Analysis of a Mollified Kinetic Equation for Granular Media

by

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ABSTRACT

We study a nonlinear kinetic model describing the interactions of particles in a granular medium, i.e. inelastic systems where kinetic energy is not conserved due to internal friction. Examples of particles that fall into this category are sand, ground coffee and many others. Originally studied by Benedetto, Caglioti and Pulvirenti in the one-dimensional setting (RAIRO Model. Math. Anal. Numér., 31(5): 615-641, (1997)) the original model contained inconsistencies later accounted for and corrected by invoking a mollifier (Modélisation Mathématique et Analyse Numérique, M2AN, Vol. 33, No 2, pp. 439441 (1999)). This thesis approximates the generalized model presented by Agueh (Arch. Rational Mech., Anal. 221, pp. 917-959 (2016)) with the added assumption of a spatial mollifier present in the kinetic equation. In dimension $d \geq 1$ this model reads as

$$\partial_t f + v \cdot \nabla_x f = \mathrm{div}_v(f([\eta_\alpha \nabla W] *_{(x,v)} f))$$

where f is a non-negative particle density function, W is a radially symmetric class C^2 velocity interaction potential, and η_α is a mollifier. A physical interpretation of this approximation is that the particles are spheres of radius $\alpha > 0$ as opposed to the original assumption of being point-masses. Properties lost by this approximation and macroscopic quantities that remain conserved are discussed in greater detail and contrasted.

The main result of this thesis is a proof of the weak global existence and uniqueness. An argument utilizing the tools of Optimal Transport allows simple construction of a weak solution to the kinetic model by transporting an initial measure under the characteristic flow curves. Concluding regularity arguments and restrictions on the velocity interaction potential ascertain that global classical solutions are obtained.

CONTENTS

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Chapter 1

Introduction

The focus of this thesis is to describe the well-posedness of a kinetic model for granular media.
The first part of this document demonstrates how the tools of optimal transport yield a unique solution to a Monge-Ampère type equation. This result is well known and follows from Brenier's theorem, a ground breaking result in the field of optimal transport [11]. We introduce the optimal mass transport model, an optimization problem which seems initially irrelevant to the Monge-Ampère equation, but will be shown, in a sense, to possess the same solution.

The remainder will be concerned with a kinetic equation that describes the evolution of an inelastic particle collision system. More specifically, the model we will discuss is a mollified form of the generalized unmollified model introduced by Agueh [3]. The reason for considering the mollified case is that the former model has not been proven to be globally well-posed unless it is restricted to a one-dimensional setting and also given some restricted conditions on the initial data [4]. We hope that the results here will assist in this open problem, and one may find a way to control the parameters in such a way that the solutions of the mollified case will, in a way, lead to solutions of the unmollified model. Attempts to this end will be further discussed in this thesis.

For clarity, we list the purpose of each chapter in greater detail below:

- **Chapters 2-3:** These chapters introduce the tools of Optimal Transport required to study the mollified kinetic model. The second chapter focuses on the functional analysis aspect (e.g. formulating the Banach space where solutions to the kinetic model lie) while the third chapter focuses on intriguing historical results.
- **Chapters 4-5:** These chapters focus on the construction and properties of solutions to the mollified kinetic model. The fourth chapter establishes macroscopic properties that the solutions possess. These justify the spaces considered in the second chapter. The fifth chapter uses the concluding remarks of the fourth chapter to construct a weak solution using the method of characteristics.

- **Chapter 6:** This chapter uses regularity arguments to show that the solution obtained in the fifth chapter is a measurable function. Further smoothness assumptions on the velocity interaction potential allow one to obtain solutions in Sobolev spaces.

- **Chapter 7:** This chapter introduces the unmollified kinetic model presented by Agueh [3]. We briefly summarize existence results that have been discovered to the unmollified kinetic model. We also explain attempts made to relate this model to the mollified kinetic model.

1.1 Notation

We clarify some of the commonly used notations in this thesis. Other notations used will be explicitly stated when presented.

1. The constant $T > 0$ represents the largest time we will consider to analyze any model or system.

2. $\overline{U}$ represents the closure of a set U.

3. For a function $T = (T_1, T_2, ..., T_n)$ we denote it's derivative by

$$\nabla T(x_1, x_2, .., x_n) = \begin{pmatrix} \frac{\partial T_1}{\partial x_1} & \frac{\partial T_2}{\partial x_1} & \cdots & \frac{\partial T_n}{\partial x_1} \\ \frac{\partial T_1}{\partial x_2} & \frac{\partial T_2}{\partial x_2} & \cdots & \frac{\partial T_n}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial T_1}{\partial x_n} & \frac{\partial T_2}{\partial x_n} & \cdots & \frac{\partial T_n}{\partial x_n} \end{pmatrix}$$

   If T has at least two distinct variables (say position and velocity) given by $T = T(x, v)$ then we denote differentiation with respect to the first component by $\nabla_x T(x, v)$, and similarly for the second.

4. Given a function T its Laplacian is represented by ΔT.

5. We define the following norms for $x = (x_1, x_2, ..., x_d) \in \mathbb{R}^d$ and $p > 0$:

$$\|x\|_p = (|x_1|^p + |x_2|^p + \ldots |x_d|^p)^{1/p}$$

$$|x| = |x_1| + |x_2| + \ldots + |x_n|.$$

   Notice that we put a special emphasis on the second norm; which satisfies $\|x\|_1 = |x|$.

6. We represent the open ball centered at $x_0 \in \mathbb{R}^d$ with radius $\delta > 0$ by

$$B(x_0, \delta) = \{x \in \mathbb{R}^d : \|x - x_0\|_2 < \delta\}.$$

   Specifically, we denote the ball of radius R centered at the origin of $\mathbb{R}^d$ by

$$B_R = \{v \in \mathbb{R}^d : \|v\|_2 \leq R\}.$$

7. Given a set U, the function χ_U is the characteristic function defined by

$$\chi_U(x) = \begin{cases} 1 & x \in U \\ 0 & \text{otherwise} \end{cases}.$$

8. Given a set U, the space $C_b(U)$ represents the space of continuous functions bounded on U.

9. For an integer k and a set U: the space $C^k(U)$ represents the space of functions that are k-times continuously differentiable on U.

10. Given a set U, the space $C_0^\infty(U)$ represents the set of functions that are infinitely continuously differentiable and have compact support in U.

11. Given a set U, the space $\mathrm{Lip}(U)$ represents the set of functions that are Lipschitz on U. Given a function $f : U \to \mathbb{R}^d$ we denote its Lipschitz constant as $\mathrm{Lip}(f)$.

12. For an integer k and set U: the Sobolev space $W^{k,\infty}(U)$ represents the set of functions that are k-times differentiable in a weak sense and have bounded derivatives on U.

13. For $p > 0$ we define the L^p-spaces on a set U by

$$L^p(U) = \left\{ f : U \to \mathbb{R} : \|f\|_{L^p(U)}^p = \int_U |f(x)|^p dx < \infty \right\}$$

with corresponding norm. For the case $p = \infty$,

$$L^\infty(U) = \left\{ f : U \to \mathbb{R} : \|f\|_{L^\infty(U)} = \mathrm{esssup}_{x \in U} |f(x)| < \infty \right\}.$$

We may often consider L^p spaces where the norms above allow integration with respect to other measures and not solely the Lebesgue measure. If so, we denote this by $L^p(\mu)$ or $L^p(d\mu)$ for a measure μ.

14. The space $C_w([0,T];\cdot)$ denotes the set of functions that are weakly continuous in time. Specifically, if we are given a function $f = f(t,\cdot)$ we say that f is weakly continuous in time provided that for each continuously bounded function ϕ and sequence $t_n \to t^*$ we have

$$\lim_{t_n \to t^*} \int \phi(x) f(t_n, x) dx = \int \phi(x) f(t^*, x) dx.$$

15. Given a set $U \subset \mathbb{R}^d$, the space $P(U)$ denotes the set of probability measures on U. For an integer $p \geq 1$ the space of finite-p^{th} moment probability measures is denoted

$$P_p(U) = \left\{ \mu \in P(U) : \int_U \|x\|_2^p d\mu(x) < \infty \right\}.$$

16. Given a set U, the space P_U denotes the set of probability measures on the set $\mathbb{R}^d \times U$ with support in U. Similarly, PF_U denotes the set of probability density functions on the set $\mathbb{R}^d \times U$ with support in U.

17. Given a measure $\mu_0 \in P_{B_R}$ we define the set

$$P^T_{\mu_0} = \left\{ \mu \in C_w([0,T]; P_{B_R}) : \mu(t=0,\cdot) = \mu_0(\cdot) \right\}.$$

    Similarly, given a function $f_0 \in PF_{B_R}$ we define the set

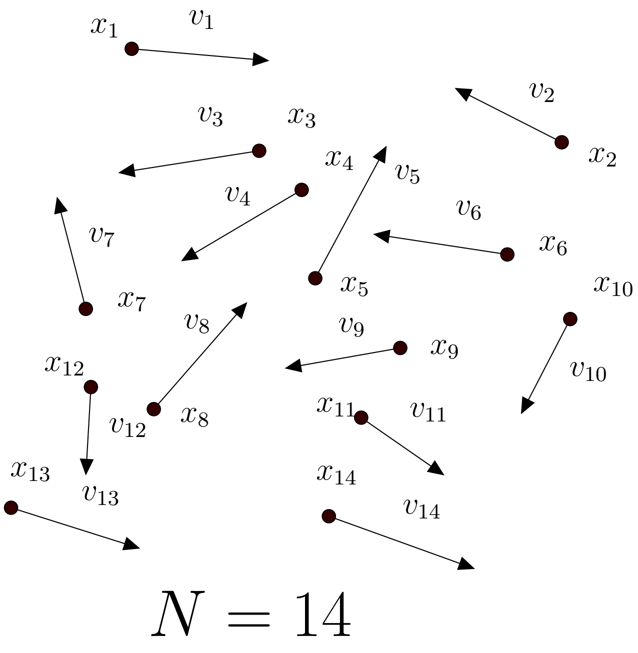
$$PF^T_{f_0} = \left\{ f \in C_w([0,T]; PF_{B_R}) : f(t=0,\cdot) = f_0(\cdot) \right\}.$$

18. Given probability measures μ and ν we denote the set of transference plans between them by $\Pi(\mu,\nu)$.

1.2 The Mollified Kinetic Model For Granular Media

We study the multi-dimensional case of particle systems undergoing inelastic collisions in a granular medium. Indeed, a granular media is a collection of discrete macroscopic solid particles that, upon collision, result in a loss of energy due to forces of friction. Many particles can be characterized by this feature. Some examples of this are grain, sand, and powders. The nature of such particles is in essence easy to describe, while their behavior is quite complex. This makes well-posedness results difficult to obtain.
This process follows the outline of [1], [3] as follows.

Let $d \in \mathbb{N}$ and consider a system of N particles occupying a region of $\mathbb{R}^d$ with the system normalized so that every particle has mass $1/N$. Let $i = 1, ..., N$ and denote by $x_i(t)$ and $v_i(t)$ the respective position and velocity of the ith particle at time $t \geq 0$. Furthermore, let (x_i^0, v_i^0) denote the initial position and velocity of the ith particle. We assume that every granular particle flows freely, as given by their velocity, until they occupy the position of another particle; then they collide inelastically.



The particles are further assumed to be indistinguishable. The proposed model of this system was first formulated by Benedetto, Caglioti and Pulvirenti [1] in 1997. They considered the one-dimensional case and were able to establish local existence. The model has been further generalized in more recent literature to $\mathbb{R}^d$, specifically by Agueh [3] in 2016 as the system of ordinary differential equations

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = \varepsilon \sum_{j=1}^{N} \delta(x_i - x_j)(v_j - v_i)\|v_i - v_j\|_2 \\ (x_i(0), v_i(0)) = (x_i^0, v_i^0) \end{cases} \tag{1.1}$$

where δ denotes the Dirac measure centered at the origin, $i = 1, ..., N$, and ε is the degree of inelasticity.

There is a conceptual error in this formulation because the right-hand side of the second equation is a measure while the left-hand side is a vector. This error has been accounted for by Benedetto, Caglioti, and Pulvirenti in [2] where they have replaced the Dirac measure with a net of mollifiers η_α indexed by $\alpha > 0$, that approximates the measure in the sense that

- For every $x \in \mathbb{R}^d$ we have $\eta_\alpha(x) \geq 0$,
- η_α is radially symmetric,
- η_α lies in the class $C_0^\infty \cap P(\mathbb{R}^d)$,
- We have $\eta_\alpha \to \delta$ as $\alpha \to 0^+$ in the distributional sense.

Remark. *Such functions exist and are given in many introductory texts; one example being*

$$\eta_\alpha(x) = \alpha^{-1} \exp\left(\frac{\alpha^2}{\alpha^2 - |x|^2}\right) \chi_{B_\alpha}(x)$$

where χ_{B_α} *is the characteristic function of the ball of radius* α *centered at the origin,* B_α*.*

The new system of equations becomes by a simple matter of replacement,

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = \varepsilon \sum_{j=1}^{N} \eta_\alpha(x_i - x_j)(v_j - v_i)\|v_i - v_j\|_2 \\ (x_i(0), v_i(0)) = (x_i^0, v_i^0) \end{cases} \tag{1.2}$$

for $i = 1, ..., N$ and $\alpha > 0$. The physical interpretation of this is that the particles are now spheres of radius $\alpha > 0$ instead of point masses. We still assume this model is justifiable in the hope that it may be used to approximate solutions to (1.1) by sending $\alpha \to 0^+$, yielding the original model.

In both models the particles change velocities according to an interaction rule given by a potential

$$\phi : \mathbb{R}^d \to [0, \infty)$$
$$v \mapsto \frac{1}{3}\|v\|_2^3$$

in the sense that $\nabla\phi(v) = v\|v\|_2$ for all $v \in \mathbb{R}^d$. We will make the following alteration to allow further generality: the particles collide and will alter their velocities according to a class C^2, strictly convex and radially symmetric interaction potential

$$W : \mathbb{R}^d \to \mathbb{R}$$

on the velocity space. It is a trivial exercise to show that ϕ satisfies all the properties listed for W. The reformulated model (1.1) now reads as

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = \varepsilon \sum_{j=1}^{N} \eta_\alpha(x_i - x_j)\nabla W(v_j - v_i) \\ (x_i(0), v_i(0)) = (x_i^0, v_i^0) \end{cases} \tag{1.3}$$

for $i = 1, ..., N$ and $\alpha > 0$.

Since the number of particles is very large, it is easier to approximate (1.3) by taking $N \to \infty$ and describe the motion of the system at a kinetic level in terms of an associated density function. That is, let $f(t, x, v)$ denote the density of the particles occupying position $x \in \mathbb{R}^d$ and moving with velocity $v \in \mathbb{R}^d$ at time $t > 0$. Accordingly let $f_0(x, v)$ denote the initial density at time $t = 0$. The process of finding the associated kinetic equation is formally described in several ways in [1], [3], [4]. Benedetto, Caglioti and Pulvirenti for instance [1], derive the equation by studying the Liouville equation for the time evolution of the system. In the following section we will derive the kinetic model by means of the mean-field limit as in [3].
By an appropriate scaling and taking $N \to \infty$, we will show that the associated kinetic equation to (1.3) is given by

$$\frac{\partial f}{\partial t}(t, x, v) + v \cdot \nabla_x f(t, x, v) = \mathrm{div}_v(f(t, x, v)F_{f,\alpha}(t, x, v)) \tag{1.4}$$

where the force term is described by

$$F_{f,\alpha}(t, x, v) = \eta_\alpha(x)\nabla W(v) *_{(x,v)} f(t, x, v) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \eta_\alpha(x - y)\nabla W(v - u)f(t, y, u)dydu.$$

This model is our primary focus and is referred to as the mollified kinetic equation for granular media. The unmollified model, described by [1, 2, 3, 4] is obtained by taking the limit as $\alpha \to 0^+$ in (1.4); this yields

$$\frac{\partial f}{\partial t}(t, x, v) + v \cdot \nabla_x f(t, x, v) = \mathrm{div}_v(f(t, x, v)F_f(t, x, v)) \tag{1.5}$$

where in contrast, the force term is described by

$$F_f(t, x, v) = \nabla W(v) *_v f(t, x, v) = \int_{\mathbb{R}^d} \nabla W(v - u)f(t, x, u)du$$

using the mollifier properties described above. The singularity in space prompts the difficulty when establishing global existence of the model (1.5). We will focus solely on the model (1.4) and establish the well-posedness property of it.

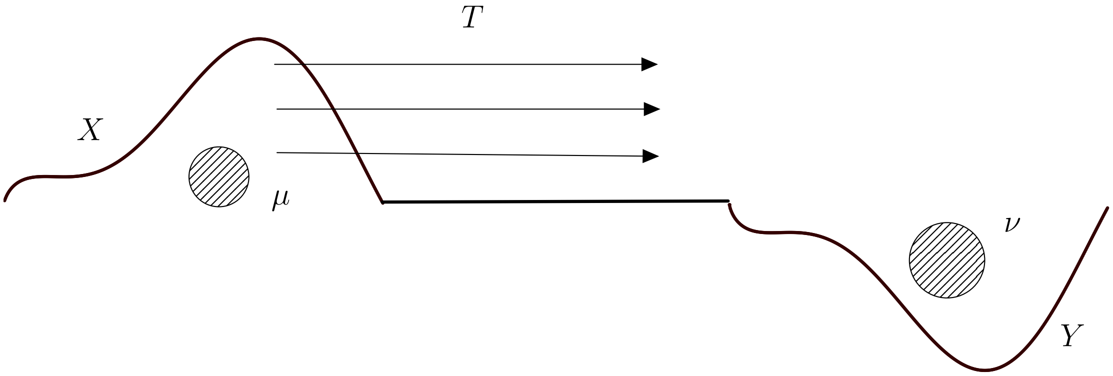
Chapter 2

Optimal Transport

In the following we will give a brief overview of the tools used in optimal transport theory and how they apply to the Monge-Ampère equation. Although the subject of optimal transport seems, at first glance, unrelated to nonlinear differential equations, we demonstrate how its rigorous development is in fact intrinsic to this field.

2.1 Introduction to Optimal Transport

Optimal mass transport is a field in the *Calculus of Variations* concerned with allocating mass from a space X to a space Y. The problem was first considered by Gaspard Monge in his paper "Mémoire sur la théorie des déblais et des remblais" in 1781 [9]. Due to recent connections with modern research, the study of optimal transport has flourished dramatically within the last thirty years. To be precise, it was Yann Brenier's paper "Décomposition polaire et réarrangement des champs de vecteurs" in 1987 [11] that gave light to the connection the problem has with partial differential equations, fluid mechanics, geometry, probability theory and functional analysis. This connection has provided a great number of astounding results, given very recently, in these fields. This result is known today as *Brenier's theorem* and describes the form the solution to the (yet to be described) quadratic optimal transport problem.



We shall state (informally) the problem of optimal transport as follows (mimicking the outline of [12]). Let us consider a measurable space X, representing a pile of sand, and we wish to move this to a measurable space Y, representing a hole which the sand fills. Clearly a requirement we need is that the size of the space X and Y must be equivalent in order for the problem to have a solution. Let us represent the density of the sand in X by a measure μ and the density of the hole in Y by a measure ν; the above condition requires that

$$\int_X d\mu = \int_Y d\nu$$

and for sake of convenience we will re-normalize the above so that μ and ν represent probability densities. Now, moving the sand around is not free, it requires us to hire people to move everything from the space X to Y. Let us say that this cost is given by a measurable function $c : X \times Y \to \overline{\mathbb{R}}$, where we are using the notation $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, thereby allowing the possibility that the cost may take infinite values. Realistically, the cost should also be non-negative as it is not reasonable to assume somebody will pay us to also work for us (otherwise this would all be too easy). To elaborate, for $x \in X$ and $y \in Y$ the value $c(x, y)$ represents the cost of moving a (point mass) grain of sand from position x to position y. When the model was first proposed, Monge chose the specific cost function $c(x, y) = \|x - y\|_2$ given by the Euclidean distance [9]. This setup bears the question that truly is the heart of optimal transport:

“How do we transport all the sand with the least possible cost?”

This should not confuse the reader in thinking we are minimizing the cost function. The cost function is fixed and not up for negotiation. The minimization will be over all the possible ways we may transport grains of sand from X into Y. We generally represent a transportation map as $T : X \to Y$. One requirement that we should have is

$$\nu(U) = \mu(T^{-1}(U)) \tag{2.1}$$

for all measurable subsets $U \subset Y$. This requirement is referred to as T pushes μ forward to ν and is traditionally denoted $\nu = T\#\mu$. It means that the quantity of sand piled up in a subregion U of Y is exactly equal to the amount of sand taken from the corresponding region $T^{-1}(U)$ of X through the transportation map T. The push-forward can be reformulated functionally as:

$$\int_{T^{-1}(Y)} \psi(T(x)) d\mu(x) = \int_Y \psi(y) d\nu(y) \tag{2.2}$$

for every test function $\psi \in L^1(\mu) \cap L^1(\nu)$. This particular formulation will be useful in the following chapter. The above question may now be reformulated mathematically:

> "Solve the following minimization problem:
>
> $$\min_T \left\{ \int_X c(x, T(x)) d\mu(x) \right\}$$
>
> over the set of all measurable maps $T : X \to Y$ such that $T\#\mu = \nu$."

This problem is referred to as *Monge's transport problem.* Although the problem seems nicely formulated, it can be ill-posed for the following reasons:

1. No admissible T may exist.
2. The constraint $\nu = T\#\mu$ is highly nonlinear, hence the set of admissible transport maps may not be weakly-sequentially closed with respect to any reasonable weak topology.

The following are common examples of these issues.

Proposition 1. *Let $x \in X$ be a fixed point. No admissible transport maps exist for Monge's transport problem if $\nu \neq \delta_y$ for any $y \in X$ and $\mu = \delta_x$, where δ_x is the Dirac measure centered at x.*

Proof. Let $U \subset Y$ be a measurable set with respect to the measure ν. Furthermore, assume that there exists a measurable transport map T such that $\nu = T\#\delta_x$. Using formulation (2.1) of the push-forward we must have

$$\begin{aligned} \nu(U) = \delta_x(T^{-1}(U)) &= \begin{cases} 1 & x \in T^{-1}(U) \\ 0 & x \notin T^{-1}(U) \end{cases} \\ &= \begin{cases} 1 & T(x) \in U \\ 0 & T(x) \notin U \end{cases} \\ &= \delta_{T(x)}(U). \end{aligned}$$

but this is a contradiction as ν is not a Dirac measure. Therefore no admissible transport maps exist. ☐

Proposition 2. *Let $X = Y = \mathbb{R}$. Furthermore, let μ be given in differential form (pointwise) by $d\mu(x) = dx \restriction_{[0,1]}$ (i.e. the Lebesgue measure concentrated on $[0,1]$) and let $\nu = \frac{1}{2}(\delta_1 + \delta_{-1})$. Under these conditions the set of transport maps between μ and ν is not weakly closed.*

Proof. The proof of this claim will use formulation (2.2) of the push-forward. Define the 1-periodic function $f : \mathbb{R} \to \{-1, 1\}$ pointwise by

$$f(x) = \begin{cases} 1 & \text{if } x \in [n, (2n+1)/2) \\ -1 & \text{if } x \in [(2n+1)/2, n+1) \end{cases}$$

for $n \in \mathbb{N}$. Now, define a sequence of functions pointwise by $f_m(x) := f(mx)$ for every $m \in \mathbb{N}$. We now see that for any fixed $m \in \mathbb{N}$ and test function $\psi \in C_b(\mathbb{R})$ we have

$$\begin{aligned}
\int_{\mathbb{R}} \psi(f_m(x))d\mu(x) &= \int_0^1 \psi(f(mx))dx \\
&= \sum_{k=0}^{m-1} \left(\int_{k/m}^{(2k+1)/2m} \psi(1)dx + \int_{(2k+1)/2m}^{(k+1)/m} \psi(-1)dx \right) \\
&= \sum_{k=0}^{m-1} \left(\psi(1) \left(\frac{2k+1}{2m} - \frac{k}{m} \right) + \psi(-1) \left(\frac{k+1}{m} - \frac{2k+1}{2m} \right) \right) \\
&= \sum_{k=0}^{m-1} \frac{1}{2m} \left(\psi(1) + \psi(-1) \right) \\
&= \frac{1}{2} \left(\psi(1) + \psi(-1) \right)
\end{aligned}$$

while on the other hand, we may easily compute

$$\begin{aligned}
\int_{\mathbb{R}} \psi(y)d\nu(y) &= \frac{1}{2} \left(\int_{\mathbb{R}} \psi(y)d\delta_1(y) + \int_{\mathbb{R}} \psi(y)d\delta_{-1}(y) \right) \\
&= \frac{1}{2} \left(\psi(1) + \psi(-1) \right)
\end{aligned}$$

where equality of the above implies by (2.2) that $\nu = f_m \# \mu$ for every $m \in \mathbb{N}$. However we may check that $f_m \rightharpoonup 0$ (weakly with respect to μ) as $m \to \infty$. Indeed, for any measurable test function ψ we have

$$\lim_{m \to \infty} \int_{\mathbb{R}} f_m(x)\psi(x)d\mu(x) = \lim_{m \to \infty} \frac{1}{m} \int_0^1 f(x)\psi\left(\frac{x}{m}\right) dx = 0.$$

If it were the case that the weak limit, $f_m \rightharpoonup 0$, were a transport map, then for every test function ψ we obtain

$$\begin{aligned}
\frac{1}{2} \left(\psi(1) + \psi(-1) \right) &= \int_{\mathbb{R}} \psi(y)d\nu(y) \\
&= \int_{\mathbb{R}} \psi(0)d\mu(x) \\
&= \int_0^1 \psi(0)dx \\
&= \psi(0)
\end{aligned}$$

a contradiction as this does not hold for every continuous bounded function. Thus the set of transport maps is not weakly closed.

□

This result is troublesome. If we do not even possess a (weakly) closed set to work in we are unable to follow any standard procedure that involves taking a minimizing sequence. Thus we need to expand the space of transport maps to something more general, this being the set of "transference plans".

Now instead of a transportation map uniquely sending a grain of sand at position $x \in X$ to a vacant position $y \in Y$ we consider a probability measure on the product space $X \times Y$. Informally, we state that a transference plan $\pi \in P(X \times Y)$, given pointwise by $\pi(x, y)$, measures the amount of sand transfered from a point $x \in X$ to the point $y \in Y$. Furthermore, to put it point blank, for such a plan to be admissible it is necessary that all the mass taken from the point $x \in X$ to coincide with the $\mu(x)$ and similarly all mass that is transferred to the point $y \in Y$ coincides with $\nu(y)$. This is equivalent to the requirement

$$\int_X d\pi(x,y) = d\nu(y) \qquad \int_Y d\pi(x,y) = d\mu(x)$$

and we say that such a probability measure π has first and second *marginals* μ and ν respectively. More rigorously, we require that for all test functions $\phi \in L^1(d\mu)$ and $\psi \in L^1(d\nu)$ that

$$\int_{X\times Y} (\phi(x) + \psi(y))\, d\pi(x,y) = \int_X \phi(x) d\mu(x) + \int_Y \psi(y) d\nu(y). \tag{2.3}$$

Before continuing further it will be important to characterize the spaces X and Y so that we may work in them.

Definition 1. *A topological space is called a Polish space provided it is a complete and separable metric space.*

Remark. *It can be shown by using a density argument that when working in a Polish space, we may define the push-forward by use of continuous and bounded test functions instead of being only measurable. The justification of this is to use a simple density argument.*

We are now able to expand the set of push-forward transport maps to the following,

Definition 2. *Let X and Y be measurable Polish spaces and let $\mu \in P(X)$ and $\nu \in P(Y)$. We denote*

$$\Pi(\mu,\nu) = \left\{\pi \in P(X \times Y) : (2.3) \text{ holds for every } (\phi,\psi) \in L^1(\mu) \times L^1(\nu)\right\}$$

and call it the set of transference plans.

Note that we have immediately lifted the first problem that was posed earlier. The set of transference plans is non-empty because the tensor product $\mu \otimes \nu$ lies in $\Pi(\mu,\nu)$. Indeed, for test functions ϕ, ψ the following holds,

$$\begin{aligned}\int_{X\times Y}\left(\phi(x)+\psi(y)\right)d\mu(x)\otimes d\nu(y) &= \int_X \phi(x)d\mu(x)\int_Y d\nu(y) + \int_Y \psi(y)d\nu(y)\int_X d\mu(x)\\ &= \int_X \phi(x)d\mu(x) + \int_Y \psi(y)d\nu(y)\end{aligned}$$

where we have used the fact that $\mu \in P(X)$ and $\nu \in P(Y)$.
We conclude this section with the alternative mass transportation problem:

> "Solve the following minimization problem
>
> $$\min_{\pi}\int_{X\times Y} c(x,y)d\pi(x,y)$$
>
> over all transference plans $\pi \in \Pi(\mu,\nu)$."

This problem is referred to as *Kantorovich's transport problem.* As speculated, this optimal transport plan will contain Monge's optimal transport problem as a particular sub-case. Indeed, if T is a solution to Monge's problem then the measure defined by $d\pi(x,y) = d\mu(x)\otimes d\delta_{T(x)}(y)$ is admissible in the Kantorovich problem. In this sense, the transport maps are "nested within" the set of transference plans. The following proposition recovers the solution to Monge's problem from Kantorovich's problem. For details we refer to [12].

