEOREM 2.3. *Let* $A \geq 0$ *be an* n-by-n *irreducible matrix, and define* $S_k = \left\{s\colon\ a_{ss}^{(k)} = \max_{i \in S_{k-1}} \left\{a_{ii}^{(k)}\right\}\right\}$ *for* $1 \leq k \leq n-1$, *where* $S_0 = \{1, \ldots, n\}$. *If* k *is the smallest integer such that* $|S_k| = 1$, *then for all sufficiently large* $t > 0$, $\mu(A;t)$ *occurs at* *the index* $s \in S_k$, i.e., *index* s *dominates* j *for* $j \neq s$. *The elements of* S_{n-1} *are the terminal winners.*

Proof. As remarked at the beginning of this section,

$\mu(A;t) = \max_{1 \le i \le n} \rho_i(t) = t \max_{1 \le i \le n} \rho(E_{ii} + \varepsilon A)$. Thus, for sufficiently large t (sufficiently small ε) if $|S_1| = 1$ then $\mu(A;t)$ occurs at the index $s \in S_1$ since $\lambda_{1,i} = a_{ii}$. In this case, s dominates all other indices for sufficiently large t. Alternatively, if $|S_1| > 1$, then further terms in the expansion must be considered. From Corollary 2.2, $\lambda_{2,i} = a_{ii}^{(2)} - a_{ii}^2$. But a_{ii}^2 is equal for all $i \in S_1$, thus differences in $\lambda_{2,i}$ depend only on differences in $a_{ii}^{(2)}$. So if $|S_1| > 1$ and $|S_2| = 1$, then $\mu(A;t)$ occurs at the index $s \in S_2$. From the form of $\lambda_{k,i}$ in Theorem 2.1, it is clear that the term corresponding to the principal partition gives the only possible difference at each stage, hence the definition of S_k. The procedure terminates with S_{n-1}, since by the Cayley-Hamilton theorem A^n is a linear combination of lower powers. If none of the S_k have cardinality 1, then all elements of S_{n-1} tie for sufficiently large t. □

If A has a unique largest diagonal entry (say a_{ii}), then $|S_1| = 1$ and i is the terminal winner. If $|S_1| > 1$, then diagonal entries of higher powers of A must be considered.

Example 2.4. Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 10 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that all $a_{ii} = 0$. The result of Theorem 2.3 shows that 2 is the terminal winner as it dominates all other indices for sufficiently large values of t. This can be easily seen from A^2 by noting that

$$S_2 = \{2\} = \left\{ s\colon \; a_{ss}^{(2)} = \max_{i \in \{1,\dots,5\}} \left\{ a_{ii}^{(2)} \right\} \right\},$$

or directly by observing that $\lambda_{2,i} = \sum_{j \neq i} a_{ij}\, a_{ji}$ for $1 \le i \le 5$. (We will see in Section 3 that index 1 dominates all other indices for sufficiently small t. Thus the functions $\rho_1(t)$ and $\rho_2(t)$ switch at some value of $t > 0$). □

3. SMALL t: DETERMINATION OF THE INITIAL WINNER(S).

To investigate $\rho_i(t)$ for small values of t, we note that $\rho(A + tE_{ii}) = \rho(A) + \max_{\lambda \in \sigma(-Q + tE_{ii})} \text{Re}\, \lambda$ in which $Q = \rho(A)I - A$, so that Q is an n-by-n singular, irreducible M-matrix. We denote the (i,i) entry of $(Q^{\#})^{\ell}$ by $q_{ii}^{\#(\ell)}$ for $\ell \geq 1$. The ith diagonal product of $Q^{\#}$ corresponding to a partition $\phi = (r_1, r_2, ..., r_s) \in \pi_m$ is defined to be

$$z_{\phi,i} = q_{ii}^{\#(r_1)} q_{ii}^{\#(r_2)} \cdots q_{ii}^{\#(r_s)}.$$

We use Theorem 1.1 to obtain the following result.

