

C^* -algebras constructed from factor groupoids and their analysis through relative
 K -theory and excision

by

Mitch Haslehurst

MMath, University of Waterloo, 2016

BA (Honours), Nipissing University, 2015

A Dissertation Submitted in Partial Fulfillment of the
Requirements for the Degree of

DOCTOR OF PHILOSOPHY

in the Department of Mathematics and Statistics

© Mitch Haslehurst, 2022
University of Victoria

All rights reserved. This dissertation may not be reproduced in whole or in part, by
photocopying or other means, without the permission of the author.

C^* -algebras constructed from factor groupoids and their analysis through relative
 K -theory and excision

by

Mitch Haslehurst

MMath, University of Waterloo, 2016

BA (Honours), Nipissing University, 2015

Supervisory Committee

Dr. Ian F. Putnam, Supervisor
(Department of Mathematics and Statistics)

Dr. Marcelo Laca, Departmental Member
(Department of Mathematics and Statistics)

Dr. Heath Emerson, Departmental Member
(Department of Mathematics and Statistics)

Dr. Michel Lefebvre, Outside Member
(Department of Physics and Astronomy)

ABSTRACT

We address the problem of finding groupoid models for C^* -algebras given some prescribed K -theory data. This is a reasonable question because a groupoid model for a C^* -algebra reveals much about the structure of the algebra. A great deal of progress towards solving this problem has been made using constructions with inductive limits, subgroupoids, and dynamical systems. This dissertation approaches the question with a more specific methodology in mind, with factor groupoids.

In the first part, we develop a portrait of relative K -theory for C^* -algebras using the general framework of Banach categories and Banach functors due to Max Karoubi. The purpose of developing such a portrait is to provide a means of analyzing the K -theory of an inclusion of C^* -algebras, or more generally of a $*$ -homomorphism between two C^* -algebras. Another portrait may be obtained using a mapping cone construction and standard techniques (it is shown that the two presentations are naturally and functorially isomorphic), but for many examples, including the ones considered in the second part, the portrait obtained by Karoubi's construction is more convenient.

In the second part, we construct examples of factor groupoids and analyze their C^* -algebras. A factor groupoid setup (two groupoids with a surjective groupoid homomorphism between them) induces an inclusion of two C^* -algebras, and therefore the portrait of relative K -theory developed in the first part, together with an excision theorem, can be used to elucidate the structure. The factor groupoids are obtained as quotients of AF-groupoids and certain extensions of Cantor minimal systems using iterated function systems. We describe the K -theory in both cases, and in the first case we show that the K -theory of the resulting C^* -algebras can be prescribed through the factor groupoids.

Contents

Supervisory Committee	ii
Abstract	iii
Contents	iv
List of Figures	vi
Acknowledgements	vii
Dedication	ix
Introduction	1
Chapter 1 Preliminaries	5
1.1 C*-algebras	5
1.1.1 Examples and constructions	5
1.1.2 Spectral theory and continuous functional calculus	9
1.2 K-theory	11
1.2.1 The group $K_0(A)$	11
1.2.2 The group $K_1(A)$	14
1.2.3 Stability, suspensions, exact sequences, and the Künneth theorem	15
1.2.4 The Elliott classification program	20
1.3 Groupoids	21
1.3.1 AF-groupoids and Cantor minimal systems	27
Chapter 2 A Karoubian portrait of relative K-theory for C*-algebras	33
2.1 The main results	35
2.2 Banach categories, Banach functors, and an alternate picture of $K_1(A)$	38
2.3 A portrait of $K_*(\phi)$	41
2.3.1 Projections, unitaries, and partial isometries	49

2.4	The proof of Theorem 2.1.1	53
2.5	The proof of Theorem 2.1.2	57
2.5.1	Half-exactness and split exactness	57
2.5.2	The index map	60
2.5.3	Bott periodicity and the exponential map	66
2.6	The proof of Theorem 2.1.3	68
2.7	Composition of $*$ -homomorphisms	71
2.8	Formulas for Δ , β_ϕ , and ∂_0	73
2.9	The excision theorem for factor groupoids	75
2.10	Summary and examples	77
Chapter 3 Factor groupoids of AF-groupoids: noncommutative Cantor functions		82
3.1	The main results	83
3.2	The spaces X_ξ	85
3.3	The groupoids R_ξ	91
3.4	The K -theory and tracial states of $C_r^*(R_\xi)$	96
Chapter 4 Factor groupoids from non-homogeneous extensions of Cantor minimal systems		103
4.1	The main results	104
4.2	The (modified) DPS construction	105
4.3	The K -theory of $C_r^*(G)$	108
4.4	Cubes, the Cantor set, and the Sierpiński triangle	112
Chapter 5 Conclusions and further directions		115
5.1	Axiomatization: generalized relative homology	115
5.2	Further examples of factor groupoids	118
Bibliography		119

List of Figures

Figure 1.1	A Bratteli diagram	28
Figure 3.1	The Bratteli diagram (V, E) in Example 3.2.10	91
Figure 3.2	Geometric approximations of X_ξ from Example 3.2.11	92
Figure 4.1	Iterative construction of the Sierpiński triangle.	113

ACKNOWLEDGEMENTS

I tried my best to organize everyone I wanted to thank into separate paragraphs based on how/when/where I met them and the sort of impact they had. Inevitably, there was some overlap, so feel free to attribute more than one of the personalized “thank you” statements to yourself accordingly.

I would like to begin by thanking all of my fellow graduate students and sessional instructors at the University of Victoria, Chris Bruce, Dan Hudson, Anna Duwenig, Dina Buric, Emily Rose Korfanty, Joseph Horan, Brittany Halverson-Duncan, Mark Piraino, Jeremy Hume, Tyler Schulz, Brendan Steed, Gavin Goerke, Mackenzie Wheeler, Flora Bowditch, Mackenzie Carr, Felicia Halliday, Tom Thompson, Amanda Malloch, George Lee, Todd Schmid, Sonya Romanovska, Kate Taylor, Trishala Bhasin, Leonardo Machado, Saeed Rezvani, Moloud Mazaheri, Kian Rahimabadi, Arash Isapour, Mahsa Mahtab, and Neil Barney: thank you for the engaging conversations, countless coffee breaks, nights at the pub, and fun social events.

To my fellow young researchers in the field of operator algebras, Boyu Li, Jamie Gabe, Ali Miller, Kari Eifler, Sam Kim, Shen Lu, Nick Manor, Luca Gazdag, Ulrik Enstad, Michael Francis, Johannes Christensen, Thiago Raszeja, Lucas Affonso, and Adam Dor-On: thank you for all of the wonderful interactions, insightful mathematical discussions, and fun social events that occurred through numerous conferences, seminars, and visiting positions that gave me steam to keep going. There is something magical about sharing a beer and having a conversation about interesting research with people who are just as driven to find the answers as you are.

To my fellow participants in the Learning and Teaching in Higher Education (LATHE) program at the University of Victoria, Janice Niemann, Natalie Boldt, Liz Williams, Tasha Jarisz, Mohammed Elsayed, Pierre Iachetti, and Jeremy Witringer: thank you for the insightful discussion about pedagogy, and for strong friendships that continue to this day.

To my friends and fellow math students during my undergraduate time at Nipissing University, Jacob Baisley, Niki Lapp, Alyssa Di Luca, Layla Tomkinson, Michael Anderson, Adam Morin, Heather-Lynn Topping, Bradley Lalonde, Beau Bedard, Zac Cluett, and Alex Rothwell: thank for amazing first post-secondary years. Those assignments were rough at times, thank goodness for all those parties on the weekends.

To my professors and mentors over the years, Marcelo Laca, Heath Emerson, Michel Lefebvre, Rogerio de Sousa, Chris Eagle, Brian Forrest, Nico Spronk, Ken Davidson,

Murat Tuncali, Alex Karassev, Vesko Valov, and Logan Hoehn: thank you for the inspiration to pursue fun, challenging problems, and for giving me plenty of clever ideas for how to solve many of them. I owe so much of my insight to you all. I am particularly grateful to my PhD supervisor Ian Putnam: thank you for five lucrative years working with you. I could not have asked for a more knowledgeable and humble mentor. It seems to me that you possess an endless supply of two things, and I hope some of both of them rubbed off on me: ingenious mathematical ideas and humorous anecdotes.

Finally, to my parents Shane and Lorianne Haslehurst, my siblings Justin, Cameron, and Jamielee Haslehurst, my paternal grandparents Darwin and Isabel Haslehurst, and my maternal grandparents Jack and Mary Ann Jones: thank you for shaping the person I am today. It is from you all that I learned to be kind, assertive, and hard-working, and it is thanks to you all that I have made it as far as I have.

In short, to my wonderful colleagues, mentors, family, and friends: it has been, and continues to be, a pleasure knowing all of you, and I hope that all of us continue to accomplish great things in the years to come.

DEDICATION

To my late grandmother Mary Ann Jones, who passed away on September 10, 2021. Though you will not witness the culmination, you were with me for the journey, and for that I am eternally grateful. Though it may have seemed a passing comment at the time, I recall a particular phrase you spoke to me that will remain in my memory for as long as I live. “I’m always proud of you.”

Introduction

Since the seminal papers of Murray and von Neumann [20], the study of operator algebras has long been intertwined with the study of dynamics. This is hardly surprising, one reason for which is that operator algebras provide a robust mathematical framework for physical systems in quantum mechanics, and one would be hard pressed to do anything interesting in physics without doing any dynamics. But more abstractly, operator algebras appear to have an ability to capture symmetry very well, which may explain their fruitful interactions with numerous areas of mathematics, including group theory, number theory, graph theory, ergodic theory, logic, knot theory, probability, and statistics, to name a few. For many of these interactions, the notion of a groupoid provides a very useful tool with which one may analyze the structure of the objects involved.

Groupoids are algebraic objects that provide an extremely general framework for studying mathematical phenomena that exhibit properties or behaviour that is locally symmetric (as opposed to globally symmetric, for which groups are very useful). This makes them an ideal lens through which to study dynamical systems, as well as situations arising in noncommutative geometry and physics such as foliations, tilings, and quasicrystals [22]. It was in 1980 that Renault formally introduced étale groupoids to the theory of C^* -algebras in his thesis [31], and this relationship has only grown since. In fact, it has grown so much that groupoids have actually begun to compete with C^* -algebras for the title “noncommutative spaces”. This is illustrated by two facts. First, if a C^* -algebra has a groupoid model (that is, it is isomorphic to a groupoid C^* -algebra), much of the information about the C^* -algebra can be deduced from information about the groupoid. Second, C^* -algebras that arise from a great many types of constructions have groupoid models, including commutative C^* -algebras, approximately finite dimensional (AF) C^* -algebras, group C^* -algebras, crossed products from locally compact groups acting on spaces, ring C^* -algebras, graph C^* -algebras, and more.

It is therefore not surprising that the problem of finding groupoid models for C^* -algebras has been, and continues to be, under a vast amount of investigation. If we restrict our attention to C^* -algebras that are classifiable in the Elliott program (see subsection 1.2.4), this reduces the problem to finding a groupoid model for some given

K -theory and tracial data, all the while not so restrictive that we are left with quite a wealth of interesting examples. An almost complete solution to this problem (for classifiable C^* -algebras) is provided by Xin Li in [19], where it is shown that every classifiable C^* -algebra has a groupoid model. (“Almost” refers to the fact that one needs a twist on the groupoid in order to deal with the case when the K_0 -group has torsion; determining when the twist can be removed and torsion retained is an active area of research.) The construction is essentially to mimic an earlier construction of George Elliott in [13] using inductive limits, modifying the inductive systems so that the connecting maps preserve Cartan subalgebras, resulting in a groupoid model for the limit. In another vein, Ian Putnam [29] discovered a method of constructing a minimal equivalence relation on a Cantor set such that the resulting C^* -algebra is tracially AF, and can exhaust all Elliott invariants where the K_0 -group is a simple, acyclic dimension group, and the K_1 -group is a countable, torsion-free abelian group. The idea behind the proof is to enlarge an AF-groupoid to regard it as an open subgroupoid of a larger groupoid, which produces two C^* -algebras, one contained in the other as a subalgebra. The K -theory is computed using relative K -theory and an excision result. Other subgroupoids constructed for similar purposes may be found in [9]. The methods in [9] improve upon those in [19] and [29] in that it is possible to prescribe K -theory with torsion without a twist.

In both [29] and [9], the crucial tool in computing the K -theory is the excision theorem of [27]. This excision theorem has recently been generalized in [30], no longer requiring the groupoids in the intended applications to be principal and étale, and without a cocycle. The key to the generalization seemed to be the choice of a different portrait of relative K -theory for C^* -algebras which is drawn from a general framework of Banach categories and Banach functors pioneered by Max Karoubi [18], which we will spend a great deal of time addressing in the first portion of this project.

When trying to cook up a groupoid whose C^* -algebra will have a prescribed Elliott invariant, the general line of attack is to start with some algebra(s) with familiar K -theory, then use techniques with tensor products, inductive limits, and subalgebras to produce examples such that the end result has the desired K -theory. Regarding two of these techniques, computing the K -theory is usually cut-and-dried. For tensor products, one has the Künneth Theorem (at least for “reasonably nice” C^* -algebras, see Theorem 1.2.19). For inductive limits, K -theory is continuous; it does not distinguish between elements that are nearby in norm, therefore the K -theory of an inductive limit is completely contained in the K -theory of the approximating subalgebras. Analyzing the K -theory associated with an inclusion of a C^* -subalgebra demands a bit more attention; this leads to the subject of relative K -theory.

In topological K -theory, given a compact space X and a closed subspace Y , one can form the relative group $K^0(X, Y)$ using vector bundles. The noncommutative version of this construction is to consider a unital C^* -algebra A , an ideal I of A , and pairs of

projections in matrix algebras over A that are equivalent via partial isometries modulo I . The relative group (usually denoted $K_0(A, I)$, see [4]) thus captures homological data of the quotient map from A to A/I . Generalizing this situation to a more general $*$ -homomorphism (for example, an inclusion map) requires a bit more structure. One method is to use the mapping cone, which is used in [27], and another is to use Karoubi’s approach, as in [30]. It is the latter definition that we consider in this project, and we show that both definitions give the same theory in a natural way and produce the same six-term exact sequences, though Karoubi’s approach takes a great deal more work to reach the desired results. The main trade-off is that the proof of the more general excision theorem appears to rely quite heavily on the presentation in [18]. There are a few other benefits to the latter portrait, which we discuss further in Chapter 2.

Once the Karoubian portrait of relative K -theory is in hand, the excision theorem in [30] will be at our disposal. This theorem, while stated in a great deal of generality, has two intended uses for groupoids. The first is the so-called “subgroupoid” situation which is used in [29] and [9]. The second, which has seen little exploration to date and which we will focus on, is the “factor groupoid” situation. This situation requires two groupoids G and G' with a surjective groupoid homomorphism $\pi : G \rightarrow G'$ satisfying some natural hypotheses, which results in an inclusion between their C^* -algebras, $C_r^*(G') \subseteq C_r^*(G)$. The term comes from the theory of dynamical systems via factor and extension systems, and therefore factor groupoids may be considered somewhat intuitively as the noncommutative (or dynamical) version of the categorical equivalence between quotient maps $\pi : X \rightarrow Y$ of compact Hausdorff spaces and inclusions $C(Y) \subseteq C(X)$, see Proposition 1.3.9. Due to the assumptions we place on π , factor groupoids are, in a sense, much more topological in nature than noncommutative, given that much of the “collapsing” happens at the level of the unit space.

We will consider two constructions of factor groupoids, both with implications to the problem of finding groupoid models for classifiable C^* -algebras. The first construction follows the same set-up as in [29] with AF-groupoids and embeddings of Bratteli diagrams. Our method differs in that, rather than extending an equivalence relation and retaining the unit space, we take a quotient of the unit space and the equivalence relation follows along. The result is a recipe for a principal groupoid model for any C^* -algebra with Elliott invariant containing a simple, acyclic K_0 -group and a countable, torsion-free K_1 -group. The unit spaces that come out of the quotient operation are quite fascinating, the connected components being either single points or circles. Many of the spaces resemble fractals (or are fractals, depending on your definition of fractal), and have particularly regular self-symmetry when the Bratteli diagrams involved have simple descriptions.

The second construction considers minimal extensions of Cantor minimal systems due to Robin Deeley, Ian Putnam, and Karen Strung. The original Cantor system is a factor groupoid of the extension, and hence we may use relative K -theory and the

excision theorem to analyze the K -theory. The result, while less structured than the first involving the quotient spaces, expresses the K -theory of the extensions in terms of the topological K -theory of attractors of compact, invertible iterated function systems.

In sum, this dissertation may be divided into two main parts. The first part adapts Karoubi's portrait of relative K -theory for C^* -algebras and shows that it is naturally isomorphic to that given by the mapping cone. The second part explores the problem of finding groupoid models for C^* -algebras through the lens of factor groupoids, using relative K -theory and excision to simplify the calculations. We end with speculation. Specifically, we surmise the possibility of axiomatizing relative K -theory on a suitable class of C^* -algebras by mimicking (in fact, extending) the original axiomatization due to Joachim Cuntz in [7]. We also speculate further possibilities for factor groupoid constructions.

Chapter 1

Preliminaries

We outline the basic background in C^* -algebras, K -theory, groupoids, and Bratteli diagrams. We also prove a handful of simple results that will be needed in later chapters.

1.1 C^* -algebras

We begin with an outline of the theory of C^* -algebras, emphasizing the ideas that will be needed most. We refer to the encyclopedic reference [5] for more information.

Definition 1.1.1. A C^* -algebra A is a normed algebra over the complex numbers \mathbb{C} , with a conjugate linear involution, that satisfies the following properties.

- (i) The involution $a \mapsto a^*$ satisfies the property that, if a and b are in A , then $(ab)^* = b^*a^*$.
- (ii) A is complete in its norm, and if a and b are in A , then $\|ab\| \leq \|a\|\|b\|$, and if a is in A , then $\|a^*a\| = \|a\|^2$.

If there is a multiplicative identity 1_A , then A is called *unital*. If $ab = ba$ for any a and b in A , then A is called *commutative*.

1.1.1 Examples and constructions

There are several important examples of C^* -algebras that we will need throughout, as well as methods of constructing new C^* -algebras from old ones. We summarize them in this subsection.

Example 1.1.2. The algebra $C_0(X)$, all continuous complex valued functions on the locally compact Hausdorff space X which vanish at infinity, with pointwise operations and supremum norm, is a C^* -algebra. X is compact if and only if $C_0(X)$ is unital, and in this case we denote the algebra by $C(X)$. It is always commutative, and every commutative C^* -algebra is of this form (this is the Gelfand-Naimark Theorem).

Example 1.1.3. The algebra $\mathcal{B}(\mathcal{H})$, all bounded linear operators on the Hilbert space \mathcal{H} , is a C*-algebra. The adjoint a^* of a is the unique operator such that $\langle a^*\xi, \eta \rangle = \langle \xi, a\eta \rangle$ for all ξ and η in \mathcal{H} . This example is always unital, the unit $1_{\mathcal{B}(\mathcal{H})}$ being the identity operator defined by $1_{\mathcal{B}(\mathcal{H})}\xi = \xi$ for all ξ in \mathcal{H} , but never commutative unless the dimension of \mathcal{H} is 1 (in which case $\mathcal{B}(\mathcal{H})$ is isomorphic to \mathbb{C}). If \mathcal{H} has dimension n , where n is a positive integer, we may fix an orthonormal basis $\{\xi_1, \dots, \xi_n\}$ and identify $\mathcal{B}(\mathcal{H})$ with $M_n(\mathbb{C})$: the (i, j) entry of a is $\langle a\xi_i, \xi_j \rangle$. The algebraic operations are then the usual ones on $M_n(\mathbb{C})$, and the involution is the complex conjugate transpose. If \mathcal{H} is separable (that is, has a countable dense subset) and infinite-dimensional, it is isomorphic to l^2 , the space of all sequences $\xi = \{\xi(n)\}_{n=1}^\infty$ such that the norm $\|\xi\|_2 = (\sum_{n=1}^\infty |\xi(n)|^2)^{1/2}$ is finite. The space l^2 has the standard orthonormal basis δ_n , where $\delta_n(k) = 1$ if $n = k$ and $\delta_n(k) = 0$ otherwise.

If \mathcal{H} is a Hilbert space, every subalgebra of $\mathcal{B}(\mathcal{H})$ that is closed under the norm and the adjoint operation is a C*-algebra. The converse is remarkably true: for every C*-algebra A , there exists a Hilbert space \mathcal{H} and an isometric (norm-preserving) *-homomorphism $\phi : A \rightarrow \mathcal{B}(\mathcal{H})$ (the Gelfand-Naimark Theorem). The definition of a *-homomorphism is given below.

We will also need the following two examples of concrete C*-algebras.

Example 1.1.4. The algebra $\mathcal{K}(\mathcal{H})$ (or just \mathcal{K} if \mathcal{H} is separable and infinite-dimensional) of all *compact operators* on \mathcal{H} (that is, the operators a in $\mathcal{B}(\mathcal{H})$ such that the set $\overline{a\{\xi \in \mathcal{H} \mid \|\xi\| \leq 1\}}$ is compact in \mathcal{H} or, equivalently, operators that are limits of operators with finite dimensional range) is a C*-algebra. \mathcal{K} can also be described as the completion of the algebra $M_\infty(\mathbb{C}) = \bigcup_{n=1}^\infty M_n(\mathbb{C})$. The latter union is viewed as increasing by viewing small matrices as larger ones via “top left corner blocks”. We will discuss this more precisely in the section on K -theory.

Example 1.1.5. The *Toeplitz algebra*, denoted \mathcal{T} , is the universal C*-algebra generated by an isometry. It may be concretely realized as the C*-algebra of operators generated by the unilateral shift on l^2 , that is, the smallest C*-subalgebra of $\mathcal{B}(l^2)$ that contains the operator v defined by $v\delta_n = \delta_{n+1}$ for $n \geq 1$.

Example 1.1.6. If $\{A_i\}_{i \in I}$ is a collection of C*-algebras, the *direct sum* $A = \bigoplus_{i \in I} A_i$ is a C*-algebra with the obvious component-wise operations: it consists of all tuples $a = \{a_i\}_{i \in I}$ such that a_i is in A_i for all i in I , and $\lim_{i \in I} \|a_i\|_{A_i} = 0$.

Example 1.1.7. If A is any C*-algebra, denote by \tilde{A} the *unitization* of A : the set of all formal sums $a + \lambda$ with a in A and λ in \mathbb{C} with component-wise addition and scalar multiplication. The product and involution are given by

$$(a + \lambda)(b + \omega) = ab + \lambda b + \omega a + \lambda\omega \quad (a + \lambda)^* = a^* + \bar{\lambda}$$

and the norm is $\|a + \lambda\| = \sup\{\|ba + \lambda b\| \mid b \in A \text{ and } \|b\| \leq 1\}$. \tilde{A} is unital, with unit $1_{\tilde{A}} = 0 + 1$. For $a = b + \lambda$ in \tilde{A} , we let \dot{a} denote the complex number λ . We also regard \dot{a} as an element of \tilde{A} by identifying it with $\lambda 1_{\tilde{A}}$. If A is unital then \tilde{A} is isomorphic to $A \oplus \mathbb{C}$ via the map $A \oplus \mathbb{C} \rightarrow \tilde{A} : (a, \lambda) \mapsto a + \lambda(1_{\tilde{A}} - 1_A)$. The unitization $C_0(X)^\sim$ is isomorphic as a C*-algebra to $C(X^+)$, where $X^+ = X \cup \{\infty\}$ is the one-point compactification of X .

Example 1.1.8. We will also need to consider tensor products of C*-algebras. The general theory is filled with technical difficulties, but thankfully, given any tensor product in this project, at least one of the algebras involved will be *nuclear*, which means there is no ambiguity in defining the norm (see II.9.4.1 of [5]).

If X is a locally compact Hausdorff space and A is a C*-algebra, the tensor product $C_0(X) \otimes A$ is isomorphic to $C_0(X, A)$, all continuous functions $f : X \rightarrow A$ which vanish at infinity. The algebraic operations are pointwise, and the norm is the supremum norm, as with $C_0(X)$. The unitization $C_0(X, A)^\sim$ may be viewed as the subalgebra of all functions in $C(X^+, \tilde{A})$ such that there is some λ in \mathbb{C} with $f(\infty) = \lambda$ and $\dot{f}(x) = \lambda$ (notice the dot) for all x in X^+ . We write SA for $C_0((0, 1)) \otimes A = C_0((0, 1), A)$, the *suspension* of A , and CA for $C_0((0, 1]) \otimes A = C_0((0, 1], A)$, the *cone* of A .

If A is a C*-algebra and n is a positive integer, the tensor product $A \otimes M_n(\mathbb{C})$ is isomorphic to $M_n(A)$, all $n \times n$ matrices with entries in A , with the usual matrix operations. The norm is obtained by representing A faithfully on a Hilbert space \mathcal{H} (this is always possible by the Gelfand-Naimark Theorem) and considering an element a in $M_n(A)$ as an operator in $\mathcal{B}(\mathcal{H}^{\oplus n})$ in the obvious way. As long as the representation is faithful, the norm is well defined.

The tensor product $A \otimes \mathcal{K}$ is often called the *stabilization* of A , and it may be viewed as the completion of the *-algebra $M_\infty(A) = \bigcup_{n=1}^\infty M_n(A)$.

Definition 1.1.9. If A and B are C*-algebras, a *-homomorphism $\phi : A \rightarrow B$ is a linear map such that $\phi(ab) = \phi(a)\phi(b)$ and $\phi(a^*) = \phi(a)^*$ for all a and b in A .

A *-homomorphism $\phi : A \rightarrow B$ extends naturally to the unitizations and tensor products described above: $\tilde{\phi} : \tilde{A} \rightarrow \tilde{B}$ is defined by $\tilde{\phi}(a + \lambda) = \phi(a) + \lambda$. The map $\text{id}_{C_0(X)} \otimes \phi$ “applies ϕ to a function pointwise”, and the map $\phi \otimes \text{id}_{M_n(\mathbb{C})}$ “applies ϕ to each entry”. We will frequently abuse notation and denote all of these extensions by ϕ , except when clarity is necessary to avoid confusion. We will sometimes use $S\phi$ and $C\phi$ to denote the respective induced homomorphisms $SA \rightarrow SB$ and $CA \rightarrow CB$.

A C*-algebra A is called *contractible* if for each t in the interval $[0, 1]$ there is a *-homomorphism $\phi_t : A \rightarrow A$ such that, for all a in A , $t \mapsto \phi_t(a)$ is continuous, $\phi_1(a) = a$, and $\phi_0(a) = 0$. The cone CA is contractible for every C*-algebra A , by taking $\phi_t(f) = f_t$, where $f_t(s) = f(ts)$.

We now name some important properties that elements of a C*-algebra might have, as well as some relations between them.

Definition 1.1.10. An element a of a unital C^* -algebra is called

- (i) *invertible* if it has a multiplicative inverse a^{-1} , that is, $a^{-1}a = aa^{-1} = 1$,
- (ii) a *unitary* if $a^*a = aa^* = 1$,
- (iii) an *isometry* if $a^*a = 1$.

An element a of a (not necessarily unital) C^* -algebra is called

- (i) *idempotent* if $a^2 = a$,
- (ii) *self-adjoint* if $a = a^*$,
- (iii) a *projection* if $a^*a = a$,
- (iv) a *partial isometry* if $aa^*a = a$, and
- (v) *normal* if $a^*a = aa^*$.

Projections satisfy $a = a^2 = a^*$ (in fact, this is an equivalent definition). If a is a partial isometry, then both a^*a and aa^* are projections (this is also an equivalent definition), and they are respectively called the *source projection* and the *range projection*.

Definition 1.1.11. Let A be a C^* -algebra.

- (i) We say that two idempotents (resp. projections, invertibles, unitaries) a and b in A are *homotopic* if there is a continuous path of idempotents (resp. projections, invertibles, unitaries) a_t for $0 \leq t \leq 1$ with $a_0 = a$ and $a_1 = b$.
- (ii) Two idempotents e and f in A are called (*algebraically*) *equivalent* if there are elements a and b in A such that $ba = e$ and $ab = f$.
- (iii) Two idempotents e and f in A are called *similar* if there is an invertible z in \tilde{A} such that $f = zez^{-1}$.
- (iv) Two projections p and q in A are called *Murray-von Neumann equivalent* if there is a partial isometry v in A such that $v^*v = p$ and $vv^* = q$.
- (v) Two projections p and q in A are called *unitarily equivalent* if there is a unitary u in \tilde{A} such that $q = upu^*$.

1.1.2 Spectral theory and continuous functional calculus

Most clever tricks in K -theory rely on spectral theory and continuous functional calculus. For a in A , we define the spectrum $\sigma(a)$ to be

$$\sigma(a) = \{\lambda \in \mathbb{C} \mid a - \lambda 1 \text{ is not invertible in } \tilde{A}\}$$

The spectrum is always compact, and nonempty thanks to Liouville's Theorem in complex analysis. If a is normal, there is an isomorphism from $C(\sigma(a))$ to $C^*(a, 1)$ (where $C^*(a, 1)$ is the smallest C^* -subalgebra of \tilde{A} containing a and 1). The isomorphism can be roughly described as follows: for a polynomial $p(z) = c_n z^n + \cdots + c_1 z + c_0$ in $C(\sigma(a))$,

$$p(a) = c_n a^n + \cdots + c_1 a + c_0 1$$

and one obtains everything else in $C(\sigma(a))$ and $C^*(a, 1)$ using the Stone-Weierstrass Theorem. An important example is the exponential function, which has the familiar series expansion

$$\exp(a) = e^a = \sum_{n=0}^{\infty} \frac{a^n}{n!}$$

Another important example is the square root function: if $\sigma(a) \subseteq [0, \infty)$ and $f(x) = x^{1/2}$, we denote $f(a)$ by $a^{1/2}$.

We record the following fairly standard facts that will be needed for K -theoretic reasons. We sketch the proofs for convenience.

Proposition 1.1.12. *Let A be a C^* -algebra, and let a , u , and v be elements of A .*

- (i) *If A is contractible, then every unitary in $M_n(\tilde{A})$ is homotopic to 1.*
- (ii) *If a is self-adjoint, then e^{ia} is a unitary in \tilde{A} and homotopic to 1.*
- (iii) *If u is a self-adjoint unitary, then $u = e^{i\pi(1-u)/2}$, hence u is homotopic to 1.*
- (iv) *If v is a partial isometry in \tilde{A} , then*

$$\begin{bmatrix} v & 1 - vv^* \\ 1 - v^*v & v^* \end{bmatrix}$$

is a unitary in $M_2(\tilde{A})$ and is homotopic to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Proof. (i) If u is a unitary in $M_n(\tilde{A}) = \tilde{A} \otimes M_n(\mathbb{C})$ and ϕ_t is as in the definition of contractible, then $\tilde{\phi}_t \otimes \text{id}_{M_n(\mathbb{C})}(u)$ is a path of unitaries from u to a unitary in $M_n(\mathbb{C})$. Since the unitary group of $M_n(\mathbb{C})$ is connected, the conclusion follows.

(ii) One can use the power series for \exp to show that $e^{a+b} = e^a e^b$ when a and b commute. Therefore, for any b in A , e^b is invertible with inverse $(e^b)^{-1} = e^{-b}$. So if a is self-adjoint, $(e^{ia})^* = e^{(ia)^*} = e^{-ia} = (e^{ia})^{-1}$. For the homotopy, take the path e^{ita} for $0 \leq t \leq 1$.

(iii) We have $u^2 = 1$, so $(1-u)^n = 2^{n-1}(1-u)$ by induction. Then

$$e^{i\pi(1-u)/2} = \sum_{n=0}^{\infty} \frac{(i\pi(1-u)/2)^n}{n!} = 1 + \frac{1}{2}(1-u)(e^{i\pi} - 1) = u$$

and u is homotopic to 1 by part (ii).

(iv) That the matrix is unitary is an easy computation. Write

$$\begin{bmatrix} v & 1 - vv^* \\ 1 - v^*v & v^* \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 - v^*v & v^* \\ v & 1 - vv^* \end{bmatrix}$$

and observe that both matrices on the right are self-adjoint unitaries. The result follows from part (iii). \square

In fact, one can show that every unitary homotopic to 1 is equal to a finite product of elements of the form e^{ia} where a is self-adjoint (see, for example, Proposition 3.4.5 of [4]). We use this fact in the proof below. If $\phi : A \rightarrow B$ is a surjective *-homomorphism and b is in B , an element a in A such that $\phi(a) = b$ is called a *lift* of b .

Proposition 1.1.13. *Suppose that A and B are unital and $\phi : A \rightarrow B$ is a surjective *-homomorphism. Then*

(i) *Every self-adjoint element in B lifts to a self-adjoint element in A .*

(ii) *Every unitary in B that is homotopic to 1_B lifts to a unitary in A that is homotopic to 1_A . In particular, if u is any unitary in B , then there is a unitary w in $M_2(A)$*

homotopic to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ such that

$$\phi(w) = \begin{bmatrix} u & 0 \\ 0 & u^* \end{bmatrix}$$

Proof. Letting c be any element with $\phi(c) = b$ and taking $a = \frac{1}{2}(c + c^*)$ shows (i). To prove (ii), find self-adjoint elements a_1, a_2, \dots, a_k with $u = e^{ia_1} e^{ia_2} \dots e^{ia_k}$. By (i) we may lift each a_i to a self-adjoint b_i , and let $w = e^{ib_1} e^{ib_2} \dots e^{ib_k}$. The last claim follows from taking $v = u$ in part (iv) of Proposition 1.1.12. \square

1.2 K -theory

Here we set out notation and record the basic definitions and results in C^* -algebra K -theory. Proofs of all results in this section may be found in [4]. Other useful references are [32] and [36].

Let A be a C^* -algebra. We need to consider arbitrarily large matrix algebras over \tilde{A} . We denote the union $\bigcup_{n=1}^{\infty} M_n(\tilde{A})$ by $M_{\infty}(\tilde{A})$. This union is sometimes viewed as disjoint, but we will view it as increasing: if n and k are positive integers and a is in $M_n(\tilde{A})$, we can regard a as an element of $M_{n+k}(\tilde{A})$ by “adding zeros down the diagonal”, that is, by identifying it with the matrix

$$\begin{bmatrix} a & 0 \\ 0 & 0_k \end{bmatrix}$$

If a and b are in $M_{\infty}(\tilde{A})$ we define the “block diagonal sum”

$$a \oplus b = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

There is some ambiguity here because, as we just described, a and b may be regarded as matrices of arbitrarily large size. However, since K -theory doesn’t distinguish elements that are “moved down the diagonal”, there will be negligible harm done by ignoring this technical issue. If $m \geq n$, we denote by 1_n the matrix in $M_m(\tilde{A})$ with n ones down the diagonal, and zeros elsewhere.

1.2.1 The group $K_0(A)$

Definition 1.2.1. (i) Let A be a unital C^* -algebra. The group $K_0(A)$ is defined to be the quotient of the free abelian group generated by all symbols of the form $[p]$ for a projection p in $M_{\infty}(A)$ subject to the relations

(a) $[p] + [q] = [p \oplus q]$.

(b) $[0] = 0$.

(c) If there exists an integer $n \geq 1$ such that p and q are Murray-von Neumann equivalent in $M_n(A)$, then $[p] = [q]$.

(ii) If A is non-unital, then $K_0(A)$ is defined to be the subgroup of $K_0(\tilde{A})$ consisting of all differences $[p] - [q]$ such that $[\dot{p}] = [\dot{q}]$ in $K_0(\mathbb{C})$ (recall that \dot{p} denotes the scalar part of p).

When A is unital, it is a convenient fact (see [32]) that $K_0(A)$ may be defined via (ii) above as well as via (i), and the resulting group is the same. Therefore, we will take (ii) to be the standard definition of $K_0(A)$ whether A is unital or not.

For every element x of $K_0(A)$ we may find positive integers m and n such that $m \geq n$ and a projection p in $M_m(\tilde{A})$ such that \dot{p} is Murray-von Neumann equivalent (or even equal) to 1_n and $x = [p] - [1_n]$ (6.2.7 of [36]).

Murray-von Neumann equivalence could also be replaced by unitary equivalence or homotopy, as these equivalences are all the same, provided one chooses a large enough matrix algebra (4.3.1 and 4.4.1 of [4]).

We could alternatively define $K_0(A)$ using idempotents in place of projections, and algebraic equivalence instead of Murray-von Neumann equivalence. Indeed, this allows us to define the K_0 -group of a more general Banach algebra A , but when A is a C^* -algebra, projections are usually preferable because they are self-adjoint (hence normal) and have nicer spectral theory.

Definition 1.2.2. An *ordered abelian group* G is an abelian group with a distinguished subsemigroup G^+ , called the *positive cone of G* , such that

- (i) $G^+ + G^+ \subseteq G^+$,
- (ii) $G^+ \cap (-G^+) = \{0\}$,
- (iii) $G = G^+ - G^+$.

We write $g \geq 0$ to mean that g is in G^+ and $g \geq h$ to mean that $g - h \geq 0$. An element u of the ordered abelian group G is called an *order unit* if, for every g in G , there is an integer n such that $-nu \leq g \leq nu$ (where $nu = u + u + \dots + u$, n times).

A unital C^* -algebra A is called *finite* if every isometry in A is a unitary, and it is called *stably finite* if $M_n(A)$ is finite for all $n \geq 1$.

Proposition 1.2.3. *If A is unital and stably finite, then $K_0(A)$ is an ordered abelian group with positive cone*

$$K_0(A)^+ = \{[p] \mid p \text{ a projection in } M_\infty(A)\}$$

and order unit $[1_A]$.

Example 1.2.4. (i) $K_0(\mathbb{C}) \cong K_0(M_n(\mathbb{C})) \cong \mathbb{Z}$, for all $n \geq 1$. A generator of the group is $[1_{\mathbb{C}}]$, or the class of any rank one projection. The positive cone is identified with $\mathbb{Z}^+ = \{0, 1, 2, 3, \dots\}$. If X is a compact Hausdorff space, let $C(X, \mathbb{Z})$ denote the continuous functions from X to \mathbb{Z} , where the integers have the discrete topology. $C(X, \mathbb{Z})$ is a group under pointwise addition. There is a surjective

group homomorphism

$$\dim : K_0(C(X)) \rightarrow C(X, \mathbb{Z})$$

defined, for a projection p in $M_n(C(X)) = C(X, M_n(\mathbb{C}))$, by $\dim([p])(x) = \text{Tr}(p(x))$, where Tr is the usual trace on $M_n(\mathbb{C})$. If X is totally disconnected, then \dim is injective, hence an isomorphism. If X is contractible, then $C(X, \mathbb{Z}) = \mathbb{Z}$, since the only continuous functions from a connected space into a discrete space are constant. Moreover, \dim is injective if X is compact and contractible (as a space), and therefore $K_0(C(X)) \cong \mathbb{Z}$, a generator given by $[1_{C(X)}]$. Again, the positive cone is identified with \mathbb{Z}^+ .

- (ii) If X is a compact Hausdorff space, $K_0(C(X))$ is isomorphic to $K^0(X)$, the *topological K-theory* of X (we will sometimes use this alternative notation). Topological K -theory is defined using things called *vector bundles* and the aforementioned isomorphism is the content of Swan's theorem (I.6.18 of [18]).
- (iii) $K_0(\mathcal{K}) \cong \mathbb{Z}$, generated by the class of any rank one projection.
- (iv) $K_0(C_0(\mathbb{R}^2)) \cong \mathbb{Z}$. To find a generator of the group, let $\mathbb{D} = \{z \in \mathbb{C} \mid |z| \leq 1\}$, and identify $C_0(\mathbb{R}^2)^\sim$ with functions in $C(\mathbb{D})$ that are constant on the unit circle $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$. Then a generator of $K_0(C_0(\mathbb{R}^2))$ is given by $[p] - [1 \oplus 0]$, where p in $C(\mathbb{D}, M_2(\mathbb{C}))$ is the Bott projection:

$$p(z) = \begin{bmatrix} |z|^2 & z\sqrt{1-|z|^2} \\ \bar{z}\sqrt{1-|z|^2} & 1-|z|^2 \end{bmatrix}$$

Tracial data is intimately tied with ordered K_0 -theory. We will need the following definitions in order to describe the Elliott invariant in subsection 1.2.4.

Definition 1.2.5. A *trace* on a C^* -algebra A is a bounded linear functional $\tau : A \rightarrow \mathbb{C}$ such that $\tau(ab) = \tau(ba)$ for all a and b in A . We say τ is a *tracial state* if τ is positive ($\tau(a^*a) \geq 0$ for all a in A) and $\|\tau\| = 1$. We let $T(A)$ denote the space of all tracial states on A .

Definition 1.2.6. Let A be a unital and stably finite C^* -algebra. Define the map r_A from $T(A)$ to $S(K_0(A))$, the space of all states on the ordered abelian group $K_0(A)$, by

$$r_A(\tau)([p] - [q]) = \tau(p) - \tau(q)$$

where, if $a = (a_{i,j})$ in $M_n(A)$, then $\tau(a) = \sum_{i=1}^n \tau(a_{i,i})$.

1.2.2 The group $K_1(A)$

Definition 1.2.7. If A is a unital C^* -algebra and $n \geq 1$, we let $U_n(A)$ denote the set of unitaries in $M_n(A)$, and let $U_\infty(A) = \bigcup_{n=1}^\infty U_n(A)$.

When regarding unitaries as arbitrarily large matrices, we add ones down the diagonal instead of zeros. So if u is in $U_n(\tilde{A})$, we view it as

$$\begin{bmatrix} u & 0 \\ 0 & 1_k \end{bmatrix}$$

in $U_{n+k}(\tilde{A})$.

Definition 1.2.8. (i) Let A be a unital C^* -algebra. The group $K_1(A)$ is defined to be the quotient of the free abelian group generated by all symbols of the form $[u]$ for u in $U_\infty(A)$ subject to the relations

(a) $[u] + [v] = [u \oplus v]$.

(b) $[1] = 0$.

(c) If there exists $n \geq 1$ such that u and v are homotopic in $U_n(A)$, then $[u] = [v]$.

(ii) If A is non-unital, then $K_1(A)$ is defined to be $K_1(\tilde{A})$.

Conveniently, we have $-[u] = [u^*]$ for all unitaries u in $U_\infty(\tilde{A})$, so there is no need for formal differences as with $K_0(A)$. If u and v are in $U_n(\tilde{A})$, we have the formula $[u] + [v] = [uv]$. Every element of $K_1(A)$ is represented by a class $[u]$ for a unitary u in $U_n(\tilde{A})$ with $\dot{u} = 1_n$.

Like the K_0 -group, we could define $K_1(A)$ in a more general way, using invertibles instead of unitaries. Again, it is usually preferable to stick with unitaries due to their nice spectral theory. In Chapter 2, we will see an even more general definition using what may be called “partially invertible” elements.

Example 1.2.9. (i) $K_1(\mathcal{B}(\mathcal{H})) = 0$ for any Hilbert space \mathcal{H} .

(ii) $K_1(C(X)) = 0$ if X is compact and contractible.

(iii) The standard example that yields a nontrivial group is $K_1(C(\mathbb{T})) \cong \mathbb{Z}$, which is generated by the class of the function $f(z) = z$ on \mathbb{T} .

Henceforth, we will economically denote the function $f(z) = z$ in $C(\mathbb{T})$ by z .

K_0 and K_1 are functors from the category of C^* -algebras to the category of abelian groups: if $\phi : A \rightarrow B$ is a $*$ -homomorphism, the induced group homomorphisms $\phi_* : K_0(A) \rightarrow K_0(B)$ and $\phi_* : K_1(A) \rightarrow K_1(B)$ satisfy the property that

$$\phi_*([p]) = [\tilde{\phi}(p)] \text{ and } \phi_*([u]) = [\tilde{\phi}(u)]$$

for all projections p and all unitaries u in $M_\infty(\tilde{A})$. By homotopy invariance, $K_0(A) = K_1(A) = 0$ if A is contractible.

Remark. A small word on notation: sometimes K_0 and K_1 are amalgamated and studied simultaneously by defining $K_*(A) = K_0(A) \oplus K_1(A)$ and regarding the theory as so-called \mathbb{Z}_2 -graded, see V.1.3.8 of [5]. We will not make use of any grading, but we will use the notation K_* when discussing anything about the K -groups that does not depend on the value of the subscript. We will do the same for the relative K -groups in Chapter 2.

1.2.3 Stability, suspensions, exact sequences, and the Künneth theorem

If A is any C^* -algebra and p is any rank one projection in \mathcal{K} , the $*$ -homomorphism $\kappa_A : A \rightarrow A \otimes \mathcal{K}$ given by $\kappa_A(a) = a \otimes p$ induces an isomorphism $(\kappa_A)_* : K_*(A) \rightarrow K_*(A \otimes \mathcal{K})$. This property of K -theory is often referred to as *stability*. Strictly speaking, while κ_A depends on the choice of p , $(\kappa_A)_*$ does not. Therefore, for all K -theoretic purposes, we may sweep p under the rug.

If A is any C^* -algebra, there are natural (in a categorical sense) isomorphisms $\theta_A : K_1(A) \rightarrow K_0(SA)$ and $\beta_A : K_0(A) \rightarrow K_1(SA)$ (Theorem 8.2.2 and Theorem 9.2.1 of [4], respectively). The map β_A is called the *Bott map* for A . The map θ_A may be computed as follows. For a class $[u]$, where u is a unitary in $M_n(\tilde{A})$ with $\dot{u} = 1_n$, find a unitary v in $C([0, 1]) \otimes M_{2n}(\tilde{A})$ such that $v(0) = 1_{2n}$, $v(1) = u \oplus u^*$, and $\dot{v}(t) = 1_{2n}$ for all $0 \leq t \leq 1$ (such a v exists by part (ii) of Proposition 1.1.13). Then

$$\theta_A([u]) = [v(1_n \oplus 0_n)v^*] - [1_n \oplus 0_n]$$

The Bott map may be computed as follows. If p and q are projections in $M_n(\tilde{A})$ with $[p] = [q]$ in $K_0(\mathbb{C})$,

$$\beta_A([p] - [q]) = [f_p f_q^*]$$

where $f_p(t) = e^{2\pi i t p}$ for $0 \leq t \leq 1$.

