HAMILTON-JACOBI THEORY FOR A GENERALIZED OPTIMAL STOPPING TIME PROBLEM

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1 Introduction

We study an optimal stopping time problem with an extended-valued lower semicontinuous stopping cost. We prove that the optimal value associated with such a problem as a function of the initial time and state is the unique lower semicontinuous solution of a generalized Hamilton-Jacobi equation (H-J equation for short) involving the proximal subgradients and derive a verification theorem that uses lower semicontinuous verification functions.

Allowing the stopping cost to be extended-valued considerably generalizes the ordinary stopping time problem. The generalized optimal stopping time problem considered in this paper encompasses stopping time problems, fixed time problems, free time problems, infinite horizon problems and some types of exit time problems as special cases.

By the classical Hamilton-Jacobi theory if the *value function* (i.e. the optimal value associated with the optimal control problems as a function of the initial time and state) is smooth, then it is a solution of the H-J equation and the H-J equation provides a tool for verifying an admissible control is acturally optimal (see e.g. Fleming and Rishel [13]). In general, however, the value function is nonsmooth and therefore can not satisfy the H-J equation in

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the classical sense. Several approaches have been taken in the control theory literature to cope with this difficulty. Boltyanski [5] (see also Fleming and Rishel [13]) restricted the class of controls so that the value function becomes piecewise smooth. Vinter and Lewis [22] characterized optimality through a sequence of C^1 subsolutions of the H-J equation. If the value function is Lipschitz continuous, Havelock [17] and Clarke [6] characterized the value function as a solution of the H-J equation involving the Clarke generalized gradient. Crandall and Lions [11] introduced the concept of viscosity solutions for uniformly continuous functions. A uniformly continuous function is called a *viscosity solution* of the H-J equation if and only if it satisfies both the H-J sub- and supinequalities involving sub- and supdifferentials respectively. When the value function is uniformly continuous it is then a viscosity solution of the H-J equation. Recently the notion of a viscosity solution has been extended to discuss discontinuous solutions by Ishii [19, 20] and Barles and Perthame [1]. According to their definition, a discontinuous function is a viscosity solution of H-J equation if and only if its upper semicontinuous envelope satisfies the H-J subinequality *and* its lower semicontinuous envelope satisfied the H-J supinequality. The value function is then characterized as a viscosity solution through its lower *and* upper semicontinuous envelopes. In order to treat semicontinuous functions, Barron and Jensen [3, 4] modified this two-sided approach and define the concept of viscosity solution for semicontinuous functions by using an one-sided approach. They charaterized the value function of the relaxed control problem as a viscosity solution of the H-J equation involving *only* the subdifferential. Recently Clarke et. al. [9] introduced the concept of proximal solutions of the H-J equation and characterized the lower semicontinuous value function as a solution to the H-J equation involving the proximal subgradient. When the value function is continuous, it was shown in [8] that the proximal solutions are equivalent to viscosity solutions. The H-J equation involving the proximal subgradient has the advantage over the viscosity solution in that only the the set of points, where the proximal subgradient exists which is a proper subset of the points where the subdifferential exists, are needed to be checked.

The various kinds of Hamilton-Jocobi theories that we discussed above have been extended to treat some nonstandard problems such as the exit time problem (or the so-called optimal control problem with a boundary condition). The exit time problem is difficult to solve since the value function is in general discontinuous even all the problem data are Lipschitz continuous unless some nontangency condition is imposed on the boundary (see e.g. [15, 24, 12] for the Lipschitz continuity of the value function). Gonzalez and Rofman [16] proved that the value function is an upper bound of a suitable set of subsolutions of the H-J equation. Dempster and Ye [12] charaterized the Lipschitz optimal value function as a solution of the H-J equation involving the Clarke generalized gradient. Discontinuous viscosity solutions for the exit time problem have been treated by Ishii in [19], Barles and Perthame in [1, 2]. The purpose of this paper is to extend the H-J theory using the proximal subgradient as in Clarke et. al. [9] to the stopping time problem with endpoint constraints, free time problem, infinite horizon problem and exit time problem. For the exit time problem, under a condition in the boundary of the state space that is similar to but different from the one introduced by Barles and Perthame [1], we are able to show that the value function is lower semicontinuous and therefore the proximal subgradient approach can be used.

The verification technique is an improtant part of the dynamical programming method. In the classical theory a verification function must be smooth. This is a severe restriction. For verification theorems that allow Lipschitz verification functions we refer to [6, 7, 10, 25] and the references therein. Since the value function is always a verification function and in many important problems the value functions are only lower semicontinuous it is important to find verification theorems that use lower semicontinuous verification functions. We prove in §3 that a candidate for a solution to our generalized stopping time problem is actually optimal if there exists a lower semicontinuous verification function (defined by a Hamilton-Jacobi inequality). We also discuss an example where a lower semicontinuous verification function can be used to verify the optimal solution.

We derive our results by using both forward and backward information of the optimal principle. This approach determines a generalized H-J equation (described in terms of the proximal subgradient) that characterizes the lower semicontinuous value function. A verification theorem is derived as a by product (using only the backward information). The technique of treating semicontinuous solutions to the H-J equation by using both forward and backward information of the optimal principle was used in [9, 14]. The technical tools for proximal subgradients are contained in [9] and also from there many of our arguments owe their origin.

We arrange the paper as follows: in the next section we state our stopping time problem and our main result that the value function of the stopping time problem is uniquely determined by a generalized H-J equation. In §3 we discuss verification theorems and examples. We discuss various special cases of our stopping time problem in §4. Longer proofs are contained in §5.

2 A generalized optimal stopping time problem and H-J equation

Let U be a compact subset of R^m and $Prob(U)$ the set of all Borel probability measures on U. Consider $Prob(U)$ as a subset of the dual of $C(U)$ endowed with the weak star topology, where $C(U)$ is the Banach space of continuous functions on U with the supremum norm. For any $\phi \in C(U)$ and $u \in Prob(U)$, we denote the pairing of ϕ and u by $\phi(u) := \int_U \phi(r)u(dr)$. Let $\mathcal{U}$ be the set of all Lebesgue measurable mappings from R to $Prob(U)$. For finite real numbers $a < b$, define $\mathcal{U}_{[a,b]} := \{u|_{[a,b]} : u \in \mathcal{U}\}$. Then $\mathcal{U}_{[a,b]}$ is a weak star compact subset of $L^1([a,b]; C(U))^*$. Elements of $\mathcal{U}_{[a,b]}$ are often called relaxed controls. We endow $\mathcal{U}$ with the following topology: u^n converges to u in $\mathcal{U}$ provided that $u^n|_{[a,b]}$ converges to $u|_{[a,b]}$ in $\mathcal{U}_{[a,b]}$ for any finite real number $a < b$. We refer to [23] for more details. Assume that $g : R \times R^d \times U \to R^d$ satisfies the condition

(H1) $g(t,x,u)$ is continuous, bounded by $M_g > 0$ and Lipschitz in t, x uniformly in $u \in U$ with rank $L_g > 0$.

Under such a condition, for each $u \in \mathcal{U}$, the differential equation

$$\dot{y}(s) = g(s, y(s), u(s)) \quad a.e.$$

has a unique solution defined on R that satisfies the side condition $y(t) = x$. We denote such a solution by $y[t,x,u](\cdot)$ to indicate its dependence on t, x and u. Consider the following generalized optimal stopping time problem

$$P_{t,x} \quad \text{minimize} \quad J(t,x,u,r) := \int_t^r f(s, y[t,x,u](s), u(s))ds + h(r, y[t,x,u](r))$$
$$\text{subject to} \quad r \in [t, \infty], u \in \mathcal{U},$$

where $f : R \times R^d \times U \to R$, $h : R \times R^d \to (-\infty, \infty]$. In this paper for a function $\psi : R \times R^d \to R$, we use the following convention for the value of $\psi(r, y[t,x,u](r))$ at $r = \infty$:

$$\psi(\infty, y[t,x,u](\infty)) := \begin{cases} \lim_{\tau\to\infty} \psi(\tau, y[t,x,u](\tau)) & \text{if the limit exists,} \\ \infty & \text{otherwise} \end{cases}$$

and we define

$$\int_t^\infty f(s, y[t,x,u](s), u(s))ds := \begin{cases} \lim_{T\to\infty} \int_t^T f(s, y[t,x,u](s), u(s))ds & \text{if the limit exists,} \\ \infty & \text{otherwise.} \end{cases}$$

Remark 2.1 Allowing the stopping cost h to be discountinous and extended valued and the intergral in the cost function to be divergent at infinity significantly generalizes the usual optimal stopping time problem. Under the current setting the generalized optimal stopping time problem encompasses stopping time, fixed time, free time, infinite horizon and some exit time problems as special cases. The definition of the cost functional at $r = \infty$ is a device for handling both finite and infinite horizon problem at the same time with minimum notations.