Proposition 3. *Suppose there exists a transference plan that solves the Kantorovich transport problem and has the form* $d\pi(x,y) = d\mu(x)d\delta_{T(x)}(y)$ *for* $x \in X$*,* $y \in Y$ *and some map* $T : X \to Y$*. If* $\nu = T\#\mu$ *then* T *solves Monge's transport problem.*

2.2 Reformulating Kantorovich's Problem by the Wasserstein Distance

At this point, we have defined probability measures but have not given much insight on the topology for it. We clarify this by showing that the space of probability measures $P(X)$ of a metric space X is metrizable by the Wasserstein metric. This metric was first conceptualized by the Russian mathematician Leonid Vaseršteĭn, in 1969. It was a year later that it was first named and incorporated in the research of optimal transport by Roland Lvovich Dobrushin (July 20, 1929 - November 12, 1995) in his paper "Definition of a system of random variables by conditional distributions". It has since then been an overwhelmingly powerful tool [19], as we shall see in the following chapters. It also has a rich geometric interpretation that relates to the Kantorovich problem. Specifically, the interpretation of this metric, representing the 'distance' between two probability measures μ and ν, also physically represents the cost of transporting a pile of sand quantified by μ into holes quantified by ν.

Let X be a metric space with associated metric d. We use transference plans to describe the Wasserstein metric through the use of probability measures with some finite moments and their transference plans.

Definition 3 (p-Wasserstein Metric)**.** *Let μ, $\nu \in P_p(X)$ be measures with finite p^{th} moments and let $\Pi(\mu,\nu)$ be the set of transference plans between them. Then the p-Wasserstein metric is defined by*

$$W_p(\mu,\nu) = \left(\inf_{\pi \in \Pi(\mu,\nu)} \int_{X \times X} d(x,y)^p d\pi(x,y) \right)^{1/p}.$$

One should immediately see at this point that the p-Wasserstein metric is directly related to the Kantorovich transport problem. That is, we have the following description of the problem for the case when $X = Y$ and the cost is described by a distance function d,

> "Determine the existence of the quantity $W_p(\mu,\nu)$ for measures μ, $\nu \in P_p(X)$ where the associated cost of the model is $c = d^p$."

Inheriting the metric properties that d provides will prove beneficial for us further on. We will be incorporating this formulation in the thesis further on. The proof that W_p defines a metric is nontrivial and we refer the reader to Villani [12].

It will be shown that this reformulation of the Kanotorovich transport problem not only yields a solution, but in fact is given by a convex density function. However, the existence of a minimizer is not dependent on the fact that the cost is given by a metric. That is, lower semi-continuity of the cost function is sufficient. Trivially, the distance functions are lower-semicontinuous over a Polish space implying that the following results will still hold in our problem.

The argument that a minimizer exists utilizes two standard ingredients:

- The map $\pi \in \Pi(\mu, \nu) \mapsto \int_{\mathbb{R}^{2d}} c(x,y) d\pi(x,y)$ is lower semi-continuous.
- The set $\Pi(\mu, \nu)$ is weakly compact.

This being, we are working in the framework of the weak* (also known as narrow) topology.

Theorem 1. *Let $c : \mathbb{R}^{2d} \to [0, \infty]$ be a lower-semicontinuous cost function. Then there exists a minimizer of the Kantorovich transport problem for $X = \mathbb{R}^d = Y$.*

Lower-Semicontinuity

Prior to this proof we recall the definition of lower-semicontinuity for topological spaces. This will ease the clutter involved in the main proof.

Definition 4. *Let X be a topological space and $c : X \to \overline{\mathbb{R}}$. The function c is defined to be lower-semicontinuous at a point x_0 if for every $\varepsilon > 0$ there exists a neighbourhood U of x_0 such that*

$$c(x_0) \leq c(x) + \varepsilon$$

for every $x \in U$. Furthermore, the function c is said to be lower-semicontinuous if it is lower-semicontinuous at every point.

Lemma 1. *Let (X, d) be a metric space and $c : X \to \overline{\mathbb{R}}$ be a lower-semicontinuous function bounded below. Define a sequence of nondecreasing continuous bounded functions c_n by*

$$c_n(x) = \inf_{y \in X} (c(y) + nd(x,y))$$

for every $n \in \mathbb{N}$. Then this sequence satisfies

$$\lim_{n \to \infty} c_n(x) = c(x)$$

for every $x \in X$.

Proof. The case that $c = \infty$ is trivial, thus we may assume that c is a proper function. We have for every x, y and $z \in X$,

$$\begin{aligned} c_n(x) &\leq c(z) + nd(x,z) \\ &= c(z) + nd(y,z) + n\left(d(x,z) - d(y,z)\right) \\ &\leq c(z) + nd(y,z) + nd(x,y). \end{aligned}$$

Then by taking the infimum over $x \in X$, we obtain

$$c_n(x) \leq \inf_{z \in X} (c(z) + nd(y, z)) + nd(x, y) = c_n(y) + nd(x, y).$$

Switching x and y and using the symmetry $d(x, y) = d(y, x)$ yields

$$|c_n(x) - c_n(y)| \leq nd(x, y),$$

thereby showing that c_n is continuous as it is n-Lipschitz. Furthermore, for $n \leq m \in \mathbb{N}$ and $x \in X$, we have

$$c_n(x) = \inf_{y \in X}(c(y) + nd(x, y)) \leq \inf_{y \in X}(c(y) + md(x, y)) = c_m(x).$$

Clearly, we also have $c_n(x) \leq c(x)$. This shows that the sequence is nondecreasing and its elements are continuous. It is also easily seen that the sequence is bounded below from its definition. To infer the pointwise convergence, let $x_0 \in X$ and $\varepsilon > 0$. By the lower-semicontinuity of c at x, there exists $\delta > 0$ such that for all $x \in B(x_0, \delta) = \{y \in \mathbb{R}^d : \|y - x_0\|_2 < \delta\}$, we have

$$c(x_0) \leq c(x) + \varepsilon.$$

Let $n \in \mathbb{N}$ be such that $n\delta \leq c(x_0) - \inf_{y \in X} c(y)$. Then we obtain for all $z \in X - B(x_0, \delta)$ the inequality

$$c(z) + nd(x_0, z) \geq \inf_{y \in X}(c(y) + nd(x_0, z)) \geq \inf_{y \in X} c(y) + n\delta \geq c(x_0).$$

While for any $z \in B(x_0, \delta)$, we have by the lower-sermicontinuity of c,

$$c(z) + nd(x_0, z) + \varepsilon \geq c(z) + \varepsilon \geq c(x_0).$$

Thus the sequence converges pointwise to c. □

Using this lemma, we are now ready to prove the first ingredient required to prove Theorem 1. Another notable change that we are invoking is to restrict the space to $X = \mathbb{R}^d$ as the models we are considering are defined in this ambient space.

Lemma 2. *Let* $c : \mathbb{R}^{2d} \to [0, \infty]$ *be a lower-semicontinuous function. For every* $n \in \mathbb{N}$ *let* $\pi_n \in P(\mathbb{R}^{2d})$ *be such that* $\pi_n \rightharpoonup \pi$ *for some* $\pi \in P(\mathbb{R}^{2d})$. *Then it follows that*

$$\int_{\mathbb{R}^{2d}} c(x, y) d\pi(x, y) \leq \liminf_{n \to \infty} \int_{\mathbb{R}^{2d}} c(x, y) d\pi_n(x, y).$$

Proof. By the previous lemma let c_m be the pointwise approximation of nondecreasing continuous functions to c. By the monotone convergence theorem we obtain

$$\begin{aligned}\int_{\mathbb{R}^{2d}} c(x,y)d\pi(x,y) &= \lim_{m\to\infty}\int_{\mathbb{R}^{2d}} c_m(x,y)d\pi(x,y)\\ &= \lim_{m\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^{2d}} c_m(x,y)\pi_n(x,y)\\ &\leq \liminf_{n\to\infty}\int_{\mathbb{R}^{2d}} c(x,y)d\pi_n(x,y),\end{aligned}$$

where we have used that fact that c_m is a nondecreasing sequence. □

Weak Compactness

The main tool to prove compactness will be *Prokhorov's theorem.* As we are restricting our ambient space to $\mathbb{R}^d$ we may reformulate the way this theorem is typically expressed to meet the requirements we desire while making the proofs easier. We require the following lemma.

Lemma 3. *If K is a compact subset of $\mathbb{R}^n$, then $P(K)$ is also compact.*

Proof. As K is assumed compact then any continuous function on it is bounded and must have compact support. Accordingly, this implies that we have the following set equalities

$$C_b(K) = C_0(K) = C(K).$$

Moreover, it is well known that $C(K)$ is a Banach space equipped with the supremum norm. Denote the operator norm in the dual space $C(K)^*$ of $C(K)$ by

$$\|\phi\|_{\text{op}} = \sup_{\|f\|_\infty\leq 1} |\phi(f)|.$$

Then by the Banach-Alaoglu theorem, the dual unit ball

$$B^* = \{\phi\in C(K)^* : \|\phi\|_{\text{op}} \leq 1\}$$

is compact in the weak* topology. Now consider the subset

$$B^*_+ = \{\phi\in B^* : \phi(1) = 1, \phi(f)\geq 0 \text{ for all } f\in C(K) \text{ such that } f\geq 0\}\,.$$

It is easy to see that this set is a weak* closed subset of B^*. Now define the map $T : P(K)\to B^*_+$ pointwise by

$$T(\eta)(f) = \int_K f(x)d\eta(x),$$

for every $\eta\in P(K)$ and $f\in C(K)$.

By the Riesz representation theorem this is a bijective mapping. Moreover, by the way the map T is defined, we infer that the weak topology on $P(K)$ is represented by the dual of $C_b(K)$, thus it follows that the map is a homeomorphism. Since T is a homeomorphism between $P(K)$ and a compact space, then it must also be compact.

□

The core idea for proving that $\Pi(\mu,\nu)$ is compact relies on showing that the set is 'tight', and then use Prokhorov's theorem. First, we recall the definition of tightness.

Definition 5. *A subset $F \subset P(\mathbb{R}^d)$ is called tight provided for any $\varepsilon > 0$ there is a compact set $K_\varepsilon \subset \mathbb{R}^d$ such that $\eta(\mathbb{R}^d - K_\varepsilon) \leq \varepsilon$ for every $\eta \in F$.*

By invoking Prokhorov's theorem we have all the necessary tools to prove compactness of the set of transference plans.

Lemma 4. $\Pi(\mu,\nu)$ *is weakly compact.*

Proof. Clearly the singleton subsets $\{\mu\}$ and $\{\nu\}$ are tight. Accordingly, for every $\varepsilon > 0$ there is a compact set $K_\varepsilon \subset \mathbb{R}^d$ such that

$$\mu(K_\varepsilon) \geq 1 - \varepsilon, \qquad \nu(K_\varepsilon) \geq 1 - \varepsilon.$$

Let $\pi \in \Pi(\mu,\nu)$ be a transference plan. From the above it is easy to see that

$$\begin{aligned}
\pi(K_\varepsilon \times K_\varepsilon) &\geq 1 - \pi\left(\left(\mathbb{R}^d - K_\varepsilon\right) \times \mathbb{R}^d\right) - \pi\left(\mathbb{R}^d \times \left(\mathbb{R}^d - K_\varepsilon\right)\right) \\
&= 1 - \mu(\mathbb{R}^d - K_\varepsilon) - \nu(\mathbb{R}^d - K_\varepsilon) \\
&\geq 1 - 2\varepsilon
\end{aligned}$$

where we have used the fact that π has marginals μ and ν. Now, as $K_\varepsilon \times K_\varepsilon$ is compact under the product topology, we have shown that $\Pi(\mu,\nu)$ is tight. By Prokhorov's theorem the set of transference plans is also weakly precompact in $P(\mathbb{R}^{2d})$. We now show that it is also weakly compact. Let π_n be a sequence of transference plans such that $\pi_n \rightharpoonup \pi$ for some $\pi \in P(\mathbb{R}^{2d})$. To show that π is a transference plan we must demonstrate that it has marginals μ and ν. This is easy to show because for every open set $U \subset \mathbb{R}^d$,

$$\begin{aligned}
\pi\left(U \times \mathbb{R}^d\right) &\leq \liminf_{n\to\infty} \pi_n\left(U \times \mathbb{R}^d\right) \\
&= \liminf_{n\to\infty} \mu\left(U\right) \\
&= \mu\left(U\right).
\end{aligned}$$

By the same process we obtain $\pi\left(\mathbb{R}^d \times U\right) \leq \nu(U)$. Thus π is a transference plan between μ and ν, implying that $\Pi(\mu,\nu)$ is weakly compact as desired.

$\square$

With the results provided by Lemmas 2 and 4 we have all the standard tools to prove the existence of a minimizer to the Kantorovich transport problem. That is, we take a minimizing sequence and use the weak compactness result to extract a convergent subsequence, yielding a minimizer.

Proof of Theorem 1. Let π_n be a sequence of transference plans such that

$$\int_{\mathbb{R}^{2d}} c(x,y)d\pi_n(x,y) \leq \inf_{\pi\in\Pi(\mu,\nu)} \int_{\mathbb{R}^{2d}} c(x,y)d\pi(x,y) + \frac{1}{n}$$

By Lemma 4 the set of transference plans is weakly compact, thus there exists a subsequence π_{n_k} such that $\pi_{n_k} \rightharpoonup \pi^*$ for some $\pi^* \in \Pi(\mu,\nu)$. Using the lower-semicontinuity of the cost function, we see that by Lemma 2:

$$\begin{aligned}\int_{\mathbb{R}^{2d}} c(x,y)d\pi^*(x,y) &\leq \liminf_{k\to\infty} \int_{\mathbb{R}^{2d}} c(x,y)d\pi_{n_k}(x,y)\\ &\leq \liminf_{k\to\infty}\left(\inf_{\pi\in\Pi(\mu,\nu)} \int_{\mathbb{R}^{2d}} c(x,y)d\pi^*(x,y) + \frac{1}{n_k}\right)\\ &= \inf_{\pi\in\Pi(\mu,\nu)} \int_{\mathbb{R}^{2d}} c(x,y)d\pi(x,y).\end{aligned}$$

This implies that π^* is a minimizer of the Kantorovich transport problem. $\square$

2.3 The Wasserstein Metric For The Kinetic Model

The definition of W_p is useful when we are considering measures dependent only on a spatial component. However, in the kinetic model we will also be working with a dependency on time and velocity; hence we must expand the metric to apply to this model. We will be considering functions with compact support in its spatial component equipped with the 1-Wasserstein distance, this assumption will be justified further on.

Proposition 4. *The space* (P_{B_R}, W_1) *is complete where*

$$P_{B_R} := \left\{\mu \in P(\mathbb{R}^{2d}) : \mathit{supp}\ \mu(x,\cdot) \subset B_R \text{ for every } x \in \mathbb{R}^d\right\}.$$

Proof. This follows from the fact that $(P(\mathbb{R}^d), W_1)$ is complete and P_{B_R} is a closed subset.
Let μ_n be a sequence in P_{B_R} such that $\mu_n \to \mu$ for some $\mu \in P(\mathbb{R}^{2d})$. Accordingly, for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that whenever $n > N$ we have $W_1(\mu_n, \mu) < \varepsilon$. Using the Kantorovich-Rubinstein theorem (Theorem 3) the 1-Wasserstein metric is equivalent to the bounded Lipschitz distance, i.e.

$$W_1(\mu_n,\mu) = \sup_{\|\phi\|_{\mathrm{Lip}}\le 1}\left|\int_{\mathbb{R}^{2d}} \phi(x)d\mu_n(x) - \int_{\mathbb{R}^{2d}} \phi(y)d\mu(y)\right| < \varepsilon$$

for every $n > N$. As $\mu_n \rightharpoonup \mu$ and $\mu \in P_{B_R}$ we obtain

$$\begin{aligned}
W_1(\mu_n,\mu) &= \sup_{\|\phi\|_{\mathrm{Lip}}\le 1}\left|\int_{\mathbb{R}^d\times B_R} \phi(x)d\mu_n(x) - \int_{\mathbb{R}^{2d}} \phi(y)d\mu(y)\right| \\
&= \sup_{\|\phi\|_{\mathrm{Lip}}\le 1}\left|\left(\int_{\mathbb{R}^d\times B_R} \phi(x)d\mu_n(x) - \int_{\mathbb{R}^d\times B_R} \phi(y)d\mu(y)\right) - \int_{\mathbb{R}^d\times\left(\mathbb{R}^d - B_R\right)} \phi(y)d\mu(y)\right| \\
&= \sup_{\|\phi\|_{\mathrm{Lip}}\le 1}\left|\int_{\mathbb{R}^d\times B_R} \phi(x)d(\mu_n-\mu)(x) - \int_{\mathbb{R}^d\times\left(\mathbb{R}^d - B_R\right)} \phi(y)d\mu(y)\right| \\
&\to \sup_{\|\phi\|_{\mathrm{Lip}}\le 1}\left|\int_{\mathbb{R}^d\times\left(\mathbb{R}^d - B_R\right)} \phi(y)d\mu(y)\right|
\end{aligned}$$

as $n \to \infty$. On the other hand $W_1(\mu_n, \mu) \to 0$, which implies

$$\int_{\mathbb{R}^d\times\left(\mathbb{R}^d - B_R\right)} \phi(y)d\mu(y) = 0$$

for every 1-Lipschitz function. Choosing $\phi = 1$ yields

$$\mu(\mathbb{R}^d \times (\mathbb{R}^d - B_R)) = \mu(\left\{(x,v) \in \mathbb{R}^d \times \mathbb{R}^d : \|v\|_2 > R\right\}) = 0.$$

This implies supp $\mu(x,\cdot) \subset B_R$, hence $\mu \in P_{B_R}$. As $P(\mathbb{R}^{2d})$ is complete and P_{B_R} is a closed subset then it must also be complete.

□

If the measures are absolutely continuous with respect to the Lebesgue measure, we define their density function space

$$PF_{B_R} = \left\{ f \in L^1(\mathbb{R}^{2d}) : \text{supp } f(x,\cdot) \subset B_R \text{ for every } x \in \mathbb{R}^d, \|f\|_{L^1(\mathbb{R}^{2d})} = 1, f \geq 0 \right\}.$$

In other words, it is the set of probability density functions with velocity support in the ball B_R.
The following chapters will also consider measures that not only depend on a spatial component in $\mathbb{R}^d$ but also a time component in $[0,T] \subset \mathbb{R}$. We next define the β-Weighted 1-Wasserstein distance. The following preliminary definition will be needed.

Definition 6 (Set of Admissible Functions)**.** *Let $T > 0$ and $\mu_0 \in P_{B_R}$. We define*

$$P^T_{\mu_0} = \{\mu \in C_w([0,T]; P_{B_R}) : \mu(t=0,\cdot) = \mu_0(\cdot)\}. \tag{2.4}$$

In the event that the measures are absolutely continuous with respect to the Lebesgue measure, we define the corresponding space

$$PF^T_{f_0} = \{f \in C_w([0,T]; PF_{B_R}) : f(t=0,\cdot) = f_0(\cdot)\} \tag{2.5}$$

where $f_0 \in PF_{B_R}$.

The justification for considering these spaces stems from properties of (yet to be mentioned) macroscopic quantities.

Definition 7 (β-Weighted Wasserstein Metric)**.** *Let $\mu_0 \in P_{B_R}$ and β, $T > 0$. We define a metric on the space $P^T_{\mu_0}$ by*

$$W_1^\beta(\mu,\nu) = \sup_{0 \leq t \leq T} W_1(\mu(t,\cdot),\nu(t,\cdot))e^{-\beta t}$$

and we refer to it as the β-weighted Wasserstein metric.

Proposition 5. *Let $\mu_0 \in P_{B_R}$ and β, $T > 0$. Then it follows that $(P^T_{\mu_0}, W_1^\beta)$ is a complete metric space.*

Proof. Let μ_n be a Cauchy sequence in $(P^T_{\mu_0}, W_1^\beta)$, then given $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all n, $m > N$, we have

$$W_1(\mu_n(t,\cdot),\mu_m(t,\cdot))e^{-\beta t} \leq W_1^\beta(\mu_n,\mu_m) < \epsilon$$

for every $t \in [0,T]$. Thus we find that for any fixed $t \in [0,T]$ that the sequence $\mu_n(t,\cdot)$ is a Cauchy sequence in (P_{B_R}, W_1) and thus has a limit as the space is complete. Define the pointwise limit for every $t \in [0,T]$ of the sequence above as the function

$$\mu(t,\cdot) := \lim_{n\to\infty} \mu_n(t,\cdot).$$

Our aim now is to prove that $\mu \in P^T_{\mu_0}$. It is clear from the definition of μ that the velocity support is contained in B_R and $\mu(t,\cdot) \in P(\mathbb{R}^d \times \mathbb{R}^d)$ for every fixed $t \in [0,T]$. Thus we need only show it is weakly continuous.

Let $t \in [0,T]$ be chosen and let t_i be a sequence in $[0,T]$ with $t_i \to t$ as $i \to \infty$. It is sufficient to show that $W_1(\mu(t_i,\cdot),\mu(t,\cdot)) \to 0$ as $i \to \infty$. This is because convergence in the Wasserstein metric implies that

$$\int_{\mathbb{R}^{2d}} \phi(x,v) d\mu(t_i,x,v) \to \int_{\mathbb{R}^{2d}} \phi(x,v) d\mu(t,x,v)$$

as $i \to \infty$ for any $\phi \in C_b(\mathbb{R}^d \times \mathbb{R}^d)$. Let $n \in \mathbb{N}$. First, we expand the Wasserstein distance to obtain

$$0 \le W_1(\mu(t_i,\cdot),\mu(t,\cdot)) \le W_1(\mu(t_i,\cdot),\mu_n(t_i,\cdot)) + W_1(\mu_n(t_i,\cdot),\mu_n(t,\cdot)) + W_1(\mu_n(t,\cdot),\mu(t,\cdot))$$

Now let $\epsilon > 0$. By the convergence of μ_n to μ, we may find an $N \in \mathbb{N}$ so that for $n > N$, we have

$$W_1(\mu(t_i,\cdot),\mu(t,\cdot)) < \epsilon/3 \qquad \text{and} \qquad W_1(\mu_n(t,\cdot),\mu(t,\cdot)) < \epsilon/3.$$

Now, by the weak continuity of μ_n and using the fact that for every fixed $(x',v') \in \mathbb{R}^d \times B_R$ we have

$$\int_{\mathbb{R}^{2d}} |(x,v)-(x',v')| d\mu_n(t_i,x,v) \to \int_{\mathbb{R}^{2d}} |(x,v)-(x',v')| d\mu_n(t,x,v)$$

as $i \to \infty$, we conclude that we must have

$$W_1(\mu_n(t_i,\cdot),\mu_n(t,\cdot)) \to 0$$

as $i \to \infty$. Thus for large enough $N > 0$, we may take the limit $i \to \infty$ to obtain

$$W_1(\mu(t_i,\cdot),\mu(t,\cdot)) \to 0,$$

implying that μ is weakly continuous. Accordingly we have proved that $\mu \in P^T_{\mu_0}$ as desired. $\square$

Chapter 3

Characterization of the Transport Map

The next section is dedicated to establishing the Kantorovich duality principle, a theorem that reformulates the Kantorovich transport minimization problem in terms of a maximization problem. Specifically, it allows one to decompose the cost function as the direct sum of measurable functions. The proof invokes the standard 'minimax' technique used in optimization theory. We will state it without proof, but reference the reader to Villani [12].

3.1 Kantorovich Duality Principle

We give an intuitive description of what this principle describes. The following explanation was provided by Caffarelli and is referred to as the "shipper's problem" (c.f. [12]):

Shipper's Problem

In the Kantorovich-Monge transport problem, we described the situation by the desire to transport sand from a pile to a hole according to a cost $c(x, y)$ for $x \in X$ (the pile) and $y \in Y$ (the hole). In this situation, suppose that the hole is far enough away, and the cost describes hiring trucks that transport sand from X to Y. While we attempt to find an optimal transference plan $\pi(x, y)$ that tells us how much sand at x should be shipped to y (at minimal cost) a mathematician approaches us and says:

"Don't worry my friend, you don't need to hire trucks to move the sand everywhere. I will take care of all of this for you. I'll charge you a cost $\phi(x)$ per unit mass for loading the sand at source x and $\psi(y)$ for unloading it at location y. Moreover, I gaurantee you that the price per unit mass transferring the sand from x to y, $\phi(x)+\psi(y)$, will be less than your original cost $c(x, y)$ (i.e. $\phi(x)+\psi(y) < c(x, y)$)."

In the Shipper's problem, the mathematician now faces the conundrum of maximizing the loading and unloading costs, $\phi(x)$ and $\psi(y)$ respectively.

As these shipping costs are crucial, we define the set containing them for easy reference.

Definition 8. *Let X and Y be polish spaces. Given a cost function $c : X \times Y \to \overline{\mathbb{R}}$, define the set*

$$\Phi_c = \left\{(\phi, \psi) \in L^1(\mu) \times L^1(\nu) : \phi(x) + \psi(y) \leq c(x, y) \quad \mu \otimes \nu \text{ a.e.}\right\}.$$

We refer to the elements of Φ_c as "shipping costs" in reference to the 'shipper's problem' metaphor.

The Kantorovich duality principle states that the mathematician need not worry now. If smart enough, there is a way for him to find the loading and unloading costs that would match the cost the employer would originally have to pay.

Theorem 2 (Kantorovich Duality Principle)**.** *Let X and Y be polish spaces, $\mu \in P(X)$, $\nu \in P(Y)$, and let $c : X \times Y \to \overline{\mathbb{R}}$ be a non-negative lower semi-continuous function. For $\pi \in P(X \times Y)$ and $\phi \in L^1(\mu)$, $\nu \in L^1(\nu)$ define the functionals $I : P(X \times Y) \to \overline{\mathbb{R}}$, $J : L^1(\mu) \times L^1(\nu) \to \overline{\mathbb{R}}$ by*

$$I(\pi) = \int_{X\times Y} c(x, y) d\pi(x, y), \qquad J(\phi, \psi) = \int_X \phi(x) d\mu(x) + \int_Y \psi(y) d\nu(y). \tag{3.1}$$

It follows that

$$\inf_{\pi\in\Pi(\mu,\nu)} I(\pi) = \sup_{(\phi,\psi)\in\Phi_c} J(\phi, \psi).$$

Moreover it is sufficient to restrict to $(\phi, \psi) \in \Phi_c \cap (C_b(X) \times C_b(Y))$.

Proof. The proof traditionally follows four main steps:

- **Step 1:** Show one side of the inequality: $\sup_{(\phi,\psi)\in\Phi_c} J(\phi, \psi) \leq \inf_{\pi\in\Pi(\mu\nu)} I(\pi)$.
- **Step 2:** Restrict the theorem by assuming the space X and Y are compact and that the cost is a continuous function. Then invoke a 'minimax' principle to obtain the result.
- **Step 3:** Relax the compactness assumption on X and Y, but still assume the cost is uniformly continuous and bounded on $X \times Y$.
- **Step 4:** Relax the bounded and uniformly continuous assumption on the cost function.

A detailed proof may be found in Villani [12]. □

Remark. *A useful but subtle trick in the proof of above is the 'double convexification'. In detail, we may further restrict the set over which we take the supremum on the functional J. Define the c-conjugates of a function $\phi : X \to \overline{\mathbb{R}}$ by*

$$\phi^c(y) = \inf_{x \in X}(c(x,y) - \phi(x)) \quad \phi^{cc}(x) = \inf_{y \in Y}(c(x,y) - \phi^c(x)).$$

It can be shown that the c-conjugates improve the value of the functional J. That is,

$$\sup_{(\phi,\psi)\in\Phi_c} J(\phi,\psi) = \sup_{\phi \in L^1(\mu)} J(\phi^{cc},\phi^c).$$

Moreover, if we assume that the cost is bounded then we can further restrict the functions in the supremum of J to pairs (ϕ,ψ) bounded as

$$0 \leq \phi \leq \|c\|_\infty$$

$$-\|c\|_\infty \leq \psi \leq 0,$$

which is a consequence of the method used in the Kantorovich duality proof where one is able to obtain the bounds:

$$-\sup\phi \leq \phi^c \leq \|c\|_\infty - \sup\phi$$
$$-\sup\phi^c \leq \phi^{cc} \leq \|c\|_\infty - \sup\phi^c.$$

We finish this section with an important application. A classical and useful tool to the theory of partial differential equations is the *Kantorovich-Rubinstein distance*, or otherwise referred to as the *bounded Lipschitz distance*. Recent advancement in the theory has shown that we may infer the same results as previously obtained with an alternate expression, specifically by use of the 1-Wasserstein metric. This is obtained by an application of the Kantorovich duality principle.

Theorem 3 (Kantorovich-Rubinstein Theorem)**.** *Let X be a Polish space with a lower-semicontinuous metric d. Define the bounded Lipschitz distance*

$$\Delta(\mu,\nu) := \sup_{\phi \in Lip_1(X)\cap L^1(|\mu-\nu|)} \int_X \phi(x) d(\mu-\nu)(x)$$

for all μ, $\nu \in P(X)$. Then it follows that

$$\Delta(\mu,\nu) = W_1(\mu,\nu)$$

for all μ, $\nu \in P(X)$.

Proof. This proof follows three main steps.

1. (Apply the Kantorovich duality principle). Using the above theorem we have an alternate way to describe the 1-Wasserstein metric,

$$W_1(\mu,\nu) = \sup_{(\phi,\psi)\in\Phi_d} J(\phi,\psi)$$

and thus it remains only to show that this simplifies to the bounded Lipschitz distance.