THEOREM 3.1. *Let $A \geq 0$ be an n-by-n irreducible matrix, $Q = \rho(A)I - A$, and u and v, respectively, be left and right Perron vectors of A with $u^T v = 1$. Then for fixed i, $1 \leq i \leq n$, and sufficiently small $t > 0$*

$$\rho(A + tE_{ii}) = \rho(A) + \sum_{k=1}^{\infty} \lambda_{k,i}\, t^k,$$

in which $\lambda_{1,i} = u_i v_i = \alpha_i$, $w_{1,i} = Q^{\#} E_{ii} v$, and for $k \geq 2$, $\lambda_{k,i} = u^T E_{ii} w_{k-1,i}$ and

$$w_{k,i} = \sum_{j=1}^{k-2} \left(u^T E_{ii} w_{k-j-1,i}\right)\left(-Q^{\#}\right) w_{j,i} - Q^{\#}\left(\alpha_i I_n - E_{ii}\right) w_{k-1,i}.$$

Moreover, for any $k \geq 1$, $\lambda_{k,i}$ *can be written in the form* $\sum_{\phi \in \pi_{k-1}} d_{\phi,i}\, z_{\phi,i}$. *Here, if* $k = 1$, *define* $d_{\phi,i} = \alpha_i$ *and* $z_{\phi,i} = 1$ *for the unique partition* $\phi = (0)$ *of* 0. *If* $k \geq 2$, *each* $d_{\phi,i}$ *is a polynomial in* α_i *with real coefficients that depend only on* ϕ, *and not on* $Q^{\#}$, u, v *or* i; *in particular, the coefficient corresponding to the principal partition* $\phi_0 = (k-1)$ *of* π_{k-1} *is equal to* $(-1)^k \alpha_i^{k-1}$.

Proof. Let $1 \leq i \leq n$. We use Theorem 1.1 with $F = A$, $\lambda = \rho(A)$, $G = E_{ii}$ and $\delta = t$. The expressions for $\lambda_{k,i}$, $w_{k,i}$ follow immediately. For the case $k = 1$, $\lambda_{1,i} = u_i v_i = \alpha_i$, by definition. We now claim that for integers $k \geq 2$

(3) $$\lambda_{k,i} = u^T E_{ii} w_{k-1,i} = \sum_{\phi \in \pi_{k-1}} d_{\phi,i}\, z_{\phi,i},$$

and for integers $k \geq 1$, $p \geq 1$

(4) $$u^T E_{ii} (Q^{\#})^p\, w_{k,i} = \sum_{\phi \in \pi_{k+p}} \tilde{d}_{\phi,i}\, z_{\phi,i},$$

where the $d_{\phi,i}$ in (3) satisfy the conditions in the theorem statement, and the $\tilde{d}_{\phi,i}$ in (4) satisfy similar properties. Moreover, in the right side of (4), the coefficient of the principal partition ϕ_0 of π_{k+p} is $(-1)^{k+1} \alpha_i^k$. Also

$$u^T E_{ii}(Q^\#)^p\ w_{1,i} = u^T E_{ii}(Q^\#)^p\ Q^\# E_{ii} v = u^T e^{(i)} e^{(i)T} (Q^\#)^{p+1} e^{(i)} e^{(i)T} v = u_i v_i q_{ii}^{\#(p+1)},$$

which is (4) for $k = 1$. For $k = 2$, $\lambda_{2,i} = u^T E_{ii} w_{1,i} = u^T E_{ii} Q^\# E_{ii} v = u_i v_i q_{ii}^\#$, which is (3) for $k = 2$. So suppose claims (3), (4) are true up to $k - 1$. Consider the case k for claim (3). Then

$$\lambda_{k,i} = u^T E_{ii} w_{k-1,i}$$

$$= u^T E_{ii}\left[-\sum_{j=1}^{k-3} \left(u^T E_{ii} w_{k-j-2,i}\right) Q^\# w_{j,i} - Q^\#\left(\alpha_i I_n - E_{ii}\right) w_{k-2,i}\right]$$

$$= -\sum_{j=1}^{k-3} \left(u^T E_{ii} w_{k-j-2,i}\right)\left(u^T E_{ii} Q^\# w_{j,i}\right) - u^T E_{ii}\ Q^\# \alpha_i I_n w_{k-2,i}$$

$$+\ u^T E_{ii}\ Q^\# E_{ii} w_{k-2,i}$$

$$= -\sum_{j=1}^{k-3} \left(u^T E_{ii} w_{k-j-2,i}\right)\left(u^T E_{ii} Q^\# w_{j,i}\right) - \alpha_i u^T E_{ii} Q^\# w_{k-2,i} + q_{ii}^\#\left(u^T E_{ii} w_{k-2,i}\right).$$