Definition 1.2.10. Let X , Y , and Z be modules over a ring (such as C^* -algebras over \mathbb{C} , or abelian groups over \mathbb{Z}) and let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be module homomorphisms. We say that the sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is *exact* (at Y) if the image of f is exactly¹ equal to the kernel of g . If such a sequence of modules and homomorphisms has five terms (modules) and the first and last are

¹I suspect this is the reason for the terminology.

zero, we say that the sequence is *short exact*. If such a sequence is cyclic and has six-terms, we say that it is a *six-term exact sequence*.

The next result is the most important standard tool in K -theory.

Proposition 1.2.11. *If the sequence*

$$0 \longrightarrow I \xrightarrow{\iota} A \xrightarrow{\pi} A/I \longrightarrow 0$$

of C^ -algebras is exact, then there are group homomorphisms $\delta_0 : K_0(A/I) \rightarrow K_1(I)$ and $\delta_1 : K_1(A/I) \rightarrow K_0(I)$ (called the exponential map and the index map, respectively) that make the sequence*

$$\begin{array}{ccccc} K_0(I) & \xrightarrow{\iota_*} & K_0(A) & \xrightarrow{\pi_*} & K_0(A/I) \\ \delta_1 \uparrow & & & & \downarrow \delta_0 \\ K_1(A/I) & \xleftarrow{\pi_*} & K_1(A) & \xleftarrow{\iota_*} & K_1(I) \end{array}$$

exact.

The map δ_1 may be computed as follows. For a class $[u]$, where u is a unitary in $M_n(\widetilde{A/I})$, find a unitary w in $M_{2n}(\widetilde{A})$ with $\pi(w) = u \oplus u^*$ (use part (ii) of Proposition 1.1.13). Then

$$\delta_1([u]) = [w(1_n \oplus 0_n)w^*] - [1_n \oplus 0_n]$$

If there is a partial isometry v in $M_n(\widetilde{A})$ with $\pi(v) = u$, then by taking w to be the unitary in part (iv) of Proposition 1.1.12, the index map has the following alternative description that forgoes doubling matrix sizes:

$$\delta_1([u]) = [1_n - v^*v] - [1_n - vv^*]$$

The map δ_0 may be computed as follows. For integers $m \geq n$ and a projection p in $M_m(\widetilde{A/I})$ with \dot{p} equivalent to 1_n , find a self-adjoint element a in $M_m(\widetilde{A})$ with $\pi(a) = p$ (use part (i) of Proposition 1.1.13). Then

$$\delta_0([p] - [1_n]) = -[e^{2\pi ia}]$$

Both maps δ_1 and δ_0 are natural in a categorical sense.

The next two six-term exact sequences involve a useful homological construction called the mapping cone.

Definition 1.2.12. Let $\phi : A \rightarrow B$ be a $*$ -homomorphism. The *mapping cone* C_ϕ is defined to be the C^* -subalgebra of $A \oplus CB$ consisting of all pairs (a, f) in $A \oplus CB$ such that $f(1) = \phi(a)$.

Such a C*-algebra is often referred to as a *pullback* of two *-homomorphisms, the two *-homomorphisms in this instance being $\phi : A \rightarrow B$ and $\pi_B : CB \rightarrow B$ defined by $\pi_B(f) = f(1)$.

Lemma 1.2.13. (i) Define the maps $\iota : SB \rightarrow C_\phi$ and $\pi : C_\phi \rightarrow A$ by $\iota(f) = (0, f)$ for f in SB and $\pi(a, f) = a$ for (a, f) in C_ϕ . Then the sequence

$$0 \longrightarrow SB \xrightarrow{\iota} C_\phi \xrightarrow{\pi} A \longrightarrow 0 \quad (1.1)$$

is short exact.

(ii) Let δ_1 and δ_0 be the boundary maps given by Proposition 1.2.11 associated to the short exact sequence (1.1). Then the diagrams

$$\begin{array}{ccc} K_1(A) & \xrightarrow{\phi_*} & K_1(B) \\ & \searrow \delta_1 & \downarrow \theta_B \\ & & K_0(SB) \end{array} \qquad \begin{array}{ccc} K_0(A) & \xrightarrow{\phi_*} & K_0(B) \\ & \searrow \delta_0 & \downarrow \beta_B \\ & & K_1(SB) \end{array}$$

are commutative up to sign.

Proof. Part (i) is straightforward and we omit the proof. For part (ii), we consider the left diagram first. Let u be a unitary in $M_n(\tilde{A})$ with $\dot{u} = 1_n$. Find a unitary v in $C([0, 1]) \otimes M_{2n}(\tilde{B})$ with $v(0) = 1_{2n}$, $v(1) = \phi(u) \oplus \phi(u)^*$, and $\dot{v}(t) = 1_{2n}$ for all t . Then $(u \oplus u^*, v)$ is a unitary in $M_{2n}(\tilde{C}_\phi)$ that lifts $u \oplus u^*$, so by the definitions of the maps δ_1 and θ_B ,

$$\delta_1([u]) = [v(1_n \oplus 0_n)v^*] - [1_n \oplus 0_n] = \theta_B([\phi(u)]) = \theta_B(\phi_*([u]))$$

Now we consider the diagram on the right. Let $n \leq m$ and p be a projection in $M_m(\tilde{A})$ with \dot{p} equivalent to 1_n . Let a be the self-adjoint element of $C([0, 1]) \otimes M_m(\tilde{B})$ defined by $a(t) = t\phi(p) + (1-t)1_n$. Then (p, a) is in $M_n(\tilde{C}_\phi)$ and is a self-adjoint lift of p , so

$$\delta_0([p] - [1_n]) = -[e^{2\pi ia}]$$

Also observe that

$$e^{2\pi ia(t)} = e^{2\pi i(t\phi(p) + (1-t)1_n)} = e^{2\pi it\phi(p)} e^{2\pi i1_n} e^{-2\pi it1_n} = f_{\phi(p)}(t) f_{1_n}(t)^*$$

since $e^{2\pi i1_n} = 1_m$ in $M_m(\tilde{B})$. So by definition of the map β_B ,

$$\beta_B(\phi_*([p] - [1_n])) = \beta_B([\phi(p)] - [1_n]) = [f_{\phi(p)} f_{1_n}^*] = [e^{2\pi ia}] = -\delta_0([p] - [1_n]). \quad \square$$

Applying Proposition 1.2.11 to the short exact sequence in part (i) of Lemma 1.2.13 and using the commutative diagrams in part (ii) of the same lemma, we obtain the following result.

Proposition 1.2.14. *The sequence*

$$\begin{array}{ccccc}
K_1(B) & \xrightarrow{\iota_* \circ \theta_B} & K_0(C_\phi) & \xrightarrow{\pi_*} & K_0(A) \\
\uparrow \phi_* & & & & \downarrow \phi_* \\
K_1(A) & \xleftarrow{\pi_*} & K_1(C_\phi) & \xleftarrow{\iota_* \circ \beta_B} & K_0(B)
\end{array}$$

is exact.

The following six-term exact sequence may be regarded as a “standard” six-term exact sequence for mapping cones.

Proposition 1.2.15. *Suppose that*

$$\begin{array}{ccccccc}
0 & \longrightarrow & I & \xrightarrow{\iota_A} & A & \xrightarrow{\pi_A} & A/I \longrightarrow 0 \\
& & \downarrow \psi & & \downarrow \phi & & \downarrow \gamma \\
0 & \longrightarrow & J & \xrightarrow{\iota_B} & B & \xrightarrow{\pi_B} & B/J \longrightarrow 0
\end{array}$$

is a commutative diagram with exact rows. Then the sequence

$$0 \longrightarrow C_\psi \xrightarrow{\iota} C_\phi \xrightarrow{\pi} C_\gamma \longrightarrow 0 \quad (1.2)$$

where $\iota(a, f) = (\iota_A(a), C\iota_B(f))$ for (a, f) in C_ψ and $\pi(a, f) = (\pi_A(a), C\pi_B(f))$ for (a, f) in C_ϕ , is a short exact sequence, and therefore the sequence

$$\begin{array}{ccccc}
K_0(C_\psi) & \xrightarrow{\iota_*} & K_0(C_\phi) & \xrightarrow{\pi_*} & K_0(C_\gamma) \\
\uparrow \delta_1 & & & & \downarrow \delta_0 \\
K_1(C_\gamma) & \xleftarrow{\pi_*} & K_1(C_\phi) & \xleftarrow{\iota_*} & K_1(C_\psi)
\end{array}$$

is exact.

Proof. The six-term exact sequence follows immediately from applying Proposition 1.2.11 to (1.2), and the only non-trivial part of proving that (1.2) is exact is in showing that π is surjective. The diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & SB & \longrightarrow & C_\phi & \longrightarrow & A \longrightarrow 0 \\
& & \downarrow S\pi_B & & \downarrow \pi & & \downarrow \pi_A \\
0 & \longrightarrow & S(B/J) & \longrightarrow & C_\gamma & \longrightarrow & A/I \longrightarrow 0
\end{array}$$

is clearly commutative, and it has exact rows by exactness of (1.1). The map π_A is surjective by assumption, and $S\pi_B$ is surjective by Proposition 10.1.2 of [32]. The conclusion then follows from the five lemma (see, for example, Section 10.5, Proposition 24 of [10]). \square

It is because of the six-term exact sequences in Proposition 1.2.14 and Proposition 1.2.15 that $K_*(C_\phi)$ may be taken to be the definition of the relative K -theory of ϕ . Proposition 1.2.14 provides a way of comparing the K -theory of A with the K -theory of B via ϕ , and Proposition 1.2.15 provides a way of comparing relative K -groups related to each other through what homological algebraists would refer to as a “homomorphism of short exact sequences”. Chapter 2 will be devoted to a more basic definition that has two six-term exact sequences that naturally correspond to those above.

The final six-term exact sequence in this subsection involves crossed products of C^* -algebras by \mathbb{Z} , and is originally due to M. Pimsner and D. Voiculescu [24]. We will not give an overview of crossed products as the ones we need will have groupoid models (namely, those that arise as actions of \mathbb{Z} on a compact metric space via a single homeomorphism), but the sequence in Proposition 1.2.17 will be of use to us.

Theorem 1.2.16. *Let A be a unital C^* -algebra and $\alpha : A \rightarrow A$ a $*$ -automorphism. There is a C^* -algebra, denoted $A \times_\alpha \mathbb{Z}$, that is generated by an isomorphic copy of A and a unitary element u such that $uau^* = \alpha(a)$ for all a in A . Moreover, this C^* -algebra is universal in the following sense: if B is any C^* -algebra containing a unitary v and if there is a $*$ -homomorphism $\phi : A \rightarrow B$ such that $v\phi(a)v^* = \phi(\alpha(a))$ for all a in A , then there is a unique $*$ -homomorphism $\psi : A \times_\alpha \mathbb{Z} \rightarrow B$ such that $\phi = \psi \circ \iota$ (where ι is the inclusion map $\iota : A \rightarrow A \times_\alpha \mathbb{Z}$) and $\psi(u) = v$.*

Proposition 1.2.17. *Let A be a unital C^* -algebra and $\alpha : A \rightarrow A$ a $*$ -automorphism. There is a six-term exact sequence*

$$\begin{array}{ccccc}
 K_0(A) & \xrightarrow{\text{id}-\alpha_*^{-1}} & K_0(A) & \xrightarrow{\iota_*} & K_0(A \times_\alpha \mathbb{Z}) \\
 \uparrow \delta_1 & & & & \downarrow \delta_0 \\
 K_1(A \times_\alpha \mathbb{Z}) & \xleftarrow{\iota_*} & K_1(A) & \xleftarrow{\text{id}-\alpha_*^{-1}} & K_1(A)
 \end{array}$$

We give a brief outline of a proof based on lecture notes of Allan Yashinski, which is in turn based on the original proof in [24]. Recall that \mathcal{T} denotes the Toeplitz algebra, and let \mathcal{T}_α be the C^* -subalgebra of $(A \times_\alpha \mathbb{Z}) \otimes \mathcal{T}$ generated by $\iota(A) \otimes 1$ and $u \otimes v$, where v is the unilateral shift. There is a short exact sequence

$$0 \longrightarrow A \otimes \mathcal{K} \xrightarrow{\nu} \mathcal{T}_\alpha \xrightarrow{\pi} A \times_\alpha \mathbb{Z} \longrightarrow 0$$

where $\pi(\iota(a) \otimes 1) = \iota(a)$ for all a in A and $\pi(u \otimes v) = u$. Furthermore, if we define $j : A \rightarrow \mathcal{T}_\alpha$ by $j(a) = \iota(a) \otimes 1$, then the induced map $j_* : K_*(A) \rightarrow K_*(\mathcal{T}_\alpha)$ is an isomorphism and the diagram

$$\begin{array}{ccccc} K_*(A) & \xrightarrow{\text{id}-\alpha_*^{-1}} & K_*(A) & \xrightarrow{\iota_*} & K_*(A \times_\alpha \mathbb{Z}) \\ \downarrow (\kappa_A)_* & & \downarrow j_* & & \parallel \\ K_*(A \otimes \mathcal{K}) & \xrightarrow{\nu_*} & K_*(\mathcal{T}_\alpha) & \xrightarrow{\pi_*} & K_*(A \times_\alpha \mathbb{Z}) \end{array}$$

commutes. Applying Proposition 1.2.11 to the given short exact sequence above and using the commutative diagram, we obtain the exact sequence in Proposition 1.2.17. (We are abusing notation a bit to be more succinct: δ_n in the theorem is actually $(\kappa_A)_*^{-1} \circ \delta_n$ for $n = 0, 1$.)

Lemma 1.2.18. $\delta_1([u]) = -[1_A]$.

Proof. The unitary u lifts to the isometry $u \otimes v$, so, using the partial isometry description of the index map,

$$\delta_1([u]) = [1_A \otimes 1_{\mathcal{T}} - (u^*u) \otimes (v^*v)] - [1_A \otimes 1_{\mathcal{T}} - (uu^*) \otimes (vv^*)] = -[1_A \otimes (1_{\mathcal{T}} - vv^*)]$$

and since $1_{\mathcal{T}} - vv^*$ is a rank one projection, $(\kappa_A)_*([1_A]) = [1_A \otimes (1_{\mathcal{T}} - vv^*)]$. \square

Finally, we need to know how to compute the K -theory of tensor products. The *bootstrap class* is the class of all separable, nuclear C^* -algebras that contains \mathbb{C} and is closed under countable inductive limits, short exact sequences, and KK -equivalence. The only thing we need to know is that this class contains the Toeplitz algebra, all AF-algebras, and all commutative C^* -algebras. The following is a simplified version of 23.1.3 of [4].

Theorem 1.2.19. *Suppose that A is in the bootstrap class and that at least one of $K_*(A)$ and $K_*(B)$ is torsion free. Then there are isomorphisms*

$$K_0(A \otimes B) \cong (K_0(A) \otimes K_0(B)) \oplus (K_1(A) \otimes K_1(B))$$

$$K_1(A \otimes B) \cong (K_0(A) \otimes K_1(B)) \oplus (K_1(A) \otimes K_0(B))$$

1.2.4 The Elliott classification program

An immense area of research in C^* -algebra theory is to determine the extent to which K -theory can distinguish C^* -algebras. After proving his seminal result in 1976 [12] that the K_0 -group, along with its order structure, is a complete invariant for AF-algebras (C^* -algebras that are inductive limits of finite-dimensional C^* -algebras, or, alternatively, isomorphic to $C_r^*(R_E)$ for some groupoid R_E as defined in the next section),

George Elliott conjectured in 1990 that K -theory is enough to distinguish C^* -algebras that are unital, separable, simple (no non-trivial ideals), and nuclear. This conjecture initiated a vast amount of work by many mathematicians over the next few decades. The conjecture as it stood did not hold, and the list of assumptions needed to grow slightly in order to accommodate pathological examples such as the Jiang-Su algebra \mathcal{Z} [17] (which is unital, separable, simple, nuclear, has the same K -theory as \mathbb{C} , but is not isomorphic to \mathbb{C}). Added assumptions include finite nuclear dimension and the Universal Coefficient Theorem (abbreviated UCT, see 23.1.1 of [4]). Tracial data also needed to be added alongside K -theory in the invariant. Ultimately, the Elliott invariant for a C^* -algebra A is

$$\text{Ell}(A) = (K_0(A), K_0(A)^+, [1_A], K_1(A), T(A), r_A)$$

We say that $\text{Ell}(A)$ and $\text{Ell}(B)$ are isomorphic if there is an order preserving isomorphism $\gamma_0 : K_0(A) \rightarrow K_0(B)$ (that is, $\gamma_0(K_0(A)^+) = K_0(B)^+$ and $\gamma_0([1_A]) = [1_B]$), an isomorphism $\gamma_1 : K_1(A) \rightarrow K_1(B)$, and a homeomorphism $\gamma_T : T(B) \rightarrow T(A)$ that intertwines the maps r_A and r_B , that is, $r_A \circ \gamma_T = (\cdot \circ \gamma_0) \circ r_B$ (if f is a state on $K_0(B)$, then $f \circ \gamma_0$ is a state on $K_0(A)$).

Theorem 1.2.20 (Theorem 14.2.1 of [34]). *Suppose that A and B are two unital, simple, separable, infinite-dimensional C^* -algebras with finite nuclear dimension that satisfy the UCT. If $\text{Ell}(A)$ is isomorphic to $\text{Ell}(B)$, then A is isomorphic to B .*

1.3 Groupoids

We will only be concerned with groupoids that are locally compact, Hausdorff, and étale, and for a more extensive covering of the preliminary material here, see [34].

Definition 1.3.1. A groupoid is a set G together with a distinguished subset $G^{(2)} \subseteq G \times G$, a multiplication map $(x, y) \mapsto xy$ from $G^{(2)}$ to G , and an inverse map $x \mapsto x^{-1}$ from G to G such that

- (i) $(x^{-1})^{-1} = x$ for all x in G ;
- (ii) if (x, y) and (y, z) belong to $G^{(2)}$, then (xy, z) and (x, yz) belong to $G^{(2)}$, and $(xy)z = x(yz)$; and
- (iii) (x, x^{-1}) belong to $G^{(2)}$ for all x in G , and for all (x, y) in $G^{(2)}$, we have $x^{-1}(xy) = y$ and $(xy)y^{-1} = x$.

We denote $G^{(0)} = \{x^{-1}x \mid x \in G\}$, and call it the *unit space* of G . The maps r and s from G to $G^{(0)}$, respectively called the *range* and *source* maps, are defined by $r(x) = xx^{-1}$ and $s(x) = x^{-1}x$. For u in $G^{(0)}$, we define $G^u = r^{-1}(u)$ and $G_u = s^{-1}(u)$.

Another definition of a groupoid, famed for its pithiness, is that it is a small category where every morphism is invertible. To compare with more familiar terminology, a group may be defined as a small category where every morphism is invertible, and there is only one object (and therefore any two morphisms are composable).

A topological groupoid is a groupoid with a topology such that the product and inverse operations are continuous. We say that a topological groupoid is *étale* if the map r is a local homeomorphism, and in this case, $G^{(0)}$ is open, and the relative topologies on G^u and G_u are discrete, for all u in $G^{(0)}$. An open subset U of G such that $r(U)$ and $s(U)$ are open and $r|_U : U \rightarrow r(U)$ and $s|_U : U \rightarrow s(U)$ are homeomorphisms is called a *open bisection*. More generally, a *bisection* is any set that is contained in an open bisection.

A *groupoid homomorphism* $\pi : G \rightarrow H$ is a map such that $\pi \times \pi(G^{(2)}) \subseteq H^{(2)}$ and $\pi(xy) = \pi(x)\pi(y)$ for every pair (x, y) in $G^{(2)}$ (so, with the category definition, it is a functor). If G and H are topological groupoids, we will require that π be continuous. We have $\pi(x^{-1}) = \pi(x)^{-1}$ for all x in G , and $\pi(G^{(0)}) \subseteq H^{(0)}$, with equality if π is surjective. We also have $\pi \circ r = r \circ \pi$ and $\pi \circ s = s \circ \pi$. If π is a homeomorphism, we say it is a *groupoid isomorphism*, and that G and H are *isomorphic as groupoids*.

The two standard examples of groupoids which will be of interest here are equivalence relations and group actions.

Example 1.3.2. If $R \subseteq X \times X$ is an equivalence relation on a set X , R has the structure of a groupoid with product and inverse

$$(x, y)(x', y') = (x, y') \quad (x, y)^{-1} = (y, x)$$

where the product is defined only when $y = x'$. The range and source maps are easily checked to be $r(x, y) = (x, x)$ and $s(x, y) = (y, y)$, respectively. Thus the unit space $R^{(0)}$ is $\{(x, x) \mid x \in X\}$, which can be identified with X in an obvious way. In general, if G is a groupoid, the set $R_G := \{(r(x), s(x)) \mid x \in G\}$ is an equivalence relation on $G^{(0)}$. The map $G \rightarrow R_G : x \mapsto (r(x), s(x))$ is a surjective groupoid homomorphism, and if it is injective, G is called *principal*. Every equivalence relation is principal, and, conversely, every principal groupoid is isomorphic to an equivalence relation. (Under the categorical definition of a groupoid, a principal groupoid is a small category where every morphism is invertible and, for every pair of objects a and b , there is at most one morphism from a to b .) If X is a space and every equivalence class is dense in X , then R is called *minimal*.

Example 1.3.3. Let X be a set and Γ a group. Recall that a (*left*) *action* of Γ on X is a map $X \times \Gamma \rightarrow X : (x, g) \mapsto g \cdot x$ such that $g \cdot (h \cdot x) = (gh) \cdot x$ for all g and h in Γ and all x in X , and $1_\Gamma \cdot x = x$ for all x in X , where 1_Γ is the identity of the group. In

this situation, the set $X \times \Gamma$ has the structure of a groupoid with product and inverse

$$(x, g)(y, h) = (x, gh) \quad (x, g)^{-1} = (g \cdot x, g^{-1})$$

for x and y in X and g and h in Γ , but the product is defined only when $y = g \cdot x$. If the action is free, then $X \times \Gamma$ is principal, as defined in the previous example. If X and Γ carry topologies and Γ acts continuously by homeomorphisms, the groupoid $X \times \Gamma$ is étale in the product topology if and only if Γ is discrete. If, for every x , the orbit $\{g \cdot x\}_{g \in \Gamma}$ is dense in X , we say $X \times \Gamma$ is *minimal*.

Definition 1.3.4. Let G be a locally compact, Hausdorff, étale groupoid. We denote by $C_c(G)$ the vector space of all continuous complex-valued functions on G with compact support (the *support* of a function f is the closure of the set $f^{-1}(\mathbb{C} - \{0\})$).

$C_c(G)$ is a $*$ -algebra with the operations

$$(f \star g)(x) = \sum_{y \in G^r(x)} f(y)g(y^{-1}x) \quad f^*(x) = \overline{f(x^{-1})}$$

For u in $G^{(0)}$, let δ_x be the element in $l^2(G_u)$ that is 1 at x and 0 elsewhere. Define the representation $\phi_u : C_c(G) \rightarrow \mathcal{B}(l^2(G_u))$ by

$$\phi_u(f)\delta_x = \sum_{y \in G_r(x)} f(y)\delta_{yx}$$

The representation $\phi_G = \bigoplus_{u \in G^{(0)}} \phi_u$ is well-defined and faithful, and it is called the *left regular representation* of $C_c(G)$.

Definition 1.3.5. The closure of $\phi_G(C_c(G))$ in $\bigoplus_{u \in G^{(0)}} \mathcal{B}(l^2(G_u))$ is called the *reduced C^* -algebra* of G , and we denote it by $C_r^*(G)$.

We will denote the norm in $C_r^*(G)$ by $\|\cdot\|_r$, so for f in $C_c(G)$ we have

$$\|f\|_r = \|\phi_G(f)\| = \sup_{u \in G^{(0)}} \|\phi_u(f)\|$$

Example 1.3.6. (i) If $X = \{1, 2, \dots, n\}$ and $R = X \times X$ (the trivial equivalence relation), then $C_r^*(R)$ is isomorphic to $M_n(\mathbb{C})$. More generally, if X is any set and $R = X \times X$ with the discrete topology, then $C_r^*(R)$ is isomorphic to $\mathcal{K}(l^2(X))$.

(ii) If X is a locally compact Hausdorff space and $R = \{(x, x) \mid x \in X\}$ (the co-trivial equivalence relation), then $C_r^*(R)$ is isomorphic to $C_0(X)$.

(iii) If \mathbb{Z} acts on the compact Hausdorff space X by a homeomorphism $\sigma : X \rightarrow X$, then $\alpha(f) = f \circ \sigma^{-1}$ is an automorphism of $C(X)$, and $C_r^*(X \times \mathbb{Z})$ is isomorphic to $C(X) \times_{\alpha} \mathbb{Z}$ (9.3.8 of [34]). Under this identification, the inclusion

$\iota : C(X) \rightarrow C(X) \times_{\alpha} \mathbb{Z}$ is given by $\iota(f)(x, n) = f(x)$ if $n = 0$ and $\iota(f)(x, n) = 0$ otherwise. The unitary u implementing the action is given by $\chi_{X \times \{1\}}$ (where χ_A is the characteristic function of the set A).

Many properties of $C_r^*(G)$ may be deduced from properties of G . In the summary below, we collect conditions on G that will ensure that $C_r^*(G)$ is classifiable. The result mentions “dynamic asymptotic dimension” for groupoids. We will not need the actual definition of this term due to Proposition 1.3.8.

Proposition 1.3.7. *Let G be a locally compact Hausdorff étale groupoid.*

- (i) *If $G^{(0)}$ is compact, then $C_c(G)$ and $C_r^*(G)$ are unital.*
- (ii) *If G is second-countable, then $C_r^*(G)$ is separable.*
- (iii) *If G is principal and minimal, then $C_r^*(G)$ is simple.*
- (iv) *If G is principal and has finite dynamic asymptotic dimension, then G is amenable, and $C_r^*(G)$ is nuclear and satisfies the UCT. If, in addition, $G^{(0)}$ has finite covering dimension, then $C_r^*(G)$ has finite nuclear dimension.*

Therefore, if G satisfies all the hypotheses in (i) through (iv) above, then $C_r^*(G)$ is classified by its Elliott invariant.

Proof. (i) Since $G^{(0)}$ is open and compact, the characteristic function $\chi_{G^{(0)}}$ is in $C_c(G)$, and it is easily checked to be the unit in both $C_c(G)$ and $C_r^*(G)$.

- (ii) If G is second-countable, then $C_c(G)$ is separable in the inductive limit topology by Lemma C.2 of [37]. The conclusion then follows from Lemma 9.2.4 of [34].
- (iii) This follows from Theorem 10.3.3 of [34].
- (iv) If G is principal and has finite dynamic asymptotic dimension, then G is amenable by Corollary 8.25 of [15]. The fact that G is amenable then implies that $C_r^*(G)$ is nuclear by Proposition 6.1.8 of [1], and it also implies that $C_r^*(G)$ satisfies the UCT by a result of Tu, see [35]. Finally, if $G^{(0)}$ has finite covering dimension, then Theorem 8.6 of [15] implies that $C_r^*(G)$ has finite nuclear dimension. \square

The following result is Example 5.3 and Theorem 1.3(i) of [15].

Proposition 1.3.8. (i) *If G is equal to the union of compact open subgroupoids, then G has dynamic asymptotic dimension zero.*

- (ii) *If X is a compact Hausdorff space and \mathbb{Z} acts freely and minimally on X , then $X \times \mathbb{Z}$ has dynamic asymptotic dimension one.*

We now set the stage for our “factor groupoid” situation. As we mentioned earlier, a factor groupoid may be regarded as a “noncommutative quotient space”. It may be helpful to recall that, given a continuous proper surjective map $\pi : X \rightarrow Y$ of locally compact Hausdorff spaces, the map $\alpha : C_0(Y) \rightarrow C_0(X)$ defined by $\alpha(f) = f \circ \pi$ is an injective $*$ -homomorphism.

Proposition 1.3.9. *Let G and G' be locally compact Hausdorff étale groupoids and $\pi : G \rightarrow G'$ a continuous proper surjective groupoid homomorphism. Suppose also that for all u in $G^{(0)}$, the map $\pi|_{G^u} : G^u \rightarrow (G')^{\pi(u)}$ is bijective. Then the map $\alpha : C_c(G') \rightarrow C_c(G)$ defined by $\alpha(f) = f \circ \pi$ is an injective $*$ -homomorphism, and it extends to an injective $*$ -homomorphism from $C_r^*(G')$ to $C_r^*(G)$.*

Proof. That $\alpha(f)$ is continuous and compactly supported follows from the fact that π is continuous and proper. We have

$$\alpha(f \star g)(x) = (f \star g)(\pi(x)) = \sum_{z \in (G')^{\pi(r(x))}} f(z)g(z^{-1}\pi(x))$$

and

$$(\alpha(f) \star \alpha(g))(x) = \sum_{y \in G^{r(x)}} f(\pi(y))g(\pi(y^{-1}x))$$

for all f and g in $C_c(G')$ and all x in G . Each z in the first sum corresponds to one and only one y in the second sum via the bijective map $\pi|_{G^{r(x)}}$, and thus both sums have precisely the same terms. We also have

$$\alpha(f^*)(x) = f^*(\pi(x)) = \overline{f(\pi(x)^{-1})} = \overline{f(\pi(x^{-1}))} = \overline{\alpha(f)(x^{-1})} = \alpha(f)^*(x)$$

If $\alpha(f)(x) = f(\pi(x)) = 0$ for all x in G , then $f(y) = 0$ for all y in G' because π is surjective, hence α is injective. Lastly, for each u in $G^{(0)}$, the operator $V_u : l^2(G_u) \rightarrow l^2(G'_{\pi(u)})$ defined by $V_u(\delta_x) = \delta_{\pi(x)}$ is unitary. Now take u in $G^{(0)}$, x' in $G'_{\pi(u)}$, and f in $C_c(G')$ to be arbitrary, and let x be the unique element in G_u with $\pi(x) = x'$ (note that $\pi|_{G_u}$ is also bijective for every u in $G^{(0)}$). We then have

$$\begin{aligned} V_u \phi_u(\alpha(f)) V_u^* \delta_{x'} &= V_u \phi_u(\alpha(f)) \delta_x \\ &= V_u \sum_{y \in G_{r(x)}} f(\pi(y)) \delta_{yx} \\ &= \sum_{y \in G_{r(x)}} f(\pi(y)) \delta_{\pi(y)x'} \\ &= \sum_{y' \in G'_{r(x')}} f(y') \delta_{y'x'} \\ &= \phi_{\pi(u)}(f) \delta_{x'} \end{aligned}$$

where we used the same trick with the bijective map $\pi|_{G_r(x)}$ to re-index the sum. Therefore $V = \bigoplus_{u \in G^{(0)}} V_u$ implements a unitary equivalence between $\phi_G \circ \alpha$ and the representation $\bigoplus_{u \in G^{(0)}} \phi_{\pi(u)}$. The latter representation gives elements of $C_c(G')$ the same reduced norm since

$$\sup_{u \in G^{(0)}} \|\phi_{\pi(u)}(f)\| = \sup_{u' \in G'^{(0)}} \|\phi_{u'}(f)\| = \|f\|_r$$

for f in $C_c(G')$, and therefore produces a completion isomorphic to $C_r^*(G')$. Thus we may identify $C_r^*(G')$ with the closure of $\phi_G(\alpha(C_c(G')))$, which is contained in $C_r^*(G)$. \square

If π and α are as in Proposition 1.3.9, it is worth recording the easy but useful observation that

$$\alpha(C_c(G')) = \{f \in C_c(G) \mid \pi(x) = \pi(x') \text{ implies that } f(x) = f(x')\} \quad (1.3)$$

So we may regard $C_c(G')$ as all functions in $C_c(G)$ that are constant on the fibres of π .

Remark. The hypothesis that $\pi|_{G^u}$ be bijective is necessary. To see this, let G and G' be the full equivalence relations on $\{0, 1\}$ and $\{0\}$, respectively, with $\pi(i, j) = 0$ for $i, j = 0, 1$. Then $C_c(G')$ and $C_c(G)$ may be respectively regarded as \mathbb{C} and $M_2(\mathbb{C})$ and the resulting linear map $\alpha : \mathbb{C} \rightarrow M_2(\mathbb{C})$ is then regarded as

$$\alpha(1) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

which is not multiplicative.

If G and H are locally compact Hausdorff étale groupoids, then $G \times H$ is too, with component-wise operations and the product topology. The following result will be useful. Although it appears rather standard, a proof is difficult to find, so we supply one. As stated, the C*-algebras may not be nuclear, so we use the minimal tensor product to avoid ambiguity, see II.9.1.3 of [5]. If at least one of G and H is amenable, then specifying the tensor product is not necessary.

Proposition 1.3.10. *Suppose G and H are locally compact Hausdorff étale groupoids. Then $C_r^*(G \times H) \cong C_r^*(G) \otimes_{\min} C_r^*(H)$.*

Proof. By the definition of the minimal tensor product, $C_r^*(G) \otimes_{\min} C_r^*(H)$ is the completion of the algebraic tensor product $C_r^*(G) \odot C_r^*(H)$ under the representation $\phi_G \otimes \phi_H$, since the left regular representations are faithful. Notice that $(G \times H)^{(0)} = G^{(0)} \times H^{(0)}$ and, if u is in $G^{(0)}$ and v is in $H^{(0)}$, $(G \times H)_{(u,v)} = G_u \times H_v$. The representation

$$\phi : C_c(G \times H) \rightarrow \mathcal{B}(l^2(G_u) \otimes l^2(H_v))$$

defined by

$$\phi(h)(\delta_x \otimes \delta_y) = \sum_{w \in G_r(x), z \in H_r(y)} h(w, z)(\delta_{wx} \otimes \delta_{zy})$$

is unitarily equivalent to $\phi_{(u,v)}$ via the unitary $U_{(u,v)} : l^2(G_u \times H_v) \rightarrow l^2(G_u) \otimes l^2(H_v)$ defined by $U_{(u,v)}(\delta_{(x,y)}) = \delta_x \otimes \delta_y$. Moreover, if $h = f \odot g$, where f is in $C_c(G)$ and g is in $C_c(H)$ and $f \odot g(w, z) = f(w)g(z)$, then it is clear from the formula above that $\phi(h) = \phi_u(f) \otimes \phi_v(g)$. Taking the direct sum over all u and v , we obtain that $\phi_{G \times H}$ is unitarily equivalent to $\phi_G \otimes \phi_H$. \square

We finish this section by providing a characterization of traces on C^* -algebras of principal groupoids, a proof of which may be found in [31]. If G is a locally compact Hausdorff étale groupoid and μ is a regular Borel measure on $G^{(0)}$, μ is called *invariant* if $\mu(r(U)) = \mu(s(U))$ for every open bisection U .

Proposition 1.3.11. *Let G be a locally compact Hausdorff étale groupoid and let μ be an invariant regular Borel measure on $G^{(0)}$. Then the formula*

$$\tau(f) = \int_{G^{(0)}} f d\mu$$

defines a trace on $C_c(G)$ and extends to a trace on $C_r^(G)$. If G is principal, then all traces on $C_r^*(G)$ arise in this way.*

1.3.1 AF-groupoids and Cantor minimal systems

All of the groupoids we will construct involve, in some way or another, a special class of groupoids called *AF-groupoids*. These groupoids are built from combinatorial objects known as *Bratteli diagrams*. A Bratteli diagram is an infinite directed graph consisting of a set of vertices V and a set of edges E with initial and terminal maps $i : E \rightarrow V$ and $t : E \rightarrow V$. V and E are partitioned into countably many non-empty finite subsets V_n for $n \geq 0$ and E_n for $n \geq 1$ in such a way that the following hold.

- (i) V_0 consists of a single vertex v_0 .
- (ii) If e is in E_n , then $i(e)$ is in V_{n-1} and $t(e)$ is in V_n .
- (iii) $i^{-1}(v)$ is nonempty for each vertex v and $t^{-1}(v)$ is nonempty for each vertex v other than v_0 .

We say that (V, E) has full edge connections if, for every $n \geq 0$ and pair of vertices v in V_n and w in V_{n+1} , there is an edge e in E_{n+1} with $i(e) = v$ and $t(e) = w$.

Definition 1.3.12. We denote by X_E the path space of (V, E) : the set of all infinite paths, that is, all sequences $x = (x_1, x_2, x_3, \dots)$, where x_n is in E_n and $t(x_n) = i(x_{n+1})$ for all $n \geq 1$.

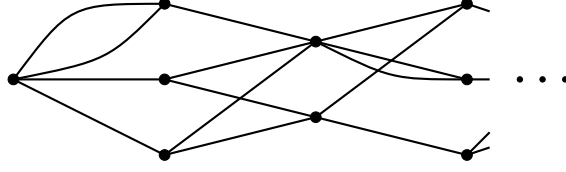


Figure 1.1: A Bratteli diagram

For a finite sequence of edges $p = (p_1, p_2, \dots, p_n)$ where p_j is in E_j and $t(p_j) = i(p_{j+1})$, we denote

$$C(p) = C(p_1, p_2, \dots, p_n) = \{x \in X_E \mid x_j = p_j \text{ for } 1 \leq j \leq n\}$$

called the *cylinder set* of the finite path p . For any p and q , either $C(p)$ and $C(q)$ are disjoint, or one is contained in the other. The path space X_E is a totally disconnected compact metric space in the metric

$$d_E(x, y) = \inf\{1, 2^{-n} \mid x_j = y_j \text{ for } 1 \leq j \leq n\}$$

and the cylinder sets $C(p)$ are closed and open (clopen) in this metric. If (V, E) has full edge connections and X_E is infinite, then X_E has no isolated points and is thus homeomorphic to the Cantor set.

Definition 1.3.13. The equivalence relation $R_E \subseteq X_E \times X_E$ is defined by $(x, y) \in R_E$ if and only if there exists some $n \geq 1$ such that $x_k = y_k$ for all $k \geq n$, also known as *tail-equivalence*. For finite paths p and q with $t(p) = t(q) \in V_n$, let

$$\gamma(p, q) = \{(x, y) \in R_E \mid x \in C(p), y \in C(q), x_k = y_k \text{ for all } k \geq n + 1\}$$

Proposition 1.3.14. *The sets $\gamma(p, q)$ defined above form a base of compact open bisections which makes R_E into a second-countable locally compact Hausdorff étale groupoid.*

Proof. Given any two such sets, their intersection is either empty, or a set of the same form. It follows that they give a base for a topology.

If p and q are finite paths with $t(p) = t(q)$, it is not difficult to see that the range map r implements a bijection between $\gamma(p, q)$ and $\gamma(p, p)$. To see that it is a homeomorphism, note that if p' and q' are finite paths with $\gamma(p', q') \subseteq \gamma(p, q)$, then $r(\gamma(p', q')) = \gamma(p', p')$, which is an open subset of $\gamma(p, p)$, and the topology of $\gamma(p, q)$ is generated by all such sets $\gamma(p', q')$. An analogous proof applies to the source map s .

Let p and q be finite paths with $t(p) = t(q)$. To see that $\gamma(p, q)$ is compact, it suffices to see that $\gamma(p, p)$ is compact, since these two sets are homeomorphic. $\gamma(p, p)$ can be identified with $C(p)$ in an obvious way, and this identification is a homeomorphism:

a finite path p' satisfies $\gamma(p', p') \subseteq \gamma(p, p)$ if and only if $C(p') \subseteq C(p)$. Since $C(p)$ is compact, so is $\gamma(p, p)$.

The collection of sets $\gamma(p, q)$ is countable, therefore R_E is second-countable in the topology it receives from this base. It is also locally compact because the sets $\gamma(p, q)$ are compact. To see that R_E is Hausdorff, let (x, y) and (x', y') be distinct pairs in R_E , and without loss of generality, choose $n \geq 1$ such that $x_n \neq x'_n$. Then, choosing $N \geq n$ large enough so that $x_k = y_k$ and $x'_k = y'_k$ for all $k \geq N$, and letting $p = (x_1, \dots, x_N)$, $p' = (x'_1, \dots, x'_N)$, $q = (y_1, \dots, y_N)$, and $q' = (y'_1, \dots, y'_N)$, the sets $\gamma(p, q)$ and $\gamma(p', q')$ are disjoint and separate (x, y) and (x', y') . \square

With the étale topology from the sets $\gamma(p, q)$, we call R_E the *AF-groupoid associated to the Bratteli diagram (V, E)* . If (V, E) has full edge connections, then R_E is minimal. The topology is concretely metrizable: write

$$R_n = \{(x, y) \in R_E \mid x_k = y_k \text{ for all } k \geq n\}$$

for $n \geq 1$ and $R_0 = \emptyset$, and note that $R_E = \bigcup_{n=0}^{\infty} (R_{n+1} - R_n)$, which is a disjoint union of compact open sets. Let $d_E^{(2)}((x, y), (x', y')) = \max\{d_E(x, x'), d_E(y, y')\}$ if (x, y) and (x', y') are both in $R_{n+1} - R_n$ for some n , and $d_E^{(2)}((x, y), (x', y')) = 1$ otherwise.

The C*-algebra $C_r^*(R_E)$ is called an *AF-algebra*. Usually AF-algebras are defined as inductive limits of finite-dimensional C*-algebras, but this equivalent definition provides a concrete groupoid model. If a topological groupoid G is isomorphic to R_E for some Bratteli diagram (V, E) , we say that G is an *AF-groupoid*.

The characteristic functions $\chi_{\gamma(p, q)}$ of the compact open sets $\gamma(p, q)$ satisfy certain “matrix unit” formulas in $C_c(R_E)$ that will be useful. We summarize these properties below. We omit the proof, as it requires only routine computations.

Proposition 1.3.15. *Let p, q, p' , and q' be finite paths with $t(p), t(q), t(p')$, and $t(q')$ in V_n , and $t(p) = t(q)$ and $t(p') = t(q')$. Then*

$$(i) \chi_{\gamma(p, q)}^* = \chi_{\gamma(q, p)}.$$

$$(ii) \chi_{\gamma(p, q)} \star \chi_{\gamma(p', q')} = 0, \text{ unless } q = p', \text{ in which case } \chi_{\gamma(p, q)} \star \chi_{\gamma(p', q')} = \chi_{\gamma(p, q')}.$$

We describe the K -theory of the C*-algebras of AF-groupoids, which are precisely the abelian groups known as dimension groups.

Definition 1.3.16 (7.2.4 of [32]). A *dimension group* G is an ordered abelian group which is the inductive limit of the sequence of ordered abelian groups

$$\mathbb{Z}^{n_1} \xrightarrow{\alpha_1} \mathbb{Z}^{n_2} \xrightarrow{\alpha_2} \mathbb{Z}^{n_3} \xrightarrow{\alpha_3} \dots$$

where $n_k \geq 1$ for all $k \geq 1$, each \mathbb{Z}^{n_k} is given the standard order

$$(\mathbb{Z}^{n_k})^+ = \{(l_1, l_2, \dots, l_{n_k}) \mid l_j \geq 0 \text{ for all } 1 \leq j \leq n_k\},$$

and α_k is a positive group homomorphism for all $k \geq 1$ (that is, α_k sends positive elements to positive elements).

Dimension groups have the following intrinsic description due to Effros, Handelmann, and Shen [11].

Theorem 1.3.17. *An ordered group (G, G^+) is a dimension group if and only if the following three properties hold.*

- (i) G is countable.
- (ii) G is unperforated, that is, if g is in G and $ng \geq 0$ (where $ng = g + g + \cdots + g$ (n times)), then $g \geq 0$.
- (iii) G has the Riesz interpolation property, that is, if g_1, g_2, h_1, h_2 are four elements in G such that $g_i \leq h_j$ for $i, j = 1, 2$, then there exists an element k in G such that $g_i \leq k \leq h_j$ for $i, j = 1, 2$.

Note that dimension groups are torsion-free; this follows from property (ii) above. The following result may be found in [29]. We provide a detailed proof here.

Proposition 1.3.18. *Every countable torsion-free abelian group may be endowed with an order under which it is a dimension group.*

Proof. Let G be a countable, torsion-free abelian group. The sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

is short exact, and since G is torsion-free, it is flat as a \mathbb{Z} -module (Corollary 18, 17.1 of [10]), hence the sequence

$$0 \longrightarrow G \longrightarrow G \otimes \mathbb{Q} \longrightarrow G \otimes (\mathbb{Q}/\mathbb{Z}) \longrightarrow 0$$

is also short exact, the important fact we draw from which is that the map $i : G \rightarrow G \otimes \mathbb{Q}$ defined by $i(g) = g \otimes 1$ is injective. Now observe that $G \otimes \mathbb{Q}$ is a vector space over \mathbb{Q} (the scalar multiplication being $x(g \otimes y) = g \otimes (xy)$ for g in G and x, y in \mathbb{Q} , and extending by linearity), as is \mathbb{R} . The dimension of \mathbb{R} as a vector space over \mathbb{Q} is uncountable and $G \otimes \mathbb{Q}$ is countable, thus there is an injective \mathbb{Q} -linear map $j : G \otimes \mathbb{Q} \rightarrow \mathbb{R}$. Endow G with the order from \mathbb{R} , that is, set $G^+ = \{g \in G \mid j(i(g)) \geq 0\}$. This is a total order, and hence, with it, (G, G^+) is unperforated and satisfies the Riesz interpolation property. The result then follows from Theorem 1.3.17. \square

For a proof of the following result, see 7.2.8 of [32].

Proposition 1.3.19. *If A is a unital AF-algebra, then $(K_0(A), K_0(A)^+, [1_A])$ is a dimension group with order unit $[1_A]$. Conversely, if (G, G^+) is a dimension group with order unit u , there exists a unital AF-algebra A and an isomorphism $\alpha : K_0(A) \rightarrow G$ such that $\alpha(K_0(A)^+) = G^+$ and $\alpha([1_A]) = u$.*

Briefly, the proof involves writing (G, G^+, u) as a sequence as in Definition 1.3.16, encoding the sequence in a Bratteli diagram (V, E) , from which one obtains A as $C_r^*(R_E)$.

