SOME NEW RESULTS
FOR RADIATION–FIELD PROBLEMS

By

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1. INTRODUCTION

In the present investigation, we define a family of integrals by

$$(1.1)\qquad S_m^{(p,q)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_p),\ (\beta_q)\end{matrix}\right]$$

$$= \frac{\sigma a}{4\pi}\int_0^b x^{\lambda}(x^m+c)^{-\alpha}\ {}_{p+1}F_q\left[\begin{matrix}\alpha,\ \alpha_1,\cdots,\alpha_p; \\ \beta_1,\cdots,\beta_q;\end{matrix}\ -\frac{a^m}{x^m+c}\right]dx,$$

where $\min\{\mathrm{Re}(a), \mathrm{Re}(b), \mathrm{Re}(c)\} > 0$, $\lambda \in (-1,1)$, b stands as the upper limit of the definite integral, and the integrand is the product of an algebraic function and a generalized hypergeometric function given by the series:

$$(1.2)\qquad {}_pF_q\left[\begin{matrix}\alpha_1,\cdots,\alpha_p; \\ \beta_1,\cdots,\beta_q;\end{matrix}\ z\right] = \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(\alpha_i)_n}{\prod_{j=1}^{q}(\beta_j)_n}\frac{z^n}{n!},$$

the factorial function $(a)_n$ being defined in terms of the Gamma functions by

$$(1.3)\qquad (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

Thus $(a)_n$ is the same as the well-known Pochhammer symbol defined by

$$
(1.4)\qquad \begin{cases} (a)_n = \prod_{k=1}^{n} (a+k-1) = a(a+1)\cdots(a+n-1), & n \in \{1,2,3,\cdots\}, \\ (a)_0 = 1. \end{cases}
$$

Throughout this investigation, we will use many known theorems and formulas involving special functions including the Beta function

$$
(1.5)\qquad B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad \mathrm{Re}(\alpha) > 0, \ \mathrm{Re}(\beta) > 0.
$$

It is worth mentioning here that the generalized hypergeometric series, also known as the generalized Gauss (and Kummer) series, in (1.2) is (absolutely) convergent whenever $p \leqq q$ and $|z| < \infty$, or $p = q + 1$ and $|z| < 1$ (*cf.* [11], p. 43). In the following sections, we evaluate the general integral (1.1) and give three different formats of the main results. The first result is associated with the Kampé de Fériet function, the second one is involved with the generalized hypergeometric function, and the third result is given in terms of the Gauss hypergeometric function. Special cases of our integral occur in radiation–field problems. Many special cases could be considered from our general results. Quite a few series expansions are given along with the behavior of such functions when $b \to \infty$. For some selected values of the parameters, the integral (1.1) can be computed numerically. (We have left these calculations as an exercise.) At the end many contiguous functions and recurrence relations are established. These results obtained by us are fairly general in character. By suitably specializing the parameters involved, our general results will also yield the corresponding formulas for each of the classical orthogonal polynomials, for example, Jacobi polynomials, Gegenbauer (or ultraspherical) polynomials, Legendre (or spherical) polynomials, and Tchebycheff polynomials of the first and second kind, and many more.

2. MAIN RESULTS

By the application of operational technique, and changing the order of integration and summation in (1.1), as a process of augmentation of parameters, we readily obtain the following three main formulas:

Result 1

(2.1) $$S_m^{(p,q)}\begin{bmatrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_p),\ (\beta_q) \end{bmatrix}$$

$$= \frac{\sigma a}{4\pi(\lambda+1)}\, b^{\lambda+1} c^{-\alpha}\ F^{1:p;1}_{0:q;1}\begin{bmatrix} \alpha:\ \alpha_1,\cdots,\alpha_p; & \frac{\lambda+1}{m}; & \\ & & -\frac{a^m}{c},\ -\frac{b^m}{c} \\ -:\ \beta_1,\cdots,\beta_q; & \frac{\lambda+1}{m}+1; & \end{bmatrix},$$

$$\min\{\text{Re}(a),\ \text{Re}(b),\ \text{Re}(c)\} > 0; \quad -1 < \lambda < 1,$$

where the right-hand side function is a double hypergeometric function due to Kampé de Fériet introduced as long ago as 1921 (*cf.* [6] and [2]); the convergence of such series is given in [11, p. 27] and [12, p. 64].

Result 2

(2.2) $$S_m^{(p,q)}\begin{bmatrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_p),\ (\beta_q) \end{bmatrix}$$

$$= \frac{\sigma a}{4\pi(\lambda+1)} b^{\lambda+1} c^{-\alpha} \sum_{n=0}^{\infty} \frac{(\alpha)_n \left[\frac{\lambda+1}{m}\right]_n}{\left[\frac{\lambda+m+1}{m}\right]_n n!} \left[-\frac{b^m}{c}\right]^n$$

$$\cdot {}_{p+1}F_q\left[\begin{matrix}\alpha+n,\ \alpha_1,\cdots,\alpha_p; \\ \beta_1,\cdots,\beta_q;\end{matrix}\ -\frac{a^m}{c}\right].$$

RESULT 3

(2.3) $$S_m^{(p,q)}\left[\begin{matrix}a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_p),\ (\beta_q)\end{matrix}\right]$$

$$= \frac{\sigma a}{4\pi(\lambda+1)} b^{\lambda+1} c^{-\alpha} \sum_{n=0}^{\infty} \frac{(\alpha)_n(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n\ n!} \left[-\frac{a^m}{c}\right]^n$$

$$\cdot {}_2F_1\left[\alpha+n,\ \frac{\lambda+1}{m};\ \frac{\lambda+1}{m}+1;\ -\frac{b^m}{c}\right].$$

The second result in (2.2) involves the generalized hypergeometric function, and (2.3) contains the Gauss hypergeometric function.

3. SERIES EXPANSIONS

We establish the following series expansion for the integral in (1.1) by using the well–known Kummer's and Euler's theorems given in the literature [10]. For the reader's convenience, first we will state these known theorems.

