Generation of Novel Wavelet Transformations towards applications in Tensor Network Algorithms

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Introduction

A wavelet is a function, Ψ(t), of finite duration which can be scaled and shifted as Ψ(at + b), where a, b ∈ ℝ, to create a family of wavelets.

A wavelet can be applied to a signal, f(t), in a convolution to give the overlap between those functions, c.

$$c(a,b) = \int f(t)\psi(at+b)dt$$

Wavelets in a family can encode information at different scales to form a multiresolution analysis, allowing any signal to be transformed to arbitrary precision [1].

Some wavelets, such as any Daubechies wavelet, can form a complete orthonormal basis in $L^2(\mathbb{R})$ with the set

$$\{\psi_{k,n}(t) = \sqrt{2}^{-k}\psi(2^{-k}t - n) \mid k, n \in \mathbb{Z}\}$$

A discrete wavelet transformation (DWT) is a signal processing tool which transforms a signal into information about the frequency (scale) over many different intervals of the signal.

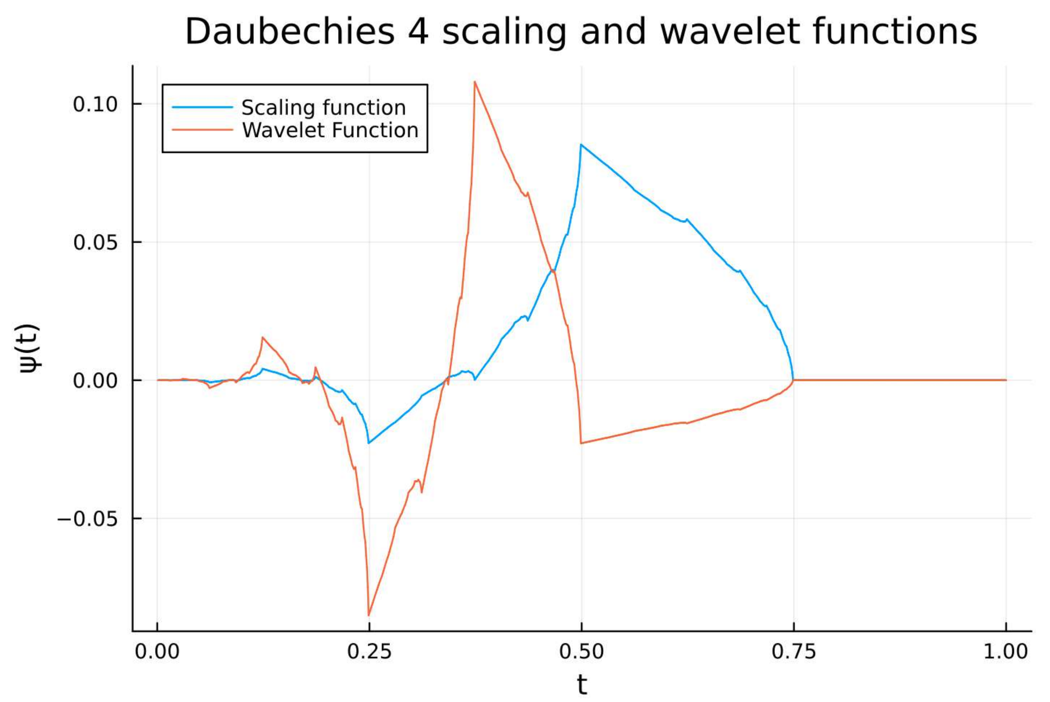

Figure 1: Daubechies 4 scaling and wavelet functions.

A gapless system is a system with a continuous or near-continuous spectrum of resonant frequencies.

Gapless systems are notoriously difficult to solve analytically and numerically.

Certain materials, like graphene and some metals, can be modeled as gapless systems. The Ising model is an example of a gapped system.

The multi-scale entanglement renormalization ansatz (MERA) is a tensor network algorithm used in finding the ground state energy of a quantum system in Hamiltonian form [2].

MERA can be applied to solve for the ground state energy of gapless quantum systems by using wavelet gates.