The characteristic functions $\chi_{\gamma(p,p)}$, for a finite path p in the diagram (V, E) , are projections in $C_c(R_E)$. The classes of these elements generate the dimension group $K_0(C_r^*(R_E))$.

We will also need a dynamical construction involving Bratteli diagrams, which has a close relationship to AF-groupoids; see [14], [25], and [28].

Definition 1.3.20. An *ordered Bratteli diagram* (V, E) is a Bratteli diagram where the edges are endowed with an order, and two edges e and f are comparable if and only if $t(e) = t(f)$.

If there is only one infinite path in X_E that consists of only maximal edges and only one infinite path in X_E that consists of only minimal edges, we say that the diagram (V, E) is *properly ordered*, and in this case, we denote the path of all maximal (resp. minimal) edges in X_E by x^{\max} (resp. x^{\min}).

Definition 1.3.21. A *Bratteli-Vershik system* is a pair (X_E, ϕ_E) , where X_E is the path space of a properly ordered Bratteli diagram, and $\phi_E : X_E \rightarrow X_E$ is a map that is defined as follows. If $x = (x_1, x_2, x_3, \dots)$ is in X_E and n is the smallest positive integer such that x_n is not maximal (if such an n exists), then

$$\phi_E(x) = (y_1, y_2, \dots, y_{n-1}, y_n, x_{n+1}, x_{n+2}, \dots)$$

where $(y_1, y_2, \dots, y_{n-1})$ is the unique finite path of minimal edges ending at $t(y_{n-1})$, and y_n is the successor of x_n . If no such n exists, then $x = x^{\max}$, and we set $\phi_E(x^{\max}) = x^{\min}$.

The map ϕ_E is well-defined and a homeomorphism [28], and therefore we obtain an action of \mathbb{Z} on X_E , and from this action, a groupoid $X_E \times \mathbb{Z}$. Notice that there are only two pairs of infinite paths in a properly ordered Bratteli diagram that are orbit equivalent under the map ϕ_E but not tail-equivalent: the pairs (x^{\min}, x^{\max}) and (x^{\max}, x^{\min}) .

Proposition 1.3.22 (Theorem 5.1 of [28]). *Every Cantor minimal system is topologically conjugate to a Bratteli-Vershik system. In other words, if X is a Cantor set and $\phi : X \rightarrow X$ is a minimal homeomorphism, there is a Bratteli-Vershik system (X_E, ϕ_E) and a homeomorphism $\psi : X \rightarrow X_E$ such that $\psi \circ \phi = \phi_E \circ \psi$.*

Proposition 1.3.23 (Theorem 1.1 and Theorem 4.1 of [25]). *Let (X_E, ϕ_E) be a Bratteli-Vershik system on the properly ordered Bratteli diagram (V, E) .*

- (i) *Let R_E be the AF-groupoid associated to the Bratteli diagram (V, E) . Then R_E is an open subgroupoid of $X_E \times \mathbb{Z}$, and this gives rise to an inclusion $C_r^*(R_E) \subseteq$*

$C_r^*(X_E \times \mathbb{Z})$. Moreover, this inclusion induces an order isomorphism $K_0(C_r^*(R_E)) \cong K_0(C_r^*(X_E \times \mathbb{Z}))$.

(ii) $K_1(C_r^*(X_E \times \mathbb{Z})) \cong \mathbb{Z}$, generated by the class of the unitary implementing the action.

Chapter 2

A Karoubian portrait of relative K -theory for C^* -algebras

In this chapter we develop a portrait of relative K -theory for C^* -algebras by following an approach due to Karoubi that uses Banach categories and Banach functors. For every $*$ -homomorphism $\phi : A \rightarrow B$ between C^* -algebras A and B , we produce two abelian groups $K_0(\phi)$ and $K_1(\phi)$ that give information about how the K -theory of A and B are related through ϕ . In fact, the assignments $\phi \mapsto K_0(\phi)$ and $\phi \mapsto K_1(\phi)$ may be regarded as functors in a natural way, and through this we obtain a homology theory that satisfies Bott periodicity (the long exact sequence being the one in Theorem 2.1.2), and the connections between the K -theory of A and B , the induced map $\phi_* : K_*(A) \rightarrow K_*(B)$, and the relative data $K_*(\phi)$ are contained in the six-term exact sequence in Theorem 2.1.1.

Karoubi's approach to defining a relative theory is to describe the elements of the groups using certain triples, sometimes referred to as "relative K -cycles". One may notice that this resembles the construction of $K^0(X, Y)$ via triples of the form (E, F, α) , where E and F are vector bundles over X and $\alpha : E|_Y \rightarrow F|_Y$ is an isomorphism between the bundles E and F when restricted to Y . This approach may appear somewhat classical at first, particularly because it is at odds with the mapping cone, a shortcut seen in both topological and operator K -theory. As we saw in Chapter 1, the mapping cone furnishes the exact sequences needed to define a relative theory and therefore obtain the excision theorem of [30]. Moreover, Theorem 2.1.3 states that the two portraits are isomorphic in a natural way. It is therefore a reasonable question to ask why one would employ an alternative portrait at all.

First, there is more freedom in selecting the elements from the algebras that represent elements of the relative groups via the setup in [18], which makes viewing and working with the groups easier in many situations. Second, certain maps in the six-term exact sequences are easier to compute (specifically μ_0 in Theorem 2.1.1 and ∂_1 in Theorem 2.1.2). Third, there is a clear and growing desire for an alternative portrait (at least for C^* -algebras) based on the literature, especially for relative K_0 -theory. For

example, the proof of the excision theorem in [30], which is a significant generalization of its predecessor in [27] (where the mapping cone was used), rests heavily on the new presentation, particularly on the notions of “isomorphism” of triples and “elementary” triples, to address certain fine, technical details. A portrait using partial isometries is developed in [27], and in fact, the map κ constructed there is essentially the same as Δ in Theorem 2.1.3, the difference being that Δ is a functorially induced group isomorphism, while κ is a bijective map constructed concretely. The characterization of the relative group in [27] later appeared in [2] in order to study exact sequences associated to Cuntz-Pimsner algebras. Also worth mentioning is recent work on groupoid homology [6], where a portrait of the relative K_0 -group using triples is presented in order to elucidate the connections between K -theory and homology (this portrait was inspired by [27]). An isomorphism between the resulting monoid and the K_0 -group of the mapping cone is constructed under the assumption that A and B are unital and ϕ is unit-preserving. The unital assumption can be done away with if ϕ is nondegenerate and A contains an approximate identity of projections. Here, the general description of the relative groups allows us to do away with these assumptions. To provide further justification, we elaborate on the K_0 - and K_1 -groups separately.

The three usual notions of equivalence of projections, Murray-von Neumann, unitary, and homotopy, are all stably equivalent; in other words, they are the same modulo passing to matrix algebras. In constructing a relative K_0 -group, it is therefore necessary to select a notion of equivalence with which to build the elements. The mapping cone C_ϕ is made from paths of projections in B with one endpoint equal to a scalar projection and the other endpoint in the image of ϕ . Therefore, in effect, $K_0(C_\phi)$ catalogues projections arising from A that are homotopic when moved to B via ϕ . It is often more desirable to describe equivalences of projections using partial isometries (see Example 2.10.2), from which the newer portrait is built.

As for K_1 , homotopy is a much more natural equivalence and hence the portrait of the relative K_1 -group gets less of a makeover than that of K_0 . In fact, the two portraits are more or less the same. However, we draw a useful property from [18] which deserves to be mentioned. It is possible to define (ordinary and relative) K_1 -groups more generally using partial unitaries (elements which are partial isometries and normal) rather than unitaries alone. This more general representation of group elements is especially convenient if one of (or both) A and B are not unital but contain nontrivial projections, such as \mathcal{K} , the compact operators on a separable Hilbert space (see Example 2.10.3).

Therefore, the goal is to develop a comprehensive portrait of Karoubi’s relative K -theory for C^* -algebras, as an alternative to the mapping cone. We do not claim that one portrait is superior to the other, but rather that either portrait can be used, and one may select that which is most convenient for the situation at hand. We remark that a preliminary development of the picture may be found in [30], where the relative

K_0 -group of an inclusion $A' \subseteq A$ is described using this approach. We also remark that, although the intention in [18] is mainly to develop topological K -theory, the setup lends itself quite well to C^* -algebras. In fact, one could pursue these results more generally using Banach algebras, but we will not conduct such a pursuit beyond the definitions and preliminary results.

We begin this chapter with a survey of Karoubi's approach to K -theory using Banach categories and Banach functors, and verify that this approach lends itself well to C^* -algebras. The survey is conducted in a great deal of generality using idempotents and invertibles. We then refine the resulting portrait using the C^* -algebra structure, allowing restriction to projections and unitaries. Next, we prove the existence of two six-term exact sequences that are analogous to those in Proposition 1.2.14 and Proposition 1.2.15. Finally, we show that the two portraits of relative K -theory (Karoubian and mapping cone) are isomorphic in a natural way. Beyond the main results, we also provide formulas for certain maps, a six-term exact sequence of relative groups involving composition of $*$ -homomorphisms, and we summarize the excision theorem of [30] in the context of factor groupoids. We end with examples.

A final remark is likely in order before we begin. It is probably possible to prove Theorem 2.1.3 directly and deduce Theorem 2.1.1 and Theorem 2.1.2 from it with the help of Proposition 1.2.14 and Proposition 1.2.15. The merit in proving Theorem 2.1.1 and Theorem 2.1.2 first is that this route helps one become accustomed to the Karoubian picture. Moreover, once the first two results are in hand, the proof of the third result becomes pleasantly slick.

Throughout this chapter, $\phi : A \rightarrow B$ will denote an arbitrary $*$ -homomorphism between C^* -algebras A and B .

2.1 The main results

In order to properly state the main results, we provide an initial picture of the relative groups $K_0(\phi)$ and $K_1(\phi)$ via Karoubi's approach. Proofs will be given in later sections.

The group $K_0(\phi)$ is made from triples (p, q, v) , where p and q are projections in some matrix algebras over \tilde{A} , the unitization of A , and v is an element in a matrix algebra over \tilde{B} such that $v^*v = \phi(p)$ and $vv^* = \phi(q)$ (recall that we may view them in the same matrix algebra by adding zeros down the diagonal if necessary). The triples are sorted into equivalence classes that are denoted $[p, q, v]$ (the equivalence relation will be rigorously defined in the Section 2.3), and are given a well-defined group operation by the usual block diagonal sum,

$$[p, q, v] + [p', q', v'] = [p \oplus p', q \oplus q', v \oplus v']$$

For two triples (p, q, v) and (p', q', v') to yield the same equivalence class, p and p'

must be (at least stably) Murray-von Neumann equivalent in \tilde{A} , as must be q and q' . Moreover, elements c and d implementing such equivalences must play well with v and v' in that we require $\phi(d)v = v'\phi(c)$. More generally, we may use idempotents instead of projections to represent the elements of $K_0(\phi)$.

$K_1(\phi)$ is made from triples (p, u, g) , where p is a projection in $M_\infty(\tilde{A})$, u is a unitary in $pM_\infty(\tilde{A})p$, and g is a unitary in $C([0, 1]) \otimes \phi(p)M_\infty(\tilde{B})\phi(p)$ such that $g(0) = \phi(p)$ and $g(1) = \phi(u)$. The triples are sorted into equivalence classes, denoted $[p, u, g]$, and are given a well-defined group operation by diagonal sum as before, although we have the formula

$$[p, u, g] + [p', u', g'] = [p, uu', gg']$$

if $p = p'$. For two triples (p, u, g) and (p', u', g') to yield the same equivalence class, p and p' must be (at least stably) Murray-von Neumann equivalent in \tilde{A} , and a partial isometry v implementing such an equivalence must satisfy $vu = u'v$ and $\phi(v)g(s) = g'(s)\phi(v)$ for $0 \leq s \leq 1$. The equivalence may also be described as stable homotopy: u and u' must be (at least stably) homotopic, as must be g and g' . Moreover, such homotopies u_t and g_t must satisfy $g_t(1) = \phi(u_t)$ for $0 \leq t \leq 1$. In general, the projection p may be replaced by an idempotent, u with an invertible in $pM_n(\tilde{A})p$, and g with a path of invertibles through $\phi(p)M_n(\tilde{B})\phi(p)$.

Remark. It may happen that $p = 0$, in which case we are speaking of invertible, or unitary, elements in $pM_n(\tilde{A})p = \{0\}$. To circumvent this small detail, we will regard 0 as an invertible element of the C*-algebra $\{0\}$.¹

We now state the first main result concerning the relative groups.

Theorem 2.1.1. *Suppose that $\phi : A \rightarrow B$ is a *-homomorphism. Then there is a six-term exact sequence*

$$\begin{array}{ccccc} K_1(B) & \xrightarrow{\mu_0} & K_0(\phi) & \xrightarrow{\nu_0} & K_0(A) \\ \uparrow \phi_* & & & & \downarrow \phi_* \\ K_1(A) & \xleftarrow{\nu_1} & K_1(\phi) & \xleftarrow{\mu_1} & K_0(B) \end{array}$$

The maps ν_0 and ν_1 are given by the formulas

$$\nu_0([p, q, v]) = [p] - [q] \quad \nu_1([p, u, g]) = [u + 1_n - p]$$

for triples (p, q, v) and (p, u, g) . The maps μ_0 and μ_1 are given by the formulas

$$\mu_0([u]) = [1_n, 1_n, u] \quad \mu_1([p] - [q]) = [1_n, 1_n, f_p f_q^*]$$

¹All complaints about this convention may be sent directly to the author, after which they will be promptly ignored.

where u is a unitary in $M_n(\tilde{B})$, and $f_p(t) = e^{2\pi itp}$ for a projection p .

Before we state the second theorem, it will be useful to have the following functorial property in mind (see Proposition 2.3.12 for a proof). If there is a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & C \\ \downarrow \phi & & \downarrow \psi \\ B & \xrightarrow{\beta} & D \end{array} \quad (2.1)$$

of C^* -algebras and $*$ -homomorphisms, then there are well-defined group homomorphisms $(\alpha, \beta)_* : K_j(\phi) \rightarrow K_j(\psi)$ for $j = 0, 1$ that satisfy $(\alpha, \beta)_*([p, q, v]) = [\alpha(p), \alpha(q), \beta(v)]$ and $(\alpha, \beta)_*([p, u, g]) = [\alpha(p), \alpha(u), \beta(g)]$.

Theorem 2.1.2. *Suppose that*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \xrightarrow{\iota_A} & A & \xrightarrow{\pi_A} & A/I & \longrightarrow & 0 \\ & & \downarrow \psi & & \downarrow \phi & & \downarrow \gamma & & \\ 0 & \longrightarrow & J & \xrightarrow{\iota_B} & B & \xrightarrow{\pi_B} & B/J & \longrightarrow & 0 \end{array}$$

is a commutative diagram with exact rows. Then there is a six-term exact sequence

$$\begin{array}{ccccc} K_0(\psi) & \xrightarrow{\iota_*} & K_0(\phi) & \xrightarrow{\pi_*} & K_0(\gamma) \\ \uparrow \partial_1 & & & & \downarrow \partial_0 \\ K_1(\gamma) & \xleftarrow{\pi_*} & K_1(\phi) & \xleftarrow{\iota_*} & K_1(\psi) \end{array}$$

The boundary maps ∂_1 and ∂_0 are natural in a functorial sense, as well as in the sense that they behave well with the boundary maps associated with the two short exact sequences of C^* -algebras in the given diagram. This will be stated more precisely later.

The third result establishes a strong connection between the Karoubian portrait of relative K -theory and that which is given by the mapping cone (recall Definition 1.2.12 and the results and discussion shortly thereafter).

Theorem 2.1.3. *Let $\phi : A \rightarrow B$ be a $*$ -homomorphism. There are group isomorphisms $\Delta : K_*(\phi) \rightarrow K_*(C_\phi)$ that are natural in the sense that if we have a commutative diagram as in (2.1), then the following diagram is commutative*

$$\begin{array}{ccc} K_*(\phi) & \xrightarrow{(\alpha, \beta)_*} & K_*(\psi) \\ \downarrow \Delta & & \downarrow \Delta \\ K_*(C_\phi) & \xrightarrow{(\alpha \oplus C\beta)_*} & K_*(C_\psi) \end{array} \quad (2.2)$$

Moreover, the isomorphisms Δ intertwine the six-term exact sequence in Theorem 2.1.1 with the six-term exact sequence in Proposition 1.2.14, and they intertwine the six-term exact sequence in Theorem 2.1.2 with the six-term exact sequence in Proposition 1.2.15.

Readers well-versed in category theory or homology may notice that the commutative diagram in Theorem 2.1.3 makes the isomorphism Δ look an awful lot like a natural transformation. We will return to this idea in Chapter 5.

2.2 Banach categories, Banach functors, and an alternate picture of $K_1(A)$

Karoubi's construction of K -theory and relative K -theory owes its generality to the notions of Banach categories and Banach functors. In this section we outline these notions and verify that they work well in the C^* -algebra setting. We also gain a slightly more general picture of $K_1(A)$.

A robust knowledge of category theory is not necessary here; the basic material in Appendix II in [10] suffices.

Notation. If \mathcal{C} is a category, we denote the class of objects of \mathcal{C} by $\text{Ob}(\mathcal{C})$, and if a and b are two objects, we denote the morphisms from E to F by $\text{hom}(E, F)$.

If \mathcal{C} is an additive category (that is, each collection of morphisms has the structure of an abelian group, and composition of morphisms is bilinear with respect to the group operation) that admits direct sums of objects, then one defines the group $K(\mathcal{C})$ (or $K^0(\mathcal{C})$ to be specific) to be the Grothendieck group of isomorphism classes of objects in \mathcal{C} , with direct sum as the monoid operation. In topological K -theory the objects in question are vector bundles over a compact space X , and the group so obtained is denoted $K^0(X)$. This is enough to define the first K -group, but to form further K -groups and the relative groups, another notion is needed.

Definition 2.2.1 (II.2.1 of [18]). A *Banach category* \mathcal{C} is an additive category such that every collection of morphisms has the structure of a Banach space, and such that, if E, F , and G are in $\text{Ob}(\mathcal{C})$, the composition map

$$\text{hom}(E, F) \times \text{hom}(F, G) \rightarrow \text{hom}(E, G)$$

is bilinear and continuous.

It is the Banach space structure on the morphisms that allows us to introduce the notion of homotopy.

Definition 2.2.2. Let A be a C^* -algebra. Define the category \mathcal{C}_A (we will prove that it is indeed a category below) whose objects are idempotents in $M_\infty(\tilde{A})$ and, if e and f are idempotents in $M_\infty(\tilde{A})$, the set $\text{hom}(e, f)$ consists of all elements a in $M_\infty(\tilde{A})$ such that $a = fae$. Composition of morphisms is given by multiplication.

Proposition 2.2.3. \mathcal{C}_A is a Banach category, and if A is unital, then $K_0(A)$ is isomorphic to $K^0(\mathcal{C}_A)$.

Proof. If a is in $\text{hom}(e, f)$ and b is in $\text{hom}(f, g)$, notice that $fa = f(fae) = f^2ae = fae = a$, so $ba = (gbf)(fae) = gb f^2ae = gbfae = gb(fa)e = gbae$. We also have $e = eee$, so e is in $\text{hom}(e, e)$, and $ae = (fae)e = a$, so e is the identity morphism. If e and f are in $\text{Ob}(\mathcal{C}_A)$ and n is large enough so that e and f are both in $M_n(\tilde{A})$, then $\text{hom}(e, f) = fM_n(\tilde{A})e$, which is a closed linear subspace of the Banach space $M_n(\tilde{A})$, hence is a Banach space itself. Since multiplication is bilinear and continuous in $M_n(\tilde{A})$ for any n , \mathcal{C}_A satisfies Definition 2.2.1. It follows directly from the definitions that e and f in $\text{Ob}(\mathcal{C}_A)$ are isomorphic if and only if there are elements a and b with $ba = e$ and $ab = f$. This is just the usual algebraic equivalence of idempotents, which aligns with the usual definition of $K_0(A)$ when A is unital. \square

Through the above result, we do not gain much more generality in the picture of $K_0(A)$ using Banach categories. We do, however, gain some in the picture of $K_1(A)$.

The following construction is II.3.3 of [18]. Let $\Gamma(A)$ denote all pairs (e, a) where e is in $\text{Ob}(\mathcal{C}_A)$ and a is an automorphism of e ; in other words, it is an invertible element of the Banach algebra $eM_\infty(\tilde{A})e$. Notice that (e, e) is in $\Gamma(A)$ for an idempotent e in $M_\infty(\tilde{A})$. Define a direct sum operation on these pairs by $(e, a) \oplus (e', a') = (e \oplus e', a \oplus a')$. Say that two pairs (e, a) and (e', a') are *isomorphic*, written $(e, a) \cong (e', a')$, if there is an isomorphism b from e to e' such that $ba = a'b$. We say a pair (e, a) is *elementary* if there is a continuous path from a to e through the isomorphisms of e . We say that two pairs (e, a) and (e', a') in $\Gamma(A)$ are *equivalent*, written $(e, a) \sim (e', a')$, if there exist elementary pairs (f, b) and (f', b') such that $(e, a) \oplus (f, b) \cong (e', a') \oplus (f', b')$. Denote by $[e, a]$ the equivalence class of the pair (e, a) via the relation \sim . $K^{-1}(\mathcal{C}_A)$ is defined to be the quotient of $\Gamma(A)$ by the relation \sim .

The following facts are II.3.4, II.3.5, II.3.6, and II.3.7 of [18].

Lemma 2.2.4. (i) $K^{-1}(\mathcal{C}_A)$ is an abelian group, and $-[e, a] = [e, a^{-1}]$, where a^{-1} is the inverse of a as an automorphism.

(ii) If a and a' are homotopic as automorphisms of e , then $[e, a] = [e, a']$.

(iii) We have the formula $[e, a] + [e, a'] = [e, aa'] = [e, a'a]$.

(iv) $[e, a] = 0$ if and only if there is an idempotent g such that $a \oplus g$ and $e \oplus g$ are homotopic as automorphisms of $e \oplus g$.

Using the algebraic structure of the objects, we can say a bit more.

Lemma 2.2.5. (i) If (e, a) and (e', a') are pairs in $\Gamma(A)$ and $ee' = e'e = 0$, then $(e, a) \oplus (e', a') \cong (e + e', a + a')$.

(ii) Every element of $K^{-1}(\mathcal{C}_A)$ is represented by a pair in $\Gamma(A)$ of the form $(1_n, a)$ for some integer n .

(iii) $[1_n, a] = 0$ if and only if there is an integer k such that $a \oplus 1_k$ and 1_{n+k} are homotopic as invertibles in $M_{n+k}(\tilde{A})$.

Proof. (i) The element $b = \begin{bmatrix} e & e' \\ 0 & 0 \end{bmatrix}$ is an isomorphism from $e \oplus e'$ to $(e + e') \oplus 0$ and satisfies $b(a \oplus a') = ((a + a') \oplus 0)b$.

(ii) If e is any idempotent in $M_n(\tilde{A})$, then $[1_n - e, 1_n - e] = 0$ since $(1_n - e, 1_n - e)$ is elementary. So by part (i), $[e, a] = [e, a] + [1_n - e, 1_n - e] = [1_n, a + 1_n - e]$.

(iii) By part (iv) of Lemma 2.2.4, we can find an idempotent g such that $a \oplus g$ and $1_n \oplus g$ are homotopic as automorphisms of $1_n \oplus g$ (let a_t denote such a homotopy). Choose k so that g is in $M_k(\tilde{A})$. Then $a_t + 0 \oplus (1_k - g)$ is a homotopy of invertibles between $a \oplus 1_k$ and 1_{n+k} . \square

Proposition 2.2.6. The map $\Omega_A : K_1(A) \rightarrow K^{-1}(\mathcal{C}_A)$ defined by $\Omega_A([u]) = [1_n, u]$ (for a unitary u in $M_n(\tilde{A})$) is a natural isomorphism.

Proof. For surjectivity, by the previous lemma we may find a preimage of some $[1_n, a]$. Being an invertible element in $M_n(\tilde{A})$, a is homotopic to the unitary $a(a^*a)^{-1/2}$ via the path $a(a^*a)^{-t/2}$ for $0 \leq t \leq 1$, so $\Omega_A([a(a^*a)^{-1/2}]) = [1_n, a]$ by part (ii) of Lemma 2.2.4. For injectivity, use part (iii) of the previous lemma combined with the standard fact that if a unitary is homotopic to the identity as an invertible, it is homotopic as a unitary (see, for example, Proposition 2.1.8 of [32]). \square

We now introduce the notion of a Banach functor in order to define the relative groups.

Definition 2.2.7 (II.2.6 of [18]). Let \mathcal{C} and \mathcal{C}' be Banach categories, and $\phi : \mathcal{C} \rightarrow \mathcal{C}'$ a covariant functor.

(i) ϕ is called a *Banach functor* if, for each pair of objects E and F in $\text{Ob}(\mathcal{C})$, the induced map from $\text{hom}(E, F)$ to $\text{hom}(\phi(E), \phi(F))$ is linear and continuous.

(ii) ϕ is called *quasi-surjective* if, for every object F in $\text{Ob}(\mathcal{C}')$, there is an object E in $\text{Ob}(\mathcal{C})$ and an object G in $\text{Ob}(\mathcal{C}')$ such that $\phi(E)$ is isomorphic to $F \oplus G$.

In topological K -theory, if X is a compact space and Y is a closed subspace of X , the Banach functor in question is the restriction functor; that is, $\phi(E) = E|_Y$, where E is a vector bundle over X . This produces the well-known relative group $K^0(X, Y)$. The quasi-surjectivity of the restriction functor follows from the fact that every vector bundle over Y is (isomorphic to) a direct summand of the trivial bundle over Y (I.6.5 of [18]), which is the image of the trivial bundle over X through ϕ .

Proposition 2.2.8. *Let $\phi : A \rightarrow B$ be a $*$ -homomorphism. Then the induced map from \mathcal{C}_A to \mathcal{C}_B is a quasi-surjective Banach functor.*

Proof. Since ϕ is linear, continuous, and preserves multiplication, the fact that it induces a Banach functor is straightforward. If f is in $\text{Ob}(\mathcal{C}_B)$, choose n so that f is in $M_n(\tilde{B})$. Then $\phi(1_n) = 1_n$ is isomorphic to $f \oplus (1_n - f)$: let

$$a = \begin{bmatrix} f & 1_n - f \\ 0 & 0 \end{bmatrix} \quad b = \begin{bmatrix} f & 0 \\ 1_n - f & 0 \end{bmatrix}$$

so that $ab = 1_n \oplus 0_n$ and $ba = f \oplus (1_n - f)$. This shows that ϕ is quasi-surjective. \square

2.3 A portrait of $K_*(\phi)$

We are now ready to define the relative groups $K_*(\phi)$ and discuss some of their properties. Recall that $\phi : A \rightarrow B$ extends to the unitizations $\tilde{\phi} : \tilde{A} \rightarrow \tilde{B}$, and we will usually denote the extension by ϕ .

Definition 2.3.1. Define $\Gamma_0(\phi)$ to be the set of all triples (e, f, b) where e and f are idempotents in $M_\infty(\tilde{A})$ and b is in $M_\infty(\tilde{B})$ and is an isomorphism from $\phi(e)$ to $\phi(f)$. For brevity, we will often denote these triples by the symbols σ and τ .

- (i) Define the direct sum operation \oplus on $\Gamma_0(\phi)$ by

$$(e, f, b) \oplus (e', f', b') = (e \oplus e', f \oplus f', b \oplus b').$$

- (ii) We say that two such triples (e, f, b) and (e', f', b') are *isomorphic*, written $(e, f, b) \cong (e', f', b')$, if there exist isomorphisms c and d from e to e' and from f to f' , respectively, that intertwine b and b' , that is, $\phi(d)b = b'\phi(c)$. It is often helpful to convey this fact by saying that the diagram

$$\begin{array}{ccc} \phi(e) & \xrightarrow{b} & \phi(f) \\ \downarrow \phi(c) & & \downarrow \phi(d) \\ \phi(e') & \xrightarrow{b'} & \phi(f') \end{array}$$

is commutative.

- (iii) A triple (e, f, b) is called *elementary* if $e = f$ and there is a homotopy b_t for $0 \leq t \leq 1$ such that $b_0 = \phi(e)$ and $b_1 = b$, and b_t is invertible in $\text{hom}(\phi(e), \phi(e))$ for all t .
- (iv) Say that two triples σ and σ' in $\Gamma_0(\phi)$ are *equivalent*, written $\sigma \sim \sigma'$, if there exist elementary triples τ and τ' such that $\sigma \oplus \tau \cong \sigma' \oplus \tau'$.

Denote by $[\sigma]$, or $[e, f, b]$, the equivalence class of the triple $\sigma = (e, f, b)$ via the relation \sim .

Definition 2.3.2. $K_0(\phi)$ is defined to be the quotient of $\Gamma_0(\phi)$ by the relation \sim , that is,

$$\{[\sigma] \mid \sigma \in \Gamma_0(\phi)\} = \Gamma_0(\phi) / \sim$$

Lemma 2.3.3. *All triples below are in $\Gamma_0(\phi)$.*

- (i) *If $\sigma_1 \cong \sigma_2$ and $\sigma_3 \cong \sigma_4$, then $\sigma_1 \oplus \sigma_3 \cong \sigma_2 \oplus \sigma_4$.*
- (ii) *For any two triples σ and σ' , we have $\sigma \oplus \sigma' \cong \sigma' \oplus \sigma$.*
- (iii) *If σ and σ' are elementary, then so is $\sigma \oplus \sigma'$.*
- (iv) *All elementary triples are equivalent to each other, and two isomorphic triples are equivalent.*
- (v) *If (e, f, b) is any triple in $\Gamma_0(\phi)$, then the triple*

$$\left(\left[\begin{array}{cc} e & 0 \\ 0 & f \end{array} \right], \left[\begin{array}{cc} e & 0 \\ 0 & f \end{array} \right], \left[\begin{array}{cc} 0 & b^{-1} \\ -b & 0 \end{array} \right] \right)$$

is elementary.

Proof. (i) through (iv) are straightforward. For (v), The homotopy

$$b_t = \left[\begin{array}{cc} \phi(e) & tb^{-1} \\ 0 & \phi(f) \end{array} \right] \left[\begin{array}{cc} \phi(e) & 0 \\ -tb & \phi(f) \end{array} \right] \left[\begin{array}{cc} \phi(e) & tb^{-1} \\ 0 & \phi(f) \end{array} \right],$$

for $0 \leq t \leq 1$, does the job. □

Proposition 2.3.4. $K_0(\phi)$ is an abelian group when equipped with the binary operation

$$[\sigma] + [\sigma'] = [\sigma \oplus \sigma']$$

where the identity element is given by $[0, 0, 0]$ and the inverse of $[e, f, b]$ is given by $[f, e, b^{-1}]$, for (e, f, b) in $\Gamma_0(\phi)$.

Proof. That $K_0(\phi)$ is an abelian group and that $[0, 0, 0] = 0$ follows quite readily from the observations above. To prove the last statement, note that

$$[e, f, b] + [f, e, b^{-1}] = [e \oplus f, f \oplus e, b \oplus b^{-1}]$$

and the triple $(e \oplus f, f \oplus e, b \oplus b^{-1})$ is isomorphic to the triple

$$\left(\left[\begin{array}{cc} e & 0 \\ 0 & f \end{array} \right], \left[\begin{array}{cc} e & 0 \\ 0 & f \end{array} \right], \left[\begin{array}{cc} 0 & -b^{-1} \\ b & 0 \end{array} \right] \right)$$

since

$$\left[\begin{array}{cc} 0 & -\phi(f) \\ \phi(e) & 0 \end{array} \right] \left[\begin{array}{cc} b & 0 \\ 0 & b^{-1} \end{array} \right] = \left[\begin{array}{cc} 0 & -b^{-1} \\ b & 0 \end{array} \right] \left[\begin{array}{cc} \phi(e) & 0 \\ 0 & \phi(f) \end{array} \right].$$

By part (v) of Lemma 2.3.3, the latter triple is elementary. \square

We collect some useful properties of the elements of $K_0(\phi)$.

Proposition 2.3.5. (i) *Given two invertible morphisms b and b' from $\phi(e)$ to $\phi(f)$, if b and b' are homotopic throughout the isomorphisms from $\phi(e)$ to $\phi(f)$, then $[e, f, b] = [e, f, b']$.*

(ii) *For two triples (e, f, b) and (e', f', b') in $\Gamma_0(\phi)$, if $f = e'$ then we have*

$$[e, f, b] + [e', f', b'] = [e, f', b'b].$$

(iii) *Let (e, f, b) and (e', f', b') be two triples in $\Gamma_0(\phi)$. If $ee' = e'e = 0$, then*

$$(e, f, b) \oplus (e', f', b') \cong \left(\left[\begin{array}{cc} e + e' & 0 \\ 0 & 0 \end{array} \right], \left[\begin{array}{cc} f & 0 \\ 0 & f' \end{array} \right], \left[\begin{array}{cc} b & 0 \\ b' & 0 \end{array} \right] \right)$$

If $ff' = f'f = 0$, then

$$(e, f, b) \oplus (e', f', b') \cong \left(\left[\begin{array}{cc} e & 0 \\ 0 & e' \end{array} \right], \left[\begin{array}{cc} f + f' & 0 \\ 0 & 0 \end{array} \right], \left[\begin{array}{cc} b & b' \\ 0 & 0 \end{array} \right] \right)$$

If $ee' = e'e = ff' = f'f = 0$, then

$$(e, f, b) \oplus (e', f', b') \cong (e + e', f + f', b + b')$$

(iv) *$[e, f, b] = 0$ if and only if there exist idempotents g and h in $M_\infty(\tilde{A})$ and isomorphisms x and y in $M_\infty(\tilde{A})$ from $e \oplus g$ to h and $f \oplus g$ to h , respectively, such that $\phi(y)(b \oplus \phi(g))\phi(x^{-1})$ is homotopic to $\phi(h)$ through the automorphisms of $\phi(h)$.*

Proof. (i) This is essentially II.2.15 of [18]; we repeat the proof here. We compute

$$[e, f, b] - [e, f, b'] = [e \oplus f, f \oplus e, b \oplus b'^{-1}]$$

and the triple $(e \oplus f, f \oplus e, b \oplus b'^{-1})$ is isomorphic to

$$\left(\begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}, \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}, \begin{bmatrix} 0 & -b'^{-1} \\ b & 0 \end{bmatrix} \right)$$

similarly as in the proof of Proposition 2.3.4. Since b is homotopic to b' , the two matrices

$$\begin{bmatrix} 0 & -b'^{-1} \\ b & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & -b^{-1} \\ b & 0 \end{bmatrix}$$

are homotopic. It follows that the latter triple is elementary.

(ii) This is essentially II.2.16 of [18]; we repeat the proof here. We compute $[e, f, b] + [f, f', b'] = [e \oplus f, f \oplus f', b \oplus b']$ and observe that the triple $(e \oplus f, f \oplus f', b \oplus b')$ is isomorphic to the triple

$$\left(\begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}, \begin{bmatrix} f' & 0 \\ 0 & f \end{bmatrix}, \begin{bmatrix} 0 & -b' \\ b & 0 \end{bmatrix} \right)$$

since

$$\begin{bmatrix} 0 & -\phi(f') \\ \phi(f) & 0 \end{bmatrix} \begin{bmatrix} b & 0 \\ 0 & b' \end{bmatrix} = \begin{bmatrix} 0 & -b' \\ b & 0 \end{bmatrix} \begin{bmatrix} \phi(e) & 0 \\ 0 & \phi(f) \end{bmatrix}.$$

We also have that $[e, f', b'b] = [e \oplus f, f' \oplus f, b'b \oplus \phi(f)]$, since $(f, f, \phi(f))$ is elementary. Now

$$\begin{bmatrix} 0 & -b' \\ b & 0 \end{bmatrix} \begin{bmatrix} b^{-1}b'^{-1} & 0 \\ 0 & \phi(f) \end{bmatrix} = \begin{bmatrix} 0 & -b' \\ b'^{-1} & 0 \end{bmatrix}$$

and the matrix on the right is homotopic to $\phi(f') \oplus \phi(f)$. It follows that the two matrices

$$\begin{bmatrix} 0 & -b' \\ b & 0 \end{bmatrix} \quad \begin{bmatrix} b'b & 0 \\ 0 & \phi(f) \end{bmatrix}$$

are homotopic, and hence, by part (i), the triples are equivalent.

(iii) If $ee' = e'e = 0$, then $b\phi(e') = b\phi(e)\phi(e') = b\phi(ee') = 0$, and similarly $b'\phi(e) = 0$.

Thus we have

$$\begin{bmatrix} \phi(f) & 0 \\ 0 & \phi(f') \end{bmatrix} \begin{bmatrix} b & 0 \\ 0 & b' \end{bmatrix} = \begin{bmatrix} b & 0 \\ b' & 0 \end{bmatrix} \begin{bmatrix} \phi(e) & \phi(e') \\ 0 & 0 \end{bmatrix}$$

so the triples are isomorphic. The other two claims are similar.

- (iv) It is a direct consequence of the definitions that $[e, f, b] = 0$ if and only if there are elementary triples (g, g, c) and (h, h, d) such that

$$(e, f, b) \oplus (g, g, c) \cong (h, h, d).$$

This is true if and only if there are isomorphisms x and y in $M_\infty(\tilde{A})$ such $d\phi(x) = \phi(y)(b \oplus c)$. Then $d = \phi(y)(b \oplus c)\phi(x^{-1})$, and since d is homotopic to $\phi(h)$ and c is homotopic to $\phi(g)$, we have the conclusion. \square

Now we construct $K_1(\phi)$. The construction of higher relative groups using Banach functors uses Clifford algebras. We offer an alternative definition which is a sort of middle ground between Karoubi's definition and the mapping cone.

Definition 2.3.6. Define $\Gamma_1(\phi)$ to be the set of all triples (e, a, g) where e is an idempotent in $M_\infty(\tilde{A})$, a is an invertible element of $eM_\infty(\tilde{A})e$ and g is an invertible element of

$$C([0, 1]) \otimes \phi(e)M_\infty(\tilde{B})\phi(e)$$

such that $g(0) = \phi(e)$ and $g(1) = \phi(a)$. For brevity, we will often denote these triples by the symbols σ and τ .

- (i) Define the direct sum operation \oplus on $\Gamma_1(\phi)$ by

$$(e, a, g) \oplus (e', a', g') = (e \oplus e', a \oplus a', g \oplus g')$$

for two triples (e, a, g) and (e', a', g') in $\Gamma_1(\phi)$.

- (ii) We say that two such triples (e, a, g) and (e', a', g') are *isomorphic*, written $(e, a, g) \cong (e', a', g')$, if there is an isomorphism from e to e' such that $ba = a'b$ and $\phi(b)g(s) = g'(s)\phi(b)$, for $0 \leq s \leq 1$.
- (iii) A triple (e, a, g) is called *elementary* if there are continuous paths (a_t) and (g_t) such that $a_1 = a$, $g_1 = g$, $a_0 = e$, $g_0(s) = \phi(e)$ for $0 \leq s \leq 1$, and (e, a_t, g_t) is in $\Gamma_1(\phi)$ for $0 \leq t \leq 1$.
- (iv) Say that two triples σ and σ' in $\Gamma_1(\phi)$ are *equivalent*, written $\sigma \sim \sigma'$, if there exist elementary triples τ and τ' such that $\sigma \oplus \tau \cong \sigma' \oplus \tau'$.

Denote by $[\sigma]$, or $[e, a, g]$, the equivalence class of the triple $\sigma = (e, a, g)$ via the relation \sim .

Definition 2.3.7. $K_1(\phi)$ is defined to be the set

$$\{[\sigma] \mid \sigma \in \Gamma_1(\phi)\} = \Gamma_1(\phi) / \sim .$$

It is easily checked that, like $\Gamma_0(\phi)$, the direct sum operation of triples in $\Gamma_1(\phi)$ behaves well with respect to the notions of isomorphism and elementary.

Proposition 2.3.8. $K_1(\phi)$ is an abelian group when equipped with the binary operation

$$[\sigma] + [\sigma'] = [\sigma \oplus \sigma']$$

where the identity element is given by $[0, 0, 0]$ and the inverse of $[e, a, g]$ is given by $[e, a^{-1}, g^{-1}]$, for (e, a, g) in $\Gamma_1(\phi)$.

Proof. We verify the last claim. We have

$$[e, a, g] + [e, a^{-1}, g^{-1}] = [e \oplus e, a \oplus a^{-1}, g \oplus g^{-1}].$$

A concrete path a_t of invertibles from $a \oplus a^{-1}$ to $e \oplus e$ is

$$a_t = \begin{bmatrix} e & -ta \\ 0 & e \end{bmatrix} \begin{bmatrix} e & 0 \\ ta^{-1} & e \end{bmatrix} \begin{bmatrix} e & -ta \\ 0 & e \end{bmatrix} \begin{bmatrix} e & -te \\ 0 & e \end{bmatrix} \begin{bmatrix} e & 0 \\ te & e \end{bmatrix} \begin{bmatrix} e & -te \\ 0 & e \end{bmatrix}$$

For $0 \leq s, t \leq 1$, let $g_t(s)$ be the product of matrices above, but with every occurrence of e replaced with $\phi(e)$ and every occurrence of a replaced with $g(s)$. Then (g_t) is a path with $g_t(1) = \phi(a_t)$ for all t , $g_1 = g \oplus g^{-1}$ and $g_0(s) = \phi(e) \oplus \phi(e)$ for all s . Thus $(e \oplus e, a \oplus a^{-1}, g \oplus g^{-1})$ is elementary. \square

The following result is similar to Proposition 2.3.5, so we omit the proof.

Proposition 2.3.9. (i) Suppose we have two triples (e, a, g) and (e', a', g') and that $e = e'$. If e is in $M_n(\tilde{A})$ and (a_t) is a path of invertibles from a to a' in $eM_n(\tilde{A})e$ and (g_t) is a path of invertibles from g to g' in $C([0, 1]) \otimes \phi(e)M_n(\tilde{B})\phi(e)$ such that $g_t(1) = \phi(a_t)$ for all t , then $[e, a, g] = [e, a', g']$.

(ii) If $e = e'$, we have

$$[e, a, g] + [e', a', g'] = [e, aa', gg'] = [e, a'a, g'g].$$

(iii) If (e, a, g) and (e', a', g') are two triples in $\Gamma_1(\phi)$ such that $ee' = e'e = 0$, then $(e, a, g) \oplus (e', a', g') \cong (e + e', a + a', g + g')$.

(iv) If e is in $M_n(\tilde{A})$, $[e, a, g] = 0$ if and only if there is an integer $k \geq 1$, an idempotent e' in $M_k(\tilde{A})$ and paths of invertibles a_t in $(e \oplus e')M_{n+k}(\tilde{A})(e \oplus e')$ and g_t in $C([0, 1]) \otimes (\phi(e) \oplus \phi(e'))M_{n+k}(\tilde{B})(\phi(e) \oplus \phi(e'))$ such that $a_0 = e \oplus e'$, $a_1 = a \oplus e'$, $g_0 = e \oplus \phi(e')$, $g_1 = g \oplus \phi(e')$, and $g_t(1) = \phi(a_t)$ for all t .

We now collect some properties that hold for both relative groups. The first is a universal property which is useful for defining maps.

Proposition 2.3.10. *Suppose that G is an abelian group and $\nu : \Gamma_j(\phi) \rightarrow G$ is a map that satisfies*

$$(i) \quad \nu(\sigma \oplus \tau) = \nu(\sigma) + \nu(\tau),$$

$$(ii) \quad \nu(\sigma) = 0 \text{ if } \sigma \text{ is elementary, and}$$

$$(iii) \quad \text{if } \sigma \cong \tau, \text{ then } \nu(\sigma) = \nu(\tau).$$

Then ν induces a unique group homomorphism $\mu : K_j(\phi) \rightarrow G$ such that $\mu([\sigma]) = \nu(\sigma)$ for all σ in $\Gamma_j(\phi)$.

Proof. If $\sigma \sim \sigma'$, find elementary triples τ and τ' such that $\sigma \oplus \tau \cong \sigma' \oplus \tau'$. Then

$$\nu(\sigma) = \nu(\sigma) + \nu(\tau) = \nu(\sigma \oplus \tau) = \nu(\sigma' \oplus \tau') = \nu(\sigma') + \nu(\tau') = \nu(\sigma')$$

So the map $\mu([\sigma]) := \nu(\sigma)$ is well-defined. It is a group homomorphism because

$$\mu([\sigma] + [\sigma']) = \mu([\sigma \oplus \sigma']) = \nu(\sigma \oplus \sigma') = \nu(\sigma) + \nu(\sigma') = \mu([\sigma]) + \mu([\sigma']). \quad \square$$

If $\phi : A \rightarrow B$ and $\psi : C \rightarrow D$ are $*$ -homomorphisms, we denote by $\phi \oplus \psi$ the component-wise $*$ -homomorphism $A \oplus C \rightarrow B \oplus D$.

Proposition 2.3.11. *Suppose $\phi : A \rightarrow B$ and $\psi : C \rightarrow D$ are $*$ -homomorphisms. Then there are natural isomorphisms $K_*(\phi \oplus \psi) \rightarrow K_*(\phi) \oplus K_*(\psi)$ that satisfy*

$$[(e, e'), (f, f'), (b, b')] \mapsto ([e, f, b], [e', f', b'])$$

in the case of K_0 , and

$$[(e, e'), (a, a'), (g, g')] \mapsto ([e, a, g], [e', a', g'])$$

in the case of K_1 .