2. (Assume d is bounded). Assuming d is bounded, define the sequence

$$d_n = \frac{d}{1+dn^{-1}}.$$

Clearly the sequence is bounded and $d_n \to d$ as $n \to \infty$ pointwise monotonically. Moreover, from how the sequence is defined we have the set inclusion $\mathrm{Lip}_1(X,d_n) \subset \mathrm{Lip}_1(X,d)$.

Thus we obtain for $\phi \in \mathrm{Lip}_1(X,d_n) \cap L^1(|\mu-\nu|)$,

$$\lim_{n\to\infty}\sup \int_X \phi(x) d_n(\mu-\nu)(x) = \sup \int_X \phi(x) d(\mu-\nu)(x).$$

Thus if we are able to prove that

$$\sup_{(\phi,\psi)\in\Phi_{d_n}} J(\phi,\psi) = \sup_{\phi\in\mathrm{Lip}_1(X,d_n)\cap L^1(|\mu-\nu|)} \int_X \phi(x) d(\mu-\nu)(x)$$

in the following step, then we know that

$$\sup_{(\phi,\psi)\in\Phi_d} J(\phi,\psi) = \lim_{n\to\infty} \sup_{(\phi,\psi)\in\Phi_{d_n}} J(\phi,\psi) = \Delta(\mu,\nu).$$

3. (Use the above remark). If we still assume d to be bounded, then $\phi \in \mathrm{Lip}_1(X,d)$ implies $\phi \in L^1(\mu) \cap L^1(\nu)$. Hence it is sufficient for us to show that

$$\sup_{(\phi,\psi)\in\Phi_d} J(\phi,\psi) = \sup_{\phi\in\mathrm{Lip}_1(X,d)} \int_X \phi(x) d(\mu-\nu)(x).$$

By the remark and the process of step 2 in the Kantorovich duality principle, we have

$$\sup_{(\phi,\psi)} J(\phi,\psi) = \sup_{\phi\in L^1(\mu)} J(\phi^{dd},\phi^d).$$

By the definition of ϕ^{dd}, we see that $\phi^{dd} \leq -\phi^d$. Moreover, $\phi^d \in \mathrm{Lip}_1(X,d)$. Indeed, by taking a sequence x_n in X such that $d(x_n,x) - \phi(x_n) \to \phi^d(x)$ we have for x, $y \in X$,

$$\begin{aligned}\phi^d(y) &\leq d(x_n, y) - \phi(x_n)\\ &= d(x_n, y) - d(x_n, x) + d(x_n, x) - \phi(x_n)\\ &\leq d(x, y) + d(x_n, x) - \phi(x_n)\end{aligned}$$

where we have use the triangle inequality. By taking the limit $n \to \infty$ we obtain

$$\phi^d(y) - \phi^d(x) \leq d(x, y).$$

Then by interchanging x and y and using the symmetry of d we obtain that ϕ^d is 1-Lipschitz. A simple argument can be made from this to show $-\phi^d \leq \phi^{dd}$. This proves that $\phi^{dd} = -\phi^d$ and it follows that

$$\begin{aligned}\sup_{(\phi,\psi)\in\Phi_d} J(\phi, \psi) &= \sup_{\phi\in L^1(\mu)} J(\phi^{dd}, \phi^d)\\ &= \sup_{\phi\in L^1(\mu)} J(-\phi^d, \phi^d)\\ &\leq \sup_{\phi\in \mathrm{Lip}_1(X,d)} J(\phi, -\phi)\\ &\leq \sup_{(\phi,\psi)\in\Phi_d} J(\phi, \psi)\end{aligned}$$

proving the desired equality.

□

3.2 Brenier's Theorem

We conclude our introduction to optimal transport with Brenier's theorem, a result that describes optimal transference plans in terms of (and demonstrates existence thereof) a transport map T satisfying $\nu = T\#\mu$. For this, one constructs the *Quadratic transport problem*, a dual formulation of Kantorovich's transport problem, under the assumption that the cost is given by the square of the Euclidean distance, i.e.

$$c(x, y) = \|x - y\|_2^2$$

for all x, $y \in \mathbb{R}^d$. From this, one is able to exploit the inner product structure to relate optimal transference plans to convex functions and thereby represent them in such a form. A further treatise of this may be found in Villani [12].

Theorem 4 (Knott-Smith optimality criterion.). *Let μ, $\nu \in P_2(\mathbb{R}^d)$. Then π is an optimal transference plan of Kantorovich's transport plan with Euclidean cost if and only if there exists a convex map $\phi : \mathbb{R}^d \to \overline{\mathbb{R}}$ such that*

$$supp\ \pi \subset \mathbb{R}^d \times \partial\phi,$$

where we define the subgradient at each point $x_0 \in \mathbb{R}^d$ by

$$\partial\phi(x_0) = \{v \in \mathbb{R}^d : \phi(x) \geq \phi(x_0) + v \cdot (x - x_0) \textit{ for every } x \in \mathbb{R}^d\}.$$

Theorem 5 (Brenier's Theorem). *Let μ, $\nu \in P_2(\mathbb{R}^d)$ such that μ does not give mass to sets of Lebesgue measure zero. Furthermore, let π be an optimal transference plan of the Kantorovich transport problem. Then the following hold,*

1. *There exists an optimal transport map T to the Monge transport problem, i.e. $d\pi(x, y) = d\mu(x)\delta(y = T(x))$. Furthermore there exists a convex function $\phi : \mathbb{R}^d \to \overline{\mathbb{R}}$ such that $T = \nabla\phi$ for μ almost everywhere. Such a map is called a Brenier map.*

2. *The map T is uniquely determined almost everywhere with respect to μ.*

3. *The support of ν is represented by*
$$supp\ \nu = \overline{\nabla\phi(supp\ \mu)}.$$

4. *If ν is absolutely continuous with respect to the Lebesgue measure then for $\mu \otimes \nu$ almost every $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$,*
$$\nabla\phi^* \circ \nabla\phi = id_{\mathbb{R}^d} = \nabla\phi \circ \nabla\phi^*,$$

   where $\phi^*(y) = \sup_x(x \cdot y - \phi(x))$ for $y \in \mathbb{R}^d$.

   Moreover, the map $\nabla\phi^*$ is the unique optimal transport map from ν to μ for ν almost everywhere. In other words, the map $T = \nabla\phi$ is invertible μ almost everywhere with inverse $T^{-1} = \nabla\phi^*$.

Proof. The proof is surprisingly simple and follows from the Knott-Smith optimality criterion. By the previous theorem, the optimal transference plan π is supported in the graph of the subgradient of a convex function ϕ. Letting $\mathrm{Gr}(\partial\phi)$ be the graph of $\partial\phi$, we have in particular that

$$I(\pi) = \int_{\mathrm{Gr}(\partial\phi)} \|x - y\|_2^2 \, d\pi(x, y).$$

Now, by Rademacher's theorem the function ϕ is differentiable almost everywhere. So, the transport map $T = \nabla\phi$ is well defined on the set $\mathbb{R}^d - U$ where it is assumed $\mu(U) = 0$. We then make the disjoint decomposition $\mathrm{Gr}(\partial\phi) = G_1 \cup G_2$ where G_1 is the gradient of T over the set $\mathbb{R}^d - U$. By definition, we have

$$\int_{G_2} f(x, y) d\pi(x, y) = \int_{\mathbb{R}^d - U} f(x, T(x)) d\mu(x)$$

for any $f \in C(\mathbb{R}^{2d})$ and hence

$$\int_{G_1} d\pi(x, y) = \mu(\mathbb{R}^d - U) = 1.$$

It then follows that π gives no mass on G_2 and thus

$$I(\pi) = \int_{\mathbb{R}^d - U} f(x, T(x)) d\mu(x)$$

for any $f \in C(\mathbb{R}^{2d})$. Moreover, the minimizing property of π shows that T is a minimizing transport map, thus solving Monge's transport problem. This shows that we have the existence of a transport map T such that $\nu = T\#\mu$. To verify uniqueness, assume that the gradients of ϕ and of a function ψ satisfy the pushforward property. Represent π_ϕ to be the optimal transference plan described by ϕ and let ψ be the optimizing function for the quantity $J(\psi, \psi^*)$. Repeating the proof of the Knott-Smith optimality criterion yields that the support of π_ϕ is contained in the graph of $\partial\psi$. However, by definition the support of π_ϕ is contained in the graph of $\partial\phi$. Since ϕ and ψ are differentiable for μ almost everywhere, we conclude that $\nabla\phi = \nabla\psi$ for μ almost everywhere. □

3.3 The Monge-Ampère Equation

3.3.1 History of the Monge-Ampère Equation

Before we formulate the type of Monge-Ampère equation we will be concerned with, we will give a brief introduction to the field of *general* Monge-Ampère type equations so that the reader may understand the motivation for studying them.

Let $d \in \mathbb{N}$. The Monge-Ampère equation (in its general form) is a fully nonlinear second order partial differential equation describing the nature of the Hessian of a function $u : \mathbb{R}^d \to \mathbb{R}$. As suggested by its name, the equation was first developed by two mathematicians around the late half of the 18th century. The first to introduce the equation was Gaspard Monge, Comte de Péluse (May 9th 1746 - July 28th 1818) in his paper "*Sur le calcul intégral des équations aux differences partielles*" submitted to Mémoires de l'Académie des Sciences in 1784 [8]. Gaspard Monge is known for being the inventor of descriptive geometry (a branch concerned with representing three dimensional objects in two dimensions) and the father of differential geometry; which is the field that originally motivated the study of the Monge-Ampère equation. The second contribution was made by André-Marie Ampère (January 20th 1775 - June 10th 1836) is his submission "*Mémoire contenant l'applicatioin de la théorie*" submitted to Journal de l'École Royal Polytechnique in 1820 [10].

In its earlier formulation, the equation was restricted to a particular two-dimensional case whose prototype is given by

$$A(u_{xx}(x,y)u_{yy}(x,y) - u_{xy}(x,y)^2) + Bu_{xx}(x,y) + 2Cu_{xy}(x,y) + Du_{yy}(x,y) + E = 0 \tag{3.2}$$

where A, B, C, D, and E are functions depending on x, y, $u(x,y)$ and $\nabla u(x,y)$ with $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ being the unknown. This equation has been further developed and reconsidered in a more general premise. Let $\Omega \subset \mathbb{R}^d$ be an open set. Given a symmetric matrix function $A : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbf{Sym}_n(\mathbb{R})$ and a function $f : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, the general Monge-Ampère equation is given by

$$\det(\nabla^2 u(x) - A(x,u,\nabla u)) = f(x,u,\nabla u) \tag{3.3}$$

where the unknown is $u : \Omega \to \mathbb{R}$, and $\nabla^2 u$ is the Hessian matrix associated to u, i.e.

$$\nabla^2 u(x) = \begin{pmatrix} u_{x_1x_1} & u_{x_1x_2} & \cdots & u_{x_1x_n} \\ u_{x_2x_1} & u_{x_2x_2} & \cdots & u_{x_2x_n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{x_nx_1} & u_{x_nx_2} & \cdots & u_{x_nx_n} \end{pmatrix}$$

where $x = (x_1, x_2, \ldots, x_n)$.

It can be shown that if A is positive definite and we restrict the solution u to be convex, then the general

Monge-Ampère equation is "elliptic", albeit this truly refers to the fact that its linearization is elliptic. This is merely a consequence of the fact that the convexity of u implies its Hessian matrix is positive definite. Consequently, it inherits several properties of elliptic type equations. This includes several forms of the maximum principle, including the *Alexandrov Maximum Principle* stated as follows (a proof of this is provided by Caffarelli [20]),

Definition 9 (Viscosity Solutions)**.** *Let $u \in C(\Omega)$ be a convex function and $f \in C(\Omega)$ with $f \geq 0$.*

1. *u is a viscosity subsolution of the equation $\det \nabla^2 u = f$ if for every convex function $\phi \in C(\Omega)$ if $x_0 \in \Omega$ is a local maximum of the function $u - \phi$ then*

$$\det \nabla^2 u \geq f(x_0).$$

2. *u is a viscosity supersolution of the equation $\det \nabla^2 u = f$ if for every convex function $\phi \in C(\Omega)$ if $x_0 \in \Omega$ is a local minimum of the function $u - \phi$ then*

$$\det \nabla^2 u \leq f(x_0).$$

We say that u is a viscosity solution of the equation $\det \nabla^2 u = f$ if it is both a viscosity sub and supersolution of $\det \nabla^2 u = f$.

Theorem 6 (Alexandrov Maximum Principle)**.** *Let $u : \Omega \subset \mathbb{R}^d \to \mathbb{R}$ be a viscosity solution of $\det \nabla^2 u = f$ with $u(\partial\Omega) = \{0\}$. Provided there exists a Λ, such that $det(\nabla^2 u) \leq \Lambda$ then there exists a constant $C = C(n, \Lambda, diam(\Omega), |\Omega|)$ such that*

$$|u(x)| \leq C d(x, \partial\Omega)^{\frac{1}{n}}$$

where we have

$$diam(\Omega) = \sup_{y,z\in\Omega} \|y - z\|_2, \qquad d(x, \partial\Omega) = \inf_{y\in\partial\Omega} \|x - y\|_2$$

and $|\Omega|$ is the Lebesgue measure of the the set Ω.

The Monge-Ampère equation does in fact has several applications, but none so common as its emergence in geometry. Indeed, to mention one particular and rather important example, one may consider the Gaussian curvature of a function $u : \mathbb{R}^d \to \mathbb{R}$. Letting K denote the Gaussian curvature of the graph of u, it can be shown to satisfy the equation

$$\det(\nabla^2 u(x)) = K(x)(1 + |\nabla u(x)|^2)^{(n+2)/2}.$$

Working backwards, if we have a prescribed Gaussian curvature K and wish to find the associated function u whose graph has this curvature, then we must solve the above equation (for u). This type of equation is of course a special case of the general Monge-Ampère equation when $A = 0$ and $f(x, u(x), \nabla u(x)) = K(x)(1 + |\nabla u(x)|^2)^{(n+2)/2}$. To simplify, it satisfies the *reduced* Monge-Ampère type equation

$$\det(\nabla^2 u(x)) = \frac{g(x)}{h(\nabla u(x))}. \tag{3.4}$$

where $h \circ \nabla u(\Omega)$ is assumed non-vanishing. This type of equation is what we shall refer to as the Monge-Ampère equation as it will be the primary focus of our analysis. Our goal will be to use the tools of optimal transport, and most importantly, *Brenier's Theorem* to conclude the weak existence of a convex function satisfying (3.4) for given g and h. Although we have considered our domain Ω to be any open set, we will assume, for the sake of an example, that the domain is the whole space $\mathbb{R}^d$. The general case is wonderfully elaborated upon in [12], to which the results are the same as in the whole space.

3.3.2 Solution of the Monge-Ampère Equation

In this section, we conclude our introduction to optimal transport with a simple application to the Monge-Ampère equation. The result follows immediately from the pushforward condition obtained from Brenier's theorem.

We assume that the two given probability measures μ and ν are absolutely continuous with respect to the Lebesgue measure. We denote their Radon-Nikodym derivatives by

$$\frac{d\mu}{dx} = f(x), \quad \frac{d\nu}{dy} = g(y)$$

and assume the densities f and g are smooth. Assuming ϕ to be a smooth submersion, let it have corresponding Brenier map $T = \nabla\phi$. It follows that

$$g = T\#f \iff \det(\nabla^2\phi)g(\nabla\phi) = f.$$

Indeed, if we assume $y := \nabla\phi(x)$ defines a diffeomorphism then by differentiating we obtain the one-form equation

$$dy = \det(\nabla^2\phi)(x)dx.$$

However, as $\nabla\phi$ is a diffeomorphism then the pushforward condition implies that

$$f(x)dx = g(y(x))dy(x) = g(\nabla\phi(x))\det(\nabla^2\phi)(x)dx$$

as we had desired.

If we expand to non-smooth ϕ then we further compensate by defining a weak solution. If we are given f, $g \in L^1(\mathbb{R}^d)$ we call ϕ a Brenier solution to the Monge-Ampère equation provided $g = \nabla\phi\#f$, i.e. for any $\psi \in C_b(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} \psi(y)g(y)dy = \int_{\mathbb{R}^d} \psi(\nabla\phi(x))f(x)dx$$

It can be shown however that if f and g are smooth then under some further assumption any Brenier solution is smooth and thus reduces to solving the previous case. However, without these assumptions, we still obtain the following powerful result, which we state without proof (c.f. [21]).

Definition 10 (Alexandrov Derivative)**.** *ϕ is twice differentiable at a point x_0 with Alexandrov derivative $\nabla^2\phi$ if $\nabla\phi(x_0)$ exists, and if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|x - x_0\|_2 < \delta$ and $\Lambda := \nabla^2\phi(x_0)$ imply*

$$\sup_{y \in \partial\phi(x)} \|y - \nabla\phi(x_0) - \Lambda(x - x_0)\|_2 < \varepsilon\|x - x_0\|_2.$$

Theorem 7. *Let ϕ be a Brenier solution to the Monge-Ampère equation (i.e. $g = \nabla\#f$), then ϕ is a solution to*

$$f(x) = g(\nabla\phi(x))\det(\nabla^2\phi)(x)$$

almost everywhere on dom ϕ provided $\nabla^2\phi$ is interpreted in the sense of Alexandrov.

Chapter 4

Analysis of the Kinetic Model

Now that we have begun to establish a framework, we move into the main discussion of this thesis: the kinetic model given by (1.4). We derive this model from the discrete model (1.3) by use of a mean-field limit argument and explore some of its properties.

4.1 Derivation of the Kinetic model by the Mean-Field Limit

Let N be the number of particles of the discrete model and assume that there is a solution given by $(x_i(t), v_i(t))$ for $i = 1, ..., N$ (where we are fixing a corresponding $\alpha > 0$). There is a density distribution of the time evolution associated to the discrete model (1.3) defined by

$$\mu^N(t,x,v) = \frac{1}{N}\sum_{i=1}^{N} \delta(x - x_i(t)) \otimes \delta(v - v_i(t)). \tag{4.1}$$

Denote the force term as

$$\tilde{F}_\alpha(x,v) = \eta_\alpha(x)\nabla W(v)$$

and define for any measure $\nu \in P(\mathbb{R}^d \times \mathbb{R}^d)$ and function $G \in L^1(\nu)$ the convolution given by

$$(G *_{(x,v)} \nu)(x,v) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(x-y, v-u)d\nu(y,u).$$

Given the representation of μ^N above, we have

$$\begin{aligned}(\tilde{F}_\alpha *_{(x,v)} \mu^N)(x,v) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tilde{F}_\alpha(x-y, v-u)d\mu^N(y,u)\\ &= \frac{1}{N}\sum_{i=1}^N \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tilde{F}_\alpha(x-y,v-u)d\delta(y-x_i(t))d\delta(u-v_i(t))\\ &= \frac{1}{N}\sum_{i=1}^N \tilde{F}_\alpha(x-x_i(t), v-v_i(t))\\ &= \frac{1}{N}\sum_{i=1}^N \eta_\alpha(x-x_i(t))\nabla W(v-v_i(t)).\end{aligned}$$

By comparing to the discrete model (1.3) we have for every $j = 1, ..., N$ that

$$\dot{v}_j(t) = \varepsilon \sum_{i=1}^N \eta_\alpha(x_j(t)-x_i(t))\nabla W(v_i(t)-v_j(t)) = -\varepsilon N(\tilde{F}_\alpha *_{(x,v)} \mu^N)(x_j(t), v_j(t))$$

where we have used the fact that ∇W is odd. Accordingly, we may rewrite the discrete model (1.3) as

$$\begin{cases}\dot{x}_i(t) = v_i(t)\\ \dot{v}_i(t) = -\varepsilon N(\tilde{F}_\alpha *_{(x,v)} \mu^N)(x_i(t), v_i(t))\\ (x_i(0), v_i(0)) = (x_i^0, v_i^0).\end{cases} \tag{4.2}$$

for $i = 1, ..., N$. For any measure $\nu \in P(\mathbb{R}^d \times \mathbb{R}^d)$ and function $G \in L^1(\nu)$ we establish the pairing notation in the distributional sense

$$\langle \nu, G\rangle = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(x,v)d\nu(x,v).$$

It follows that for any $\varphi \in C_0^1(\mathbb{R}^d \times \mathbb{R}^d)$ we obtain

$$\begin{aligned}\frac{d}{dt}\langle \mu^N(t,\cdot), \varphi\rangle &= \frac{d}{dt}\left(\frac{1}{N}\sum_{i=1}^N \varphi(x_i(t), v_i(t))\right)\\ &= \frac{1}{N}\sum_{i=1}^N \nabla_{x_i(t)}\varphi(x_i(t), v_i(t)) \cdot \dot{x}_i(t) + \frac{1}{N}\sum_{i=1}^N \nabla_{v_i(t)}\varphi(x_i(t), v_i(t)) \cdot \dot{v}_i(t)\\ &= \frac{1}{N}\sum_{i=1}^N \nabla_{x_i(t)}\varphi(x_i(t), v_i(t)) \cdot v_i(t) - \varepsilon\sum_{i=1}^N \nabla_{v_i(t)}\varphi(x_i(t), v_i(t)) \cdot (\tilde{F}_\alpha *_{(x,v)} \mu^N)(x_i(t), v_i(t))\\ &= \langle \mu^N(t,\cdot), \nabla_x\varphi(x,v)\cdot v\rangle - \langle \mu^N(t,\cdot), \varepsilon N\nabla_v\varphi(x,v)\cdot(\tilde{F}_\alpha *_{(x,v)} \mu^N)\rangle.\end{aligned}$$

By an integration by parts we then obtain

$$\left\langle \frac{\partial \mu^N}{\partial t} + v \cdot \nabla_x \mu^N - \varepsilon N \mathrm{div}_v(\tilde{F}_\alpha *_{x,v} \mu^N), \varphi \right\rangle = 0. \tag{4.3}$$

This describes our kinetic model (1.4) in the distributional sense. By establishing the initial density measure

$$\mu_0^N = \frac{1}{N}\sum_{i=1}^{N} \delta(x - x_i^0) \otimes \delta(v - v_i^0)$$

we formally let $\mu_0^N \rightharpoonup \mu_0$ as $N \to \infty$ where μ_0 represents the initial datum of the kinetic model. Likewise, $\mu^N \rightharpoonup \mu$ as $N \to \infty$. Simultaneously take $\varepsilon \to 0$ such that $\varepsilon N \to \lambda$ for some $\lambda > 0$. This yields the kinetic model

$$\left\langle \frac{\partial \mu}{\partial t} + v \cdot \nabla_x \mu - \lambda \mathrm{div}_v(\tilde{F}_\alpha *_{(x,v)} \mu), \varphi \right\rangle = 0$$

with pointwise initial condition $d\mu(0,x,v) = d\mu_0(x,v)$. If we further assume that μ_0 is absolutely continuous with respect to the Lebesgue measure then we represent its Radon-Nikodym derivative by

$$\frac{d\mu_0}{d(x,v)}(x,v) = f_0(x,v).$$

Upcoming results (discussed in a further section) will tell us that if μ_0 is absolutely continuous with respect to the Lebesgue measure then the solution μ will be absolutely continuous with respect to the Lebesgue measure. With this we also represent the Radon-Nikodym derivative of μ as

$$\frac{d\mu}{d(x,v)}(t,x,v) = f(t,x,v).$$

For simplicity, if we take $\lambda = 1$ integration by parts yields the weak kinetic equation for granular media:

Let $f_0 : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ be a measureable function with compact support and let $T > 0$. For every $\phi \in C_0^1([0,T] \times \mathbb{R}^d \times \mathbb{R}^d)$ find a measurable function $f : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ with $f(0,x,v) = f_0(x,v)$ satisfying

$$\int_0^T \int_{\mathbb{R}^{2d}} \left(\frac{\partial \phi}{\partial t}(t,x,v) - v \cdot \nabla_x \phi + F_{f,\alpha} \cdot \nabla_v \phi(t,x,v) \right) f(t,x,v)dxdv = -\int_{\mathbb{R}^{2d}} \phi(0,x,v) f_0(x,v)dxdv \tag{4.4}$$

for every $t \in [0,T]$ where,

$$F_{f,\alpha}(t,x,v) = \tilde{F}_\alpha *_{(x,v)} f(t,x,v) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \eta_\alpha(x-y)\nabla W(v-u) f(t,y,u)dydu.$$

We refer to this as the weak kinetic model for granular media. Assuming further regularity conditions (i.e. $f \in C^1([0,T) \times \mathbb{R}^d \times \mathbb{R}^d)$) we are able to obtain the (classical) kinetic model:

Let $f_0 \in C_0^1(\mathbb{R}^d \times \mathbb{R}^d; [0,\infty))$ and $T > 0$. Find an $f \in C^1([0,T) \times \mathbb{R}^d \times \mathbb{R}^d; [0,\infty))$ such that $f(0,x,v) = f_0(x,v)$ satisfying

$$\frac{\partial f}{\partial t}(t,x,v) + v \cdot \nabla_x f(t,x,v) = \operatorname{div}_v(f(t,x,v)F_{f,\alpha}(t,x,v)) \tag{4.5}$$

for every $t \in [0,T]$ and x, $v \in \mathbb{R}^d$.

This coincides with the model we had presented in the introduction. Initially, we will work with measure theoretic solutions to (4.4) and move into classical solutions to (4.5) once necessary regularity results are obtained.

4.2 Properties of Solutions to the Kinetic Model

As we had mentioned in the previous chapter, an admissible space to look for solutions to the weak kinetic model (4.4) is the set $P^T_{\mu_0}$ where the initial measure $\mu_0 \in P_{B_R}$is given and $T > 0$. We explore the justification of this assertion and infer other crucial properties, specifically, the behavior of macroscopic quantities and their physical relevance to the kinetic model. In the following we assume that the solution to the kinetic model possesses enough regularity (i.e. at least C^1) to solve the classical model (1.4). Furthermore, as each fixed $\alpha > 0$ yields a different solution, we obtain a net of solutions f_α. This notation will be implemented in the following computations. Moreover, we will also use the notation f_0 to represent the initial datum as in the previous subsection.

4.2.1 Conservation of Mass and Momentum

Proposition 6 (Conservation of Mass)**.** *For a fixed $\alpha > 0$ and initial datum $f_0 \in PF_{B_R} \cap C^1(\mathbb{R}^{2d})$, the corresponding solution $f_\alpha \in PF^T_{f_0} \cap C^1([0,T] \times \mathbb{R}^{2d})$ to our system satisfies*

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f_\alpha(t,x,v)dxdv = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f_0(x,v)dxdv,$$

for every $0 \leq t \leq T$.

Proof. This follows from the integration by parts:

$$\frac{d}{dt}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f_\alpha(t,x,v)dxdv = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (-\mathrm{div}_x(vf_\alpha(t,x,v)) + \mathrm{div}_v(f(t,x,v)F_{f,\alpha}(t,x,v))) = 0.$$

By integrating in time we obtain the desired result. □

Proposition 7 (Conservation of Momentum)**.** *For a fixed $\alpha > 0$ and initial datum $f_0 \in PF_{B_R} \cap C^1(\mathbb{R}^{2d})$, the corresponding solution $f_\alpha \in PF^T_{f_0} \cap C^1([0,T] \times \mathbb{R}^{2d})$ to our system satisfies*

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} vf_\alpha(t,x,v)dxdv = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} vf_0(x,v)dxdv.$$

Proof. Differentiating componentwise for some index $i = 1, ..., d$ yields

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} v_i f_\alpha(t,x,v)dxdv &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} v_i \mathrm{div}_v(f_\alpha(t,x,v)F_{f_\alpha,\alpha}(t,x,v))dxdv - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} v_i v \cdot \nabla_x f_\alpha(t,x,v)dxdv \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} v_i \mathrm{div}_v(f_\alpha(t,x,v)F_{f_\alpha,\alpha}(t,x,v))dxdv - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} v_i \mathrm{div}_x(vf_\alpha(t,x,v))dxdv \\ &= -\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \nabla_v v_i \cdot F_{f_\alpha,\alpha}(t,x,v)f_\alpha(t,x,v)dxdv \\ &= -\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \eta_\alpha(x-y)e_i \cdot \nabla W(v-u)f_\alpha(t,y,u)f_\alpha(t,x,v)dydudxdv.\end{aligned}$$

Exchanging the variables u and v and then using the fact that ∇W is odd gives

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} v_i f_\alpha(t,x,v)dxdv &= -\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \eta_\alpha(x-y)e_i\cdot\nabla W(u-v)f_\alpha(t,y,v)f_\alpha(t,x,u)dydudxdv\\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \eta_\alpha(x-y)e_i\cdot\nabla W(v-u)f_\alpha(t,y,v)f_\alpha(t,x,u)dydudxdv.\end{aligned}$$

Next, we may exchange the variables x and y. Using the fact that η_α is symmetric we obtain

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} v_i f_\alpha(t,x,v)dxdv &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \eta_\alpha(y-x)e_i\cdot\nabla W(v-u)f_\alpha(t,y,v)f_\alpha(t,x,u)dydudxdv\\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \eta_\alpha(x-y)e_i\cdot\nabla W(v-u)f_\alpha(t,x,v)f_\alpha(t,y,u)dydudxdv.\end{aligned}$$

Adding the above quantities yields the condition

$$\frac{d}{dt}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} v_i f_\alpha(t,x,v)dxdv = 0.$$

Integrating obtains

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} v_i f_\alpha(t,x,v)dxdv = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} v_i f_0(x,v)dxdv,$$

which holds for each index. □

4.2.2 Decrease of Moments and Loss of Kinetic Energy

Proposition 8 (Decrease of Moments)**.** *Let $p \geq 1$. For a fixed $\alpha > 0$ and given initial datum $f_0 \in PF_{B_R} \cap C^1(\mathbb{R}^{2d})$ the corresponding solution $f_\alpha \in PF^T_{f_0} \cap C^1([0,\infty)\times\mathbb{R}^{2d})$ to our system satisfies*

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \|v\|_2^p f_\alpha(t,x,v)dxdv \leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \|v\|_2^p f_0(x,v)dxdv,$$

for every $0 \leq t \leq T$.

