The Hot, Magnetized, Relativistic Vlasov Maxwell System

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Abstract

This master thesis is devoted to the kinetic description in phase space of strongly magnetized plasmas. It addresses the problem of stability near equilibria for magnetically confined plasmas modeled by the relativistic Vlasov Maxwell system. A small physically pertinent parameter ϵ, with $0 < \epsilon \ll 1$, related to the inverse of a gyrofrequency, governs the strength of a spatially inhomogeneous applied magnetic field given by the function $x \mapsto \epsilon^{-1}\mathbf{B}_e(x)$. Local C^1-solutions do exist. But these solutions may blow up in finite time. This phenomenon can only happen at high velocities [14] and, since ϵ^{-1} is large, standard results predict that this may occur at a time T_ϵ shrinking to zero when ϵ goes to 0. It has been proved recently in [7] that, in the case of *neutral*, *cold*, and *dilute* plasmas (like in the Earth's magnetosphere), smooth solutions corresponding to perturbations of equilibria exist on a uniform time interval $[0, T]$, with $0 < T$ independent of ϵ. We investigate here the *hot* situation, which is more suitable for the description of fusion devices. A condition is derived for which perturbed $W^{1,\infty}$-solutions with large initial momentum also exist on a uniform time interval, they remain bounded in the sup norm for well-prepared initial data, and moreover they inherit some kind of stability.

Table of Contents

Dedications

I acknowledge my supervisor, committee members, friends and family for all their support in completion of this program. The knowledge I have gained will follow me in my future endeavors.

Chapter 1

Introduction to Plasma Physics and to the Strongly Magnetized Vlasov Maxwell System

A standard description of a plasma is an “ionized gas”. A plasma can be created when a substance is heated to high enough temperatures, such that the outer electrons of atoms can be stripped away from the nuclei leaving a mixture of positive and negative charges. This can also be accomplished through the presence of a strong electromagnetic field. Plasmas are electrically conductive and subject to long range electromagnetic fields generated by particle movement. They can also be subjected to short range interactions through particle collisions. If a plasma is very concentrated, such as inside of stars, binary collisions between particles (strong, short range electromagnetic forces from nearby particles) can dominate over the mean fields of the plasma. In these regimes, the Maxwell-Boltzman (collisional plasmas) or Magnetohydrodynmic (fluid mechanics) models will apply.

In this thesis, we are interested in collisionless plasmas, on time scales for which the mean electromagnetic fields dominate the plasma behavior. This is well described mathematically by the Vlasov Maxwell system. Due to the ionization, the plasma consists of multiple types of charged particles which are passively transported, while also being subjected to self consistent electromagnetic forces. Each particle of type $i \in \{1, 2, ..., N\}$ within the plasma has an associated distribution function $\mathbf{f}_i$ described by a Vlasov equation:

$$\partial_t \mathbf{f}_i + v_i(\xi) \cdot \nabla_x \mathbf{f}_i + F_i(t, x, \xi) \cdot \nabla_\xi \mathbf{f}_i = 0,$$

where F_i is the Lorentz force given by

$$F_i(t, x, \xi) := Z_i(\mathbf{E}(t, x) + \mathbf{E}_e(t, x) + v_i(\xi) \times [\mathbf{B}(t, x) + \mathbf{B}_e(t, x)]).$$

The constant Z_i is the charge of particle i and the vector field $v_i(\cdot)$ is the relativistic velocity. Furthermore we denote the mass of the ith particle to be m_i. The functions $\mathbf{f}_i$ depend on the time t, the spatial variable $x \in \mathbb{R}^3$ and the momentum variable $\xi \in \mathbb{R}^3$. They can be physically interpreted as

$$\int_{A_x} \int_{A_\xi} \mathbf{f}_i(t, x, \xi) dx d\xi = \left\{ \begin{array}{l} \text{Number of particles of type } i \text{ at time } t \text{, inside the spatial} \\ \text{region } A_x \subset \mathbb{R}^3 \text{ and having momentum in the range } A_\xi \subset \mathbb{R}^3 \end{array} \right.$$

The fields $(\mathbf{E}, \mathbf{B})$ are the self consistent, macroscopic electromagnetic fields generated by the electrons and ions within the plasma and are governed by Maxwell's equations. These depend on all distribution functions $\mathbf{f}_i$ through the superposition principle. By “macroscopic” we mean that the charge density

$$\rho = \sum_{i=1}^{N} \left(Z_i \int \mathbf{f}_i d\xi \right)$$

and the current density

$$\mathbf{J} = \sum_{i=1}^{N} \left(Z_i \int v_i(\xi) \mathbf{f}_i d\xi \right)$$

are determined by averaging over all particle momentum.