Using the induction hypotheses for claims (3), (4)

$$\lambda_{k,i} = -\sum_{j=1}^{k-3} \left(\sum_{\phi \in \pi_{k-j-2}} d_{\phi,i} z_{\phi,i}\right)\left(\sum_{\phi \in \pi_{j+1}} \tilde{d}_{\phi,i} z_{\phi,i}\right) - \alpha_i\left(\sum_{\phi \in \pi_{k-2+1}} \tilde{d}_{\phi,i} z_{\phi,i}\right) + q_{ii}^\#\left(\sum_{\phi \in \pi_{k-2}} d_{\phi,i} z_{\phi,i}\right).$$

This is in the form needed, thus completing the proof for claim (3). For claim (4), consider case k,

$$u^T E_{ii}(Q^{\#})^p \, w_{k,i}$$

$$= u^T E_{ii}(Q^{\#})^p \left[-\sum_{j=1}^{k-2} \left(u^T E_{ii} w_{k-j-1,i}\right) Q^{\#} w_{j,i} - Q^{\#}\left(\alpha_i I_n - E_{ii}\right) w_{k-1,i} \right]$$

$$= -\sum_{j=1}^{k-2} \left(u^T E_{ii} w_{k-j-1,i}\right)\left(u^T E_{ii}(Q^{\#})^{p+1} w_{j,i}\right) - \alpha_i u^T E_{ii}(Q^{\#})^{p+1} w_{k-1,i} + u^T E_{ii}(Q^{\#})^{p+1} E_{ii} w_{k-1,i}$$

$$= -\sum_{j=1}^{k-2} \left(u^T E_{ii} w_{k-j-1,i}\right)\left(u^T E_{ii}(Q^{\#})^{p+1} w_{j,i}\right) - \alpha_i u^T E_{ii}(Q^{\#})^{p+1} w_{k-1,i} + q_{ii}^{\#(p+1)}\left(u^T E_{ii} w_{k-1,i}\right).$$

Now using the above form for $\lambda_{k,i}$ and induction hypotheses for claims (3) and (4)

$$u^T E_{ii}(Q^{\#})^p w_{k,i} = -\sum_{j=1}^{k-2} \left(\sum_{\phi \in \pi_{k-j-1}} d_{\phi,i} z_{\phi,i}\right)\left(\sum_{\phi \in \pi_{p-j+1}} \tilde{d}_{\phi,i} z_{\phi,i}\right) - \alpha_i\left(\sum_{\phi \in \pi_{k-1+p+1}} \tilde{d}_{\phi,i} z_{\phi,i}\right)$$

$$+ q_{ii}^{\#(p+1)}\left(\sum_{\phi \in \pi_{k-1}} d_{\phi,i} z_{\phi,i}\right).$$

The only term that contains the principal partition is the second one. Using the induction hypothesis, $\tilde{d}_{\phi_0,i} = -\alpha_i\left((-1)^{(k-1)+1} \alpha_i^{k-1}\right) = (-1)^{k+1} \alpha_i^k$. This completes the proof for claim (4).

From the form of $\lambda_{k,i}$ in this proof, the only term that contains the principal partition is the second one. Thus using the induction hypothesis and the properties of $\tilde{d}_{\phi_0,i}$, it can be seen that the coefficient corresponding to the principal partition is

$$-\alpha_i\left((-1)^{k-1} \alpha_i^{k-2}\right) = (-1)^k \alpha_i^{k-1}, \quad \text{for} \quad k \geq 2,$$

as desired. Thus

$$\lambda_{k,i} = (-1)^k \alpha_i^{k-1} q_{ii}^{\#(k-1)} + \sum_{\substack{\phi \in \pi_{k-1} \\ \phi \neq (k-1)}} d_{\phi,i} z_{\phi,i}. \quad \square$$

We note that the expression for $\lambda_{1,i}$ is well known in the literature, see, e.g. [EJN 82], and the term $\lambda_{2,i}$ is given in [DN 84] and [HRR 92].

The above theorem leads to an algorithm to find $\mu(A;t)$ for sufficiently small t.