Proof. We continue in the same manner as before. Differentiating yields

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\|v\|_2^p f_\alpha(t,x,v)dxdv &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\|v\|_2^p \mathrm{div}_v(f_\alpha(t,x,v)F_{f_\alpha,\alpha}(t,x,v))dxdv \\ &\quad -\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\|v\|_2^p \mathrm{div}_x(vf_\alpha(t,x,v))dxdv \\ &= -p\int_{\mathbb{R}^d-\{0\}}\int_{\mathbb{R}^d}\|v\|_2^{p-2} v\cdot F_{f_\alpha,\alpha}(t,x,v)f_\alpha(t,x,v)dxdv.\end{aligned}$$

We have removed the origin to keep the integral proper for the case of $1 \leq p < 2$. This is allowed as this set is of measure zero and the integral is otherwise convergent. Continuing by expanding the force term and removing the origin, we obtain

$$-p\int_{\mathbb{R}^d-\{0\}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d-\{0\}}\int_{\mathbb{R}^d}\eta_\alpha(x-y)\|v\|_2^{p-2}v\cdot\nabla W(v-u)f_\alpha(t,y,u)f_\alpha(t,x,v)dydudxdv.$$

We may exchange the variables v and u, then use the fact that ∇W is odd to obtain

$$p\int_{\mathbb{R}^d-\{0\}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d-\{0\}}\int_{\mathbb{R}^d}\eta_\alpha(x-y)\|u\|_2^{p-2}u\cdot\nabla W(v-u)f_\alpha(t,y,v)f_\alpha(t,x,u)dydudxdv.$$

We then exchange the variables x and y, and use the fact that η_α is symmetric to obtain

$$p\int_{\mathbb{R}^d-\{0\}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d-\{0\}}\int_{\mathbb{R}^d}\eta_\alpha(x-y)\|u\|_2^{p-2}u\cdot\nabla W(v-u)f_\alpha(t,x,v)f_\alpha(t,y,u)dydudxdv.$$

We add the above to find that the change in velocity p-moment is equivalent to

$$-\frac{p}{2}\int_{\mathbb{R}^d-\{0\}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d-\{0\}}\int_{\mathbb{R}^d}\eta_\alpha(x-y)\left(\|v\|_{l^2}^{p-2}v-\|u\|_{l^2}^{p-2}u\right)\cdot\nabla W(v-u)f_\alpha(t,x,v)f_\alpha(t,y,u)dydudxdv.$$

Using the fact that W is radially symmetric implies there exists a function w such that $W(v) = w(\|v\|_2)$. Since w is of class C^2, the map $v \mapsto \|v\|_2^p$ is convex, and $w'(\|v-u\|_2) \geq 0$ we obtain

$$\left(\|v\|_2^{p-2}v-\|u\|_2^{p-2}u\right)\cdot\nabla W(v-u) = \left(\|v\|_2^{p-2}v-\|u\|_2^{p-2}u\right)\cdot(v-u)\frac{w'(\|v-u\|_2)}{\|v-u\|_2} \geq 0$$

for $v \neq u$ by the first order convexity condition. Acknowledging that $\eta_\alpha(x-y) \geq 0$ yields

$$\frac{d}{dt}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\|v\|_2^p f_\alpha(t,x,v)dxdv \leq 0,$$

and integration gives

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\|v\|_2^p f_\alpha(t,x,v)dxdv \leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\|v\|_2^p f_0(x,v)dxdv$$

as desired.

□

Consequently, as the kinetic energy of our system is described by the second velocity moment, we see that the kinetic energy of our system decreases (as is expected for an inelastic system).

Corollary 1 (Loss of Kinetic Energy)**.** *For a fixed* $\alpha > 0$*, initial datum* $f_0 \in PF_{B_R} \cap C^1(\mathbb{R}^{2d})$*, and corresponding solution* $f_\alpha \in PF_{f_0}^T \cap C^1([0,\infty) \times \mathbb{R}^{2d})$ *to our system the kinetic energy functional*

$$E_K(f_\alpha(t,\cdot)) = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \|v\|_2^2 f_\alpha(t,x,v)dxdv,$$

satisfies

$$E_K(f_\alpha(t,\cdot)) \leq E_K(f_0).$$

4.2.3 Increase of Internal Energy

A crucial result of this kinetic model is a “reverse H-theorem”, describing the tendency of the system to cluster over time. That is, the entropy satisfies

$$H(f_\alpha(t,\cdot)) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(t,x,v)\log f(t,x,v)dxdv \geq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f_0(x,v)\log f_0(x,v)dxdv.$$

for every $0 \leq t \leq T$. This result is a direct consequence of the fact that the mapping

$$s \mapsto s\log s$$

is convex.

Proposition 9 (Increase in Internal Energy)**.** *Let* $U \in C^1([0,\infty)\,;\mathbb{R})$ *be a convex function such that* $U(0) = 0$*. For a fixed* $\alpha > 0$ *and initial datum* $f_0 \in PF_{B_R} \cap C^1(\mathbb{R}^{2d})$*, the corresponding solution* $f_\alpha \in PF_{f_0}^T \cap C^1([0,\infty) \times \mathbb{R}^{2d})$ *to our system satisfies*

$$\mathfrak{U}(f_\alpha(t,\cdot)) := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} U(f_\alpha(t,x,v))dxdv \geq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} U(f_0(x,v))dxdv.$$

for every $t \geq 0$*.*

Proof. The proof follows the same procedure as Agueh [3]. Let

$$P_U : [0,\infty) \to \mathbb{R}$$

$$s \mapsto sU'(s) - U(s)$$

be the associated pressure. As U is a convex function and $U(0) = 0$, then for any $s \geq 0$ the first order convexity condition implies

$$0 = -U(0) \leq (s-0)U'(s) - U(s) = P_U(s).$$

As $U(0) = 0$ we have $U(f_\alpha(t,x,v)) = 0$ on $(x,v) \in \mathbb{R}^d \times B_R - \text{supp } f_\alpha(t,\cdot)$ for every fixed $t \geq 0$. This means we may represent the energy functional above as

$$\mathfrak{U}(f_\alpha(t,\cdot)) = \int\int_{\text{supp } f_\alpha(t,\cdot)} U(f_\alpha(t,x,v))dxdv,$$

and as the solution f_α is non-negative we may assume that the function $U \circ f_\alpha$ is non-negative.
We may now differentiate the functional to obtain

$$\begin{aligned}
\frac{d\mathfrak{U}}{dt}(f_\alpha(t,\cdot)) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} U'(f_\alpha(t,x,v))\partial_t f_\alpha(t,x,v)dxdv \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} U'(f_\alpha(t,x,v)) \text{ div}_v(f_\alpha(t,x,v)F_{f_\alpha,\alpha}(t,x,v))dxdv \\
&\qquad - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} U'(f_\alpha(t,x,v))v \cdot \nabla_x f_\alpha(t,x,v)dxdv \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} U'(f_\alpha(t,x,v))\nabla_v f_\alpha(t,x,v) \cdot F_{f_\alpha,\alpha}(t,x,v)dxdv \\
&\qquad + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} U'(f_\alpha(t,x,v))f_\alpha(t,x,v) \text{ div}_v(F_{f_\alpha,\alpha}(t,x,v))dxdv \\
&\qquad\qquad + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \nabla_x(U'(f_\alpha(t,x,v))) \cdot v f_\alpha(t,x,v)dxdv.
\end{aligned}$$

Using the fact that

$$\begin{aligned}
\nabla_x(P_U(f_\alpha(t,x,v))) &= \nabla_x(f_\alpha(t,x,v)U'(f_\alpha(t,x,v)) - U(f_\alpha(t,x,v))) \\
&= \nabla_x(f_\alpha(t,x,v))U'(f_\alpha(t,x,v)) + f_\alpha(t,x,v)\nabla_x(U'(f_\alpha(t,x,v))) \\
&\qquad - \nabla_x(f_\alpha(t,x,v))U'(f_\alpha(t,x,v)) \\
&= f_\alpha(t,x,v)\nabla_x(U'(f_\alpha(t,x,v))),
\end{aligned}$$

we rewrite the above to obtain

$$\begin{aligned}\frac{d\mathfrak{U}}{dt}(f_\alpha(t,\cdot)) = &\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \nabla_v(U'(f_\alpha(t,x,v)))\cdot F_{f_\alpha,\alpha}(t,x,v)dxdv \\ &\quad + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} U'(f_\alpha(t,x,v))f_\alpha(t,x,v)\ \mathrm{div}_v(F_{f_\alpha,\alpha}(t,x,v))dxdv \\ &\qquad + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \nabla_x(P_U(f_\alpha(t,x,v)))\cdot vdxdv \\ = &\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} U'(f_\alpha(t,x,v))\ \mathrm{div}_v(F_{f_\alpha,\alpha}(t,x,v))dxdv \\ &\quad - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} U'(f_\alpha(t,x,v))f_\alpha(t,x,v)\ \mathrm{div}_v(F_{f_\alpha,\alpha}(t,x,v))dxdv \\ = &\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} P_U(f_\alpha(t,x,v))\ \mathrm{div}_v(F_{f_\alpha,\alpha}(t,x,v))dxdv.\end{aligned}$$

By expanding the above term we find

$$\begin{aligned}\mathrm{div}_v(F_{f_\alpha,\alpha}(t,x,v)) &= \nabla_v\cdot\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\eta_\alpha(x-y)\nabla W(v-u)f_\alpha(t,y,u)dydu \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\eta_\alpha(x-y)\Delta W(v-u)f_\alpha(t,y,u)dydu.\end{aligned}$$

Substituting this back into the above obtains

$$\frac{d\mathfrak{U}}{dt}(f_\alpha(t,\cdot)) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} P_U(f_\alpha(t,x,v))\eta_\alpha(x-y)\Delta W(v-u)f_\alpha(t,y,u)dxdvdydu. \tag{4.6}$$

Note that $\eta_\alpha(x-y)\geq 0$, $P_U(f_\alpha(t,x,v))\geq 0$ and by the second order convexity condition we have $\Delta W(v-u)\geq 0$. This implies

$$\frac{d\mathfrak{U}}{dt}(f_\alpha(t,\cdot))\geq 0$$

and integrating gives

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} U(f_\alpha(t,x,v))dxdv \geq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} U(f_0(x,v))dxdv.$$

□

Choosing $U(s)=s\log s$ in Proposition 9, we get the reverse H-theorem claimed before.

4.2.4 L^p Bounds in Finite Time

Proposition 10 (L^p Bounds on Solutions in Finite Time)**.** *Let* $f_0 \in PF_{B_R} \cap C^1(\mathbb{R}^{2d})$ *be the initial datum of our system,* $f_\alpha \in PF^T_{f_0} \cap C^1([0,\infty) \times \mathbb{R}^{2d})$ *be the corresponding solution, and* $p \geq 2$*. Given* $f_0 \in L^p(\mathbb{R}^d \times \mathbb{R}^d)$*, we have* $f_\alpha(t,\cdot) \in L^p(\mathbb{R}^d \times \mathbb{R}^d)$ *for* $t \geq 0$*. Specifically, we have*

$$\|f_\alpha(t,\cdot)\|_{L^p(\mathbb{R}^d\times\mathbb{R}^d)} \leq \exp\left(\frac{p-1}{p}\|\eta_\alpha\|_{L^\infty}\|\Delta W\|_{L^\infty(\overline{B}_{2R})}t\right)\|f_0\|_{L^p(\mathbb{R}^d\times\mathbb{R}^d)}.$$

For the case $p = \infty$ *we have*

$$\|f_\alpha(t,\cdot)\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)} \leq \exp(\|\eta_\alpha\|_{L^\infty}\|\Delta W\|_{L^\infty(\overline{B}_{2R})}t)\|f_0\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}.$$

Proof. As the function $s \mapsto s^p$ is convex we may apply our previous result. The corresponding pressure to this is

$$P_U(s) = (p-1)s^p.$$

We use (4.6) and the fact that $f_\alpha(t,\cdot) \in PF_{B_R}(\mathbb{R}^d \times \mathbb{R}^d)$. This yields

$$\begin{aligned}
\frac{d}{dt}\|f_\alpha(t,\cdot)\|^p_{L^p(\mathbb{R}^d\times\mathbb{R}^d)} &= (p-1)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\eta_\alpha(x-y)\Delta W(v-u)f_\alpha(t,x,v)^p f_\alpha(t,y,u)dxdvdydu \\
&\leq (p-1)\|\eta_\alpha\|_{L^\infty}\|\Delta W\|_{L^\infty(\overline{B}_{2R})}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f_\alpha(t,x,v)^p f_\alpha(t,y,u)dxdvdydu \\
&= (p-1)\|\eta_\alpha\|_{L^\infty}\|\Delta W\|_{L^\infty(\overline{B}_{2R})}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f_\alpha(t,x,v)^p dxdv \\
&= (p-1)\|\eta_\alpha\|_{L^\infty}\|\Delta W\|_{L^\infty(\overline{B}_{2R})}\|f_\alpha(t,\cdot)\|^p_{L^p(\mathbb{R}^d\times\mathbb{R}^d)}.
\end{aligned}$$

Gronwall's inequality gives

$$\|f_\alpha(t,\cdot)\|_{L^p(\mathbb{R}^d\times\mathbb{R}^d)} \leq \exp\left(\frac{p-1}{p}\|\eta_\alpha\|_{L^\infty}\|\Delta W\|_{L^\infty(\overline{B}_{2R})}t\right)\|f_0\|_{L^p(\mathbb{R}^d\times\mathbb{R}^d)},$$

which is the desired result. By taking $p \to \infty$ we get the latter result. □

4.2.5 Bounded Velocity Support of the System

We are now ready to justify the focus on the space $PF^T_{f_0}$ given an initial datum f_0. This follows immediately from the decrease of the p^{th} moments of solutions to our model. We explore its natural emergence in the discrete sense (1.3) first.

Theorem 8 (Bounded Velocity Support)**.** *Consider the system described in (1.3) and let*

$$R(t) = \max_{1\leq i\leq N}\|v_i(t)\|_2.$$

Assume further that $R(0)$ is finite. Then the velocity support of the system is bounded by $R(0)$ for all time $t > 0$.

Proof. We follow the same proof as Agueh [3]. For any $t > 0$ there exists a closed interval I and an index $i = 1, ..., N$ such that $R(t) = \|v_i(t)\|_2$ for all $t \in I$. Moreover, as the potential is radially symmetric there exists a function w such that $W(v) = w(\|v\|_2)$. Using

$$\nabla W(v) = \frac{w'(\|v\|_2)}{\|v\|_2} v,$$

we may differentiate $R(t)^2$ to obtain

$$\frac{d}{dt} R(t)^2 = 2v_i \cdot \dot{v}_i = 2\varepsilon \sum_{j=1}^{N} \eta_\alpha(x_i - x_j) \frac{w'(\|v_j - v_i\|_2)}{\|v_j - v_i\|_2} (v_i \cdot v_j - \|v_i\|_2^2).$$

Using the Cauchy-Schwarz inequality and the fact that $\|v_i\|_2$ is largest vector, we obtain $v_i \cdot v_j \leq \|v_i\|_2 \|v_j\|_2 \leq \|v_i\|_2^2$. Since w is nondecreasing and η_α is nonnegative, the condition $v_i \cdot v_j - \|v_i\|_2^2 \leq 0$ implies

$$\frac{d}{dt} R(t)^2 = 2R(t)\dot{R}(t) \leq 0.$$

As $R(t) \geq 0$ we obtain $\dot{R}(t) \leq 0$. Thus the velocity support must be bounded by $R(0)$.

□

This leads us to expect that the velocity support of a solution $f(t, x, v)$ to (1.4) possesses compact velocity support as described by Agueh [3]. Specifically, this is given by

$$\text{supp } f(t, x, \cdot) \subset B_R$$

provided that the initial datum f_0 satisfies

$$\text{supp } f_0(x, \cdot) \subset B_R.$$

This is seen to be true as a corollary to previously mentioned results.

Theorem 9. *Solutions to the kinetic model have bounded velocity support. In other words, for a fixed $\alpha > 0$ and initial datum $f_0 \in PF_{B_R} \cap C^1(\mathbb{R}^{2d})$, the corresponding solution $f_\alpha \in C^1([0, \infty) \times \mathbb{R}^{2d})$ is in the set $PF_{f_0}^T$.*

Proof. We know that

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \|v\|_p^p f_\alpha(t, x, v) dx dv \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \|v\|_p f_0(x, v) dx dv$$

for every $1 < p < \infty$ by the previous result. Reformulating, this means that

$$\|v\|_{L^p(f_\alpha dxdv)} \leq \|v\|_{L^p(f_0 dxdv)}.$$

Taking the limit as $p \to \infty$ gives

$$\|v\|_{L^\infty(f_\alpha dxdv)} \leq \|v\|_{L^\infty(f_0 dxdv)}.$$

As the initial datum satisfies supp $f_0(x,\cdot) \subset B_R$ for every $x \in \mathbb{R}^d$, it follows from the inequality that the velocity support in respect to f_α is bounded in the ball B_R. $\square$

4.3 Integral Curves and the Linear System

In order to construct a solution to the model, we break the kinetic equation into its characteristic system (i.e. finding a system of differential equations satisfied by integral curves of the solution). Then, we proceed to determine an associated 'flow map' that will *push* the initial datum forward over time in such a way that it yields a solution.

However, as we embrace a problem of nonlinearity in our kinetic model we alter it by replacing the force term $F_{f,\alpha}$, as to enforce linearity. A fixed point argument will recover the alteration we have made. For this approximation we define the linear model:

Let T, $\alpha > 0$. Given an initial datum $f_0 \in PF_{B_R} \cap C^1(\mathbb{R}^{2d})$ and a function $g \in PF_{f_0}^T$ we say that $f \in PF_{f_0}^T \cap C^1([0,T] \times \mathbb{R}^{2d})$ is a solution to the linear kinetic model if

$$\frac{\partial f}{\partial t}(t,x,v) + v \cdot \nabla_x f(t,x,v) = \mathrm{div}_v(f F_{g,\alpha}(t,x,v)) \tag{4.7}$$

holds for every $0 \le t \le T$ and x, $v \in \mathbb{R}^d$ where

$$F_{g,\alpha}(t,x,v) = \eta_\alpha(x) \nabla W(v) *_{(x,v)} g(t,x,v)$$

Theorem 10 (Strong Characteristic System of Linear Kinetic Model)**.** *Let $f_0 \in PF_{B_R} \cap C^1(\mathbb{R}^{2d})$ be the initial datum of our system and let $g \in PF_{f_0}^T$ be given. Furthermore, for fixed $\alpha > 0$ let $f \in PF_{f_0}^T \cap C^1([0,T] \times \mathbb{R}^{2d})$ be a solution of the classical model (4.7). For $0 \le t \le T$ it follows that if $(x(t), v(t))$ is an integral curve of the surface $f = f(t,x,v)$ then it satisfies the system*

$$\begin{cases} x'(t) & = v(t) \\ v'(t) & = -F_{g,\alpha}(t, x(t), v(t)) \\ (f(t,x(t),v(t))' & = f(t,x(t),v(t))\, div_{v(t)} F_{g,\alpha}(t,x(t),v(t)). \end{cases}$$

Proof. We may rewrite and expand the linear system above as

$$\partial_t f + \sum_{i=1}^d v_i \partial_{x_i} f - \sum_{i=1}^d \partial_{v_i} f F_{g,\alpha}^{(i)} = f \mathrm{div}_v F_{g,\alpha}.$$

From this, we may rewrite the system as the inner product

$$(1, v, -F_{g,\alpha}, f \mathrm{div}_v F_{g,\alpha}) \cdot (\partial_t f, \nabla_x f, \nabla_v f, -1) = 0.$$

This implies that the two vectors are orthogonal. Define the function $\Theta : \mathbb{R}^{2d+2} \to \mathbb{R}$ by

$$\Theta(t,x,v,\tilde{f}) = \tilde{f} - f(t,x,v),$$

where $f(t,x,v)$ represents a solution to (4.7). Note that $\tilde{f}$ is simply a variable of Θ. A calculation shows

$$-\nabla\Theta = (-\partial_t\Theta, -\nabla_x\Theta, -\nabla_v\Theta, -\partial_f\Theta) = (\partial_t f, \nabla_x f, \nabla_v f, -1).$$

However, as $-\nabla\Theta$ is orthogonal to the manifold described by $\Theta = 0$ (i.e. $\tilde{f} = f(t,x,v)$) this implies that the vector

$$(1, v, -F_{g,\alpha}, f\mathrm{div}_v F_{g,\alpha})$$

lies in the tangent space of the manifold satisfying $\tilde{f} = f(t,x,v)$. Hence, any integral curve $(t(s), x(s), v(s))$ parameterized by $s \geq 0$ satisfies the system

$$\begin{cases} t'(s) & = 1 \\ x'(s) & = v(s) \\ v'(s) & = -F_{g,\alpha}(s, x(s), v(s)) \\ (f(s,x(s),v(s))' & = f(s,x(s),v(s))\mathrm{div}_{v(s)} F_{g,\alpha}(s, x(s), v(s)). \end{cases}$$

Using the initial condition $t(0) = 0$ we obtain the desired system. □

However, it is not guaranteed (yet) that we can obtain greater regularity than a weak solution to our kinetic model. Hence, it is important to consider the system in a measure theoretic sense. We may establish a similar linear system of ordinary differential equations by fixing some $\nu \in P^T_{\mu_0}$ where $\mu_0 \in P_{B_R}$ is the initial datum. By tweaking the measure theoretic model (4.3) we establish the following:

Let T, $\alpha > 0$. Given an initial datum $\mu_0 \in P_{B_R}$ and a measure $\nu \in P^T_{\mu_0}$ we say that $\mu \in P^T_{\mu_0}$ is a linear solution provided that for every $\phi \in C_0^\infty([0,T] \times \mathbb{R}^{2d})$ the following holds

$$\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\frac{\partial \phi}{\partial t}(t,x,v) + v \cdot \nabla_x \phi - F_{\nu,\alpha} \cdot \nabla_v \phi(t,x,v) \right) d\mu(t,x,v) = -\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(0,x,v) d\mu_0(x,v) \tag{4.8}$$

where we have defined the force term in the measure-theoretic sense by

$$F_{\nu,\alpha}(t,x,v) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \nabla W(v-u) \eta_\alpha(x-y) d\nu(t,y,u). \tag{4.9}$$

Using a similar argument to the above gives the following result:

Theorem 11 (Weak Characteristic System of Linear Kinetic Model)**.** *Let* $\mu_0 \in P_{B_R}$ *be the initial datum of our system and let* $\nu \in P^T_{\mu_0}$ *be given. Furthermore, for fixed* $\alpha > 0$ *let* $\mu \in P^T_{\mu_0}$ *be a solution of the model (4.8). For* $t \geq 0$ *it follows that if* $(x(t), v(t))$ *is an integral curve then it satisfies the system*

$$\begin{cases} x'(t) & = v(t) \\ v'(t) & = -F_{\nu,\alpha}(t, x(t), v(t)). \end{cases}$$

4.4 Existence of the Characteristic Flow Transport Map

In this section we analyze the characteristic system established by Theorem 11. Our primary goal in this section will be to establish a solution to the system of ordinary differential equations. We will use the flow obtained to 'push' the initial measure μ_0 to a solution of (4.4) in a measure-theoretic sense. To prove existence of the flow we invoke the Picard-Lindelöf theorem.

4.4.1 Lipschitz Properties of the Force Term

Proposition 11. *Let $0 \leq t \leq T$ be fixed, $\nu \in P^T_{\mu_0}$, and $F_{\nu,\alpha}$ be the force term described by (4.9). Then $F_{\nu,\alpha}(t,\cdot)$ is Lipschitz with constant*

$$L_F = \|\eta_\alpha\|_{L^\infty}(Lip\ \nabla W)_{\overline{B}_{2R}} + \|\nabla W\|_{L^\infty(\overline{B}_{2R})} Lip\ \eta_\alpha$$

dependent only on α and R.

Proof. This is seen by direct evaluation. Notice that the velocity support of the system lies in a ball B_R (described by Theorem 8). Moreover, as $\nu \in P^T_{\mu_0}$, then supp $\mu(t,x,\cdot) \subset B_R$. Thus, in the integrand of

$$F_{\nu,\alpha}(t,x,v) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \eta_\alpha(x-y)\nabla W(v-u)d\nu(t,y,u),$$

the potential $\nabla W(v-u)$ may be considered as a class C^1 function over the disc $\overline{B}_{2R}$ and consequently Lipschitz. Moreover, we know that for every α, the mollifer η_α is continuously differentiable, and thus Lipschitz in x. Summarizing these results we obtain

$$|\nabla W(v-u) - \nabla W(v'-u)| \leq (\text{Lip } \nabla W)_{\overline{B}_{2R}}|v-v'|$$

$$|\eta_\alpha(x-y) - \eta_\alpha(x'-y)| \leq (\text{Lip } \eta_\alpha)|x-x'|$$

for every v, v', $u \in B_R$ and x, x', $y \in \mathbb{R}^d$. Putting these results together, gives the following Lipschitz estimate on the force term. Let x, $x' \in \mathbb{R}^d$, v, $v' \in B_R$ and let $t \in [0,T]$ be fixed. Then the force term associated to $\nu \in P^T_{\mu_0}$ satisfies

$$\begin{aligned} |F_{\nu,\alpha}(t,x,v) - F_{\nu,\alpha}(t,x',v')| &\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\eta_\alpha(x-y)\nabla W(v-u) - \eta_\alpha(x'-y)\nabla W(v'-u)|d\nu(t,y,u) \\ &\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\eta_\alpha(x-y)\nabla W(v-u) - \eta_\alpha(x-y)\nabla W(v'-u)|d\nu(t,y,u) \\ &\quad + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\eta_\alpha(x-y)\nabla W(v'-u) - \eta_\alpha(x'-y)\nabla W(v'-u)|d\nu(t,y,u) \end{aligned}$$

Using the fact that the mollifier is bounded over $\mathbb{R}^d$ and $\nu(t,\cdot) \in P(\mathbb{R}^d \times \mathbb{R}^d)$ for every fixed $t \in [0,T]$ the first integral estimates as

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\eta_\alpha(x-y)\nabla W(v-u) - \eta_\alpha(x-y)\nabla W(v'-u)| d\nu(t,y,u) \leq \|\eta_\alpha\|_{L^\infty(\mathbb{R}^d)} (\text{Lip } \nabla W)_{\overline{B}_{2R}} |v-v'|.$$

In a similar fashion, the potential $\nabla W(v-u)$ is a class C^1 function evaluated over the compact space $\overline{B}_R$, and is thus bounded. Hence, the second integral estimates as

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\eta_\alpha(x-y)\nabla W(v'-u) - \eta_\alpha(x'-y)\nabla W(v'-u)| d\nu(t,y,u) \leq \|\nabla W\|_{L^\infty(\overline{B}_{2R})} (\text{Lip } \eta_\alpha) |x-x'|.$$

Combining the above two results,

$$|F_{\nu,\alpha}(t,x,v) - F_{\nu,\alpha}(t,x',v')| \leq \|\eta_\alpha\|_{L^\infty(\mathbb{R}^d)} (\text{Lip } \nabla W)_{\overline{B}_{2R}} |v-v'| + \|\nabla W\|_{L^\infty(\overline{B}_{2R})} (\text{Lip } \eta_\alpha) |x-x'|.$$

By the l^1-norm properties we have

$$|v-v'| \leq |x-x'| + |v-v'| = |(x,v)-(x',v')|.$$

An analogous result holds for $|x-x'|$. Inputting this into the above inequality yields

$$|F_{\nu,\alpha}(t,x,v) - F_{\nu,\alpha}(t,x',v')| \leq (\|\eta_\alpha\|_{L^\infty(\mathbb{R}^d)} (\text{Lip } \nabla W)_{\overline{B}_{2R}} + \|\nabla W\|_{L^\infty(\overline{B}_{2R})} (\text{Lip } \eta_\alpha)) |(x,v)-(x',v')|.$$

Thus the force term is Lipschitz for every fixed $t \in [0,T]$ with constant dependent only on α and R. ☐

Remark. *Note that the Lipschitz constant of $F_{\nu,\alpha}(t,\cdot)$ is independent of any particular choice of $\nu \in P^T_{\mu_0}$. Due to this independence, we will abbreviate it as L_F with reminder that it depends only on α and R.*

4.4.2 Properties of the Characteristic Map

Referring to Theorem 11, we establish an associated characteristic map which defines the system. Our aim will be to use the Lipschitz property of the force term to enforce enough regularity on the characteristic system. This will ensure us a solution to (11).

Definition 11. *Let $\nu \in P^T_{\mu_0}$. We define the function $\Phi_{\nu,\alpha} : [0,T] \times \mathbb{R}^d \times B_R \to B_R \times \mathbb{R}^d$ by*

$$\Phi_{\nu,\alpha}(t,x,v) = (v, -F_{\nu,\alpha}(t,x,v)) \tag{4.10}$$

and refer to it as the characteristic map.

This map Φ is defined in such a way that if $(x(t), v(t))$ is an integral curve of the linear model (4.8), then it satisfies the system

$$(x'(t), v'(t)) = \Phi_{\nu,\alpha}(t, x(t), v(t)).$$

Establishing existence of a flow map is fundamentally tied to establishing regularity on $\Phi_{\nu,\alpha}$, in particular Lipschitz in the space-velocity component and globally continuous.