In this thesis we are concerned with a two particle system, containing electrons and a single type of positively charged ions. Remark that the ratio μ between proton mass m_p and electron mass m_e is large ($\mu \approx 1.8 \times 10^4$). For this reason, and the fact that the acceleration generated by the Lorenz force is inversely proportional to the particle mass, we are concerned with time scales for which the ion motion can be neglected and only the electrons may be viewed as free to move. The ions, however, serve as a neutralizing background for the plasma through their density ρ_i.

On the other hand, $(\mathbf{E}_e, \mathbf{B}_e)$ is an *external* electromagnetic field, which is predetermined. For instance, the Van Allen Belts are regions of space surrounding Earth consisting of a hydrogen ion plasma, for which the Vlasov Maxwell system can be used for modeling. This plasma generates it's own electromagnetic field $(\mathbf{E}, \mathbf{B})$, but is subject to the (much stronger and independent) magnetic field of the Earth. The Van Allen Belts shield Earth from cosmic rays and solar flares, protecting the atmosphere from destruction. The presence of the external magnetic field of Earth acts to confine the plasma to within a few radii of Earth. Toroidal flux surfaces of Earth's magnetic field prevent particles from escaping radially away from the earth. This leads to drift of the particles along the magnetic field lines, and then bouncing back and forth between magnetic poles.

In addition to astronomical applications, there has been significant research into fusion energy reactors. At high enough temperature and pressure, hydrogen ions can fuse exothermically to create helium and release energy. This is precisely the process which takes place in stars. Researches have continually been working to efficiently harness this energy. However, in stars the plasma is conveniently contained by very strong gravitational forces which are obviously not available in laboratory fusion reactors. A solution to this problem may be magnetic confinement. The idea, like in the Van Allen Belts, is to apply a strong magnetic field along the toroidal axis of a toroidal chamber to contain the plasma. Such devices are known as tokomaks or stellarators. It is therefore crucial to understand as exactly as possible the effects of introducing the external field $(\mathbf{E}_e, \mathbf{B}_e)$. The aim is to better understand the plasma behaviors during its confinement time, including the stability and the turbulent behaviour which may arise spontaneously or due to the fact that the initial data are ill-prepared.

The Vlasov Maxwell Cauchy problem in the absence of an external electromagnetic field, that is when $(\mathbf{E}_e, \mathbf{B}_e) \equiv (0, 0)$, has been extensively studied in the last few decades. R. Glassey and W. Strauss were the first to determine sufficient conditions for local C^1-solutions to exist. The monograph [11] is a complete review of their work. As long as the distribution function $\mathbf{f}$ has compact support in the momentum variable, local smooth solutions exist and can be extended to larger time intervals [14]. Furthermore, global C^1-solutions exist for nearly neutral ($|\rho|_{t=0}| \ll 1$) and sufficiently dilute plasmas [12, 13]. Other results (see [8] and related references) deal with global weak $L^1 \cap L^2$-solutions.

In this thesis, we are interested in the effects of a strong, inhomogeneous external magnetic field, which is denoted by $\epsilon^{-1}\mathbf{B}_e(x)$. In dimensionless units, the number ϵ is of size $\epsilon \approx 10^{-5}$ in the case of both Van Allen belts and tokamaks. From now on, we consider that $0 < \epsilon \ll 1$

is a small parameter. In practice, this number ϵ is related to the inverse of a gyrofrequency. It controls the strength of the external applied magnetic field, and thereby the function $\mathbf{B}_e(\cdot)$ has an amplitude of size one. On the other hand, the variations of the vector field $\mathbf{B}_e(\cdot)$ account for the spatial inhomogeneities coming from physical geometries inside the problem. The number ϵ may also be associated with the period at which the particles tend to wrap around the magnetic field lines.

As a matter of fact, under the action of $\epsilon^{-1}\mathbf{B}_e(x)$, the charged particles starting from the position x tend to follow deformed cylindrical paths of radius $\sim \mathcal{O}(\epsilon)$ orientated along the direction $|\mathbf{B}_e(x)|^{-1}\,\mathbf{B}_e(x)$. For longer times, the motions become much more complicated. But, for well-adjusted functions $\mathbf{B}_e(\cdot)$, they remain bounded in a compact set [5, 6]. This is what could be meant by a "dynamical particle confinement". This property plays a crucial role in fixing the Van Allen belts to Earth. It is also essential in tomokak reactors to prevent particles from escaping radially outwards.