We state some further basic assumptions:

(H2) $f(t, x, u)$ is continuous and Lipschitz in t, x uniformly in $u \in U$ with rank $L_f > 0$.

(H3) h is lower semicontinuous.

(H4) Either there exists $T > 0$ such that $h(\tau, x) = \infty$ for all $\tau > T$ or as $\tau, \tau' \to \infty$, $h(\tau, x)$ and $\int_\tau^{\tau'} f(s, y[t,x,u](s), u(s))ds$ converges to 0 uniformly for all $(t, x, u) \in R \times R^d \times \mathcal{U}$.

Remark 2.2 Assumptions (H1)-(H3) are standard and mild. Assumption (H4) is a technical one for the value function to be lower semicontinuous and attained where it is finite. It is automatically satisfied for a problem with an indicator function as a stopping cost or with discounted running and stopping costs.

We define the value function of the family of problems $P_{t,x}$ as an extended valued function $V_s : R \times R^d \to [-\infty, \infty]$ given by

$$V_s(t,x) := \inf\{\int_t^r f(s, y[t,x,u](s), u(s))ds + h(r, y[t,x,u](r)) : r \in [t,\infty], u \in \mathcal{U}\}$$

and define the Hamiltonian corresponding to problem $P_{t,x}$ by

$$H(t,x,p) := \sup\{\langle p, g(t,x,u)\rangle - f(t,x,u) : u \in U\}.$$

Our main results are:

Theorem 2.1 *Assume that f, g and h satisfy assumptions* (H1)-(H4). *Then the value function V_s is the unique lower semicontinuous function $w : R \times R^d \to (-\infty, \infty]$ that satisfies the terminal condition at ∞*

$$\lim_{s\to\infty} w(s,x) = \lim_{s\to\infty} h(s,x) \tag{1}$$

uniformly for $x \in R^d$ and the Hamilton-Jacobi equation

$$\max\{w(t,x) - h(t,x), -p_t + H(t,x,-p_x)\} = 0 \quad \forall (p_t, p_x) \in \partial_p w(t,x) \tag{2}$$

for all $(t,x) \in R \times R^d$, where $\partial_p w(t,x)$ is the proximal subgradient of w at (t,x) defined as follows (see Clarke [7] for more details): $(p_t, p_x) \in \partial_p w(t,x)$ if and only if there exist $\sigma > 0$ and $\delta > 0$ such that, for all (s,y) satisfying $\|(s,y) - (t,x)\| < \delta$, one has

$$w(s,y) - w(t,x) + \sigma\|(s,y) - (t,x)\|^2 \geq p_t(s-t) + \langle p_x, y - x\rangle.$$

Remark 2.3 The Hamilton-Jacobi equation (2) is actually a variational inequality.

The proximal subgradient of a lower semicontinuous function is in general nonempty only on a dense subset of its domain. Equation (2) in Theorem 2.1 should be interpreted in the following sense: it is checked at points where $\partial_p w(t,x) \neq \emptyset$ for $-p_t + H(t,x,-p_x)$ and in the domain of w for $w(t,x) - h(t,x)$ (The domain of a function is the set of all points where the function is finite).

The proof of Theorem 2.1 depends on the following optimality principle of the value function which is of independent interest:

Theorem 2.2 *Assume that f, g and h satisfy assumptions* (H1)-(H4). *Let $w : R \times R^d \longrightarrow (-\infty, \infty]$ be a lower semicontinuous function. Then w is equal to the value function of $P_{t,x}$ if and only if it satisfies*

(z) the terminal condition at ∞

$$\lim_{s \longrightarrow \infty} w(s, x) = \lim_{s \longrightarrow \infty} h(s, x)$$

uniformly in $x \in R^d$,

(a) the supoptimality principle

for any $(t, x) \in R \times R^d$ where $w(t, x) < \infty$ there exist $u \in \mathcal{U}$ and $r \in [t, \infty]$ such that, for all $\tau \in [t, r)$,

$$w(t, x) \geq \int_t^{\tau} f(s, y[t, x, u](s), u(s))ds + w(\tau, y[t, x, u](\tau)) \tag{3}$$

and

$$w(r, y[t, x, u](r)) = h(r, y[t, x, u](r)), \tag{4}$$

(b) the backward supoptimality principle

for any $(t, x) \in R \times R^d$,

$$w(t, x) \leq h(t, x) \tag{5}$$

and, for any $u \in \mathcal{U}$ and $\tau \leq t$,

$$w(t, x) \geq \int_t^{\tau} f(s, y[t, x, u](s), u(s))ds + w(\tau, y[t, x, u](\tau)). \tag{6}$$

Remark 2.4 The counterpart of (b) in Theorem 2.2 for the classical Mayer problem (see e.g. [13, Theorem 3.1] is that the value function evaluated along any trajectory is a nondecreasing function of time and the counterpart of (a) (see e.g. [13, Theorem 3.2]) is that the value function evaluated along any optimal trajectory is constant as a function of time.

3 Lower semicontinuous verification functions

In the classical verification theorem a verification function must be smooth. This is a severe restriction. Verification theorems that allow Lipschitz verification functions are discussed in, for example, [6, 7, 10, 12, 25]. Example 3.7.5 of [6] shows that there exists a problem that does not have any Lipschitz verification function but has a lower semicontinuous verification function (in the above example the value function is a lower semicontinuous verification function). Therefore it is desirable to extend the Lipschitz verification theorem to allow lower semicontinuous verification functions. The following result offers a verification theorem with lower semicontinuous verification function for the generalized optimal stopping time problems. The proof is defered to §5.

Theorem 3.1 *Suppose that assumptions* (H1)-(H4) *hold. Let* $(y[0,x_0,u],u)$ *be an admissible pair for* P_{0,x_0} *with a stopping time* r. *If there exists a lower semicontinuous function* $w : R \times R^d \to (-\infty,\infty]$ *that satisfies the terminal condition at* ∞

$$\lim_{s\to\infty} w(s,x) = \lim_{s\to\infty} h(s,x) \tag{7}$$

uniformly for $x \in R^d$, *the Hamilton-Jacobi inequality*

$$-p_t + H(t,x,-p_x) \leq 0 \quad \forall (p_t,p_x) \in \partial_p w(t,x), \tag{8}$$

for all $(t,x) \in R \times R^d$, *the stopping condition*

$$w(t,x) \leq h(t,x) \qquad \forall (t,x) \in R \times R^d \tag{9}$$

and the verification equality

$$w(0,x_0) = \int_0^r f(s,y[0,x_0,u](s),u(s))ds + h(r,y[0,x_0,u](r)). \tag{10}$$

Then $(y[0,x_0,u],u)$ *is optimal for* P_{0,x_0} *with optimal stopping time* r *and*

$$V_s(0,x) = w(0,x_0).$$

A lower semicontinuous function w satisfying (7), (8), (9) and (10) is called a verification function (for $(y[0,x_0,u],u)$). Unlike in Theorem 2.1, the verification function in Theorem 3.1 is not uniquely determined. Thus, we can often choose a verification function that is simpler than the value function. The following is an example:

Example 3.1 Consider the generalized optimal stopping time problem $P_{0,-1}$ with $f=0$, $g(s,y,u)=yu$, $U=[0,1]$ and

$$h(s,y)=\begin{cases}\infty & \text{if } y>2,\\ -2e^{-s}y & \text{if } y\in[1,2],\\ -e^{-s}(y+1) & \text{if } y<1.\end{cases}$$