THEOREM 3.2. *Let* $A \geq 0$ *be an* n-by-n *irreducible matrix and* u *and* v, *respectively, be left and right Perron vectors of* A *with* $u^T v = 1$. *Define*

$$T_k = \left\{ s: \ (-1)^k q_{ss}^{\#(k-1)} = \max_{i \in T_{k-1}} \left\{ (-1)^k q_{ii}^{\#(k-1)} \right\} \right\} \text{ for } 2 \leq k \leq n, \text{ with } T_0 = \{1, \ldots, n\} \text{ and}$$

$$T_1 = \left\{ s: \ u_s v_s = \max_{i \in T_0} \{u_i v_i\} \right\}.$$

If k *is the smallest integer such that* $|T_k| = 1$, *then for all sufficiently small* $t > 0$, $\mu(A;t)$ *occurs at the index* $s \in T_k$, *i.e.,* s *dominates* j *for* $j \neq s$. *The elements of* T_n *are the initial winners.*

Proof. $\mu(A,t) = \max_{1 \leq i \leq n} \rho_i(t)$, where for each i, $\rho_i(t)$ has the power series expansion given by Theorem 3.1. If $|T_1| = 1$, then $\alpha_s = \max_{i \in T_0} \{\alpha_i\}$, and s dominates as stated. If $|T_1| > 1$, then further terms in the expansion must be considered, noting that α_i are equal for all

$i \in T_1$. The results follow by arguments as in Theorem 2.3. □

It follows that the distinct entries in the Hadamard (entry-wise) product $u \circ v$ order the functions $\rho_i(t)$, $1 \le i \le n$, for small t. If $u_i v_i > u_j v_j$, then $\rho_i(t) > \rho_j(t)$ for sufficiently small $t > 0$. In particular, if $u \circ v$ has a unique largest component, then the corresponding index is the initial winner. Consider again Example 2.4. Then (to 4 decimal places) $\rho(A) = 2.3118$, and the Hadamard product of the left and right Perron vectors is

$$u \circ v = (0.3550,\ 0.0690,\ 0.0013,\ 0.2873,\ 0.2873)^T.$$

Thus, for sufficiently small $t > 0$, index 1 dominates each of $j = 2, 3, 4, 5$ and so is the initial winner; also indices 4 and 5 dominate index 2, which in turn dominates index 3. In fact, by Theorem 4.6, indices 4 and 5 tie.

4. COMPARISONS BETWEEN $\rho_i(t)$ AND $\rho_j(t)$

We now consider any pair of functions $\rho_i(t)$ and $\rho_j(t)$. When these two functions are not identical, we determine the maximum number of switches that they may have. Also, we give a characterization of the case when two such functions are identical for all $t \ge 0$, and give a characterization of the situation that all n of the functions $\rho_i(t)$, $1 \le i \le n$, are identical for all $t \ge 0$.

4.1 DISTINCT $\rho_i(t)$ AND $\rho_j(t)$

We begin with some additional definitions and notation. We denote the characteristic polynomial of $A + tE_{ii}$ by $p(A + tE_{ii};\ \lambda)$. For any n-by-n matrix B, let $E_k(B)$ denote the sum of all k-by-k principal minors of B, $1 \le k \le n$. We let $E_k(\mu_1, \mu_2, \ldots, \mu_n)$ denote the k-th elementary symmetric function of any scalars $\mu_1, \mu_2, \ldots, \mu_n$. Finally, for any $1 \le i \le n$, let $B(i)$ denote the $(n-1)$-by-$(n-1)$ matrix obtained from B be deleting row i and column i.

The proof of the following result is a straightforward computation.

LEMMA 4.1. *Let $A \ge 0$ be an n-by-n real matrix, let $1 \le i \le n$ and let t be a real scalar. Then the characteristic polynomial of $A + tE_{ii}$ is*

$$p(A + tE_{ii};\ \lambda) = \lambda^n - \big(E_1(A) + t\big)\lambda^{n-1} + \big(E_2(A) + tE_1(A(i))\big)\lambda^{n-2} - \big(E_3(A) + tE_2(A(i))\big)\lambda^{n-3} + \cdots + (-1)^n\big(E_n(A) + tE_{n-1}(A(i))\big).$$

In the following result, the maximum possible number of switches in the two distinct functions $\rho_i(t)$ and $\rho_j(t)$ is determined.