THEOREM 1. *If* $|z| < 1$, *and* $\left|\frac{z}{1-z}\right| < 1$, *then*

$$(3.1)\qquad {}_2F_1(a,b;c;z) = (1-z)^{-a}\ {}_2F_1\left[a,\ c-b;\ c;\ \frac{-z}{1-z}\right].$$

THEOREM 2. *If* $|z| < 1$, *then*

$$(3.2)\qquad {}_2F_1(a,b;c;z) = (1-z)^{c-a-b}\ {}_2F_1(c-a,c-b;c;z).$$

THEOREM 3. *If* $|z| < 1$ *and* $|1-z| < 1$, *if* $\mathrm{Re}(c) < 1$ *and* $\mathrm{Re}(c-a-b) > 0$, *and none of* a, b, c, $c-a$, $c-b$, $c-a-b$ *is an integer, then*

$$(3.3)\qquad {}_2F_1(a,b;c;z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\ {}_2F_1(a,b;\ a+b-c+1;\ 1-z)$$

$$+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\ (1-z)^{c-a-b}\ {}_2F_1(c-a,c-b;\ c-a-b+1;\ 1-z).$$

THEOREM 4. *If* $|z| > 1$ *and* $|1-z| > 1$, *and if* a, b, c *are suitably restricted, then*

$$(3.4)\qquad {}_2F_1(a,b;c;z) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}\ (-z)^{-a}\ {}_2F_1\left[a,\ a-c+1;\ a-b+1;\ \frac{1}{z}\right]$$

$$+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}\ (-z)^{-b}\ {}_2F_1\left[b,\ c-b+1;\ b-a+1;\ \frac{1}{z}\right].$$

By the help of the above mentioned theorems and (1.3), we will get (3.5), (3.6), (3.7), and (3.8) as immediate consequences of the definition (1.1).

$$(3.5)\quad S_m^{(p,q)}\begin{bmatrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_p),\ (\beta_q)\end{bmatrix} = \frac{\sigma a}{4\pi(\lambda+1)}\, b^{\lambda+1}(c+b^m)^{-\alpha}$$

$$\cdot \sum_{n=0}^{\infty} \frac{(\alpha)_n(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n\ n!}\left[-\frac{a^m}{c+b^m}\right]^n {}_2F_1\left[\alpha+n,1;\ \frac{\lambda+1}{m}+1;\ \frac{b^m}{c+b^m}\right].$$

$$(3.6)\quad S_m^{(p,q)}\begin{bmatrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_p),\ (\beta_q)\end{bmatrix} = \frac{\sigma a}{4\pi(\lambda+1)}\, b^{\lambda+1}c^{-1}(c+b^m)^{1-\alpha}$$

$$\cdot \sum_{n=0}^{\infty} \frac{(\alpha)_n(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n\ n!}\left[-\frac{a^m}{c+b^m}\right]^n {}_2F_1\left[1,\ \frac{\lambda+1}{m}-\alpha-n+1;\ \frac{\lambda+1}{m}+1;\ -\frac{b^m}{c}\right].$$

$$(3.7)\quad S_m^{(p,q)}\begin{bmatrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_p),\ (\beta_q)\end{bmatrix} = \frac{\sigma a}{4\pi(\lambda+1)}\, b^{\lambda+1}c^{-\alpha+(\lambda+1)/m}\,(c+b^m)^{-(\lambda+1)/m}$$

$$\cdot \sum_{n=0}^{\infty} \frac{(\alpha)_n(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n\ n!}\left[-\frac{a^m}{c}\right]^n {}_2F_1\left[\frac{\lambda+1}{m},\ \frac{\lambda+1}{m}-\alpha-n+1;\ \frac{\lambda+1}{m}+1;\ \frac{b^m}{c+b^m}\right].$$

$$(3.8)\quad S_m^{(p,q)}\begin{bmatrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_p),\ (\beta_q)\end{bmatrix}$$

$$= \Gamma\begin{bmatrix} \frac{\lambda+1}{m}+1,\ \alpha-\frac{\lambda+1}{m} \\ \alpha \end{bmatrix} \frac{\sigma a}{4\pi(\lambda+1)}\, c^{-\alpha+(\lambda+1)/m}$$

$$\cdot\ {}_{p+1}F_q\left[\begin{matrix}\alpha-\frac{\lambda+1}{m},\ \alpha_1,\cdots,\alpha_p; \\ \beta_1,\cdots,\beta_q;\end{matrix}\ -\frac{a^m}{c}\right]+\frac{\sigma a}{4\pi(\lambda+1-m\alpha)}\ b^{\lambda+1-m\alpha}$$

$$\cdot\sum_{n=0}^{\infty}\frac{(\alpha)_n(\alpha-(\lambda+1)/m)_n(\alpha_1)_n\cdots(\alpha_p)_n}{(\alpha-(\lambda+1)/m+1)_n(\beta_1)_n\cdots(\beta_q)_n\ n!}\left[-\frac{a^m}{b^m}\right]^n$$

$$\cdot\ {}_2F_1\left[\alpha+n,\ \alpha-\frac{\lambda+1}{m}+n;\ \alpha-\frac{\lambda+1}{m}+1+n;\ -\frac{c}{b^m}\right].$$

The behavior of $S_m^{(p,q)}[\cdots]$ defined by (1.1) is also considered when $b\to\infty$, and from equation (3.8) we easily obtain

$$\text{(3.9)}\qquad \lim_{b\to\infty}\left\{S_m^{(p,q)}\left[\begin{matrix}a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_p),\ (\beta_q)\end{matrix}\right]\right\}$$

$$=\frac{\sigma a}{4\pi}\frac{1}{\lambda+1}\ c^{(\lambda+1)/m-\alpha}\ \Gamma\left[\begin{matrix}\frac{\lambda+1}{m}+1,\ \alpha-\frac{\lambda+1}{m} \\ \alpha\end{matrix}\right]\ {}_{p+1}F_q\left[\begin{matrix}\alpha-\frac{\lambda+1}{m},\ \alpha_1,\cdots,\alpha_p; \\ \beta_1,\cdots,\beta_q;\end{matrix}\ -\frac{a^m}{c}\right],$$

$$\lambda < m\alpha-1;\quad \alpha > 1.$$

Our result (3.9) is a strong generalization of Formulas (24) and (25) in [5], which, in turn, yield results due to Andrews [1] for the detector response to an infinite strip source in a nonattenuating medium.

Many other interesting formulas can be derived for our results (3.5) to (3.9) by giving special values to different parameters. We omit the details.