This is an ordinary stopping time problem with end point constraints. The Hamilton-Jacobi inequality (8) becomes:

$$-p_t+\max\{-p_x x,0\}\le 0 \quad \forall(p_t,p_x)\in\partial_p w(t,x),\ (t,x)\in R\times R. \tag{11}$$

We can directly verify that the value function of this problem is

$$V_s(s,y)=\begin{cases}\infty & \text{if } y>2,\\ -2e^{-s}y & \text{if } y\in[1,2],\\ -e^{-s}(y+1) & \text{if } y\in[-1,1],\\ 0 & \text{if } y<-1.\end{cases}$$

Of course V_s is a verification function for this stopping time problem. We now show that the following simpler function

$$w(s,y)=\begin{cases}\infty & \text{if } y>2,\\ -(4/3)e^{-s}(y+1) & \text{if } y\in[-1,2],\\ 0 & \text{if } y<-1.\end{cases}$$

is also a verification function. First the terminal condition at infinity (7) and the stopping condition (9) are satisfied. It is obvious that $\partial_p w(s,y)=\{\nabla w(s,y)\}$ at $y\in(-\infty,-1)\cap(-1,2)$ and

$$\nabla w(s,y)=\begin{cases}((4/3)e^{-s}(y+1),-(4/3)e^{-s}) & \text{if } y\in(-1,2),\\ (0,0) & \text{if } y<-1.\end{cases}$$

Therefore, (11) is satisfied for $y\in(-\infty,-1)\cap(-1,2)$. At $y=-1$ since $\partial_p w(t,y)=\emptyset$, the inequality (11) is automatically satisfied. At $y=2$

$\partial_p w(s,2) = \{4e^{-s}\} \times (-(4/3)e^{-s}, \infty)$ and it is easily verified again that (11) is satisfied. Let $x_0 = -1$. The control $u^* \equiv 0$ and the stopping time $r^* = 0$ are feasible for $P_{0,-1}$ and satisfy the verification equality. Therefore w as defined is a verification function and $u^* \equiv 0$ and $r^* = 0$ is indeed a pair of solution to problem $P_{0,-1}$.

4 Fixed time, free time, infinite horizon and exit time problems

In this section we discuss how to uniformly treat fixed time, free time, infinite horizon and exit time problems by using the general model of the optimal stopping time problems discussed in the previous sections. We emphasize free time and exit time problems and treat fixed time and infinite horizon problems as their special cases respectively.

4.1 Free time and fixed time problems

Consider the following free time problem:

$$\begin{aligned} FREE_{t,x} \quad \text{minimize} \quad & J_f(t,x,u,T) := \int_t^T f(s, y[t,x,u](s), u(s))ds \\ & \qquad + \varphi(T, y[t,x,u](T)) \\ \text{subject to} \quad & (T, y(T)) \in S, u \in \mathcal{U}, \end{aligned}$$

where S is a compact subset of $R \times R^d$, $f : R \times R^d \times U \to R$, $g : R \times R^d \times U \to R^d$ and $\varphi : S \to R$. Let T^* denote the maximum projection of S to R, i.e.,

$$T^* := \max\{T : \exists y \in R^d \text{ such that } (T, y) \in S\}.$$

We define the value function of the family of problems $FREE_{t,x}$ as an extended valued function $V : (-\infty, \infty) \times R^d \to [-\infty, \infty]$, given by

$$V(t,x) := \begin{cases} \inf\{J_f(t,x,u,T) : (T, y[t,x,u](T)) \in S, u \in \mathcal{U}\} & \text{if } t \le T^*, \\ \infty & t > T^*. \end{cases}$$

Let the stopping cost $h : R \times R^d \to (-\infty, \infty]$ be defined as

$$h(s, y) := \delta_S(s, y) + \varphi(s, y), \tag{12}$$

where

$$\delta_A(z) := \begin{cases} 0 & \text{if } z \in A \\ \infty & \text{if } z \notin A \end{cases}$$

denote the indicate function of set A at point z. Then the free time problem $FREE_{t,x}$ is equivalent to the generalized optimal stopping problem $P_{t,x}$ in the sense that the optimal controls for the two problems are the same and the optimal terminal time of $FREE_{t,x}$ is equal to the optimal stopping time of $P_{t,x}$. We can therefore deduce from the results in §2 and §3 the corresponding H-J equation and the verification theorem for the free time problem $FREE_{t,x}$ as follows:

Corollary 4.1 *Assume that f and g satisfy assumptions* (H1)-(H2) *and φ is a lower semicontinuous function. Then the value function V is the unique lower semicontinuous extended valued function $w(\cdot, \cdot)$ on $(-\infty, \infty) \times R^d$ such that*

$$w(t, x) = \infty \quad \forall (t, x) \in (T^*, \infty) \times R^d$$

which satisfies the H-J equation

$$-p_t + H(t, x, -p_x) = 0 \quad \forall (p_t, p_x) \in \partial_p w(t, x) \tag{13}$$

for all $(t, x) \in S^c$ and the terminal condition

$$\max\{w(t, x) - \varphi(t, x), -p_t + H(t, x, -p_x)\} = 0 \quad \forall (p_t, p_x) \in \partial_p w(t, x)$$

for all $(t, x) \in S$.

Remark 4.1 Since the function w is defined on the whole space, the proximal subgradients on the boundary of the set S is then well-defined.

The above terminal condition should be understood as follows: At any $(t, x) \in S$ where $\partial_p w(t, x) \neq \emptyset$ and $w(t, x) \neq \infty$, either $w(t, x) = \varphi(t, x)$ and the H-J sub-inequality

$$-p_t + H(t, x, -p_x) \leq 0 \quad \forall (p_t, p_x) \in \partial_p w(t, x)$$

is satisfied or $w(t,x) < \varphi(t,x)$ and the H-J equation

$$-p_t + H(t,x,-p_x) = 0 \quad \forall (p_t,p_x) \in \partial_p w(t,x)$$

must be satisfied.

In the case when $S = \{1\} \times C$ and φ is independent of the time variable, the free time problem reduces to the fixed time problem. Since in the fixed time problem $V(1,x) = \varphi(x), \forall x \in C$ the terminal condition becomes $w(1,x) = \delta_C(x) + \varphi(x)$ and for all $x \in C$

$$-p_t + H(t,x,-p_x) \leq 0 \quad \forall (p_t,p_x) \in \partial_p w(1,x).$$

This type of terminal condition was discussed in [14] in the viscosity sense. Note that the uniqess of lower semicontinuous functions satisfying the H-J inequality involving proximal subgradient with the limiting type of terminal condition was first given in [9, Theorem 9.1].

Corollary 4.2 *Assume that f and g satisfy assumptions* (H1)-(H2) *and φ is a lower semicontinuous function. Then a feasible pair $(y[0,x_0,u],u)$ is a solution for problem $FREE_{0,x_0}$ with optimal terminal time $T \leq T^*$ if there exists a lower semicontinuous function $w(\cdot,\cdot)$ on $(-\infty,\infty) \times R^d$ such that*

$$\lim_{t\to\infty} w(t,x) = \infty$$

uniformly for all $x \in R^d$ which satisfies the Hamilton-Jacobi inequality,

$$-p_t + H(t,x,-p_x) \leq 0 \quad \forall (p_t,p_x) \in \partial_p w(t,x)$$

for all $(t,x) \in R \times R^d$, the terminal condition

$$w(t,x) \leq \varphi(t,x)$$

for all $(t,x) \in S$ and the verification equality

$$w(0,x_0) = \int_0^T f(s,y[0,x_0,u](s),u(s))ds + \varphi(T,y[0,x_0,u](T)).$$

The proofs of these two corollaries are merely to check conditions and to convert the conclusions of Theorems 2.1 and 3.1 to the appropriate forms. We prove only Corollary 4.1.