Now, in concrete situations, a self-consistent electromagnetic field $(\mathbf{E}, \mathbf{B})$ does appear. This phenomenon is well described through the coupling between the Vlasov equation and the Maxwell equations. This induces many extra phenomena which can completely change the preceding stabilized picture. In particular, the onset of a non trivial electric field $\mathbf{E} \not\equiv 0$ may have disruptive effects. It can be shown that the energy of the system is bounded by initial data. This is due to the fact that the magnetic field $\mathbf{B}_e$ does no work on charged particles. This is a key observation in order to obtain weak solutions, as well as a set of preliminary information (see the article [3] and related works). But this does not allow to describe sufficiently precisely the structure of the solutions $(\mathbf{f}^\epsilon, \mathbf{E}^\epsilon, \mathbf{B}^\epsilon)(t)$ when ϵ is small. It is not clear whether the external applied field $\mathbf{B}_e$ can lead to large amplitude oscillations or rapid oscillations which can degrade and destabilize the plasma (for instance through resonances). The problem is to better explain how the solutions will behave in the limit that ϵ tends to zero. This means to get a uniform lifespan T_ϵ, and to control the evolution of the solutions in adequate norms, like L^∞ or $W^{1,\infty}$ which give accurate information.

In the article [7], C. Cheverry and S. Ibrahim have initiated this program. They have derived a condition for which equilibria (or stationary solutions) $(\mathbf{f}^s, \mathbf{E}^s, \mathbf{B}^s)$ are stable in the space $C^0([0,T], W^{1,\infty})$. These authors assume that the momentum variable ξ is initially confined to a set of size ϵ. Physically, this means particle velocities are bounded far away from the speed of light. This is the notion of "coldness" of the plasma. In cold plasmas such as the Van Allen Belts, this is a reasonable assumption. Then, the perturbed solutions exist on a uniform time interval $[0,T]$ with $T \in \mathbb{R}_+^*$, implying that $T_\epsilon \geq T$ does not shrink to zero with ϵ. Moreover, they stay close to the equilibrium profile and they remain uniformly bounded (in the sense of L^∞). The goal of this thesis is to determine whether the stability conditions of [7] are necessary. In particular we are concerned with initial data with large momentum (hot plasma), which corresponds better to the case of fusion reactors.

The outline for this thesis is as follows. In Section 2, we introduce the Hot, Magnetized, Relativistic Vlasov Maxwell (HMRVM) system and its underlying physical assumptions. This section states our main result, Theorem 1 of Subsection 2.4.1, devoted to the well posedness of the HMRVM system for dilute data. As long as the initial density function rotates slowly about the frozen in applied magnetic field lines, small perturbations of dilute equilibrium remain stable. In Section 2, we also outline an open problem, namely:

Does the perturbed HMRVM system remain stable for ill-prepared data ?

Section 3 reviews a number of techniques of [7] from a slightly different perspective. One difference is we avoid the use of scalar and vector potentials in deriving representation formulas for the electromagnetic fields. Furthermore, this section pinpoints exactly the mathematical difficulty faced for the hot plasma regime. It concludes by reformulating the HMRVM system in terms of canonical cylindrical coordinates. This has the advantage of introducing a single, periodic, rapidly oscillating variable, which is useful to establish averaging procedures.

Next, Section 4 is entirely new. The approach is to completely study an associated linear Vlasov Maxwell system. The introduction of the fast, periodic variable leads to an asymptotic approximation of the characteristic curves constructed using a non-stationary phase lemma. This is key to approximating the linear system. Then, we use a bootstrap argument to approximate the non-linear system for dilute equilibrium using the linear model. In essence, the non-linear term in the HMRVM system will remain small as long as the initial data is well prepared. This is a great accomplishment, as it allows us to precisely understand hot plasma dynamics using a reduced model which possesses a derived asymptotic expansion in terms of the parameter ϵ.

