

**SOME APPLICATIONS OF  
FRACTIONAL CALCULUS**

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## ABSTRACT

The familiar (Riemann–Liouville) operator of fractional calculus is applied in order to evaluate a class of definite integrals involving products of certain multivariable hypergeometric functions. Relevant connections of the results presented here with those available in the literature are also indicated.

## 1. INTRODUCTION AND PRELIMINARIES

One of the most common definitions for the fractional derivative of order  $\mu$ , found in the literature, is based upon the familiar Riemann–Liouville integral (see, for example, [3, Chapter 13] and [5]):

$$\begin{aligned}
D_z^\mu\{f(z)\} &= \frac{1}{\Gamma(-\mu)} \int_0^z (z-\zeta)^{-\mu-1} f(\zeta) d\zeta, \quad \operatorname{Re}(\mu) < 0 \\
&= \frac{d^m}{dz^m} D_z^{\mu-m}\{f(z)\}, \quad m-1 < \operatorname{Re}(\mu) < m
\end{aligned} \tag{1.1}$$

$$(m = 1, 2, 3, \dots),$$

where the path of integration is along a line from 0 to  $z$  in the complex  $\zeta$ -plane. The following result is an easy consequence of the definition (1.1).

**THEOREM 1** (*cf.*, e.g., Srivastava and Manocha [14, p. 289, Theorem 2]). *If a function  $f(z)$ , analytic in the disk  $|z| < R$ , has the power series expansion*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad (|z| < R), \tag{1.2}$$

then

$$D_z^\mu\{z^{\lambda-1} f(z)\} = z^{\lambda-\mu-1} \sum_{n=0}^{\infty} \frac{\Gamma(\lambda+n)}{\Gamma(\lambda-\mu+n)} a_n z^n, \tag{1.3}$$

provided that  $\operatorname{Re}(\lambda) > 0$  and  $|z| < R$ .

An analogous theorem involving multiple series would follow similarly from the definition (1.1). As a matter of fact, in terms of Lauricella's hypergeometric function  $F_D^{(r)}$  of  $r$  variables and its confluent form  $\Phi_D^{(r)}$ , we have [14, p. 303, Problem 1]

$$\begin{aligned}
D_z^{\lambda-\mu} \left\{ z^{\lambda-1} \prod_{j=1}^r (1-x_j z)^{-\alpha_j} \right\} \\
= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} F_D^{(r)}[\lambda, \alpha_1, \dots, \alpha_r; \mu; x_1 z, \dots, x_r z], \quad (1.4) \\
\operatorname{Re}(\lambda) > 0; \quad \max\{|x_1 z|, \dots, |x_r z|\} < 1,
\end{aligned}$$

and [12, p. 649, Equation (3.7)]

$$\begin{aligned}
D_z^{\lambda-\mu} \left\{ z^{\lambda-1} \exp(x_r z) \prod_{j=1}^{r-1} (1-x_j z)^{-\alpha_j} \right\} \\
= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} \Phi_D^{(r)}[\lambda, \alpha_1, \dots, \alpha_{r-1}; \mu; x_1 z, \dots, x_r z], \quad (1.5) \\
\operatorname{Re}(\lambda) > 0; \quad \max\{|x_1 z|, \dots, |x_{r-1} z| < 1\}, \quad |x_r z| < \infty,
\end{aligned}$$

where, in terms of the Pochhammer symbol  $(\lambda)_n$  given by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if } n = 0, \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & \forall n \in \{1, 2, 3, \dots\}, \end{cases} \quad (1.6)$$

the multivariable hypergeometric functions  $F_D^{(r)}$  and  $\Phi_D^{(r)}$  are defined by (*cf.* [4, p. 113]; see also [1, p. 114, Equation (4)])

$$\begin{aligned}
F_D^{(r)}[\alpha, \beta_1, \dots, \beta_r; \gamma; x_1, \dots, x_r] \\
= \sum_{m_1, \dots, m_r=0}^{\infty} \frac{(\alpha)_{m_1+\dots+m_r} (\beta_1)_{m_1} \cdots (\beta_r)_{m_r}}{(\gamma)_{m_1+\dots+m_r}}
\end{aligned}$$

$$\cdot \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_r^{m_r}}{m_r!}, \quad (1.7)$$