Proof. For a triple $((e, e'), (f, f'), (b, b'))$ in $\Gamma_0(\phi \oplus \psi)$, define the map $\nu : \Gamma_0(\phi \oplus \psi) \rightarrow K_0(\phi) \oplus K_0(\psi)$ by

$$\nu((e, e'), (f, f'), (b, b')) = ([e, f, b], [e', f', b']).$$

It is straightforward to check that ν satisfies the hypotheses of Proposition 2.3.10, so we get a well-defined group homomorphism that factors ν . The fact that the group homomorphism is surjective is clear, and injectivity follows from a simple application of part (iv) of Proposition 2.3.5. The proof is similar for K_1 . \square

The next result shows that the relative K groups satisfy a kind of functoriality. This hints at the notion of a “generalized relative homology theory” for $*$ -homomorphisms. We will discuss this idea in more detail in Chapter 5.

Proposition 2.3.12. *Suppose that*

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \alpha \downarrow & & \downarrow \beta \\ C & \xrightarrow{\psi} & D \end{array}$$

is a commutative diagram of C^ -algebras and $*$ -homomorphisms. Then there are well-defined group homomorphisms $(\alpha, \beta)_* : K_j(\phi) \rightarrow K_j(\psi)$ that satisfy*

$$(\alpha, \beta)_*([e, f, b]) = [\alpha(e), \alpha(f), \beta(b)]$$

for a triple (e, f, b) in $\Gamma_0(\phi)$ and

$$(\alpha, \beta)_*([e, a, g]) = [\alpha(e), \alpha(a), \beta(g)]$$

for a triple (e, a, g) in $\Gamma_1(\phi)$. If α and β are $$ -isomorphisms, then $(\alpha, \beta)_*$ is a group isomorphism.*

Proof. For a triple (e, f, b) in $\Gamma_0(\phi)$, set $\nu(e, f, b) = [\alpha(e), \alpha(f), \beta(b)]$. Again, the hypotheses of Proposition 2.3.10 are easy to check, so ν factors to a group homomorphism that we denote by $(\alpha, \beta)_*$. If α and β are $*$ -isomorphisms, then the diagram

$$\begin{array}{ccc} C & \xrightarrow{\psi} & D \\ \alpha^{-1} \downarrow & & \downarrow \beta^{-1} \\ A & \xrightarrow{\phi} & B \end{array}$$

is commutative and the same argument works to obtain the group homomorphism $(\alpha^{-1}, \beta^{-1})_*$, which is easily seen to be the inverse of $(\alpha, \beta)_*$. Again, the proof is similar for K_1 . \square

At this point it is worth making a brief digression regarding units. The reader may notice that the only place we have used the units in \tilde{A} and \tilde{B} in this chapter so far is in the proofs of Lemma 2.2.5, Proposition 2.2.6, and Proposition 2.2.8, and these results have not been used yet. They will be used when defining certain connecting maps (specifically μ_0 and ∂_1) and the units will play a useful role when refining the

portrait of $K_*(\phi)$ in the coming subsection. However, the results we have up until now enable us to show that if A and B are already unital and $\phi(1_A) = 1_B$, one may define $K_*(\phi)$ without unitizations while remaining consistent with the results above.

Definition 2.3.13. Suppose that A and B are unital and $\phi(1_A) = 1_B$. Let $K_j^u(\phi)$ be the group defined in the same way as $K_j(\phi)$, but with \tilde{A} replaced with A and \tilde{B} replaced with B .

Notice $K_j(\phi)$ and $K_j^u(\tilde{\phi})$ are precisely the same objects, and all preceding results about $K_j(\phi)$ have analogues for $K_j^u(\phi)$. The only modifications needed are in the proofs of Lemma 2.2.5, Proposition 2.2.6, and Proposition 2.2.8, and the same arguments work by using the units already present.

Proposition 2.3.14. *If A and B are unital and $\phi(1_A) = 1_B$, then $K_j(\phi)$ and $K_j^u(\phi)$ are naturally isomorphic as groups.*

Proof. The map $\nu_A : A \oplus \mathbb{C} \rightarrow \tilde{A}$ defined by $\nu_A(a, \lambda) = a + \lambda(1_{\tilde{A}} - 1_A)$ is a *-isomorphism and the diagram

$$\begin{array}{ccc} A \oplus \mathbb{C} & \xrightarrow{\phi \oplus \text{id}_{\mathbb{C}}} & B \oplus \mathbb{C} \\ \downarrow \nu_A & & \downarrow \nu_B \\ \tilde{A} & \xrightarrow{\tilde{\phi}} & \tilde{B} \end{array}$$

is commutative. Therefore $(\nu_A, \nu_B)_* : K_j^u(\phi \oplus \text{id}_{\mathbb{C}}) \rightarrow K_j^u(\tilde{\phi})$ is an isomorphism by Proposition 2.3.12. Then

$$K_j(\phi) = K_j^u(\tilde{\phi}) \cong K_j^u(\phi \oplus \text{id}_{\mathbb{C}}) \cong K_j^u(\phi) \oplus K_j^u(\text{id}_{\mathbb{C}}) \cong K_j^u(\phi)$$

where the third isomorphism is due to Proposition 2.3.11. The fact that $K_j^u(\text{id}_{\mathbb{C}}) = 0$ is rather clear, but the skeptical reader is referred to part (ii) of Corollary 2.4.4. \square

2.3.1 Projections, unitaries, and partial isometries

We now focus on simplifying the portrait of $K_*(\phi)$ using the underlying C*-algebra structure. As discussed in Chapter 1, when one is working with C*-algebra K -theory, it is convenient to work with projections, unitaries, and partial isometries instead of idempotents and invertible morphisms, chiefly because the former types of elements behave better with respect to the adjoint operation, have much nicer spectral theory, and continuous functional calculus is available. To this end, we introduce a more specific picture using a refined definition of triples, and show that this new picture gives the same theory.

Define $\Gamma_0^*(\phi)$ to be the subset of $\Gamma_0(\phi)$ consisting of all triples (p, q, v) , where p and q are projections in $M_\infty(\tilde{A})$ and v is a partial isometry in $M_\infty(\tilde{B})$ with $v^*v = \phi(p)$,

$vv^* = \phi(q)$. We define a refined equivalence relation \sim_* on $\Gamma_0^*(\phi)$ as follows. Say that (p, q, v) and (p', q', v') are $*$ -isomorphic, written $(p, q, v) \cong_* (p', q', v')$, if there are partial isometries c and d in $M_\infty(\tilde{A})$ with $c^*c = p$, $cc^* = p'$, $d^*d = q$, $dd^* = q'$, and $\phi(d)v = v'\phi(c)$. A triple (p, q, v) is called $*$ -elementary if $p = q$ and there is a continuous path (v_t) of partial isometries such that $v_0 = \phi(p) = \phi(q)$ and $v_1 = v$, and $v_t^*v_t = v_tv_t^* = \phi(p) = \phi(q)$ for all t . Say that two triples σ and σ' in $\Gamma_0^*(\phi)$ are $*$ -equivalent, written $\sigma \sim_* \sigma'$, if there exist $*$ -elementary triples τ and τ' such that $\sigma \oplus \tau \cong_* \sigma' \oplus \tau'$. Denote by $[\sigma]_*$, or $[p, q, v]_*$, the equivalence class of the triple $\sigma = (p, q, v)$ via the relation \sim_* .

It is straightforward to verify that the set of equivalence classes forms an abelian group in the same way as for $K_0(\phi)$. To see that $-[p, q, v]_* = [q, p, v^*]_*$, notice that if two unitaries are homotopic as invertibles, they are homotopic as unitaries. The analogue of Proposition 2.3.10 also holds. We denote this new group by $K_0^*(\phi)$. The notation is not desirable, but it is temporary, as we now aim to show that the inclusion $\Gamma_0^*(\phi) \subseteq \Gamma_0(\phi)$ induces a natural isomorphism from $K_0^*(\phi)$ to $K_0(\phi)$.

Lemma 2.3.15. (i) *For every triple σ in $\Gamma_0(\phi)$, there is a triple τ in $\Gamma_0^*(\phi)$ with $\sigma \sim \tau$. Moreover, such a triple $\tau = (p, q, v)$ in $\Gamma_0^*(\phi)$ may be chosen so that one of p or q is equal to 1_n for some $n \geq 1$, and $\dot{p} = \dot{q} = \dot{v} = 1_n$.*

(ii) *Suppose $m \geq n$, p is in $M_m(\tilde{A})$, $(p, 1_n, v)$ is in $\Gamma_0^*(\phi)$, and $\dot{p} = \dot{v} = 1_n$. Then $[p, 1_n, v] = 0$ in $K_0(\phi)$ if and only if there exist $k \geq 0$ and a partial isometry w in $M_{m+k}(\tilde{A})$ with $w^*w = 1_n \oplus 0_{m-n} \oplus 1_k$ and $w w^* = p \oplus 1_k$ such that $(v \oplus 1_k)\phi(w)$ is a unitary in $(1_n \oplus 0_{m-n} \oplus 1_k)M_{m+k}(\tilde{B})(1_n \oplus 0_{m-n} \oplus 1_k)$ homotopic to $1_n \oplus 0_{m-n} \oplus 1_k$. The partial isometry w may be chosen so that $\dot{w} = 1_n \oplus 0_{m-n} \oplus 1_k$.*

Proof. (i) First, if e is an idempotent in \tilde{A} ,

$$p = ee^*(1 + (e - e^*)(e^* - e))^{-1}$$

is a projection, and $ep = p$ and $pe = e$ (see 4.6.2 of [4]). Thus, if (e, f, b) is a triple in $\Gamma_0(\phi)$, we have $(e, f, b) \cong (p, q, \phi(f)b\phi(p))$, where $q = ff^*(1 + (f - f^*)(f^* - f))^{-1}$, thanks to the commutative diagram

$$\begin{array}{ccc} \phi(e) & \xrightarrow{b} & \phi(f) \\ \downarrow \phi(e) & & \downarrow \phi(f) \\ \phi(p) & \xrightarrow{\phi(f)b\phi(p)} & \phi(q) \end{array}$$

We set $b_1 = \phi(f)b\phi(p)$.

Next, notice that $b_1 b_1^*$ is an invertible element of the C^* -algebra $\phi(q)M_\infty(\tilde{B})\phi(q)$ with inverse $(b_1^{-1})^* b_1^{-1}$. Now $(b_1 b_1^*)^{-t/2} b_1$ for $0 \leq t \leq 1$ is a homotopy from b_1 to

$(b_1 b_1^*)^{-1/2} b_1$, and each $(b_1 b_1^*)^{-t/2} b_1$ is an invertible morphism from $\phi(p)$ to $\phi(q)$ with inverse $b_1^{-1} ((b_1^{-1})^* b_1^{-1})^{-t/2}$. Set $v = (b_1 b_1^*)^{-1/2} b_1$, which is a partial isometry with $v^* v = \phi(p)$ and $v v^* = \phi(q)$. By part (i) of Proposition 2.3.5, $(p, q, b_1) \sim (p, q, v)$.

Now choose n such that p and q are in $M_n(\tilde{A})$ and v is in $M_n(\tilde{B})$. By adding the elementary triple $(1_n - p, 1_n - p, 1_n - \phi(p))$ and using part (iii) of Proposition 2.3.5,

$$(p, q, v) \sim \left(\left[\begin{array}{cc} 1_n & 0 \\ 0 & 0 \end{array} \right], \left[\begin{array}{cc} q & 0 \\ 0 & 1_n - p \end{array} \right], \left[\begin{array}{cc} v & 0 \\ 1_n - \phi(p) & 0 \end{array} \right] \right)$$

Set $q_1 = q \oplus (1_n - p)$ and $v_1 = \left[\begin{array}{cc} v & 0 \\ 1_n - \phi(p) & 0 \end{array} \right]$. Adding $1_n - q$ instead of $1_n - p$ would replace q with 1_n .

We have $\dot{v}_1^* \dot{v}_1 = 1_n$ and $\dot{v}_1 \dot{v}_1^* = \dot{q}_1$, so choose $m \geq n$ and a unitary u in $M_m(\mathbb{C})$ such that $u \dot{q}_1 u^* = 1_n$. Then $(1_n, q_1, v_1) \cong (1_n, u q_1 u^*, u v_1)$ since $u v_1 \phi(1_n) = \phi(u q_1) v_1$. The scalar part of $u q_1 u^*$ is 1_n and we now set $q_2 = u q_1 u^*$ and $v_2 = u v_1$. Lastly, \dot{v}_2 may then be regarded as a unitary in $M_n(\mathbb{C})$, so choose a homotopy v_t from \dot{v}_2 to 1_n and observe that each $v_2 v_t^*$ is a partial isometry from 1_n to $\phi(q_2)$. By part (i) of Proposition 2.3.5, $(1_n, q_2, v_2) \sim (1_n, q_2, v_2 \dot{v}_2^*)$, and the third element of the latter triple has scalar part 1_n .

(ii) For appropriate $m \geq n$, obtain elementary triples (r, r, c) and (s, s, d) such that

$$(p, 1_n \oplus 0_{m-n}, v) \oplus (r, r, c) \cong (s, s, d)$$

as in part (iv) of Proposition 2.3.5. As we saw in part (i), an elementary triple in $\Gamma_0(\phi)$ is isomorphic to an elementary triple consisting of projections, not just idempotents, so we may assume that r and s are projections. If $k \geq 0$ is large enough so that r is in $M_k(\tilde{A})$, then

$$(r, r, c) \oplus (1_k - r, 1_k - r, 1_k - \phi(r)) \cong (1_k, 1_k, c + 1_k - \phi(r))$$

by part (iii) of Proposition 2.3.5, so

$$(p, 1_n \oplus 0_{m-n}, v) \oplus (1_k, 1_k, c + 1_k - \phi(r)) \cong (s, s, d) \oplus (1_k - r, 1_k - r, 1_k - \phi(r))$$

and so we may assume that $r = 1_k$ for some $k \geq 0$. Obtain x and y as in part (iv) of Proposition 2.3.5, so that $d\phi(x) = \phi(y)(v \oplus c)$, hence $\phi(y^{-1})d\phi(y) = (v \oplus c)\phi(x^{-1}y)$. Since d is homotopic to $\phi(s)$, $\phi(y^{-1})d\phi(y)$ is homotopic to $1_n \oplus 0_{m-n} \oplus 1_k$ and also c is homotopic to 1_k . Lastly, $\phi(x^{-1}y)$ is homotopic

to $\phi((x^{-1}x^{-1*})^{-1/2}x^{-1}(yy^*)^{-1/2}y)$ as seen in part (i), and

$$(v \oplus 1_k)\phi((x^{-1}x^{-1*})^{-1/2}x^{-1}(yy^*)^{-1/2}y)$$

is then a unitary in $(1_n \oplus 0_{m-n} \oplus 1_k)M_{m+k}(\tilde{B})(1_n \oplus 0_{m-n} \oplus 1_k)$. It is still homotopic to the identity through invertibles, and it is standard that two unitaries homotopic through invertibles are homotopic through unitaries. Upon naming $w_1 = (x^{-1}x^{-1*})^{-1/2}x^{-1}(yy^*)^{-1/2}y$ and $w = w_1\dot{w}_1^*$, we have the conclusion. \square

The following analogous result holds in the K_1 case. Let $\Gamma_1^*(\phi)$ denote the subset of $\Gamma_1(\phi)$ consisting of triples $(1_n, u, g)$, where $n \geq 1$, u is a unitary in $M_n(\tilde{A})$, and g is a unitary in $C([0, 1]) \otimes M_n(\tilde{B})$ with $g(0) = 1_n$ and $g(1) = \phi(u)$.

Lemma 2.3.16. (i) For every triple σ in $\Gamma_1(\phi)$, there is a triple τ in $\Gamma_1^*(\phi)$ with $\sigma \sim \tau$. Moreover, such a triple $\tau = (1_n, u, g)$ in $\Gamma_1^*(\phi)$ may be chosen so that $\dot{u} = \dot{g}(s) = 1_n$ for all $0 \leq s \leq 1$.

(ii) $[1_n, u, g] = 0$ in $K_1(\phi)$ if and only if there exist $k \geq 0$, a path u_t of unitaries in $M_{n+k}(\tilde{A})$ and a path g_t of unitaries in $C([0, 1]) \otimes M_{n+k}(\tilde{B})$ such that $u_0 = 1_{n+k}$, $u_1 = u \oplus 1_k$, $g_0 = 1_{n+k}$, $g_1 = g \oplus 1_k$, and $g_t(1) = \phi(u_t)$ for all $0 \leq t \leq 1$.

Proof. The techniques in the proof are similar to those seen before, so we merely sketch the proof.

(i) Take a triple (e, a, g) , add the elementary triple $(1_n - e, 1_n - e, 1_n - \phi(e))$, and use part (iii) of Proposition 2.3.9 to obtain 1_n in place of e . Use the paths $(aa^*)^{-t/2}a$ and $(gg^*)^{-t/2}g$ and part (i) of Proposition 2.3.9 to obtain unitaries in place of invertibles. Replace u with \dot{u}^*u and g with \dot{g}^*g and use the fact that \dot{u} and \dot{g} are homotopic to the identity to obtain the last claim.

(ii) If $[1_n, u, g] = 0$, use part (iv) of Proposition 2.3.9 to find an integer $k \geq 0$, an idempotent e' in $M_k(\tilde{A})$ and paths of invertibles a_t in $(1_n \oplus e')M_{n+k}(\tilde{A})(1_n \oplus e')$ and g_t in $C([0, 1]) \otimes (1_n \oplus \phi(e'))M_{n+k}(\tilde{B})(1_n \oplus \phi(e'))$ such that $a_0 = 1_n \oplus e'$, $a_1 = u \oplus e'$, $g_0 = 1_n \oplus \phi(e')$, $g_1 = g \oplus \phi(e')$, and $g_t(1) = \phi(a_t)$ for all t . This path takes place in $(1_n \oplus \phi(e'))M_{n+k}(\tilde{B})(1_n \oplus \phi(e'))$, so by adding $0_n \oplus (1_k - \phi(e'))$ to everything, we obtain paths of invertibles in M_{n+k} , and using the usual polar decomposition trick, we obtain paths of unitaries. \square

Proposition 2.3.17. The inclusion $\Gamma_j^*(\phi) \subseteq \Gamma_j(\phi)$ induces a group isomorphism $\Omega_\phi : K_j^*(\phi) \rightarrow K_j(\phi)$. Moreover, given the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \alpha \downarrow & & \downarrow \beta \\ C & \xrightarrow{\psi} & D \end{array}$$

we have the commutative diagram

$$\begin{array}{ccc}
K_j^*(\phi) & \xrightarrow{(\alpha, \beta)_*} & K_j^*(\psi) \\
\downarrow \Omega_\phi & & \downarrow \Omega_\psi \\
K_j(\phi) & \xrightarrow{(\alpha, \beta)_*} & K_j(\psi)
\end{array}$$

Proof. It is straightforward to check for both $j = 0, 1$ that the map $\Gamma_j^*(\phi) \rightarrow K_j(\phi)$ satisfies the hypotheses of Proposition 2.3.10 (with $\Gamma_j(\phi)$ replaced with $\Gamma_j^*(\phi)$), so we get a group homomorphism $\Omega_\phi : K_j^*(\phi) \rightarrow K_j(\phi)$. In the case $j = 0$, surjectivity of Ω_ϕ follows from part (i) of Lemma 2.3.15, and injectivity follows from part (ii) of the same lemma. The $j = 1$ case is analogous, using Lemma 2.3.16. \square

Thus the groups $K_j^*(\phi)$ and $K_j(\phi)$ are, for all intents and purposes, identical when A and B are C^* -algebras. From this point on we will favour the portrait using projections, unitaries, and partial isometries, and we will denote both portraits using the notation $K_*(\phi)$ with little to no risk of confusion.

2.4 The proof of Theorem 2.1.1

The exact sequence of Proposition 2.4.1 below is essentially that of Theorem 3.22 in [18]. We provide a proof using our refined portrait.

Define the map $\mu_0 : K_1(B) \rightarrow K_0(\phi)$ by $\mu_0([u]) = [1_n, 1_n, u]$, where u is a unitary in $M_n(\tilde{B})$. By part (i) of Proposition 2.3.5, μ_0 is well-defined, and clearly it is a group homomorphism.

Define a map $\nu : \Gamma_0(\phi) \rightarrow K_0(A)$ by $\nu(p, q, v) = [p] - [q]$. Observe that the image of ν is indeed in $K_0(A)$ (not just $K_0(\tilde{A})$) since $\dot{v}^* \dot{v} = \dot{p}$ and $\dot{v} \dot{v}^* = \dot{q}$, hence $[p] = [q]$. It is easy to check that ν satisfies the hypotheses of Proposition 2.3.10, hence factors to a well-defined group homomorphism $\nu_0 : K_0(\phi) \rightarrow K_0(A)$ that satisfies $\nu_0([p, q, v]) = [p] - [q]$.

Proposition 2.4.1. *The sequence*

$$K_1(A) \xrightarrow{\phi_*} K_1(B) \xrightarrow{\mu_0} K_0(\phi) \xrightarrow{\nu_0} K_0(A) \xrightarrow{\phi_*} K_0(B)$$

is exact.

Proof. It is quite clear that all compositions are zero. Let us check exactness at $K_0(A)$. If $\phi_*([p] - [q]) = [\phi(p)] - [\phi(q)] = 0$, choose a large enough $n \geq 1$ and w in $M_n(\tilde{B})$ such that $w^*w = \phi(p)$ and $ww^* = \phi(q)$. Then we have

$$[p] - [q] = \nu_0([p, q, w]).$$

Next, we check exactness at $K_0(\phi)$. If $(p, 1_n, v)$ is such that p is in $M_m(\tilde{A})$ and $\nu_0([p, 1_n, v]) = [p] - [1_n] = 0$, choose $k \geq m$ and w in $M_k(\tilde{A})$ such that $w^*w = p$ and $ww^* = 1_n \oplus 0_{m-n} \oplus 1_k$. Then

$$(p \oplus 1_k, 1_n \oplus 0_{m-n} \oplus 1_k, v \oplus 1_k) \cong (1_n \oplus 0_{m-n} \oplus 1_k, 1_n \oplus 0_{m-n} \oplus 1_k, (v \oplus 1_k)\phi(w^*))$$

hence $(v \oplus 1_k)\phi(w^*) + 0_n \oplus 1_{m-n} \oplus 0_k$ is a unitary in $M_{m+k}(\tilde{B})$ and

$$[p, 1_n, v] = \mu_0([(v \oplus 1_k)\phi(w^*) + 0_n \oplus 1_{m-n} \oplus 0_k]).$$

Lastly, we check exactness at $K_1(B)$. If $\mu_0([u]) = [1_n, 1_n, u] = 0$, use part (ii) of Lemma 2.3.15 to find $k \geq 0$ and a partial isometry w such that $\phi(w)(u \oplus 1_k)$ is a unitary and homotopic to 1_{n+k} in $M_{n+k}(\tilde{B})$. Since $u \oplus 1_k$ is a unitary, so is w and $u \oplus 1_k$ is homotopic to $\phi(w^*)$. Thus

$$[u] = [u \oplus 1_k] = [\phi(w^*)] = \phi_*([w^*])$$

which completes the proof. \square

We now turn to the analogue for $K_1(\phi)$. Consider a unitary g in $C([0, 1]) \otimes M_n(\tilde{B})$ with $g(0) = g(1) = \dot{g} = 1_n$, so that g is in $M_n(\widetilde{SB})$. Define $\mu_1([g]) = [1_n, 1_n, g]$. By part (i) of Proposition 2.3.9, this is a well-defined group homomorphism $\mu_1 : K_1(SB) \rightarrow K_1(\phi)$.

For a triple (e, a, g) in $\Gamma_1(\phi)$, define the map $\nu : \Gamma_1(\phi) \rightarrow K^{-1}(\mathcal{C}_A)$ by $\nu(e, a, g) = [e, a]$ (recall the definition of $K^{-1}(\mathcal{C}_A)$ given before Lemma 2.2.4). The hypotheses of Proposition 2.3.10 are satisfied, so we get a group homomorphism $\nu_1 : K_1(\phi) \rightarrow K^{-1}(\mathcal{C}_A)$ such that $\nu_1([e, a, g]) = [e, a]$. With the unitary picture of $K_1(A)$, the formula is more simply $\nu_1([1_n, u, g]) = [u]$.

Proposition 2.4.2. *The sequence*

$$K_1(SA) \xrightarrow{(S\phi)_*} K_1(SB) \xrightarrow{\mu_1} K_1(\phi) \xrightarrow{\nu_1} K_1(A) \xrightarrow{\phi_*} K_1(B)$$

is exact.

Proof. Again, all compositions are clearly zero. Let us show exactness at $K_1(A)$. If $\phi_*([u]) = [\phi(u)] = 0$ for a unitary u in $M_n(\tilde{A})$, we may find a unitary g in $C([0, 1]) \otimes M_n(\tilde{B})$ such that $g(1) = \phi(u)$ and $g(0) = 1_n$ (after possibly increasing n). Then we have

$$[u] = \nu_1([1_n, u, g]).$$

Next, we show exactness at $K_1(\phi)$. If $(1_n, u, g)$ is a triple in $\Gamma_1(\phi)$ with $\dot{u} = \dot{g} = 1_n$ and $\nu_1([1_n, u, g]) = [u] = 0$, after increasing n if necessary we may find a unitary f in

$C([0, 1]) \otimes M_n(\tilde{A})$ such that $f(0) = \dot{f} = 1_n$ and $f(1) = u$. Set

$$\tilde{g}(s) = \begin{cases} g(2s) & 0 \leq s \leq 1/2 \\ \phi(f(2-2s)) & 1/2 \leq s \leq 1 \end{cases}.$$

Then \tilde{g} is a unitary in $C([0, 1]) \otimes M_n(\tilde{B})$ and $\tilde{g}(0) = \tilde{g}(1) = \dot{g} = 1_n$. Now for a fixed t in $[0, 1]$, the function g_t defined by

$$g_t(s) = \begin{cases} g(s(1 - \frac{1}{2}t)^{-1}) & 0 \leq s \leq 1 - \frac{1}{2}t \\ \phi(f(3 - 2s - t)) & 1 - \frac{1}{2}t \leq s \leq 1 \end{cases}$$

satisfies $g_0 = g$, $g_1 = \tilde{g}$, and $g_t(1) = \phi(f(1-t))$, and so

$$[1_n, u, g] = [1_n, 1_n, \tilde{g}] = \mu_1([\tilde{g}]).$$

Finally, we show exactness at $K_1(SB)$. If $\mu_1([g]) = [1_n, 1_n, g] = 0$ for a unitary g in $M_n(\widetilde{SB})$, after increasing n if necessary, use part (ii) of Lemma 2.3.16 to find paths (u_t) and (g_t) such that $u_0 = u_1 = 1_n$, $g_1 = g$, $g_0 = 1_n$, and $g_t(1) = \phi(u_t)$ and $\dot{u}_t = \dot{g}_t = 1_n$ for all t . Write $f(t) = u_t$ and set

$$\tilde{g}_t(s) = \begin{cases} g_t(2s) & 0 \leq s \leq 1/2 \\ \phi(f((2-2t)s + 2t - 1)) & 1/2 \leq s \leq 1 \end{cases}$$

Then $\tilde{g}_t(0) = \tilde{g}_t(1) = 1_n$ for all t and

$$\tilde{g}_1(s) = \begin{cases} g(2s) & 0 \leq s \leq 1/2 \\ 1_n & 1/2 \leq s \leq 1 \end{cases}$$

and

$$\tilde{g}_0(s) = \begin{cases} 1_n & 0 \leq s \leq 1/2 \\ \phi(f(2s-1)) & 1/2 \leq s \leq 1 \end{cases}$$

Which are homotopic to g and $S\phi(f)$, respectively. Thus

$$[g] = [\tilde{g}_1] = [\tilde{g}_0] = [S\phi(f)] = (S\phi)_*([f]). \quad \square$$

Proposition 2.4.3. *If a $*$ -homomorphism $\phi : A \rightarrow B$ satisfies $\phi(a) = 0$ for all a in A , then the sequence in Theorem 2.1.1 splits at $K_0(A)$ and $K_1(A)$. In other words, for each $j = 0, 1$ there is a group homomorphism $\lambda_j : K_j(A) \rightarrow K_j(\phi)$ such that $\nu_j \circ \lambda_j$ is the identity map on $K_j(A)$.*

Proof. If p and q are two projections in $M_\infty(\tilde{A})$ with $[p] = [q]$, let v be a partial isometry in $M_\infty(\mathbb{C})$ such that $v^*v = \dot{p}$ and $vv^* = \dot{q}$. If u is a unitary in $M_n(\tilde{A})$, let g be any

unitary in $C([0, 1]) \otimes M_n(\mathbb{C})$ such that $g(0) = 1_n$ and $g(1) = \dot{u}$. Define

$$\lambda_0([p] - [q]) = [p, q, v], \quad \lambda_1([u]) = [1_n, u, g].$$

For both $j = 0, 1$, it is straightforward to check that λ_j is well-defined, additive, independent of the choices of v and g , and that $\nu_j \circ \lambda_j$ is the identity. \square

By combining all results in this section, we obtain Theorem 2.1.1. The map μ_1 in Theorem 2.1.1 is (by abuse of notation) the composition of the Bott map β_B and μ_1 from Proposition 2.4.2. It may therefore be written as $\mu_1([p] - [q]) = [1_n, 1_n, f_p f_q^*]$, where f_p is the projection loop $f_p(s) = e^{2\pi i s p}$ in $C([0, 1]) \otimes M_n(\tilde{B})$. Since the Bott map is natural, the diagram

$$\begin{array}{ccccccc} K_0(A) & \xrightarrow{\phi_*} & K_0(B) & \overset{\mu_1}{\dashrightarrow} & K_1(\phi) & \xrightarrow{\nu_1} & K_1(A) \\ \downarrow \beta_A & & \downarrow \beta_B & & \parallel & & \parallel \\ K_1(SA) & \xrightarrow{(S\phi)_*} & K_1(SB) & \xrightarrow{\mu_1} & K_1(\phi) & \xrightarrow{\nu_1} & K_1(A) \end{array}$$

is commutative, and all vertical maps are isomorphisms. It follows that the top row is exact.

We record the following immediate and useful consequence of Theorem 2.1.1.

Corollary 2.4.4. *If $\phi_* : K_*(A) \rightarrow K_*(B)$ is an isomorphism, then $K_*(\phi) = 0$.*

The following result will also be useful.

Proposition 2.4.5. *Suppose that the diagram*

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \alpha \downarrow & & \downarrow \beta \\ C & \xrightarrow{\psi} & D \end{array}$$

is commutative. Then the diagram

$$\begin{array}{ccccc} K_{1-j}(B) & \xrightarrow{\mu_j} & K_j(\phi) & \xrightarrow{\nu_j} & K_j(A) \\ \downarrow \beta_* & & \downarrow (\alpha, \beta)_* & & \downarrow \alpha_* \\ K_{1-j}(D) & \xrightarrow{\mu_j} & K_j(\psi) & \xrightarrow{\nu_j} & K_j(B) \end{array}$$

is commutative.

Proof. For the left square, $(\alpha, \beta)_*(\mu_j([u])) = (\alpha, \beta)_*([1_n, 1_n, u]) = [1_n, 1_n, \beta(u)] = \mu_j([\beta(u)]) = \mu_j(\beta_*([u]))$. The right square is just as easy. \square

2.5 The proof of Theorem 2.1.2

Throughout this section, we will assume that

$$\begin{array}{ccccccc}
 0 & \longrightarrow & I & \xrightarrow{\iota_A} & A & \xrightarrow{\pi_A} & A/I \longrightarrow 0 \\
 & & \downarrow \psi & & \downarrow \phi & & \downarrow \gamma \\
 0 & \longrightarrow & J & \xrightarrow{\iota_B} & B & \xrightarrow{\pi_B} & B/J \longrightarrow 0
 \end{array} \tag{2.3}$$

is a commutative diagram with exact rows. We will denote the group homomorphisms $(\iota_A, \iota_B)_*$ and $(\pi_A, \pi_B)_*$ by ι_* and π_* , respectively.

2.5.1 Half-exactness and split exactness

Proposition 2.5.1. *The sequence*

$$K_0(\psi) \xrightarrow{\iota_*} K_0(\phi) \xrightarrow{\pi_*} K_0(\gamma)$$

is exact. Moreover, if $\lambda_A : A/I \rightarrow A$ and $\lambda_B : B/J \rightarrow B$ are splittings of the rows in (2.3) that keep the diagram commutative, then the sequence

$$0 \longrightarrow K_0(\psi) \xrightarrow{\iota_*} K_0(\phi) \xrightleftharpoons[\lambda_*]{\pi_*} K_0(\gamma) \longrightarrow 0$$

is split exact, where $\lambda_* = (\lambda_A, \lambda_B)_*$.

Proof. It is clear that $\pi_* \circ \iota_* = 0$. Conversely, suppose that $[1_n, q, v]$ is in the kernel of π_* , so $[1_n, \pi_A(q), \pi_B(v)] = 0$. Find (in order):

- (i) an integer $m \geq n$ so that q is in $M_m(\tilde{A})$,
- (ii) an integer $k \geq 0$ and a partial isometry w in $M_{m+k}(\widetilde{A/I})$ such that $w^*w = \pi_A(q) \oplus 1_k$ and $ww^* = 1_n \oplus 0_{m-n} \oplus 1_k$ and $\gamma(w)(\pi_B(v) \oplus 1_k)$ is homotopic to $1_n \oplus 0_{m-n} \oplus 1_k$ as a unitary in $(1_n \oplus 0_{m-n} \oplus 1_k)M_{m+k}(\widetilde{A/I})(1_n \oplus 0_{m-n} \oplus 1_k)$ (use Lemma 2.3.15),
- (iii) an integer $l \geq 0$ and a unitary z homotopic to 1_{m+k+l} in $M_{m+k+l}(\widetilde{A/I})$ such that $z(\pi_A(q) \oplus 1_k \oplus 0_l)z^* = 1_n \oplus 0_{m-n} \oplus 1_k \oplus 0_l$ and $\gamma(z)(\pi_B(v) \oplus 1_k \oplus 0_l) = (\gamma(w)(\pi_B(v) \oplus 1_k)) \oplus 0_l$. For example, one may take $l = m + k$ and

$$z = \begin{bmatrix} w & 1_{m+k} - ww^* \\ 1_{m+k} - w^*w & w^* \end{bmatrix}$$

and use part (iv) of Proposition 1.1.12.

(iv) a unitary U in $M_{m+k+l}(\tilde{A})$ such that $\pi_A(U) = z$ (use part (ii) of Proposition 1.1.13),

(v) a unitary V in $(1_n \oplus 0_{m-n} \oplus 1_k)M_{m+k}(\tilde{B})(1_n \oplus 0_{m-n} \oplus 1_k)$ homotopic to $1_n \oplus 0_{m-n} \oplus 1_k$ such that $\pi_B(V) = \gamma(w)(\pi_B(v) \oplus 1_k)$ (use part (ii) of Proposition 1.1.13 on the unitary $\gamma(w)(\pi_B(v) \oplus 1_k)$, which is homotopic to $1_n \oplus 0_{m-n} \oplus 1_k$).

Then

$$\begin{aligned}
[1_n, q, v] &= \left[\begin{bmatrix} 1_n & 0 & 0 & 0 \\ 0 & 0_{m-n} & 0 & 0 \\ 0 & 0 & 1_k & 0 \\ 0 & 0 & 0 & 0_l \end{bmatrix}, \begin{bmatrix} q & 0 & 0 \\ 0 & 1_k & 0 \\ 0 & 0 & 0_l \end{bmatrix}, \begin{bmatrix} v & 0 & 0 \\ 0 & 1_k & 0 \\ 0 & 0 & 0_l \end{bmatrix} \right] \\
&= \left[\begin{bmatrix} 1_n & 0 & 0 & 0 \\ 0 & 0_{m-n} & 0 & 0 \\ 0 & 0 & 1_k & 0 \\ 0 & 0 & 0 & 0_l \end{bmatrix}, U \begin{bmatrix} q & 0 & 0 \\ 0 & 1_k & 0 \\ 0 & 0 & 0_l \end{bmatrix} U^*, \phi(U) \begin{bmatrix} v & 0 & 0 \\ 0 & 1_k & 0 \\ 0 & 0 & 0_l \end{bmatrix} \right] \\
&= \left[\begin{bmatrix} 1_n & 0 & 0 & 0 \\ 0 & 0_{m-n} & 0 & 0 \\ 0 & 0 & 1_k & 0 \\ 0 & 0 & 0 & 0_l \end{bmatrix}, U \begin{bmatrix} q & 0 & 0 \\ 0 & 1_k & 0 \\ 0 & 0 & 0_l \end{bmatrix} U^*, \phi(U) \begin{bmatrix} v & 0 & 0 \\ 0 & 1_k & 0 \\ 0 & 0 & 0_l \end{bmatrix} \right] \\
&\quad + \left[\begin{bmatrix} 1_n & 0 & 0 & 0 \\ 0 & 0_{m-n} & 0 & 0 \\ 0 & 0 & 1_k & 0 \\ 0 & 0 & 0 & 0_l \end{bmatrix}, \begin{bmatrix} 1_n & 0 & 0 & 0 \\ 0 & 0_{m-n} & 0 & 0 \\ 0 & 0 & 1_k & 0 \\ 0 & 0 & 0 & 0_l \end{bmatrix}, \begin{bmatrix} V^* & 0 \\ 0 & 0_l \end{bmatrix} \right] \\
&= \left[\begin{bmatrix} 1_n & 0 & 0 & 0 \\ 0 & 0_{m-n} & 0 & 0 \\ 0 & 0 & 1_k & 0 \\ 0 & 0 & 0 & 0_l \end{bmatrix}, U \begin{bmatrix} q & 0 & 0 \\ 0 & 1_k & 0 \\ 0 & 0 & 0_l \end{bmatrix} U^*, \phi(U) \begin{bmatrix} v & 0 & 0 \\ 0 & 1_k & 0 \\ 0 & 0 & 0_l \end{bmatrix}, \begin{bmatrix} V^* & 0 \\ 0 & 0_l \end{bmatrix} \right].
\end{aligned}$$

To get the first equality above, we added an elementary scalar triple. To get the second, notice that the two triples are isomorphic via the unitary U . In the third equality, the new triple being added is elementary because V is homotopic to the identity. The fourth equality follows from part (ii) of Proposition 2.3.5. Regarding the elements of the latter triple, we have

$$\pi_A \left(U \begin{bmatrix} q & 0 & 0 \\ 0 & 1_k & 0 \\ 0 & 0 & 0_l \end{bmatrix} U^* \right) = \pi_B \left(\phi(U) \begin{bmatrix} v & 0 & 0 \\ 0 & 1_k & 0 \\ 0 & 0 & 0_l \end{bmatrix} \begin{bmatrix} V^* & 0 \\ 0 & 0_l \end{bmatrix} \right) = \begin{bmatrix} 1_n & 0 & 0 & 0 \\ 0 & 0_{m-n} & 0 & 0 \\ 0 & 0 & 1_k & 0 \\ 0 & 0 & 0 & 0_l \end{bmatrix}$$

from which it follows that $[1_n, q, v]$ is in the image of ι_* .

For the split exact sequence, it is clear that λ_* is a right inverse for π_* , so we need only show that ι_* is injective. Suppose that $(1_n, q, v)$ is a triple in $\Gamma_0(\psi)$ with $\dot{q} = \dot{v} = 1_n$ and $[1_n, q, v] = 0$ in $K_0(\phi)$. Choose $m \geq n$ so that $1_n \oplus 0_{m-n}$ and q are in $M_m(\tilde{I})$ and v is in $M_m(\tilde{J})$. Use Lemma 2.3.15 to find an integer $k \geq 0$ and a partial isometry w in $M_{m+k}(\tilde{A})$ with $w^*w = q \oplus 1_k$ and $ww^* = 1_n \oplus 0_{m-n} \oplus 1_k$ and $\phi(w)(v \oplus 1_k)$ is homotopic to $1_n \oplus 0_{m-n} \oplus 1_k$. Let y_t be such a homotopy, that is, $y_0 = 1_n \oplus 0_{m-n} \oplus 1_k$ and $y_1 = \phi(w)(v \oplus 1_k)$. Set $x = \lambda_A(\pi_A(w^*))w$. Then $\pi_A(x) = 1_n \oplus 0_{m-n} \oplus 1_k$ so that x is in $M_{m+k}(\tilde{I})$. We have $x^*x = q \oplus 1_k$ and $xx^* = 1_n \oplus 0_{m-n} \oplus 1_k$ and, since $\pi_B(v \oplus 1_k) = 1_n \oplus 0_{m-n} \oplus 1_k$,

$$\begin{aligned} \psi(x) \begin{bmatrix} v & 0 \\ 0 & 1_k \end{bmatrix} &= \psi(\lambda_A(\pi_A(w^*))w) \begin{bmatrix} v & 0 \\ 0 & 1_k \end{bmatrix} \\ &= \lambda_B(\pi_B(\phi(w^*)))\phi(w) \begin{bmatrix} v & 0 \\ 0 & 1_k \end{bmatrix} \\ &= \lambda_B \left(\pi_B \left(\begin{bmatrix} v & 0 \\ 0 & 1_k \end{bmatrix} \phi(w^*) \right) \right) \phi(w) \begin{bmatrix} v & 0 \\ 0 & 1_k \end{bmatrix} \\ &= \lambda_B \left(\pi_B \left(\phi(w) \begin{bmatrix} v & 0 \\ 0 & 1_k \end{bmatrix} \right)^* \right) \phi(w) \begin{bmatrix} v & 0 \\ 0 & 1_k \end{bmatrix} \end{aligned}$$

is homotopic to $1_n \oplus 0_{m-n} \oplus 1_k$ through $M_{m+k}(\tilde{J})$ via $\lambda_B(\pi_B(y_t^*))y_t$. It follows that $[1_n, q, v] = 0$ in $K_0(\psi)$. \square

Corollary 2.5.2. *The inclusions $A \rightarrow \tilde{A}$ and $B \rightarrow \tilde{B}$ induce isomorphisms $K_*(\phi) \rightarrow K_*(\tilde{\phi})$.*

Proof. Use Proposition 2.5.1 on the diagram with split exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & \tilde{A} & \xrightleftharpoons{\quad} & \mathbb{C} \longrightarrow 0 \\ & & \downarrow \phi & & \downarrow \tilde{\phi} & & \parallel \\ 0 & \longrightarrow & B & \longrightarrow & \tilde{B} & \xrightleftharpoons{\quad} & \mathbb{C} \longrightarrow 0 \end{array}$$

together with Corollary 2.4.4. \square

Of course, a similar result involving mapping cones can be obtained. Using Proposition 1.2.15 on the above diagram results in the split exact sequence

$$0 \longrightarrow C_\phi \longrightarrow C_{\tilde{\phi}} \xrightleftharpoons{\quad} C_0((0, 1]) \longrightarrow 0$$

and $C_0((0, 1])$ is contractible.

2.5.2 The index map

Now we associate an index map $\partial_1 : K_1(\gamma) \rightarrow K_0(\psi)$ to the diagram (2.3).

Definition 2.5.3. Let $(1_n, u, g)$ be a triple in $\Gamma_1(\gamma)$. Define the map $\nu : \Gamma_1(\gamma) \rightarrow K_0(\psi)$ by

$$\nu(1_n, u, g) = \left[w \begin{bmatrix} 1_n & 0 \\ 0 & 0_l \end{bmatrix} w^*, \begin{bmatrix} 1_n & 0 \\ 0 & 0_l \end{bmatrix}, \begin{bmatrix} h(1) & 0 \\ 0 & 0_l \end{bmatrix} \phi(w^*) \right]$$

where $l \geq 0$ is an integer, w is a unitary in $M_{n+l}(\tilde{A})$ such that $\pi_A(w)(1_n \oplus 0_l) = u \oplus 0_l$, and h is a unitary in $C([0, 1]) \otimes M_n(\tilde{B})$ such that $h(0) = 1_n$ and $\pi_B(h) = g$.

Observe that we may always choose such elements l , w , and h : we may take $l = n$ and w to be a lift of $u \oplus u^*$ by part (ii) of Proposition 1.1.13, and h exists because g , as a unitary in $C([0, 1]) \otimes M_n(\tilde{B}/J)$, is homotopic to 1_n , so part (ii) of Proposition 1.1.13 applies. If $h(0) \neq 1_n$, simply replace h with $h(0)^*h$, which is still a lift of g since $g(0) = 1_n$.

As changing the value of l effectively only changes the number of zeros (or ones, in the case of w) down the diagonal, the definition clearly does not depend on the choice of this integer. One could avoid the arbitrary choice altogether and simply use n instead, but the arbitrary choice simplifies the proof of Proposition 2.5.7.

Lemma 2.5.4. *The map ν in Definition 2.5.3 is independent of the choices of w and h , and if $(1_n, u, g) \sim (1_n, u', g')$ in $\Gamma_1(\gamma)$, then $\nu(1_n, u, g) = \nu(1_n, u', g')$.*

Proof. Fix a triple $(1_n, u, g)$ in $\Gamma_1(\gamma)$ and let w' be an alternative choice for w satisfying the same property. Notice that

$$x = w' \begin{bmatrix} 1_n & 0 \\ 0 & 0_l \end{bmatrix} w^*$$

satisfies $\pi_A(x) = 1_n \oplus 0_l$ and hence is a partial isometry in $M_{n+l}(\tilde{I})$. Moreover, the diagram

$$\begin{array}{ccc} \phi(w)(1_n \oplus 0_l)\phi(w^*) & \xrightarrow{(h(1) \oplus 0_l)\phi(w^*)} & 1_n \oplus 0_l \\ \downarrow \psi(x) & & \downarrow 1_n \oplus 0_l \\ \phi(w')(1_n \oplus 0_l)\phi(w'^*) & \xrightarrow{(h(1) \oplus 0_l)\phi(w'^*)} & 1_n \oplus 0_l \end{array}$$

is commutative, which shows that the triples $(w(1_n \oplus 0_l)w^*, 1_n \oplus 0_l, (h(1) \oplus 0_l)\phi(w^*))$ and $(w'(1_n \oplus 0_l)w'^*, 1_n \oplus 0_l, (h(1) \oplus 0_l)\phi(w'^*))$ are isomorphic in $\Gamma_0(\psi)$.