Finally Section 5 improves the results of Section 4 for the linear system by using Fourier analysis for homogeneous applied fields. The improvement is that uniform estimates and lifespans are established for non-dilute equilibrium. This gives insight into how instabilities may occur for spatially varying magnetic fields. However, the picture is not yet completely clear. According to the Beurling-Helson theorem, the non-linearity (in space) of the inhomogeneous, diffeomorphic flow may imply ill-posedness for the HMRVM for general initial data. However, the asymptotic analysis established in Section 4, tells us this non-linearity is small due to the rapid oscillations and converges to linear solutions. This dichotomy will be addressed in future work which hopefully will definitively state the validity of the open conjecture of Section 2.

Chapter 2

Modeling of Magnetized Plasma

This chapter is intended to defining the Hot Magnetized Relativistic Vlasov Maxwell system (HMRVM in abbreviated form). It starts in Section 2.1 by introducing the Magnetized Relativistic Vlasov Maxwell (MRVM) system for a plasma consisting of electrons and stationary ions. Section 2.2 then introduces some physically relevant assumptions pertaining to plasmas: the *hot, cold and dilute assumptions*. Improving on the work of [7], we no longer impose the cold assumption. Next, Section 2.3 defines the HMRVM system by considering perturbations of equilibrium solutions to the MRVM system in the hot regime. Finally, in Section 2.4, we state the main result of this thesis, Theorem 1, regarding stability and well possedness of the HMRVM system for dilute well prepared data, and we finish by addressing the open Conjecture 1 regarding the necessity of the dilute and well prepared data assumptions of our Theorem 1.

2.1 The MRVM System

This section is devoted to constructing our mathematical model and precisely outlining the assumptions necessary to prove the main result given by Theorem 1. The Relativistic Vlasov Maxwell system gives a kinetic description of the time evolution in the phase space of charged particles within a plasma. We work in dimension three, with spatial position $x \in \mathbb{R}^3$ and momentum $\xi \in \mathbb{R}^3$. We study properties of the Vlasov Maxwell system under the influence of a strong applied magnetic field. The strength of this inhomogeneous field is controlled by a large parameter ϵ^{-1}, with $\epsilon \in (0, 1]$. The parameter ϵ is known as the inverse gyro-frequency. As mentioned, here we consider a two particle system consisting of electrons and a singly stationary ion type. We first define the relativistic velocity as a function of the momentum ξ, for electron mass m_e as

$$v_e(\xi) := \langle \frac{\xi}{m_e c} \rangle^{-1} \frac{\xi}{m_e c}, \quad 1 \leq \langle \xi \rangle := \sqrt{1 + |\xi|^2}, \quad \forall \xi \in \mathbb{R}^3.$$

Since the ions are assumed stationary, we are free to choose units such that mass is measured in units of electron mass m_e. In other words, we simply set $m_e = 1$. Furthermore, we also take the speed of light c to be set to unity ($c = 1$). The electron velocity then reduces to

$$v(\xi) = v_e(\xi) = \frac{\xi}{\sqrt{1 + |\xi|^2}}.$$

Therefore, the Magnetized Relativistic Vlasov Maxwell (MRVM) system on the electron density $\mathbf{f}$, with charge $Z_e = -1$, is given by:

$$\partial_t \mathbf{f} + [v(\xi) \cdot \nabla_x]\mathbf{f} - \frac{1}{\epsilon}[v(\xi) \times \mathbf{B}_e(x)] \cdot \nabla_\xi \mathbf{f} = [\mathbf{E} + v(\xi) \times \mathbf{B})] \cdot \nabla_\xi \mathbf{f}, \tag{2.1}$$

$$\nabla_x \cdot \mathbf{E} = \rho_i - \rho(\mathbf{f}) \quad ; \quad \partial_t \mathbf{E} - \nabla \times \mathbf{B} = \mathbf{J}(\mathbf{f}), \tag{2.2}$$

$$\nabla_x \cdot \mathbf{B} = 0 \quad ; \quad \partial_t \mathbf{B} + \nabla_x \times \mathbf{E} = 0. \tag{2.3}$$