4. SEVERAL CASES OF REDUCING THE ORDER OF THE INTEGRAL (1.1)

In this section, we establish many interesting formulas for the integral (1.1), which are very general in character and believed to be new. We use many known results from [8] to derive our results. First of all, we make use of the well-known summation law:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n-k), \tag{4.1}$$

in our result (2.2) to obtain the following explicit expression for the integral (1.1):

$$S_m^{(p,q)}\begin{bmatrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_p),\ (\beta_q)\end{bmatrix} = \frac{\sigma a}{4\pi(\lambda+1)}\, b^{\lambda+1} c^{-\alpha} \sum_{n=0}^{\infty} \frac{(\alpha)_n(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n\ n!}\left[-\frac{a^m}{c}\right]^n \cdot {}_{q+2}F_{p+1}\begin{bmatrix} -n,\ \frac{\lambda+1}{m},\ 1-\beta_1-n,\cdots,1-\beta_q-n; \\ \frac{\lambda+1}{m}+1,\ 1-\alpha_1-n,\cdots,1-\alpha_p-n; \end{bmatrix}\frac{b^m}{a^m}\Bigg] . \tag{4.2}$$

By using the result (4.2) with $\beta_q = \alpha_p + 1$ along with the result (22) in [8, p. 439], we get the following expression for $S_m^{(p,p)}[\cdots]$:

$$S_m^{(p,p)}\begin{bmatrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_p),\ (\alpha_p+1)\end{bmatrix} = \frac{\sigma a}{4\pi}\,\frac{1}{\lambda+1}\, b^{\lambda+1} c^{-\alpha}\left[\prod_{\ell=1}^{p} \frac{m\alpha_\ell}{\lambda+1+m\alpha_\ell}\right] \tag{4.3}$$

$$\cdot \sum_{n=0}^{\infty} \left[\prod_{\ell=1}^{p} \frac{\left[\alpha_\ell + \frac{\lambda+1}{m}\right]_n}{\left[\alpha_\ell + 1 + \frac{\lambda+1}{m}\right]_n} \right] \frac{(\alpha)_n}{\left[\frac{\lambda+1}{m} + 1\right]_n} \left[-\frac{b^m}{c}\right]^n P_n^{(-(\lambda+1)/m-n,-1-n)} \left[1-2\left[\frac{a}{b}\right]^m\right]$$

$$+ \frac{\sigma a}{4\pi} b^{\lambda+1} c^{-\alpha} \sum_{k=1}^{p} \frac{1}{\lambda+1+m\alpha_k} \left[\prod_{\substack{\ell=1\\ \ell\neq k}}^{p} \frac{\alpha_k}{\alpha_\ell - \alpha_k} \right]$$

$$\cdot \sum_{n=0}^{\infty} \frac{(\alpha)_n \left[\alpha_k + \frac{\lambda+1}{m}\right]_n}{(\alpha_k+1)_n \left[\alpha_k + 1 + \frac{\lambda+1}{m}\right]_n} \left[\frac{b^m}{c}\right]^n P_n^{(\alpha_k,-1-n)} \left[1-2\left[\frac{a}{n}\right]^m\right],$$

where $P_n^{(\alpha,\beta)}(x)$ is the well-known Jacobi polynomial of order (α,β) and degree n in x (see Rainville [9] and Szegö [13]). It is interesting to observe that by suitably specializing the result (4.3) with $a = b$, we obtain

$$(4.4) \qquad S_m^{(p,p)} \left[\begin{matrix} a,\ a,\ c,\ \lambda \\ \alpha,\ (\alpha_p),\ (\alpha_p+1) \end{matrix} \right]$$

$$= \frac{\sigma}{4\pi} \frac{1}{\lambda+1} a^{\lambda+2} c^{-\alpha} \left[\prod_{\ell=1}^{p} \frac{m\alpha_\ell}{\lambda+1+m\alpha_\ell} \right] {}_{p+2}F_{p+1} \left[\begin{matrix} \alpha,\ \alpha_1 + \frac{\lambda+1}{m}, \cdots, \alpha_p + \frac{\lambda+1}{m}; \\ \frac{\lambda+1}{m},\ \alpha_1 + \frac{\lambda+1}{m} + 1, \cdots, \alpha_p + \frac{\lambda+1}{m} + 1; \end{matrix} -\frac{a^m}{c} \right]$$

$$+ \frac{\sigma}{4\pi} a^{\lambda+2} c^{-\alpha} \sum_{k=1}^{p} \frac{1}{\lambda+1+m\alpha_k} \left[\prod_{\substack{\ell=1\\ \ell\neq k}}^{p} \frac{\alpha_k}{\alpha_\ell - \alpha_k} \right] {}_3F_2 \left[\alpha, \alpha_k + \frac{\lambda+1}{m},\ 1;\ \alpha_k + 1, \alpha_k + \frac{\lambda+1}{m} + 1;\ \frac{a^m}{c} \right],$$

where the integral (1.1) is expressed as the sum of the generalized hypergeometric functions ${}_{p+2}F_{p+1}(x)$ and ${}_3F_2(x)$. In (4.4), by further setting $p = 1$ and $\alpha_p = \alpha_1$, we get an interesting result associated with special Jacobi polynomials, *viz*

$$(4.5)\quad S_m^{(1,1)}\begin{bmatrix} a,\ b,\ c,\ \lambda \\ \alpha,\ \alpha_1,\ \alpha_1+1 \end{bmatrix}$$

$$= \frac{\sigma a}{4\pi}\frac{1}{\lambda+1} b^{\lambda+1} c^{-\alpha} \frac{m\alpha_1}{\lambda+1+m\alpha_1} \sum_{n=0}^{\infty} \frac{(\alpha)_n \left[\alpha_1 + \frac{\lambda+1}{m}\right]_n}{\left[\alpha_1+1+\frac{\lambda+1}{m}\right]_n \left[\frac{\lambda+1}{m}+1\right]_n}$$

$$\cdot \left[-\frac{b^m}{c}\right]^n P_n^{(-(\lambda+1)/m-n,-1-n)}\left[1-2\left[\frac{a}{b}\right]^m\right]$$

$$+ \frac{\sigma a}{4\pi} b^{\lambda+1} c^{-\alpha} \frac{1}{\lambda+1+m\alpha_1} \sum_{n=0}^{\infty} \frac{(\alpha)_n \left[\alpha_1 + \frac{\lambda+1}{m}\right]_n}{(\alpha_1+1)_n \left[\alpha_1+1+\frac{\lambda+1}{m}\right]_n} \left[\frac{b^m}{c}\right]^n P_n^{(\alpha_1,-1-n)}\left[1-2\left[\frac{a}{b}\right]^m\right].$$