Next, suppose that h' is another lift of g in $C([0, 1]) \otimes M_n(\tilde{B})$. Then the triple

$$\sigma = \left(\left[\begin{array}{cc} 1_n & 0 \\ 0 & 0_l \end{array} \right], \left[\begin{array}{cc} 1_n & 0 \\ 0 & 0_l \end{array} \right], \left[\begin{array}{cc} h'(1)h(1)^* & 0 \\ 0 & 0_l \end{array} \right] \right)$$

is in $\Gamma_0(\psi)$, and it is elementary due to the path $t \mapsto h'(t)h(t)^*$ (note carefully that this is a path through $M_n(\tilde{J})$, not just $M_n(\tilde{B})$). By part (ii) of Proposition 2.3.5,

$$\nu(1_n, u, g) + [\sigma] = \left[w \left[\begin{array}{cc} 1_n & 0 \\ 0 & 0_l \end{array} \right] w^*, \left[\begin{array}{cc} 1_n & 0 \\ 0 & 0_l \end{array} \right], \left[\begin{array}{cc} h'(1) & 0 \\ 0 & 0_l \end{array} \right] \phi(w^*) \right]$$

as desired.

To prove the last claim, we use a mapping cone argument². Suppose that $(1_n, u, g) \sim (1_n, u', g')$ in $\Gamma_1(\gamma)$. By increasing n if necessary, we may find a path of unitaries u_t in $M_n(\widetilde{A/I})$ and a path of unitaries g_t in $C([0, 1]) \otimes M_n(\widetilde{B/J})$ such that $u_0 = u$, $u_1 = u'$, $g_0 = g$, $g_1 = g'$, and $g_t(1) = \gamma(u_t)$ for all t . Define the C*-algebras

$$C = \{(a, f) \in M_n(\tilde{A}) \oplus (C([0, 1]) \otimes M_n(\tilde{B})) \mid f(1) = \phi(a) \text{ and } f(0) \in M_n(\mathbb{C})\}$$

$$D = \{(a, f) \in M_n(\widetilde{A/I}) \oplus (C([0, 1]) \otimes M_n(\widetilde{B/J})) \mid f(1) = \gamma(a) \text{ and } f(0) \in M_n(\mathbb{C})\}$$

and observe that, applying π_A and π_B component-wise gives a surjective *-homomorphism $C \rightarrow D$. (To see this, observe that C and D are isomorphic to $M_n(\tilde{C}_{\tilde{\phi}})$ and $M_n(\tilde{C}_{\tilde{\gamma}})$, respectively, and use exactness of (1.2).) Now, (u^*u', g^*g') is a unitary in D homotopic to 1_n , and thus there is a unitary (z, k) in C with $\pi_A(z) = u^*u'$ and $\pi_B(k) = g^*g'$. By replacing (z, k) with $(k(0)^*z, k(0)^*k)$ if necessary, we may assume that $k(0) = 1_n$. Also, let z' be a unitary in $M_n(\tilde{A})$ such that $\pi_A(z') = uu'^*$. Let w and h be required lifts in the definition of ν corresponding to u and g , and set

$$w' = w \left[\begin{array}{cc} z & 0 \\ 0 & z' \end{array} \right], \quad h' = hk,$$

which satisfy $\pi_A(w') = u' \oplus u'^*$, $h'(0) = 1_n$, and $\pi_B(h') = g'$, hence are lifts corresponding to u' and g' . Moreover,

$$w \left[\begin{array}{cc} 1_n & 0 \\ 0 & 0_n \end{array} \right] w^* = w' \left[\begin{array}{cc} 1_n & 0 \\ 0 & 0_n \end{array} \right] w'^*$$

²It is rather disappointing to appeal to mapping cones at this stage, but a more elementary (no pun intended) method of producing the lift (z, k) proved to be more elusive than expected.

and

$$\begin{bmatrix} h(1) & 0 \\ 0 & 0_n \end{bmatrix} \phi(w^*) = \begin{bmatrix} h'(1) & 0 \\ 0 & 0_n \end{bmatrix} \phi(w'^*). \quad \square$$

Definition 2.5.5. We define $\partial_1 : K_1(\gamma) \rightarrow K_0(\psi)$ to be the unique group homomorphism that makes the diagram

$$\begin{array}{ccc} \Gamma_1(\gamma) & & \\ \downarrow & \searrow \nu & \\ K_1(\gamma) & \xrightarrow{\partial_1} & K_0(\psi) \end{array}$$

commutative, where ν is as in Definition 2.5.3.

Proposition 2.5.6. *The map ∂_1 is natural in the following sense. Suppose that*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I' & \longrightarrow & A' & \longrightarrow & A'/I' & \longrightarrow & 0 \\ & & \nearrow \omega_I & & \nearrow \omega_A & & \nearrow \omega_{A/I} & & \\ 0 & \longrightarrow & I & \longrightarrow & A & \longrightarrow & A/I & \longrightarrow & 0 \\ & & \downarrow \psi & & \downarrow \phi & & \downarrow \gamma & & \\ & & \downarrow \psi' & & \downarrow \phi' & & \downarrow \gamma' & & \\ 0 & \longrightarrow & J' & \longrightarrow & B' & \longrightarrow & B'/J' & \longrightarrow & 0 \\ & & \nearrow \omega_J & & \nearrow \omega_B & & \nearrow \omega_{B/J} & & \\ 0 & \longrightarrow & J & \longrightarrow & B & \longrightarrow & B/J & \longrightarrow & 0 \end{array}$$

is a commutative diagram with exact rows. Then the diagram

$$\begin{array}{ccc} K_1(\gamma) & \xrightarrow{\partial_1} & K_0(\psi) \\ \downarrow (\omega_{A/I}, \omega_{B/J})_* & & \downarrow (\omega_I, \omega_J)_* \\ K_1(\gamma') & \xrightarrow{\partial_1} & K_0(\psi') \end{array}$$

is commutative.

Proof. Let $(1_n, u, g)$ be a triple in $\Gamma_1(\gamma)$, and choose lifts w and h corresponding to this triple as in Definition 2.5.3. Then $\omega_A(w)$ and $\omega_B(h)$ are lifts corresponding to the triple $(1_n, \omega_{A/I}(u), \omega_{B/J}(g))$. \square

Proposition 2.5.7. *The sequence*

$$K_1(\phi) \xrightarrow{\pi_*} K_1(\gamma) \xrightarrow{\partial_1} K_0(\psi) \xrightarrow{\iota_*} K_0(\phi)$$

is exact and the diagram

$$\begin{array}{ccccc}
K_1(S(B/J)) & \xrightarrow{\mu_1} & K_1(\gamma) & \xrightarrow{\nu_1} & K_1(A/I) \\
\downarrow \theta_J^{-1} \circ \delta_2 & & \downarrow \partial_1 & & \downarrow \delta_1 \\
K_1(J) & \xrightarrow{\mu_0} & K_0(\psi) & \xrightarrow{\nu_0} & K_0(I)
\end{array}$$

is commutative.

Proof. For ease of notation we will denote

$$p = w \begin{bmatrix} 1_n & 0 \\ 0 & 0_l \end{bmatrix} w^* \quad v = \begin{bmatrix} h(1) & 0 \\ 0 & 0_l \end{bmatrix} \phi(w^*).$$

It is easy to see from the formula for ∂_1 that the right square in the diagram is commutative. For the left square, take $[f]$ in $K_1(S(B/J))$, where f is a unitary in $M_n(\widetilde{S(B/J)})$ and $f(0) = \dot{f} = 1_n$. Find a unitary h in $M_n(\widetilde{CB})$ such that $h(0) = \dot{h} = 1_n$ and $\pi_B(h) = f$. Then we have

$$\partial_1(\mu_1([f])) = \partial_1([1_n, 1_n, f]) = [1_n, 1_n, h(1)].$$

Now find g in $M_{2n}(\widetilde{SB})$ such that $g(0) = \dot{g} = 1_{2n}$ and $\pi_B(g) = f \oplus f^*$. Let

$$\tilde{g}(t) = \begin{cases} g(2t) & 0 \leq t \leq 1/2 \\ h(2t-1) \oplus h(2t-1)^* & 1/2 \leq t \leq 1 \end{cases}.$$

Then $[g(1_n \oplus 0_n)g^*] - [1_n \oplus 0_n] = [\tilde{g}(1_n \oplus 0_n)\tilde{g}^*] - [1_n \oplus 0_n]$ in $K_0(SJ)$, the latter being equal to $\theta_J([h(1)])$ since $\tilde{g}(1) = h(1) \oplus h(1)^*$. All in all, we have

$$\mu_0(\theta_J^{-1}(\delta_2([f]))) = \mu_0(\theta_J^{-1}([\tilde{g}(1_n \oplus 0_n)\tilde{g}^*] - [1_n \oplus 0_n])) = \mu_0([h(1)]) = [1_n, 1_n, h(1)]$$

which shows commutativity of the left square.

The composition $\partial_1 \circ \pi_*$ is clearly zero since everything has a unitary lift. We also have $\iota_* \circ \partial_1 = 0$ since

$$[p, 1_n \oplus 0_l, v] = [p, 1_n \oplus 0_l, v] + [1_n \oplus 0_l, 1_n \oplus 0_l, h(1)^* \oplus 0_l] = [p, 1_n \oplus 0_l, (1_n \oplus 0_l)\phi(w^*)]$$

since $(1_n \oplus 0_l, 1_n \oplus 0_l, h(1)^* \oplus 0_l)$ is elementary in $\Gamma_0(\phi)$ and $(p, 1_n \oplus 0_l, (1_n \oplus 0_l)\phi(w^*)) \cong (1_n \oplus 0_l, 1_n \oplus 0_l, 1_n \oplus 0_l)$.

Now suppose that

$$\partial_1([1_n, u, g]) = [p, 1_n \oplus 0_l, v] = [w(1_n \oplus 0_l)w^*, 1_n \oplus 0_l, (h(1) \oplus 0_l)\phi(w^*)] = 0$$

for a triple $(1_n, u, g)$ in $\Gamma_1(\phi)$. Find $k \geq 1$ and a partial isometry x in $M_{n+l+k}(\tilde{I})$ with

$xx^* = p \oplus 1_k$ and $\dot{x} = x^*x = 1_n \oplus 0_l \oplus 1_k$, and such that $(v \oplus 1_k)\psi(x)$ is homotopic to $1_n \oplus 0_l \oplus 1_k$. Let y_t be such a homotopy, with $\dot{y}_t = y_0 = 1_n \oplus 0_l \oplus 1_k$ for all t and $y_1 = (v \oplus 1_k)\psi(x)$. Set

$$z = \begin{bmatrix} 1_n & 0 & 0 \\ 0 & 0_l & 0 \\ 0 & 0 & 1_k \end{bmatrix} \begin{bmatrix} w^* & 0 \\ 0 & 1_k \end{bmatrix} x$$

and

$$h'(t) = \begin{cases} y_{2t} & 0 \leq t \leq 1/2 \\ (h(2t-1)^* \oplus 0_l \oplus 1_k)(v \oplus 1_k)\psi(x) & 1/2 \leq t \leq 1 \end{cases}$$

Then $\pi_A(z) = u \oplus 0_l \oplus 1_k$ and

$$\pi_B(h'(t)) = \begin{cases} 1_n \oplus 0_l \oplus 1_k & 0 \leq t \leq 1/2 \\ g(2t-1) \oplus 0_l \oplus 1_k & 1/2 \leq t \leq 1 \end{cases}$$

which is clearly homotopic to $g \oplus 0_l \oplus 1_k$. Moreover, $h'(1) = \phi(z)$. It follows that

$$[1_n, u, g] = [1_n \oplus 0_l \oplus 1_k, u \oplus 0_l \oplus 1_k, g \oplus 0_l \oplus 1_k] = \pi_*([1_n \oplus 0_l \oplus 1_k, z, h'])$$

Now suppose that $(p, 1_n, v)$ is a triple in $\Gamma_0(\psi)$ with $[p, 1_n, v] = 0$ in $K_0(\phi)$. Choose $m \geq n$ such that $1_n \oplus 0_{m-n}$ and p are in $M_m(\tilde{I})$ and v is in $M_m(\tilde{J})$. Find $k \geq 0$ and a partial isometry x in $M_{m+k}(\tilde{A})$ with $xx^* = p \oplus 1_k$ and $\dot{x} = x^*x = 1_n \oplus 0_{m-n} \oplus 1_k$, and such that $(v \oplus 1_k)\phi(x)$ is homotopic to $1_n \oplus 0_{m-n} \oplus 1_k$. Find a unitary U in $M_{m+k}(\mathbb{C})$ such that

$$U(1_n \oplus 0_{m-n} \oplus 1_k)U^* = 1_{n+k} \oplus 0_{m-n}$$

and let $p' = U(p \oplus 1_k)U^*$, $v' = U(v \oplus 1_k)U^*$, and $x' = UxU^*$. Clearly $(p, 1_n, v) \oplus (1_k, 1_k, 1_k) \cong (p', 1_{n+k}, v')$, $x'x'^* = p'$, $x'^*x' = 1_{n+k} \oplus 0_{m-n}$, and that $v'\phi(x')$ is homotopic to $1_{n+k} \oplus 0_{m-n}$. Let y_t be such a homotopy, with $\dot{y}_t = y_0 = 1_{n+k} \oplus 0_{m-n}$ for all t and $y_1 = v'\phi(x')$. Notice that $\pi_A(x') = \widetilde{(1_{n+k} \oplus 0_{m-n})\pi_A(x')(1_{n+k} \oplus 0_{m-n})}$, so we may regard $\pi_A(x')$ as a unitary in $M_{n+k}(\tilde{A}/I)$, and similarly we may regard y_t as a path of unitaries in $M_{n+k}(\tilde{B})$. Set $g(t) = \pi_B(y_t)$ and notice that

$$g(1) = \pi_B(\phi(x'^*))\pi_B(v'^*) = \gamma(\pi_A(x'^*))$$

so that $(1_{n+k}, \pi_A(x'^*), g)$ is a triple in $\Gamma_1(\gamma)$. Moreover, we see that its image under ∂_1 is $[p, 1_n, v]$ by using $l = 2m + k - n$,

$$w = \begin{bmatrix} x' & 1_{m+k} - x'x'^* \\ 1_{m+k} - x'^*x' & x'^* \end{bmatrix}$$

in $M_{2(m+k)}(\tilde{A})$ and $h(t) = y_t$ in the definition of ∂_1 . \square

Definition 2.5.8. Define the map $\theta_\phi : K_1(\phi) \rightarrow K_0(S\phi)$ to be the index map associated to the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & SA & \longrightarrow & CA & \longrightarrow & A \longrightarrow 0 \\ & & \downarrow S\phi & & \downarrow C\phi & & \downarrow \phi \\ 0 & \longrightarrow & SB & \longrightarrow & CB & \longrightarrow & B \longrightarrow 0 \end{array}$$

Corollary 2.5.9. *The map $\theta_\phi : K_1(\phi) \rightarrow K_0(S\phi)$ is an isomorphism. Moreover, the diagram*

$$\begin{array}{ccccc} K_1(SB) & \xrightarrow{\mu_1} & K_1(\phi) & \xrightarrow{\nu_1} & K_1(A) \\ \parallel & & \downarrow \theta_\phi & & \downarrow \theta_A \\ K_1(SB) & \xrightarrow{\mu_0} & K_0(S\phi) & \xrightarrow{\nu_0} & K_0(SA) \end{array}$$

is commutative.

Proof. CA and CB are contractible, hence $K_0(C\phi) = K_1(C\phi) = 0$ by Corollary 2.4.4. So by Proposition 2.5.7, the sequence

$$0 \longrightarrow K_1(\phi) \xrightarrow{\theta_\phi} K_0(S\phi) \longrightarrow 0$$

is exact, and the given diagram is commutative. \square

An explicit description of θ_ϕ is as follows. Let $(1_n, u, g)$ be a triple in $\Gamma_1(\phi)$, and let w be a unitary in $C([0, 1]) \otimes M_{2n}(\tilde{A})$ with $w(0) = 1_{2n}$ and $w(1) = u \oplus u^*$. Then

$$\theta_\phi([1_n, u, g]) = \left[w \begin{bmatrix} 1_n & 0 \\ 0 & 0_n \end{bmatrix} w^*, \begin{bmatrix} 1_n & 0 \\ 0 & 0_n \end{bmatrix}, \begin{bmatrix} g & 0 \\ 0 & 0_n \end{bmatrix} \phi(w^*) \right]$$

Corollary 2.5.10. *The sequence*

$$K_1(\psi) \xrightarrow{\iota_*} K_1(\phi) \xrightarrow{\pi_*} K_1(\gamma)$$

is exact. If $\lambda_A : A/I \rightarrow A$ and $\lambda_B : B/J \rightarrow B$ are splittings of the rows in (2.3) that keep the diagram commutative, then the sequence

$$0 \longrightarrow K_1(\psi) \xrightarrow{\iota_*} K_1(\phi) \xrightleftharpoons[\lambda_*]{\pi_*} K_1(\gamma) \longrightarrow 0$$

is split exact.

Proof. The map θ_ϕ is natural, so we have the commutative diagram

$$\begin{array}{ccccc}
K_1(\psi) & \xrightarrow{\iota_*} & K_1(\phi) & \xrightarrow{\pi_*} & K_1(\gamma) \\
\downarrow \theta_\psi & & \downarrow \theta_\phi & & \downarrow \theta_\gamma \\
K_0(S\psi) & \xrightarrow{(S\iota)_*} & K_0(S\phi) & \xrightarrow{(S\pi)_*} & K_0(S\gamma)
\end{array}$$

in which, by Proposition 2.5.1, the bottom row is exact. It follows that the top row is exact as well. The proof for split exactness is similar. \square

At this point we may unambiguously define higher relative groups $K_j(\phi)$ by $K_0(S^j\phi)$ and higher index maps $\partial_j : K_j(\gamma) \rightarrow K_{j-1}(\psi)$ to obtain a long exact sequence. We proceed to prove that Bott periodicity holds so that the long exact sequence collapses to the six-term exact sequence in Theorem 2.1.2.

2.5.3 Bott periodicity and the exponential map

The proof of Bott periodicity for relative K -theory is quite slick, as it essentially follows from ordinary Bott periodicity and Corollary 2.4.4. We will follow the proof in [7] that uses the Toeplitz algebra (recall Example 1.1.5). Let $\pi : \mathcal{T} \rightarrow C(\mathbb{T})$ be the $*$ -homomorphism that sends the generating isometry to the function z on \mathbb{T} . The kernel of π is isomorphic to \mathcal{K} , and by identifying $C_0((0, 1))$ with elements in $C(\mathbb{T})$ that vanish at 1 and letting $\mathcal{T}_0 = \pi^{-1}(C_0((0, 1)))$, we obtain the reduced Toeplitz extension

$$0 \longrightarrow \mathcal{K} \hookrightarrow \mathcal{T}_0 \xrightarrow{\pi} C_0((0, 1)) \longrightarrow 0 \quad (2.4)$$

By applying Proposition 1.2.11 to (2.4), we obtain that $K_*(\mathcal{T}_0) = 0$.

Lemma 2.5.11. *If C is in the bootstrap class and $K_*(C) = 0$, then $K_*(\phi \otimes \text{id}_C) = 0$. In particular, $K_*(\phi \otimes \text{id}_{\mathcal{T}_0}) = 0$.*

Proof. By Theorem 1.2.19, we have $K_*(A \otimes C) = K_*(B \otimes C) = 0$. The conclusion follows from Corollary 2.4.4. \square

Lemma 2.5.12. $(\kappa_A, \kappa_B)_* : K_*(\phi) \rightarrow K_*(\phi \otimes \text{id}_{\mathcal{K}})$ is an isomorphism.

Proof. For $j = 0, 1$, we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
K_{1-j}(A) & \xrightarrow{\phi_*} & K_{1-j}(B) & \xrightarrow{\mu_j} & K_j(\phi) & \xrightarrow{\nu_j} & K_j(A) & \xrightarrow{\phi_*} & K_j(B) \\
\downarrow (\kappa_A)_* & & \downarrow (\kappa_B)_* & & \downarrow (\kappa_A, \kappa_B)_* & & \downarrow (\kappa_A)_* & & \downarrow (\kappa_B)_* \\
K_{1-j}(A \otimes \mathcal{K}) & \xrightarrow{(\phi \otimes \text{id}_{\mathcal{K}})_*} & K_{1-j}(B \otimes \mathcal{K}) & \xrightarrow{\mu_j} & K_j(\phi \otimes \text{id}_{\mathcal{K}}) & \xrightarrow{\nu_j} & K_j(A \otimes \mathcal{K}) & \xrightarrow{(\phi \otimes \text{id}_{\mathcal{K}})_*} & K_j(B \otimes \mathcal{K})
\end{array}$$

All vertical maps except for the middle one are known to be isomorphisms. The five lemma then shows that the middle vertical arrow is an isomorphism. \square

We now produce the Bott map. By tensoring (2.4) with A and B , we obtain the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A \otimes \mathcal{K} & \longrightarrow & A \otimes \mathcal{T}_0 & \longrightarrow & SA \longrightarrow 0 \\ & & \downarrow \phi \otimes \text{id}_{\mathcal{K}} & & \downarrow \phi \otimes \text{id}_{\mathcal{T}_0} & & \downarrow S\phi \\ 0 & \longrightarrow & B \otimes \mathcal{K} & \longrightarrow & B \otimes \mathcal{T}_0 & \longrightarrow & SB \longrightarrow 0 \end{array}$$

(All C^* -algebras involved, with the exception of possibly A and B , are nuclear, hence the rows remain exact, see II.9.6.6 of [5].) Proposition 2.5.7 implies that

$$K_1(\phi \otimes \text{id}_{\mathcal{T}_0}) \longrightarrow K_1(S\phi) \longrightarrow K_0(\phi \otimes \text{id}_{\mathcal{K}}) \longrightarrow K_0(\phi \otimes \text{id}_{\mathcal{T}_0})$$

is exact, and Lemma 2.5.11 and Lemma 2.5.12 together give an isomorphism $K_0(\phi) \cong K_1(S\phi)$. We let $\beta_\phi : K_0(\phi) \rightarrow K_1(S\phi)$ denote this isomorphism. By Proposition 2.5.7 we have the commutative diagram

$$\begin{array}{ccccc} K_1(B) & \xrightarrow{\mu_0} & K_0(\phi) & \xrightarrow{\nu_0} & K_0(A) \\ \downarrow \beta_{SB \circ \theta_B} & & \downarrow \beta_\phi & & \downarrow \beta_A \\ K_1(S^2B) & \xrightarrow{\mu_1} & K_1(S\phi) & \xrightarrow{\nu_1} & K_1(SA) \end{array}$$

Now we complete the six-term exact sequence in Theorem 2.1.2. We define the exponential map $\partial_0 : K_0(\gamma) \rightarrow K_1(\psi)$ to be the group homomorphism that makes the diagram

$$\begin{array}{ccc} K_0(\gamma) & \overset{\partial_0}{\dashrightarrow} & K_1(\psi) \\ \downarrow \beta_\gamma & & \downarrow \theta_\psi \\ K_1(S\gamma) & \xrightarrow{\partial_2} & K_0(S\psi) \end{array}$$

commutative. All maps in the above diagram are natural, so the sequence in Theorem 2.1.2 is exact.

Remark. It is interesting to note that split exactness was not necessary to prove Theorem 2.1.2. An examination of the proof of Bott periodicity in [7] reveals that split exactness is pivotal in deducing that $K_*(\mathcal{T}_0) = 0$ from the isomorphism $K_*(\mathcal{T}) \cong K_*(\mathbb{C})$. Here we were able to sneak around this difficulty using Corollary 2.4.4 and the fact that $K_*(\mathcal{T}_0) = 0$.

We end this section with a straightforward application of Theorem 2.1.2, which relates $K_*(\phi)$, $K_*(\ker \phi)$, and $K_*(\iota)$, where $\iota : \phi(A) \rightarrow B$ is the inclusion map.

Notation. When A is a C^* -subalgebra of B and $\iota : A \rightarrow B$ is the inclusion map, we denote $K_*(\iota)$ by $K_*(A, B)$.

Corollary 2.5.13. *Suppose that $\phi : A \rightarrow B$ is a $*$ -homomorphism. Then there is a six-term exact sequence*

$$\begin{array}{ccccc} K_0(\ker \phi) & \longrightarrow & K_0(\phi) & \longrightarrow & K_0(\phi(A), B) \\ \uparrow & & & & \downarrow \\ K_1(\phi(A), B) & \longleftarrow & K_1(\phi) & \longleftarrow & K_1(\ker \phi) \end{array}$$

Proof. This follows immediately from applying Theorem 2.1.2 to the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker \phi & \hookrightarrow & A & \xrightarrow{\phi} & \phi(A) \longrightarrow 0 \\ & & \downarrow & & \downarrow \phi & & \downarrow \\ 0 & \longrightarrow & 0 & \longrightarrow & B & \xlongequal{\quad} & B \longrightarrow 0 \end{array}$$

□

We refer to Example 2.10.6 and Example 2.10.7 for specific instances of the six-term sequence in Corollary 2.5.13.

2.6 The proof of Theorem 2.1.3

In this section, we prove Theorem 2.1.3. Consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & SB & \xrightarrow{\iota_A} & C_\phi & \xrightarrow{\pi_A} & A \longrightarrow 0 \\ & & \parallel & & \downarrow \eta_\phi & & \downarrow \phi \\ 0 & \longrightarrow & SB & \hookrightarrow & CB & \xrightarrow{\pi_B} & B \longrightarrow 0 \end{array} \quad (2.5)$$

with exact rows, where $\iota_A(f) = (0, f)$, $\pi_A(a, f) = a$, $\pi_B(f) = f(1)$, and $\eta_\phi(a, f) = f$. Since $K_*(CB) = 0$ because CB is contractible, we have by Theorem 2.1.1 that $\nu_* : K_*(\eta_\phi) \rightarrow K_*(C_\phi)$ is an isomorphism. Applying Theorem 2.1.2 to the diagram (2.5) and using Corollary 2.4.4, we obtain that

$$\begin{array}{ccccc} 0 & \longrightarrow & K_0(\eta_\phi) & \xrightarrow{\pi_*} & K_0(\phi) \\ \uparrow & & & & \downarrow \\ K_1(\phi) & \xleftarrow{\pi_*} & K_1(\eta_\phi) & \xleftarrow{\quad} & 0 \end{array}$$

is exact, and hence $\pi_* : K_*(\eta_\phi) \rightarrow K_*(\phi)$ is an isomorphism.

Definition 2.6.1. $\Delta = \nu_* \circ \pi_*^{-1}$

We now proceed to prove that Δ has the desired properties, starting with the fact that (2.2) is commutative. To this end, assume that (2.1) is commutative. Using the commutative cube

$$\begin{array}{ccccc}
 & & C_\psi & \xrightarrow{\pi_C} & C \\
 & (\alpha \oplus C\beta) \nearrow & \downarrow \eta_\psi & \searrow \alpha & \downarrow \psi \\
 C_\phi & \xrightarrow{\pi_A} & A & & \\
 \downarrow \eta_\phi & & \downarrow \phi & & \\
 & C\beta \nearrow & CD & \xrightarrow{\pi_D} & D \\
 & & \downarrow \beta & & \\
 CB & \xrightarrow{\pi_B} & B & &
 \end{array}$$

together with Proposition 2.4.5, we see that the diagram

$$\begin{array}{ccccc}
 K_*(\phi) & \xleftarrow{\pi_*} & K_*(\eta_\phi) & \xrightarrow{\nu_*} & K_*(C_\phi) \\
 \downarrow (\alpha, \beta)_* & & \downarrow (\alpha \oplus C\beta, C\beta)_* & & \downarrow (\alpha \oplus C\beta)_* \\
 K_*(\psi) & \xleftarrow{\pi_*} & K_*(\eta_\psi) & \xrightarrow{\nu_*} & K_*(C_\psi)
 \end{array}$$

is commutative. By the definition of the map Δ , this implies that the diagram in Theorem 2.1.3 is commutative.

Next we show that the six-term exact sequence of Proposition 1.2.15 plays well with the six-term exact sequence of Theorem 2.1.2. By using naturality of the index map (Proposition 2.5.6) on the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & I & \longrightarrow & A & \longrightarrow & A/I & \longrightarrow & 0 \\
 & & \uparrow \pi_I & & \uparrow \phi & & \uparrow \pi_{A/I} & & \\
 0 & \longrightarrow & C_\psi & \longrightarrow & C_\phi & \longrightarrow & C_\gamma & \longrightarrow & 0 \\
 & & \downarrow \eta_\psi & & \downarrow \eta_\phi & & \downarrow \eta_\gamma & & \\
 0 & \longrightarrow & J & \longrightarrow & B & \longrightarrow & B/J & \longrightarrow & 0 \\
 & & \uparrow \pi_J & & \uparrow & & \uparrow \pi_{B/J} & & \\
 0 & \longrightarrow & CJ & \longrightarrow & CB & \longrightarrow & C(B/J) & \longrightarrow & 0
 \end{array}$$

and the commutative diagram in Proposition 2.5.7, we have the commutative diagram

$$\begin{array}{ccccc}
K_1(\gamma) & \xleftarrow{\pi_*} & K_1(\eta_\gamma) & \xrightarrow{\nu_1} & K_1(C_\gamma) \\
\downarrow \partial_1 & & \downarrow \partial'_1 & & \downarrow \delta_1 \\
K_0(\psi) & \xleftarrow{\pi_*} & K_0(\eta_\psi) & \xrightarrow{\nu_0} & K_0(C_\psi)
\end{array}$$

where δ_1 is the index map associated to the sequence (1.2), and ∂'_1 is the relative index map associated to the (sub-)diagram of cones above, mapping or otherwise. It follows that Δ intertwines the index maps ∂_1 and δ_1 , and hence also the exponential maps ∂_0 and δ_0 (as these are also index maps). Putting this all together, we have the following commutative diagram that intertwines the exact sequence in Proposition 1.2.15 with the one in Theorem 2.1.2:

$$\begin{array}{ccccccc}
K_0(\psi) & \xrightarrow{\iota_*} & K_0(\phi) & \xrightarrow{\pi_*} & K_0(\gamma) & & \\
\downarrow \Delta & & \downarrow \Delta & & \swarrow \Delta & & \\
K_0(C_\psi) & \xrightarrow{\iota_*} & K_0(C_\phi) & \xrightarrow{\pi_*} & K_0(C_\gamma) & & \\
\delta_1 \uparrow & & \delta_0 \downarrow & & & & \\
K_1(C_\gamma) & \xleftarrow{\pi_*} & K_1(C_\phi) & \xleftarrow{\iota_*} & K_1(C_\psi) & & \\
\Delta \uparrow & & \Delta \uparrow & & \swarrow \Delta & & \\
K_1(\gamma) & \xleftarrow{\pi_*} & K_1(\phi) & \xleftarrow{\iota_*} & K_1(\psi) & & \\
\uparrow \partial_1 & & & & \downarrow \partial_0 & & \\
& & & & & &
\end{array}$$

Finally, we prove the intertwining result for Theorem 2.1.1.

Proposition 2.6.2. *The diagram*

$$\begin{array}{ccccc}
& & K_0(\phi) & & \\
& \nearrow \mu_0 & \downarrow \Delta & \searrow \nu_0 & \\
K_1(B) & \xrightarrow{\iota_* \circ \theta_B} & K_0(C_\phi) & \xrightarrow{\pi_*} & K_0(A) \\
\uparrow \phi_* & & & & \downarrow \phi_* \\
K_1(A) & \xleftarrow{\pi_*} & K_1(C_\phi) & \xleftarrow{\iota_* \circ \beta_B} & K_0(B) \\
& \swarrow \nu_1 & \Delta \uparrow & \searrow \mu_1 & \\
& & K_1(\phi) & &
\end{array}$$

is commutative up to sign.

Proof. Commutativity of the top right triangle and the bottom left triangle can be proved in one go: their commutativity is equivalent to commutativity of the square

$$\begin{array}{ccc}
K_*(\eta_\phi) & \xrightarrow{\pi_*} & K_*(\phi) \\
\downarrow \nu_* & & \downarrow \nu_* \\
K_*(C_\phi) & \xrightarrow{\pi_*} & K_*(A)
\end{array}$$

which is immediate from Proposition 2.4.5.

Next we consider the top left triangle. Take $[u]$ in $K_1(B)$ where u is a unitary in $M_n(\tilde{B})$ with $\dot{u} = 1_n$. Then $\iota_*(\theta_B([u])) = [v(1_n \oplus 0_n)v^*] - [1_n \oplus 0_n]$ in $K_0(C_\phi)$, where v is in $C([0, 1]) \otimes M_{2n}(\tilde{B})$ such that $v(0) = 1_{2n}$, $v(1) = u \oplus u^*$, and $\dot{v}(t) = 1_{2n}$ for all t . We also have $\mu_0([u]) = [1_n, 1_n, u]$. Define

$$\sigma = (v(1_n \oplus 0_n)v^*, 1_n \oplus 0_n, (1_n \oplus 0_n)v^*)$$

which is a triple in $\Gamma_0(\eta_\phi)$. Then $\nu_*([\sigma]) = [v(1_n \oplus 0_n)v^*] - [1_n \oplus 0_n]$ and $\pi_*([\sigma]) = [1_n, 1_n, u^*] = -[1_n, 1_n, u]$.

Now for the bottom right triangle. Let p and q be projections in $M_n(\tilde{B})$ with $[p] = [q]$. Then we have $\mu_1([p] - [q]) = [1_n, 1_n, f_p f_q^*]$ in $K_1(\phi)$ and $\iota_*(\beta_B([p] - [q])) = [(1_n, f_p f_q^*)]$ in $K_1(C_\phi)$. Though it is slightly sloppy notation, for $0 \leq s \leq 1$, define $f_s(t) = f_p(st)f_q(st)^*$ and $g(s) = f_s$. Then the triple

$$\sigma = (1_n, (1_n, f_p f_q^*), g)$$

is in $\Gamma_1(\eta_\phi)$ and $\nu_*([\sigma]) = [(1_n, f_p f_q^*)]$ and $\pi_*([\sigma]) = [1_n, 1_n, f_p f_q^*]$. \square

2.7 Composition of *-homomorphisms

In this section we prove a generalized version of Theorem 2.1.1 that involves the composition of two *-homomorphisms. The proof is similar to the proof of Theorem 2.1.1, with a few additional details.

Theorem 2.7.1. *Suppose that $\phi : A \rightarrow B$ and $\psi : B \rightarrow C$ are *-homomorphisms. Then there is a six-term exact sequence*

$$\begin{array}{ccccc}
K_0(\phi) & \xrightarrow{\mu_0} & K_0(\psi \circ \phi) & \xrightarrow{\nu_0} & K_0(\psi) \\
\uparrow \partial & & & & \downarrow \partial \\
K_1(\psi) & \xleftarrow{\nu_1} & K_1(\psi \circ \phi) & \xleftarrow{\mu_1} & K_1(\phi)
\end{array}$$

The maps ν_j and μ_j for $j = 0, 1$ are given by the formulas

$$\mu_0([p, q, v]) = [p, q, \psi(v)] \quad \nu_0([p, q, v]) = [\phi(p), \phi(q), v]$$

$$\mu_1([p, u, g]) = [p, u, \psi(g)] \quad \nu_1([p, u, g]) = [\phi(p), \phi(u), g]$$

The vertical map ∂ on the left is

$$\partial([1_n, u, g]) = [1_n, 1_n, u]$$

and the vertical one on the right is the composition of $\beta_\psi : K_0(\psi) \rightarrow K_1(S\psi)$ with the map $K_1(S\psi) \rightarrow K_1(\phi)$ defined by

$$[1_n, g, h] \mapsto [1_n, 1_n, g]$$

It is straightforward to check that all of these maps are well-defined using Proposition 2.3.10. Observe that by setting $C = \{0\}$, we have $K_*(\psi \circ \phi) \cong K_*(A)$ and $K_*(\psi) \cong K_*(B)$, and we obtain the six-term exact sequence of Theorem 2.1.1 as a special case.

Proof. First observe that the diagram

$$\begin{array}{ccccccccc} K_0(\psi \circ \phi) & \xrightarrow{\nu_0} & K_0(\psi) & \xrightarrow{\partial} & K_1(\phi) & \xrightarrow{\mu_1} & K_1(\psi \circ \phi) & \xrightarrow{\nu_1} & K_1(\psi) \\ \downarrow \beta_{\psi \circ \phi} & & \downarrow \beta_\psi & & \downarrow \theta_\phi & & \downarrow \theta_{\psi \circ \phi} & & \downarrow \theta_\psi \\ K_1(S\psi \circ S\phi) & \xrightarrow{\nu_1} & K_1(S\psi) & \xrightarrow{\partial} & K_0(S\phi) & \xrightarrow{\mu_0} & K_0(S\psi \circ S\phi) & \xrightarrow{\nu_0} & K_0(S\psi) \end{array}$$

is commutative, and the bottom row is exact if and only if the top row is exact. Therefore it suffices to show exactness only at $K_0(\psi \circ \phi)$, $K_0(\phi)$, and $K_1(\psi)$. It is straightforward to check that all compositions are zero.

Suppose that $\nu_0([p, 1_n, v]) = [\phi(p), 1_n, v] = 0$. Find $k \geq 0$ and a partial isometry w in $M_{m+k}(\tilde{B})$ with $w^*w = 1_n \oplus 0_{m-n} \oplus 1_k$ and $ww^* = \phi(p) \oplus 1_k$ such that $(v \oplus 1_k)\psi(w)$ is a unitary in $(1_n \oplus 0_{m-n} \oplus 1_k)M_{m+k}(\tilde{C})(1_n \oplus 0_{m-n} \oplus 1_k)$ homotopic to $1_n \oplus 0_{m-n} \oplus 1_k$. This means that the triple $(1_n \oplus 0_{m-n} \oplus 1_k, 1_n \oplus 0_{m-n} \oplus 1_k, (v \oplus 1_k)\psi(w))$ is elementary, and we have

$$\begin{aligned} [p, 1_n, v] &= [p, 1_n, v] + [1_n \oplus 0_{m-n} \oplus 1_k, 1_n \oplus 0_{m-n} \oplus 1_k, \psi(w^*)(v^* \oplus 1_k)] \\ &= [p, 1_n, \psi(w^*)] \\ &= \mu_0([p, 1_n, w]) \end{aligned}$$

which shows exactness at $K_0(\psi \circ \phi)$.

Now assume $\mu_0([p, 1_n, v]) = [p, 1_n, \psi(v)] = 0$. Find a partial isometry w in $M_{m+k}(\tilde{A})$ with $w^*w = 1_n \oplus 0_{m-n} \oplus 1_k$ and $ww^* = p \oplus 1_k$ such that $(\psi(v) \oplus 1_k)\psi(\phi(w))$ is a unitary in $(1_n \oplus 0_{m-n} \oplus 1_k)M_{m+k}(\tilde{C})(1_n \oplus 0_{m-n} \oplus 1_k)$ homotopic to $1_n \oplus 0_{m-n} \oplus 1_k$. Let g be a unitary in $C([0, 1]) \otimes (1_n \oplus 0_{m-n} \oplus 1_k)M_{m+k}(\tilde{C})(1_n \oplus 0_{m-n} \oplus 1_k)$ such that $g(0) = 1_n \oplus 0_{m-n} \oplus 1_k$ and $g(1) = (\psi(v) \oplus 1_k)\psi(\phi(w))$. The diagram

$$\begin{array}{ccc}
p \oplus 1_k & \xrightarrow{v \oplus 1_k} & 1_n \oplus 0_{m-n} \oplus 1_k \\
\downarrow \phi(w^*) & & \downarrow 1_n \oplus 0_{m-n} \oplus 1_k \\
1_n \oplus 0_{m-n} \oplus 1_k & \xrightarrow{(v \oplus 1_k)\phi(w)} & 1_n \oplus 0_{m-n} \oplus 1_k
\end{array}$$

is commutative, showing that $(p, 1_n, v)$ is isomorphic to the triple $(1_n \oplus 0_{m-n} \oplus 1_k, 1_n \oplus 0_{m-n} \oplus 1_k, (v \oplus 1_k)\phi(w))$, and hence

$$\begin{aligned}
[p, 1_n, v] &= [1_n \oplus 0_{m-n} \oplus 1_k, 1_n \oplus 0_{m-n} \oplus 1_k, (v \oplus 1_k)\phi(w)] \\
&= \partial([1_n \oplus 0_{m-n} \oplus 1_k, (v \oplus 1_k)\phi(w), g])
\end{aligned}$$

which shows exactness at $K_0(\phi)$.

Finally, we show exactness at $K_1(\psi)$. Suppose that $\partial([1_n, u, g]) = [1_n, 1_n, u] = 0$. Find $k \geq 0$ and a partial isometry w in $M_{n+k}(\tilde{A})$ such that $\phi(w)(u \oplus 1_k)$ is homotopic to 1_{n+k} in $M_{n+k}(\tilde{B})$. Let u_t be a path of unitaries in $M_{n+k}(\tilde{B})$ with $u_0 = u$ and $u_1 = \phi(w^*)$. Let

$$h(t) = \begin{cases} g(2t) \oplus 1_k & 0 \leq t \leq 1/2 \\ \psi(u_{2t-1}) & 1/2 \leq t \leq 1 \end{cases}$$

Then $[1_n, u, g] = [1_{n+k}, u \oplus 1_k, g \oplus 1_k] = [1_{n+k}, \phi(w^*), h] = \nu_1([1_{n+k}, w^*, h])$. \square

2.8 Formulas for Δ , β_ϕ , and ∂_0

We now provide formulas for the natural isomorphisms $\Delta : K_*(\phi) \rightarrow K_*(C_\phi)$, the Bott map $\beta_\phi : K_0(\phi) \rightarrow K_1(S\phi)$, and the exponential map $\partial_0 : K_0(\gamma) \rightarrow K_1(\psi)$, where γ and ψ are as in diagram (2.3). Though all of the maps are defined for both unital and non-unital algebras, we will assume throughout this section that A and B are unital and that $\phi(1_A) = 1_B$. As is typical in K -theory, the non-unital cases can likely be dealt with via diagram chases, which we elect to forgo here for the sake of simplicity.

We begin with a useful piece of notation.

Definition 2.8.1. For a triple $(p, 1_n, v)$ in $\Gamma_0(\phi)$, choose $m \geq n$ such that p is in $M_m(A)$, and let

$$p_v(s) = w(s)^* \begin{bmatrix} 1_n & 0 \\ 0 & 0_{2m-n} \end{bmatrix} w(s)$$

where w is a path of unitaries in $M_{2m}(B)$ with $w(0) = 1_{2m}$ and

$$w(1) = \begin{bmatrix} v & 1_m - vv^* \\ 1_m - v^*v & v^* \end{bmatrix}.$$

Note that such a path w exists by part (iv) of Proposition 1.1.12.

Proposition 2.8.2. (i) Given a triple $(p, 1_n, v)$ in $\Gamma_0(\phi)$, we have

$$\Delta([p, 1_n, v]) = [(p \oplus 0_m, p_v)] - [1_n \oplus 0_{2m-n}]$$

where m and p_v are as in Definition 2.8.1.

(ii) Given a triple $(1_n, u, g)$ in $\Gamma_1(\phi)$, we have

$$\Delta([1_n, u, g]) = [(u, g)]$$

Proof. As we are assuming that A and B are unital and $\phi(1_A) = 1_B$, we make the identification

$$M_n(\tilde{C}_\phi) = \{(a, f) \in M_n(A) \oplus (C([0, 1]) \otimes M_n(B)) \mid f(1) = \phi(a) \text{ and } f(0) \in M_n(\mathbb{C})\}$$

for $n \geq 1$. For (i), notice that $(p \oplus 0_m, p_v)$ is indeed in $M_{2m}(\tilde{C}_\phi)$, and the scalar part is $p_v(0) = 1_n \oplus 0_{2m-n}$, so $[(p \oplus 0_m, p_v)] - [1_n \oplus 0_{2m-n}]$ is a well-defined element of $K_0(C_\phi)$. Consider the triple

$$\sigma = \left(\left(\left[\begin{array}{cc} p & 0 \\ 0 & 0_m \end{array} \right], p_v \right), \left[\begin{array}{cc} 1_n & 0 \\ 0 & 0_{2m-n} \end{array} \right], \left[\begin{array}{cc} 1_n & 0 \\ 0 & 0_{2m-n} \end{array} \right] w \right).$$

This is an element of $\Gamma_0(\eta_\phi)$ where η_ϕ is as in (2.5). Clearly we have $\nu_0([\sigma]) = [(p \oplus 0_m, p_v)] - [1_n \oplus 0_{2m-n}]$ and $\pi_*([\sigma]) = [p, 1_n, v]$, which completes the proof of (i).