Proposition 12. *Let $t \in [0,T]$ be fixed, $\nu \in P_{\mu_0}^T$ and $\Phi_{\nu,\alpha}$ be the associated term in the characteristic system described by (4.10). Then $\Phi_{\nu,\alpha}(t,\cdot)$ is Lipschitz with constant*

$$L_\Phi = 1 + L_F$$

dependent only on α and R.

Proof. We use the results of the previous proposition as follows. Let $x,\ x' \in \mathbb{R}^d$ and $v,\ v' \in B_R$. Then as $\Phi_{\nu,\alpha}(t,x,v) = (v, -F_{\nu,\alpha}(t,x,v))$ we obtain

$$\begin{aligned}
|\Phi_{\nu,\alpha}(t,x,v) - \Phi_{\nu,\alpha}(t,x',v')| &= |v - v'| + |F_{\nu,\alpha}(t,x,v) - F_{\nu,\alpha}(t,x',v')| \\
&\le |v - v'| + L_F|(x,v) - (x',v')| \\
&\le (1 + L_F)|(x,v) - (x',v')|.
\end{aligned}$$

It follows that $\Phi_{\nu,\alpha}(t,\cdot)$ is Lipschitz with constant depending only on α and R. □

Remark. *As above, we will use the notation L_Φ to denote the Lipschitz constant of $\Phi_{\nu,\alpha}(t,\cdot)$ for fixed $0 \le t \le T$. Reminding that we have dependency only on α and R.*

Proposition 13. *Let $\nu \in P_{\mu_0}^T$ where $\mu_0 \in P_{B_R}$ is the initial measure, then the associated characteristic map $\Phi_{\nu,\alpha}$ described by (4.10) is continuous on $[0,T] \times \mathbb{R}^d \times B_R$.*

Proof. Given $t,\ t' \in [0,T]$, $x,\ x' \in \mathbb{R}^d$ and $v,\ v' \in B_R$ we obtain

$$\begin{aligned}
|\Phi_{\nu,\alpha}(t,x,v) - \Phi_{\nu,\alpha}(t',x',v')| &\le |(v - v', F_{\nu,\alpha}(t,x,v) - F_{\nu,\alpha}(t,x',v')| + |(0, F_{\nu,\alpha}(t,x',v') - F_{\nu,\alpha}(t',x',v')| \\
&\le |v - v'| + L_F|(x,v) - (x',v')| + |F_{\nu,\alpha}(t,x',v') - F_{\nu,\alpha}(t',x',v')| \\
&\le (1 + L_F)|(x,v) - (x',v')| + \sum_{i=1}^{d} |F_{\nu,\alpha}^{(i)}(t,x',v') - F_{\nu,\alpha}^{(i)}(t',x',v')|,
\end{aligned}$$

where $F_{\nu,\alpha}^{(i)}$ is the ith component of $F_{\nu,\alpha}$. Given $\varepsilon > 0$ we consider the region $|(x,v) - (x',v')| < \varepsilon/(1 + L_F)$. Now, as each component is representable as

$$F^{(i)}_{\nu,\alpha} = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \eta_\alpha(x-y)\partial_{v_i}W(v-u)d\nu(t,x,v)$$

then the fact that $\eta_\alpha(x-y)\partial_{v_i}W(v-u)$ is bounded over the support of the integrand implies we may exploit the weak continuity of $\nu \in P^T_{\mu_0}$.
That is, given $\varepsilon/2d$ there exists a $\delta_1 > 0$ such that $|t-t'| < \delta_1$ implies $|F^{(i)}_{\nu,\alpha}(t,x',v') - F^{(i)}_{\nu,\alpha}(t',x',v')| < \varepsilon/2d$. Accordingly, given $\varepsilon > 0$ we may select $\delta = \min\{\delta_1, \varepsilon/(2(1+L_F))\}$ so that whenever $|(t,x,v)-(t',x',v')| < \delta$ we have

$$|\Phi_{\nu,\alpha}(t,x,v) - \Phi_{\nu,\alpha}(t',x',v')| < (1+L_F)\frac{\varepsilon}{2(1+L_F)} + d\frac{\varepsilon}{2d} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $\Phi_{\nu,\alpha}$ is continuous. □

Proposition 14. *Let $\nu \in P^T_{\mu_0}$ where $\mu_0 \in P_{B_R}$ is the initial measure. Consider the associated characteristic map $\Phi_{\nu,\alpha}$ described by (4.10). For every fixed $0 \le t \le T$ it follows that $\Phi_{\nu,\alpha}(t,\cdot) \in C^\infty(\mathbb{R}^d; C^1(B_R; B_R \times \mathbb{R}^d))$.*

Proof. By definition $\Phi_{\nu,\alpha}$ is represented as

$$\Phi_{\nu,\alpha}(t,x,v) = (v, -F_{\nu,\alpha}(t,x,v)) = \left(v, -\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \nabla W(v-u)\eta_\alpha(x-y)d\nu(t,y,u)\right).$$

The result follows from the fact that W is of class C^2 and η_α is of class C^∞. □

Lemma 5. *Let $\nu \in P^T_{\mu_0}$ where $\mu_0 \in P_{B_R}$ is the initial measure and $\Phi_{\nu,\alpha}$ be the associated characteristic field described by (4.10). There exists a constant $K > 0$ depending only on α and R such that for every point $(x,v) \in \mathbb{R}^d \times B_R$ and $0 \le t \le T$ we have*

$$|\Phi_{\nu,\alpha}(t,x,v)| \le K(\alpha,R)(1+|(x,v)|).$$

Proof. This is established by the following:

$$\begin{aligned}
|\Phi_{\nu,\alpha}(t,x,v)| &= |(v, -F_{\nu,\alpha}(t,x,v))| \\
&\le |v| + |F_{\nu,\alpha}(t,x,v)| \\
&\le |(x,v)| + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\eta_\alpha(x-y)\nabla W(v-u)| d\nu(t,y,u).
\end{aligned}$$

Recall that for each α the mollifier η_α is bounded. Then, as the potential $\nabla W(v-u)$ is a class C^1 function evaluated over the ball $\overline{B}_{2R}$ (in the integrand) it is bounded as well. Moreover, as $\nu(t,\cdot)$ is a probability measure we have for every fixed $0 \le t \le T$ the inequality:

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\eta_\alpha(x-y)\nabla W(v-u)|d\nu(t,y,u) \leq \|\eta_\alpha\|_{L^\infty(\mathbb{R}^d)}\|\nabla W\|_{L^\infty(\overline{B}_{2R})}$$

Combining this result with above yields

$$|\Phi_{\nu,\alpha}(t,x,v)| \leq |(x,v)| + \|\eta_\alpha\|_{L^\infty(\mathbb{R}^d)}\|\nabla W\|_{L^\infty(\overline{B}_{2R})}.$$

By setting $K = \max\{1, \|\eta_\alpha\|_{L^\infty(\mathbb{R}^d)}\|\nabla W\|_{L^\infty(\overline{B}_{2R})}\}$ we obtain

$$|\Phi_{g,\alpha}(t,x,v)| \leq K(1+|(x,v)|)$$

where K depends only on α and R. □

Lemma 6 (Lipschitz Property in $P^T_{\mu_0}$)**.** *Let ν, $\lambda \in P^T_{\mu_0}$ where $\mu_0 \in P_{B_R}$ is the initial measure. Furthermore, let $\Phi_{\nu,\alpha}$, $\Phi_{\lambda,\alpha}$ be respectively described by (4.10). There exists a constant N depending only on α and R such that for every $(x,v) \in \mathbb{R}^d \times B_R$ and $0 \leq t \leq T$ we have*

$$|\Phi_{\nu,\alpha}(t,x,v) - \Phi_{\lambda,\alpha}(t,x,v)| \leq N(\alpha,R)W_1(\nu(t,\cdot),\lambda(t,\cdot)).$$

Proof. Recall that every $\nu \in P^T_{\mu_0}$ has velocity support in B_R. By Urysohn's lemma (see Appendix) there exists a smooth function $\Omega(v) : \mathbb{R}^d \to [0,1]$ with compact support such that $\Omega|_{B_R} = 1$. This serves as a '*cutoff*' function leaving the force invariant as follows.

For fixed $0 \leq t \leq T$ let $\pi \in \Pi(\nu(t,\cdot),\lambda(t,\cdot))$ be a transference plan. It follows that

$$\begin{aligned}
&F_{\nu,\alpha}(t,x,v) - F_{\lambda,\alpha}(t,x,v)\\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \eta_\alpha(x-y)\nabla W(v-u)\Omega(u)d\nu(t,y,u) - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \eta_\alpha(x-z)\nabla W(v-w)\Omega(w)d\lambda(t,y,u)\\
&= \int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}} (\eta_\alpha(x-y)\nabla W(v-u)\Omega(u) - \eta_\alpha(x-z)\nabla W(v-w)\Omega(w))d\pi(y,u,z,w).
\end{aligned}$$

Thus $|\Phi_{\nu,\alpha}(t,x,v) - \Phi_{\lambda,\alpha}(t,x,v)| = |F_{\nu,\alpha}(t,x,v) - F_{\mu,\alpha}(t,x,v)|$ gives

$$|\Phi_{\nu,\alpha}(t,P) - \Phi_{\lambda,\alpha}(t,P)| \leq \int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}} |\eta_\alpha(x-y)\nabla W(v-u)\Omega(u) - \eta_\alpha(x-z)\nabla W(v-w)\Omega(w)|d\pi(y,u,z,w).$$

Next, define the function

$$G_{(x,v)}(y,u) = \eta_\alpha(x-y)\nabla W(v-u)\Omega(u)$$

where $y \in \mathbb{R}^d$ and $u \in \operatorname{supp}\Omega$. We seek to prove that the function is Lipschitz with constant independent of any point $(x, v) \in \mathbb{R}^d \times B_R$ chosen. Let e_i denote the vector in $\mathbb{R}^d$ with 1 in the ith position and zero elsewhere. We represent each component of the above as

$$G_{(x,v)}(y,u) \cdot e_i = \eta_\alpha(x-y)\frac{\partial W(v-u)}{\partial v_i}\Omega(u).$$

For each index k we obtain

$$\begin{aligned}\frac{\partial G_{(x,v)}(y,u) \cdot e_i}{\partial y_k} &= \frac{\partial \eta_\alpha(x-y)}{\partial y_k}\frac{\partial W(v-u)}{\partial v_i}\Omega(u)\\ &= -\frac{\partial \eta_\alpha(x-y)}{\partial(x_k-y_k)}\frac{\partial W(v-u)}{\partial v_i}\Omega(u).\end{aligned}$$

Now, for each fixed α the function η_α is of class C_0^∞. Hence, it is globally Lipschitz, implying its derivative is bounded (depending only on α). Consider the largest disc $\overline{B}_M$ containing $\operatorname{supp}\Omega$ and let $\overline{B}_U$ be the disc of radius $U = M + R$ where R is the radius of the velocity support B_R. As W is of class C^2 over the compact space $\overline{B}_U$ and Ω is of class C^∞ over its compact support then the product

$$\frac{\partial W(v-u)}{\partial v_i}\Omega(u)$$

is bounded (depending only on both R and $\operatorname{supp}\Omega$). Putting this together gives

$$\left|\frac{\partial G_{(x,v)}(y,u) \cdot e_i}{\partial y_k}\right| \leq \|\partial_k \eta_\alpha\|_{L^\infty(\mathbb{R}^d)} \|\partial_i W\|_{L^\infty(\overline{B}_U)}.$$

Continuing in this manner we obtain

$$\left|\frac{\partial G_{(x,v)}(y,u) \cdot e_i}{\partial u_k}\right| = |\eta_\alpha(x-y)| \left|\frac{\partial}{\partial u_k}\left[\frac{\partial W(v-u)}{\partial v_i}\Omega(u)\right]\right|.$$

Now, for each α the mollifier η_α is bounded with dependence only on α. As implied above, the product

$$\frac{\partial W(v-u)}{\partial v_i}\Omega(u)$$

is of class C^1 over the compact space $\overline{B}_U$ and thus is bounded with dependence on only its domain (i.e $\operatorname{supp}\Omega$ and R).

Putting this together, we obtain the bound

$$\left|\frac{\partial G_{(x,v)}(y,u) \cdot e_i}{\partial u_k}\right| \leq \|\eta_\alpha\|_{L^\infty(\mathbb{R}^d)} \left\|\frac{\partial}{\partial u_k}\left[\frac{\partial W(v-u)}{\partial v_i}\Omega(u)\right]\right\|_{L^\infty(\overline{B}_U)}$$

with dependency on only α, $\operatorname{supp}\Omega$ and R. As this choice was independent of any particular index k then the derivative of $G_{(x,v)}(y,u)$ is bounded independently of the point (x, v). Accordingly, the function is Lipschitz and there exists a constant $N > 0$ (depending only on α, $\operatorname{supp}\Omega$ and R) such that

$$|\eta_\alpha(x-y)\nabla W(v-u)\Omega(u)-\eta_\alpha(x-z)\nabla W(v-w)\Omega(w)|\leq N|(y,u)-(z,w)|$$

for all y, $z\in\mathbb{R}^d$ and u, $w\in\overline{B}_U$. We obtain

$$|\Phi_{\nu,\alpha}(t,x,v)-\Phi_{\lambda,\alpha}(t,x,v)|\leq N\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}}|(y,u)-(z,w)|\pi(y,u,z,w)dydudzdw.$$

Taking the infimum over all $\pi\in\Pi(\nu(t,\cdot),\lambda(t,\cdot))$ gives the desired result.

□

4.4.3 Properties of the Flow Map

Theorem 12. *Let $\nu\in P^T_{\mu_0}$ where $\mu_0\in P_{B_R}$ is the initial measure and $\Phi_{\nu,\alpha}$ be the characteristic map described by (4.10). Furthermore let $(x_0,v_0)\in\mathbb{R}^d\times B_R$ be a point of initial data for the integral curves. It follows that for $0\leq t\leq T$ the characteristic system*

$$\begin{cases}(x'(t),v'(t))=\Phi_{\nu,\alpha}(t,x,v)\\(x(0),v(0))=(x_0,v_0)\end{cases}$$

is well-posed.

The proof of this theorem is merely an application of using the previously mentioned properties of $\Phi_{\nu,\alpha}$ and the Picard-Lindelöf theorem. With this result established we now give a name to the flow map to easily distinguish it from other functions that we will consider.

Definition 12. *Let $\nu\in P^T_{\mu_0}$ where $\mu_0\in P_{B_R}$ is the initial measure and $\Phi_{\nu,\alpha}$ be the characteristic map described by (4.10). For any point $(x,v)\in\mathbb{R}^d\times B_R$ let $T^t_\nu:\mathbb{R}^d\times B_R\to\mathbb{R}^d\times B_R$ be the solution of the characteristics system*

$$\begin{cases}(x'(t),v'(t))=\Phi_{\nu,\alpha}(t,x,v)\\(x(0),v(0))=(x,v)\end{cases}$$

defined by

$$T^t_\nu(x,v)=(x(t),v(t))\tag{4.11}$$

for $0\leq t\leq T$. This map will be referred to as the flow map with respect to ν (or simply flow map when the measure ν is implied).

Proposition 15 (Lipschitz Continuity of the Flow Map in Time)**.** *Let $\nu\in P^T_{\mu_0}$ where μ_0 is the initial measure and T^t_ν be the associated flow map described by (4.11). Then for each $(x,v)\in\mathbb{R}^d\times B_R$ the flow map is Lipschitz in time with constant*

$$L_{T,t} = R + \|\eta_\alpha\|_{L^\infty} \|\nabla W\|_{L^\infty(\overline{B}_{2R})}$$

depending only on α and R.

Proof. Let $0 \leq t, t' \leq T$, $(x, v) \in \mathbb{R}^d \times B_R$ and $T^t_\nu(x, v) = (x(t), v(t))$ be the associated integral curves. Then bounding the force term gives

$$\begin{aligned}
|T^t_\nu(x,v) - T^{t'}_\nu(x,v)| &= \left| \int_{t'}^{t} (x'(\tau), v'(\tau))\, d\tau \right| \\
&= \left| \int_{t'}^{t} (v(\tau), -F_{\nu,\alpha}(\tau, x(\tau), v(\tau)))\, d\tau \right| \\
&\leq \int_{t'}^{t} |v(\tau)| d\tau + \int_{t'}^{t} |F_{\nu,\alpha}(\tau, x(\tau), v(\tau))| d\tau \\
&\leq R|t - t'| + \int_{t'}^{t} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\eta_\alpha(x - y) \nabla W(v - u)| d\nu(\tau, y, u) \\
&\leq (R + \|\eta_\alpha\|_{L^\infty} \|\nabla W\|_{L^\infty(\overline{B}_{2R})})|t - t'|.
\end{aligned}$$

Thus the flow map is Lipschitz in time. □

Proposition 16 (Lipschitz Continuity of the Flow Map in Space and Velocity)**.** *Let $\nu \in P^T_{\mu_0}$ where $\mu_0 \in P_{B_R}$ is the initial measure and T^t_ν be the associated flow map described by (4.11). Then for every fixed $0 \leq t \leq T$, the flow map is Lipschitz in space and velocity with constant*

$$L_{T,(x,v)} = e^{L_\Phi t}$$

depending only on α and R.

Proof. For every $(x, v) \in \mathbb{R}^d \times B_R$ the flow map satisfies

$$T^t_\nu(x, v) = (x, v) + \int_0^t \Phi_{\nu,\alpha}(\tau, T^\tau_\nu(x, v)) d\tau.$$

Letting $(x', v') \in \mathbb{R}^d \times B_R$ we have

$$\begin{aligned}
|T^t_\nu(x, v) - T^t_\nu(x', v')| &\leq |(x, v) - (x', v')| + \int_0^t |\Phi_{\nu,\alpha}(\tau, T^\tau_\nu(x, v)) - \Phi_{\nu,\alpha}(\tau, T^\tau_\nu(x', v'))|\, d\tau \\
&\leq |(x, v) - (x', v')| + L_\Phi \int_0^t |T^\tau_\nu(x, v) - T^\tau_\nu(x', v')|\, d\tau,
\end{aligned}$$

where we have used the fact that $\Phi_{\nu,\alpha}(t, \cdot)$ is Lipschitz for every fixed $0 \leq t \leq T$. By Gronwall's inequality we obtain

$$|T_\nu^t(x,v) - T_\nu^t(x',v')| \leq e^{L_\Phi t}|(x,v) - (x',v')|.$$

As L_Φ is dependent only on α and R then the same must then hold for the Lipschtitz constant $L_{T,(x,v)}$ of T_ν^t.

∎

Proposition 17 (Bijectivity of the Flow Map for Fixed Time)**.** *Let $\nu \in P_{\mu_0}^T$ where $\mu_0 \in P_{B_R}$ is the initial measure and T_ν^t is the associated flow map described by (4.11). For every fixed $0 \leq t \leq T$ the flow map is a bijection.*

Proof. Suppose there exist (x,v), $(x',v') \in \mathbb{R}^d \times B_R$ such that

$$T_\nu^t(x,v) = T_\nu^t(x',v').$$

By uniqueness of the flow map there is only one solution to the system

$$\begin{cases} (x'(t), v'(t)) = \Phi_{\nu,\alpha}(x(t), v(t)) \\ (x(t), v(t)) = T_\nu^t(x,v) = T_\nu^t(x',v'). \end{cases}$$

Thus, it follows that $(x,v) = (x',v')$; this establishes that the flow map is injective. The fact that T_ν^t is surjective follows simply from tracing the flow of a particle backwards to obtain the preimage. ∎

Proposition 18 (Regularity of the Flow Map)**.** *Let $\nu \in P_{\mu_0}^T$ where $\mu_0 \in P_{B_R}$ is the initial measure and T_ν^t is the associated flow map described by (4.11). For every fixed $0 \leq t \leq T$ the flow map is a C^1 diffeomorphism.*

Proof. By definition, the flow map satisfies the characteristic system

$$\frac{dT_\nu^t}{dt}(x,v) = \Phi_{\nu,\alpha}(T_\nu^t(x,v)).$$

By Proposition 14 the characteristic map is of class $C^\infty(\mathbb{R}^d; C^1(B_R; B_R \times \mathbb{R}^d))$. Moreover, any nonautonomous system has an equivalent autonomous representation that preserves regularity results. Hence it follows by Proposition 29 that the flow is continuously differentiable. By Lemma 17 the flow map is invertible. It may also be shown that the inverse is given by $(T_\nu^t)^{-1} = T_\nu^{-t}$ by flow map properties (one may extend the time interval to allow reverse flow). As such, the inverse satisfies the characteristic system

$$\frac{dT_\nu^{-t}}{dt}(x,v) = \Phi_{\nu,\alpha}(T_\nu^{-t}(x,v)).$$

Hence the same argument concludes that the inverse is also continuously differentiable.

∎

Lemma 7 (Linear Growth of the Flow Map)**.** *Let* B_S *be a ball of radius* S *centered at the origin,* $\nu \in P^T_{\mu_0}$ *where* $\mu_0 \in P_{B_R}$ *is the initial measure and* T^t_ν *be the associated flow map described by (4.11). Then for any fixed* $0 \leq t \leq T$ *we have*

$$T^t_\nu(B_S \times B_R) \subset B_{S+tR} \times B_R.$$

Proof. As the flow map solves the characteristic equations (Theorem 11) then for any point $(x, v) \in B_S \times B_R$ we have

$$(x'(t), v'(t)) = \frac{dT^t_\nu}{dt}(x, v) = (v(t), -F_{g,\alpha}(t, x(t), v(t))$$

where $(x(t), v(t))$ is the associated integral curve satisfying $(x(0), v(0)) = (x, v)$; in other words $T^t_\nu(x, v) = (x(t), v(t))$. Integrating the first component we obtain

$$x(t) = x(0) + \int_0^t v(\tau)d\tau = x + \int_0^t v(\tau)d\tau.$$

As the velocity support of the system lies in B_R (i.e. $v(t) \in B_R$ for all $0 \leq t \leq T$) and $x \in B_S$ we obtain

$$|x(t)| \leq |x| + \int_0^t |v(\tau)|d\tau \leq S + tR.$$

This implies $x(t) \in B_{S+tR}$. We get the bound in the second component rather trivially. It follows from noticing the velocity support of the system is bounded in the ball B_R. As the point $(x, v) \in B_S \times B_R$ is arbitrary we conclude

$$T^t_\nu(B_S \times B_R) \subset B_{S+tR} \times B_R$$

as desired. □

Lemma 8 (Characteristics are Bounded in Finite Time)**.** *Let* $\nu \in P^T_{\mu_0}$ *where* $\mu_0 \in P_{B_R}$ *is the initial measure and* T^t_ν *is the associated flow map described by (4.11). For every* $(x, v) \in \mathbb{R}^d \times B_R$ *and* $0 \leq t \leq T$ *there exists a constant* $K > 0$ *dependent only on* α *and* R *such that*

$$|T^t_\nu(x, v)| \leq (1 + |(x, v)|)e^{K(\alpha,R)t}.$$

Proof. As T^t_ν solves the characteristic system then Lemma 5 gives

$$\begin{aligned}|T_\nu^t(x,v)| &\le |(x,v)| + \int_0^t |\Phi_{\nu,\alpha}(\tau, T_\nu^\tau(x,v))| d\tau \\ &\le |(x,v)| + \int_0^t K(1 + |T_\nu^\tau(x,v)|) d\tau \\ &= |(x,v)| + Kt + K \int_0^t |T_\nu^\tau(x,v)| d\tau.\end{aligned}$$

By Gronwall's inequality (see Appendix) we obtain

$$|T_\nu^t(x,v)| \le (1 + |(x,v)|)e^{Kt} - 1 \le (1 + |(x,v)|)e^{Kt}.$$

□

Chapter 5

Measure Solutions to the Weak Kinetic Model

In the previous chapter we approximated the kinetic model (4.4) by a linear system (4.8). To relate the linear model to model (4.4) we will define a contractive operator on the space $P^T_{\mu_0}$ that, by a fixed point argument, will give a measure theoretic solution as defined below:

Let T, $\alpha > 0$. Given an initial datum $\mu_0 \in P_{B_R}$ we say that $\mu \in P^T_{\mu_0}$ is a measure solution to the kinetic model (1.4) provided that for every $\phi \in C_0^\infty([0,T] \times \mathbb{R}^{2d})$ the following holds:

$$\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\frac{\partial \phi}{\partial t}(t,x,v) + v \cdot \nabla_x \phi - F_{\mu,\alpha} \cdot \nabla_v \phi(t,x,v) \right) d\mu(t,x,v) = - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(0,x,v) d\mu_0(x,v) \tag{5.1}$$

where we have defined the force term in the measure-theoretic sense by

$$F_{\mu,\alpha}(t,x,v) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \nabla W(v-u) \eta_\alpha(x-y) d\mu(t,y,u). \tag{5.2}$$

That is, we do not perturbate the force by an arbitrary measure $\nu \in P^T_{\mu_0}$. This is where the fixed point argument will be used, to show there is a measure $\nu \in P^T_{\mu_0}$ of (4.8) that coincides with model (5.1) to yield a weak solution. We define the operator below with the assumption that the initial measure not only has compact support in velocity, but also in space. This assumption will be made clear in the upcoming arguments presented.

5.1 Properties of the Pushforward Flow Map

Definition 13 (Pushforward Flow Map)**.** *Let $\nu \in P^T_{\mu_0}$ where $\mu_0 \in P_{B_R}$ is the initial measure. Assume that μ_0 has compact support. Furthermore, let T^t_ν be the associated flow map described by (4.11). We define the pushforward flow map $\Psi : P^T_{\mu_0} \to P^T_{\mu_0}$ by*

$$\Psi(\nu)(t,x,v) = T^t_\nu \# \mu_0(x,v). \tag{5.3}$$

In other words, for every fixed $0 \le t \le T$ and $\phi \in C^\infty_0(\mathbb{R}^d \times \mathbb{R}^d)$ the map Ψ satisfies

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi(x,v) d\Psi(\nu)(t,x,v) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi\left(T^t_\nu(x,v)\right) d\mu_0(x,v). \tag{5.4}$$

It will often be implied that the initial measure is given by μ_0 in context. If another initial measure is considered we adopt the notation

$$\Psi(\nu|\mu_0)(t,x,v) = T^t_\nu \# \mu_0.$$

It will be shown later that the pushforward flow map is a well defined mapping. The motivation for considering this operator is shown by the following result.

Theorem 13 (Solutions of the Weak Measure Kinetic Model as Fixed Points)**.** *Suppose there exists a $\mu \in P^T_{\mu_0}$ (where $\mu_0 \in P_{B_R}$ is the initial measure) such that $\Psi(\mu) = \mu$. Then μ is a solution of (5.1).*

Proof. Indeed, let $\phi \in C^\infty_0([0,T) \times \mathbb{R}^{2d})$ and assume that μ is a fixed point of Ψ. Computation gives

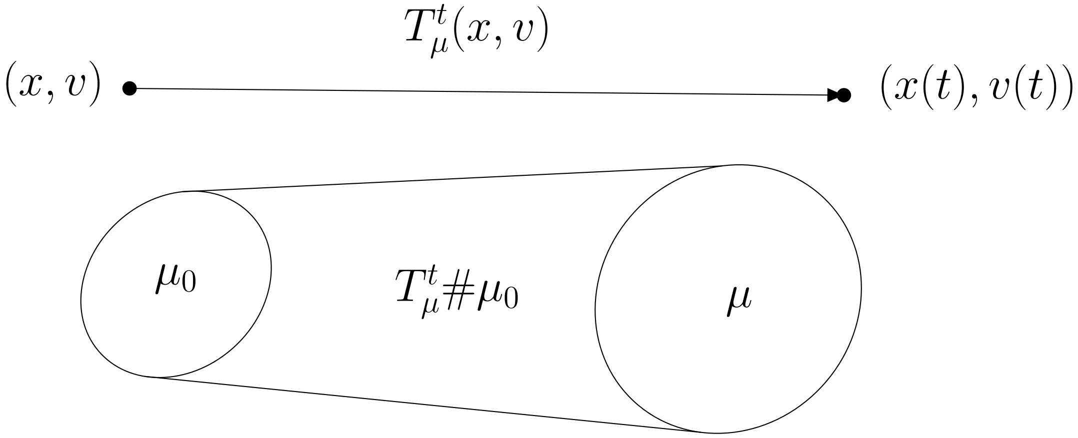
$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi(t,x,v) d\mu(t,x,v) \\
&= \frac{d}{dt}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi(t,x,v) d(T^t_\mu \# \mu_0)(x,v) \\
&\overset{\text{def}}{=} \frac{d}{dt}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi(t, T^t_\mu(x,v)) d\mu_0(x,v) \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\partial \phi}{\partial t}(t, T^t_\mu(x,v)) d\mu_0(x,v) + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \nabla_{(x,v)}\phi(t, T^t_\mu(x,v)) \frac{\partial T^t_\mu}{\partial t}(x,v) d\mu_0(x,v) \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\partial \phi}{\partial t}(t, T^t_\mu(x,v)) d\mu_0(x,v) + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \nabla_x \phi(t, T^t_\mu(x,v)) \cdot v \ \ d\mu_0(x,v) \\
&\qquad\qquad - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \nabla_v \phi(t, T^t_\mu(x,v)) \cdot F_{\mu,\alpha}(t,x,v) d\mu_0(x,v) \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\partial \phi}{\partial t}(t,x,v) d\mu(t,x,v) + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} v \cdot \nabla_x \phi(t,x,v) d\mu(t,x,v) - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F_{\mu,\alpha} \cdot \nabla_v \phi(t,x,v) d\mu(t,x,v) \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left(\frac{\partial \phi}{\partial t}(t,x,v) + v \cdot \nabla \phi(t,x,v) - F_{\mu,\alpha} \cdot \nabla_v \phi(t,x,v) \right) d\mu(t,x,v).
\end{aligned}$$

Integrating over $0 \le t \le T$ and using the fact that ϕ has compact support, we obtain

$$\int_0^T \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left(\frac{\partial \phi}{\partial t}(t,x,v) + v\cdot\nabla_x\phi - F_{\mu,\alpha}\cdot\nabla_v\phi(t,x,v)\right) d\mu(t,x,v) = -\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\phi(0,x,v)d\mu_0(x,v)$$

as desired. □

That is, the previous theorem coincides with the following intuition. As the flow map T^t_μ transports a point of initial data along integral curves, it is akin in nature that the pushforward flow map $\Psi(\mu) = T^t_\mu\#\mu_0$ will transport an initial measure to the desired weak measure solution.