Furthermore, a very special case of (4.5) is given below for $p = 1$ and $a = b$:

$$(4.6)\quad S_m^{(1,1)}\begin{bmatrix} a,\ a,\ c,\ \lambda \\ \alpha,\ \alpha_1, \alpha_1+1 \end{bmatrix}$$

$$= \frac{\sigma a}{4\pi}\frac{1}{\lambda+1} a^{\lambda+2} c^{-\alpha} \frac{m\alpha_1}{\lambda+1+m\alpha_1}\ {}_3F_2\left[\alpha, \alpha_1 + \frac{\lambda+1}{m},\ 1;\ \alpha_1+1+\frac{\lambda+1}{m},\ \frac{\lambda+1}{m}+1;\ -\frac{a^m}{c}\right]$$

$$+ \frac{\sigma}{4\pi} a^{\lambda+2} c^{-\alpha} \frac{1}{\lambda+1+m\alpha_1}\ {}_3F_2\left[\alpha, \alpha_1 + \frac{\lambda+1}{m},\ 1;\ \alpha_1+1+\frac{\lambda+1}{m},\ \alpha_1+1;\ -\frac{a^m}{c}\right].$$

Once again we take our result (4.2) and the formula (23) in [8] to get the following result:

$$(4.7)\quad S_m^{(p,p)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_p),\ (\alpha_p+m_p)\end{matrix}\right] = \frac{\sigma a}{4\pi}\frac{1}{\lambda+1}\, b^{\lambda+1} c^{-\alpha}\left[\prod_{j=1}^{p}\frac{(-1)^{m_j}}{(m_j-1)!}(\alpha_j)_{m_j}\right]$$

$$\cdot \sum_{j_1=0}^{m_1-1}\cdots\sum_{j_p=0}^{m_p-1}\left[\prod_{\ell=1}^{p}\frac{(1-m_\ell)_{j_\ell}}{j_\ell!}\right]\prod_{i=1}^{p}\frac{1}{\left[1-\alpha_i-\frac{\lambda+1}{m}-m_i+j_i\right]}$$

$$\cdot \sum_{n=0}^{\infty}\frac{(\alpha)_n\left[\alpha_i+\frac{\lambda+1}{m}+m_i-j_i-1\right]_n}{\left[\frac{\lambda+1}{m}+1\right]_n\left[\alpha_i+\frac{\lambda+1}{m}+m_i-j_i\right]_n}\left[-\frac{b^m}{c}\right]^n P_n^{(-n-(\lambda+1)/m,-1-n)}\left[1-\frac{2a^m}{b^m}\right]$$

$$+\frac{\sigma a}{4\pi}\, b^{\lambda+1}c^{-\alpha}\left[\prod_{j=1}^{p}\frac{(-1)^{m_j}}{(m_j-1)!}(\alpha_j)_{m_j}\right]$$

$$\cdot\sum_{k=1}^{p}\sum_{j_1=0}^{m_1-1}\cdots\sum_{j_p=0}^{m_p-1}\frac{1}{1-\alpha_k-m_k+j_k}\left[\prod_{\ell=1}^{p}\frac{(1-m_\ell)_{j_\ell}}{j_\ell!}\right]$$

$$\cdot\left[\prod_{\substack{i=1\\ i\neq k}}^{p}\frac{1}{\alpha_k-\alpha_i+m_k-m_i-j_k+j_i}\right]\frac{1}{\alpha_k+\frac{\lambda+1}{m}+m_k-j_k-1}$$

$$\cdot\sum_{n=0}^{\infty}\frac{(\alpha)_n\left[\alpha_k+\frac{\lambda+1}{m}+m_k-j_k-1\right]_n}{(\alpha+m_k-j_k)_n\left[\alpha_k+\frac{\lambda+1}{m}+m_k-j_k\right]_n}\left[\frac{b^m}{c}\right]^n P_n^{(\alpha_k+m_k-j_k-1,-1-n)}\left[1-\frac{2a^m}{b^m}\right],$$

where $m_n = 1,2,3,\cdots$; all α_i are different; if $\alpha_i - \alpha_k = N = 1,2,\cdots$, then $m_k < N$.

The result (4.7) as well as other results in this section can also be expressed in terms of such other orthogonal polynomials as Gegenbauer, Legendre, Tchebycheff, and many more. Formula (4.7) will also give a more special case such as

$$(4.8) \quad S_m^{(p,p)}\left[\begin{matrix} a,\ a,\ c,\ \lambda \\ \alpha,\ (\alpha_p),\ (\alpha_p+m_p) \end{matrix}\right]$$

$$= \frac{\sigma}{4\pi}\frac{1}{\lambda+1} a^{\lambda+2} c^{-\alpha}\left[\prod_{j=1}^{p}\frac{(-1)^{m_j}}{(m_j-1)!}\right](\alpha_j)_{m_j}$$

$$\cdot \sum_{j_1=0}^{m_1-1}\cdots\sum_{j_p=0}^{m_p-1}\left[\prod_{\ell=1}^{p}\frac{(1-m_\ell)_{j_\ell}}{\left[1-\alpha_\ell-\frac{\lambda+1}{m}-m_\ell+j_\ell\right]\cdot j_\ell!}\right]$$

$$\cdot\ {}_{p+2}F_{p+1}\left[\begin{matrix}\alpha,\ \alpha_1+\frac{\lambda+1}{m}+m_1-j_1-1,\cdots,\alpha_p+\frac{\lambda+1}{m}+m_p-j_p-1,1; \\ \frac{\lambda+1}{m}+1,\ \alpha_1+\frac{\lambda+1}{m}+m_1-j_1,\cdots,\alpha_p+\frac{\lambda+1}{m}+m_p-j_p;\end{matrix}\ -\frac{a^m}{c}\right]$$