Equation (2.1) is known as the Vlasov equation, and (2.2) - (2.3) are Maxwell's equations governing propagation of the fields. The constant $\rho_i \in \mathbb{R}_+$ represents the background ion charge density. The current and charge densities of the electrons are defined respectively as

$$\mathbf{J}(\mathbf{f})(t,x) := \int v(\xi)\mathbf{f}(t,x,\xi)d\xi, \tag{2.4}$$

$$\rho(\mathbf{f})(t,x) := \int \mathbf{f}(t,x,\xi)d\xi. \tag{2.5}$$

The unknown in the above system is $\mathbf{U} := {}^t(\mathbf{f}, \mathbf{E}, \mathbf{B})$. We impose here a strong inhomogeneous exterior magnetic field that is non-vanishing, divergence free, and curl free. More specifically, for any compact set $K \subset \mathbb{R}^3$, there exists a constant $c(K) > 0$ such that

$$\forall x \in K, \quad c(K) \leq b_e(x) \leq c(K)^{-1}, \quad b_e(x) := |\mathbf{B}_e(x)| \tag{2.6}$$

and

$$\forall x \in K, \quad \nabla_x \cdot \mathbf{B}_e(x) \equiv 0, \quad \nabla_x \times \mathbf{B}_e(x) = 0. \tag{2.7}$$

The article [7] gives an extensive treatment of uniform estimates with respect to $\epsilon \in (0,1]$, as well as stability of $\mathbf{U}$ under particular technical assumptions related to a perturbed regime about stationary solutions. The aim of this thesisis to prove similar stability when these assumptions are removed.

2.2 Assumptions

Before stated the physical assumptions, we first introduce a family of equilibria denoted by $\mathbf{U}^s := (\mathbf{f}^s, \mathbf{E}^s, \mathbf{B}^s)$, which have the form

$$\mathbf{f}^s(t,x,\xi) := M_\epsilon(|\xi|), \quad \mathbf{E}^s := 0, \quad \mathbf{B}^s := 0. \tag{2.8}$$

Fix any non-negative function $M_\epsilon \in C_c^1(\mathbb{R}_+;\mathbb{R}_+)$. We can always adjust ρ_i in such a way that $\rho_i := \rho(M_\epsilon)$. Then, the expression $\mathbf{U}^s$ is sure to solve (2.1)-(2.3). Thus, it is a stationary solution of (2.1)-(2.3), hence the superscript"s" while the subscript "ϵ" is put to mark a possible dependence on ϵ.

The goal is to perturb the stationary solutions $\mathbf{U}^s$, and to examine their stability. To this end, we need to impose constraints on the data ρ_i and M_ϵ. In [7], the plasma was supposed to be *globally neutral*, *cold* and *dilute*. In Subsections 2.2.1, 2.2.2 and 2.2.3 below, we come back to the definitions of these three key assumptions.

2.2.1 Global Neutrality Assumption

The first important assumption is the *neutrality* assumption which describes the apparent charge neutrality of a plasma overall. This property is widely used when looking at plasmas. It is sometimes qualified as quasi-neutrality because, at smaller scales, the positive and negative charges may give rise to charged regions and electric fields. In the present context, for each equilibrium profile M, this means to fix the constant $\rho_i := \rho_i(M) = ||M||_{L^1}$ in such a way that

$$\rho(\mathbf{f}^s) = \rho_i. \text{ (Neutral background).} \tag{2.9}$$

2.2.2 Coldness Assumption

The next assumption that is involved in [7] is the notion of *coldness*. After rescaling, this condition limits particle momentum to be concentrated near the origin, that is for $|\xi| \sim \mathcal{O}(\epsilon)$. This may be achieved by looking at equilibria such as

$$\mathbf{f}^s(t, x, \xi) = M_\epsilon(|\xi|) := \epsilon^{-2} M(\epsilon^{-1}|\xi|),$$

where $M \in C_c^1(\mathbb{R}^3)$ is adjusted in such a way that (for some constant R_M)

$$\operatorname{supp}(M) \subset \{\xi \in \mathbb{R}^3 \mid |\xi| \leq R_M\}. \tag{2.10}$$

Next, we seek perturbed solutions having the form

$$\mathbf{f}(t, x, \xi) = \epsilon^{-2} [M(\epsilon^{-1}|\xi|) + \bar{f}(t, x, \epsilon^{-1}\xi)]. \tag{2.11}$$

Recall that a sufficient condition for local existence of smooth solutions of (2.1)-(2.3) to exist on $[0, T]$ with $0 < T$ is that $\mathbf{f}(t, x, \cdot)$ has compact support in the variable ξ for $t \in [0, T]$. With this in mind, in [7], local C^1-solutions satisfying (for some constants R_x and R_ξ)

$$\operatorname{supp}(\bar{f}(t, \cdot, \cdot)) \subset \{(x, \xi) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid |x| \leq R_x, \text{ and } |\xi| \leq R_\xi\} \tag{2.12}$$

were constructed on $[0, T]$. The restriction (2.11) meant that for $|\xi| \geq \epsilon \max\{R_\xi, R_M\}$, there was $\mathbf{f}(t, x, \xi) = 0$.