Proposition 19 (Conservation of Mass)**.** *Let Ψ be the pushforward flow map described by (5.3). For any $\nu\in P^T_{\mu_0}$ and fixed time $0\le t\le T$ (where $\mu_0\in P_{B_R}$ is the initial measure) we have $\Psi(\nu)(t,\cdot)\in P(\mathbb{R}^d\times\mathbb{R}^d)$ (i.e. the image is a probability measure in space and velocity).*

Proof. The proof of this statement follows from Definition (5.4). Choosing a bounded function $\phi(x,v)\in C_b(\mathbb{R}^d\times\mathbb{R}^d)$ by $\phi(x,v)=1$ gives

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} d\Psi(\nu)(t,x,v) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} d\left(T^t_\nu\#\mu_0\right)(x,v)\\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} d\mu_0.\end{aligned}$$

As $\mu_0\in P(\mathbb{R}^d\times\mathbb{R}^d)$ we infer that $\Psi(\nu)(t,\cdot)\in P(\mathbb{R}^d\times\mathbb{R}^d)$ as desired. □

Proposition 20 (Image of the Pushforward Flow Map is Weakly Continuous in Time)**.** *Let Ψ be the pushforward flow map described by (5.3). For any $\nu\in P^T_{\mu_0}$ (where $\mu_0\in P_{B_R}$ is the initial measure) we have $\Psi(\nu)\in C_w([0,T];P(\mathbb{R}^d\times\mathbb{R}^d))$ (i.e. the image in weakly continuous in time).*

Proof. Let $\phi\in C_b(\mathbb{R}^d\times\mathbb{R}^d)$ and $t,\ t'\in[0,T]$. Then Definition (5.4) yields

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} d\Psi(\nu)(t,x,v) - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi(x,v)d\Psi(\nu)(t',x,v)\right| \\
&= \left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi(x,v)d\left(T_\nu^t\#\mu_0\right)(x,v) - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi(x,v)d\left(T_\nu^{t'}\#\mu_0\right)(x,v)\right| \\
&= \left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi(T_\nu^t(x,v))d\mu_0(x,v) - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi(T_\nu^{t'}(x,v))d\mu_0(x,v)\right| \\
&= \left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (\phi(T_\nu^t(x,v)) - \phi(T_\nu^{t'}(x,v)))d\mu_0(x,v)\right| \\
&\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left|\phi(T_\nu^t(x,v)) - \phi(T_\nu^{t'}(x,v))\right| d\mu_0(x,v).
\end{aligned}$$

Note that by Proposition 16 the flow map is continuous for fixed time $0 \leq t \leq T$ (i.e. $T_\nu^t \in C(\mathbb{R}^d \times \mathbb{R}^d)$). Furthermore, as we assumed that $\phi \in C_b(\mathbb{R}^d \times \mathbb{R}^d)$ then the composition $\phi \circ T_\nu^t$ is continuous in space and velocity. It follows that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $(x,v) \in \mathbb{R}^d \times B_R$ we have

$$|t-t'| = |(t,x,v)-(t',x,v)| < \delta \qquad \text{implies} \qquad \left|\phi(T_\nu^t(x,v)) - \phi(T_\nu^{t'}(x,v))\right| < \varepsilon.$$

Since $\mu_0 \in P(\mathbb{R}^d \times \mathbb{R}^d)$ this gives

$$\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} d\Psi(\nu)(t,x,v) - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi(x,v)d\Psi(\nu)(t',x,v)\right| < \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \varepsilon d\mu_0(x,v) = \varepsilon.$$

This implies the pushforward flow map Ψ is weakly continuous in time. □

Lemma 9. *Let Ψ be the pushforward flow map described by (5.3). For any $\nu \in P_{\mu_0}^T$ and $0 \leq t \leq T$, the initial measure $\mu_0 \in P_{B_R}$ has the differential form presentation*

$$d\mu_0(x,v) = \left|\det\left(\nabla_{(x,v)}T_\nu^t(x,v)\right)\right| d\Psi(\nu)(t,T_\nu^t(x,v)),$$

where $(x,v) \in \mathbb{R}^d \times B_R$ and T_ν^t is the flow map described by (4.11).

Proof. By Definition (5.4) we have for every $\phi \in C_b(\mathbb{R}^d \times \mathbb{R}^d)$ that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi(x,v)d\Psi(\nu)(t,x,v) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi(x,v)d(T_\nu^t\#\mu_0)(x,v) \overset{\text{def}}{=} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi(T_\nu^t(x,v))d\mu_0(x,v).$$

By a change of variables we find the above is equivalent to

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi(T_\nu^t(x,v))|\det\left(\nabla_{(x,v)}T_\nu^t(x,v)\right)|d\Psi(\nu)(t,T_\nu^t(x,v)) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi(T_\nu^t(x,v))\mu_0(x,v).$$

As this holds for every function $\phi \in C_b(\mathbb{R}^d \times \mathbb{R}^d)$ we conclude that

$$d\mu_0(x,v) = \left|\det\left(\nabla_{(x,v)}T_\nu^t(x,v)\right)\right| d\Psi(\nu)(t,T_\nu^t(x,v))$$

holds for every $(x,v) \in \mathbb{R}^d \times B_R$. □

Proposition 21 (Bounded Velocity Support of the Flow Operator)**.** *Let $\nu \in P^T_{\mu_0}$ (where $\mu_0 \in P_{B_R}$ is the initial measure) and assume μ_0 has compact support given by*

$$\textit{supp } \mu_0 \subset B_S \times B_R$$

for some $S > 0$. Then, the pushforward flow operator Ψ described by (5.3) satisfies

$$\textit{supp } \Psi(\nu)(t,\cdot) \subset B_{S+tR} \times B_R$$

for every $0 \le t \le T$.

Proof. By Lemma 9 we know that for every $0 \le t \le T$ and $(x,v) \in B_S \times B_R$ the following holds:

$$d\mu_0(x,v) = \left|\det\left(\nabla_{(x,v)} T^t_\nu(x,v)\right)\right| d\Psi(\nu)(t, T^t_\nu(x,v)).$$

By Proposition 18 the flow is a class C^1 diffeomorphism in fixed time, hence its derivative is non-singular. Accordingly, this implies

$$d\Psi(\nu)(t, T^t_g(x,v)) \neq 0 \qquad \Longleftrightarrow \qquad d\mu_0(x,v) \neq 0.$$

.

It follows that

$$T^t_\nu(x,v) \in \{(y,u) : d\Psi(\nu)(t,y,u) \neq 0\} \qquad \Longleftrightarrow \qquad (x,v) \in \{(x,v) : d\mu_0(x,v) \neq 0\}$$

holds for $(x,v) \in B_S \times B_R$. This further implies

$$(x,v) \in (T^t_\nu)^{-1}\left(\{(y,u) : d\Psi(t,y,u) \neq 0\}\right) \qquad \Longleftrightarrow \qquad (x,v) \in \{(x,v) : d\mu_0(x,v) \neq 0\}.$$

Thus,

$$(T^t_\nu)^{-1}\left(\{(y,u) : \Psi(\nu)(t,y,u) \neq 0\}\right) \subset \{(x,v) : d\mu_0(x,v) \neq 0\}$$

and accordingly

$$\{(y,u) : d\Psi(\nu)(t,y,u) \neq 0\} \subset T^t_\nu\left(\{(x,v) : d\mu_0(x,v) \neq 0\}\right).$$

Taking the closure gives

$$\text{supp } \Psi(\nu)(t,\cdot) = \overline{\{(y,u) : d\Psi(\nu)(t,y,u) \neq 0\}} \subset \overline{T^t_\nu(\{(x,v) : d\mu_0(x,v) \neq 0\})}.$$

Lastly, since $\{(x,v) : d\mu_0(x,v) \neq 0\} \subset \text{supp } \mu_0$ then

$$\begin{aligned}\text{supp } \Psi(\nu)(t\cdot) &\subset \overline{T^t_\nu(\text{supp } \nu_0)}\\ &\subset T^t_\nu(B_S \times B_R)\\ &\subset B_{S+tR} \times B_R\end{aligned}$$

using Lemma 7. ☐

Corollary 2. *The pushforward flow map described by (5.3) is well defined (i.e. the image of any measure under the pushforward flow map lies in $P^T_{\mu_0}$ where $\mu_0 \in P_{B_R}$ is the initial measure).*

Proof. To verify that the flow map is well defined it needs to satisfy the following properties:

- The image of any measure in $P^T_{\mu_0}$ under the pushforward flow map is weakly continuous in time.
- The image of any measure in $P^T_{\mu_0}$ under the pushforward flow map is a probability measure.
- For fixed time the image of any measure in $P^T_{\mu_0}$ under the pushforward map has compact velocity support in the ball B_R.

Each point has been proven respectively by Propositions 20, 19 and 21. ☐

5.2 Wasserstein Bounds of the Pushforward Flow Map

Lemma 10. *Let ν, $\mu \in P^T_{\mu_0}$ (where $\mu_0 \in P_{B_R}$ is the initial measure) and consider the pushforward flow map Ψ described by (5.3). Furthermore, assume that the initial measure has compact support given by supp $\mu_0 \subset B_S \times B_R$ for some $S > 0$. Then for any $0 \le t \le T$ it follows that*

$$W_1(\Psi(\nu)(t,\cdot), \Psi(\mu)(t,\cdot)) \le \|T^t_\nu - T^t_\mu\|_{L^\infty(supp\ \mu_0)}.$$

Proof. Consider the transference plan $\pi \in \Pi(\Psi(\nu)(t,\cdot), \Psi(\mu)(t,\cdot))$ defined by $\pi := (T^t_\nu \times T^t_\mu)\#\mu_0$. First, we will verify that this selection is appropriate as a transference plan (i.e. has marginals $\Psi(\nu)(t,\cdot) = T^t_\nu\#\mu_0$ and $\Psi(\mu)(t,\cdot) = T^t_\mu\#\mu_0$). By definition, we have for every $\phi \in C_b(\mathbb{R}^{2d} \times \mathbb{R}^{2d})$ that

$$\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}} \phi(x,v,x',v')d\pi(x,v,x',v') \overset{\text{def}}{=} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi(T^t_\nu(x,v), T^t_\mu(x,v))d\mu_0(x,v).$$

Letting $\phi(x,v,x',v') = \phi_1(x,v)$ (a bounded function dependent only on the first component) we obtain

$$\begin{aligned}\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}} \phi_1(x,v)d\pi(x,v,x',v') &\overset{\text{def}}{=} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi_1(T^t_\nu(x,v))d\mu_0(x,v)\\ &\overset{\text{def}}{=} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi_1(x,v)d(T^t_\nu\#\mu_0)(x,v)\\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi_1(x,v)d(\Psi(\nu))(t,x,v).\end{aligned}$$

This implies π has first marginal $T^t_\nu\#\mu_0$.

In the same manner, setting $\phi(x,v,x',v') = \phi_2(x',v')$ (a bounded function dependent only on the second variable) shows that π has second marginal $\Psi(\mu)(t\cdot) = T^t_\mu\#\mu_0$. As the Wasserstein metric is the infimum over all transference plans in $\Pi(\Psi(\nu)(t,\cdot),\Psi(\mu)(t,\cdot))$, then π gives the following upper bound:

$$\begin{aligned}W_1(\Psi(\nu)(t,\cdot),\Psi(\mu)(t,\cdot)) &= \inf_{\kappa\in\Pi(\Psi(\nu)(t,\cdot),\Psi(\mu)(t,\cdot))}\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}} |(x,v)-(x',v')|d\kappa(x,v,x',v')\\ &\leq \int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}} |(x,v)-(x',v')|d\pi(x,v,x',v')\\ &\overset{\text{def}}{=} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left|T^t_\nu(x,v)-T^t_\nu(x,v)\right| d\mu_0(x,v).\end{aligned}$$

It is now clear that we must show $|T^t_\nu(x,v)-T^t_\mu(x,v)|$ is finite. Let $(x,v)\in \operatorname{supp}\mu_0 \subset B_S\times B_R$, then $T^t_\nu(x,v) = (x(t),v(t))$ where $x(t) = x+tv(t)$ and $|v(t)|\leq R$. Thus the inequality

$$|x(t)| \leq S + tR$$

implies

$$|T^t_\nu(x,v)| = |(x(t),v(t)| \leq S+(t+1)R.$$

Using this result, for every $\nu\in P^T_{\mu_0}$ and fixed $0\leq t\leq T$ the above quantity is bounded.

Now, as $|T^t_\nu(x,v)-T^t_\mu(x,v)| = |T^t_\nu(x,v)|+|T^t_\mu(x,v)|$ this implies the quantity is finite. In conjuction with $\mu_0\in P(\mathbb{R}^d\times\mathbb{R}^d)$ we obtain

$$W_1(\Psi(\nu)(t,\cdot),\Psi(\mu)(t,\cdot)) \leq \|T^t_\nu - T^t_\mu\|_{L^\infty(\operatorname{supp}\mu_0)}.$$

as desired.

□

Proposition 22. *Let ν, $\mu \in P_{\mu_0}^T$ where $\mu_0 \in P_{B_R}$ is the initial measure. Furthermore assume that μ_0 has compact support. Then for any $0 \leq t \leq T$ the pushforward flow map Ψ described by (5.3) satisfies*

$$W_1(\Psi_\nu(t,\cdot), \Psi_\mu(t,\cdot)) \leq N e^{L_\Phi t} \int_0^t W_1(\mu(\tau,\cdot), \nu(\tau,\cdot)) e^{-L_\Phi \tau} d\tau,$$

where $N = N(\alpha, R) > 0$ is given by Lemma 6 and L_Φ is the Lipschitz bound given by Lemma 12.

Proof. Let B_V be the smallest disc containing $\operatorname{supp} \mu_0$, then the following inequalities hold by the previous Lemmas

$$\begin{aligned} W_1(\Psi(\nu)(t,\cdot), \Psi(\mu)(t,\cdot)) &\leq \|T_\nu^t - T_\mu^t\|_{L^\infty(\operatorname{supp}\mu_0)} \\ &\leq \|T_\nu^t - T_\mu^t\|_{L^\infty(B_V)}. \end{aligned}$$

As $T_\nu^t(x,v) - T_\mu^t(x,v)$ is a continuous map over the compact space B_V (for $(x,v) \in B_V$ and fixed time) then a maximal point is obtained. That is, there exists an $(x,v) \in \mathbb{R}^d \times B_R$ such that

$$\|T_\mu^t - T_\nu^t\|_{L^\infty(B_V)} = |T_\nu^t(x,v) - T_\mu^t(x,v)|.$$

Now, by Lemma 8 the integral curves are bounded by $W = (1+V)e^{Kt}$. Now, consider the ball B_W. As the characteristic map is Lipschitz we obtain

$$\begin{aligned} |T_\nu^t(x,v) - T_\mu^t(x,v)| &\leq \int_0^t |\Phi_{\nu,\alpha}(\tau, T_\nu^\tau(x,v)) - \Phi_{\mu,\alpha}(\tau, T_\mu^\tau(x,v))| d\tau \\ &\leq \int_0^t |\Phi_{\nu,\alpha}(\tau, T_\nu^\tau(x,v)) - \Phi_{\nu,\alpha}(\tau, T_\mu^\tau(x,v))| d\tau \\ &\quad + \int_0^t |\Phi_{\nu,\alpha}(\tau, T_\mu^\tau(x,v)) - \Phi_{\mu,\alpha}(\tau, T_\mu^\tau(x,v))| d\tau \\ &\leq \int_0^t L_\Phi |T_\nu^\tau(x,v) - T_\mu^\tau(x,v)| d\tau \\ &\quad + \int_0^t \|\Phi_{\nu,\alpha}(\tau,\cdot) - \Phi_{\mu,\alpha}(\tau,\cdot)\|_{L^\infty(B_W)} d\tau. \end{aligned}$$

This gives the following inequality:

$$\|T_\nu^t - T_\mu^t\|_{L^\infty(B_V)} \leq \int_0^t L_\Phi \|T_\nu^t - T_\mu^t\|_{L^\infty(B_V)} d\tau + \int_0^t \|\Phi_{\nu,\alpha}(\tau,\cdot) - \Phi_{\mu,\alpha}(\tau,\cdot)\|_{L^\infty(B_W)} d\tau.$$

Using Gronwall's inequality we obtain

$$\|T^t_\nu - T^t_\mu\|_{L^\infty(B_V)} \leq e^{L_\Phi t} \int_0^t \|\Phi_{\nu,\alpha}(\tau,\cdot) - \Phi_{\mu,\alpha}(\tau,\cdot)\|_{L^\infty(B_W)} e^{-L_\Phi \tau} d\tau.$$

Using Lemma 6 we have

$$|\Phi_{\nu,\alpha}(t,\cdot) - \Phi_{\mu,\alpha}(t,\cdot)| \leq N W_1(\nu(t,\cdot), \mu(t,\cdot)).$$

Substituting this into the above gives the result. □

Lemma 11. *Let* $\nu \in C_w([0,T]; P(\mathbb{R}^d \times \mathbb{R}^d))$ *with* $\nu(t,\cdot) \in P_{B_R}$ *for all* $0 \leq t \leq T$ *(i.e. has bounded velocity support). Furthermore, let* μ_0*,* $\nu_0 \in P(\mathbb{R}^d \times \mathbb{R}^d)$ *have compact support. Then the pushforward flow map* Ψ *described by (5.3) satisfies*

$$W_1(\Psi(\nu|\mu_0)(t,\cdot), \Psi(\nu|\nu_0)(t,\cdot)) \leq L_{T,t} W_1(\mu_0, \nu_0)$$

for all $0 \leq t \leq T$*, where* $L_{T,t}$ *is the Lipschitz constant described by Lemma 15.*

Proof. Let π be the optimal transport plan between μ_0 and ν_0 and define the measure $\pi' = (T^t_\nu \times T^t_\nu)\#\pi$. We now show it is true that $\pi' \in \Pi(\Psi(\nu|\mu_0)(t,\cdot), \Psi(\nu|\nu_0)(t,\cdot))$.

Indeed, we have for every $\phi \in C_b(\mathbb{R}^{2d} \times \mathbb{R}^{2d})$ that

$$\int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} \phi(x,v,x',v') d\pi'(x,v,x',v') = \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} \phi(T^t_\nu(x,v), T^t_\nu(x',v')) d\pi(x,v,x',v').$$

Then by setting $\phi(x,v,x',v') = \phi_1(x,v)$ (a function of the first component alone) gives

$$\begin{aligned}
\int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} \phi(T^t_\nu(x,v), T^t_\nu(x',v')) d\pi(x,v,x',v') &= \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} \phi_1(T^t_\nu(x,v)) d\pi(x,v,x',v') \\
&= \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} \phi_1(T^t_\nu(x,v)) d\mu_0(x,v) \\
&\overset{\text{def}}{=} \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} \phi_1(x,v) d(T^t_\nu \# \mu_0)(x,v) \\
&= \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} \phi_1(x,v) d(\Psi(\nu|\mu_0))(t,x,v).
\end{aligned}$$

Thus π' has first marginal $\Psi(\nu|\mu_0)(t,\cdot) = T^t_\nu \# \mu_0$. Similarly, we set $\phi(x,v,x',v') = \phi_2(x',v')$ (a function of the second component alone) to find that π' has second marginal $\Psi(\nu|\nu_0)(t,\cdot) = T^t_\nu \# \nu_0$. Now, we use the fact that ν_0 and μ_0 have compact support. Let $B_{R_x} \times B_{R_v}$ be the smallest product of balls containing the support of both μ_0 and ν_0. We obtain

$$\begin{aligned} W_1(\Psi(\nu|\mu_0)(t,\cdot),\Psi(\nu|\nu_0)(t,\cdot)) &= \inf_{\kappa\in\Pi(\Psi(\nu|\mu_0)(t,\cdot),\Psi(\nu|\nu_0)(t,\cdot))}\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}}|(x,v)-(x',v')|d\kappa(x,v,x',v') \\ &\leq \int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}}|(x,v)-(x',v')|d\pi'(x,v,x',v'). \end{aligned}$$

Notice that $\Psi(\nu|\mu_0)(t,\cdot)$ and $\Psi(\nu|\nu_0)(t,\cdot)$ have support lying in $(B_{R_x}+tB_{R_v})\times B_{R_v}$. Accordingly, given $\pi'=(T_\nu^t\times T_\nu^t)\#\pi$ where π is the optimal transportation plan between μ_0 and ν_0, the function

$$\psi(x,v,x',v')=|(x,v)-(x',v')|$$

must be bounded in the integrand above (as its support is absorbed into that of π').
Using the definition of the pushforward gives

$$\begin{aligned} \int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}}|(x,v)-(x',v')|d\pi'(x,v,x',v') &= \int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}}|T_\nu^t(x,v)-T_\nu^t(x',v')|d\pi(x,v,x',v') \\ &\leq L_{T,t}\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}}|(x,v)-(x',v')|d\pi(x,v,x',v') \\ &= L_{T,t}W_1(\mu_0,\nu_0), \end{aligned}$$

where we have used the Lipschitz condition of the flow map and the fact that π is the optimal plan between μ_0 and ν_0. □

5.3 Well-Posedness of the Weak Kinetic Model

Theorem 14 (Fixed Point Argument on the Pushforward Flow Map)**.** *The pushforward flow map Ψ described by (5.3) has a unique fixed point. In other words, there exists a $\mu\in P_{\mu_0}^T$ (where $\mu_0\in P_{B_R}$ is the initial measure assumed to have compact support) such that $\Psi(\mu)=\mu$.*

Proof. By Corollary 2 we know that the pushforward flow map Ψ on $(W_1^\beta, P_{\mu_0}^T)$ is well defined. We will conclude that the map is contractive for a suitable choice of β.
Let ν, $\mu\in P_{\mu_0}^T$. By Lemma 11 we see that

$$
\begin{aligned}
W_1^{\beta}(\Psi(\nu),\Psi(\mu)) &\overset{\text{def}}{=} \sup_{0\le t\le T} W_1(\Psi(\nu)(t,\cdot),\Psi(\mu)(t,\cdot))e^{-\beta t}\\
&\le \sup_{0\le t\le T} Ne^{(L_\Phi-\beta)t}\int_0^t W_1(\nu(\tau,\cdot),\mu(\tau,\cdot))e^{-L_\Phi\tau}d\tau\\
&\le \sup_{0\le t\le T} Ne^{(L_\Phi-\beta)t}\int_0^t W_1^{\beta}(\nu,\mu)e^{(\beta-L_\Phi)\tau}d\tau\\
&= W_1^{\beta}(\nu,\mu)\sup_{0\le t\le T} Ne^{(L_\Phi-\beta)t}\frac{1}{\beta-L_\Phi}\left(e^{(\beta-L_\Phi)t}-1\right)\\
&= W_1^{\beta}(\nu,\mu)\sup_{0\le t\le T}\frac{N}{\beta-L_\Phi}\left(1-e^{(L_\Phi-\beta)t}\right)\\
&\le W_1^{\beta}(\nu,\mu)\frac{N}{\beta-L_\Phi}.
\end{aligned}
$$

Hence the above is a contraction provided that the coefficient satisfies

$$\frac{N}{\beta-L_\Phi}<1.$$

By taking $\beta > N + L_\Phi$ we obtain that the map is contractive . As the map is contractive and the space $(W_1^{\beta}, P_{\mu_0}^T)$ is complete (Proposition 5) this guarantees us a unique fixed point by the Banach fixed point theorem.

□

Corollary 3 (Existence of a Unique Solution)**.** *The weak kinetic model described by (5.1) has a unique measure solution for every $T > 0$.*

Proof. This result is an immediate consequence of Theorems 11 and 13. □

Proposition 23 (Continuous Dependence of Weak Solutions on the Initial Data)**.** *Let $\mu \in P_{\mu_0}^T$, $\nu \in P_{\nu_0}^T$ be measure solutions of the kinetic model (5.1) with initial measures μ_0, $\nu_0 \in P_{B_R}$ respectively. Furthermore, assume that the initial measures have compact support. Then for every fixed $0 \le t \le T$ it holds that*

$$W_1(\mu(t,\cdot),\nu(t,\cdot)) \le e^{1+L_\Phi t}W_1(\mu_0,\nu_0).$$

where L_Φ is the Lipschitz constant given by (12).

Proof. As μ and ν are measure solutions we may represent them respectively as $\mu(t,\cdot) = \Psi(\mu|\mu_0)(t,\cdot) = T_\mu^t\#\mu_0$ and $\nu(t,\cdot) = \Psi(\nu|\nu_0) = T_\nu^t\#\nu_0$ (Proposition 13). By the preceeding Lemmas 22 and 11 we obtain

$$\begin{aligned} W_1(\mu(t,\cdot),\nu(t,\cdot)) &= W_1(\Psi(\mu|\mu_0)(t,\cdot),\Psi(\nu|\nu_0)(t,\cdot)) \\ &\leq W_1(\Psi(\mu|\mu_0)(t,\cdot),\Psi(\nu|\mu_0)(t,\cdot)) + W_1(\Psi(\nu|\mu_0)(t,\cdot),\Psi(\nu|\nu_0)(t,\cdot)) \\ &\leq Ne^{L_\Phi t}\int_0^t W_1(\mu(\tau,\cdot),\nu(\tau,\cdot))e^{-L_\Phi \tau}d\tau + L_{T,t}W_1(\mu_0,\nu_0). \end{aligned}$$

Using the fact that $L_{T,t} = e^{L_\Phi t}$ we may rewrite this as

$$e^{-L_\Phi t}W_1(\mu(t,\cdot),\nu(t,\cdot)) \leq N\int_0^t e^{-L_\Phi \tau}W_1(\mu(\tau,\cdot),\nu(\tau,\cdot))d\tau + W_1(\mu_0,\nu_0).$$

Thus by Gronwall's inequality we obtain

$$W_1(\mu(t,\cdot),\nu(t,\cdot)) \leq e^{1+L_\Phi t}W_1(\mu_0,\nu_0).$$

□

Theorem 15. *Let T, $\alpha > 0$. Given an initial datum $\mu_0 \in P_{B_R}$ with compact support and any $\phi \in C_0^\infty([0,T]\times\mathbb{R}^d\times\mathbb{R}^d)$ the model*

$$\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left(\frac{\partial\phi}{\partial t}(t,x,v) + v\cdot\nabla_x\phi - F_{\mu,\alpha}\cdot\nabla_v\phi(t,x,v)\right)d\mu(t,x,v) = -\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\phi(0,x,v)d\mu_0(x,v)$$

taken over $\mu \in P_{\mu_0}^T$ is well-posed. Above we have defined the force term in the measure-theoretic sense by

$$F_{\mu,\alpha}(t,x,v) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\nabla W(v-u)\eta_\alpha(x-y)d\mu(t,y,u).$$

Proof. This is an immediate consequence of Corollary 3 and Proposition 23. □

Chapter 6

Regularity

We will attempt to improve the regularity of measure solutions we obtained in the previous section. This is achieved by exploiting the property that any solution is representable as $\mu = T^t_\mu \# \mu_0$ (Proposition 13), thus it solves a Monge-Ampère equation as demonstrated by Lemma 9. First, we show that the measure is represented by a measurable function.

6.1 Measurable Solutions

Lemma 12. *Let $\mu \in P^T_{\mu_0}$ be a measure solution described by Theorem 15. If the initial measure μ_0 is absolutely continuous with respect to the Lebesgue measure then μ is absolutely continuous with respect to the Lebesgue measure.*

Proof. Consider the set

$$E_\mu = \{(x,v) \in \mathbb{R}^d \times \mathbb{R}^d : (\nabla (T^t_\mu)^{-1})(x,v) \text{ doesn't exist or } \nabla(T^t_\mu)((T^t_\mu)^{-1}(x,v)) \text{ doesn't exist } \}.$$

By the regularity of the flow map we have $E_\mu = \emptyset$ allowing us to construct the following calculations without much difficulty.
Taking $(x,v) \in (\mathbb{R}^d \times \mathbb{R}^d)$ and using the fact that T^t_μ is a diffeomorphism we obtain

$$(\nabla T^t_\mu((T^t_\mu)^{-1}(x,v))(\nabla (T^t_\mu)^{-1}(x,v)) = \nabla (T^t_\mu \circ (T^t_\mu)^{-1}) = I,$$

implying that by taking the determinant both Jacobians are non-vanishing on $\mathbb{R}^d \times \mathbb{R}^d$. Let A be a Borel measureable set in $\mathbb{R}^d \times B_R$.
Using the assumption that μ_0 is absolutely continuous with respect to the Lebesgue measure, there exists an $f_0 \in L^1(\mathbb{R}^d \times \mathbb{R}^d)$ such that

$$\frac{d\mu_0}{d(x,v)} = f_0(x,v).$$

Substituting this into the pushforward condition $\mu = T^t_\mu \# \mu_0$ gives

$$\begin{aligned}\int\int_A d\mu &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \chi_A(T^t_\mu(x,v))d\mu_0(x,v)\\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \chi_A(T^t_\mu(x,v))f_0(x,v)dxdv\\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \chi_A(y,u)f_0((T^t_\mu)^{-1}(y,u))|\nabla(T^t_\mu)^{-1}(y,u)|dydu\\ &= \int\int_A f_0((T^t_\mu)^{-1}(y,u))|\nabla(T^t_\mu)^{-1}(y,u)|dydu\end{aligned}$$

and if the Jacobian $|\nabla(T^t_\mu)^{-1}(x,v)|$ is uniformly bounded on $\mathbb{R}^d \times B_R$ then the integral is finite. Moreover, if A is of Lebesgue measure zero then (by the absolute continuity of μ_0),

$$\|f_0\|_{L^1(A)} = \int\int_A f_0(x,v)dxdv = 0.$$

By Proposition 18, the map T^t_μ is a class C^1 diffeomorphism. This implies that the composition $f_0 \circ (T^t_\mu)^{-1}$ is a measurable function that vanishes when integrated over a set of measure zero.