$$\max\{|x_1|, \dots, |x_r|\} < 1,$$

and (cf. [11, p. 373, Equation (12)])

$$\begin{aligned} & \Phi_D^{(r)}[\alpha, \beta_1, \dots, \beta_{r-1}; \gamma; x_1, \dots, x_r] \\ = & \sum_{m_1, \dots, m_r=0}^{\infty} \frac{(\alpha)_{m_1+\dots+m_r} (\beta_1)_{m_1} \cdots (\beta_{r-1})_{m_{r-1}}}{(\gamma)_{m_1+\dots+m_r}} \\ & \cdot \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_r^{m_r}}{m_r!}, \quad (1.8) \end{aligned}$$

$$\max\{|x_1|, \dots, |x_{r-1}|\} < 1, \quad |x_r| < \infty,$$

respectively. Since [13, p. 35, Equation (15)]

$$\begin{aligned} & \Phi_D^{(r)}[\alpha, \beta_1, \dots, \beta_{r-1}; \gamma; x_1, \dots, x_r] \\ = & \lim_{|\beta_r| \rightarrow \infty} \left\{ F_D^{(r)}[\alpha, \beta_1, \dots, \beta_r; \gamma; x_1, \dots, x_{r-1}, x_r/\beta_r] \right\}, \quad (1.9) \end{aligned}$$

the fractional derivative formula (1.5) would follow readily as a confluent case of (1.4) when  $x_r$  is replaced by  $x_r/\alpha_r$  and  $|\alpha_r| \rightarrow \infty$  (see also [12, p. 649]).

The familiar Leibniz rule for ordinary derivatives admits itself of a generalization [5, p. 76, Equation (5.5.2)]:

$$D_z^\mu \{f(z)g(z)\} = \sum_{n=0}^{\infty} \begin{bmatrix} \mu \\ n \end{bmatrix} D_z^{\mu-n} \{f(z)\} D_z^n \{g(z)\} \quad (1.10)$$

in terms of fractional derivatives. Formula (1.10) suffers from an apparent drawback in the sense that the interchange of  $f(z)$  and  $g(z)$  on the right-hand side is not obvious. A further generalization of (1.10), considered by Watanabe [15] and Osler ([6], [7]), *without* such a drawback, is given by

$$D_z^\mu \{f(z)g(z)\} = \sum_{n=-\infty}^{\infty} \begin{bmatrix} \mu \\ n+\nu \end{bmatrix} D_z^{\mu-n-\nu} \{f(z)\} D_z^{n+\nu} \{g(z)\}, \quad (1.11)$$

where  $\nu$  is an arbitrary real or complex number.

Of our main concern here is the following integral analog of (1.10) due to Osler [8]:

$$D_z^\mu \{f(z)g(z)\} = \int_{-\infty}^{\infty} \begin{bmatrix} \mu \\ \omega \end{bmatrix} D_z^{\mu-\omega} \{f(z)\} D_z^\omega \{g(z)\} d\omega, \quad (1.12)$$

in which the interchange of  $f(z)$  and  $g(z)$  is obviously permissible. We shall illustrate how this integral analog of the generalized Leibniz rule (1.10) can be applied, in conjunction with such fractional derivative formulas as (1.4) and (1.5), with a view to evaluating a class of definite integrals involving products of multivariable hypergeometric functions. Our main result (2.1) below would unify and extend several integral formulas given by Osler [8] and, more recently, by Arora and Koul [2].

## 2. MAIN RESULTS

Our main results are contained in the following theorems.

**THEOREM 2.** *If  $\operatorname{Re}(\xi+\eta) > 1$ , and if*

$$\max\{|x_1|, \dots, |x_r|, |y_1|, \dots, |y_s|\} < 1,$$

*then*

$$\begin{aligned} & \int_{-\infty}^{\infty} \begin{bmatrix} \mu \\ \omega \end{bmatrix} \frac{d\omega}{\Gamma(\xi-\mu+\omega) \Gamma(\eta-\omega)} \\ & \quad \cdot F_D^{(r)}[\xi, \alpha_1, \dots, \alpha_r; \xi-\mu+\omega; x_1, \dots, x_r] \\ & \quad \cdot F_D^{(s)}[\eta, \beta_1, \dots, \beta_s; \eta-\omega; y_1, \dots, y_s] \\ & = \frac{\Gamma(\xi+\eta-1)}{\Gamma(\xi) \Gamma(\eta) \Gamma(\xi+\eta-\mu-1)} \\ & \quad \cdot F_D^{(r+s)}[\xi+\eta-1, \alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s; \xi+\eta-\mu-1; \\ & \quad \quad \quad x_1, \dots, x_r, y_1, \dots, y_s]. \end{aligned} \quad (2.1)$$