THEOREM 4.2. *Let $A \ge 0$ be an n-by-n irreducible matrix, let $1 \le i, j \le n$ with $i \ne j$ and suppose that $p(A + tE_{ii};\ \lambda)$ is not identically equal to $p(A + tE_{jj};\ \lambda)$ Then there exist at most $n-2$ positive values of t such that $\rho_i(t) = \rho_j(t)$.*

Proof. Suppose that $t_0 > 0$ and $\rho_i(t_0) = \rho_j(t_0)$. For simplicity, let $\rho_0 \equiv \rho_i(t_0)$. Thus $p(A + t_0 E_{ii};\ \rho_0) = 0$ and $p(A + t_0 E_{jj};\ \rho_0) = 0$. On subtracting these two equations and using Lemma 4.1, we obtain

$$t_0\Big\{\big(E_1(A(j)) - E_1(A(i))\big)\rho_0^{n-2} - \big(E_2(A(j)) - E_2A(i))\big)\rho_0^{n-3} + \\ \dots + (-1)^n\big(E_{n-1}(A(j)) - E_{n-1}(A(i))\big)\Big\} = 0.$$

It follows that ρ_0 is a root of the equation

$$(5) \qquad \big(E_1(A(j)) - E_1(A(i))\big)\lambda^{n-2} - \big(E_2(A(j)) - E_2(A(i))\big)\lambda^{n-3} + \dots \\ + (-1)^n\big(E_{n-1}(A(j)) - E_{n-1}(A(i))\big) = 0.$$

The polynomial in (5) is not identically 0 since $p(A + tE_{ii};\ \lambda) \neq p(A + tE_{jj};\ \lambda)$ and thus equation (5) has at most $n-2$ roots. In order to complete the proof as well as to establish in Example 4.3 the sharpness of the bound $n - 2$, we prove the following claim.

If $\rho_0 > \rho(A)$ *and* ρ_0 *is a root of* (5), *then there exists a unique* $t_0 > 0$, *such that* $\rho_0 = \rho_i(t_0) = \rho_j(t_0)$.

To prove this claim, consider $\rho_i(t)$ for $t \geq 0$. We have $\rho_i(0) = \rho(A)$ and $\rho_i(t) \to \infty$ as $t \to \infty$, since $\rho_i(t) \geq a_{ii} + t \geq t$. Since $\rho_i(t)$ is continuous, there exists

$t > 0$ such that $\rho_0 = \rho_i(t_0)$. Hence $p(A + t_0 E_{ii};\ \rho_0) = 0$ and, since ρ_0 satisfies (5), we also have $p(A + t_0 E_{jj};\ \rho_0) = 0$. Thus ρ_0 is an eigenvalue of $A + t_0 E_{jj}$. We claim that $\rho_0 = \rho_j(t_0)$. To this end, consider the set of values t for which $p(A + tE_{jj};\ \rho_0) = 0$. This condition takes the form $at + b = 0$, and $a \neq 0$ as ρ_0 cannot be an eigenvalue of $A(j)$ (since $\rho_0 > \rho(A)$). So there is a unique t such that $p(A + tE_{jj};\ \rho_0) = 0$. We also know that there is a unique t such that $\rho_0 = \rho_j(t)$. It follows that $\rho_0 = \rho_j(t_0)$. □

Note that (for $\rho_i(t) \neq \rho_j(t)$) this theorem gives an upper bound on the number of values t such that $\rho_i(t) = \rho_j(t)$. The number of switches is less than or equal to this value. Using the claim in the proof of Theorem 4.2, the following example shows that the bound of $n - 2$ is the best possible. Without loss of generality, we let $i = 1$ and $j = 2$, and first consider n odd.