2.2.3 Dilute Assumption

The last assumption given in [7] is the *dilute* assumption which is given by the condition $\rho_i = \mathcal{O}(\epsilon)$. This may be viewed as a direct consequence of (2.9) and (2.10) since we have

$$\rho_i = ||\epsilon^{-2} M(\epsilon^{-1}|\cdot|)||_{L^1} = \epsilon ||M||_{L^1} = \mathcal{O}(\epsilon). \tag{2.13}$$

2.3 The Actual Framework

The global neutrality condition is physically relevant at the scales under consideration. It is therefore unavoidable, and we keep it. By contrast, the cold assumption is not suitable in the case of many applications like fusion devices. Here we remove this condition so that for most of the plasma (in the sense of L^1) we have $|\xi| \sim \mathcal{O}(1)$. We typically consider the two following distinct equilibrium profiles

$$(\mathbf{f}^s, \mathbf{E}^s, \mathbf{B}^s) = (M(|\xi|), 0, 0), \quad \rho_i = ||M||_{L^1} \quad \text{(Neutral, Hot and Dense)}, \tag{2.14}$$

$$(\mathbf{f}^s, \mathbf{E}^s, \mathbf{B}^s) = (\epsilon M(|\xi|), 0, 0), \quad \rho_i = \epsilon ||M||_{L^1} \quad \text{(Neutral, Hot and Dilute)}. \tag{2.15}$$

We abuse notation and write $M_\epsilon(|\xi|)$ to denote either $\epsilon M(|\xi|)$ or $M(|\xi|)$. These distinctions will be made clear when distinguishing results for the dilute or dense settings.

2.3.1 The HMRVM System

We consider a perturbation of the equilibrium solution as indicated below:

$$\mathbf{f}(t,x,\xi) := M_\epsilon(|\xi|) + \epsilon\bar{f}(t,x,\xi), \quad \mathbf{E}(t,x) := \epsilon E(t,x), \quad \mathbf{B}(t,x) := \epsilon B(t,x). \tag{2.16}$$

For a complete mathematical explanation of this particular scaling see Section 5.5. Let

$$U^{in} := (\bar{f}^{in}, E^{in}, B^{in}) \in C_c^1(\mathbb{R}^3 \times \mathbb{R}^3) \times C_c^2(\mathbb{R}^3) \times C_c^2(\mathbb{R}^3) \tag{2.17}$$

be some initial functions. Consider the system (2.1)-(2.3) with initial data given by

$$\mathbf{f}|_{t=0} = \mathbf{f}^{in} := M_\epsilon(|\xi|) + \epsilon\bar{f}^{in}(x,\xi), \tag{2.18}$$

$$\mathbf{E}|_{t=0} = \mathbf{E}^{in} := \epsilon E^{in}(x), \tag{2.19}$$

$$\mathbf{B}|_{t=0} = \mathbf{B}^{in} := \epsilon B^{in}(x). \tag{2.20}$$

Substituting the expression (2.16) into the system (2.1)-(2.3) leads to the new system on the perturbation $\bar{f}$ given by

$$\begin{aligned} \partial_t\bar{f} + [v(\xi)\cdot\nabla_x]\bar{f} & -\epsilon^{-1}[v(\xi)\times\mathbf{B}_e(x)]\cdot\nabla_\xi\bar{f} \\ & -\epsilon[E + v(\xi)\times B]\cdot\nabla_\xi\bar{f} = M_\epsilon'(|\xi|)\frac{E\cdot\xi}{|\xi|}, \end{aligned} \tag{2.21}$$

together with

$$(\bar{f}, E, B)(0,\cdot) = (\bar{f}^{in}, E^{in}, B^{in})(\cdot).$$

From the global neutrality condition, Maxwell's equations become

$$\nabla_x\cdot E = -\rho(\bar{f}) \quad ; \quad \partial_t E - \nabla\times B = J(\bar{f}), \tag{2.22}$$

$$\nabla_x\cdot B = 0 \quad ; \quad \partial_t B + \nabla_x\times E = 0, \tag{2.23}$$

with the following current and charge densities

$$\rho(\bar{f}) := \int \bar{f}(t,x,\xi)d\xi, \tag{2.24}$$

$$J(\bar{f}) := \int v(\xi)\bar{f}(t,x,\xi)d\xi. \tag{2.25}$$

Denote the system (2.21)-$\cdots$-(2.25) as the Hot, Magnetized, Relativistic Vlasov Maxwell system (HMRVM). This is the main focus of the thesis.