For (ii), clearly (u, g) is in $M_n(\tilde{C}_\phi)$ and therefore $[(u, g)]$ is a well-defined element of $K_1(C_\phi)$. Define h in $C([0, 1]) \otimes M_n(\widetilde{CB})$ by $h(t) = g_t$, where $g_t(s) = g(st)$. Then $\sigma = (1_n, (u, g), h)$ is an element of $\Gamma_1(\eta_\phi)$, and $\nu_1([\sigma]) = [(u, g)]$ in $K_1(C_\phi)$ and $\pi_*([\sigma]) = [1_n, u, g]$ in $K_1(\phi)$ (notice that $\pi(h)(t) = g_t(1) = g(t)$ for all t). \square

Proposition 2.8.3. Given a triple $(p, 1_n, v)$ in $\Gamma_0(\phi)$, we have the formula $\beta_\phi([p, 1_n, v]) = [1_{2m}, u, g]$ where

$$u(t) = \exp \left(2\pi i t \begin{bmatrix} p & 0 \\ 0 & 0_m \end{bmatrix} \right) \exp \left(-2\pi i t \begin{bmatrix} 1_n & 0 \\ 0 & 0_{2m-n} \end{bmatrix} \right)$$

and

$$g(s, t) = \exp(2\pi i t p_v(s)) \exp \left(-2\pi i t \begin{bmatrix} 1_n & 0 \\ 0 & 0_{2m-n} \end{bmatrix} \right).$$

Proof. Since the Bott map is constructed as an index map, we have by (2.6) the commutative diagram

$$\begin{array}{ccc}
K_0(\phi) & \xrightarrow{\beta_\phi} & K_1(S\phi) \\
\downarrow \Delta & & \downarrow \Delta \\
K_0(C_\phi) & \xrightarrow{\beta_{C_\phi}} & K_1(C_{S\phi})
\end{array}$$

(observe that SC_ϕ and $C_{S\phi}$ are isomorphic in a canonical way). Due to the simplicity of the formula for Δ given in part (ii) of Proposition 2.8.2, the remainder of the proof amounts to using the formula for Δ given in part (i) of Proposition 2.8.2 and the formula for the ordinary Bott map β_{C_ϕ} . \square

Proposition 2.8.4. *Given a triple $(p, 1_n, v)$ in $\Gamma_0(\gamma)$, choose m and p_v as in Definition 2.8.1. Let a in $M_m(A)$ be such that $a = a^*$, $\pi_A(a) = p$, and let f in $M_{2m}(\widehat{CB})$ be such that $f(t) = f(t)^*$ for all t , $\pi_B(f) = p_v$, and $f(1) = \phi(a) \oplus 0_m$. Then we have*

$$\partial_0([p, 1_n, v]) = -[1_{2m}, \exp(2\pi i(a \oplus 0_m)), \exp(2\pi i f)]$$

Proof. The proof is almost identical to that of Proposition 2.8.3, with the commutative diagram there replaced with the commutative diagram

$$\begin{array}{ccc}
K_0(\gamma) & \xrightarrow{\partial_0} & K_1(\psi) \\
\downarrow \Delta & & \downarrow \Delta \\
K_0(C_\gamma) & \xrightarrow{\delta_0} & K_1(C_\psi)
\end{array}$$

where δ_0 is the exponential map associated to the sequence (1.2). \square

2.9 The excision theorem for factor groupoids

We now record the set-up and statement of the excision theorem in [30] in the context of factor groupoids. This could have been placed in Chapter 1, but it seems prudent to place it here since the notation for relative K -theory is established.

Suppose the hypotheses of Proposition 1.3.9 hold; that is, let G and G' be locally compact Hausdorff étale groupoids, $\pi : G \rightarrow G'$ a continuous proper surjective groupoid homomorphism, and suppose that for all u in $G^{(0)}$, the map $\pi|_{G^u} : G^u \rightarrow (G')^{\pi(u)}$ is bijective. Suppose also that G has a metric d_G yielding its topology. Let

$$H' = \{x' \in G' \mid \#\pi^{-1}(x') \neq 1\}$$

and $H = \pi^{-1}(H')$. Give H' and H the metrics

$$d_{H'}(x', y') = d_G(\pi^{-1}(x'), \pi^{-1}(y'))$$

(via the Hausdorff distance) and

$$d_H(x, y) = d_G(x, y) + d_{H'}(\pi(x), \pi(y)).$$

Definition 2.9.1 (7.7 of [30]). $\pi : G \rightarrow G'$ is *regular* if, for all x' in H' and all $\varepsilon > 0$, there is an open set $U' \subseteq G'$ with x' in U' such that if y' is in U' , either

$$d_G(\pi^{-1}(x'), \pi^{-1}(y')) < \varepsilon$$

or

$$\text{diam } \pi^{-1}(y') < \varepsilon.$$

There is also a necessary notion of “measure regularity” for π . We will not record the definition because in our cases it will follow from regularity.

Definition 2.9.2 (5.5 of [30]). Let G be a locally compact, Hausdorff, étale groupoid, and let H be a subgroupoid of G that is also locally compact, Hausdorff, and étale. Let $C \geq 1$ be a real number. We say that a subset $X \subseteq H$ which is closed in G has the *C-extension property* if, for any f in $C_c(H)$ with support in X , there exists \tilde{f} in $C_c(G)$ such that $\tilde{f}|_X = f|_X$ and $\|\tilde{f}\|_r \leq C\|f\|_r$, where the left norm is on $C_c(G)$ and the right is on $C_c(H)$.

The following will be an important tool for computing the K -theory of the factor groupoids. The assumptions are somewhat stronger than necessary, but sufficient for our purposes.

Theorem 2.9.3 (7.9, 7.18 and 7.19 of [30]). *Let $\pi : G \rightarrow G'$ be as in Proposition 1.3.9 and assume it is regular and measure regular. Then H' and H , with the metrics $d_{H'}$ and d_H , are locally compact, Hausdorff, étale groupoids with finer topologies than the relative topologies received from G' and G , respectively, and $\pi|_H : H \rightarrow H'$ also satisfies the hypotheses of Proposition 1.3.9. Moreover, if there exists a $C \geq 1$ such that every subset $X \subseteq H$ which is closed in G has the C -extension property, then*

$$K_*(C_r^*(G'), C_r^*(G)) \cong K_*(C_r^*(H'), C_r^*(H)).$$

The moral of Theorem 2.9.3 is that in order to compute the relative K -theory of the inclusion $C_r^*(G') \subseteq C_r^*(G)$, one need only pay attention to the parts of G and G' where π is not one-to-one. In the examples we will encounter, the inclusion $C_r^*(H') \subseteq C_r^*(H)$ will be simpler than the inclusion $C_r^*(G') \subseteq C_r^*(G)$ in the sense that $C_r^*(H)$ will be either a direct product or tensor product, and the “pieces” will be $C_r^*(H')$. For this reason, $K_*(C_r^*(H'), C_r^*(H))$ will be easier to compute than $K_*(C_r^*(G'), C_r^*(G))$.

It should be pointed out, however, that the isomorphism in the final conclusion of Theorem 2.9.3 is somewhat difficult to describe. It involves representing $C_r^*(G)$

and $C_r^*(H)$ on a suitable Hilbert space \mathcal{H} in such a way that $C_r^*(G)$ acts as multipliers of $C_r^*(H)$, and realizing the factor groupoid C^* -algebras $C_r^*(G')$ and $C_r^*(H')$ as those elements in $C_r^*(G)$ and $C_r^*(H)$ (respectively) that lie in the kernel of a certain $*$ -derivation $\delta : C_r^*(G) + C_r^*(H) \rightarrow \mathcal{B}(\mathcal{H})$. Then, with the inclusion maps $i : C_r^*(H) \rightarrow C_r^*(G) + C_r^*(H)$ and $j : C_r^*(G) \rightarrow C_r^*(G) + C_r^*(H)$, we draw the diagram

$$\begin{array}{ccc}
 K_*(C_r^*(G'), C_r^*(G)) & \xrightarrow{j_*} & K_*(\ker(\delta), C_r^*(G) + C_r^*(H)) \\
 & \searrow \text{---} & \uparrow i_* \\
 & & K_*(C_r^*(H'), C_r^*(H))
 \end{array}$$

The map i_* is always an isomorphism (even without regularity of π and the C -extension property), and the bulk of the work in [30] is in showing that $(i_*)^{-1} \circ j_*$ (the dashed arrow) is an isomorphism under the additional assumptions. Consequently, even if one has a good idea of what the elements of $K_*(C_r^*(H'), C_r^*(H))$ look like, determining which elements in $K_*(C_r^*(G'), C_r^*(G))$ they correspond to under the inverse of an induced map can be tricky, and may involve some guesswork. One approach is, given x in $K_*(C_r^*(H'), C_r^*(H))$, to find y in $K_*(C_r^*(G'), C_r^*(G))$ with $j_*(y) - i_*(x)$ equal to something trivial, such as the class of an elementary triple. It is here that an advantage of the Karoubian portrait becomes visible: the simplicity of the presentation of the elements in the relative groups makes finding such a y easier.

2.10 Summary and examples

We now discuss some simple examples to begin to illustrate the utility of Theorem 2.1.1 and Theorem 2.1.2.

Example 2.10.1. The following four examples are rather standard.

- (i) If $K_*(B) = 0$, the maps ν_j in Theorem 2.1.1 give isomorphisms $K_j(\phi) \cong K_j(A)$ for $j = 0, 1$. In particular, if $A \subseteq \mathcal{B}(\mathcal{H})$ where \mathcal{H} is an infinite dimensional Hilbert space and $\phi : A \rightarrow \mathcal{B}(\mathcal{H})$ is the inclusion map, the K -theory of A is the same as the relative K -theory as a subalgebra of $\mathcal{B}(\mathcal{H})$.
- (ii) If $K_*(A) = 0$, the maps μ_j in Theorem 2.1.1 give isomorphisms $K_j(\phi) \cong K_{1-j}(B)$ for $j = 0, 1$.
- (iii) If I is an ideal in A and $\phi : A \rightarrow A/I$ is the quotient map, then $K_j(\phi) \cong K_j(I)$ for $j = 0, 1$. This is the standard “strong excision” result from topological K -theory, and it follows immediately from Corollary 2.5.13. See also 5.4.2 of [4].

(iv) If I is an ideal in A , then the boundary maps in Theorem 2.1.2 associated to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \hookrightarrow & A & \longrightarrow & A/I & \longrightarrow & 0 \\ & & \downarrow & & \parallel & & \downarrow & & \\ 0 & \longrightarrow & A & \xlongequal{\quad} & A & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

give isomorphisms $K_j(I, A) \cong K_{1-j}(A/I)$ for $j = 0, 1$.

Example 2.10.2. Let D be any C^* -algebra, $B = M_2(D)$, and

$$A = \left\{ \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \mid d_1, d_2 \in D \right\}.$$

Since $K_*(A) \cong K_*(D) \oplus K_*(D)$ and $K_*(B) \cong K_*(D)$, we may write the six-term exact sequence of Theorem 2.1.1 as

$$\begin{array}{ccccc} K_1(D) & \longrightarrow & K_0(A, B) & \longrightarrow & K_0(D) \oplus K_0(D) \\ \uparrow \phi_* & & & & \downarrow \phi_* \\ K_1(D) \oplus K_1(D) & \longleftarrow & K_1(A, B) & \longleftarrow & K_0(D) \end{array}$$

The vertical maps are both $\phi_*(g, h) = g + h$. Exactness implies that $K_*(A, B) \cong \ker \phi_* \cong K_*(D)$.

A slight variant of this example that has seen a great deal of attention in constructions with subgroupoids, such as in [9] and [26], is as follows. Let \mathcal{H} be a separable Hilbert space of dimension at least 2, and \mathcal{M} a closed subspace such that $\mathcal{M} \neq \{0\}$ and $\mathcal{M} \neq \mathcal{H}$. Let $A = \mathcal{K}(\mathcal{M}) \oplus \mathcal{K}(\mathcal{M}^\perp)$, where $\mathcal{K}(\mathcal{M})$ is the C^* -algebra of compact operators on \mathcal{M} . We regard A as the subalgebra of $B = \mathcal{K}(\mathcal{H})$ consisting of operators that leave \mathcal{M} and \mathcal{M}^\perp invariant. Then $K_0(A, B) \cong \mathbb{Z}$ and $K_1(A, B) = 0$. If we fix a unit vector ξ in \mathcal{M} , a unit vector η in \mathcal{M}^\perp , and a partial isometry v in B with source subspace $\text{span}\{\xi\}$ and range subspace $\text{span}\{\eta\}$, the group $K_0(A, B)$ is generated by the class of the triple (v^*v, vv^*, v) .

Example 2.10.3. Let D be any C^* -algebra and consider $A = D$ as a subalgebra of $B = D \oplus D$ via the embedding $d \mapsto (d, d)$. The six-term exact sequence of Theorem 2.1.1 becomes

$$\begin{array}{ccccc} K_1(D) \oplus K_1(D) & \longrightarrow & K_0(A, B) & \longrightarrow & K_0(D) \\ \uparrow \phi_* & & & & \downarrow \phi_* \\ K_1(D) & \longleftarrow & K_1(A, B) & \longleftarrow & K_0(D) \oplus K_0(D) \end{array}$$

This time the vertical maps are $\phi_*(g) = (g, g)$, which are injective, whence exactness implies $K_0(A, B) \cong K_1(D)$ and $K_1(A, B) \cong K_0(D)$. In the case that $D = \mathcal{K}$, the group $K_1(A, B) \cong \mathbb{Z}$ is generated by the class of the triple (p, p, g) , where p is a rank one projection in \mathcal{K} and $g(s) = (e^{2\pi is}p, p)$. Observe that we do not need to consider the unit in the unitization $\tilde{\mathcal{K}}$ to describe the group $K_1(A, B)$.

Example 2.10.4. Define the *-homomorphism $\phi : C([0, 1]) \rightarrow M_2(\mathbb{C})$ by $\phi(f) = f(0) \oplus f(1)$. Since $K_0(C([0, 1])) \cong K_0(M_2(\mathbb{C})) \cong \mathbb{Z}$ and $K_1(C([0, 1])) = K_1(M_2(\mathbb{C})) = 0$, we may write the six-term exact sequence of Theorem 2.1.1 as

$$\begin{array}{ccccc} 0 & \longrightarrow & K_0(\phi) & \longrightarrow & \mathbb{Z} \\ \uparrow & & & & \downarrow \phi_* \\ 0 & \longleftarrow & K_1(\phi) & \xleftarrow{\mu_1} & \mathbb{Z} \end{array}$$

where $\phi_*(n) = 2n$ because the class of $1_{C(X)}$ gets sent to the class of a rank two projection. It follows that $K_0(\phi) = 0$ and $K_1(\phi) \cong \mathbb{Z}/2\mathbb{Z}$. Since the class of the projection $1 \oplus 0$ is not in the image of ϕ_* , the nontrivial element of $K_1(\phi)$ is given by triple $\mu_1([1 \oplus 0]) = [1_2, 1_2, g]$, where

$$g(s) = \begin{bmatrix} e^{2\pi is} & 0 \\ 0 & 1 \end{bmatrix}$$

Example 2.10.5. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_0((0, 1)) & \xrightarrow{\iota} & C(\mathbb{T}) & \xrightarrow{\pi_1} & \mathbb{C} & \longrightarrow & 0 \\ & & \parallel & & \downarrow \phi & & \downarrow \gamma & & \\ 0 & \longrightarrow & C_0((0, 1)) & \hookrightarrow & C([0, 1]) & \xrightarrow{\pi_2} & \mathbb{C} \oplus \mathbb{C} & \longrightarrow & 0 \end{array}$$

where $\iota(f)(e^{2\pi it}) = f(t)$ for $0 < t < 1$ and $\iota(f)(1) = 0$, $\pi_1(f) = f(1)$, $\pi_2(f) = (f(0), f(1))$, $\phi(f)(t) = f(e^{2\pi it})$, and $\gamma(z) = (z, z)$. Since ϕ and γ are injective, we will identify $C(\mathbb{T})$ and \mathbb{C} with their images in $C([0, 1])$ and $\mathbb{C} \oplus \mathbb{C}$, respectively. Theorem 2.1.2 gives the six-term exact sequence

$$\begin{array}{ccccc} 0 & \longrightarrow & K_0(C(\mathbb{T}), C([0, 1])) & \xrightarrow{\pi_*} & K_0(\mathbb{C}, \mathbb{C} \oplus \mathbb{C}) \\ \uparrow & & & & \downarrow \\ K_1(\mathbb{C}, \mathbb{C} \oplus \mathbb{C}) & \xleftarrow{\pi_*} & K_1(C(\mathbb{T}), C([0, 1])) & \xleftarrow{\pi_*} & 0 \end{array}$$

So that the maps π_* are isomorphisms. Example 2.10.3 tells us that $K_0(\mathbb{C}, \mathbb{C} \oplus \mathbb{C}) = 0$ and $K_1(\mathbb{C}, \mathbb{C} \oplus \mathbb{C}) \cong \mathbb{Z}$, the latter generated by $(1, 1, g)$, where $g(s) = (e^{2\pi is}, 1)$. It is

not difficult to find an element of $K_1(C(\mathbb{T}), C([0, 1]))$ that maps to the class of $(1, 1, g)$ (and is therefore a generator): take the class of the triple $(1, z, h)$, where $h(s) = f_s$ and $f_s(t) = e^{2\pi ist}$ for $0 \leq s, t \leq 1$.

The previous example (and diagram used in the proof of Theorem 2.1.3) is clearly a specific instance of a more general phenomenon, that if A has an ideal I such that ϕ is injective on I and $\phi(I)$ is an ideal in B , then the relative K -theory of ϕ is the same as the relative K -theory when dropped to the quotients A/I and $B/\phi(I)$.

The final two examples illustrate that the boundary maps in Theorem 2.1.2 need not be trivial.

Example 2.10.6. Consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_0(\mathbb{R}^2) & \longrightarrow & C(\mathbb{D}) & \xrightarrow{\pi} & C(\mathbb{T}) \longrightarrow 0 \\ & & \downarrow & & \downarrow \phi & & \downarrow \gamma \\ 0 & \longrightarrow & 0 & \longrightarrow & C([0, 1]) & \xlongequal{\quad} & C([0, 1]) \longrightarrow 0 \end{array}$$

where $\mathbb{D} = \{z \in \mathbb{C} \mid |z| \leq 1\}$ and \mathbb{R}^2 is identified with $\mathbb{D}^\circ = \{z \in \mathbb{C} \mid |z| < 1\}$. The $*$ -homomorphisms are $\pi(f) = f|_{\mathbb{T}}$, $\gamma(f)(t) = f(e^{2\pi it})$, and $\phi = \gamma \circ \pi$. We have $K_0(\phi) = K_1(\phi) = 0$ since $K_1(C([0, 1])) = K_1(C(\mathbb{D})) = 0$ (see Corollary 2.4.4) and the induced map $\phi_* : K_0(C(\mathbb{D})) \rightarrow K_0(C([0, 1]))$ is an isomorphism. The six-term exact sequence of Theorem 2.1.2 becomes

$$\begin{array}{ccccc} K_0(C_0(\mathbb{R}^2)) & \longrightarrow & 0 & \longrightarrow & K_0(C(\mathbb{T}), C([0, 1])) \\ \uparrow \partial_1 & & & & \downarrow \partial_0 \\ K_1(C(\mathbb{T}), C([0, 1])) & \longleftarrow & 0 & \longleftarrow & K_1(C_0(\mathbb{R}^2)) \end{array}$$

By Example 2.10.5, $K_1(C(\mathbb{T}), C([0, 1])) \cong \mathbb{Z}$ is generated by the class of $(1, z, g)$ where $g(s) = f_s$ and $f_s(t) = e^{2\pi ist}$. Using the notation in Definition 2.5.3, let $l = 1$,

$$w = \begin{bmatrix} z & -(1 - |z|^2)^{1/2} \\ (1 - |z|^2)^{1/2} & \bar{z} \end{bmatrix}$$

and $h = g$. Then

$$\partial_1([1, z, g]) = \left[\begin{bmatrix} |z|^2 & z(1 - |z|^2)^{1/2} \\ \bar{z}(1 - |z|^2)^{1/2} & 1 - |z|^2 \end{bmatrix} \right] - \left[\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right]$$

i.e., the class of the Bott projection.

Example 2.10.7. Let D be the diagonal matrices in $M_2(\mathbb{C})$. Consider the diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & C_0((0, 1)) & \hookrightarrow & C([0, 1]) & \xrightarrow{\pi} & D \longrightarrow 0 \\
& & \downarrow & & \downarrow \phi & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & M_2(\mathbb{C}) & \xlongequal{\quad} & M_2(\mathbb{C}) \longrightarrow 0
\end{array}$$

Where $\phi(f) = \pi(f) = \begin{bmatrix} f(0) & 0 \\ 0 & f(1) \end{bmatrix}$. We have $K_0(C_0((0, 1))) = 0$ and $K_1(D; M_2(\mathbb{C})) = 0$, the latter by Example 2.10.2. The six-term exact sequence of Theorem 2.1.2 becomes

$$\begin{array}{ccccc}
0 & \longrightarrow & K_0(\phi) & \longrightarrow & K_0(D, M_2(\mathbb{C})) \\
\uparrow & & & & \downarrow \partial_0 \\
0 & \longleftarrow & K_1(\phi) & \longleftarrow & K_1(C_0((0, 1)))
\end{array}$$

By Example 2.10.4, $K_0(\phi) = 0$ and $K_1(\phi) \cong \mathbb{Z}/2\mathbb{Z}$, with the nontrivial element in $K_1(\phi)$ given by the class of the triple $(1, 1, g)$, where

$$g(s) = \begin{bmatrix} e^{2\pi i s} & 0 \\ 0 & 1 \end{bmatrix}$$

The map ∂_0 is therefore injective and takes a generator of $K_0(D, M_2(\mathbb{C})) \cong \mathbb{Z}$ to twice a generator of $K_1(C_0((0, 1))) \cong \mathbb{Z}$. More concretely,

$$\partial_0 \left(\left(\left[\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right] \right) = -[z^2].$$

Chapter 3

Factor groupoids of AF-groupoids: noncommutative Cantor functions

In this chapter we introduce our first method of producing factor groupoids. We begin with a motivating example that is essentially the bedrock of the entire construction. Consider the function

$$\phi : \{0, 1\}^{\mathbb{N}} \rightarrow \mathbb{T}$$

defined by $\phi(\{j_n\}) = \exp(2\pi i \sum_{n=1}^{\infty} j_n 2^{-n})$, which is nothing but the Cantor ternary function in a rather uncommon guise. We may view this function geometrically as “closing all the gaps” in the Cantor set to form the circle. The space $\{0, 1\}^{\mathbb{N}}$ can be identified with the path space of the Bratteli diagram with one vertex at each level and two edges at each level, and thus may be endowed with the tail-equivalence relation R_E and given the étale topology described in Chapter 1. If we let $T = (\phi \times \phi)(R_E)$ and endow T with the quotient topology from R_E , T is a familiar equivalence relation on \mathbb{T} with an equally familiar étale topology, namely,

$$T = \{(w, z) \in \mathbb{T} \times \mathbb{T} \mid w = e^{2\pi i \theta} z \text{ for some } \theta \in \mathbb{Z}[\frac{1}{2}]\}$$

with basic open sets $U_{W, \theta} = \{(z, e^{2\pi i \theta} z) \mid z \in W\}$, where W is open in \mathbb{T} and θ is in $\mathbb{Z}[\frac{1}{2}]$. T is isomorphic, as a topological groupoid, to the the group action $\mathbb{T} \times \Gamma$, where the (discrete) group $\Gamma = \{e^{2\pi i \theta} \mid \theta \in \mathbb{Z}[\frac{1}{2}]\}$ acts on \mathbb{T} by multiplication. The C*-algebra $C_r^*(T)$ is then recognizable as the Bunce–Deddens algebra of type 2^∞ , which is typically constructed via weighted shifts, or as an inductive limit of circle algebras. This puts the K -theory of $C_r^*(T)$ within reach: it is known that

$$K_0(C_r^*(T)) \cong \mathbb{Z}[\frac{1}{2}] \quad K_1(C_r^*(T)) \cong \mathbb{Z}$$

where $K_0(C_r^*(T))$ has the usual order from \mathbb{R} . The key observation to make is that, while the K_0 -group is unchanged passing from R_E to T , the K_1 -group becomes non-trivial, essentially due to the fact that we passed from the Cantor set to the circle;

the generator of $K_1(C_r^*(T))$ is given by the class of the function z on the unit space. The goal of this chapter is to generalize this construction, replacing $\{0, 1\}^{\mathbb{N}}$ with the path space of a Bratteli diagram (V, E) , and the map ϕ with a quotient map ρ whose fibres depend on two embedded copies of a second Bratteli diagram (W, F) in (V, E) . In doing so, we obtain a method of producing groupoid models for a sizeable collection of classifiable C^* -algebras, and these groupoids have interesting unit spaces whose connected components are either points or circles. Depending on the diagrams (V, E) and (W, F) , the unit spaces may possess some self-similar or fractal-like structure.

3.1 The main results

The construction begins with the same set-up as in [29]: two Bratteli diagrams (V, E) and (W, F) and two graph embeddings of (W, F) into (V, E) satisfying some specified conditions. The goal is to create a groupoid whose ordered K_0 -group is the dimension group of (V, E) , and whose K_1 -group is the dimension group of (W, F) . While the construction in [29] enlarges the usual tail equivalence relation R_E on the path space X_E of (V, E) to include pairs of paths in the two embeddings (which creates a new groupoid over the same unit space), we actually collapse these paths in X_E and pass R_E through the resulting quotient map, creating an equivalence relation on an entirely new space, and we denote the space and equivalence relation by X_ξ and R_ξ , respectively. It should also be noted that, unlike in [29], obtaining the étale topology and amenability on the new groupoids is straightforward (these properties essentially carry over right from R_E). It is the new unit spaces, rather than the new groupoids, that require more attention in order to describe them. We address this in section 3.2.

Theorem 3.1.1. *Let (V, E) and (W, F) be two Bratteli diagrams satisfying the embedding conditions described in section 3.2. There exists a quotient space X_ξ of X_E that satisfies the following.*

- (i) X_ξ is compact and metrizable.
- (ii) Every connected component of X_ξ is either a single point or homeomorphic to \mathbb{T} .
- (iii) X_ξ has covering dimension one.

Moreover, the resulting factor groupoid R_ξ of R_E satisfies the following.

- (i) R_ξ is second-countable, locally compact, Hausdorff, étale, and principal with dynamic asymptotic dimension zero.
- (ii) If R_E is minimal, then so is R_ξ , and hence $C_r^*(R_\xi)$ is classifiable.
- (iii) $K_0(C_r^*(R_\xi))$ is order isomorphic to $K_0(C_r^*(R_E))$.

(iv) $K_1(C_r^*(R_\xi))$ is isomorphic to $K_0(C_r^*(R_F))$.

(v) The map $\tau \mapsto \tau \circ \alpha$, where τ is a tracial state on $C_r^*(R_E)$ and α is as in Proposition 1.3.9, is an affine homeomorphism from $T(C_r^*(R_E))$ to $T(C_r^*(R_\xi))$.

While the order isomorphism between the K_0 -groups in Theorem 3.1.1 is induced by the inclusion $C_r^*(R_\xi) \subseteq C_r^*(R_E)$, the K_1 -group appears to receive its structure from the unit space of R_ξ , whose connected components are either points or circles. Indeed, the isomorphism between $K_1(C_r^*(R_\xi))$ and $K_0(C_r^*(R_F))$ is essentially given by a Bott map (the map μ_1 in Theorem 2.1.1), and the elements of the group $K_0(C_r^*(R_F))$ seem to correspond to winding numbers on (at least a portion of) the space X_ξ . It is interesting to note that these C^* -algebras are isomorphic (by classification considerations) to those constructed in [29] using subgroupoid methods, but the two methods produce non-isomorphic Cartan subalgebras, as the unit spaces are not homeomorphic.

From Theorem 3.1.1 we obtain the following result of prescribed K -theory.

Corollary 3.1.2. *Let G_0 be a simple acyclic dimension group and G_1 a countable torsion-free abelian group. Then there exist Bratteli diagrams (V, E) and (W, F) satisfying the embedding conditions described in section 4.2 such that the resulting R_ξ from Theorem 3.1.1 has $K_0(C_r^*(R_\xi)) \cong G_0$ as ordered groups with order unit and $K_1(C_r^*(R_\xi)) \cong G_1$.*

The proof is essentially to find Bratteli diagrams (V, E) and (W, F) whose dimension groups are G_0 and G_1 , respectively (using Proposition 1.3.18 on G_1). By telescoping and symbol splitting the diagram (V, E) , one can arrange that the embeddings described in section 3 exist without altering the dimension groups. Details may be found in [29].

Admittedly, the collection of Elliott invariants that are obtainable via Theorem 3.1.1 and Corollary 3.1.2 is not very extensive, as we are restricted to K -groups that are dimension groups. However, the simplicity of the construction should not be overlooked; indeed, the groupoids R_ξ are obtained by passing an AF-groupoid through what is essentially a generalized Cantor ternary function, and the desirable properties of R_ξ (second-countable, locally compact, Hausdorff, étale, principal, dynamic asymptotic dimension zero) are inherited directly from R_E through the quotient map. In contrast, endowing the groupoid R from [29] with an étale topology is a subtle issue. This raises the question as to what K -theory data may be obtained with more complicated factor maps, or with additional structure(s) such as twists.

We also obtain a result on AF-embeddability. We say that a C^* -algebra is *AF-embeddable* if it can be embedded into an AF-algebra. There has been a considerable amount of research conducted into determining when a C^* -algebra is AF-embeddable, initiated by the result of Pimsner and Voiculescu that every irrational rotation algebra is AF-embeddable [23]. Possibly the most substantial recent result is the following,

due to Schafhauser [33]: every separable, exact C^* -algebra satisfying the Universal Coefficient Theorem (UCT) and possessing a faithful, amenable trace is AF-embeddable. Though the following result is weaker, it seems worth stating because it follows immediately from Theorem 3.1.1 and Corollary 3.1.2, and because the embedding is given concretely by a factor map.

Corollary 3.1.3. *Suppose that A is a classifiable C^* -algebra, $(K_0(A), K_0(A)^+, [1_A])$ is a simple, acyclic dimension group, and $K_1(A)$ is a countable, torsion-free abelian group. Then A is AF-embeddable.*

3.2 The spaces X_ξ

In this section we describe the equivalence relation, denoted \sim_ξ , on the path space X_E of a Bratteli diagram (V, E) that results in the desired quotient space of X_E .

Let (V, E) and (W, F) be two Bratteli diagrams, (V, E) with full edge connections, such that there exist two graph embeddings ξ^0 and ξ^1 of (W, F) into (V, E) . We regard ξ^0 and ξ^1 as functions on both W and F . More precisely, for both $j = 0, 1$, the following hold.

- (i) $\xi^j(w)$ is in V_n for all w in W_n and all $n \geq 0$.
- (ii) $\xi^j(f)$ is in E_n for all f in F_n and all $n \geq 1$.
- (iii) $i(\xi^j(f)) = \xi^j(i(f))$ and $t(\xi^j(f)) = \xi^j(t(f))$ for all f in F .
- (iv) $\xi^j(f) \neq \xi^j(f')$ for all $f \neq f'$ in F .

We also require that

- (v) $\xi^0(w) = \xi^1(w)$ for all w in W .
- (vi) $\xi^0(F) \cap \xi^1(F) = \emptyset$.

Due to (v), the functions $\xi^0, \xi^1 : W \rightarrow V$ are identical, so we may denote them both by ξ . Observe that (iii) and (v) imply that, for any f in F , $\xi^0(f)$ and $\xi^1(f)$ have the same initial vertices and terminal vertices. By (vi), they are distinct.

Definition 3.2.1. The equivalence relation \sim_ξ on X_E is defined as follows. For $x = (x_1, x_2, x_3, \dots)$ in X_E , suppose that there is a j in $\{0, 1\}$, an $n_0 \geq 0$, and edges z_n in F_n such that $x_n = \xi^j(z_n)$ for $n \geq n_0 + 1$. Moreover, suppose n_0 is the least integer with this property.

- (i) If $n_0 = 0$, then

$$x \sim_\xi (\xi^{1-j}(z_1), \xi^{1-j}(z_2), \xi^{1-j}(z_3), \dots)$$

(ii) If $n_0 \geq 1$ and x_{n_0} is not in $\xi^{1-j}(F)$, then

$$x \sim_\xi (x_1, x_2, \dots, x_{n_0}, \xi^{1-j}(z_{n_0+1}), \xi^{1-j}(z_{n_0+2}), \dots)$$

(iii) If $n_0 \geq 1$ and there is some f in F such that $x_{n_0} = \xi^{1-j}(f)$, then

$$x \sim_\xi (x_1, x_2, \dots, x_{n_0-1}, \xi^j(f), \xi^{1-j}(z_{n_0+1}), \xi^{1-j}(z_{n_0+2}), \dots)$$

If none of the above occurs for x , then $x \sim_\xi x$ only. If $x \sim_\xi x'$ and $x \neq x'$, we will refer to the edge level E_m , where m is the least integer such that $x_m \neq x'_m$, as the *splitting level* of the equivalence class $\{x, x'\}$. We denote the quotient space X_E / \sim_ξ by X_ξ and let $\rho : X_E \rightarrow X_\xi$ denote the quotient map.

Observe that each equivalence class under \sim_ξ consists of either one or two infinite paths in X_E . Speaking a bit more loosely, each path in X_E that has edges eventually in $\xi^0(F)$ is “partnered up” with a corresponding path with edges eventually in $\xi^1(F)$, and vice versa. Every other path is in an equivalence class on its own.

If the two embeddings ξ^0 and ξ^1 cover every edge in (V, E) , then the quotient space has a simple description, as the following proposition shows.

Proposition 3.2.2. *If $E = \xi^0(F) \cup \xi^1(F)$, then X_E is homeomorphic to $X_F \times \{0, 1\}^\mathbb{N}$, and X_ξ is homeomorphic to $X_F \times \mathbb{T}$.*

Proof. Define the map $\phi : \{0, 1\}^\mathbb{N} \rightarrow \mathbb{T}$ by $\phi(\{j_n\}) = \exp(2\pi i \sum_{n=1}^\infty j_n 2^{-n})$ as in the discussion at the beginning of this chapter. Notice that two distinct sequences in $\{0, 1\}^\mathbb{N}$ have the same image under ϕ if and only if they are $(0, 0, 0, \dots)$ and $(1, 1, 1, \dots)$, or of the form

$$(j_1, j_2, \dots, j_m, 1, 0, 0, 0, 0, \dots) \text{ and } (j_1, j_2, \dots, j_m, 0, 1, 1, 1, 1, \dots)$$

By our assumption and properties (iv) and (vi) of the embeddings, each x in X_E is uniquely determined by a path z_x in X_F and a sequence $j_x = \{j_n\}$ in $\{0, 1\}^\mathbb{N}$ with $x_n = \xi^{j_n}((z_x)_n)$ for all n . This gives a canonical continuous bijection from X_E to $X_F \times \{0, 1\}^\mathbb{N}$, hence a homeomorphism. The composition of this homeomorphism with $\text{id}_{X_F} \times \phi$ is a quotient map whose fibres are identical to the fibres of ρ . \square

If the inclusion $E \supseteq \xi^0(F) \cup \xi^1(F)$ is proper, describing the space X_ξ takes a bit more work, and it may have a bit more self-similar structure. Over the next few results, we provide a complete abstract description of the spaces based on the diagram (V, E) and the location of the embeddings of (W, F) inside (V, E) .

Definition 3.2.3. Let $p = (p_1, p_2, \dots, p_n)$ be a finite path in (V, E) . Denote

$$D_j(p) = \{x \in C(p) \mid x_k \in \xi^j(F) \text{ for all } k \geq n+1\}$$

for $j = 0, 1$.

It is not difficult to see that such a set $D_j(p)$ is closed: if x is not in $D_j(p)$, then there is an integer $k \geq n + 1$ with x_n not in $\xi^j(F)$. Then the cylinder set $C(x_1, x_2, \dots, x_n)$ is an open neighbourhood of x disjoint from $D_j(p)$.

In the following lemma, we describe some open sets in X_E that remain open when passed to X_ξ . If a map $f : X \rightarrow Y$ is surjective, we call a subset V of X *saturated* if there is a subset U of Y with $V = f^{-1}(U)$. It follows directly from the definition of the quotient topology that open sets in the quotient are precisely the images of open saturated sets in the ambient space.

Lemma 3.2.4. *Let $p = (p_1, p_2, \dots, p_n)$ be a finite path in (V, E) .*

- (i) *If p_n is not in $\xi^0(F) \cup \xi^1(F)$, then $C(p)$ is clopen and saturated (with respect to the map ρ).*
- (ii) *If p_n is in $\xi^0(F) \cup \xi^1(F)$, then*

$$U_p = C(p) - (D_0(p) \cup D_1(p))$$

is open and saturated.

- (iii) *Suppose that $k \geq m + 1$ and p and q are two finite paths either of the form*

$$p = (\xi^0(z_1), \xi^0(z_2), \dots, \xi^0(z_k))$$

$$q = (\xi^1(z_1), \xi^1(z_2), \dots, \xi^1(z_k))$$

or of the form

$$p = (x_1, x_2, \dots, x_{m-1}, x_m, \xi^0(z_{m+1}), \xi^0(z_{m+2}), \dots, \xi^0(z_k))$$

$$q = (x_1, x_2, \dots, x_{m-1}, x'_m, \xi^1(z_{m+1}), \xi^1(z_{m+2}), \dots, \xi^1(z_k))$$

where x_m is not in $\xi^0(F)$, x'_m is not in $\xi^1(F)$, $x_m = x'_m$ if x_m is not in $\xi^1(F)$, and $x'_m = \xi^0(f)$ if $x_m = \xi^1(f)$ for some f in F . Then

$$V_{p,q} = (C(p) - D_1(p)) \cup (C(q) - D_0(q))$$

is open and saturated.

Proof. (i) If x is in $C(p)$, $x \neq x'$, and $x \sim_\xi x'$, then the splitting level of $\{x, x'\}$ must be past level E_n since p_n is not in $\xi^0(F) \cup \xi^1(F)$, so x' is in $C(p)$.

- (ii) If x is in U_p , $x \neq x'$, and $x \sim_\xi x'$, the removal of the sets $D_j(p)$ for $j = 0, 1$ forces the splitting level of $\{x, x'\}$ to be past level E_n , so x and x' coincide on the first n edges and x' is thus in U_p . Since $D_j(p)$ is closed, U_p is open.

- (iii) The proof that $V_{p,q}$ is open is analogous to that for (ii). If $x \in C(p) - D_1(p)$, $x \sim_\xi x'$, and the splitting level of $\{x, x'\}$ is past level k , then x' is in $C(p) - D_1(p)$ similarly as in (ii). Otherwise, the splitting level must be either E_m or E_{m+1} , but then x' must be in $C(q) - D_0(q)$. \square

Definition 3.2.5. (i) Let \mathcal{P} denote all finite paths $p = (p_1, p_2, \dots, p_n)$ in (V, E) such that p_n is not in $\xi^0(F) \cup \xi^1(F)$ but $t(p_n)$ is in $\xi(W)$.

- (ii) For p in \mathcal{P} , define

$$C_\xi(p) = \{x \in C(p) \mid x_k \in \xi^0(F) \cup \xi^1(F) \text{ for all } k \geq n+1\}$$

To simplify notation, we will assume that there is an “empty path” p_0 in \mathcal{P} with the property that $C_\xi(p_0) = \{x \in X_E \mid x_k \in \xi^0(F) \cup \xi^1(F) \text{ for all } k \geq 1\}$.

- (iii) For $w \in W_n$, define

$$X_F^{(w)} = \{z = (z_{n+1}, z_{n+2}, \dots) \mid z_k \in F_k, t(z_k) = i(z_{k+1}) \text{ for all } k \geq n+1, \text{ and } i(z_{n+1}) = w\}$$

that is, all infinite paths in (W, F) that start at w .

Note that $C_\xi(p)$ is not equal to $D_0(p) \cup D_1(p)$, since edges in paths in $C_\xi(p)$ may alternate between the two embeddings. Regarding (iii) above, by fixing any finite path $p = (p_1, p_2, \dots, p_n)$ in (W, F) with $t(p_n) = w$, we may identify $X_F^{(w)}$ with $C(p)$ by associating z with pz (the concatenation of p and z) and endow $X_F^{(w)}$ with the metric it receives from this identification. This makes $X_F^{(w)}$ a totally disconnected compact metric space.

Lemma 3.2.6. *Let $p = (p_1, p_2, \dots, p_n)$ be a finite path in \mathcal{P} with $t(p_n) = \xi(w)$.*

- (i) $C_\xi(p)$ is closed and saturated.
- (ii) $C_\xi(p)$ is homeomorphic to $X_F^{(w)} \times \{0, 1\}^{\mathbb{N}}$ and $\rho(C_\xi(p))$ is homeomorphic to $X_F^{(w)} \times \mathbb{T}$.
- (iii) If $p \neq q$ in \mathcal{P} , there are disjoint clopen saturated sets U and V in X_E such that $C_\xi(p) \subseteq U$ and $C_\xi(q) \subseteq V$; in particular, $\rho(C_\xi(p))$ and $\rho(C_\xi(q))$ are disjoint.
- (iv) For every x in X_E and $\delta > 0$, there is an open saturated subset U of X_E such that $\rho^{-1}(\rho(x)) \subseteq U \subseteq B(\rho^{-1}(\rho(x)), \delta)$.
- (v) If x is in $X_E - \bigcup_{q \in \mathcal{P}} C_\xi(q)$, then U from part (iv) can be chosen to be clopen; it follows that $\{\rho(x)\}$ is a connected component of X_ξ in this case.

Proof. (i) For $m \geq n + 1$, let A_m be the finite union of all cylinder sets of the form

$$C(p_1, p_2, \dots, p_n, q_{n+1}, q_{n+2}, \dots, q_m)$$

where q_k is in $\xi^0(F) \cup \xi^1(F)$ for $n + 1 \leq k \leq m$. Each A_m is closed and $C_\xi(p) = \bigcap_{m=n+1}^{\infty} A_m$. Suppose that x is in $C_\xi(p)$, $x \neq x'$, and $x \sim_\xi x'$. Since p_n is not in $\xi^0(F) \cup \xi^1(F)$, the splitting level of $\{x, x'\}$ is past level n , so x' is in $C(p)$. But since x_k is in $\xi^0(F) \cup \xi^1(F)$ for all $k \geq n + 1$, we must also have x'_k in $\xi^0(F) \cup \xi^1(F)$ for all $k \geq n + 1$, hence x' is in $C_\xi(p)$.

(ii) Each x in $C_\xi(p)$ is uniquely determined by a path in $X_F^{(w)}$ and a sequence in $\{0, 1\}^{\mathbb{N}}$, and the proof then proceeds analogously to that of Proposition 3.2.2.

(iii) If $C(p) \cap C(q) = \emptyset$, then $U = C(p)$ and $V = C(q)$ suffice (they are saturated by Lemma 3.2.4). If $C(p) \subseteq C(q)$, take $U = C(p)$ and $V = X_E - C(p)$.

(iv) We consider two cases.

(a) Suppose that the \sim_ξ equivalence class of x is $\{x\}$. Choose $k \geq 1$ such that $2^{-k} < \delta$, let $p = (x_1, \dots, x_k)$, and let $U = U_p$ as in part (ii) of Lemma 3.2.4. Observe that x is not in $D_0(p) \cup D_1(p)$ because otherwise $\rho^{-1}(\rho(x))$ would consist of two points.

(b) If $\rho^{-1}(\rho(x)) = \{x, x'\}$ and E_m is the splitting level, choose $k \geq m$ with $2^{-k} < \delta$, let $p = (x_1, \dots, x_k)$ and $q = (x'_1, \dots, x'_k)$, and let $U = V_{p,q}$ from part (iii) of Lemma 3.2.4.

(v) There are infinitely many edges x_{n_1}, x_{n_2}, \dots of x that are not in $\xi^0(F) \cup \xi^1(F)$. Choose $k \geq 1$ such that $2^{-n_k} < \delta$, let $U = C(x_1, \dots, x_{n_k})$, and use part (i) of Lemma 3.2.4. \square

We now establish the properties of X_ξ claimed in Theorem 3.1.1.

Proposition 3.2.7. (i) *The quotient map $\rho : X_E \rightarrow X_\xi$ is closed.*

(ii) *The space X_ξ is second-countable, compact, and Hausdorff, hence metrizable.*

(iii) *Every connected component of X_ξ is either a single point or homeomorphic to \mathbb{T} .*

(iv) *The covering dimension of X_ξ is one.*

Proof. (i) Let F be closed in X_E and y a point not in $\rho(F)$. Then $F \cap \rho^{-1}(y) = \emptyset$, so $F \cap B(\rho^{-1}(y), \delta) = \emptyset$ for some $\delta > 0$. By part (iv) of Lemma 3.2.6, we can find an open saturated set U of X_E with $\rho^{-1}(y) \subseteq U$ and $F \cap U = \emptyset$. Then $\rho(U)$ is a neighbourhood of y disjoint from $\rho(F)$.

- (ii) A closed continuous surjective map with compact fibres (sometimes referred to as a *perfect map*) preserves the Hausdorff property and the second-countable property [21]. Metrizable follows from the Urysohn metrization theorem.
- (iii) Let Y be a connected component of X_ξ . If Y consists of more than one point, it must be contained in $\bigcup_{p \in \mathcal{P}} \rho(C_\xi(p))$ by part (v) of Lemma 3.2.6. By part (iii) of Lemma 3.2.6, it must be contained in a single $\rho(C_\xi(p))$. The conclusion then follows from part (ii) of Lemma 3.2.6 since $X_F^{(w)}$ is totally disconnected.
- (iv) By part (ii) there is a metric that gives the topology of X_ξ ; choose such a metric d . The space X_ξ contains at least one homeomorphic copy of \mathbb{T} , so it suffices to prove that $\dim X_\xi \leq 1$. We can show this by showing that, for any $\varepsilon > 0$, we can find two collections \mathcal{U}_0 and \mathcal{U}_1 of pairwise disjoint open saturated subsets of X_E such that $\mathcal{U}_0 \cup \mathcal{U}_1$ is a cover of X_E and $\text{diam } \rho(U) < \varepsilon$ for all U in $\mathcal{U}_0 \cup \mathcal{U}_1$. Since X_E and X_ξ are compact, ρ is uniformly continuous with respect to the metrics d_E and d , so choose $n \geq 1$ such that $d_E(x, y) < 2^{-n}$ implies that $d(\rho(x), \rho(y)) < \frac{\varepsilon}{2}$. Consider a finite path $p = (p_1, \dots, p_n)$. If p_n is not in $\xi^0(F) \cup \xi^1(F)$, put $C(p)$ in \mathcal{U}_0 . Otherwise, put the set U_p from part (ii) of Lemma 3.2.4 in \mathcal{U}_0 . By the definition of the sets U_p , the remaining paths possibly not covered by the elements of \mathcal{U}_0 are the two-point equivalence classes whose splitting levels are among the levels E_1, E_2, \dots, E_n . To cover these, let \mathcal{U}_1 consist of all sets of the form $V_{p,q}$ from part (iii) of Lemma 3.2.4, where $k = n$ and $1 \leq m \leq n - 1$. \square

We now consider some examples to help realize the spaces X_ξ geometrically.