**THEOREM 3.** *If  $\operatorname{Re}(\xi+\eta) > 1$ ,  $|x_r| < \infty$ ,  $|y_s| < \infty$ , and if*

$$\max\{|x_1|, \dots, |x_{r-1}|, |y_1|, \dots, |y_{s-1}|\} < 1,$$

*then*

$$\begin{aligned}
& \int_{-\infty}^{\infty} \left[ \begin{matrix} \mu \\ \omega \end{matrix} \right] \frac{d\omega}{\Gamma(\xi-\mu+\omega) \Gamma(\eta-\omega)} \\
& \quad \cdot \Phi_D^{(r)}[\xi, \alpha_1, \dots, \alpha_{r-1}; \xi-\mu+\omega; x_1, \dots, x_r] \\
& \quad \cdot \Phi_D^{(s)}[\eta, \beta_1, \dots, \beta_{s-1}; \eta-\omega; y_1, \dots, y_s] \\
& = \frac{\Gamma(\xi+\eta-1)}{\Gamma(\xi) \Gamma(\eta) \Gamma(\xi+\eta-\mu-1)} \\
& \quad \cdot \Phi_D^{(r+s-1)}[\xi+\eta-1, \alpha_1, \dots, \alpha_{r-1}, \beta_1, \dots, \beta_{s-1}; \xi+\eta-\mu-1; \\
& \quad \quad \quad x_1, \dots, x_{r-1}, y_1, \dots, y_{s-1}, x_r+y_s]. \quad (2.2)
\end{aligned}$$

**Proof of Theorem 2.** The assertion (2.1) of Theorem 2 follows when we set

$$f(z) = z^{\xi-1} \prod_{j=1}^r \left\{ (1-x_j z)^{-\alpha_j} \right\} \quad (2.3)$$

and

$$g(z) = z^{\eta-1} \prod_{j=1}^s \left\{ (1-y_j z)^{-\beta_j} \right\} \quad (2.4)$$

in (1.12), apply the fractional derivative formula (1.4), and then set the scaling factor  $z = 1$ .

**Proof of Theorem 3.** To prove the assertion (2.2) of Theorem 3, we choose

$$f(z) = z^{\xi-1} \exp(x_r z) \prod_{j=1}^{r-1} \left\{ (1-x_j z)^{-\alpha_j} \right\} \quad (2.5)$$

and

$$g(z) = z^{\eta-1} \exp(y_s z) \prod_{j=1}^{s-1} \left\{ (1-y_j z)^{-\beta_j} \right\}$$

in (1.12), apply the fractional derivative formula (1.5), and then set the scaling factor  $z = 1$ .

Alternatively, in view of the limit relationship (1.9), Theorem 3 can be derived as a confluent case of Theorem 2 when  $x_r$  and  $y_s$  are replaced by  $x_r/\alpha_r$  and  $y_s/\beta_s$ , respectively, and

$$\min\{|\alpha_r|, |\beta_s|\} \rightarrow \infty.$$

By appealing to a special case of the elementary series identity [9, p. 4, Equation (12)]:

$$\begin{aligned} \sum_{m_1, \dots, m_r=0}^{\infty} \Delta(m_1 + \dots + m_r) \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_r^{m_r}}{m_r!} \\ = \sum_{m=0}^{\infty} \Delta(m) \frac{(x_1 + \dots + x_r)^m}{m!}, \end{aligned} \quad (2.7)$$

the second member of (2.1) can be reduced, in this limit process, to the right-hand side of (2.2).