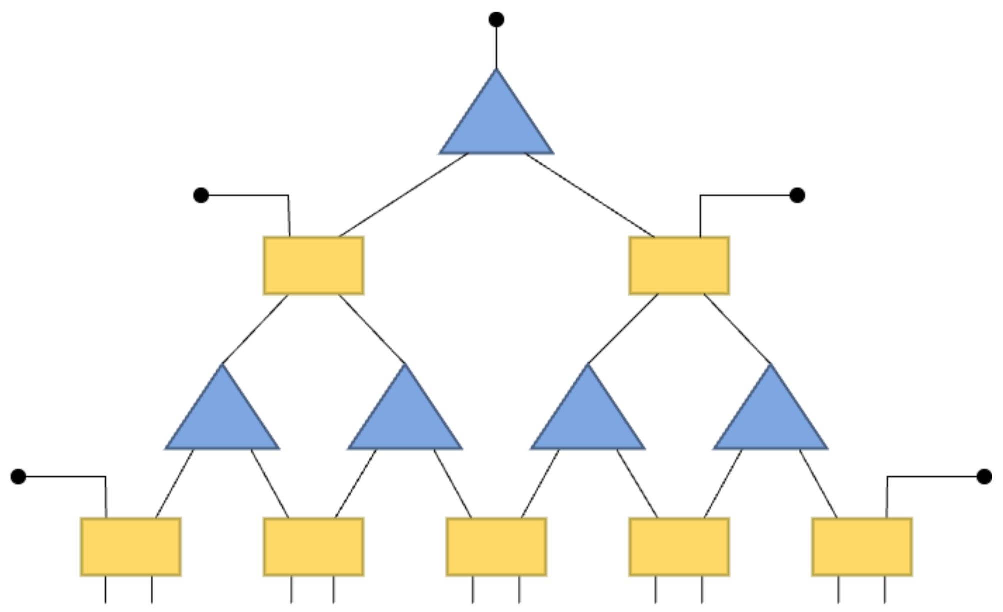

Figure 2: MERA tensor network.

Novel Wavelet Generation

G. Evenbly and S. R. White have shown that orthogonal wavelets can be constructed with the following pyramidal structure of 2-by-2 unitary rotation matrices [3]:

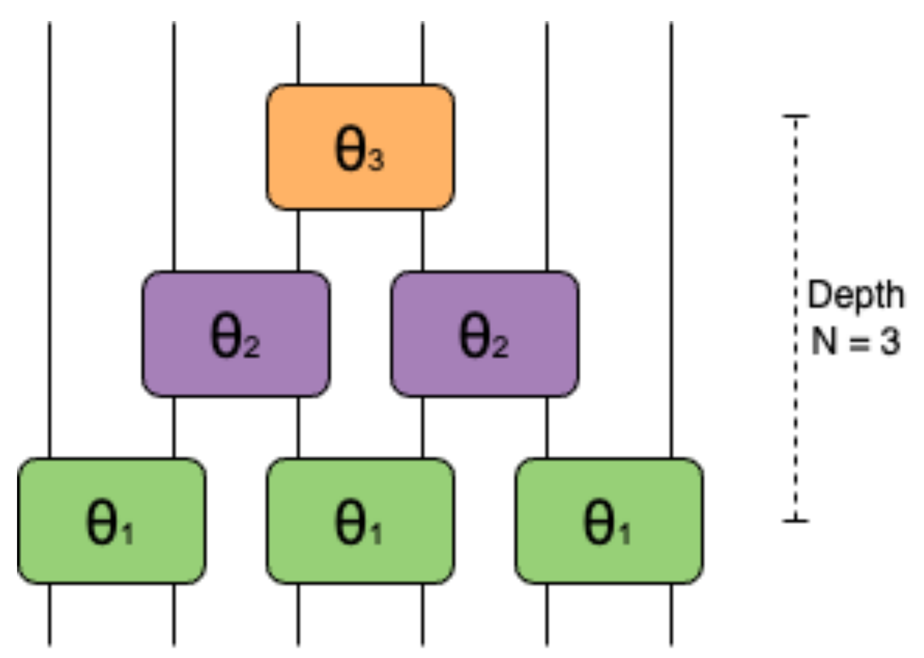

Figure 3: Unitary circuit structure for wavelet generation.

This is done be satisfying the vanishing moment equations:

$$\sum_{r=1}^{2N} (r^{\alpha} g_r) = 0$$

Where N is the level of the circuit, α = [0, 1, . . . , N − 1], and **g** = [g_1, g_2, ..., g_{2N}] is the (N+1)th column vector of the whole unitary circuit.

By applying the Nelder-Mead minimization algorithm over a cost function of the vanishing moments, each θ_n value can be solved for small N.

It can be shown this construction method is flawed in that barren plateau-like features exist in the solution space of the cost function.

This causes the circuit to become harder to solve numerically with increasing depth, N.