Combining these two results would prove that μ is absolutely continuous with respect to the Lebesgue measure.
For every fixed $0 \le t \le T$ we know by that T^t_μ is a bijective class C^1-diffeomorphism and thus a Lipschitz homeomorphism. Accordingly, at every fixed point the Jacobian $|\nabla(T^t_\mu)^{-1}(x,v)|$ is equal to the product of its eigenvalues, to which each eigenvalue is bounded above by the Lipschitz constant of $(T^t_\mu)^{-1}$. Thus we obtain

$$|\nabla(T^t_\mu)^{-1}(x,v)| \le (\text{Lip }(T^t_\mu)^{-1})^d$$

for every $x \in \mathbb{R}^d$ and $v \in B_R$. □

Theorem 16. *Let T, $\alpha > 0$. Given an initial datum $f_0 \in PF_{B_R}$ with compact support and any $\phi \in C_0^\infty([0,T]\times\mathbb{R}^d\times\mathbb{R}^d)$ the model*

$$\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left(\frac{\partial\phi}{\partial t}(t,x,v) + v\cdot\nabla_x\phi - F_{f,\alpha}\cdot\nabla_v\phi(t,x,v)\right)f(t,x,v)dxdv = -\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\phi(0,x,v)f_0(x,v)dxdv$$

taken over $f \in PF^T_{\mu_0}$ is well-posed. Above we have defined the force term by

$$F_{f,\alpha}(t,x,v) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\nabla W(v-u)\eta_\alpha(x-y)f(t,y,u)dydu.$$

Proof. This is an immediate consequence of Lemma 12. By Theorem 15 there exists a measure solution $\mu \in P^T_{\mu_0}$ given an initial measure $\mu_0 \in P_{B_R}$ with compact support. By selecting this initial measure in such a way that it is absolutely continuous with respect to the Lebesgue measure and has Radon-Nikodym derivative given by

$$\frac{d\mu_0}{dx}(x, v) = f_0(x, v)$$

it follows that the measure solution μ is absolutely continuous with respect to the Lebesgue measure. Accordingly, it is associated with a density $f \in PF_{f_0}^{T}$ that solves the above model. All other previously established results still hold and thus the model is also well-posed. □

6.2 Sobolev Regularity

Proposition 24. *Assume that the interaction potential has regularity* $W \in C^k(\mathbb{R}^d)$ *for* $k \geq 2$ *and consider the flow map* T^t_μ *described by (4.11). For fixed time, the flow is a class* $W^{k-1,\infty}(\mathbb{R}^d \times B_R) \cap C^{k-1}(\mathbb{R}^d \times B_R)$ *diffeomorphism.*

Proof. By definition the flow map satisfies the following system of characteristics

$$\frac{dT^t_\mu}{dt}(x,v) = \Phi_{\mu,\alpha}(t, T^t_\mu(x,v)).$$

It follows that if the characteristic map $\Phi_{\mu,\alpha}(t,\cdot)$ is of class $C^{k-1}(\mathbb{R}^d \times B_R)$ then T^t_μ is of class $C^{k-1}(\mathbb{R}^d \times B_R)$ by Proposition 29. Since the characteristic map is given explicitly by

$$\Phi(t,x,v) = \left(v, -\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \nabla W(v-u)\eta_\alpha(x-y)d\mu(t,y,u)\right)$$

then $W \in C^k(\mathbb{R}^d)$ implies that $\nabla W \in C^{k-1}(\mathbb{R}^d;\mathbb{R}^d)$. To show that the derivatives are bounded we see that

$$\nabla T^t_\mu(x,v) = I + \int_0^t \nabla \Phi_{\mu,\alpha}(\tau, T^\tau_\mu(x,v))\nabla T^\tau_\mu(x,v)d\tau.$$

By taking the norm of both sides we obtain the inequality

$$\|\nabla T^t_\mu(x,v)\|_2 \leq \sqrt{d} + \int_0^t \|\nabla \Phi_{\mu,\alpha}(\tau, T^\tau_\mu(x,v))\|_2 \|\nabla T^\tau_\mu(x,v)\|_2 d\tau.$$

An application of Gronwall's inequality yields

$$\|\nabla T^t_\mu(x,v)\|_2 \leq \sqrt{d}\left(1 + \int_0^t \|\nabla \Phi_{\mu,\alpha}(\tau, T^\tau_\mu(x,v))\|_2 e^{\int_0^\tau \|\nabla \Phi_{\mu,\alpha}(\kappa, T^\kappa_\mu(x,v))\|_2 d\kappa} d\tau\right).$$

In the above expression for the characteristic map $\Phi_{\mu,\alpha}$, note that η_α and its derivatives are all bounded. Furthermore, the vector potential is evaluated over a compact space meaning the derivatives of the characteristic map are bounded. This implies $T^t_\mu \in W^{1,\infty}(\mathbb{R}^d \times B_R)$. Using a bootstrap argument on this iteration yields the desired result. The inverse map obtains the same properties as it satisfies the same characteristic system. □

Proposition 25. *Assume that the interaction potential has regularity* $W \in C^k(\mathbb{R}^d)$ *for* $k \geq 2$*. Also, for the initial datum* $f_0 \in PF_{B_R}$ *with compact support let* $f \in PF^T_{f_0}$ *be the corresponding solution to Theorem 16. It follows that if* $f_0 \in W^{k-2,\infty}(\mathbb{R}^d \times \mathbb{R}^d) \cap PF_{B_R}$ *and has compact support, then* $f \in W^{1,\infty}([0,T]; W^{k-2,\infty}(\mathbb{R}^d \times B_R)) \cap PF^T_{f_0}$*.*

Proof. By construction we have for every $\phi \in C^\infty_0(\mathbb{R}^d \times \mathbb{R}^d)$ and $0 \leq t \leq T$ the solution satisfies the pushforward property

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi(x,v)f(t,x,v)dxdv = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi(T^t_f(x,v))f_0(x,v)dxdv.$$

By substitution we obtain

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\phi(x,v)f(t,x,v)dxdv=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\phi(x,v)f_0(T_f^{-t}(x,v))|\det\nabla T_f^{-t}(x,v)|dxdv$$

and accordingly, we obtain the Monge-Ampère equation (almost everywhere)

$$f(t,x,v)=f_0(T_f^{-t}(x,v))|\det\nabla T_f^{-t}(x,v)|. \tag{6.1}$$

This is because the integral equality holds for every test function $\phi\in C_0^\infty(\mathbb{R}^d\times\mathbb{R}^d)$. By the Liouville-Ostrogradski formula the Jacobian of the flow map satisfies the equation

$$\begin{aligned}\det\nabla T_f^{-t}(x,v)&=\exp\left(\int_0^t \operatorname{div}_{(x,v)}\Phi_{f,\alpha}(\tau,T_f^{-\tau}(x,v))d\tau\right)\\&=\exp\left(-\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Delta W(\pi_v(T_f^{-\tau}(x,v))-u)\eta_\alpha(\pi_x(T_f^{-\tau}(x,v))-y)f(t,y,u)dydud\tau\right),\end{aligned}$$

where $\pi_x:\mathbb{R}^d\times B_R\to\mathbb{R}^d$ and $\pi_v:\mathbb{R}^d\times B_R\to B_R$ are the projection maps onto the space and velocity components respectively. By Proposition 24 the flow map is a class C^{k-1} diffeomorphism with bounded derivatives. Similarly, the mollifier, vector potential and projections are all of class C^{k-1} with bounded derivatives. Thus the Jacobian of the flow map is of class C^{k-2} as it is capped by the fact that ΔW is of class C^{k-2}. By the Monge-Ampère equation (6.1) all products and compositions displayed are class $W^{k-2,\infty}(\mathbb{R}^d\times B_R)\cap C^{k-2}(\mathbb{R}^d\times B_R)$ implying that $f(t,\cdot)$ is as well (almost everywhere). To verify that $f(\cdot,x,v)$ is (almost everywhere) continuously differentiable, it follows from the fact that the flow map and its Jacobian are continuously differentiable in time. □

6.2.1 Classical Solutions

Theorem 17. *Let T, $\alpha>0$ and $W\in C^k(B_R)$ for $k\geq 3$. Given an initial datum $f_0\in W^{k-2}(\mathbb{R}^d\times\mathbb{R}^d)\cap PF_{B_R}$ with compact support, the model*

$$\frac{\partial f}{\partial t}(t,x,v)+v\cdot\nabla_x f(t,x,v)=div_v(f(t,x,v)F_{f,\alpha}(t,x,v))$$

is well-posed with solution $f\in W^{1,\infty}([0,T];W^{k-2,\infty}(\mathbb{R}^d\times B_R))\cap PF_{f_0}^T$. Here the force term is given by

$$F_{f,\alpha}(t,x,v)=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\nabla W(v-u)\eta_\alpha(x-y)f(t,y,u)dydu.$$

Chapter 7

The Unmollified Model

7.1 Derivation of the Unmollified Kinetic Model

The unmollified model considers the case where interacting particles in the kinetic system are no longer spheres of radius α, but rather point-masses. This is done taking the limit where $\alpha \to 0^+$ in the mean-field limit argument previously done.

Formally, we may derive this model as follows. Considering the discrete-particle system

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = \varepsilon \sum_{j=1}^{N} \eta_\alpha(x_i - x_j) \nabla W(v_j - v_i) \\ (x_i(0), v_i(0)) = (x_i^0, v_i^0) \end{cases} \tag{7.1}$$

the previous mean-field limit argument derived the alternative expression (4.2)

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = -\varepsilon N (F_\alpha *_{x,v} \mu^N)(x_i(t), v_i(t)) \\ \mu^N(t=0, x, v) = \mu_0^N(x, v). \end{cases} \tag{7.2}$$

where we have denoted

$$F_\alpha(x, v) = \nabla W(v) \eta_\alpha(x)$$

and the described probability measures are given by

$$\mu^N(t, x, v) = \frac{1}{N} \sum_{j=1}^{N} \delta(x - x_j(t)) \otimes \delta(v - v_j(t))$$

$$\mu_0^N(x, v) = \frac{1}{N} \sum_{j=1}^{N} \delta(x - x_j^0) \otimes \delta(v - v_j^0).$$

We wish to explore the limit $\alpha \to 0^+$. Formally, abstaining from taking the limit $N \to \infty$ and writing

$$\begin{aligned}\lim_{\alpha\to 0^+} F_\alpha(x,v) *_{(x,v)} \mu^N(t,x,v) &= \int_{\mathbb{R}^d} \nabla W(v-u) \lim_{\alpha\to 0^+} \left(\int_{\mathbb{R}^d} \eta_\alpha(x-y) d\mu^N(t,y,u) \right) \\ &= \int_{\mathbb{R}^d} \nabla W(v-u) d\mu^N(t,x,u) \\ &= \nabla W(v) *_v \mu^N(t,x,v),\end{aligned}$$

we arrive at a limit system, in *weak* form

$$\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\frac{\partial \phi}{\partial t}(t,x,v) + v \cdot \nabla_x \phi - F_{\mu^N} \cdot \nabla_v \phi(t,x,v) \right) d\mu^N(t,x,v) = -\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(0,x,v) d\mu_0^N(x,v). \tag{7.3}$$

In the above, we have defined the force term in the measure-theoretic sense by

$$F_{\mu^N}(t,x,v) = \int_{\mathbb{R}^d} \nabla W(v-u) d\mu^N(t,x,u) \tag{7.4}$$

where x and t are fixed. Expanding the left hand side of the above gives

$$\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\frac{\partial \phi}{\partial t}(t,x,v) + v \cdot \nabla_x \phi - \nabla W(v-u) \cdot \nabla_v \phi(t,x,v) \right) d\mu^N(t,x,u) d\mu^N(t,x,v). \tag{7.5}$$

This is where an inconsistency becomes transparent. Recall that for the Dirac delta $\delta(x-\bar{x})$, the square $\delta(x-\bar{x})^2$ is undefined. Hence, the product

$$d\mu^N(t,x,u) d\mu^N(t,x,v) = \sum_{i=1}^N \sum_{j=1}^N d\delta(x-x_i) d\delta(x-x_j) d\delta(u-u_i) d\delta(v-v_j),$$

having the same spatial argument x in both factors is undefined. One way to arrive at a noble limiting equation is to let $N \to \infty$ *first*, then $\alpha \to 0^+$. Furthermore, assume that

$$d\mu^N(t,x,v) \rightharpoonup f(t,x,v) dx dv$$

with a sufficiently smooth f. Likewise, assume that

$$d\mu_0^N(x,v) \rightharpoonup f_0(x,v) dx dv$$

with appropriate smoothness assumptions.

Then clearly the term (7.5) is defined, and we get the unmollified equation

$$\begin{cases} \partial_t f(t,x,v) + v\cdot\nabla_x f(t,x,v) = \mathrm{div}_v(f(t,x,v)(\nabla W(v) *_v f(t,x,v))) \\ f(0,x,v) = f_0(x,v) \\ t \geq 0, x\in\mathbb{R}^d, v\in B_R \end{cases} \tag{7.6}$$

Here we assume that as $N\to\infty$ and $\varepsilon\to 0$ we have $\varepsilon N \to \lambda$; we choose $\lambda = 1$ for simplicity.

Remark. *The key difference between trying to relate the unmollified and mollified model is subtle. In the above we have taken the limiting case $\alpha\to 0^+$ and then derived the unmollified using the mean field limit. However, in our previous analysis we have abstained from taking this limit and have proven well-posedness to obtain a net of solutions $\{f_\alpha\}$ corresponding to each $\alpha > 0$. It is not clear that one may then find a limit to the net that will yield a solution to the unmollified model.*

7.2 Local Solutions to the Unmollified Model

Following the work of Benedetto et al. [1] and Agueh [3], the unmollified kinetic system has been proven to have a local solution. Unfortunately, the local solution is restrictive and depends drastically on its initial conditions.

Definition 14. *Let $f_0 \in L^1(\mathbb{R}^d\times B_R)$ be non-negative. We say that a function $f\in L^1([0,\infty)\times\mathbb{R}^d\times B_R)$ is a weak solution to the unmollified model (7.6) in the interval $[0,T_0)$ for some $T_0 > 0$, if for every test function $\phi\in C_0^\infty(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)$ with time support in $[-T,T]$ for $T\in(0,T_0)$, we have*

$$\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left(\partial_t\phi + v\cdot\nabla_x\phi - (\nabla W *_v f)\cdot\nabla_v\phi\right) f(t,x,v)dxdvdt = -\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f_0(x,v)\phi(0,x,v)dxdv$$

It is shown in [1], [3] that a local solution can be found if the initial daturm $f_0\in L^1\cap L^\infty(\mathbb{R}^d\times B_R)$; such a solutions is a measurable function satisfying $f(t,\cdot)\in L^1\cap L^\infty(\mathbb{R}^d\times B_R)$. Here there is the restriction $t\in[0,T^*]$, where $T^* < (C\|f_0\|_L^\infty)^{-1}$ and C is a constant depending on d and R.

7.3 Remarks on Attempted Convergence in the Birnbaum-Orlicz space

By the previous remark it has been made clear that even if we do obtain a limit of the net $\{f_\alpha\}$ as $\alpha\to 0^+$ it is not clear whether or not this will yield a solution to the unmollified model. Regardless, we proceed to discuss what has been attempted to find a limit $f_\alpha\to f$ as $\alpha\to 0^+$ in the hope that further results will clarify that f is a solution to the unmollified model. A notable attempt is one done on the Birnbaum-Orlicz space.

Definition 15. *The Birnbaum-Orlicz space, which we denote by* Γ, *is the space of functions*

$$\Gamma = \left\{ g \in L^1(\mathbb{R}^d \times \mathbb{R}^d) : g \text{ non-negative}, \ \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(x,v)\log^+(g(x,v))dxdv < \infty \right\}$$

This space satisfies the set inclusion

$$L^p(\Omega) \subset \Gamma \subset L^1(\Omega)$$

for all $p > 1$ where $\Omega \subset \mathbb{R}^d \times \mathbb{R}^d$ is a bounded region. Thus it serves as an intermediate space in which to do analysis on solutions to the mollified model. The issue present is that earlier we obtained an unsatisfactory bound in the L^p space, this being that solutions to the mollified model satisfied

$$\|f_\alpha(t,\cdot)\|_{L^p(\mathbb{R}^d\times\mathbb{R}^d)} \leq \exp\left(\frac{p-1}{p}\|\eta_\alpha\|_{L^\infty}\|\Delta W\|_{L^\infty(\overline{B}_{2R})}t\right)\|f_0\|_{L^p(\mathbb{R}^d\times\mathbb{R}^d)}$$

which does not remain bounded as we take the limit $\alpha \to 0^+$. Thus the Birnbaum-Orlicz space appears to be a reasonable space to attempt a uniform bound on the the net of solutions $\{f_\alpha\}$ for $\alpha > 0$.

By taking the case where the particle dynamics take place on a line, i.e. $d = 1$, we attempt to obtain sequential compactness on the set $\{f_\alpha\}$.
Agueh, Carlier and Illner [4] have shown that in the unmollified model that a uniform bound is obtainable provided that the potential W is of class C^2 and its second derivative satisfies some quadratic growth near the origin.

Lemma 13. *The collision term in the unmollified model have vanishing zeroeth and first moment in velocity. That is, if* $f(t,x,v)$ *is a solution to (7.6) where* $d = 1$ *then*

$$\int_{\mathbb{R}} \partial_v(f(t,x,v)(W'(v) *_v f(t,x,v)))dv = 0$$

$$\int_{\mathbb{R}} v\partial_v(f(t,x,v)(W'(v) *_v f(t,x,v)))dv = 0$$

It was then shown from this result that the following is obtained.

Theorem 18. *Assume that there exists a* $\delta > 0$ *and* $M > 0$ *such that the interaction kernel satisfies*

$$W''(v) \leq Mv^2$$

for every $v \in [-\delta, \delta]$. *Furthermore, if the initial condition satisfies* $f_0 \in \Gamma$ *then any solution to the unmollified kinetic model (7.6) satisfies*

$$f(t,\cdot) \in \Gamma \qquad \text{for every } t \in [0, T^*)$$

and

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(t,x,v)\ln f(t,x,v)dxdv \leq C$$

where C is a constant independent of t.

The process used in [4] to produce this result uses the foregoing lemma to conjure a Bony functional. The dependency on the previous lemma is crucial to establish this. However, we obtain the rather unsatisfactory result.

Theorem 19. *The collision term in the mollified model has vanishing zeroeth and nonvanishing first moment in velocity. That is, if $f_\alpha(t,x,v)$ is a solution to (1.4) where $d = 1$ then*

$$\int_{\mathbb{R}} \partial_v(f_\alpha(t,x,v)(W'(v) *_v f_\alpha(t,x,v)))dv = 0$$

$$\int_{\mathbb{R}} v\partial_v(f_\alpha(t,x,v)(W'(v) *_v f_\alpha(t,x,v)))dv \neq 0$$

Proof. Although it is true that for every solution f_α in the mollified case a simple computation yields

$$\int_{\mathbb{R}} \partial_v(f_\alpha(t,x,v)(\eta_\alpha(x)\partial_v W(v) *_{x,v} f_\alpha(t,x,v)))dv = 0$$

we find that for that the first moment

$$\int_{\mathbb{R}} v\, \partial_v(f_\alpha(t,x,v)(\eta_\alpha(x)\partial_v W(v) *_{x,v} f_\alpha(t,x,v)))dv \neq 0$$

does not hold as is seen by the following argument. Using integration by parts on a compactly support space we find that

$$\begin{aligned}\int_{\mathbb{R}} v\, \partial_v(f_\alpha(t,x,v)(\eta_\alpha(x)W'(v) *_{x,v} f_\alpha(t,x,v)))dv &= -\int_{\mathbb{R}} f_\alpha(t,x,v)(\eta_\alpha(x)W'(v) *_{x,v} f_\alpha(t,x,v))dv \\ &= -\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \eta_\alpha(x-y)W'(v-u)f_\alpha(t,x,v)f_\alpha(t,y,u)dvdydu \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \eta_\alpha(x-y)W'(v-u)f_\alpha(t,x,u)f_\alpha(t,y,v)dvdydu\end{aligned}$$

where we have interchanged variables and used the fact that $\partial_v W$ is odd. Adding the two previous lines means that the first moment is equivalent to

$$\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \eta_\alpha(x-y)W'(v-u)(f_\alpha(t,x,u)f_\alpha(t,y,v) - f_\alpha(t,x,v)f_\alpha(t,y,u))dvdydu$$

and this is non-vanishing due to the convolution in position. □

7.4 Two Points of Contrast Between the Mollified and Unmollified Model

Following the previous chapter many of the macroscopic properties obtained by Agueh [3] remain true, thus showing that the mollified kinetic model works as a good parallel to the unmollified kinetic model. To this point, two noteworthy differences between them have been established, these being the nature of how solutions act on transport curves and how the L^p-norms are more concrete in the mollified case.

7.4.1 Decreasing Nature Along Transport Curves Fails in the Mollified Case

Let $\alpha > 0$ and f_α the solution to the mollified model. Consider a convex function $\phi \in C^1(\mathbb{R}^d \times B_R)$ and define

$$I_\alpha : [0,T] \to \mathbb{R}$$

by

$$I_\alpha(t) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi(x - tv, v) f_\alpha(t,x,v)dxdv,$$

for all $t \in [0,T]$.

An interesting result obtained in the unmollified case is that the associated integral I is nonincreasing; that is, the solutions decrease along the free transport curves. However the mollified case loses this property. Assuming (for the moment) smoothness of the integrand we obtain

$$\dot{I}_\alpha(t) = -\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \nabla_{x-tv}\phi(x-tv,v)\cdot v f_\alpha(t,x,v)dxdv + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi(x-tv,v)\partial_t f_\alpha(t,x,v)dxdv$$

To which we will expand the second integral separately. We have by using that fact that f_α is a solution to our system that

$$\begin{aligned}
\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi(x-tv,v)\partial_t f_\alpha(t,x,v)dxdv &= -\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi(x-tv,v) v\cdot\nabla_x f_\alpha(t,x,v)dxdv \\
&\quad + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi(x-tv,v)\operatorname{div}_v(f_\alpha(t,x,v)F_{f_\alpha,\alpha}(t,x,v))dxdv \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \nabla_{x-tv}\phi(x-tv,v)\cdot v f_\alpha(t,x,v)dxdv \\
&\quad - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \nabla_v\phi(x-tv,v)\cdot\left(f_\alpha(t,x,v)F_{f_\alpha,\alpha}(t,x,v)\right)dxdv.
\end{aligned}$$

We substitute this into the above to obtain

$$
\begin{aligned}
\dot{I}(t) &= -\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \nabla_v \phi(x-tv,v)\cdot\left(f_\alpha(t,x,v)F_{f_\alpha,\alpha}(t,x,v)\right)dxdv \\
&= -\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left(\nabla_v\phi(x-tv,v)\cdot\nabla W(v-u)\right)\eta_\alpha(x-y)f_\alpha(t,y,u)f_\alpha(t,x,v)dxdvdydu \\
&= -\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left(\nabla_u\phi(x-tu,u)\cdot\nabla W(u-v)\right)\eta_\alpha(x-y)f_\alpha(t,y,v)f_\alpha(t,x,u)dxdvdydu \\
&= -\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left(\nabla_u\phi(y-tu,u)\cdot\nabla W(u-v)\right)\eta_\alpha(x-y)f_\alpha(t,x,v)f_\alpha(t,y,u)dxdvdydu
\end{aligned}
$$

Where we have used the fact that η_α is symmetric. Thus by using the fact that ∇W is odd we may add the equalities together to obtain

$$
\dot{I}(t) = -\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \nabla W(v-u)\cdot\left(\nabla_v\phi(x-tv,v)-\nabla_u\phi(y-tu,u)\right)\eta_\alpha(x-y)f_\alpha(t,y,v)f_\alpha(t,x,u)dxdvdydu.
$$

At this point we would, as before, use the radial representation of W to obtain

$$
\nabla W(v-u) = w'(\|v-u\|_2)\frac{v-u}{\|v-u\|_2}
$$

and as the terms $\eta_\alpha(x-y)f_\alpha(t,x,v)f_\alpha(t,y,u)$ and $w'(\|v-u|_2)$ are nonnegative we focus our interest to the term

$$
(v-u)\cdot\left(\nabla_v\phi(x-tv,v)-\nabla_u\phi(y-tu,u)\right)
$$

that appears in the integrand. This is almost the first order convexity condition. As a consequence of mollifying this is an example of difficulty that surfaces when attempting to establish invariant properties for the mollified equation. Thus we obtain the following result,

Proposition 26. *Let f be a solution, with initial datum f_0, of the unmollified system (1.5). Furthermore, let $\phi \in C^1(\mathbb{R}^d\times\mathbb{R}^d)$ be a convex function. Then for every fixed $t\in[0,T^*]$ and $x\in\mathbb{R}$ we have*

$$
\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\phi(x-tv,v)f(t,x,v)dxdv \le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\phi(x,v)f_0(x,v)dxdv.
$$

In contrast, this result no longer holds in the mollified case.

The proof of this proposition follows the same steps described above.

7.4.2 Dependency on Bounds of the L^p-norm

A noteworthy difference between the mollified and unmollified kinetic model is a bound on their L^p-norm. For the mollified case, we have for $t \geq 0$,

$$\|f_\alpha(t,\cdot)\|_{L^p(\mathbb{R}^d\times\mathbb{R}^d)} \leq \exp\left(\frac{p-1}{p}\|\eta_\alpha\|_{L^\infty}\|\Delta W\|_{L^\infty(\overline{B}_{2R})}t\right)\|f_{0,\alpha}\|_{L^p(\mathbb{R}^d\times\mathbb{R}^d)}.$$

As mentioned above, when $\alpha \to 0^+$, $\|\eta_\alpha\|_{L^\infty} \to \infty$ so that the estimate no longer holds for the unmollified equation. But in contrast, it is shown in [3] that provided the spatial density

$$\rho(t,x) = \int_{\mathbb{R}^d} f(t,x,v)dv$$

is bounded, a similar estimate can be obtained for the unmollified kinetic equation. More precisely, we have

Proposition 27. *Let $p \geq 2$ and f be a solution, with initial datum $f_0 \in L^p(\mathbb{R}^d \times \mathbb{R}^d)$, of the unmollified system (7.6). Assume that the spatial density*

$$\rho(t,x) = \int_{\mathbb{R}^d} f(t,x,v)dv$$

is uniformly bounded. Then for fixed $t \in [0,T^*]$ we have

$$||f(t,\cdot)||_{L^p(\mathbb{R}^d\times\mathbb{R}^d)} \leq \exp\left(\frac{p-1}{p}||\rho(t,x)||_{L^\infty([0,T^*]\times\mathbb{R}^d)}||\Delta W||_{L^\infty(B_{2R})}t\right)||f_0||_{L^p(\mathbb{R}^d\times\mathbb{R}^d)}.$$

As stated in [3], the lack of a uniform L^∞-bound of the spatial density $\rho(t,x)$ in $(0,T^*) \times \mathbb{R}^d$ is what prevents us from obtaining (in general) a global existence result for the unmollified kinetic equation.

Chapter 8

Conclusion

8.1 Concluding Remarks

The mollified kinetic model (1.4) exhibits all the desired properties of an inelastic system. The macroscopic quantities obtained in the fourth chapter show that the approximation by a spatial mollifier has altered the significant properties minimally. The model has better behaviour compared to its predecessor (the unmollified model (1.5)), as shown by the estimates in the fifth chapter.

The approximation by a mollifier allows a measure-theoretic approach that is impossible in the original model. This is demonstrated in the seventh chapter by a mean-field limit argument, which fails because the product of Dirac masses is undefined. In contrast, the mollified model exhibits the tensor product

$$d\mu^N(t,y,u)d\mu^N(t,x,v) = \sum_{i=1}^{N}\sum_{j=1}^{N} d\delta(y-y_i)d\delta(x-x_j)d\delta(u-u_i)d\delta(v-v_j),$$

which is well-defined. This allows us to formulate a measure-theoretic representation (5.1) and solve the mollified kinetic model using the method of characteristics. Using the β-weighted Wasserstein distance for $\beta > L_\Phi(\alpha,R) + N(\alpha,R)$, we are able to formulate a solution with a fixed point argument. That is, given $\mu_0 \in P_{B_R}$, find a measure $\mu \in P^T_{\mu_0}$ such that $\mu = T^t_\mu \# \mu_0$. This was shown to hold for every $T > 0$, but unfortunately not in the limit $T \to \infty$.