Proof of Corollary 4.1. Since φ is lower semicontinuous and S is a closed set, the stopping cost h defined by (12) is lower semicontinuous (i.e., (H3) is satisfied). Since $h(\tau, x) = \infty \quad \forall \tau > T^*$, it also satisfies assumption (H4). By Theorem 2.1, the value function $V_s(t,x)$ of the generalized optimal stopping time problem is the unique lower semicontinous function on $R \times R^d$ which satisfies the boundary condition at ∞:

$$\lim_{t \to \infty} V_s(t,x) = \lim_{t \to \infty} h(t,x)$$

uniformly for all $x \in R^d$ and the Hamilton-Jacobi equation (2). For any $y \in R^d$ since $h(t,x) = \infty, \forall t > T^*$, we have

$$V_s(t,x) = V(t,x) = \infty, \forall t > T^*, y \in R^d.$$

The conclusion of the corollary follows from the above observation and the fact that for all $(t,x) \in S^c$ such that $V(t,x) < \infty$ we have $V(t,x) - h(t,x) = -\infty$ and, therefore, (2) becomes (13).

4.2 Exit time and infinite horizon problems

Throughout this section E is an open domain in R^d with boundary ∂E and closure $\bar{E}$. For $(t,x) \in R \times R^d$ and $u \in \mathcal{U}$ we denote the first exit time of the trajectory $y[t,x,u]$ from E as

$$\theta = \theta[t,x,u] := \inf\{s \geq t : y[t,x,u] \notin E\}.$$

Note that the first exit time is defined for any x starting from inside the state space $\bar{E}$ as well as from outside the state space E. $\theta[t,x,u]$ as defined is equal to zero for any $x \notin E$.

Given any $x \in \bar{E}$ consider the exit time problem from E:

$$\begin{aligned} EXIT_{t,x} \quad \text{minimize} \quad & J_e(t,x,u) := \int_t^\theta f_0(y[t,x,u](s), u(s))e^{-s}ds \\ & \qquad + e^{-\theta}\varphi(y[t,x,u](\theta)) \\ \text{subject to} \quad & u \in \mathcal{U}, \end{aligned}$$

where $\varphi : \bar{E}^c \to R$, $g : R \times \bar{E} \times U \to R^d$ and $f_0 : \bar{E} \times U \to R$ satisfies

(H2') $f_0(x,u)$ is continuous, bounded by $M_{f_0} > 0$ and Lipschitz in x uniformly in $u \in U$ with rank $L_{f_0} > 0$.

To convert this problem to the generalized stopping time problem we need the following technical assumption which is somewhat different from the one proposed by Barles and Perthame in [1].

(H3') $\varphi : \bar{E}^c \to R$ is a lower semicontinuous function bounded by M_φ and there exist extensions g and f_0 to $R \times R^d \times U$ and $R^d \times U$, respectively, satisfying (H1) and (H2) such that

$$e^{-t}\varphi(x) \leq \int_t^r f_0(y[t,x,u](s), u(s))e^{-s}ds + e^{-r}\varphi(y[t,x,u](r)),$$

for all $(t,x) \in R \times \partial E$, all controls $u \in \mathcal{U}$ and $r \geq t$ such that $y[t,x,u](r) \in \partial E$ or $r = \infty$.

Remark 4.2 In its essentials, assumption (H3') requires that the cost of a trajectory leaving the boundary of E after hitting E for the first time is larger than those staying at the boundary.

When $E = R^d$ and $\varphi = 0$ problem $EXIT_{t,x}$ becomes an infinite horizon problem and the assumption (H3') is satisfied vacuously.

We define the value function of the family of problems $EXIT_{t,x}$ as an extended valued function $V : R \times R^d \to [-\infty, \infty]$, given by

$$V(t,x) := \begin{cases} \inf\{J_e(t,x,u) : u \in \mathcal{U}\} & \text{if } x \in \bar{E}, \\ e^{-t}\varphi(x) & \text{if } x \in \bar{E}^c. \end{cases}$$

Under assumptions (H1), (H2') and (H3') we can convert the exit time problem to the generalized optimal stopping time problem. Then Theorems 2.1 and 3.1 lead to a H-J equation for the value function of problem $EXIT_{t,x}$. Precisely, we have

Corollary 4.3 *Assume that the extended valued functions f and g and $\varphi, \mathcal{U}$ satisfy assumptions* (H1), (H2') *and* (H3'). *Then the value function V is the unique lower semicontinuous function $w(\cdot,\cdot)$ on $R \times R^d$ such that*

$$w(t,x) = e^{-t}\varphi(x) \quad \forall (t,x) \in R \times \bar{E}^c$$

and

$$\lim_{t\to\infty} w(t,x) = 0$$

uniformly for $x \in \bar{E}$ which satisfies the Hamilton-Jacobi equation

$$-p_t + H(t,x,-p_x) = 0 \quad \forall (p_t,p_x) \in \partial_p w(t,x)$$

for all $(t,x) \in R \times E$ and the exit condition

$$w(t,x) = e^{-t}\varphi(x)$$

and

$$-p_t + H(t,x,-p_x) \leq 0 \quad \forall (p_t,p_x) \in \partial_p w(t,x)$$

for all $(t,x) \in R \times \partial E$.

Corollary 4.4 *Assume that the extended valued functions f and g and $\varphi, \mathcal{U}$ satisfy assumptions* (H1), (H2') *and* (H3'). *Let $x_0 \in \bar{E}$. Then a feasible pair $(y[0,x_0,u],u)$ is a solution for problem $EXIT_{0,x_0}$ with optimal exit time θ^* if there exists a lower semicontinuous function $w(\cdot,\cdot)$ on $R \times R^d$ such that*

$$\lim_{t\to\infty} w(t,x) = 0$$

uniformly for $x \in R^d$ which satisfies the Hamilton-Jacobi inequality,

$$-p_t + H(t,x,-p_x) \leq 0 \quad \forall (p_t,p_x) \in \partial_p w(t,x)$$

for all $(t,x) \in R \times R^d$, the exit condition

$$w(t,x) \leq \begin{cases} 2e^{-s}(M_{f_0} + M_\varphi) & \text{if } x \in E \cup \bar{E}^c, \\ e^{-s}\varphi(x) & \text{if } x \in \partial E. \end{cases}$$

for all $(t,x) \in R \times R^d$ and the verification equality

$$w(0,x_0) = \int_0^{\theta^*} f(s,y[0,x_0,u](s),u(s))ds + e^{-\theta^*}\varphi(y[0,x_0,u](\theta^*)).$$

Proof of Corollary 4.3. Define the stopping cost

$$h(s,y) := \begin{cases} 2e^{-s}(M_{f_0} + M_\varphi) & \text{if } y \in E \cup \bar{E}^c, \\ e^{-s}\varphi(y) & \text{if } y \in \partial E. \end{cases}$$

and let $f(t,x,u) = e^{-t} f_0(x,u)$. Then we can easily check that g, f and h satisfy assumptions (H1)-(H4).

We now prove that the generalized stopping time problem $P_{t,x}$ is equivalent to the exit time problem $EXIT_{t,x}$. Let $(y[t,x,u],u)$ be a solution pair of $P_{t,x}$ with the optimal stopping time $r \in [t,\infty]$ and $V_s(t,x)$ the value function corresponding to the generalized stopping time problem. Then

$$V_s(t,x) = \int_t^r f_0(y[t,x,u](s),u(s))e^{-s}ds + e^{-r}\varphi(y[t,x,u](r)).$$

The definition of h implies that either $r < \infty$ and $y[t,x,u](r) \in \partial E$ or $r = \infty$. Indeed, if $r < \infty$ then $y[t,x,u](r)$ cannot belong to E because if that is the case then we can take a $\eta > 0$ very small such that $y[t,x,u](r+\eta) \in E$. This leads to

$$\begin{aligned} & J(t,x,u,r+\eta) - J(t,x,u,r) \\ = & \int_r^{r+\eta} f_0(y[t,x,u](s),u(s))e^{-s}ds + 2(M_{f_0} + M_\varphi)(e^{-(r+\eta)} - e^{-r}) \\ = & \int_r^{r+\eta} [f_0(y[t,x,u](s),u(s)) - 2(M_{f_0} + M_\varphi)]e^{-s}ds < 0, \end{aligned}$$

a contradiction. For the same reason it cannot belong to $\bar{E}^c$ either. We may assume $r = \theta = \theta[t,x,u]$. In fact, it is obvious that $r \geq \theta[t,x,u]$. If $r > \theta$ by assumption (H3') we have

$$\begin{aligned} V_s(t,x) &= \int_t^r f_0(y[t,x,u](s),u(s))e^{-s}ds + e^{-r}\varphi(y[t,x,u](r)) \\ &= \int_t^\theta f_0(y[t,x,u](s),u(s))e^{-s}ds \\ &\quad + \int_\theta^r f_0(y[t,x,u](s),u(s))e^{-s}ds + e^{-r}\varphi(y[t,x,u](r)) \\ &\geq \int_t^\theta f_0(y[t,x,u](s),u(s))e^{-s}ds + e^{-\theta}\varphi(y[t,x,u](\theta)) \\ &\geq V_s(t,x). \end{aligned}$$

Therefore we can take $r = \theta$. Thus, $(y[t,x,u],u)$ is also an optimal solution pair for $EXIT_{t,x}$. The converse can be proved similarly.