2.3.2 Conditions on the Initial Data

Select $(R_x^0, R_\xi^0) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$, and define

$$R^0 := \max\{R_x^0, R_\xi^0\}\,.$$

Impose

$$\operatorname{supp}(\bar{f}^{in}) \subset \{(x,\xi) \mid |x| \le R_x^0 \text{ and } |\xi| \le R_\xi^0\}. \tag{2.26}$$

Remark that if $\bar{f}^{in}$ is compactly supported in x, then by the relation (2.18) implies $\mathbf{f}^{in}$ is not, since for large enough $|x| > R_x^0$ we must have $\mathbf{f}^{in}(x,\xi) = M_\epsilon(|\xi|)$. To guarantee the neutrality at time $t = 0$, we have to adjust $\bar{f}^{in}$ in such a way that

$$\forall x \in \mathbb{R}^3, \qquad \int \bar{f}^{in}(x,\xi)\, d\xi = 0. \tag{2.27}$$

We also pay special attention to initial data that are prepared in the following sense.

Definition 1. *Initial data, $\bar{f}^{in} \equiv \bar{f}^{in}_\epsilon$, is said to be* ***prepared*** *if there exists some $C > 0$ such that for all $\epsilon \in (0,1]$*

$$||[v(\xi) \times \mathbf{B}_e(x)] \cdot \nabla_\xi \bar{f}^{in}_\epsilon||_{L^\infty_{x,\xi}} \leq C\, \epsilon. \tag{2.28}$$

Remark that this condition arises naturally from equation (2.21) which yields

$$\partial_t \bar{f}|_{t=0} = \epsilon^{-1}\, [v(\xi) \times \mathbf{B}_e(x)] \cdot \nabla_\xi \bar{f}^{in} + \mathcal{O}(1). \tag{2.29}$$

Thus, in the absence of (2.28), the time derivative of $\bar{f}$ at time $t = 0$ is large. This means that (2.28) is a necessary condition for uniform estimates in the Lipschitz norm.
Given $\bar{f}^{in}$ as above, we have to assume that the initial data E^{in} and B^{in} satisfy at time $t = 0$ the necessary compatibility conditions:

$$\nabla_x \cdot E^{in} = \rho(\bar{f}^{in}), \quad \nabla_x \cdot B^{in} = 0. \tag{2.30}$$

2.3.3 Compact Support Assumption

Finally, we work under the classic Glassey-Strauss momentum condition. As in [7], this means to look at a time interval $[0,T]$, with T below the maximal lifetime of $(\bar{f}, E, B)(t,\cdot)$ solving the HMRVM system, such that

$$\forall t \in [0,T],\ \operatorname{supp}(\bar{f}(t,\cdot)) \subset \{(x,\xi) \mid |x| \leq R_x^T,\ \text{and } |\xi| \leq R_\xi^T\} \tag{2.31}$$

for some $R_x^T > 0$ and $R_\xi^T > 0$. It is easy to show (in the relativistic context) that we may take $R_x^T = R_x^0 + T$. But there is no such evident control concerning R_ξ^T. In particular, we would like to show a uniform (in ϵ) positive lower bound for T, as well as a uniform (in ϵ) upper bound for R_ξ^T. This is what has been done in [7]. Define

$$R^T := \max\{R_x^T, R_\xi^T\},$$

and consider the set

$$\mathcal{A}_T := \{(y,\eta) \mid |y| \leq R_x^0 + T,\ \text{and } |\eta| \leq R_\xi^T\}. \tag{2.32}$$

Note that, in the case of spherical symmetry, the very recent result [17] states that the compact support assumption (2.31) is not necessary for global well possedness of the RVM system. The assumption of spherical symmetry is a way to reduce the dimension, and this is in general not compatible with the magnetized context. Here, we work in dimension three, and with an external magnetic field $\mathbf{B}_e(\cdot)$ which may be non constant and not invariant under rotations.

2.4 Main Result and Possible Extensions

In Subsection 2.4.1, we state our main result. Then, in Subsection 2.4.2, we present some associated perspectives.