**Remark.** A direct proof of each of the assertions (2.1) and (2.2), *without* using fractional calculus, can be constructed by expressing the multivariable hypergeometric functions on the left-hand side in terms of their multiple series given by (1.7) or (1.8), and then integrating the resulting multiple series termwise by means of the known integral [3, p. 300, Equation (21)]:

$$\begin{aligned}
& \int_{-\infty}^{\infty} \frac{d\omega}{\Gamma(\alpha+\omega) \Gamma(\beta-\omega) \Gamma(\gamma+\omega) \Gamma(\delta-\omega)} \\
&= \frac{\Gamma(\alpha+\beta+\gamma+\delta-3)}{\Gamma(\alpha+\beta-1) \Gamma(\beta+\gamma-1) \Gamma(\gamma+\delta-1) \Gamma(\delta+\alpha-1)}, \\
& \quad \text{Re}(\alpha+\beta+\gamma+\delta) > 3. \tag{2.8}
\end{aligned}$$

### 3. APPLICATIONS

In the special case of Theorem 2 when  $r = s$ , if we further set

$$x_j = y_j \quad (j = 1, \dots, r),$$

and apply the particular case  $p = q = 1$  of a reduction formula given by Srivastava [10, p. 3083, Equation (21)], we shall obtain

$$\begin{aligned}
& \int_{-\infty}^{\infty} \left[ \begin{matrix} \mu \\ \omega \end{matrix} \right] \frac{d\omega}{\Gamma(\xi-\mu+\omega) \Gamma(\eta-\omega)} \\
& \quad \cdot F_D^{(r)}[\xi, \alpha_1, \dots, \alpha_r; \xi-\mu+\omega; x_1, \dots, x_r] \\
& \quad \cdot F_D^{(r)}[\eta, \beta_1, \dots, \beta_r; \eta-\omega; x_1, \dots, x_r] \\
&= \frac{\Gamma(\xi+\eta-1)}{\Gamma(\xi) \Gamma(\eta) \Gamma(\xi+\eta-\mu-1)} \\
& \quad \cdot F_D^{(r)}[\xi+\eta-1, \alpha_1+\beta_1, \dots, \alpha_r+\beta_r; \xi+\eta-\mu-1; x_1, \dots, x_r], \\
& \quad \text{Re}(\xi+\eta) > 1; \quad \max\{|x_1|, \dots, |x_r|\} < 1. \tag{3.1}
\end{aligned}$$

Similarly, we find from Theorem 3 with  $r = s$  and

$$x_j = y_j \quad (j = 1, \dots, r-1)$$

that

$$\begin{aligned} & \int_{-\infty}^{\infty} \begin{bmatrix} \mu \\ \omega \end{bmatrix} \frac{d\omega}{\Gamma(\xi - \mu + \omega) \Gamma(\eta - \omega)} \\ & \quad \cdot \Phi_D^{(r)}[\xi, \alpha_1, \dots, \alpha_{r-1}; \xi - \mu + \omega; x_1, \dots, x_r] \\ & \quad \cdot \Phi_D^{(r)}[\eta, \beta_1, \dots, \beta_{r-1}; \eta - \omega; x_1, \dots, x_{r-1}, y_r] \\ & = \frac{\Gamma(\xi + \eta - 1)}{\Gamma(\xi) \Gamma(\eta) \Gamma(\xi + \eta - \mu - 1)} \\ & \quad \cdot \Phi_D^{(r)}[\xi + \eta - 1, \alpha_1 + \beta_1, \dots, \alpha_{r-1} + \beta_{r-1}; \xi + \eta - \mu - 1; x_1, \dots, x_{r-1}, x_r + y_r], \\ & \quad \operatorname{Re}(\xi + \eta) > 1; \max\{|x_1|, \dots, |x_{r-1}|\} < 1; \\ & \quad \max\{|x_r|, |y_r|\} < \infty. \end{aligned} \tag{3.2}$$

Formulas (3.1) and (3.2) can indeed be proven directly by using (1.12) in conjunction with (1.4) or (1.5) appropriately. The special case  $r = 2$  of (3.1), and the special cases of (3.2) when  $r = 2$  and  $r = 3$ , happen to be among the main results of Arora and Koul [2]. Moreover, (3.1) and (3.2) when  $r = 1$  are essentially equivalent to certain integral formulas given earlier by Osler [8, p. 913, Entry 9; p. 915, Entry 14].

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