$$+\frac{\sigma}{4\pi} a^{\lambda+2} c^{-\alpha}\left[\prod_{j=1}^{p}\frac{(-1)^{m_j}}{(m_j-1)!}(\alpha_j)_{m_j}\right]$$

$$\cdot\sum_{k=1}^{p}\sum_{j_1=0}^{m_1-1}\cdots\sum_{j_p=0}^{m_p-1}\frac{1}{(1-\alpha_k-m_k+j_k)\left[\alpha_k+\frac{\lambda+1}{m}+m_k-j_k-1\right]}\left[\prod_{\ell=1}^{p}\frac{(1-m_\ell)_{j_\ell}}{j_\ell!}\right]$$

$$\cdot \left[\prod_{\substack{i=1\\ i\neq k}}^{p} \frac{1}{\alpha_k-\alpha_i+m_k-m_i-j_k+j_i}\right] {}_3F_2\left[\begin{matrix}\alpha,\ \alpha_k+\frac{\lambda+1}{m}+m_k-j_k-1,1; \\ \alpha_k+m_k-j_k,\alpha_k+\frac{\lambda+1}{m}+m_k-j_k;\end{matrix}\ \frac{a^m}{c}\right],$$

where the integral in (1.1) is expressed as the sum of ${}_{p+2}F_{q+1}(x)$ and its reduced form ${}_3F_2(x)$. By specializing parameters, one can obtain many special cases that are scattered throughout the literature.

In a similar manner, our explicit expression (4.2) and several formulas (for instance, (14) through (21)) will yield the following interesting results for reducing the order of the integral in (1.1). These are listed below:

$$(4.9)\qquad S_m^{(p,q)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_{p-1}),\ \beta_q+1,\ (\beta_q)\end{matrix}\right]$$

$$= S_m^{(p-1,q-1)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_{p-1}),\ (\beta_{q-1})\end{matrix}\right] - \frac{a^m\alpha}{\beta_1\cdots\beta_q} S_m^{(p-1,q-1)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha+1,\ (\alpha_{p-1}+1),\ (\beta_{q-1}+1)\end{matrix}\right].$$

$$(4.10)\qquad S_m^{(p,q)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_{p-1}),\ \beta_q+s,\ (\beta_q)\end{matrix}\right]$$

$$= \sum_{k=0}^{s} \begin{bmatrix} s \\ k\end{bmatrix} \frac{(\alpha)_k \prod_{j=1}^{p-1} (\alpha_j)_k}{\prod_{j=1}^{q} (\beta_j)_k} (-a^m)^k\ S_m^{(p-1,q-1)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha+k,\ (\alpha_{p-1}+k),\ (\beta_{q-1}+k)\end{matrix}\right].$$

$$(4.11)\qquad S_m^{(p,q)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_{p-2}),\ \beta_{q-1}+s,\ \beta_q+t,\ (\beta_q)\end{matrix}\right]$$

$$= \sum_{j=0}^{s} \sum_{k=0}^{t} \frac{(-s)_j (-t)_k}{j!k!} \frac{(\alpha)_{j+k} \prod_{\ell=1}^{p-2} (\alpha_\ell)_{j+k} (\beta_q+t)_j}{\prod_{\ell=1}^{q} (\beta_\ell)_{j+k}} a^{m(j+k)}$$

$$\cdot S_m^{(p-2,q-2)} \left[\begin{matrix} a, b, c, \lambda \\ \alpha+j+k, (\alpha_{p-2}+j+k), (\beta_{q-2}+j+k) \end{matrix} \right].$$

(4.12) $$S_m^{(p,q)} \left[\begin{matrix} a, b, c, \lambda \\ \alpha, (\alpha_{p-1}), 1, (\beta_{q-1}), 2 \end{matrix} \right]$$

$$= -\frac{(\beta_1-1)\cdots(\beta_{q-1}-1)}{(\alpha-1)(\alpha_1-1)\cdots(\alpha_{p-1}-1)} a^{-m} S_m^{(p-1,q-1)} \left[\begin{matrix} a, b, c, \lambda \\ \alpha, (\alpha_{p-1}-1), (\beta_{q-1}-1) \end{matrix} \right]$$

$$+ \frac{\sigma}{4\pi(\lambda+1)} a^{1-m} b^{\lambda+1} c^{1-\alpha} \frac{(\beta_1-1)\cdots(\beta_{q-1}-1)}{(\alpha-1)(\alpha_1-1)\cdots(\alpha_{p-1}-1)} F\left[\alpha-1, \frac{\lambda+1}{m}; \frac{\lambda+1}{m}+1; -\frac{b^m}{c}\right].$$

(4.13) $$S_m^{(p,q)} \left[\begin{matrix} a, b, c, \lambda \\ \alpha, (\alpha_{p-1}), 1, (\beta_{q-1}), 3 \end{matrix} \right]$$

$$= -\frac{2(\beta_1-2)_2\cdots(\beta_{q-1}-2)_2}{(\alpha-2)_2(\alpha_1-2)_2\cdots(\alpha_{p-1}-2)_2} a^{-2m} S_m^{(p-1,q-1)} \left[\begin{matrix} a, b, c, \lambda \\ \alpha, (\alpha_{p-2}-2), (\beta_{q-2}-2) \end{matrix} \right]$$

$$+ \frac{\sigma}{4\pi(\lambda+1)} a^{1-m} b^{\lambda+1} c^{1-\alpha} \frac{(\beta_1-1)\cdots(\beta_{q-1}-1)}{(\alpha-1)(\alpha_1-1)\cdots(\alpha_{p-1}-1)} {}_2F_1\left[\alpha-1, \frac{\lambda+1}{m}; \frac{\lambda+1}{m}+1; -\frac{b^m}{c}\right]$$

$$- \frac{\sigma}{2\pi(\lambda+1)} a^{1-2m} b^{\lambda+1} c^{2-\alpha} \frac{(\beta_1-2)_2\cdots(\beta_{q-1}-2)_2}{(\alpha-2)_2(\alpha_1-2)_2\cdots(\alpha_{p-1}-2)_2} {}_2F_1\left[\alpha-2, \frac{\lambda+1}{m}; \frac{\lambda+1}{m}+1; -\frac{b^m}{c}\right].$$

$$(4.14)\quad S_m^{(p,q)}\begin{bmatrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_{p-1}),\ 1,\ (\beta_{q-1}),\ s+1 \end{bmatrix}$$