The third chapter demonstrates that solutions to $\mu = T^t_\mu \# \mu_0$ also satisfy a Monge-Ampère type equation. This guarantees enough regularity for classical solutions and confirms that the system is well-posed; although, we were required to restrict the interaction potential W to be at least C^3. This is more restrictive than the original assumption of being class C^2, but no significant (macroscopic) properties are lost. Specifically, assuming that the initial datum satisfies $f_0 \in W^{k-2}(\mathbb{R}^d \times \mathbb{R}^d) \cap PF_{B_R}$ (with compact support) and the interaction potential W is of class $C^k(\mathbb{R}^d)$, then $f \in W^{1,\infty}([0,T]; W^{k-2}(\mathbb{R}^d \times \mathbb{R}^d)) \cap PF^T_{f_0}$. The assumption that f_0 has compact support is unfortunate but necessary to guarantee solutions. However, solutions to the kinetic model naturally possess compact velocity support, so this assumption is not unreasonable.

Prior work by Agueh and Benedetto et al. [1, 2, 3, 4] have proven weak existence to the unmollified model (1.5) provided that the initial satisfies $f_0 \in L^1(\mathbb{R}^d \times \mathbb{R}^d) \cap L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$. The solutions to this model exist for time $0 < t < T^*$ where $T^* = (C\|f_0\|_{L^\infty})^{-1}$ for some constant $C > 0$. In contrast, the mollified model possesses solutions for all time $T > 0$ and dimension $d \geq 1$. These solutions are classical with reasonable assumptions on the initial datum and interaction potential.

8.2 Open Problems

1. By Corollary 3 the unmollified kinetic model (1.4) is well-posed for every $T > 0$. Unfortunately, this result has not been verified for $T \to \infty$. Assuming that the initial datum satisfies $f_0 \in C_0^1(B_S \times B_R)$ implies

   $$T_{f_0}^t \subset (B_S T + B_R) \times B_R.$$

   For the pushforward flow map Ψ to be well-defined it was mandatory that the spatial component of the flow remains bounded; this is given by Proposition 21. This means that we have a solution in the interval $t \in [0, T]$ for every selected $T > 0$ but not $t \in [0, \infty)$. It is hopeful that more concise global solutions are obtained in the future.

2. The attempts to relate the mollified model (1.4) to the unmollified model (1.5) by using the Birnbaum-Orlicz space have proven unsuccessful, due to not meeting the first order convexity condition requirements (chapter seven). Two problems remain unsolved when attempting to solve the unmollified model using the mollified model:

   - Show that the net $\{f_\alpha\}$ converges.
   - If $f_\alpha \to f$ as $\alpha \to 0^+$, show that f solves (1.5).

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Appendix A

Appendix

A.1 Complications In The Origin Of The Kinetic Model

As we have previously mentioned, the system of differential equations (1.1) that describe the evolution of colliding particles involves a conceptual error. Although a heuristic adjustment has been made to eliminate this issue there is still a mathematical gap of justifiable work in deriving this model.

The derivation of the model proposed in [1] is as follows. Assume we are observing a two-particle inelastic system in one-dimension with velocities v_1 and v_2. Such particles will flow freely until a moment of collision, in which they change velocities according to the rule

$$v_1^* = v_2 + \varepsilon(v_1 - v_2), \qquad v_2^* = v_1 - \varepsilon(v_1 - v_2) \tag{A.1}$$

where v_1^*, v_2^* are the post-collision velocities of particles with respective pre-collision velocities v_1, v_2 and $0 \leq \varepsilon \leq 1$ represents the degree of inelasticity. As both particles are normalized to have mass $1/2$ the post-collision momentum of the system is given by

$$\begin{aligned} p_{\text{post}} = \frac{1}{2}v_1^* + \frac{1}{2}v_2^* &= \frac{1}{2}\left(v_2 + \varepsilon(v_1 - v_2)\right) + \frac{1}{2}\left(v_1 - \varepsilon(v_1 - v_2)\right) \\ &= \frac{1}{2}v_1 + \frac{1}{2}v_2 \\ &= p_{\text{pre}} \end{aligned}$$

and equals the momentum of the system before collision, thus the model is shown to conserve momentum. However, the key defining property of inelastic systems is their loss in kinetic energy after collision. By another simple calculation, the kinetic energy of the system after collision is given by

$$\begin{aligned} E_{\text{post}} = \frac{1}{4}(v_1^*)^2 + \frac{1}{4}(v_2^*)^2 &= \frac{1}{4}(v_2 + \varepsilon(v_1 - v_2))^2 + \frac{1}{4}(v_1 - \varepsilon(v_1 - v_2))^2 \\ &= \frac{1}{4}\left(v_2^2 + 2v_2\varepsilon(v_1 - v_2) + \varepsilon^2(v_1 - v_2)^2 + v_1^2 - 2v_1\varepsilon(v_1 - v_2) + \varepsilon^2(v_1 - v_2)^2\right) \\ &= \frac{1}{4}\left(v_1^2 + v_2^2 - 2\varepsilon(v_1 - v_2)^2(1 - \varepsilon)\right) \\ &= E_{\text{pre}} - \frac{1}{2}\varepsilon(v_1 - v_2)^2(1 - \varepsilon) \\ &\leq E_{\text{pre}}, \end{aligned}$$

as $0 \leq \varepsilon \leq 1$. Thus the kinetic energy of the system behaves exactly as desired.

We then assume that both particles are indistinguishable; this step is justifiable. As both point masses will assume the same position at the moment of collision, we cannot properly infer what velocity each post-collision particle had before collision. This allows us to replace the velocity dynamics of (A.1) with the following

$$v_1^* = v_1 - \varepsilon(v_1 - v_2), \qquad v_2^* = v_2 + \varepsilon(v_1 - v_2). \tag{A.2}$$

At this point a leap of faith is made; where the model (1.1) is presented in the one-dimensional case,

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = \varepsilon \sum_{j=1}^{N} \delta(x_i - x_j)(v_j - v_i)\|v_i - v_j\|_2 \\ (x_i(0), v_i(0)) = (x_i^0, v_i^0). \end{cases} \tag{A.3}$$

It is not clear how this model coincides with this derivation. For instance, if we examine the case where $N = 2$ and assume that both particles are occupying the same position we obtain from (A.3)

$$\dot{v}_1(t) = -\varepsilon(v_1(t) - v_2(t))|v_1(t) - v_2(t)|, \qquad \dot{v}_2(t) = \varepsilon(v_1(t) - v_2(t))|v_1(t) - v_2(t)|.$$

Now assume that a single collision occurs within the time interval (t_0, t_1); upon integration we obtain

$$\begin{aligned} v_1^* &= v_1 - \varepsilon \int_{t_0}^{t_1} (v_1(\tau) - v_2(\tau))|v_1(\tau) - v_2(\tau)|d\tau \\ v_2^* &= v_2 + \varepsilon \int_{t_0}^{t_1} (v_1(\tau) - v_2(\tau))|v_1(\tau) - v_2(\tau)|d\tau \end{aligned}$$

which does not have any evident relation to the above dynamics of the velocity. Moreover, it is also dimensionally inconsistent prior to introducing a mollifier. Nonetheless, this is still a decent model for studying the dynamics of an inelastic system. That is, the following properties still hold in this model.

Proposition 28. *In the two-particle case the momentum of the system is conserved. Furthermore, the kinetic energy is decreasing.*

Proof. The conservation of momentum is seen by adding the two previous equations. To demonstrate the loss in energy consider the kinetic energy function

$$E_K(t) = \frac{1}{4}v_1(t)^2 + \frac{1}{4}v_2(t)^2.$$

By differentiating we obtain

$$\begin{aligned}
\dot{E}_K(t) &= \frac{1}{2}v_1(t)\dot{v}_1(t) + \frac{1}{2}v_2(t)\dot{v}_2(t) \\
&= \frac{1}{2}\left(v_1(t)\dot{v}_1(t) - v_2(t)\dot{v}_1(t)\right) \\
&= \frac{1}{2}\dot{v}_1(t)\left(v_1(t) - v_2(t)\right) \\
&= -\frac{1}{2}\varepsilon(v_1(t) - v_2(t))|v_1(t) - v_2(t)|\left(v_1(t) - v_2(t)\right) \\
&= -\frac{1}{2}\varepsilon(v_1(t) - v_2(t))^2|v_1(t) - v_2(t)| \\
&\le 0
\end{aligned}$$

as desired.

□

A.2 Common Theorems and Longer Proofs

In this section we expand upon some of the cluttering and well-known trivial results that interrupt the flow of this thesis.

Lemma 14. *Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be Lipschitz with constant L_f. Consider the autonomous differential system*

$$\begin{cases} \frac{dy}{dt} = f(y) \\ y(0) = y_0 \end{cases} \tag{A.4}$$

where $T_0 \le t \le T_1$ and $y_0 \in \mathbb{R}^d$. Furthermore, define a sequence $\{y_n\}$ in $C^1([0,T];\mathbb{R}^d)$ in such a way that it satisfies

$$\frac{dy_n}{dt} = \begin{cases} f(y_n\left(\frac{j}{n}\right)) & \text{if } \frac{j}{n} < t < \frac{j+1}{n} \text{ for } j \ge 0 \\ f(y_n\left(\frac{j+1}{n}\right)) & \text{if } \frac{j}{n} < t < \frac{j+1}{n} \text{ for } j < 0. \end{cases}$$

It follows that the sequence is Cauchy and its limit solves (A.4).

Proof. We begin by attempting to establish equicontinuity of the sequence. If we can establish such a result then we may enfore the Arzelà-Ascoli theorem to obtain a convergent subsequence. To obtain this it is sufficient to show that the sequence $y_n^j := y_n(j/n)$ is uniformly bounded for $|j/m| < M$ where $M > 0$ is a constant. Indeed, if $j > 0$ then we obtain

$$\begin{aligned}
|y_n^j - y_0| &\leq |y_n^j - y_n^{j-1}| + |y_n^{j-1} - y_0| \\
&= \frac{1}{n}|f(y_n^{j-1})| + |y_n^{j-1} - y_0| \\
&\leq \frac{1}{n}|f(y_0)| + \frac{1}{n}|f(y_n^{j-1}) - f(y_0)| + |y_n^{j-1} - y_0| \\
&\leq \frac{1}{n}|f(y_0)| + \left(\frac{L_f}{n} + 1\right)|y_n^{j-1} - y_0| \\
&\leq \frac{1}{n}|f(y_0)| \sum_{k=0}^{j-1} \left(\frac{L_f}{n} + 1\right)^k \\
&= \frac{|f(y_0)|}{L_f}\left(\left(\frac{L_f}{n} + 1\right)^j - 1\right) \\
&\leq \frac{|f(y_0)|}{L_f}\left(e^{\frac{L_f j}{n}} - 1\right) \\
&\leq \frac{|f(y_0)|}{L_f}\left(e^{L_f M} - 1\right)
\end{aligned}$$

and a similar argument holds for $j < 0$. From this we obtain that the sequence is equicontinuous. By applying the Arzelà-Ascoli theorem we may find a convergent subsequence of $\{y_n\}$ to which we also will relabel as $\{y_n\}$. Furthermore, we note that for $\frac{j}{n} < t < \frac{j+1}{n}$ and $j \geq 0$ we have

$$\begin{aligned}
\left|\frac{dy_n}{dt} - f(y_n(t))\right| &= \left|f\left(y_n\left(\frac{j}{n}\right)\right) - f(y_n(t))\right| \\
&\leq L_f \left|y_n\left(\frac{j}{n}\right) - y_n(t)\right| \\
&\leq \frac{L_f}{n}\left|f\left(y_n\left(\frac{j}{n}\right)\right)\right|
\end{aligned}$$

Thus we have that the approximating sequence is given by

$$y_n(t) = y_0 + \int_0^t f(y_n(\tau))d\tau + O\left(\frac{1}{n}\right)$$

where $O(1/n)$ denotes terms of order $1/n$ and greater. By taking the limit $y_n \to y$ as $n \to \infty$ we obtain

$$y(t) = y_0 + \int_0^t f(y(\tau))d\tau$$

as desired. The case for negative time is treated similarly.

□

Proposition 29 (Regularity of the Flow Map)**.** *Consider the autonomous system (A.4). If f is of class $C^k(\mathbb{R}^d)$ then the flow map T^t is of class $C^k(\mathbb{R}^d)$ for every $T_1 \leq t \leq T_2$.*

Proof. Using the same reasoning as in the previous Lemma 14, we construct a sequence approximating the derivative of T^t. Define a sequence $\{\nabla T_n^t\}$ in such a way that it satisfies

$$\frac{d}{dt}\nabla T_n^t(x) = \begin{cases} \nabla f\left(T_n^{j/n}(x)\right)\nabla T_n^{j/n}(x) & \text{if } \frac{j}{n} < t < \frac{j+1}{n} \text{ for } j \geq 0 \\ \nabla f\left(T_n^{(j+1)/n}(x)\right)\nabla T_n^{(j+1)/n}(x) & \text{if } \frac{j}{n} < t < \frac{j+1}{n} \text{ for } j < 0 \end{cases}.$$

By Lemma 14 we note that if $|j/n|$, $\|x\| \leq M$ for a constant $M > 0$ then we have the bound

$$\left|T_n^{j/n}(x)\right| \leq \left(\sup_{\|x\|\leq M} |f(x)|\right)\left(e^{L_f M} - 1\right) + M =: M'.$$

This implies that we have the following constraint

$$\begin{aligned} \left|\nabla T_n^{j/n}(x)\right| &\leq \left|\nabla T_n^{j/n}(x) - \nabla T_n^{(j-1)/n}(x)\right| + \left|\nabla T_n^{(j-1)/n}(x)\right| \\ &\leq \frac{1}{n}\|\nabla f\|_{L^\infty(\|x\|\leq M')}\left|\nabla T_n^{(j-1)/n}(x)\right| + \left|\nabla T_n^{(j-1)/n}(x)\right| \\ &= \left(1 + \frac{1}{n}\|\nabla f\|_{L^\infty(\|x\|\leq M')}\right)\left|\nabla T_n^{(j-1)/n}(x)\right| \\ &\leq \left(1 + \frac{1}{n}\|\nabla f\|_{L^\infty(\|x\|\leq M')}\right)^j \\ &\leq e^{j\|\nabla f\|_{L^\infty(\|x\|\leq M')}/n} \\ &\leq e^{M\|\nabla f\|_{L^\infty(\|x\|\leq M')}} \end{aligned}$$

it follow from this that we deduce equicontinuity of ∇T_n^t in time, accordingly we may take the limit along a convergent subsequence. Let us relabel this convergent subsequence as $\nabla T_n^t(x)$ and let $\phi(\cdot, x)$ be its limit for fixed $x \in \mathbb{R}^d$. Accordingly we obtain that the system

$$\begin{cases} \frac{d\phi}{dt}(t,x) = \nabla f(T^t(x))\phi(t,x) \\ \phi(0,x) = I \end{cases}$$

where I is the identity matrix. This system has a unique solution, hence the sequence ∇T_n^t converges uniformly to $\phi(\cdot, x)$ for each fixed $x \in \mathbb{R}^d$. However, by Lemma 14 the sequence T_n^t converges uniformly to

T^t. As the sequence of derivatives converge to ϕ, it is clear that the derivative ∇T^t exists and coincides with ϕ. Accordingly the flow map is differentiable for each fixed $T_0 \leq t \leq T_1$. To show that $T^t \in C^1(\mathbb{R}^d)$ we only remain to show that $\nabla T^t = \phi$ is continuous. As $f \in C^1(\mathbb{R}^d)$ we obtain the estimate

$$\|\phi(x) - \phi(y)\| \leq \left\| \int_{t_0}^{t} \left(\nabla f(T^\tau(x))\phi(\tau, x) - \nabla f(T^\tau(y))\phi(\tau, x)\right) d\tau \right\| + \|\phi(t_0, x) - \phi(t_0, y)\|$$

for x, $y \in \mathbb{R}^d$. Accordingly it follows that

$$\begin{aligned}\|\phi(t, x) - \phi(t, y)\| \leq \|\phi(t_0, x) - \phi(t_0, y)\| + \|\nabla f(T^\tau(x))\|_{L^\infty(|\tau|, \|x\| \leq M)} \int_{t_0}^{t} \|\phi(\tau, x) - \phi(\tau, y)\| d\tau \\ + \|\phi(\tau, x)\|_{L^\infty(|\tau|, \|x\| \leq M)} \int_{t_0}^{t} \|\nabla f(T^\tau(x)) - \nabla f(T^\tau(y))\| d\tau\end{aligned}$$

Now, as f is Lipschitz we obtain the estimate

$$|T^\tau(x) - T^\tau(y)| \leq e^{L_f|\tau|}|x - y|.$$

Furthermore, as ∇f is continuous we have for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|x - y| < \delta \Longrightarrow \|\nabla f(T^\tau(x)) - \nabla f(T^\tau(y))\|$$

for $|\tau| \leq M$. Accordingly, for $|x - y| < \delta$ we obtain

$$\begin{aligned}\|\phi(t, x) - \phi(t, y)\| < \|\phi(t_0, x) - \phi(t_0, y)\| + \|\nabla f(T^\tau(x))\|_{L^\infty(|\tau|, \|x\| \leq M)} \int_{t_0}^{t} \|\phi(\tau, x) - \phi(\tau, y)\| d\tau \\ + \varepsilon \|\phi(\tau, x)\|_{L^\infty(|\tau|, \|x\| \leq M)}(t - t_0).\end{aligned}$$

Using Gronwall's inequality we obtain

$$\begin{aligned}\|\phi(t, x) - \phi(t, y)\| \leq \|\phi(t_0, x) - \phi(t_0, y)\| e^{|t - t_0| \|\nabla f(T^\tau(x))\|_{L^\infty(|\tau|, \|x\| \leq M)}} \\ + \varepsilon \frac{\|\phi(\tau, x)\|_{L^\infty(|\tau|, \|x\| \leq M)}}{\|\nabla f(T^\tau(x))\|_{L^\infty(|\tau|, \|x\| \leq M)}} \left(e^{|t - t_0| \|\nabla f(T^\tau(x))\|_{L^\infty(|\tau|, \|x\| \leq M)}} - 1\right)\end{aligned}$$

Setting $t_0 = 0$ we use the fact that $\phi(0, x) = \phi(0, y) = I$ to obtain

$$\|\phi(t, x) - \phi(t, y)\| \leq \varepsilon \frac{\|\phi(\tau, x)\|_{L^\infty(|\tau|, \|x\| \leq M)}}{\|\nabla f(T^\tau(x))\|_{L^\infty(|\tau|, \|x\| \leq M)}} \left(e^{|t| \|\nabla f(T^\tau(x))\|_{L^\infty(|\tau|, \|x\| \leq M)}} - 1\right)$$

implying that $\phi(t, \cdot) = \nabla T^t(\cdot)$ is continuous for every $T_0 \leq t \leq T_1$. To obtain higher order differentiability assume that $k = 2$ and follow a standard 'bootstrap' argument. Consider the system

$$\begin{cases} \dfrac{dx}{dt} = f(x) \\ \\ \dfrac{dy}{dt} = \nabla f(x) y \end{cases} .$$

Accordingly, if $f \in C^2(\mathbb{R}^d)$ then the map $x \mapsto (f(x), \nabla f(x))$ is of class $C^1(\mathbb{R}^d \times \mathbb{R}^{d^2})$. Accordingly, it possesses a class C^1 flow map given by $\overline{T}^t$ with $\overline{T}^t(x_0, I) = (T^t(x_0), \nabla T^t(x_0))$ for any initial point $x_0 \in \mathbb{R}^d$. As we have $y = \nabla T^t$ we see that for each fixed time that the flow T^t is of class C^2. Continuing this process by induction we obtain that T^t is of class C^k as desired. □

Lemma 15 (Gronwall's Inequality). *Let $(x, v) \in \mathbb{R}^d \times B_R$, $\nu \in P^T_{\mu_0}$ where $\mu_0 \in P_{B_R}$ is the initial measure and let T^t_ν be the associated flow map by (4.11). Furthermore let $K > 0$ be a constant dependent only on α and R. Provided the flow map satisfies for every $0 \le t \le T$ the inequality*

$$|T^t_\nu(x, v)| \le |(x, v)| + Kt + \int_0^t K|T^s_\nu(x, v)|ds$$

then it also satisfies

$$|T^t_\nu(x, v)| \le (|(x, v)| + 1)e^{Kt} - 1.$$

Proof. Let $v : [0, T] \to \mathbb{R}$ be the function defined by

$$v(t) = e^{-Kt} \int_0^t K|T^s_\nu(x, v)|ds.$$

Then upon differentiating we obtain

$$\begin{aligned} \dot{v}(t) &= -K^2 e^{-Kt} \int_0^t |T^s_\nu(x, v)|ds + Ke^{-Kt}|T^t_\nu(x, v)| \\ &\le -K^2 e^{-Kt} \int_0^t |T^s_\nu(x, v)|ds + Ke^{-Kt} \left(|(x, v)| + Kt + \int_0^t K|T^s_\nu(x, v)|ds \right) \\ &= (|(x, v)| + Kt)\, Ke^{-Kt} \end{aligned}$$

where we have used the assumed inequality. By acknowledging that $v(0) = 0$ we obtain upon integrating the above

$$\begin{aligned}
v(t) = v(t) - v(0) &\leq \int_0^t \left(|(x,v)| + Ks\right) K e^{-Ks} ds \\
&= \int_0^t K|(x,v)|e^{-Ks} ds + \int_0^t K^2 s e^{-Ks} ds \\
&= -|(x,v)| \left(e^{-Kt} - 1\right) + K^2 \left(\frac{-t}{K} e^{-Kt} + \frac{1}{K} \int_0^t e^{-Ks} ds \right) \\
&= -|(x,v)| \left(e^{-Kt} - 1\right) + K^2 \left(\frac{-t}{K} e^{-Kt} + \frac{1}{K^2} - \frac{1}{K^2} e^{-Kt} \right) \\
&= -|(x,v)|e^{-Kt} + |(x,v)| - Kte^{-Kt} + 1 - e^{-Kt} \\
&= -\left(|(x,v)| + Kt + 1\right) e^{-Kt} + |(x,v)| + 1.
\end{aligned}$$

Thus by substituting $v(t)$ in we obtain the following relation

$$e^{-Kt} \int_0^t K|T_\nu^s(x,v)| ds \leq -\left(|(x,v)| + Kt + 1\right) e^{-Kt} + |(x,v)| + 1$$

or rather we have

$$\int_0^t K|T_\nu^s(x,v)| ds \leq -\left(|(x,v)| + Kt + 1\right) + (|(x,v)| + 1)e^{Kt}.$$

Finally, we combine this inequality with the one given above to obtain

$$\begin{aligned}
|T_\nu^t(x,v)| &\leq |(x,v)| + Kt + \int_0^t K|T_\nu^s(x,v)| ds \\
&\leq |(x,v)| + Kt - \left(|(x,v)| + Kt + 1\right) + (|(x,v)| + 1)e^{Kt} \\
&= \left(|(x,v)| + 1\right) e^{Kt} - 1.
\end{aligned}$$

□

Theorem 20 (Smooth Urysohn's Lemma in $\mathbb{R}^d$). *If K is a compact subset of $\mathbb{R}^d$ and U is an open neighbourhood of K, there is a function $f \in C_0^\infty(\mathbb{R}^d; [0,1])$ such that $f(K) = 1$ and supp $f \subset U$.*

Proof. This proof is divided into five main steps; they are as follows.

1. The function $h : \mathbb{R} \to [0,1)$ given by $h(t) = e^{-1/t^2} \chi_{\mathbb{R}-\{0\}}(t)$ is smooth.

   The proof of this is step simple enough. For $t \neq 0$ the function $t \mapsto e^{-1/t^2}$ is continuous. Moreover, the limit of this mapping as $t \to 0$ vanishes. Let $k \in \mathbb{N}$. Repeated differentiation shows that the k^{th} order derivative is given by

$$\frac{d^k}{dt^k}h(t) = h(t)Q(t)$$

for some rational function Q whose only residue lies at $t = 0$. Thus for $t \neq 0$ the k^{th} order derivative of h is continuous. Moreover, as $h(t)$ is of exponential decay we have $h(t)Q(t) \to 0$ as $t \to \infty$. As $k \in \mathbb{N}$ was arbitrary we conclude that h is smooth.

2. The functions $h^+(t) = e^{-1/t^2}\chi_{(0,\infty)}(t)$ and $h^-(t) = e^{-1/t^2}\chi_{(-\infty,0)}(t)$ is smooth.

   This step is obvious; the argument is the same as above.

3. Let a, $b \in \mathbb{R}$ such that $a < b$. The function $H : \mathbb{R} \to [0, 1)$ defined by $H(t) = h^+(t - a)h^-(t - b)$ is smooth and $H(t) > 0$ for all $t \in (a, b)$.

   The fact that H is smooth is clear. This is because the translation mapping $t \mapsto t - a$ is smooth, thus the composition $t \mapsto h^+(t - a)$ is also smooth. Similarly the map $t \mapsto h^-(t - b)$ is smooth. As the product of smooth functions is also smooth we must have that H is smooth as well. Lastly, we see that both $h^+(t - a)$ and $h^-(t - b)$ are positive for $t \neq a$, b. Thus the product of them, H, is positive in (a, b).

4. For $i = 1, \cdots, d$ let a_i, $b_i \in \mathbb{R}$ with $a_i < b_i$. Let $R \subset \mathbb{R}^d$ be the open rectangle defined by

   $$R := \prod_{i=1}^{d}(a_i, b_i).$$

   There exists a smooth function $g : \mathbb{R}^d \to [0, 1)$ such that $g(x) > 0$ for all $x \in R$.

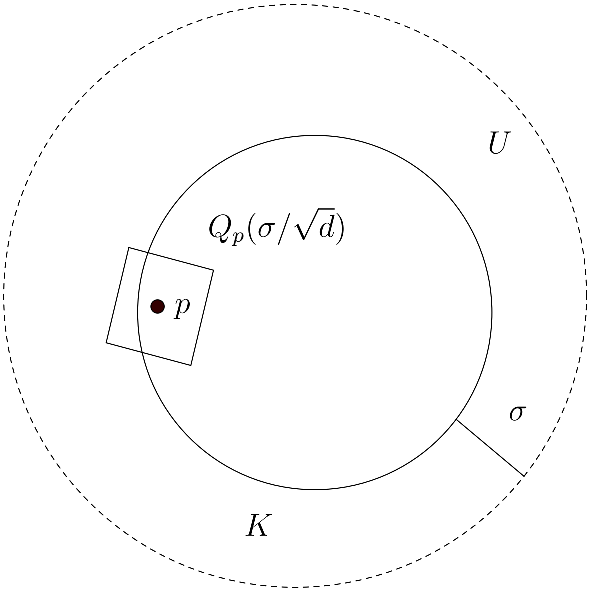
   This is an immediate consequence of the previous point. In accordance to above denote $H_i(x_i) = h^+(x_i - a_i)h^-(x_i - b_i)$ for $i = 1, \cdots, d$. Then we naturally define g by the tensor product

   $$g(x) = H_1(x_1) \otimes H_2(x_2) \otimes \cdots \otimes H_d(x_d)$$

   for $x = (x_1, ..., x_d) \in \mathbb{R}^d$. Then, as H_i is smooth for every $i = 1, ..., d$ then so must be the tensor product of them, i.e. g is smooth. Lastly as $H_i(x_i) > 0$ for every $x_i \in (a_i, b_i)$ and $i = 1, ..., d$ we must have that the tensor product of them is positive in R, i.e. g is also positive in R.

5. Conclude the main result of the theorem.

   As K is compact and $\mathbb{R}^d - U$ is closed then the Euclidean distance between them, $\sigma := d(K, \mathbb{R}^d - U)$, is positive. Consider then an arbitrary point $p \in K$. We may cover each point with an open cube centered at p with sides of length $\sigma/\sqrt{d}$; this we denote by $Q_p(\sigma/\sqrt{d})$.



It is easy to see that every point on $Q_p(\sigma/\sqrt{d})$ is within distance $\sigma/2$. Now, covering every point $p \in K$ with a cube $Q_p(\sigma/\sqrt{d})$ we may use the compactness of K to select a finite subcover of these cubes, given by the set

$$Q = \left\{ Q_{p_i}(\sigma/\sqrt{d}) : 1 \leq i \leq N \right\}.$$

Now define the functions f_i for $i = 1, ..., N$ as follows. Let g_i denote the function in the previous step defined on the open cube $Q_{p_i}(2\sigma/\sqrt{d})$. We define each f_i by

$$f_i(x) = \frac{g_i(x)}{\min g_i \left(\overline{Q}_{p_i} \left(\sigma/\sqrt{d} \right) \right)}.$$

We see then that each such f_i satisfies supp $f_i \subset Q_{p_i}(2\sigma/\sqrt{d})$ and $f_i(x) \geq 1$ for all $x \in Q_{p_i}(\sigma/\sqrt{d})$. Define the function $\phi : \mathbb{R} \to [0, 1]$ by

$$\phi(x) = \frac{h^+(x)}{h^+(x) + h^+(1 - x)}.$$

Note that ϕ is smooth and $\phi(x) = 0$ for $x \leq 0$ and $\phi(x) = 1$ for $x \geq 1$.

We then note that by above that $f_1(x) + f_2(x) + \cdots + f_N(x) \geq 1$ for $x \in K$ and is supported in U. Thus the composition $f := \phi \circ (f_1 + f_2 + \cdots + f_N)$ is supported in U and $f(K) = 1$ as desired.

□