Example 3.2.8. Let (V, E) be the Bratteli diagram with $\#V_n = 1$ for all n and $\#E_n = 2$ for all n (the 2^∞ diagram), while (W, F) is the diagram with $\#W_n = \#F_n = 1$ for all n . Embed one copy of (W, F) on the left and the other copy on the right. Then X_ξ is homeomorphic to \mathbb{T} by Proposition 3.2.2.

Example 3.2.9. Let (V, E) be the Bratteli diagram with $\#V_n = 1$ for all n and $\#E_n = 4$ for all n (the 4^∞ diagram), while (W, F) is the diagram with $\#W_n = 1$ and $\#F_n = 2$ for all n . Embed one copy of (W, F) on the left and the other copy on the right. Then X_ξ is homeomorphic to $X_F \times \mathbb{T}$ by Proposition 3.2.2. By identifying X_F with the middle thirds Cantor set $X \subseteq [1, 2]$ (shifted to the right from its usual position), we can identify X_ξ with the planar set $\bigcup_{x \in X} x\mathbb{T}$, where $x\mathbb{T} = \{xz \mid z \in \mathbb{T}\}$.

Example 3.2.10. Consider the Bratteli diagram in Figure 3.1. Let (W, F) be the diagram with $\#W_n = \#F_n = 1$ for all n . Embed (W, F) down the bottom edges of (V, E) . Then X_ξ is homeomorphic to $(\bigcup_{n=0}^\infty 2^{-n}\mathbb{T}) \cup \{0\}$. Indeed, any finite path ending in a diagonal edge is in \mathcal{P} , and $\rho(C_\xi(p))$ is clopen and homeomorphic to \mathbb{T} for each such path. The only path in X_E that is not in $\bigcup_{p \in \mathcal{P}} C_\xi(p)$ is the path along the top of the diagram, which corresponds to the point 0.

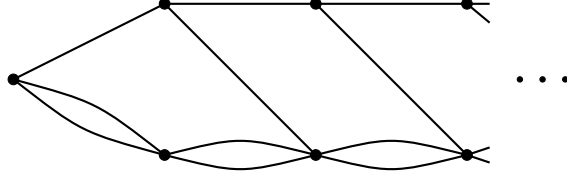


Figure 3.1: The Bratteli diagram (V, E) in Example 3.2.10

Example 3.2.11. Let (V, E) be the Bratteli diagram with $\#V_n = 1$ for all n and $\#E_n = 3$ for all n (the 3^∞ diagram), while (W, F) is the diagram with $\#W_n = \#F_n = 1$ for all n . For every finite path $p = (p_1, p_2, \dots, p_n)$ in \mathcal{P} , $\rho(C_\xi(p))$ is homeomorphic to \mathbb{T} by part (ii) of Lemma 3.2.6. To help illustrate X_ξ , we note that $\bigcup_{p \in \mathcal{P}} C_\xi(p)$ is dense in X_E and construct a continuous map from $\bigcup_{p \in \mathcal{P}} C_\xi(p)$ to the plane \mathbb{C} which is constant precisely on the fibres of ρ . We will identify X_E with $\{0, 1, 2\}^{\mathbb{N}}$ and regard the edges labelled with j the edges in $\xi^j(F)$ for $j = 0, 1$.

First define $\theta_2 = 0$ and for a finite path $p = (p_1, p_2, \dots, p_n)$ in $\{0, 1\}^{n-1} \times \{2\}$ for $n \geq 2$, set $\theta_p = 2^{-n} + \sum_{k=1}^{n-1} p_k 2^{-k}$. Define the function $f_p : \mathbb{C} \rightarrow \mathbb{C}$ by

$$f_p(z) = e^{2\pi i \theta_p} (2^{-n} z + (1 + 2^{-(n-1)}))$$

Now suppose that p in \mathcal{P} is arbitrary, and partition $p = p^{(1)} p^{(2)} \dots p^{(m)}$ where each $p^{(k)}$ is in $\{0, 1\}^{l_k} \times \{2\}$ for some integer $l_k \geq 0$ (note that such a partition exists and is unique). Define

$$f_p = f_{p^{(1)}} \circ f_{p^{(2)}} \circ \dots \circ f_{p^{(m)}}$$

If p_0 is the empty path, let $f_{p_0}(z) = z$. If $x = (x_1, x_2, \dots)$ is in $C_\xi(p)$, define

$$\rho(x) = f_p \left(\exp \left(2\pi i \sum_{k=n+1}^{\infty} x_k 2^{-k} \right) \right)$$

Letting $X_n = \bigcup_{\text{length}(p) \leq n-1} C_\xi(p)$ (where $\text{length}(p)$ is the number of edges in p with the convention that $\text{length}(p_0) = 0$), we have $\rho(X_1)$ the unit circle and $\rho(X_2)$ is the disjoint union of the unit circle and the circle of radius $\frac{1}{2}$ centred at -2 . Figure 3.2 illustrates $\rho(X_n)$ for $n = 3, 4, 5, 6$.

3.3 The groupoids R_ξ

We now turn to the factor groupoids R_ξ . Unlike the spaces X_ξ , their definition is quite straightforward.

Definition 3.3.1. We let $R_\xi = \rho \times \rho(R_E)$ and endow it with the quotient topology induced by the étale topology on R_E . We will denote $\pi : R_E \rightarrow R_\xi$ to be the restriction

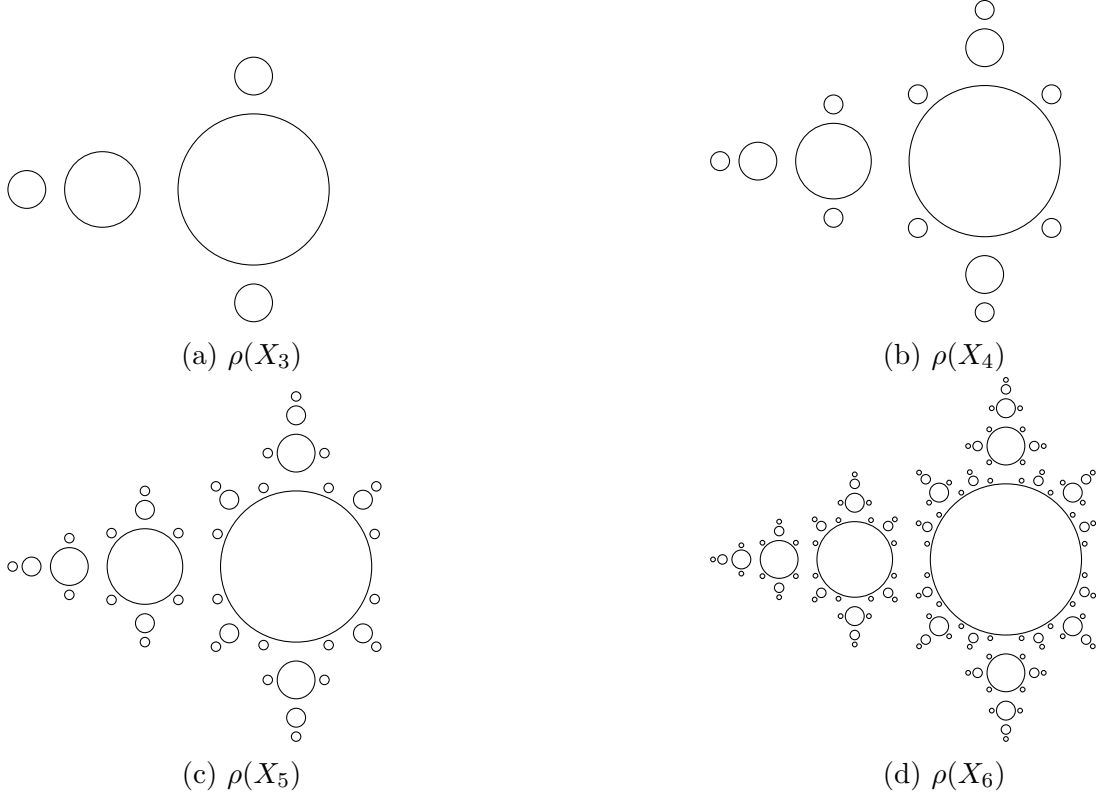


Figure 3.2: Geometric approximations of X_ξ from Example 3.2.11

of $\rho \times \rho$ to R_E .

Clearly π is a continuous surjective groupoid homomorphism, and since tail-equivalent paths are never in the same fibre of ρ , π is bijective on the fibres of the range map $r : R_E \rightarrow R_E^{(0)}$. We observe that the sets R_n and $R_{n+1} - R_n$ are saturated with respect to π , for all $n \geq 1$.

Proposition 3.3.2. (i) R_ξ is a second-countable, locally compact, Hausdorff, étale groupoid.

(ii) The map $\pi : R_E \rightarrow R_\xi$ satisfies the hypotheses in Proposition 1.3.9, and is regular and measure regular.

Proof. We begin by proving that R_ξ is Hausdorff. Notice that π is continuous with respect to the relative product topologies from $X_E \times X_E$ and $X_\xi \times X_\xi$. It follows that the quotient topology R_ξ receives from R_E is finer than the relative topology from $X_\xi \times X_\xi$. The space X_ξ is Hausdorff by Proposition 3.2.7, thus R_ξ is too.

To see that R_ξ is locally compact, notice that $R_E = \bigcup_{n=1}^{\infty} R_n$ and each R_n is compact, open, and saturated. Therefore each element of R_ξ is contained in the compact open set $\pi(R_n)$ for some n .

Let $\mu_E : R_E^{(2)} \rightarrow R_E$ and $\mu_\xi : R_\xi^{(2)} \rightarrow R_\xi$ be the multiplication maps. We will show that μ_ξ is continuous. It is readily verified that $\pi \circ \mu_E = \mu_\xi \circ (\pi \times \pi)$. If U is open in R_ξ , then $\mu_E^{-1}(\pi^{-1}(U)) = (\pi \times \pi)^{-1}(\mu_\xi^{-1}(U))$ is open in $R_E^{(2)}$ since μ_E and π are continuous. Since R_E and R_ξ are locally compact Hausdorff and $R_E^{(2)}$ is a closed subset of $R_E \times R_E$ which is saturated (because $(\pi \times \pi)^{-1}(R_\xi^{(2)}) = R_E^{(2)}$), the map $\pi \times \pi|_{R_E^{(2)}}$ is a quotient map, so $\mu_\xi^{-1}(U) = \pi \times \pi((\pi \times \pi)^{-1}(\mu_\xi^{-1}(U)))$ is open in $R_\xi^{(2)}$. Continuity of inversion may be proven similarly.

To see that R_ξ is étale, first observe that, since $R_E^{(0)} = \pi^{-1}(R_\xi^{(0)})$, the groupoid R_ξ is r -discrete (that is, $R_\xi^{(0)}$ is open), so by Proposition 1.2.9 of [37], it suffices to show that r is an open map. Let $U' \subseteq R_\xi$ be open. Then $U = \pi^{-1}(U')$ is open by continuity of π , and $r(U)$ is open since R_E is étale. Now $\pi(r(U)) = r(\pi(U)) = r(U')$, so we need only show that $r(U)$ is saturated with respect to π . If (x, x) is in $r(U)$ and (x', x') is in $R_E^{(0)}$ with $\rho(x) = \rho(x')$, pick y in X_E with (x, y) in U . Then there is some y' in X_E with $\rho(y) = \rho(y')$. Then (x', y') is in R_E and U is saturated, so (x', y') is in U . Then $(x', x') = r(x', y')$ is in $r(U)$.

Regarding the properties of π , the only nontrivial one to check is that π is proper. Let $K \subseteq R_\xi$ be compact. As noted earlier, R_E is the union of the compact open saturated sets R_n . It follows that $K \subseteq \pi(R_n)$ for some n , so $\pi^{-1}(K) \subseteq R_n$. Being a closed subset of a compact set, $\pi^{-1}(K)$ is compact.

At this point, we may conclude that π is a closed map, and hence that R_ξ is second-countable, using Lemma 7.5 of [30] and the proof of parts (i) and (ii) of Proposition 3.2.7.

Finally, we show that π is regular. Let (x, y) and (x', y') be two distinct pairs in R_E with $\pi(x, y) = \pi(x', y')$ and $\varepsilon > 0$. Let $n \geq 0$ be the integer such that (x, y) and (x', y') are in $R_{n+1} - R_n$, let E_m be the splitting level of $\{x, x'\}$, and let E_l be the splitting level of $\{y, y'\}$. As before, we assume that the edges of x and y are eventually in $\xi^0(F)$ and the edges of x' and y' are eventually in $\xi^1(F)$. Choose $k \geq \max\{l, m, n+1\}$ so that $2^{-k} < \varepsilon$. Let $p = (x_1, \dots, x_k)$, $p' = (x'_1, \dots, x'_k)$, $q = (y_1, \dots, y_k)$, and $q' = (y'_1, \dots, y'_k)$, and let

$$U = (\gamma(p, q) \cup \gamma(p', q')) \cap (V_{p, p'} \times V_{q, q'})$$

We must show that

- (i) U is open,
- (ii) U is saturated, and
- (iii) $\pi(U)$ satisfies the definition of regularity.

For (i), since the étale topology on R_E is finer than the relative product topology, $V_{p, p'} \times V_{q, q'}$ is open in R_E , and therefore so is U .

(ii) and (iii) can be proved simultaneously, in two cases. Let (x'', y'') be in U (suppose, without loss of generality, that it is in $\gamma(p, q)$) and (x''', y''') be another pair with $\pi(x'', y'') = \pi(x''', y''')$. We must show that (x''', y''') is also in U and that either

$$d_E^{(2)}(\{(x, y), (x', y')\}, \{(x'', y''), (x''', y''')\}) < \varepsilon \quad (3.1)$$

(in the Hausdorff metric) or

$$d_E^{(2)}((x'', y''), (x''', y''')) < \varepsilon. \quad (3.2)$$

The first case considers when $x_j'' = y_j''$ is in $\xi^0(F)$ for all $j \geq k + 1$. Here, the splitting level of $\{x'', x'''\}$ must be E_m and the splitting level of $\{y'', y'''\}$ must be E_l . This forces (x''', y''') into $\gamma(p', q') \cap (V_{p,p'} \times V_{q,q'}) \subseteq U$. Also, we have

$$\begin{aligned} d_E^{(2)}(\{(x, y), (x', y')\}, \{(x'', y''), (x''', y''')\}) &= \max\{d_E^{(2)}((x, y), (x'', y'')), d_E^{(2)}((x', y'), (x''', y'''))\} \\ &= \max\{d_E(x, x''), d_E(y, y''), d_E(x', x'''), d_E(y', y''')\} \\ &\leq 2^{-k} \\ &< \varepsilon \end{aligned}$$

and hence (3.1) holds.

The second case considers when $x_j'' = y_j''$ is not in $\xi^0(F)$ for some $j \geq k + 1$. This assumption, together with (x'', y'') being in $V_{p,p'} \times V_{q,q'}$, forces the splitting levels of $\{x'', x'''\}$ and $\{y'', y'''\}$ to be level E_{k+1} or further down. In this case, (x''', y''') is also in $\gamma(p, q) \cap (V_{p,p'} \times V_{q,q'}) \subseteq U$. Moreover, we have

$$\begin{aligned} d_E^{(2)}((x'', y''), (x''', y''')) &= \max\{d_E(x'', x'''), d_E(y'', y''')\} \\ &\leq 2^{-k} \\ &< \varepsilon \end{aligned}$$

and hence (3.2) holds. Finally, π is measure regular by Proposition 7.16 of [30]. \square

Proposition 3.3.3. *The dynamic asymptotic dimension of R_ξ is zero. It follows that R_ξ is amenable, and that the nuclear dimension of $C_r^*(R_\xi)$ is at most one.*

Proof. Since R_ξ is equal to the union of the compact open subgroupoids $\pi(R_n)$, its dynamic asymptotic dimension is zero by part (i) of Proposition 1.3.8. The remaining conclusions follow from part (iv) of Proposition 3.2.7 and part (iv) of Proposition 1.3.7. \square

All in all, the groupoids R_ξ have compact unit space, are separable, and have finite dynamic asymptotic dimension. If, in addition, R_E is minimal, then so is R_ξ because ρ is continuous and surjective, hence dense equivalence classes in X_E map to dense

equivalence classes in X_ξ . We therefore obtain the following result from Proposition 1.3.7.

Corollary 3.3.4. *If R_E is minimal, then the C^* -algebra $C_r^*(R_\xi)$ is classified by its Elliott invariant.*

We now prove a noncommutative version of Proposition 3.2.2.

Proposition 3.3.5. *If $E = \xi^0(F) \cup \xi^1(F)$, then $C_r^*(R_E)$ is isomorphic to $C_r^*(R_F) \otimes M_{2^\infty}$, and $C_r^*(R_\xi)$ is isomorphic to $C_r^*(R_F) \otimes B$, where B is the Bunce–Deddens algebra of type 2^∞ .*

Proof. We first establish the simplest case described in the beginning of the chapter. Let S denote the tail-equivalence relation on $\{0, 1\}^\mathbb{N}$ with its usual étale topology, so that $C_r^*(S) \cong M_{2^\infty}$ (Example 9.2.6 of [34]). Let

$$T = \{(w, z) \in \mathbb{T} \times \mathbb{T} \mid w = e^{2\pi i\theta} z \text{ for some } \theta \in \mathbb{Z}[\frac{1}{2}]\}$$

Give T the topology with basic open sets $U_{W,\theta} = \{(z, e^{2\pi i\theta} z) \mid z \in W\}$, where $W \subseteq \mathbb{T}$ is open and $\theta \in \mathbb{Z}[\frac{1}{2}]$. This makes T a second-countable locally compact Hausdorff étale principal groupoid, and $C_r^*(T) \cong B$, see 10.11.4 of [4]. We show that this topology is the same as the quotient topology received from $\tau : S \rightarrow T$, where $\tau = (\phi \times \phi)|_S$ and ϕ is as in Proposition 3.2.2. Suppose that $U \subseteq S$ is open and saturated; we must show that $\tau(U)$ is open. Select a pair (w, z) in $\tau(U)$.

Suppose first that w and z are not dyadic. Then $\tau^{-1}(w, z) = \{(x, y)\}$ for a pair (x, y) in S , and U is open, so there is a positive integer k and finite strings p and q in $\{0, 1\}^k$ such that $(x, y) \in \gamma(p, q) \subseteq U$. Set $V = \gamma(p, q) \cap (U_p \times U_q)$. Then

$$(w, z) \in \tau(V) = U_{W,\theta} \subseteq \tau(U)$$

where $W = \phi(U_p)$ and $\theta = \sum_{n=1}^\infty (y_n - x_n)2^{-n} = \sum_{n=1}^k (q_n - p_n)2^{-n}$.

If w and z are dyadic, then $\tau^{-1}(w, z) = \{(x, y), (x', y')\}$ for two pairs (x, y) and (x', y') in S . This time find $\gamma(p, q)$ and $\gamma(p', q')$, such that $\tau^{-1}(w, z) \subseteq \gamma(p, q) \cup \gamma(p', q') \subseteq U$ and set $V = (\gamma(p, q) \cup \gamma(p', q')) \cap (V_{p,p'} \times V_{q,q'})$. Then

$$(w, z) \in \tau(V) = U_{W,\theta} \subseteq \tau(U)$$

where $W = \phi(V_{p,p'})$ and $\theta = \sum_{n=1}^\infty (y_n - x_n)2^{-n}$.

Define the map $\Psi : R_E \rightarrow R_F \times S$ by $\Psi(x, y) = ((z_x, z_y), (j_x, j_y))$, using the notation in Proposition 3.2.2. It is straightforward to check that this is an isomorphism of groupoids (which proves the first statement of the proposition, by Proposition 1.3.10). The composition of this isomorphism with $\text{id}_{R_F} \times \tau$ is a quotient map whose fibres are identical to the fibres of π , which therefore factors to an isomorphism from R_ξ to $R_F \times T$. Apply Proposition 1.3.10 again. \square

3.4 The K -theory and tracial states of $C_r^*(R_\xi)$

Up until now we have proven parts (i) and (ii) of the second list in Theorem 3.1.1. In this section we turn to the remaining parts, and finish with some examples.

We begin by analyzing the subgroupoids $H \subseteq R_E$ and $H' \subseteq R_\xi$ introduced in section 2.9. We have

$$H' = \{(x', y') \in R_\xi \mid \#\pi^{-1}(x', y') = 2\}$$

and $H = \pi^{-1}(H')$. Clearly (x, y) is in H if and only if both x and y are eventually in $\xi^0(F)$ or eventually in $\xi^1(F)$. The metrics on H and H' are given by

$$d_{H'}((x', y'), (w', z')) = d_E^{(2)}(\pi^{-1}(x', y'), \pi^{-1}(w', z'))$$

(with the Hausdorff distance on the right hand side) and

$$d_H((x, y), (w, z)) = d_E^{(2)}((x, y), (w, z)) + d_E^{(2)}(\pi^{-1}(\pi(x, y)), \pi^{-1}(\pi(w, z))).$$

The next result follows from the description of d_H above. We use the notation $x^{(n)}$ for $n \geq 1$ to denote a sequence in X_E , and reserve the subscript to label the edges.

Proposition 3.4.1. *Suppose that $j \in \{0, 1\}$ is fixed, (x, y) is a pair in H , and that k is the least integer such that x_m and y_m are in $\xi^j(F)$ for $m \geq k$. A sequence $(x^{(n)}, y^{(n)}) \rightarrow (x, y)$ in d_H if and only if the following two conditions hold.*

(i) $x^{(n)} \rightarrow x$ and $y^{(n)} \rightarrow y$ in d_E .

(ii) There is an integer l such that k is the least integer such that $x_m^{(n)}$ and $y_m^{(n)}$ are in $\xi^j(F)$ for all $n \geq l$ and all $m \geq k$.

Lemma 3.4.2. *There is a $*$ -isomorphism $\beta : C_r^*(H') \oplus C_r^*(H') \rightarrow C_r^*(H)$ such that the diagram*

$$\begin{array}{ccc} C_r^*(H') & \xrightarrow{\gamma} & C_r^*(H') \oplus C_r^*(H') \\ \parallel & & \downarrow \beta \\ C_r^*(H') & \xrightarrow{\alpha} & C_r^*(H) \end{array}$$

is commutative, where α is as in Proposition 1.3.9, and $\gamma(a) = (a, a)$.

Proof. Set

$$H_j = \{(x, y) \in H \mid x, y \in \xi^j(F) \text{ eventually}\}$$

for $j = 0, 1$. Then $H_0 \cup H_1$ is a partition of H into two nonempty disjoint sets, and by Proposition 3.4.1, they are clopen. H_0 and H_1 are therefore locally compact Hausdorff

étale subgroupoids of H . Moreover, if f_j is in $C_c(H)$ with support contained in H_j for $j = 0, 1$, then for any (x, y) in H ,

$$(f_0 \star f_1)(x, y) = \sum_{(x, z) \in R_E} f_0(x, z) f_1(z, y) = 0$$

since all (x, z) and (z, y) lie in one and only one H_j . It follows that $C_c(H) = C_c(H_0) \oplus C_c(H_1)$. By partitioning the direct sum

$$\bigoplus_{x \in H^{(0)}} l^2(H_x) = \left(\bigoplus_{x \in H_0^{(0)}} l^2(H_x) \right) \oplus \left(\bigoplus_{x \in H_1^{(0)}} l^2(H_x) \right)$$

and representing $C_c(H_0)$ (respectively $C_c(H_1)$) on the left (respectively right) factor, it is apparent that $C_r^*(H) = C_r^*(H_0) \oplus C_r^*(H_1)$ as well. Define

$$\beta' : C_c(H') \oplus C_c(H') \rightarrow C_c(H_0) \oplus C_c(H_1)$$

by $\beta'(f, g) = (f \circ (\pi|_{H_0}), g \circ (\pi|_{H_1}))$. Similarly as in Proposition 1.3.9, this is a *-isomorphism which extends to the reduced algebras. It is straightforward to check that the diagram commutes on continuous compactly supported functions, and therefore on the completions. \square

Lemma 3.4.3. $C_r^*(H')$ is Morita equivalent to $C_r^*(R_F)$, where R_F is tail-equivalence on the diagram (W, F) . It follows that $K_*(C_r^*(H'))$ is isomorphic to $K_*(C_r^*(R_F))$.

Proof. Let H_0 be as in the previous lemma, and let $J = \{(x, y) \in H_0 \mid x_n, y_n \in \xi^0(F) \text{ for all } n \geq 1\}$, endowed with the relative topology from H . The map $R_F \rightarrow J$ which sends (x, y) to $(\xi^0(x), \xi^0(y))$ (applying ξ^0 to each edge of x and y in the obvious way) is clearly an isomorphism of groupoids. $J^{(0)}$ is thus compact and open (in H), hence $\chi_{J^{(0)}}$ is a projection in $C_c(H_0)$. Then $\chi_{J^{(0)}} C_r^*(H_0)$ is an equivalence bimodule between $C_r^*(H_0) \cong C_r^*(H')$ and $\chi_{J^{(0)}} C_r^*(H_0) \chi_{J^{(0)}} \cong C_r^*(J) \cong C_r^*(R_F)$. \square

Lemma 3.4.4. The map $\alpha_* : K_0(C_r^*(R_\xi)) \rightarrow K_0(C_r^*(R_E))$ is surjective.

Proof. Since $C_r^*(R_E)$ is a unital AF-algebra, it suffices to show that, for a finite path $p = (p_1, p_2, \dots, p_n)$ in the diagram (V, E) , the characteristic function $\chi_{\gamma(p, p)}$ is equivalent to a projection in $\alpha(C_c(R_\xi))$. If p_n is not in $\xi^0(F) \cup \xi^1(F)$, then $\chi_{\gamma(p, p)}$ is in $\alpha(C_c(R_\xi))$, so there is nothing to do. Otherwise, let k be the least integer in $\{1, 2, \dots, n\}$ such that p_j is in $\xi^0(F) \cup \xi^1(F)$ for all $k \leq j \leq n$. For a string $\omega = (j_1, j_2, \dots, j_{n-k+1})$ in $\{0, 1\}^{n-k+1}$, let

$$q_\omega = (p_1, p_2, \dots, p_{k-1}, \xi^{j_1}(z_k), \xi^{j_2}(z_{k+1}), \dots, \xi^{j_{n-k+1}}(z_n))$$

and let

$$v = \frac{1}{\sqrt{2^{n-k+1}}} \sum_{\omega \in \{0,1\}^{n-k+1}} \chi_{\gamma(p,q_\omega)}.$$

The element v is a partial isometry in $C_c(R_E)$, and using the formulas in Proposition 1.3.15, we obtain that $vv^* = \chi_{\gamma(p,p)}$ and $v^*v = \frac{1}{2^{n-k+1}} \chi_L$, where

$$L = \bigcup_{\omega, \eta \in \{0,1\}^{n-k+1}} \gamma(q_\omega, q_\eta).$$

L is saturated, so v^*v is in $\alpha(C_c(R_\xi))$. \square

Readers with a background in quantum information theory may notice a similarity between the formula for v in the above proof and the one for the n -fold tensor product of the Hadamard operator \tilde{H} acting on a computational basis state (written in ket notation),

$$\tilde{H}^{\otimes n} |\omega\rangle_n = \frac{1}{\sqrt{2^n}} \sum_{\eta \in \{0,1\}^n} (-1)^{\omega \cdot \eta} |\eta\rangle_n$$

where $\omega \cdot \eta$ is the componentwise dot product computed mod 2. Indeed, if one identifies the functions $\chi_{\gamma(p,q)}$ with matrix units as described in Proposition 1.3.15, the matrices vv^* and v^*v are respectively identified with the projections

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \frac{1}{2^n} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

in $M_{2^n}(\mathbb{C})$. A unitary equivalence between them is implemented by $\tilde{H}^{\otimes n}$.

Finally, we verify that every subset X of H that is closed in R_E satisfies the C -extension property, with $C = 1$.

Proposition 3.4.5. *Every subset $X \subseteq H$ which is closed in R_E satisfies the C -extension property with $C = 1$.*

Proof. Suppose $X \subseteq H$ is closed in R_E and f is in $C_c(H)$ with support in X . We seek a function \tilde{f} in $C_c(R_E)$ such that $\tilde{f}|_X = f|_X$ and $\|\tilde{f}\|_r \leq \|f\|_r$. In fact, we find such a function \tilde{f} such that $\|\tilde{f}\|_r = \|f\|_r$.

As the support of f is compact in H , it is also compact in R_E , hence the support is contained in R_n for some n . Denote $K = X \cap R_n$ and $\tilde{K} = R_n K R_n$ (that is, all possible products $\alpha\beta\gamma$ where β is in K and α and γ are in R_n). Notice that $K \subseteq \tilde{K}$, $\tilde{K} \subseteq H$ and $\tilde{K} \subseteq R_n$.

Claim 1. \tilde{K} is a compact subgroupoid of R_n .

Proof. Let $K^{(0)} = \bigcup_{(x,y) \in K} [x]_{R_n}$, where $[x]_{R_n}$ denotes the R_n -equivalence class of the path x . We will prove that $\tilde{K} = \{(x, y) \in R_n \mid x \in K^{(0)}\}$; this will imply that \tilde{K} is a subgroupoid.

Suppose first that (w, z) is in \tilde{K} . Then there are paths x and y such that (w, x) and (y, z) are in R_n and (x, y) is in K . By the definition of $K^{(0)}$, x is in $K^{(0)}$, which means w is in $K^{(0)}$ as well, because (w, x) is in R_n . This implies that (w, z) is in $\{(x, y) \in R_n \mid x \in K^{(0)}\}$.

Conversely, suppose that (x, y) is in R_n and x is in $K^{(0)}$. By the definition of $K^{(0)}$, we may find a path y' such that (x, y') is in K . Then (y', y) is in R_n because $(y', y) = (y', x)(x, y) \in R_n R_n = R_n$, and $(x, y) = (x, x)(x, y')(y', y) \in R_n K R_n = \tilde{K}$.

Finally, $K = X \cap R_n$ is compact because X is closed in R_E and R_n is compact, and $\tilde{K} = R_n K R_n$ is compact because multiplication is continuous in R_E . \square

Now consider the map $C(R_n) \rightarrow C(\tilde{K})$ that simply restricts a function to \tilde{K} . Since \tilde{K} is a subgroupoid of R_n , this map is a surjective *-homomorphism. Therefore, by restricting f to \tilde{K} , we may find a function \tilde{f} in $C(R_n)$ that lifts f , that is, $\tilde{f}|_{\tilde{K}} = f|_{\tilde{K}}$, and has the same norm. Since R_n is compact and open in R_E , we may regard \tilde{f} as an element of $C_c(R_E)$ (with the same norm) by extending it to be zero on $R_E - R_n$. To check that $\tilde{f}|_X = f|_X$, first suppose that (x, y) is in K . Then $\tilde{f}(x, y) = f(x, y)$ because \tilde{f} and f agree on \tilde{K} , and $K \subseteq \tilde{K}$. If (x, y) is in X but not in K , then (x, y) is in $R_E - R_n$, but then $\tilde{f}(x, y) = f(x, y) = 0$ as both of their supports are contained in R_n . \square

Remark. The important property of R_E in the above proof is the existence of the compact, open subgroupoids R_n for $n \geq 1$. The proof could therefore be adapted to groupoids that are equal to the union of compact, open subgroupoids.

We are now ready to compute the K -theory of $C_r^*(R_\xi)$. Theorem 2.1.1 furnishes the six-term exact sequence

$$\begin{array}{ccccc}
 K_1(C_r^*(R_E)) & \longrightarrow & K_0(C_r^*(R_\xi), C_r^*(R_E)) & \longrightarrow & K_0(C_r^*(R_\xi)) \\
 \uparrow & & & & \downarrow \\
 K_1(C_r^*(R_\xi)) & \longleftarrow & K_1(C_r^*(R_\xi), C_r^*(R_E)) & \longleftarrow & K_0(C_r^*(R_E))
 \end{array}$$

Using (in order) Theorem 2.9.3, Lemma 3.4.2, Example 2.10.3, and Lemma 3.4.3, we

obtain

$$\begin{aligned}
K_j(C_r^*(R_\xi); C_r^*(R_E)) &\cong K_j(C_r^*(H'), C_r^*(H)) \\
&\cong K_j(C_r^*(H'), C_r^*(H') \oplus C_r^*(H')) \\
&\cong K_{1-j}(C_r^*(H')) \\
&\cong K_{1-j}(C_r^*(R_F)).
\end{aligned}$$

The exact sequence may therefore be simplified to

$$\begin{array}{ccccc}
0 & \longrightarrow & 0 & \longrightarrow & K_0(C_r^*(R_\xi)) \\
\uparrow & & & & \downarrow \\
K_1(C_r^*(R_\xi)) & \longleftarrow & K_0(C_r^*(R_F)) & \longleftarrow & K_0(C_r^*(R_E))
\end{array}$$

By Lemma 3.4.4, the vertical map on the right is an order isomorphism. Exactness implies that the following map is zero, which in turn implies that the map after that is an isomorphism.

We have thus proven parts (iii) and (iv) of the second list in Theorem 3.1.1; it remains to prove part (v). As these groupoids are principal, we may appeal to Proposition 1.3.11 and examine the invariant Borel measures on their unit spaces. It suffices to show that any invariant measure cannot see the non-one-to-one parts of π .

Proposition 3.4.6. (i) *If μ is an invariant measure on X_E then $\mu(H^{(0)}) = 0$.*

(ii) *If μ is an invariant measure on X_ξ then $\mu(H'^{(0)}) = 0$.*

Proof. (i) Every such measure μ is uniquely determined by a function $\nu : V \rightarrow [0, 1]$ such that $\nu(v_0) = 1$ and $\nu(v) = \sum_{e \in i^{-1}(v)} \nu(t(e))$, the correspondence being $\nu(v) = \mu(C(p))$, where p is any finite path ending at v , see Theorem 11.31 of [28]. It suffices to show that the set Y of all paths whose edges are eventually in $\xi^0(F)$ has measure zero; the proof is analogous for those paths eventually in $\xi^1(F)$, and $H^{(0)}$ is the union of these. Let

$$Y_n = \{x \in X_E \mid x_m \in \xi^0(F) \text{ for all } m \geq n\}$$

We have $Y = \bigcup_{n=1}^{\infty} Y_n$, so it further suffices to show that $\mu(Y_n) = 0$ for all n . To this end, fix n and, for $m \geq n+1$, let A_m be the union of all cylinder sets of the form $C(p_1, p_2, \dots, p_n, q_{n+1}, q_{n+2}, \dots, q_m)$ where q_k is in $\xi^0(F)$ for $n+1 \leq k \leq m$. For any cylinder set $C(p)$ in the union A_m , there are at least $2^{m-(n+1)}$ cylinder sets in total passing through the same vertices as the path p , due to the edges in $\xi^1(F)$ (and possibly more). Thus, by the sum given by ν , we have $\mu(A_m) \leq 2^{-m}$. Since $Y_n \subseteq A_m$ for every $m \geq n+1$, this completes the proof of (i).

(ii) The set $H^{(0)}$ is contained in the countable union $\bigcup_{p \in \mathcal{P}} \rho(C_\xi(p))$, so it suffices to show that $\mu(H^{(0)} \cap \rho(C_\xi(p))) = 0$ for every p in \mathcal{P} . Using part (ii) of Lemma 3.2.6, we may identify $\rho(C_\xi(p))$ with $X_F^{(w)} \times \mathbb{T}$ for some w in W . Let $n \geq 1$ and let γ be any open arc on \mathbb{T} of normalized Lebesgue length 2^{-n} . By invariance of μ , the disjoint sets $X_F^{(w)} \times e^{2\pi i k/2^n} \gamma$ for $k = 0, 1, \dots, 2^n - 1$ are each contained in an open set in X_ξ of equal measure, so we have

$$\mu(X_F^{(w)} \times \gamma) \leq 2^{-n} \mu(X_F^{(w)} \times \mathbb{T}).$$

Thus, we obtain that $\mu(X_F^{(w)} \times \{z\}) = 0$, for every z in \mathbb{T} . As $H^{(0)} \cap \rho(C_\xi(p))$ is identified with $X_F^{(w)} \times \exp(2\pi i \mathbb{Z}[\frac{1}{2}])$ and $\exp(2\pi i \mathbb{Z}[\frac{1}{2}])$ is countable, we are done. \square

Example 3.4.7. Consider the situation in Example 3.2.8. By Theorem 3.1.1, we have $K_0(C_r^*(R_\xi)) \cong \mathbb{Z}[\frac{1}{2}]$ with its usual order, and $K_1(C_r^*(R_\xi)) \cong \mathbb{Z}$ with generator given by the class of the identity function z in $C(\mathbb{T}) = C(R_\xi^{(0)}) \subseteq C_r^*(R_\xi)$. We illustrate how to use the excision theorem to spot the generator of $K_1(C_r^*(R_\xi))$.

Since (W, F) consists of a single path, we have $C_r^*(R_F) = \mathbb{C}$, and under the Morita equivalence to $C_r^*(H')$, the generator $[1_{\mathbb{C}}]$ of $K_0(C_r^*(R_F))$ corresponds to the projection $\chi_{\{(0,0,0,\dots)\}}$ in $C_c(H)$, which corresponds to the projection

$$p = \chi_{\{(0,0,0,\dots)\}} + \chi_{\{(1,1,1,\dots)\}}$$

in $C_c(H')$ under the isomorphism in Lemma 3.4.2. Under the map $\mu_1 : K_0(C_r^*(H)) \rightarrow K_1(C_r^*(H'), C_r^*(H))$ in Theorem 2.1.1, we obtain

$$\mu_1([\chi_{\{(0,0,0,\dots)\}}]) = [p, p, g]$$

where $g(s) = \chi_{\{(0,0,0,\dots)\}} + e^{2\pi i s} \chi_{\{(1,1,1,\dots)\}}$ for $0 \leq s \leq 1$. Here comes the guesswork: we need an element in $K_1(C_r^*(R_\xi), C_r^*(R_E))$ that corresponds to the triple (p, p, g) in $K_1(C_r^*(H'), C_r^*(H))$. A natural choice is to take the triple $(1_{C_r^*(R_\xi)}, f, h)$, where $h(s) = f_s$ and $f_s(\{j_n\}) = \exp(2\pi i s \sum_{n=1}^{\infty} j_n 2^{-n})$ for $\{j_n\}$ in $\{0, 1\}^{\mathbb{N}} = R_E^{(0)}$ and $0 \leq s \leq 1$. By the main result of [30], the two classes $[p, p, g]$ and $[1_{C_r^*(R_\xi)}, f, h]$ both live in $K_1(\ker(\delta), C_r^*(R_E) + C_r^*(H))$ as discussed in section 2.9, and thus we have

$$[1_{C_r^*(H')}, 1_{C_r^*(H')}, g + (1_{C_r^*(H)} - p)] - [1_{C_r^*(R_\xi)}, f, h] = [1_{C_r^*(H')}, f^*, (g + (1_{C_r^*(H)} - p))h^*]$$

by part (ii) of Proposition 2.3.9. A consequence of the results of [30] is that the elements of $C_r^*(R_E)$ act as multipliers on $C_r^*(H)$, and thus we regard the elements of the latter triple above as living in $C_r^*(H)$. For a fixed $0 \leq s \leq 1$, the functions $g(s)$ and $h(s)$

agree on the set $\{(0, 0, 0, \dots)\} \cup \{(1, 1, 1, \dots)\}$, so we obtain

$$(g + (1_{C_r^*(H)} - p))h^* = gh^* + (1_{C_r^*(H)} - p)h^* = p + (1_{C_r^*(H)} - p)h^*$$

and thus

$$\begin{aligned} [1_{C_r^*(H')}, f^*, (g + (1_{C_r^*(H)} - p))h^*] &= [1_{C_r^*(H')}, f^*, p + (1_{C_r^*(H)} - p)h^*] \\ &= [1_{C_r^*(H')} - p, (1_{C_r^*(H')} - p)f^*, (1_{C_r^*(H)} - p)h^*] \end{aligned}$$

and the latter triple is elementary. It follows that the triples $[p, p, g]$ and $[1_{C_r^*(R_\xi)}, f, h]$ correspond to one another under the excision isomorphism. Finally, under the map $\nu_1 : K_1(C_r^*(R_\xi), C_r^*(R_E)) \rightarrow K_1(C_r^*(R_\xi))$, we have $\nu_1([1_{C_r^*(R_\xi)}, f, h]) = [f]$, which is the usual generator with winding number 1.

Example 3.4.8. Consider the situation in Example 3.2.9. By Theorem 3.1.1, we have $K_0(C_r^*(R_\xi)) \cong K_1(C_r^*(R_\xi)) \cong \mathbb{Z}[\frac{1}{2}]$ with the usual order in the K_0 case. One can use a similar argument as in the previous example to show that the K_1 -group is generated by the partial unitaries $\chi_U \otimes z$ in $C(X_F) \otimes C(\mathbb{T}) = C(R_\xi^{(0)})$, where U is a clopen subset of X_F .

Example 3.4.9. Consider the situation in Example 3.2.10. By Theorem 3.1.1, we have $K_0(C_r^*(R_\xi)) \cong \mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z}$ with lexicographic order on the direct sum and the usual order on both factors, and $K_1(C_r^*(R_\xi)) \cong \mathbb{Z}$.

Example 3.4.10. Consider the situation in Example 3.2.11. By Theorem 3.1.1, we have $K_0(C_r^*(R_\xi)) \cong \mathbb{Z}[\frac{1}{3}]$ with its usual order, and $K_1(C_r^*(R_\xi)) \cong \mathbb{Z}$.

Chapter 4

Factor groupoids from non-homogeneous extensions of Cantor minimal systems

In this chapter we discuss a second construction of factor groupoids due to Robin Deeley, Ian Putnam, and Karen Strung. Unlike the examples in the previous chapter, the fibres of the factor map will be either a single point, or infinite; specifically, the infinite fibres will be homeomorphic to the attractor of an iterated function system (detailed definitions to follow). The main result of [8] is the following.

Theorem 4.0.1 (Theorem 1.5 of [8]). *Let (C, d_C, \mathcal{F}) be a compact invertible iterated function system and let (X, ϕ) be a minimal homeomorphism of the Cantor set. There exists a minimal extension $(\tilde{X}, \tilde{\phi})$ of (X, ϕ) with a factor map $\pi : (\tilde{X}, \tilde{\phi}) \rightarrow (X, \phi)$ such that, for each x in X , $\pi^{-1}\{x\}$ is a single point or homeomorphic to C . Moreover, both possibilities occur.*

The original intention for this theorem was to provide a means of tying together minimal dynamical systems with spaces of arbitrarily large finite covering dimension in order to produce examples of minimal, principal groupoids whose K -theory can be prescribed. It is a standard result in topology that any compact metrizable space with covering dimension $n \geq 0$ can be embedded in $[0, 1]^{2n+1}$ [21]. The main example of interest, therefore, is to take C to be equal to a cube in Euclidean space. One then may find a closed subset Y of C , and construct groupoids whose K -theory essentially depends on the space Y . It follows from the fact that C is contractible that the factor map induces an isomorphism between the K -theory of $(\tilde{X}, \tilde{\phi})$ and the K -theory of (X, ϕ) (this is Corollary 1.6 of [8], and also Corollary 4.1.2 below).

Our goal in this chapter is to use Theorem 2.9.3 to obtain a description of the K -theory of the C^* -algebras of the groupoids $\tilde{X} \times \mathbb{Z}$, for more general C . As discussed in Section 2.9, the excision theorem allows one to view the relative K -theory of the inclusion $\alpha : C_r^*(X_E \times \mathbb{Z}) \rightarrow C_r^*(\tilde{X} \times \mathbb{Z})$ through the non-one-to-one parts of the factor map, and hence, by Theorem 4.0.1, one expects the K -theory of the extensions to depend on the topological K -theory of C .

4.1 The main results

Before we state the main result, we remark that we must modify the construction of the extension $(\tilde{X}, \tilde{\phi})$ in Theorem 4.0.1 slightly in order to apply the excision theorem. We will describe the modification in detail later, but loosely speaking, we will require that the fibres of the factor map “shrink” quickly enough so that Definition 2.9.1 is satisfied. The dynamics are not affected in any profound way by this modification.

We refer to the extension $(\tilde{X}, \tilde{\phi})$ in Theorem 4.0.1 as a *DPS extension* in acknowledgement of the authors.

Theorem 4.1.1. *Let (X, d_X, \mathcal{F}) be an iterated function system and let C denote its attractor. Let (X_E, ϕ_E) a Bratteli-Vershik system on a properly ordered Bratteli diagram. Then there is a modified version of the associated DPS extension $(\tilde{X}, \tilde{\phi})$ of (X_E, ϕ_E) , an AF-algebra A , a short exact sequence*

$$0 \longrightarrow K_0(C_r^*(R_E)) \longrightarrow K_0(C_r^*(\tilde{X} \times \mathbb{Z})) \longrightarrow K_0(A) \otimes (K^0(C)/\mathbb{Z}) \longrightarrow 0$$

and a short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow K_1(C_r^*(\tilde{X} \times \mathbb{Z})) \longrightarrow K_0(A) \otimes K^{-1}(C) \longrightarrow 0.$$

We are using the notation for the topological K -theory of a compact, Hausdorff space X , that is, denoting $K_j(C(X))$ by $K^{-j}(X)$ for $j = 0, 1$. By $K^0(C)/\mathbb{Z}$ we mean the quotient of $K^0(C)$ by the canonical copy of \mathbb{Z} generated by the unit of $C(C)$.

We refine the conclusion of Theorem 4.1.1 in some specific cases of interest, such as when C is a closed cube in Euclidean space (this case actually motivated the original construction in [8]).