$$= (-1)^{p+q+s+1}\ s!\ \frac{(1-\beta_1)_s\cdots(1-\beta_{1-q})_s}{(1-\alpha)_s(1-\alpha_1)_s\cdots(1-\alpha_{p-1})_s}\ a^{-sm}$$

$$\cdot\left\{S_m^{(p-1,q-1)}\begin{bmatrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_{p-2}-s),\ (\beta_{q-2}-s) \end{bmatrix}\right.$$

$$-\frac{\sigma a}{4\pi(\lambda+1)}\ a^{-sm}\ b^{\lambda+1}\sum_{\ell=0}^{s-1}\frac{(\alpha-s)_\ell(\alpha_1-s)_\ell\cdots(\alpha_{p-1}-s)_\ell}{(\beta_1-s)_\ell\cdots(\beta_{q-1}-s)_\ell\ \ell!}\ c^{s-\alpha-\ell}$$

$$\left.\cdot\ {}_2F_1\left[\alpha-s+\ell,\ \frac{\lambda+1}{m};\ \frac{\lambda+1}{m}+1;\ -\frac{b^m}{c}\right]\right\}.$$

$$(4.15)\quad S_m^{(p,q)}\begin{bmatrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_p),\ (\beta_{q-2}),\ \alpha_{p-1}+1,\ \alpha_p+1 \end{bmatrix}$$

$$= \frac{\alpha_p}{\alpha_p-\alpha_{p-1}}\ S_m^{(p-1,q-1)}\begin{bmatrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_{p-1}),\ (\beta_{q-2}),\ \alpha_{p-1}+1 \end{bmatrix}$$

$$-\frac{\alpha_{p-1}}{\alpha_p-\alpha_{p-1}}\ S_m^{(p-1,q-1)}\begin{bmatrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_{p-2}),\ \alpha_p,\ (\beta_{q-2}),\ \alpha_p+1 \end{bmatrix}.$$

$$(4.16)\quad S_m^{(p,q)}\begin{bmatrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_p),\ (\beta_{q-2}),\ \alpha_{p-1}+1,\ \alpha_p+k \end{bmatrix}$$

$$= \frac{(\alpha_p)_k}{(\alpha_p-\alpha_{p-1})_k} S_m^{(p-1,q-1)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_{p-2}),\ \alpha_{p-1},\ (\beta_{q-2}),\ \alpha_{p-1}+1 \end{matrix}\right]$$

$$- \frac{(\alpha_p)_k \alpha_{p-1}}{(\alpha_p-\alpha_{p-1})_k} \sum_{j=1}^{k} \frac{(\alpha_p-\alpha_{p-1})_{j-1}}{(\alpha_{p-1})_j} S_m^{(p-1,q-1)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_{p-1}),\ (\beta_{q-2}),\ \alpha_p+j \end{matrix}\right].$$

All the aforementioned results are believed to be new and are very general in nature.

5. CONTIGUOUS FUNCTIONS AND RECURRENCE RELATIONS

Recollecting the basic concept of the contiguous functions, it is known that two hypergeometric functions are said to be contiguous if they are alike except for one pair of parameters, and these parameters differ by unity. For example: ${}_2F_1(a,b;c;z)$ is contiguous to the six functions

$${}_2F_1(a\pm1,b;c;z),\quad {}_2F_1(a,b\pm1;c;z),\quad \text{and}\quad {}_2F_1(a,b;c\pm1;z).$$

Any two of three of these functions can be connected by a linear relation in z. Such a relationship is called a recurrence (or contiguous function) relation. These relationships are of great use in extending numerical tables of the function. Since $S_m^{(p,q)}[\cdots]$ includes more general formulas for orthogonal polynomials like Jacobi, Legendre, Gegenbauer, Tchebycheff, and many others, by choosing particular values for (α_p) and (β_q), one can express many such relations for various orthogonal polynomials. We omit the details.

We establish the following recurrence relations for $S_m^{(p,q)}[\cdots]$, where the upper parameters are different from the corresponding lower parameters by an integer. Our main tools to produce such recurrence relations for our integral are based on Equation (4.2) and Formulas (25) through (37) in [8]. We list the following interesting results without going into the hit-and-miss process. The entire section is by means of manipulative skill and the relations are simple and of pure nature. The first nine results are two-term recurrence relations, and at the end some three-term recurrence relations are listed.

(5.1)
$$S_m^{(p,q)}\begin{bmatrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_p),\ (\beta_q) \end{bmatrix} = \frac{\alpha_p}{\alpha_p-\alpha_{p-1}} S_m^{(p,q)}\begin{bmatrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_{p-1}),\ \alpha_p+1,\ (\beta_q) \end{bmatrix}$$
$$+\frac{\alpha_{p-1}}{\alpha_p-\alpha_{p-1}} S_m^{(p,q)}\begin{bmatrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_{p-2}),\ \alpha_{p-1}+1,\ \alpha_p,\ (\beta_q) \end{bmatrix}.$$

(5.2)
$$S_m^{(p,q)}\begin{bmatrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_p),\ (\beta_q) \end{bmatrix}$$
$$= \frac{\beta_q-1}{\beta_q-\alpha_p-1} S_m^{(p,q)}\begin{bmatrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_p),\ (\beta_{q-1}),\ \beta_q-1 \end{bmatrix}$$
$$-\frac{\alpha_p}{\beta_q-\alpha_p-1} S_m^{(p,q)}\begin{bmatrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_{p-1}),\ \alpha_p+1,\ (\beta_q) \end{bmatrix}.$$