Invoking Theorem 2.1 and noticing that, for $(t,x) \in R \times E$, $h(t,x) > V(t,x)$ we obtain

$$-p_t + H(t,x,-p_x) = 0 \quad \forall (p_t,p_x) \in \partial_p V(t,x), \forall (t,x) \in R \times E.$$

Also by (H3'), for all $(t,x) \in R \times \partial E$, $V(t,x) = e^{-t}\varphi(x)$. Thus, for all $(t,x) \in R \times \partial E$,

$$-p_t + H(t,x,-p_x) \leq 0 \quad \forall (p_t,p_x) \in \partial_p V(t,x).$$

■

We now turn to a time independent version of Corollary 4.3. Assume that g does not depend on t then the value function $V(t,x)$ corresponding to $EXIT_{t,x}$ is of the form $V(t,x) = e^{-t}V_0(x)$ where $V_0(x) := V(0,x)$.

Lemma 4.1 *Assume that f and φ are nonnegative. Then $(p_t,p_x) \in \partial_p V(t,x)$ if and only if $e^t p_x \in \partial_p V_0(x)$ and $p_t = -e^{-t}V_0(x)$.*

Proof. "Only if" part. Let $(p_t,p_x) \in \partial_p V(t,x)$. Then there exist $\sigma > 0$ and a neighborhood $N(t,x)$ of (t,x) such that, for all $(s,y) \in N(t,x)$,

$$e^{-s}V_0(y) - e^{-t}V_0(x) + \sigma((s-t)^2 + \|y-x\|^2) \geq p_t(s-t) + \langle p_x, y-x\rangle. \quad (14)$$

Setting $s = t$ in (14) yields

$$e^{-t}(V_0(y) - V_0(x)) + \sigma\|y-x\|^2 \geq \langle p_x, y-x\rangle.$$

That is to say $e^t p_x \in \partial_p V_0(x)$. Similarly, setting $y = x$ in (14) yields

$$(e^{-s} - e^{-t})V_0(x) + \sigma(s-t)^2 \geq \langle p_t, s-t\rangle,$$

i.e.

$$-e^{-t}V_0(x)(s-t) + o((s-t)^2) \geq \langle p_t, s-t\rangle.$$

Thus, $p_t = -e^{-t}V_0(x)$.

"If" part. Since f and φ are nonnegative so is V_0. Let $e^t p_x \in \partial_p V_0(x)$ and $p_t = -e^{-t}V_0(x)$. Then there exists $\sigma_1 > 0$ such that

$$V_0(y) - V_0(x) + \sigma_1\|y-x\|^2 \geq \langle e^t p_x, y-x\rangle.$$

Also since e^{-s} is a C^2 function there exists a $\sigma_2 > 0$ such that

$$e^{-s} - e^{-t} + \sigma_2(s-t)^2 \geq -e^{-t}(s-t).$$

Therefore

$$\begin{aligned} & e^{-s}V_0(y) - e^{-t}V_0(x) \\ = \; & e^{-s}(V_0(y) - V_0(x)) + (e^{-s} - e^{-t})V_0(x) \\ \geq \; & e^{-s}\langle e^t p_x, y-x\rangle - e^{-s}\sigma_1\|y-x\|^2 \\ & -V_0(x)e^{-t}(s-t) - \sigma_2 V_0(x)(s-t)^2 \\ \geq \; & \langle p_x, y-x\rangle - V_0(x)e^{-t}(s-t) \\ & -\sigma_1\|y-x\|^2 - \sigma_2 V_0(x)(s-t)^2 + (e^{t-s}-1)\langle p_x, y-x\rangle. \end{aligned}$$

Observe that there exists $\sigma_3 > 0$ such that

$$\begin{aligned} & |(e^{t-s}-1)\langle p_x, y-x\rangle| \leq \|p_x\||e^{t-s}-1|\|y-x\| \\ \leq \; & (\|p_x\|/2)(\|y-x\|^2 + \sigma_3(s-t)^2). \end{aligned}$$

Let $M := \sigma_1 + \sigma_2 V_0(x) + (\|p_x\|/2)(1+\sigma_3)$. Then we have

$$e^{-s}V_0(y) - e^{-t}V_0(x) + M((s-t)^2 + \|y-x\|^2) \geq \langle p_x, y-x\rangle - V_0(x)e^{-t}(s-t).$$

Thus, $(p_x, p_t) \in \partial V(t,x)$. ■

Define

$$H_0(x,p) := sup_{u\in U}\{\langle p, g(x,u)\rangle - f_0(x,u)\}.$$

Then $H(t,x,e^{-t}p) = e^{-t}H_0(x,p)$. Invoking Lemma 4.1 we derive the following time independent form of Corollary 4.3.

Corollary 4.5 *Assume that the extended functions f and g and $\varphi, \mathcal{U}$ satisfy assumptions* (H1), (H2') *and* (H3') *and f and φ are nonnegative. Then the value function V_0 corresponding to problem $EXIT_{0,x}$ is the unique bounded lower semicontinuous function $w(\cdot)$ on R^d such that*

$$w(x) = \varphi(x) \quad \forall x \in \bar{E}^c$$

which satisfies the Hamilton-Jacobi equation

$$w(x) + H_0(x, -p_x) = 0 \quad \forall p_x \in \partial_p w(x)$$

for all $x \in E$ and the boundary condition

$$w(x) = \varphi(x)$$

and

$$w(x) + H_0(x, -p_x) \leq 0 \quad \forall p_x \in \partial_p w(x)$$

for all $x \in \partial E$.

A time independent version of Corollary 4.4 can be derived similarly.

5 Proofs of the main results

We break the proofs of our main results into a series of lemmas.

Lemma 5.1 *Under assumptions* (H1)-(H4) *suppose that $(t^n, x^n) \in R \times R^d$ converges to (t, x) where $V_s(t, x) < \infty$. Then there exist a control $u \in \mathcal{U}$ and a stopping time $r \in [t, \infty]$ such that*

$$\liminf_{n\to\infty} V_s(t^n, x^n) \geq \int_t^r f(s, y[t, x, u](s), u(s))ds + h(r, y[t, x, u](r)).$$

Proof. For each n, there exists $u^n \in \mathcal{U}$ and a stopping time $r^n \geq t^n$ such that

$$V_s(t^n, x^n) + \frac{1}{n} \geq \int_{t^n}^{r^n} f(s, y[t^n, x^n, u^n](s), u^n(s))ds + h(r^n, y[t^n, x^n, u^n](r^n)). \tag{15}$$

Without loss of generality we may assume that $t^n \geq t - 1$ and consider two cases:

Case 1. The sequence $\{r^n\}$ is bounded. Without loss of generality we may assume that r^n converges to $r (\geq t)$, $r^n \in [t-1, r+1]$ and $u^n|_{[t-1,r+1]}$ converges to $u \in \mathcal{U}_{[t-1,r+1]}$ in the topology of $\mathcal{U}_{[t-1,r+1]}$. Then $y[t^n, x^n, u^n](s)$ uniformly converges to $y[t, x, u](s)$ in $[t-1, r+1]$. Taking liminf in (15) when $n \to \infty$ yields by virtue of lower semicontinuity of h

$$\liminf_{n\to\infty} V_s(t^n, x^n) \geq \int_t^r f(s, y[t, x, u](s), u(s))ds + h(r, y[t, x, u](r)).$$