Example 4.3. Let $n = 2m + 1 \geq 5$, let $\mu_1, \mu_2, \ldots, \mu_{n-2}$ be arbitrary positive numbers, and denote $E_k(\mu_1, \mu_2, \ldots, \mu_{n-2})$ by E_k, $1 \leq k \leq n - 2$. Let

$$A = \begin{bmatrix} 0 & 0 & a_{13} & 0 & a_{15} & 0 & a_{17} & 0 & \cdots & 0 & a_{1,2m+1} \\ 0 & \varepsilon & 0 & a_{24} & 0 & a_{26} & 0 & a_{28} & \cdots & a_{2,2m} & 0 \\ a_{31} & a_{32} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & a_{43} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & a_{54} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{65} & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & a_{2m+1,2m} & 0 \end{bmatrix},$$

where $\varepsilon > 0$ and the nonzero a_{ij} are defined as follows:

$$a_{31} = a_{32} = a_{43} = a_{54} = a_{65} = \cdots = a_{2m+1,2m} = E_1 \varepsilon^{1/2m};$$

$$a_{1k} = \frac{E_{k-2}}{E_1^{k-2}} \varepsilon^{(2m-k+2)/2m}, \quad \text{for} \quad k = 3, 5, \cdots, 2m+1;$$

$$a_{2k} = \frac{E_{k-2}}{E_1^{k-2}} \varepsilon^{(2m-k+2)/2m}, \quad \text{for} \quad k = 4, 6, \cdots, 2m.$$

Then

(a) the roots of (5), with $i = 1$ and $j = 2$, are $\mu_1, \mu_2, \cdots, \mu_{n-2}$; and

(b) if ε is sufficiently small, then $\rho(A) < \mu_k$ for all $1 \le k \le n-2$.

Proof. It suffices to prove (a), since it is clear that as $\varepsilon \to 0$, $A \to 0$ and $\rho(A) \to 0$.

The proof of (a) is computational. The coefficients in equation (5) are as follows:

$$E_1(A(2)) - E_1(A(1)) = a_{11} - a_{22};$$

$$-(E_2(A(2)) - E_2(A(1))) = a_{13}\,a_{31};$$

$$E_k(A(2)) - E_k(A(1)) = -a_{2,k+1}\,a_{32}\,a_{43}\,a_{54}\cdots a_{k+1,k} \quad \text{for} \quad k = 3, 5, \cdots, 2m-1;$$

$$-(E_k(A(2)) - E_k(A(1))) = a_{1,k+1}\,a_{31}\,a_{43}\,a_{54}\cdots a_{k+1,k} \quad \text{for} \quad k = 4, 6, \cdots, 2m.$$

Equation (5) becomes

$$-\varepsilon\lambda^{n-2} + \varepsilon E_1\lambda^{n-3} - \varepsilon E_2\lambda^{n-4} + \varepsilon E_3\lambda^{n-5} - \cdots - \varepsilon E_{n-3}\lambda + \varepsilon E_{n-2} = 0,$$

and indeed its roots are $\mu_1, \mu_2, \cdots, \mu_{n-2}$. □

Note that the leading principal submatrix of order $2m$ of the matrix A in Example 4.3 above shows that the bound of $n-2$ in Theorem 4.2 is also best possible for $n \geq 4$ and even.

Example 4.4. Let

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 0.5 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

This matrix exhibits the maximum number of switches over all pairs $\rho_i(t)$, $\rho_j(t)$, namely 3 switches (by Theorem 4.2). Numerically the 3 distinct values of t at which a switch occurs are $t_1 = 0.90$, $t_2 = 3.21$ and $t_3 = 4.81$, with $\rho_1(t_1) = \rho_2(t_1)$, $\rho_1(t_2) = \rho_3(t_2)$ and $\rho_2(t_3) = \rho_3(t_3)$. Index 1 dominates for small t $(t < t_1)$, so is the initial winner; index 2 dominates for intermediate t $(t_1 < t < t_3)$; and index 3 dominates for large t $(t > t_3)$, so is the terminal winner. Note that A is symmetric with Perron vector $(.6538, .6318, .4165)^T$, so Theorem 3.1 gives the initial winner. Also, $\max\{a_{ii}\}$ occurs at index 3 so Theorem 2.1 gives the terminal winner. □

Once the set of initial winners T_n is determined from Theorem 3.2, equation (5) can be invoked to determine indices j such that $\rho_i(t_0) = \rho_j(t_0) > \rho(A)$, for $i \in T_n$ and $j \notin T_n$. The smallest such t_0 at which a switch occurs can be found as in the proof of the claim in Theorem 4.2. As was explicitly done in Example 4.4, this process can be repeated to determine the finite number of intervals on which there are different winners. Using the result of Theorem 4.2, if $\rho_i(t) = \rho_j(t)$ on an *interval*, then indices i and j tie. Thus from Theorems 2.3 and 3.2, if $|S_{n-1}| > 1$, then the elements of S_{n-1} all tie, and if $|T_n| > 1$, then the elements of T_n all tie.