(5.3)
$$S_m^{(p,q)}\begin{bmatrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_p),\ (\beta_q) \end{bmatrix} = \frac{\beta_q-1}{\beta_q-\beta_{q-1}} S_m^{(p,q)}\begin{bmatrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_p),\ (\beta_{q-1}),\ \beta_q-1 \end{bmatrix}$$

$$-\frac{\beta_{q-1}-1}{\beta_q-\beta_{q-1}} S_m^{(p,q)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_p),\ (\beta_{q-2}),\ \beta_{q-1}-1,\ \beta_q \end{matrix}\right].$$

$$(5.4)\quad S_m^{(p,q)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_p),\ (\beta_q) \end{matrix}\right] = \frac{\alpha_{p-1}(\beta_q-\alpha_p)}{\beta_q(\alpha_{p-1}-\alpha_p)} S_m^{(p,q)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha,(\alpha_{p-2}),\alpha_{p-1}+1,\alpha_p,(\beta_{q-1}),\beta_q+1 \end{matrix}\right]$$

$$-\frac{\alpha_p(\beta_q-\alpha_{p-1})}{\beta_q(\alpha_{p-1}-\alpha_p)} S_m^{(p,q)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha,(\alpha_{p-1}),\alpha_p+1,(\beta_{q-1}),\beta_q+1 \end{matrix}\right].$$

$$(5.5)\quad S_m^{(p,q)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_p),\ (\beta_q) \end{matrix}\right] = \frac{(\beta_{q-1}-\alpha_p)(\beta_q-1)}{(\alpha_p-1)(\beta_q-\beta_{q-1})} S_m^{(p,q)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha,(\alpha_{p-1}),\alpha_p-1,(\beta_{q-1}),\beta_q-1 \end{matrix}\right]$$

$$-\frac{(\beta_q-\alpha_p)(\beta_{q-1}-1)}{(\alpha_p-1)(\beta_q-\beta_{q-1})} S_m^{(p,q)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha,(\alpha_{p-1}),\alpha_p-1,(\beta_{q-2}),\beta_{q-1}-1,\beta_q \end{matrix}\right].$$

$$(5.6)\quad S_m^{(p,q)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_p),\ (\beta_q) \end{matrix}\right] = S_m^{(p,q)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_{p-1}),\ \alpha_p+1,\ (\beta_q) \end{matrix}\right]$$

$$+\frac{\alpha\alpha_1\cdots\alpha_{p-1}}{\beta_1\cdots\beta_q}\ \frac{a^m}{c} S_m^{(p,q)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha+1,\ (\alpha_p+1),\ (\beta_q+1) \end{matrix}\right].$$

$$(5.7)\quad S_m^{(p,q)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_p),\ (\beta_q) \end{matrix}\right] = S_m^{(p,q)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_p),\ (\beta_{q-1}),\ \beta_q+1 \end{matrix}\right]$$

$$-\frac{\alpha\alpha_1\cdots\alpha_{p-1}}{\beta_1\cdots\beta_q(\beta_q+1)}\ \frac{a^m}{c} S_m^{(p,q)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha+1,\ (\alpha_p+1),\ (\beta_{q-1}+1),\ (\beta_q+2) \end{matrix}\right].$$

$$(5.8)\quad S_m^{(p,q)}\begin{bmatrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_p),\ (\beta_q) \end{bmatrix} = S_m^{(p,q)}\begin{bmatrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_{p-2}),\ \alpha_{p-1}+1,\ \alpha_p-1,\ (\beta_q) \end{bmatrix}$$

$$+\frac{\alpha\alpha_1\cdots\alpha_{p-2}}{\beta_1\cdots\beta_q}(\alpha_p-\alpha_{p-1}-1)\frac{a^m}{c}S_m^{(p,q)}\begin{bmatrix} a,\ b,\ c,\ \lambda \\ \alpha+1,\ (\alpha_{p-1}+1),\ \alpha_p,\ (\beta_q+2) \end{bmatrix}.$$

$$(5.9)\quad S_m^{(p,q)}\begin{bmatrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_p),\ (\beta_q) \end{bmatrix} = S_m^{(p,q)}\begin{bmatrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_{p-1}),\ \alpha_p+1,\ (\beta_{q-1}),\ \beta_q+1 \end{bmatrix}$$

$$+\frac{\alpha\alpha_1\cdots\alpha_p}{\beta_1\cdots\beta_q(\beta_q+1)}(\beta_q-\alpha_p)\frac{a^m}{c}S_m^{(p,q)}\begin{bmatrix} a,\ b,\ c,\ \lambda \\ \alpha+1,\ (\alpha_p+1),\ (\beta_{q-1}+1),\ \beta_1+2 \end{bmatrix}.$$

$$(5.10)\quad S_m^{(p,q)}\begin{bmatrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_p),\ (\beta_q) \end{bmatrix}$$

$$=\frac{\alpha_{p-1}}{\beta_q}S_m^{(p,q)}\begin{bmatrix} a,\ b,\ c,\ \lambda \\ \alpha,(\alpha_{p-2}),\alpha_{p-1}+1,\alpha_p+1,(\beta_{q-1}),\beta_q+1 \end{bmatrix}$$

$$+\frac{\beta_q-\alpha_{p-1}}{\beta_q}S_m^{(p,q)}\begin{bmatrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_{p-1}),\ \alpha_p+1,\ (\beta_{q-1}),\ \beta_q+1 \end{bmatrix}$$

$$+\frac{\alpha\alpha_1\cdots\alpha_{p-1}}{\beta_q}\frac{a^m}{c}S_m^{(p,q)}\begin{bmatrix} a,\ b,\ c,\ \lambda \\ \alpha+1,\ (\alpha_p+1),\ (\beta_q+1) \end{bmatrix}.$$

$$(5.11)\quad S_m^{(p,q)}\begin{bmatrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_p),\ (\beta_q) \end{bmatrix} = \frac{(\beta_{q-2}-1)(\beta_{q-1}-1)(\alpha_{p-1}-\beta_q)(\alpha_p-\beta_q)}{(\alpha_{p-1}-1)(\alpha_p-1)(\beta_{q-2}-\beta_q)(\beta_{q-1}-\beta_q)}$$

$$\cdot\, S_m^{(p,q)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_{p-2}),\ \alpha_{p-1}-1,\ \alpha_p-1,\ (\beta_{q-3}),\ \beta_{q-2}-1,\ \beta_{q-1}-1,\ \beta_q \end{matrix}\right]$$

$$+\frac{(\beta_{q-2}-1)(\beta_q-1)(\alpha_{p-1}-\beta_{q-1})(\alpha_p-\beta_{q-1})}{(\alpha_{p-1}-1)(\alpha_p-1)(\beta_{q-2}-\beta_{q-1})(\beta_q-\beta_{q-1})}$$