Case 2. The sequence $\{r^n\}$ is unbounded. Without loss of generality we may assume that $r^n \to \infty$. For each integer $m > t$, consider the restriction of u^n to $[t-1, m]$. Since $\mathcal{U}_{[t-1,m]}$ is compact we can extract a convergent subsequence from $\{u^n|_{[t-1,m]}\}$. Using the diagonal method we can choose a subsequence u^{n_i} of u^n and an element $u \in \mathcal{U}$ such that $u^{n_i}|_{[t-1,m]}$ converges to $u|_{[t-1,m]}$ in $\mathcal{U}|_{[t-1,m]}$ for any m. We may assume that

$$\lim_{i\to\infty} V_s(t^{n_i}, x^{n_i}) = \liminf_{n\to\infty} V_s(t^n, x^n).$$

Since

$$\int_{t^{n_i}}^{r^{n_i}} f(s, y[t^{n_i}, x^{n_i}, u^{n_i}](s), u^{n_i}(s))ds + h(r^{n_i}, y[t^{n_i}, x^{n_i}, u^{n_i}](r^{n_i}))$$
$$\leq V_s(t^{n_i}, x^{n_i}) + \frac{1}{n_i} < \infty,$$

by assumption (H4) for any given $\varepsilon > 0$ there exist m_0 and i_0 such that, for $i > i_0$ and $m \in [m_0, r^{n_i}]$

$$\left| \int_m^{r^{n_i}} f(s, y[t^{n_i}, x^{n_i}, u^{n_i}](s), u^{n_i}(s))ds + h(r^{n_i}, y[t^{n_i}, x^{n_i}, u^{n_i}](r^{n_i})) \right| < \varepsilon. \quad (16)$$

Since $r^{n_i} > m_0$ when $i > i_0$, by (15) we have for $i > i_0$ and $m \in [m_0, r^{n_i}]$,

$$\begin{aligned}
&V_s(t^{n_i}, x^{n_i}) + \frac{1}{n_i} \\
&\geq \int_{t^{n_i}}^{r^{n_i}} f(s, y[t^{n_i}, x^{n_i}, u^{n_i}](s), u^{n_i}(s))ds + h(r^{n_i}, y[t^{n_i}, x^{n_i}, u^{n_i}](r^{n_i})) \\
&= \int_{t^{n_i}}^{m} f(s, y[t^{n_i}, x^{n_i}, u^{n_i}](s), u^{n_i}(s))ds \\
&\quad + \int_m^{r^{n_i}} f(s, y[t^{n_i}, x^{n_i}, u^{n_i}](s), u^{n_i}(s))ds + h(r^{n_i}, y[t^{n_i}, x^{n_i}, u^{n_i}](r^{n_i})) \\
&\geq \int_{t^{n_i}}^{m} f(s, y[t^{n_i}, x^{n_i}, u^{n_i}](s), u^{n_i}(s))ds - \varepsilon, \text{ by virtue of (16).}
\end{aligned}$$

Taking limits when $i \to \infty$ yields

$$\liminf_{n\to\infty} V_s(t^n, x^n) = \lim_{i\to\infty} V_s(t^{n_i}, x^{n_i}) \geq \int_t^m f(s, y[t,x,u](s), u(s))ds - \varepsilon.$$

Observe that by assumption (H4), (16) implies that $h(\infty, y[t,x,u](\infty)) = 0$ and the limit

$$\lim_{m\to\infty} \int_t^m f(s, y[t,x,u](s), u(s))ds.$$

exists. Since ε and $m > m_0$ are arbitrary, one has

$$\begin{aligned}\liminf_{n\to\infty} V_s(t^n, x^n) &\geq \int_t^\infty f(s, y[t,x,u](s), u(s))ds \\ &= \int_t^\infty f(s, y[t,x,u](s), u(s))ds + h(\infty, y[t,x,u](\infty)).\end{aligned}$$

■

Lemma 5.2 *Under assumptions* (H1)-(H4), *for any* $(t,x) \in R \times R^d$ *such that* $V(t,x) < \infty$, *the problem* $P_{t,x}$ *has at least one solution pair* $(y[t,x,u], u)$ *with an optimal stopping time* $r \in [t, \infty]$.

Proof. The result follows from Lemma 5.1 by taking $(t^n, x^n) = (t, x)$ $\forall n$.

■

The proofs of Lemmas 5.3-5.8 below basically follows the technique in [9].

Lemma 5.3 *Assume that* f, g *and* h *satisfy assumptions* (H1)-(H4). *Then the value function* V_s *is a lower semicontinuous extended valued function* $w(\cdot)$ *on* $R \times R^d$ *which satisfies the terminal condition at* ∞ *(z), the supoptimality principle (a) and the backward supoptimality principle (c).*

Proof. The lower semicontinuity of V_s follows from Lemma 5.1.

Proof of V_s satisfying (z). Suppose there exists T such that $h(t,x) = \infty$ for all $t > T$ and $x \in R^d$. Then $V_s(t,x) = \infty$ for all $t > T$ and $x \in R^d$. Therefore in this case

$$\lim_{t\to\infty} V_s(t,x) = \lim_{t\to\infty} h(t,x) = \infty.$$

If such a T does not exist then by assumption (H4), $h(s,y) \longrightarrow 0$ uniformly for all $x \in R^d$ as $s \to \infty$ and for any $\varepsilon > 0$ there exists t_0 such that

$$|\int_t^r f(s, y[t,x,u](s), u(s))ds + h(r, y[t,x,u](r))| < \varepsilon,$$

for all $x \in R^d, u \in \mathcal{U}$ and $r, t > t_0$. Therefore in this case

$$\lim_{t\to\infty} V_s(t,x) = \lim_{t\to\infty} h(t,x) = 0.$$

Proof of V_s satisfying (a). By Lemma 5.2, for any $(t,x) \in R \times R^d$ where $V_s(t,x) < \infty$ there exists a solution pair $(y[t,x,u],u)$ corresponding to the problem $P_{t,x}$ with an optimal stopping time $r \in [t,\infty]$. For any $\tau \in [t,r)$, since $(y[t,x,u],u)$ is also feasible for problem $P_{\tau,y[t,x,u](\tau)}$, one has

$$\begin{aligned} V_s(t,x) &= \int_t^r f(s, y[t,x,u](s), u(s))ds + h(r, y[t,x,u](r)) \\ &= \int_t^\tau f(s, y[t,x,u](s), u(s))ds \\ &\quad + \int_\tau^r f(s, y[t,x,u](s), u(s))ds + h(r, y[t,x,u](r)) \\ &\geq \int_t^\tau f(s, y[t,x,u](s), u(s))ds + V_s(\tau, y[t,x,u](\tau)). \end{aligned}$$

That is, V_s satisfies (3). If $r = \infty$, (4) is always satisfied by V_s by virtue of (z). If $r < \infty$ and (4) is not satisfied by V_s then

$$V_s(r, y[t,x,u](r)) < h(r, y[t,x,u](r)) \tag{17}$$

since one always has $V_s(t,x) \leq h(t,x)$ for all $(t,x) \in R \times R^d$. Applying Lemma 5.2 to problem P_{r,x_r} where $x_r = y[t,x,u](r)$ yields a solution pair $(y[r,x_r,v],v)$ to problem P_{r,x_r} with an optimal stopping time $r' > r$. (17) implies

$$\begin{aligned} &\int_t^r f(s, y[t,x,u](s), u(s))ds + h(r, y[t,x,u](r)) \\ &> \int_t^r f(s, y[t,x,u](s), u(s))ds + V_s(r, y[t,x,u](r)) \\ &= \int_t^r f(s, y[t,x,u](s), u(s))ds + \int_r^{r'} f(s, y[t,x,v](s), v(s))ds \\ &\quad + h(r', y[t,x,v](r')) \end{aligned}$$

which implies that the pair $(y[t,x,u],u)$ and the stopping time r is not an optimal solution of $P_{t,x}$. That is a contradiction. Hence (4) has to be safisfied by V_s.