We discovered that the cost function can be altered to give novel wavelets by introducing a parameter, β ∈ ℝ, in the exponent of the vanishing moment equations.

$$\sum_{r=0}^{2N} (r^{\beta\alpha} g_r) = 0$$

The barren plateau-like features in the original formulation (β =1) have their basins widened for β < 1 and become much easier to solve numerically.

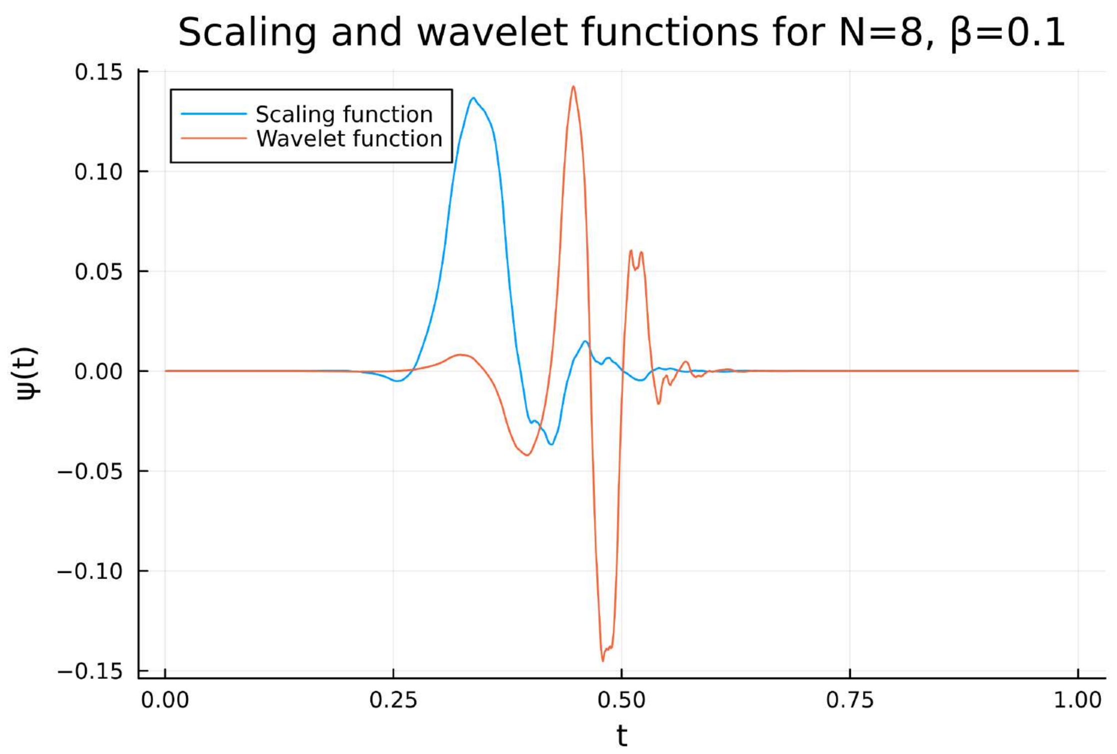

Figure 4: Generated wavelet of a depth N=8 unitary circuit for β=0.1.

Results

Reimplementation of the wavelet generation method created by Evenbly and White (β=1 always) yields precise wavelet coefficients up to circuit depth of N=4 then the cost function increases significantly.

Generated Wavelet Data				
N	Unitary Circuit Angles	Generated Wavelet Coefficients	Known Wavelet Coefficients	Cost Function
1	$\theta_1 = -5.4978$	$g_{1gen} = -0.7071$ $g_{2gen} = 0.7071$	$g_{1known} = -0.7071$ $g_{2known} = 0.7071$	$3.80 \cdot 10^{-65}$
2	$\theta_1 = 0.2618$ $\theta_2 = 1.0240$	$g_{1gen} = -0.4830$ $g_{2gen} = 0.8365$ $g_{3gen} = -0.2241$ $g_{4gen} = -0.1294$	$g_{1known} = -0.4830$ $g_{2known} = 0.8365$ $g_{3known} = -0.2241$ $g_{4known} = -0.1294$	$3.25 \cdot 10^{-64}$
3	$\theta_1 = -5.2106$ $\theta_2 = 6.2582$ $\theta_3 = -5.8921$	$g_{1gen} = -0.0352$ $g_{2gen} = -0.0854$ $g_{3gen} = 0.1350$ $g_{4gen} = 0.4599$ $g_{5gen} = -0.8069$ $g_{6gen} = 0.3327$	$g_{1known} = 0.0352$ $g_{2known} = 0.0854$ $g_{3known} = -0.1350$ $g_{4known} = -0.4599$ $g_{5known} = 0.8069$ $g_{6known} = -0.3327$	$5.99 \cdot 10^{-61}$

Figure 5: Table of generated wavelet circuit angles, coefficients, and final cost function value for β=1, with comparison to Daubechies wavelet coefficients.