4.2. IDENTICAL FUNCTIONS $\rho_i(t)$ AND $\rho_j(t)$

Recalling that $p(A(i); \lambda)$ denotes the characteristic polynomial of $A(i)$, it is easily seen from Lemma 4.1 and Theorem 4.2 that $\rho_i(t) = \rho_j(t)$ for all $t \geq 0$ if and only if

(6) $$E_\ell(A(i)) = E_\ell(A(j)), \qquad \ell = 1, 2, \ldots, n-1;$$

that is, if and only if $p(A(i);\ \lambda)$ and $p(A(j);\ \lambda)$ are identical. In the following theorem, we give other conditions that are equivalent to the above; we first prove a general lemma.

LEMMA 4.5. *Let A be an n-by-n real matrix, r be a fixed real number, and $R = rI - A$ be such that $R^\#$ exists. Then for fixed indices i, j the following are equivalent.*

(i) $p(A(i);\ \lambda) = p(A(j);\ \lambda)$

(ii) $a_{ii}^{(\ell)} = a_{jj}^{(\ell)}$ *for all* $1 \le \ell \le n-1$

(iii) $r_{ii}^{(\ell)} = r_{jj}^{(\ell)}$ *for all* $1 \le \ell \le n-1$

(iv) $r_{ii}^{\#(\ell)} = r_{jj}^{\#(\ell)}$ *for all* $1 \le \ell \le n-1$

Proof. By the Cayley-Hamilton theorem, if $a_{ii}^{(\ell)} = a_{jj}^{(\ell)}$ for $1 \le \ell \le n-1$, then it is true for all $\ell \ge 1$. Suppose (i) holds. Let $B = \lambda I - A$ with $\lambda > 0$ sufficiently large so that B is invertible and the Neumann series

$$B^{-1} = (\lambda I - A)^{-1} = \frac{1}{\lambda} \sum_{k=0}^{\infty} \lambda^{-k} A^k$$

converges. Then $b_{ii}^{(-1)} = p(A(i);\ \lambda)/\det B = p(A(j);\ \lambda)/\det B = b_{jj}^{(-1)}$. Comparing coefficients in the series, we conclude that (ii) holds. Assume (ii) holds. Let $B = \lambda I - A$, where λ is an indeterminate, and thus B is a matrix over the field of rational functions in λ with real coefficients. It is easily seen that $b_{ii}^{(\ell)} = b_{jj}^{(\ell)}$, $\ell = 1, 2, \ldots, n-1$, since B is a polynomial in A. Furthermore, B is an invertible matrix with entries in the field of rational functions in λ with real coefficients (because $\det B \not\equiv 0$), and therefore B^{-1} can be written as a polynomial of degree $\leq n-1$ in B. Hence, we also have $b_{ii}^{(-1)} = b_{jj}^{(-1)}$, which implies that the (i,i) and (j,j) entries of adj B are equal. Thus, $p(A(i);\ \lambda) = p(A(j);\ \lambda)$, and (i) holds. The equivalence of (ii) and (iii) is clear. Since $R^{\#}$ is a polynomial of degree at most $n-1$ in R and conversely [CM, p. 130], the equivalence of (iii) and (iv) follows. □

We note that the above four equivalences can be proven over a general field.

THEOREM 4.6. *Let* $A \geq 0$ *be an* n-by-n *irreducible matrix,* $Q = \rho(A)I - A$ and i, j *be fixed. Then the following are equivalent.*

(i) $p(A(i);\ \lambda) = p(A(j);\ \lambda)$

(ii) $\rho_i(t) = \rho_j(t)$ *for all* $t \geq 0$

(iii) $a_{ii}^{(\ell)} = a_{jj}^{(\ell)}$ *for all* $1 \leq \ell \leq n-1$

(iv) $q_{ii}^{\#(\ell)} = q_{jj}^{\#(\ell)}$ *for all* $1 \leq \ell \leq n-1$.