Proof of V_s satisfying (b). (5) is easily seen to be satisfied by the value function V_s. Now we prove the backward suboptimality principle (6). If $V_s(t,x) = \infty$ then there is nothing to prove since (6) is always satisfied. Suppose $V_s(t,x) < \infty$ and let $(y[t,x,v],v)$ be a solution pair to the problem $P_{t,x}$ with an optimal stopping time $r \in [t,\infty]$, i.e.,

$$V(t,x) = \int_t^r f(s,y[t,x,v](s),v(s))ds + h(r,y[t,x,v](r)).$$

Consider any $u \in \mathcal{U}$ and $\tau < t$. Define a control $u' \in \mathcal{U}$ by

$$u'(s) := \begin{cases} u(s) & \text{if } s \in [\tau,t) \\ v(s) & \text{if } s \in [t,r]. \end{cases}$$

Then $y[t,x,u](\tau) = y[t,x,u'](\tau)$ and one has

$$\begin{aligned} V_s(\tau,y[t,x,u](\tau)) &= V_s(\tau,y[t,x,u'](\tau)) \\ &\leq \int_\tau^r f(s,y[t,x,u](s),u'(s))ds + h(r,y[t,x,u'](r)) \\ &= \int_\tau^t f(s,y[t,x,u](s),u(s))ds \\ &\quad + \int_t^r f(s,y[t,x,v](s),v(s))ds + h(r,y[t,x,v](r)) \\ &= \int_\tau^t f(s,y[t,x,u](s),u(s))ds + V_s(t,x). \end{aligned}$$

That, is V_s satisfies (6). ■

Lemma 5.4 *Assume that f,g and h satisfy assumptions* (H1)-(H4).*Suppose $w : R \times R^d \to (-\infty,\infty]$ is a lower semicontinuous function which satisfies the terminal condition at ∞ (z) and the suboptimality principle (a).*

Then

$$w(t,x) \geq V_s(t,x).$$

Proof. If $w(t,x) = \infty$ at (t,x), then there is nothing to prove. For any $(t,x) \in R \times R^d$ where $w(t,x) < \infty$ since w satisfies the suboptimality, there exits $u \in \mathcal{U}$ and $r \in [t, \infty]$ such that for any $\tau \in [t, r)$ one has

$$w(t,x) \geq \int_t^{\tau} f(s, y[t,x,u](s), u(s))ds + w(\tau, y[t,x,u](\tau)) \tag{18}$$

and

$$w(r, y[t,x,u](r)) = h(r, y[t,x,u](r)),$$

Taking liminf on the right hand side of (18) as $\tau \to r$ yields by virtue of assumption (H4) and the lower semicontinuity of w that

$$\begin{aligned} w(t,x) &\geq \int_t^r f(s, y[t,x,u](s), u(s))ds + w(r, y[t,x,u](r)) \\ &= \int_t^r f(s, y[t,x,u](s), u(s))ds + h(r, y[t,x,u](r)) \\ &\geq V_s(t,x). \end{aligned}$$

■

Lemma 5.5 *Assume that f, g and h satisfy assumptions* (H1)-(H4). *Suppose $w : R \times R^d \to R \cup \{\infty\}$ is a lower semicontinuous function which satisfies the terminal condition at ∞ (z) and the backward suboptimality principle (b). Then*

$$w(t,x) \leq V_s(t,x).$$

Proof. If $V_s(t,x) = \infty$, then there is nothing to prove. Suppose $V_s(t,x) < \infty$ and let $(y[t,x,u], u)$ be a solution pair of $P_{t,x}$ with an optimal stopping time $r \in [t, \infty]$. If $r < \infty$ then by (b)

$$\begin{aligned} V_s(t,x) &= \int_t^r f(s, y[t,x,u](s), u(s))ds + h(r, y[t,x,u](r)) \\ &\geq \int_t^r f(s, y[t,x,u](s), u(s))ds + w(r, y[t,x,u](r)) \\ &\geq w(t, y[t,x,u](t)) = w(t,x). \end{aligned}$$

If $r = \infty$ then since $V_s(t,x) < \infty$ by assumption (H4)

$$\lim_{t' \to \infty} h(t', x) = 0$$

uniformly in $x \in R^d$and one has

$$V_s(t,x) = \lim_{t'\to\infty} \{\int_t^{r'} f(s, y[t,x,u](s), u(s))ds + h(t', y[t,x,u](t'))\}.$$

Therefore for any $\varepsilon > 0$ there exists a $r_\varepsilon < r$ such that, for all $r' \in [r_\varepsilon, \infty)$,

$$V_s(t,x) + \varepsilon \geq \int_t^{r'} f(s, y[t,x,u](s), u(s))ds + h(r', y[t,x,u](t').$$

Hence by the backward suboptimality of w, one has

$$\begin{aligned} V_s(t,x) + \varepsilon &\geq \int_t^{r'} f(s, y[t,x,u](s), u(s))ds + h(r', y[t,x,u](r') \\ &\geq \int_t^{r'} f(s, y[t,x,u](s), u(s))ds + w(r', y[t,x,u](r')) \\ &\geq w(t, y[t,x,u](t)) = w(t,x). \end{aligned}$$

Letting $\varepsilon \to 0$ we are done. ■

For the convenience of the exposition, we denote the epigraph of a function w defined on $R \times R^d$ by $E(w)$, i.e.

$$E(w) := epi(w) := \{(s,y,z) : z \geq w(s,y)\}.$$

We will also use the proximal normal cone $N_A^p(x)$ for a closed set A at point $x \in A$ defined by

$$N_A^p(x) := \{p : \text{there exists } M > 0 \text{ such that } \langle p, x'-x\rangle \leq M\|x'-x\| \ \forall x' \in A\}.$$

Lemma 5.6 *Assume that f, g and h satisfy assumptions* (H1)-(H4). *Let $w : R \times R^d \to (-\infty, \infty]$ be a lower semicontinuous function that satisfies*

$$-p_t + H(t,x,-p_x) \geq 0 \quad \forall (p_t, p_x) \in \partial_p w(t,x),$$

for any $(t,x) \in R \times R^d$. Then, for any $(t,x) \in R \times R^d$ there exist $\eta > 0$ and $u \in \mathcal{U}_{[t,t+\eta]}$ such that, for all $\tau \in [t, t+\eta]$,

$$(\tau, y[t,x,u](\tau), z(\tau)) \in E(w) \tag{19}$$

where

$$z(\tau) := w(t,x) + \int_\tau^t f(s, y[t,x,u](s), u(s))ds.$$

Proof. Define

$$F(s,y,z) := \{(1, g(s,y,v), -f(s,y,v)) : v \in Prob(U)\}.$$

Then the conclusion of the lemma is equivalent to saying that, for some $\eta > 0$, $(F, E(w))$ is *weakly invariant* on $[t, t+\eta]$ as defined in [9, Section 2]. By Theorem 2.1 amd Remark 2.3 of [9] this is in turn equivalent to

$$\max_{\xi \in N^p_{E(W)}(t,x,z)} \min_{v \in F(t,x)} \langle \xi, v \rangle \leq 0 \quad (t,x) \in [t, t+\eta] \times R^d. \tag{20}$$

Using the same argument as in the proof of [9, Theorem 2.2] (20) is equivalent to

$$-p_t + H(t,x,-p_x) \geq 0 \quad \forall (p_t, p_x) \in \partial_p w(t,x)$$

for any $(t,x) \in R \times R^d$. ■

Lemma 5.7 *Assume that f, g and h satisfy assumptions* (H1)-(H4). *Let $w : R \times R^d \to (-\infty, \infty]$ be a lower semicontinuous function which satisfies the terminal condition at ∞ (z). Then the suboptimality principle (a) is equivalent to the* H-J subinequality

(c) for any $(t,x) \in R \times R^d$, either

$$w(t,x) = h(t,x)$$

or

$$-p_t + H(t,x,-p_x) \geq 0 \quad \forall (p_t, p_x) \in \partial_p w(t,x).$$

Proof.