For a circuit of depth N=2, the generation algorithm was run over a set of β-values. Each numerically solved set of θ-values in the solution space falls on a linear line.

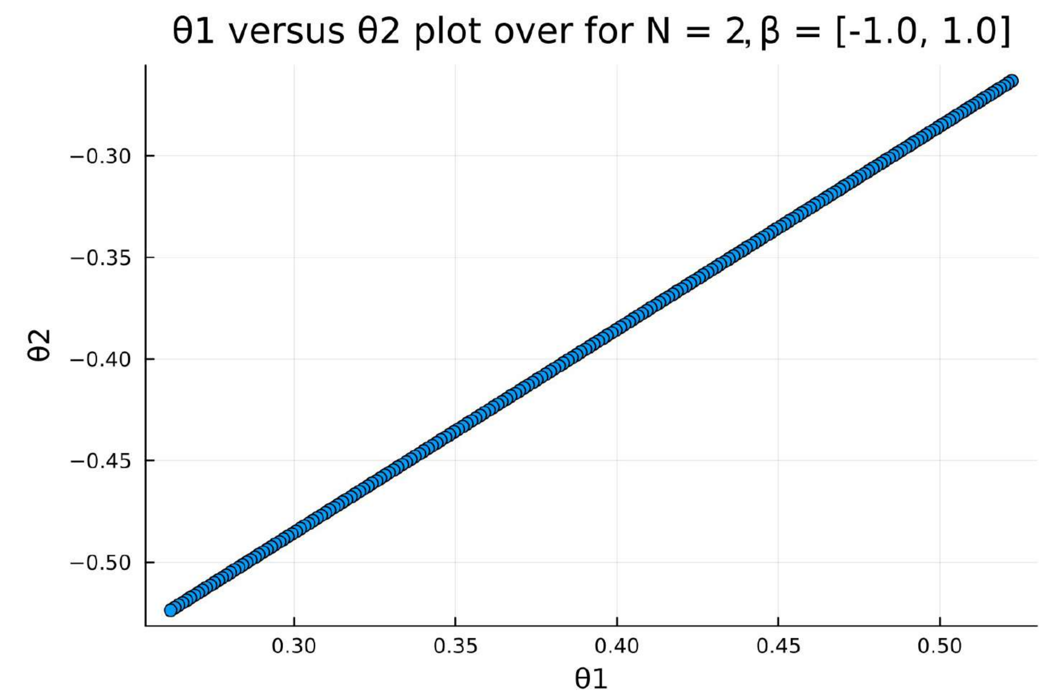

Figure 6: θ-values of a depth N=2 circuit over β =[-1, 1].

Further Research

An improved numerical algorithm for solving unitary wavelet circuits will be developed to solve for larger circuit-depths. It will run as follows:

1. Set β << 1 and choose a point in the solution space to solve around.
2. Minimize the β-dependant cost function starting at the chosen initial point and record the resulting θ-values.
3. Increase β, use the θ-values as the new initial point, and repeat step 2 until β=1.

The MERA tensor network algorithm and its wavelet-gate variation for solving gapless systems will be implemented.

Conclusions

- It was confirmed that wavelets can be constructed in a formulation of 2-by-2 unitary rotation matrices.
- Barren plateau-like features exist in this formulation for wavelets, making it difficult to solve for larger circuit depths.
- The vanishing moment equations can be altered to widen the basins of the barren plateau-like features.
- The solutions to the β-dependant vanishing moment equations exist on a continuum for at least the depth N=2 circuit.

Acknowledgements

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References

[1] Burke Hubbard, B. (1998). *The world according to wavelets: The story of a mathematical technique in the making*. Peters.

[2] Evenbly, G. (2018, December 29). *Mera*. Tensors.net. https://www.tensors.net/mera

[3] Evenbly, G., & White, S. R. (2018). Representation and design of wavelets using unitary circuits. *Physical Review A*, *97*(5). https://doi.org/10.1103/physreva.97.052314