Corollary 4.1.2. *In the context of Theorem 4.1.1, we have the following.*

- (i) *If C is contractible, then the injective maps in both sequences are isomorphisms.*
- (ii) *If X_E consists of only one path (x_1, x_2, \dots) of edges such that $f_{x_n} = \text{id}_C$ for all $n \geq 1$, then $K_0(A) \cong \mathbb{Z}$ and hence*

$$K_0(C_r^*(\tilde{X} \times \mathbb{Z}))/K_0(C_r^*(R_E)) \cong K^0(C)/\mathbb{Z} \quad K_1(C_r^*(\tilde{X} \times \mathbb{Z}))/\mathbb{Z} \cong K^{-1}(C)$$

At this point it is not clear whether or not the conclusions of Theorem 4.1.1 and Corollary 4.1.2 can be improved upon without knowing more about the dynamics of the iterated function system and the topology of the attractor. For example, it is not clear precisely when the short exact sequences split, and hence when the quotients in Corollary 4.1.2 can be replaced with direct sums. Of course, this is trivially the case in part (i) of the corollary, and also when both $K^0(C)$ and $K^{-1}(C)$ are free abelian, such as when C is the Sierpiński triangle.

4.2 The (modified) DPS construction

We begin with the definition of an iterated function system. For more information on this matter, we refer to [3].

Definition 4.2.1. An *iterated function system* (X, d_X, \mathcal{F}) is a complete metric space (X, d_X) together with a finite set \mathcal{F} of functions $f : X \rightarrow X$ such that there exists a real number λ with $0 < \lambda < 1$ and $d_X(f(x), f(y)) \leq \lambda d_X(x, y)$ for all points x and y in X and all f in \mathcal{F} .

The following result is standard; see Theorem 7.1 and Definition 7.2 of [3].

Theorem 4.2.2. *Let (X, d_X, \mathcal{F}) be an iterated function system. There exists a unique non-empty compact subset C of X such that*

$$\bigcup_{f \in \mathcal{F}} f(C) = C.$$

The set C is called the attractor of the iterated function system.

The proof is essentially to show that the assignment

$$Y \mapsto \bigcup_{f \in \mathcal{F}} f(Y)$$

is a contraction mapping on the space of all compact subsets of X with the Hausdorff metric. A fixed point theorem of Hutchinson (Theorem 6.1 of [3]) then guarantees the existence and uniqueness of C .

We now outline the construction of the extension in [8]. First, we may assume that the Cantor minimal system is given by a Bratteli-Vershik system (X_E, ϕ_E) on a properly ordered Bratteli diagram by Proposition 1.3.22. By telescoping the Bratteli diagram, we may assume that there are at least $\#\mathcal{F} + 1$ many edges between any two vertices in consecutive levels. We will also assume that each f in \mathcal{F} is injective. Now, to each edge e in the diagram we assign a function f_e in $\mathcal{F} \cup \{\text{id}_C\}$ such that the following properties hold.

- (i) If e is either maximal or minimal, then $f_e \neq \text{id}_C$.
- (ii) For every v in V , we have $\bigcup_{t(e)=v, f_e \neq \text{id}_C} f_e(C) = C$.
- (iii) The set $\{e \mid f_e = \text{id}_C\}$ contains an infinite path.

We then define

$$X_n = \{(x, c) \in X_E \times C \mid c \in f_{x_1} \circ \cdots \circ f_{x_n}(C)\}$$

and $\tilde{X} = \bigcap_{n=1}^{\infty} X_n$. The factor map $\tilde{\pi} : \tilde{X} \rightarrow X_E$ (we reserve the letter π for our groupoid homomorphism) is defined by $\tilde{\pi}(x, c) = x$. Notice that the space \tilde{X} consists of two types of points, as follows.

- (i) If (x, c) is in \tilde{X} and $f_{x_n} \neq \text{id}_C$ for infinitely many n , then the fibre $\tilde{\pi}^{-1}(\tilde{\pi}(x, c))$ consists of a single point (x, c_x) due to the contractive factor λ causing the diameters of the sets $f_{x_1} \circ \cdots \circ f_{x_n}(C)$ to shrink.
- (ii) If there is an m with $f_{x_n} = \text{id}_C$ for all $n \geq m$, then the fibre $\tilde{\pi}^{-1}(\tilde{\pi}(x, c))$ is equal to

$$\{x\} \times f_{x_1} \circ \cdots \circ f_{x_m}(C)$$

and is thus homeomorphic to C since each function f in \mathcal{F} is injective.

The self map $\tilde{\phi}$ of \tilde{X} is then defined as follows, based on these two types of points.

- (i) If x has infinitely many n with $f_{x_n} \neq \text{id}_C$, there is a unique point c_x in C with $\tilde{\pi}(x, c_x) = x$. In this case, $\phi_E(x)$ also has infinitely many such n , and we set $\tilde{\phi}(x, c) = (\phi_E(x), c_{\phi_E(x)})$.
- (ii) If there is an m with $f_{x_n} = \text{id}_C$ for all $n \geq m$, then

$$\phi_E(x) = (y_1, y_2, \dots, y_m, x_{m+1}, x_{m+2}, \dots)$$

for some edges y_1, \dots, y_m with $t(x_m) = t(y_m)$. In this case, we set

$$\tilde{\phi}(x, c) = (\phi_E(x), f_{y_1} \circ \cdots \circ f_{y_m} \circ f_{x_m}^{-1} \circ \cdots \circ f_{x_1}^{-1}(c)).$$

We now turn to the modification of the extension needed to apply the excision theorem. We provide an example to show that it is necessary: take $C = [0, 1]$ with the usual metric and \mathcal{F} to consist of the two functions $f(x) = \frac{1}{2}x$ and $g(x) = \frac{1}{2}x + \frac{1}{2}$. Let (V, E) be the 3^∞ diagram with the canonical odometer action. Order the edges from left to right at each level, assign the left edge the function f , the centre edge id_C , and the right edge g . Take $\varepsilon = \frac{1}{2}$ and x to be the path consisting of the centre edge at each level. Any open set containing x will contain a path y consisting of the centre edge at each level except for one. Then

$$d(\pi^{-1}(x), \pi^{-1}(y)) = \frac{1}{2} \quad \text{diam } \pi^{-1}(y) = \frac{1}{2}$$

and thus Definition 2.9.1 cannot hold.

The modification is as follows. For $n \geq 1$ we set

$$\mathcal{F}^{(n)} = \{f_1 \circ f_2 \circ \cdots \circ f_n \mid f_j \in \mathcal{F} \text{ for } 1 \leq j \leq n\}$$

adjust the edge assignment so that an edge e in E_n is assigned to a function f_e in $\mathcal{F}^{(n)} \cup \{\text{id}_C\}$ (note that this may require further telescoping of the diagram). The action $\tilde{X} \times \mathbb{Z}$ is constructed in the same way.

Definition 4.2.3. Let $G = \tilde{X} \times \mathbb{Z}$ and $G' = X_E \times \mathbb{Z}$, and define the continuous surjective groupoid homomorphism $\pi : G \rightarrow G'$ by $\pi = \tilde{\pi} \times \text{id}_{\mathbb{Z}}$.

Proposition 4.2.4. (i) G and G' are second-countable, locally compact, Hausdorff, étale groupoids.

(ii) The map $\pi : G \rightarrow G'$ is regular and satisfies the hypotheses of Proposition 1.3.9.

Proof. Both G' and G have the product topology and \mathbb{Z} is discrete, so (i) is immediate. The only nontrivial thing to prove for (ii) is that π is regular. Since π does nothing to \mathbb{Z} , it suffices restrict our attention to $\tilde{\pi}$. By scaling the metric d_C , we may assume that $\text{diam } C = 1$. Fix $\varepsilon > 0$ and x in X_E ; we seek an open set U of X_E such that for all y in U , either $d(\tilde{\pi}^{-1}(x), \tilde{\pi}^{-1}(y)) < \varepsilon$ or $\text{diam } \tilde{\pi}^{-1}(y) < \varepsilon$.

First suppose x has infinitely many n with $f_{x_n} \neq \text{id}_C$. Choose N such that $\lambda^N < \varepsilon$ and $f_{x_N} \neq \text{id}_C$, and set $U = C(x_1, x_2, \dots, x_N)$. If y is in U and (y, c) and (y, d) are in $\tilde{\pi}^{-1}(y)$, then c and d are both in $f_{y_1} \circ \dots \circ f_{y_N}(C) = f_{x_1} \circ \dots \circ f_{x_N}(C)$ and since f_{x_N} has a contractive factor of at most λ^N by the modification above, we have $d_C(c, d) \leq \lambda^N < \varepsilon$ and hence $\text{diam } \tilde{\pi}^{-1}(y) < \varepsilon$.

Second, suppose that $f_{x_n} = \text{id}_C$ eventually. Choose N such that $2^{-N} < \varepsilon$, $\lambda^N < \varepsilon$, and $f_{x_n} = \text{id}_C$ for all $n \geq N$, and set $U = C(x_1, x_2, \dots, x_N)$. Now let y be in U ; if $f_{y_n} \neq \text{id}_C$ for some $n \geq N$, then $\text{diam } f_{y_1} \circ \dots \circ f_{y_n}(C) \leq \lambda^n \leq \lambda^N < \varepsilon$, so $\text{diam } \tilde{\pi}^{-1}(y) < \varepsilon$ as in the first case. Otherwise, $f_{x_n}(C) = f_{y_n}(C)$ for all $n \geq 1$, and $d_E(x, y) \leq 2^{-N} < \varepsilon$, which implies that $d(\tilde{\pi}^{-1}(x), \tilde{\pi}^{-1}(y)) < \varepsilon$. \square

π is also measure regular in this construction, but it will be easier to prove it later (Proposition 4.3.4).

Proposition 4.2.5. *The dynamic asymptotic dimension of G is one, and if the covering dimension of C is finite, then $C_r^*(G)$ has finite nuclear dimension.*

Proof. This is immediate from Theorem 1.3(i) of [15] and Theorem 8.6 of [15]. \square

Similarly as with Corollary 3.3.4, we obtain

Corollary 4.2.6. *If the covering dimension of C is finite, then the C^* -algebra $C_r^*(G)$ is classified by its Elliott invariant.*

4.3 The K -theory of $C_r^*(G)$

We now prove Theorem 4.1.1. In this case we have

$$H' = \{(x, k) \mid f_{x_n} = \text{id}_C \text{ eventually}\} \subseteq X_E \times \mathbb{Z}$$

and

$$H = \{(x, c, k) \mid f_{x_n} = \text{id}_C \text{ eventually}\} \subseteq \tilde{X} \times \mathbb{Z} \subseteq X_E \times C \times \mathbb{Z}$$

We give a sequential description of the topologies on H' and H .

Proposition 4.3.1. (i) $(x^{(n)}, k_n) \rightarrow (x, k)$ in H' if and only if $k_n = k$ eventually, $x^{(n)} \rightarrow x$ in X_E , and if i_0 is the least integer such that $f_{x_i} = \text{id}_C$ for all $i \geq i_0$, then i_0 is the least integer such that there exists an n_0 with $f_{x_i^{(n)}} = \text{id}_C$ for all $i \geq i_0$ and $n \geq n_0$

(ii) $(x^{(n)}, c_n, k_n) \rightarrow (x, c, k)$ in H if and only if $(x^{(n)}, k_n) \rightarrow (x, k)$ in H' and $c_n \rightarrow c$ in C .

The following lemma describes H in terms of H' . The proof is straightforward.

Lemma 4.3.2. (i) The map $\Psi : H \rightarrow H' \times C$ defined by

$$\Psi(x, c, k) = ((x, k), f_{x_n}^{-1} \circ \dots \circ f_{x_1}^{-1}(c))$$

is well-defined as long as $f_{x_i} = \text{id}_C$ for $i \geq n$, and is an isomorphism of groupoids (C is regarded as the cotrivial groupoid $\{(c, c) \mid c \in C\}$).

(ii) If $\pi^C : H' \times C \rightarrow H'$ is defined by $\pi^C((x, k), c) = (x, k)$, then $\pi^C \circ \Psi = \pi$.

Lemma 4.3.3. $C_r^*(H')$ is Morita equivalent to an AF-algebra.

Proof. The proof is similar to that of Lemma 3.4.3. Consider the subdiagram (W, F) of (V, E) formed by all infinite paths x such that $f_{x_n} = \text{id}_C$ for all $n \geq 1$, and let R_F be the AF-groupoid associated to (W, F) . Note that there is at least one such path by property (iii) of the edge assignment. Let

$$J = \{(x, k) \mid f_{x_n} = f_{y_n} = \text{id}_C \text{ for all } n \geq 1, \text{ where } y = \phi_E^k(x)\}.$$

It is straightforward to verify that J is an open subgroupoid of H' . The map $J \rightarrow R_F$ defined by $(x, k) \mapsto (x, \phi_E^k(x))$ is an isomorphism of groupoids. $J^{(0)}$ is thus compact and open (in H'), hence $\chi_{J^{(0)}}$ is a projection in $C_c(H')$. Then $\chi_{J^{(0)}} C_r^*(H')$ is an equivalence bimodule between $C_r^*(H')$ and $\chi_{J^{(0)}} C_r^*(H') \chi_{J^{(0)}} \cong C_r^*(J) \cong C_r^*(R_F)$. \square

Proposition 4.3.4. $\pi : G \rightarrow G'$ is measure regular.

Proof. By Lemma 4.3.2 we may show instead that π^C is measure regular. Fix a point c_0 in C and define the map $\mu : H' \rightarrow H' \times C$ by $\mu(x, k) = ((x, k), c_0)$. Clearly μ is a continuous groupoid homomorphism and $\pi^C \circ \mu = \text{id}_{H'}$, so Proposition 7.15 from [30] applies. \square

We also need to verify that the C -extension property in Theorem 2.9.3 is not an obstacle.

Proposition 4.3.5. *Every subset $X \subseteq H$ which is closed in G satisfies the C -extension property with $C = 1$.*

Proof. Breaking the orbit $\{\phi_E^n(x^{\min})\}_{n \in \mathbb{Z}}$ at x^{\min} and letting the two sets

$$\{\phi_E^n(x^{\min})\}_{n \geq 0} \quad \{\phi_E^n(x^{\min})\}_{n < 0}$$

be two separate equivalence classes results in a subgroupoid which is open in G' and isomorphic to R_E [25]. Since H' does not intersect the orbit of x^{\min} , H' is contained in $R_E = \bigcup_{n=1}^{\infty} R_n$. Thus $X \subseteq \bigcup_{n=1}^{\infty} \pi^{-1}(R_n)$ and since π is continuous and proper, any f in $C_c(H)$ with support contained in X has support contained in some compact open subgroupoid $\pi^{-1}(R_n)$. The proof then proceeds analogously to that of Proposition 3.4.5. \square

Lemma 4.3.6. *We have*

$$K_0(C_r^*(G'), C_r^*(G)) \cong K_0(C_r^*(H')) \otimes K^{-1}(C)$$

and

$$K_1(C_r^*(G'), C_r^*(G)) \cong K_0(C_r^*(H')) \otimes (K^0(C)/\mathbb{Z})$$

Proof. By Theorem 2.9.3 and Lemma 4.3.2, we have

$$K_*(C_r^*(G'), C_r^*(G)) \cong K_*(C_r^*(H'), C_r^*(H)) \cong K_*(C_r^*(H'), C_r^*(H' \times C))$$

By combining the isomorphism $C_r^*(H' \times C) \cong C_r^*(H') \otimes C(C)$ of Proposition 1.3.10 with Theorem 1.2.19, we have an isomorphism

$$K_*(C_r^*(H' \times C)) \cong K_0(C_r^*(H')) \otimes K^*(C)$$

under which the K_0 map induced by the inclusion $C_r^*(H') \subseteq C_r^*(H' \times C)$ becomes $K_0(C_r^*(H')) \rightarrow K_0(C_r^*(H')) \otimes K^0(C) : g \mapsto g \otimes 1$. We have the six-term exact sequence

$$\begin{array}{ccccc}
K_0(C_r^*(H')) \otimes K^{-1}(C) & \longrightarrow & K_0(C_r^*(H'), C_r^*(H' \times C)) & \longrightarrow & K_0(C_r^*(H')) \\
\uparrow & & & & \downarrow \\
K_1(C_r^*(H')) & \longleftarrow & K_1(C_r^*(H'), C_r^*(H' \times C)) & \longleftarrow & K_0(C_r^*(H')) \otimes K^0(C)
\end{array}$$

As C is compact, there is a short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow K^0(C) \longrightarrow K^0(C)/\mathbb{Z} \longrightarrow 0$$

and since $C_r^*(H')$ is Morita equivalent to an AF-algebra, $K_0(C_r^*(H'))$ is a dimension group, hence torsion free. Thus we may tensor the sequence above with $K_0(C_r^*(H'))$ and obtain that

$$0 \longrightarrow K_0(C_r^*(H')) \longrightarrow K_0(C_r^*(H')) \otimes K^0(C) \longrightarrow K_0(C_r^*(H')) \otimes (K^0(C)/\mathbb{Z}) \longrightarrow 0$$

is exact. In particular, the map $K_0(C_r^*(H')) \rightarrow K_0(C_r^*(H')) \otimes K^0(C)$ is injective, so the right vertical arrow in the six-term sequence is injective. Combining the fact that $K_1(C_r^*(H')) = 0$ with exactness gives the conclusion. \square

Lemma 4.3.7. *The map $\alpha_* : K_*(C_r^*(G')) \rightarrow K_*(C_r^*(G))$ is injective.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccccc}
& & C(X_E, \mathbb{Z}) & \xrightarrow{\text{id}-\phi_E} & C(X_E, \mathbb{Z}) & & \\
& & \uparrow \text{dim} & & \uparrow \text{dim} & & \\
K_1(C_r^*(G')) & \xrightarrow{\delta_1} & K^0(X_E) & \xrightarrow{\text{id}-\phi_E^*} & K^0(X_E) & \xrightarrow{\iota_*} & K_0(C_r^*(G')) \longrightarrow 0 \\
\downarrow \alpha_* & & \downarrow \tilde{\pi}^* & & \downarrow \tilde{\pi}^* & & \downarrow \alpha_* \\
K_1(C_r^*(G)) & \xrightarrow{\delta_1} & K^0(\tilde{X}) & \xrightarrow{\text{id}-\tilde{\phi}_*} & K^0(\tilde{X}) & \xrightarrow{\iota_*} & K_0(C_r^*(G)) \\
& & \downarrow \text{dim} & & \downarrow \text{dim} & & \\
& & C(\tilde{X}, \mathbb{Z}) & \xrightarrow{\text{id}-\tilde{\phi}} & C(\tilde{X}, \mathbb{Z}) & &
\end{array}$$

The second and third rows are extracted from Theorem 1.2.17, therefore they are exact. By the map $\alpha : C(X_E, \mathbb{Z}) \rightarrow C(\tilde{X}, \mathbb{Z})$, we mean $\alpha(f) = f \circ \tilde{\pi}$, which is clearly an injective group homomorphism.

We consider the $j = 0$ case first. Let x be in $K_0(C_r^*(G'))$ such that $\alpha_*(x) = 0$. Since the top ι_* is surjective (by exactness), there is a y in $K^0(X_E)$ with $\iota_*(y) = x$. We have $\iota_*(\tilde{\pi}^*(y)) = \alpha_*(\iota_*(y)) = \alpha_*(x) = 0$, so by exactness again, there is some z in $K^0(\tilde{X})$ such that $(\text{id} - \tilde{\phi}_*)(z) = \tilde{\pi}^*(y)$. Denote $f = \text{dim}(y)$ and $g = \text{dim}(z)$. Commutativity of the diagram implies that $f \circ \tilde{\pi} = g - g \circ \tilde{\phi}^{-1}$ in $C(\tilde{X}, \mathbb{Z})$, which means that $g - g \circ \tilde{\phi}^{-1}$ is constant on the fibres of $\tilde{\pi}$. Thus, if w and w' are points in \tilde{X} with $\tilde{\pi}(w) = \tilde{\pi}(w')$, we have

$$g(w) - g(w') = g(\tilde{\phi}^{-1}(w)) - g(\tilde{\phi}^{-1}(w')).$$

The above equation implies that the difference $g(w) - g(w')$ remains constant as w and w' run through their orbits under $\tilde{\phi}$ in tandem. Since $\tilde{\phi}$ is minimal and there are fibres that consist of a single point, we must have $g(w) - g(w') = 0$, that is, g must also be constant on the fibres of $\tilde{\pi}$. Therefore there is a g_0 in $C(X_E, \mathbb{Z})$ with $g = g_0 \circ \tilde{\pi}$. Then we have

$$f \circ \tilde{\pi} = g - g \circ \tilde{\phi}^{-1} = g_0 \circ \tilde{\pi} - g_0 \circ \tilde{\pi} \circ \tilde{\phi}^{-1} = (g_0 - g_0 \circ \phi_E^{-1}) \circ \tilde{\pi}.$$

Since $\alpha : C(X_E, \mathbb{Z}) \rightarrow C(\tilde{X}, \mathbb{Z})$ is injective, we have $f = g_0 - g_0 \circ \phi_E^{-1} = (\text{id} - \phi_E)(g_0)$, and since $\text{dim} : K^0(X_E) \rightarrow C(X_E, \mathbb{Z})$ is an isomorphism (X_E is totally disconnected), there is one and only one y' in $K^0(X_E)$ with $\text{dim}(y') = g_0$. By commutativity, we have

$$(\text{id} - \phi_{E*})(y') = \text{dim}^{-1} \circ (\text{id} - \phi_E) \circ \text{dim}(y') = \text{dim}^{-1} \circ (\text{id} - \phi_E)(g_0) = \text{dim}^{-1}(f) = y.$$

Finally, we have

$$x = \iota_*(y) = \iota_*((\text{id} - \phi_{E*})(y')) = 0$$

by exactness.

The $j = 1$ case is simpler, since $K_1(C_r^*(G')) \cong \mathbb{Z}$ is generated by the class of the unitary u in $C_r^*(G')$ implementing the action, so we need only show that $\alpha_*([u])$ is nonzero. By Lemma 1.2.18, we have

$$\tilde{\pi}^*(\delta_1([u])) = \tilde{\pi}^*(-[1_{C(X_E)}]) = -[1_{C(\tilde{X})}]$$

and both $[1_{C(X_E)}]$ and $[1_{C(\tilde{X})}]$ are non-zero since both X_E and \tilde{X} are compact. Thus $\delta_1(\alpha_*([u])) = \tilde{\pi}^*(\delta_1([u]))$ is non-zero and hence $\alpha_*([u])$ cannot be zero. \square

Returning to the six-term exact sequence with the relative groups $K_*(C_r^*(G'), C_r^*(G))$,

$$\begin{array}{ccccc} K_1(C_r^*(G)) & \longrightarrow & K_0(C_r^*(G'), C_r^*(G)) & \longrightarrow & K_0(C_r^*(G')) \\ \uparrow & & & & \downarrow \\ K_1(C_r^*(G')) & \longleftarrow & K_1(C_r^*(G'), C_r^*(G)) & \longleftarrow & K_0(C_r^*(G)) \end{array}$$

Lemma 4.3.6 and the fact that $K_0(C_r^*(G')) \cong K_0(C_r^*(R_E))$ and $K_1(C_r^*(G')) \cong \mathbb{Z}$ (Proposition 1.3.23) simplify this to

$$\begin{array}{ccccc} K_1(C_r^*(G)) & \longrightarrow & K_0(C_r^*(H')) \otimes K^{-1}(C) & \longrightarrow & K_0(C_r^*(R_E)) \\ \uparrow & & & & \downarrow \\ \mathbb{Z} & \longleftarrow & K_0(C_r^*(H')) \otimes (K^0(C)/\mathbb{Z}) & \longleftarrow & K_0(C_r^*(G)) \end{array}$$

By Lemma 4.3.7 and exactness, the top right and bottom left maps are zero, giving the conclusion of Theorem 4.1.1.

4.4 Cubes, the Cantor set, and the Sierpiński triangle

We conclude this chapter with some applications of Theorem 4.1.1 using some classical examples of iterated function systems. In all examples, the vertex sets in the Bratteli diagrams satisfy $\#V_n = 1$ for all $n \geq 1$, and there will only be one infinite path x in X_E such that $f_{x_n} = \text{id}_C$, so that $K_0(C_r^*(H')) \cong \mathbb{Z}$.

Example 4.4.1. Let $m \geq 1$ be an integer, $C = [0, 1]^m$, and define the functions $f_\delta : C \rightarrow C$ by $f_\delta(x) = \frac{1}{2}(x + \delta)$ for δ in $\{0, 1\}^m$. Let (V, E) be the Bratteli diagram with $\#V_n = 1$ for all n , and $\#E_n = 2^{mn} + 1$ for all n . Any order on the edges will do, and, for a fixed $n \geq 1$, assign each function in $\mathcal{F}^{(n)} \cup \{\text{id}_C\}$ to an edge in E_n , making sure that id_C is assigned neither the maximal edge nor the minimal edge. Since C is contractible, part (i) of Corollary 4.1.2 implies that the inclusion map $\alpha : C_r^*(X_E \times \mathbb{Z}) \rightarrow C_r^*(\tilde{X} \times \mathbb{Z})$ induced by the factor map induces isomorphisms $\alpha_* : K_0(C_r^*(R_E)) \rightarrow K_0(C_r^*(\tilde{X} \times \mathbb{Z}))$ and $\alpha_* : \mathbb{Z} \rightarrow K_1(C_r^*(\tilde{X} \times \mathbb{Z}))$.

Example 4.4.2. Let $X = [0, 1]$ with the usual metric from \mathbb{R} and define the functions $f_j : X \rightarrow X$ by $f_j(x) = \frac{1}{3}(x + 2j)$ for $j = 0, 1$. The attractor C of this system is the middle-thirds Cantor set. Let (V, E) be the Bratteli diagram with $\#V_n = 1$ for all n and $\#E_n = 2^n + 1$ for all n . Any order on the edges will do, and, for a fixed $n \geq 1$, assign each function in $\mathcal{F}^{(n)} \cup \{\text{id}_C\}$ to an edge in E_n , making sure that id_C is assigned neither the maximal edge nor the minimal edge. We have $K^{-1}(C) = 0$, so by part (ii) of Corollary 4.1.2, the inclusion map $\alpha : C_r^*(X_E \times \mathbb{Z}) \rightarrow C_r^*(\tilde{X} \times \mathbb{Z})$ induces an isomorphism $\alpha_* : \mathbb{Z} \rightarrow K_1(C_r^*(\tilde{X} \times \mathbb{Z}))$ and

$$K_0(C_r^*(\tilde{X} \times \mathbb{Z}))/K_0(C_r^*(R_E)) \cong C(C, \mathbb{Z})/\mathbb{Z}$$

where we identify \mathbb{Z} with the subgroup of $C(C, \mathbb{Z})$ of constant functions.

More generally, if the iterated function system (X, d_X, \mathcal{F}) satisfies the *strong separation condition*, that is, if C is the attractor and $f(C) \cap f'(C) = \emptyset$ whenever $f \neq f'$ in \mathcal{F} , then C is necessarily homeomorphic to the Cantor set, and we are in the situation in the above example (with C replaced with $\{0, 1, 2, \dots, m\}^{\mathbb{N}}$, where $m = \#\mathcal{F} - 1$).

Example 4.4.3. Let \triangle denote the solid equilateral triangle in \mathbb{R}^2 with corners at

$(0, 0)$, $(1, 0)$, and $(\frac{1}{2}, \frac{\sqrt{3}}{2})$. For $j = 0, 1, 2$, define the three functions $f_j : \Delta \rightarrow \Delta$ by

$$f_0(x) = \frac{1}{2}x \quad f_1(x) = \frac{1}{2}x + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} \quad f_2(x) = \frac{1}{2}x + \begin{bmatrix} 1/4 \\ \sqrt{3}/4 \end{bmatrix}$$

The attractor C is called the Sierpiński triangle, and is obtained by successively removing a countable number of disjoint open triangles from Δ , see Figure 4.1.¹



Figure 4.1: Iterative construction of the Sierpiński triangle.

Let (V, E) be the Bratteli diagram with $\#V_n = 1$ for all n and $\#E_n = 3^n + 1$ for all n . As before, any order on the edges will do, and assign the functions similarly. Before we apply Theorem 4.1.1, we verify that $K^0(C) \cong \mathbb{Z}$ and $K^{-1}(C) \cong \mathbb{Z}^\infty$, where \mathbb{Z}^∞ denotes the group of all sequences of integers that are eventually zero.

Let U denote the union of open triangles removed from the solid equilateral triangle (each open triangle is homeomorphic to \mathbb{R}^2 , hence U is homeomorphic to a countable disjoint union of copies of \mathbb{R}^2). There is a short exact sequence

$$0 \longrightarrow C_0(U) \longrightarrow C(\Delta) \longrightarrow C(C) \longrightarrow 0$$

Since $K^0(\mathbb{R}^2) \cong K^0(\Delta) \cong \mathbb{Z}$ and $K^{-1}(\mathbb{R}^2) = K^{-1}(\Delta) = 0$, Proposition 1.2.11 gives

$$\begin{array}{ccccc} \mathbb{Z}^\infty & \longrightarrow & \mathbb{Z} & \longrightarrow & K^0(C) \\ \uparrow & & & & \downarrow \\ K^{-1}(C) & \longleftarrow & 0 & \longleftarrow & 0 \end{array}$$

The top right arrow is surjective by exactness, and since $K^0(\Delta) \cong \mathbb{Z}$ is generated by the class of $1_{C(\Delta)}$ and this is mapped to the class of $1_{C(C)}$, it is also injective. Thus $K^0(C) \cong \mathbb{Z}$, which implies that $K^{-1}(C) \cong \mathbb{Z}^\infty$ by exactness.

Theorem 4.1.1 therefore implies that $\alpha_* : K_0(C_r^*(R_E)) \rightarrow K_0(C_r^*(\tilde{X} \times \mathbb{Z}))$ is an isomorphism, and $K_1(C_r^*(\tilde{X} \times \mathbb{Z})) \cong \mathbb{Z} \oplus \mathbb{Z}^\infty$ (\mathbb{Z}^∞ is free abelian, therefore the short exact sequence splits).

¹My thanks to Emily Korfanty for the image.

Example 4.4.4. Let S denote the solid cube in \mathbb{R}^3 with corners at (i, j, k) for i, j, k in $\{0, 1\}$. For each δ in $\{0, 1, 2\}^3 - \{(1, 1, 1)\}$, define the function $f_\delta : S \rightarrow S$ by $f_\delta(x) = \frac{1}{3}(x + \delta)$. We let C be the attractor of this iterated function system.²

Let (V, E) be the Bratteli diagram with $\#V_n = 1$ for all n and $\#E_n = 26^n + 1$ for all n . As before, any order on the edges will do, and assign the functions similarly. Before we apply Theorem 4.1.1, we verify that $K^0(C) \cong \mathbb{Z} \oplus \mathbb{Z}^\infty$ and $K^{-1}(C) = 0$.

C is obtained from S by removing a countable number of open cubes, each of which is homeomorphic to \mathbb{R}^3 . Let U denote the union of these open cubes. There is a short exact sequence

$$0 \longrightarrow C_0(U) \longrightarrow C(S) \longrightarrow C(C) \longrightarrow 0$$

Since $K^{-1}(\mathbb{R}^3) \cong K^0(S) \cong \mathbb{Z}$ and $K^0(\mathbb{R}^3) = K^{-1}(S) = 0$, Proposition 1.2.11 gives

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & K^0(C) \\ & & & & \downarrow \\ & \uparrow & & & \mathbb{Z}^\infty \\ K^{-1}(C) & \longleftarrow & 0 & \longleftarrow & \mathbb{Z}^\infty \end{array}$$

and therefore $K^0(C) \cong \mathbb{Z} \oplus \mathbb{Z}^\infty$ and $K^{-1}(C) = 0$.

Theorem 4.1.1 therefore implies that $K_0(C_r^*(\tilde{X} \times \mathbb{Z})) \cong K_0(C_r^*(R_E)) \oplus \mathbb{Z}^\infty$, and that $\alpha_* : \mathbb{Z} \rightarrow K_1(C_r^*(\tilde{X} \times \mathbb{Z}))$ is an isomorphism.

²I do not know if this attractor has a name. ‘‘Sierpiński cube’’ seems to be already taken by the Menger curve.

Chapter 5

Conclusions and further directions

In this final chapter we indicate some possibilities for further work. Though we do provide some definitions, everything from this point on is speculative, and we do not provide proofs.

5.1 Axiomatization: generalized relative homology

In [7], it is shown that K -theory is essentially the only “reasonable” homology theory for C^* -algebras, a fact that is sometimes colloquially referred to as “Cuntz’s trap”. More specifically, if one has a covariant, stable, half-exact, homotopy invariant, continuous (i.e., preserves inductive limits) functor E from the smallest subcategory of separable C^* -algebras that

- (i) includes \mathbb{C} ,
- (ii) is closed under countable inductive limits,
- (iii) is closed under extensions (short exact sequences), and
- (iv) is closed under KK -equivalence

to the category of abelian groups, and $E(\mathbb{C}) \cong \mathbb{Z}$, then there is a natural transformation between E and K_0 . In case the reader, like the author¹, is unfamiliar with KK -theory, “for suitably nice separable C^* -algebras A and B , KK -equivalence is the same as $K_*(A) \cong K_*(B)$ ” [36].

We have seen two approaches to defining relative K -theory; one via Banach categories, and the other via mapping cones. Both of these constructions satisfy similar axioms, and we have seen that they are isomorphic in a natural way. Therefore it is not unreasonable to surmise that an axiomatization, analogous to that in [7], of relative K -theory exists. In this section we outline a possible such axiomatization.

¹It is speculated by the author that KK -theory is so named because it is twice as frightening as K -theory, and therefore deserves twice as many K ’s.

Consider the category ***-homo**, where the objects are *-homomorphisms $\phi : A \rightarrow B$ between C^* -algebras, and a morphism $\Phi = (\alpha, \beta)$ from $\phi : A \rightarrow B$ to $\psi : C \rightarrow D$ is a pair of *-homomorphisms $\alpha : A \rightarrow C$ and $\beta : B \rightarrow D$ such that

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow \alpha & & \downarrow \beta \\ C & \xrightarrow{\psi} & D \end{array}$$

commutes. If $\Phi_0 = (\alpha_0, \beta_0)$ and $\Phi_1 = (\alpha_1, \beta_1)$ are both morphisms from $\phi : A \rightarrow B$ to $\psi : C \rightarrow D$, then we'll say that $\Phi_t = (\alpha_t, \beta_t)$ for $0 \leq t \leq 1$ is a homotopy from Φ_0 to Φ_1 if Φ_t is a morphism from ϕ to ψ for all t and $t \mapsto \alpha_t(a)$ and $t \mapsto \beta_t(b)$ are continuous for all a in A and all b in B .

Proposition 5.1.1. (i) *The map $\phi \mapsto K_*(\phi)$ is a functor from ***-homo** to the category of abelian groups.*

(ii) *The map $\phi \mapsto K_*(C_\phi)$ is a functor from ***-homo** to the category of abelian groups.*

(iii) *The map Δ is a natural transformation between the functor $\phi \mapsto K_*(\phi)$ and the functor $\phi \mapsto K_*(C_\phi)$.*

(iv) *The category of all *-homomorphisms of the form $\phi : \{0\} \rightarrow B$ is a subcategory of ***-homo**, is equivalent as a category to **C*-alg**, and Δ is equivalent to the transformations θ and β .*

Now suppose that E is a covariant functor from this category into the category of abelian groups. Consider the properties

(i) (Stability) The morphism

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow \kappa_A & & \downarrow \kappa_B \\ A \otimes \mathcal{K} & \xrightarrow{\phi \otimes \text{id}_{\mathcal{K}}} & B \otimes \mathcal{K} \end{array}$$

gives an isomorphism $(\kappa_A, \kappa_B)_* : E(\phi) \rightarrow E(\phi \otimes \text{id}_{\mathcal{K}})$.

(ii) (Half-exactness) If

$$\begin{array}{ccccccc} 0 & \longrightarrow & I & \longrightarrow & A & \longrightarrow & A/I & \longrightarrow & 0 \\ & & \downarrow \psi & & \downarrow \phi & & \downarrow \gamma & & \\ 0 & \longrightarrow & J & \longrightarrow & B & \longrightarrow & B/J & \longrightarrow & 0 \end{array}$$

is commutative with exact rows, then

$$E(\psi) \longrightarrow E(\phi) \longrightarrow E(\gamma)$$

is exact, and

- (iii) (Homotopy invariance) if Φ and Ψ are homotopic morphisms from $\phi : A \rightarrow B$ to $\psi : C \rightarrow D$ as defined above, then $\Phi_* = \Psi_*$.

Conjecture. If E satisfies (i), (ii), and (iii), then we have Bott periodicity, that is, for each $*$ -homomorphism ϕ , there is an isomorphism $\beta_\phi : E(\phi) \rightarrow E(S^2\phi)$ that is natural in the sense that if $\Phi : \phi \rightarrow \psi$ is a morphism, the diagram

$$\begin{array}{ccc} E(\phi) & \xrightarrow{\Phi_*} & E(\psi) \\ \downarrow \beta_\phi & & \downarrow \beta_\psi \\ E(S^2\phi) & \xrightarrow{(S^2\Phi)_*} & E(S^2\psi) \end{array}$$

commutes.

To get a relative version of Cuntz's trap, it is likely that we need a relative version of continuity. A reasonable candidate seems to be the following: if $\varinjlim A_n = A$ and $\varinjlim B_n = B$, and if

$$\begin{array}{ccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & \cdots & \longrightarrow & A \\ \downarrow \phi_1 & & \downarrow \phi_2 & & \downarrow \phi_3 & & & & \downarrow \phi \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & \cdots & \longrightarrow & B \end{array}$$

commutes then $\varinjlim E(\phi_n) \cong E(\phi)$. This gives a variant of countable inductive limits for relative groups. Of course, one would need to show first that the category $*$ -**homo** admits inductive limits.

Lastly, it is likely that we need some trivial cases to hold in order to imitate the "bootstrap" axiom in [7]. Based on Example 2.10.2 and Example 2.10.3, the following guess seems reasonable.

Conjecture. Suppose that E is a covariant functor from the category $*$ -**homo** to the category of abelian groups, and that it satisfies (i), (ii), (iii), and is continuous. Define the $*$ -homomorphisms ϕ and ψ by

$$\phi : \mathbb{C} \oplus \mathbb{C} \rightarrow M_2(\mathbb{C}) : (w, z) \mapsto \begin{bmatrix} w & 0 \\ 0 & z \end{bmatrix}$$

$$\psi : \mathbb{C} \rightarrow \mathbb{C} \oplus \mathbb{C} : z \mapsto (z, z)$$

Then

- (i) If $E(\phi) \cong \mathbb{Z}$ and $E(\psi) = 0$, then $E = K_0$.
- (ii) If $E(\phi) = 0$ and $E(\psi) \cong \mathbb{Z}$, then $E = K_1$.

If we define $E(A)$ to be the relative group $E(\phi)$ where $\phi : A \rightarrow \{0\}$, and if the above conjecture holds, it seems likely that one would obtain Cuntz’s result as a special case from the subclass of $*$ -homomorphisms that map into the zero C^* -algebra.

5.2 Further examples of factor groupoids

We end with a short discussion on general factor groupoids.

We have seen that a certain subcollection of classifiable C^* -algebras have factor groupoid models, specifically as factor groupoids of AF-groupoids. We have also seen how to use the topological K -theory of attractors of iterated function systems to elucidate the possible K -theory of minimal integer actions on compact metric spaces. It seems yet to be seen just how “universal” factor groupoids are. In fact, one could ask a more specific, but equally interesting, question given the following two facts.

- (i) The unit spaces of AF-groupoids are totally disconnected and often Cantor sets, which have the universal property of possessing a continuous surjective map onto any compact metric space.
- (ii) There appears to be some interest in knowing when a C^* -algebra may be embedded into an AF-algebra.

In fact, the question proposed below is essentially a rephrasing of (ii) above in groupoid language.

Question. Which C^* -algebras arise from a factor groupoid of an AF-groupoid?

Corollary 3.1.2 implies that among all such C^* -algebras are all classifiable C^* -algebras A with $K_0(A)$ a simple, acyclic dimension group, $K_1(A)$ a countable, torsion-free abelian group, and standard pairing between K -theory and traces.

Bibliography

- [1] C. ANANTHARAMAN-DELAROCHE, J. RENAULT, *Amenable groupoids*, L'Enseignement Mathématique, 2000.
- [2] F. ARICI, A. RENNIE, The Cuntz-Pimsner extension and mapping cone exact sequences, *Mathematica Scandinavica* **125**(2)(2019), 291–319.
- [3] M. F. BARNESLEY, *Fractals Everywhere*, Academic Press, Inc., Cambridge, MA, 1993.
- [4] B. BLACKADAR, *K-Theory for Operator Algebras*, Mathematical Sciences Research Institute Publications, Springer-Verlag New York, 1986.
- [5] B. BLACKADAR, *Operator Algebras: Theory of C^* -Algebras and von Neumann Algebras*, Springer-Verlag Berlin Heidelberg, 2006.
- [6] C. BONICKE; C. DELL'AIERA; J. GABE, R. WILLETT, Dynamic asymptotic dimension and Matui's HK conjecture, preprint arXiv:2104.05885.
- [7] J. CUNTZ, *K-Theory and C^* -algebras*, *Lecture Notes in Math.*, 1046, Springer Berlin, 1982, 55-79.
- [8] R. J. DEELEY, I. F. PUTNAM, K. R. STRUNG, Non-homogeneous extensions of Cantor minimal systems, *Proc. Amer. Math. Soc.*, to appear.
- [9] R. J. DEELEY, I. F. PUTNAM, K. R. STRUNG, Classifiable C^* -algebras from minimal \mathbb{Z} -actions and their orbit-breaking subalgebras, preprint arXiv:2012.10947.
- [10] D. S. DUMMIT, R. M. FOOTE, *Abstract Algebra*, John Wiley & Sons, Inc., 2004.
- [11] E. G. EFFROS, D. E. HANDELMAN, C.-L. SHEN, Dimension groups and their affine representations, *Amer. J. Math.* **102**(1980), 385–407.
- [12] G. A. ELLIOTT, On the classification of inductive limits of sequences of semisimple finite-dimensional algebras, *J. Algebra* **38**(1976), 29–44.

- [13] G. A. ELLIOTT, An invariant for simple C^* -algebras, *Canadian Mathematical Society* **3**(1996), 61–90.
- [14] T. GIORDANO, I. F. PUTNAM, C. F. SKAU, Affable equivalence relations and orbit structure of Cantor dynamical systems, *Ergod. Th. and Dynam. Sys.*, **24**(2004), 441–476.
- [15] E. GUENTNER, R. WILLETT, G. YU, Dynamic asymptotic dimension: relation to dynamics, topology, coarse geometry, and C^* -algebras, *Math. Ann.*, **367**(2017), 785–829.
- [16] N. HIGSON, J. ROE, *Analytic K -Homology*, Oxford Mathematical Monographs, Oxford University Press, 2000.
- [17] X. JIANG, H. SU. On a simple unital projectionless C^* -algebra, *Amer. J. Math.*, **121**(2)(1980), 359–413.
- [18] M. KAROUBI, *K -Theory: An Introduction*, Springer-Verlag Berlin Heidelberg New York, 1978.
- [19] X. LI, Every classifiable simple C^* -algebra has a Cartan subalgebra, *Invent. Math.*, **219**(2020), 653–699.
- [20] F. J. MURRAY, J. VON NEUMANN, On rings of operators. IV, *Ann. Math.*, **44**(4)(1943), 716–808.
- [21] J. R. MUNKRES, *Topology*, Prentice Hall, Inc., Upper Saddle River, NJ, 2000.
- [22] A. L. T. PATERSON, *Groupoids, Inverse Semigroups, and their Operator Algebras*, Birkhauser Boston, 1999.
- [23] M. PIMSNER, D. VOICULESCU, Imbedding the irrational rotation algebra into an AF-algebra, *J. Operator Theory*, **4**(1)(1980), 201–210.
- [24] M. PIMSNER, D. VOICULESCU, Exact sequences for K -groups and Ext-groups of certain cross-product C^* -algebras, *J. Operator Theory*, **4**(1)(1980), 93–118.
- [25] I. F. PUTNAM, The C^* -algebras associated with minimal homeomorphisms of the Cantor set, *Pac. J. Math*, **136**(2)(1989), 329–353.
- [26] I. F. PUTNAM, On the K -theory of C^* -algebras of principal groupoids, Rocky M. Journal, 1996.
- [27] I. F. PUTNAM, An excision theorem for the K -theory of C^* -algebras, *J. Operator Theory*, **38**(1997), 151–171.

- [28] I. F. PUTNAM, *Cantor Minimal Systems*, Amer. Math. Soc. Lecture Series 70, Providence, RI, 2018.
- [29] I. F. PUTNAM, Some classifiable groupoid C^* -algebras with prescribed K -theory, *Mathematische Annalen*, **370**(2018), 1361–1387.
- [30] I. F. PUTNAM, An excision theorem for the K -theory of C^* -algebras, with applications to groupoid C^* -algebras, *Münster J. Math.*, **14**(2021), 349–402.
- [31] J. RENAULT, *A Groupoid Approach to C^* -algebras*, Lect. Notes Math. 739, Springer-Verlag Berlin Heidelberg New York, 1980.
- [32] M. RØRDAM, F. LARSEN, N. J. LAUSTEN, *An Introduction to K -Theory for C^* -Algebras*, London Mathematical Society Student Texts 49, Cambridge University Press, 2000.
- [33] C. SCHAFHAUSER, Subalgebras of simple AF-algebras, *Ann. Math.*, **192**(2)(2020), 309–352.
- [34] A. SIMS, G. SZABÓ, D. WILLIAMS, *Operator Algebras and Dynamics: Groupoids, Crossed Products, and Rokhlin Dimension*, Springer Nature, Switzerland, 2020.
- [35] J. L. TU, La conjecture de Baum–Connes pour les feuilletages moyennables, *K-Theory* **17**(1999), 215–264.
- [36] N. E. WEGGE-OLSEN, *K -Theory and C^* -algebras*, Oxford University Press, 1993.
- [37] D. WILLIAMS, *A Toolkit for Groupoid C^* -algebras*, Amer. Math. Soc. Mathematical Surveys and Monographs 241, Providence, RI, 2019.