(a) implies (c). Let w be a lower semicontinuous function which satisfies the terminal condition at ∞ and the suboptimality principle (a). The H-J subinequality is satisfied vacuously at any point (t,x) where $w(t,x) = \infty$. For any $(t,x) \in R \times R^d$ where $w(t,x) < \infty$, by (a) there exist $u \in \mathcal{U}$ and $r \in [t, \infty]$ such that, for all $\tau \in [t, r)$,

$$w(t,x) \geq \int_t^\tau f(s, y[t,x,u](s), u(s))ds + w(\tau, y[t,x,u](\tau))$$

and

$$w(r, y[t,x,u](r)) = h(r, y[t,x,u](r)).$$

If $r = t$ then $w(t,x) = h(t,x)$ and (c) is satisfied. We turn to the case when $r > t$ and $(p_t, p_x) \in \partial_p w(t,x)$ (when $\partial_p w(t,x) = \emptyset$ (c) is satisfied vacuously). By the definition of the proximal subgradient, there exist $\sigma > 0$ and $\eta > 0$ such that

$$w(s,y) - w(t,x) + \sigma\|(s,y) - (t,x)\|^2 \geq p_t(s-t) + \langle p_x, y - x\rangle, \qquad (21)$$

for all (s,y) such that $\|(s,y) - (t,x)\| < \eta$. Therefore, for $\tau > t$ sufficiently close to t set $s := \tau$ and $y := y[t,x,u](\tau)$ in (21) yields

$$\begin{aligned} & p_t(\tau - t) + \langle p_x, y[t,x,u](\tau) - x\rangle \\ \leq\ & w(\tau, y[t,x,u](\tau)) - w(t,x) + \sigma\|(\tau, y[t,x,u](\tau)) - (t,x)\|^2 \\ \leq\ & \int_\tau^t f(s, y[t,x,u](s), u(s))ds + \sigma\|(\tau, y[t,x,u](\tau)) - (t,x)\|^2. \qquad (22) \end{aligned}$$

We can rewrite (22) as

$$\begin{aligned} 0 \ \leq\ & -p_t(\tau - t) + \int_t^\tau [\langle -p_x, g(s, y[t,x,u](s), u(s))\rangle \\ & -f(s, y[t,x,u](s), u(s))]ds + \sigma\|(\tau, y[t,x,u](\tau)) - (t,x)\|^2 \\ \leq\ & -p_t(\tau - t) + \int_t^\tau H(s, y[t,x,u](s), -p_x)ds \\ & +\sigma\|(\tau, y[t,x,u](\tau)) - (t,x)\|^2. \qquad (23) \end{aligned}$$

Dividing both sides of (23) by $\tau - t$ and letting $\tau \to t$, noticing that the Hamiltonian in the integration is continuous in its first two components, we obtain

$$-p_t + H(t,x,-p_x) \geq 0.$$

(c) implies (a). Let $(t,x) \in R \times R^d$. If $w(t,x) = h(t,x)$ then there is nothing to prove. Assume that $w(t,x) < h(t,x)$. Then by Lemma 5.6, the H-J sub inequality implies the existence of η and $u \in \mathcal{U}_{[t,t+\eta]}$ such that for all $\tau \in [t, t+\eta]$,

$$w(t,x) \geq \int_t^\tau f(s, y[t,x,u](s), u(s))ds + w(\tau, y[t,x,u](\tau)). \qquad (24)$$

If for some $r \in [t, t+\eta]$,

$$w(r, y[t,x,u](r)) = h(r, y[t,x,u](r))$$

then we are done. Otherwise applying Lemma 5.6 again to $(t+\eta, y[t,x,u](t+\eta))$ we can extend the intervel on which (24) is valid. Repeating this process one conclude that either there exist an $r<\infty$ such that (24) is valid on $[t,r]$ and

$$w(r, y[t,x,u](r)) = h(r, y[t,x,u](r))$$

or the intervel on which (24) is valid can be extend to $[t,\infty)$. ■

Lemma 5.8 *Assume that f, g and h satisfy assumptions* (H1)-(H4). *Let $w : R \times R^d \to (-\infty, \infty]$ be a lower semicontinuous function which satisfies the terminal condition at ∞. Then the backward suboptimality principle (b) is equivalent to the* H-J sup-inequality

(d) for any $(t,x) \in R \times R^d$,

$$w(t,x) \le h(t,x)$$

and

$$-p_t + H(t,x,-p_x) \le 0 \quad \forall (p_t, p_x) \in \partial_p w(t,x).$$

Proof.

(b) implies (d). Assume that w satisfies (z) and (b). Let $(p_t, p_x) \in \partial_p w(t,x)$. Then there exist $\sigma > 0$ and $\eta > 0$ such that

$$w(s,y) - w(t,x) + \sigma \|(s,y) - (t,x)\|^2 \ge p_t(s-t) + \langle p_x, y-x \rangle, \tag{25}$$

for all (s,y) with $\|(s,y)-(t,x)\| < \eta$. Let v be an arbitrary element of U and define $u \in \mathcal{U}$ by $u(s) = v$, $\forall s \in R$. Then when $\tau < t$ is sufficiently close to t set $s := \tau$ and $y := y[t,x,u](\tau)$ in (25) yields

$$\begin{aligned} & p_t(\tau - t) + \langle p_x, y[t,x,u](\tau) - x \rangle \\ \le\ & w(\tau, y[t,x,u](\tau)) - w(t,x) + \sigma \|(\tau, y[t,x,u](\tau)) - (t,x)\|^2 \\ \le\ & \int_\tau^t f(s, y[t,x,u](s), u(s))ds + \sigma \|(\tau, y[t,x,u](\tau)) - (t,x)\|^2. \end{aligned} \tag{26}$$

Dividing both sides of (26) by $t-\tau$ and letting $\tau \to t$ yield

$$-p_t + \langle -p_x, g(t,x,v) \rangle - f(t,x,v) \le 0. \tag{27}$$

Since $v \in U$ is arbitrary, taking sup in (27) we obtain

$$-p_t + H(t, x, -p_x) \leq 0.$$

(d) implies (b). Assume w satisfies (z) and (d). We need to show that, for any $(t, x) \in R \times R^d$, $u \in \mathcal{U}$ and $\tau \leq t$,

$$w(t,x) \geq \int_t^\tau f(s, y[t,x,u](s), u(s))ds + w(\tau, y[t,x,u](\tau)).$$

The above inequality is obviously equivalent to

$$(\tau, y[t,x,u](\tau), z(\tau)) \in E(w) \tag{28}$$

where

$$z(\tau) := w(t,x) + \int_\tau^t f(s, y[t,x,u](s), u(s))ds.$$

To deduce (28) from

$$-p_t + H(t,x,p_x) \leq 0 \quad \forall (p_t, p_x) \in \partial_p w(t,x)$$

we apply [9, Theorem 7.2] with set $E(w)$ and multifunction

$$\{1\} \times \overline{co}\{(g(t,x,u), f(t,x,u)) : u \in U\}.$$

■

Proof of Theorem 2.2 The "only if" part is given in Lemma 5.3. The "if" part follows from Lemmas 5.4 and 5.5. ■

Proof of Theorem 2.1 Follows directly from Theorem 2.2, Lemma 5.7 and Lemma 5.8. ■

Proof of Theorem 3.1

Let w be a lower semicontinuous function satisfying the requirement of theorem 3.1. Then by Lemma 5.8 W has property (b) of Lemma 5.8. Invoking Lemma 5.5 yields

$$V_s(0, x_0) \geq w(0, x_0) = \int_0^r f(s, y[0, x_0, u](s), u(s))ds + h(r, y[0, x_0, u](r)).$$

But by definition $V_s(0, x_0)$ cannot be greater than the right hand side of the above inequality. Thus,

$$V_s(0, x_0) = \int_0^r f(s, y[0, x_0, u](s), u(s))ds + h(r, y[0, x_0, u](r))$$

and $(y[0, x_0, u], u)$ is a solution pair to problem P_{0,x_0} with an optimal stopping time r. ■

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