$$\cdot\, S_m^{(p,q)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_{p-2}),\ \alpha_{p-1}-1,\ \alpha_p-1,\ (\beta_{q-3}),\ \beta_{q-2}-1,\ \beta_{q-1},\ \beta_q-1 \end{matrix}\right]$$

$$+\frac{(\beta_{q-1}-1)(\beta_q-1)(\alpha_{p-1}-\beta_{q-2})(\alpha_p-\beta_{q-2})}{(\alpha_{p-1}-1)(\alpha_p-1)(\beta_{q-1}-\beta_{q-2})(\beta_q-\beta_{q-2})}$$

$$\cdot\, S_m^{(p,q)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_{p-2}),\ \alpha_{p-1}-1,\ \alpha_p-1,\ (\beta_{q-2}),\ \beta_{q-1}-1,\ \beta_q-1 \end{matrix}\right].$$

$$(5.12)\quad S_m^{(p,q)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_p),\ (\beta_q) \end{matrix}\right] = \frac{\alpha_{p-1}\alpha_p(\beta_{q-1}-\alpha_p)(\beta_q-\alpha_p)}{\beta_{q-1}\beta_q(\alpha_{p-2}-\alpha_p)(\alpha_{p-1}-\alpha_p)}$$

$$\cdot\, S_m^{(p,q)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_{p-3}),\ \alpha_{p-2}+1,\ \alpha_{p-1}+1,\ \alpha_p,\ (\beta_{q-2}),\ \beta_{q-1}+1,\ \beta_q+1 \end{matrix}\right]$$

$$+\frac{\alpha_{p-2}\alpha_p(\beta_{q-1}-\alpha_{p-1})(\beta_q-\alpha_{p-1})}{\beta_{q-1}\beta_q(\alpha_{p-2}-\alpha_{p-1})(\alpha_p-\alpha_{p-1})}$$

$$\cdot\, S_m^{(p,q)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_{p-3}),\ \alpha_{p-2}+1,\ \alpha_{p-1},\ \alpha_p+1,\ (\beta_{q-2}),\ \beta_{q-1}+1,\ \beta_q+1 \end{matrix}\right]$$

$$+\frac{\alpha_{p-1}\alpha_p(\beta_{q-1}-\alpha_{p-2})(\beta_q-\alpha_{p-2})}{\beta_{q-1}\beta_q(\alpha_{p-1}-\alpha_{p-2})(\alpha_p-\alpha_{p-2})}$$

$$\cdot\, S_m^{(p,q)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_{p-2}),\ \alpha_{p-1}+1,\ \alpha_p+1,\ (\beta_{q-2}),\ \beta_{q-1}+1,\ \beta_q+1 \end{matrix}\right].$$

6. RELATIONS OF THE SPECIAL FORMS

We list here four more three–term recurrence relations for $S_m^{(p,q)}[\cdots]$. From Equation (4.2) and Formulas (38) to (41) in [8], it is easy to establish the following relations that are independent of any multipliers.

$$(6.1)\qquad S_m^{(p,q)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_p),\ (\beta_{q-1}),\ 1-\beta_{q-1} \end{matrix}\right] + S_m^{(p,q)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_p),\ (\beta_{q-2}),\ -\beta_{q-1},\ \beta_{q-1}+1 \end{matrix}\right]$$

$$= 2S_m^{(p,q)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_p),\ (\beta_{q-2}),\ 1-\beta_{q-1},\ \beta_{q-1}+1 \end{matrix}\right].$$

$$(6.2)\qquad S_m^{(p,q)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_p),\ (\beta_{q-2}),\ -\alpha_p,\ \alpha_p+1 \end{matrix}\right] - 2S_m^{(p,q)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_p),\ (\beta_{q-2}),\ 1-\alpha_p,\ \alpha_p+1 \end{matrix}\right]$$

$$= -\,S_m^{(p-1,q-1)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_{p-1}),\ (\beta_{q-2}),\ 1-\alpha_p \end{matrix}\right].$$

$$(6.3)\qquad S_m^{(p,q)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_p),\ (\beta_{q-1}),\ \alpha_p+1 \end{matrix}\right] + S_m^{(p,q)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_{p-1}),\ -\alpha_p,\ (\beta_{q-1}),\ 1-\alpha_p \end{matrix}\right]$$

$$= 2S_m^{(p+1,q+1)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_p),\ -\alpha_p,\ (\beta_{q-1}),\ 1-\alpha_p,\ \alpha_p+1 \end{matrix}\right].$$

(6.4) $$S_m^{(p,q)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_p),\ 1-\alpha_{p-1},\ (\beta_q) \end{matrix}\right] + S_m^{(p,q)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_{p-2}),\ -\alpha_{p-1},\ \alpha_{p-1}+1,\ (\beta_q) \end{matrix}\right]$$

$$= 2S_m^{(p,q)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha,\ (\alpha_{p-2}),\ -\alpha_{p-1},\ \alpha_{p-1},\ (\beta_q) \end{matrix}\right].$$

7. SPECIAL CASES AND APPLICATIONS

One immediate special case of our integral is when $p = 1 = q$ and $m = 2$. It gives

$$S_2^{(1,1)}\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha,\ \beta,\ \gamma \end{matrix}\right] = H\left[\begin{matrix} a,\ b,\ c,\ \lambda \\ \alpha,\ \beta,\ \lambda \end{matrix}\right],$$

which is defined and studied by Kalla *et al.* (*cf.* [5], p. 221). As a matter of fact, Kalla's results are a slight generalization of those of Hubbell, Bach, and Lamkin [4]. It is also true that by selecting suitable values for different parameters, (1.1) can be reduced to several integrals with potential application in radiation-field problems of specific configuration of source, barrier, and detector. Engineers use such results in illumination and heat-exchange programs. It is needless to mention here that many other known results of this nature can be deduced as particular cases of our integral (1.1).

8. COMPUTATIONS

For $p = 1 = q$ and $m = 2$, Equation (2.3) will correspond to a known result (20) of Kalla *et al.* [5]. Hence for various values of a, b and c, for $\lambda = 0$, $\alpha = 1$, $\alpha_1 = 0.5$, $\beta_1 = 1.5$, one can get a table analogous to Table I in [5]. Similarly, Table II and Table III of [5] can also be extended appropriately. We leave the preparation of these tables as exercises for the interested readers. One can also use Simpson's rule to tabulate our general integral (1.1).

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