Proof. The equivalence of (i) and (ii) follows from the remarks at the beginning of this section. With $R = Q$ the other equivalences follow from Lemma 4.5. □

Example 4.7. Let

$$A = \begin{bmatrix} a & b & c & d \\ b & a & d & c \\ e & f & a & g \\ f & e & g & a \end{bmatrix},$$

where $a, b, c, d, e, f, g > 0$. A straightforward computation shows that $a_{11}^{(\ell)} = a_{22}^{(\ell)}$, $\ell = 1, 2, 3$. Thus, $\rho_1(t) = \rho_2(t)$ for all $t \geq 0$. Furthermore, when $g = b$ in A, then it can be shown that $a_{11}^{(\ell)} = a_{22}^{(\ell)} = a_{33}^{(\ell)} = a_{44}^{(\ell)}$, for $\ell = 1, 2, 3$, so that all four indices tie. □

Example 4.8. If A is any n-by-n matrix satisfying the conditions of Theorem 4.6 for all i, j (i.e., all n indices tie), then the n^2-by-n^2 matrix $J \otimes A$ also satisfies these conditions, where J is the n-by-n matrix with all entries equal to 1 and $\otimes$ denotes the Kronecker product. Thus all n^2 indices tie. □

A sufficient condition for (iii) in Theorem 4.6 to hold for all pairs i, j is given in the following result.

THEOREM 4.9. *Let* $A \geq 0$ *be an* n-by-n *irreducible matrix, and suppose that there exists a diagonal matrix* $D = [d_{ij}]$ *with all* $d_{ii} > 0$ *such that* $C \equiv D^{-1}AD$ *is a circulant matrix. Then* $a_{ii}^{(\ell)} = a_{jj}^{(\ell)}$ *for all pairs* i, j *and for* $\ell = 1, 2, \ldots, n-1$. *Thus all indices tie.*

Proof. Since $A = DCD^{-1}$, for each $\ell = 1, 2, \ldots, n-1$ we have $A^{\ell} = DC^{\ell}D^{-1}$. It is well known that C^{ℓ} is also a circulant, so the result follows from Theorem 4.6. □

The matrix A of Example 4.7 with $g = b$ shows that the converse of Theorem 4.9 does not hold. It is easily shown that such a matrix A is not in general diagonally similar to a circulant matrix.

5. ALL $t > 0$: DETERMINING THE UNIVERSAL WINNER(S)

In Sections 2 and 3, the technology was developed to determine the terminal and initial winners (which always exist and there may be ties). There may, of course, be no universal winner, but we may use the ideas of the last section to determine if there are universal winners and, if so, which indices are universal winners. First of all, if i is to be a universal winner then it must be both an initial winner and a terminal winner. These two occurrences may be checked (using Theorems 3.2 and 2.3). For general n, suppose that i is both an initial and a terminal winner. Then i is a universal winner if and only if $\rho_i(t) > \rho_j(t)$, for all $t > 0$

and all j such that $\rho_i(t) \neq \rho_j(t)$, i.e., the polynomial in (5) has no root greater than $\rho(A)$. Thus, the roots of at most $(n-1)$ polynomials, each of degree $n-2$, will determine if i is a universal winner.

When $n = 2$, equation (5) reduces to a constant, so either both indices tie or there is a unique universal winner, see Ex. 1.2. However, as the following example shows, it is possible for an index to be both an initial winner and a terminal winner but not a universal winner.

Example 5.1. Let

$$A = \begin{bmatrix} 0 & 5 & 1 & 0 \\ 5 & 1 & 0 & 0.2494 \\ 21 & 21 & 0 & 0 \\ 0 & 0 & 21 & 0 \end{bmatrix}.$$

Clearly index 2 is the terminal winner and it is also the initial winner. However, $\rho_1(t)$ and $\rho_2(t)$ switch at $t = 2.38$ and $t = 4.63$. □

We note that the initial and terminal orderings are both determined by diagonal entries of powers of A up to $(n-1)^2$. For the terminal ordering, this is clear from Theorem 2.3; for the initial ordering, this follows from Theorem 3.2 and the facts [DN] [CM] that $I - QQ^{\#} = vu^T$ and $(Q^{\#})^k$ is a polynomial of degree $k(n-1)$ in A.

Acknowledgement The authors wish to thank Shaun Fallat for discussions on this paper

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