2.4.1 Well Posedness of HMRVM for Dilute Data

In the new context of neutral, hot and dilute plasmas, we can show that solutions having a uniform lifespan and satisfying sup-norm or Lipschitz estimates do persist.

Theorem 1. *Let $(\bar{f}^{in}, E^{in}, B^{in})$ come from the space described in (2.17) and satisfy (2.26), (2.27), and (2.30). Assume the dilute condition*

$$||M'_\epsilon||_{L^\infty} \leq \epsilon C. \tag{2.33}$$

Then, there exists $T > 0$ and $\epsilon_0 \in (0, 1]$ such that for all $\epsilon \in (0, \epsilon_0]$, there is a unique solution $(f_\epsilon, E_\epsilon, B_\epsilon) \in C^0([0, T]; L^\infty_{x,\xi})$ to the HMRVM system (2.21)-(2.25). This solution is subject to (2.31) for some $R^T_x > 0$ and $R^T_\xi > 0$, and it is such that, for all $t \in [0, T]$:

$$||(f_\epsilon, \epsilon E_\epsilon, \epsilon B_\epsilon)(t, \cdot, \cdot)||_{L^\infty_{x,\xi}(\mathcal{A}_T)} \leq C(T, R^T, ||\mathbf{B}_e||_{W^{1,\infty}}, ||(\bar{f}^{in}, E^{in}, B^{in})||_{L^\infty_{x,\xi}(\mathcal{A}_T)}) < \infty. \tag{2.34}$$

Furthermore, if $\bar{f}^{in}$ is prepared in the sense of Definition 1, then

$$||(f_\epsilon, E_\epsilon, B_\epsilon)(t, \cdot, \cdot)||_{L^\infty_{x,\xi}(\mathcal{A}_T)} \leq C(T, R^T, ||\mathbf{B}_e||_{W^{1,\infty}}, ||(\bar{f}^{in}, E^{in}, B^{in})||_{L^\infty_{x,\xi}(\mathcal{A}_T)}) < \infty. \tag{2.35}$$

In particular, the dilute equilibrium given by (2.15) is a stable solution to the HMRVM system under perturbed initial data. Observe that Example 1 in Section 4 gives a situation showing that the prepared data assumption is necessary to ensure the uniform C^1-estimate stated in (2.35).

To prove Theorem 1, we consider a linear version of (2.21), be setting the order ϵ non-linear term $\epsilon[E + v(\xi) \times B] \cdot \nabla_\xi \bar{f} = 0$. It can then be argued that, for prepared data, this linear model serves as a good approximation (of order ϵ) to the non-linear system in the variable $\bar{f}$ only. This linear approximation is then used to overcome the difficulty that will be described in the next chapter.

2.4.2 Prospects for Progress

The goal of the thesis is also to advance in the direction of the following open problem.

Conjecture 1. *Let $(\bar{f}^{in}, E^{in}, B^{in})$ come from the space described in (2.17) and satisfy (2.26), (2.27), and (2.30). Assume that*

$$||M'_\epsilon||_{L^\infty} \leq C \tag{2.36}$$

Then, there exists $T > 0$ and $\epsilon_0 \in (0, 1]$ such that for all $\epsilon \in (0, \epsilon_0]$, there is a unique solution $(f_\epsilon, E_\epsilon, B_\epsilon) \in C^0([0, T]; L^\infty_{x,\xi})$ to the HMRVM system (2.21)-(2.25). This solution is subject to (2.31) for some $R^T_x > 0$ and $R^T_\xi > 0$, and it is such that, for all $t \in [0, T]$:

$$||(f_\epsilon, E_\epsilon, B_\epsilon)(t, \cdot, \cdot)||_{L^\infty_{x,\xi}(\mathcal{A}_T)} \leq C(T, R^T, ||\mathbf{B}_e||_{W^{1,\infty}}, ||(\bar{f}^{in}, E^{in}, B^{in})||_{L^\infty_{x,\xi}(\mathcal{A}_T)}) < \infty, \tag{2.37}$$

and Lipschitz norm,

$$\begin{aligned}||\epsilon\partial_t(f_\epsilon, E_\epsilon, B_\epsilon)(t,\cdot,\cdot)||_{L^\infty_{x,\xi}(\mathcal{A}_T)} + ||\epsilon\nabla_x(f_\epsilon, E_\epsilon, B_\epsilon)(t,\cdot,\cdot)||_{L^\infty_{x,\xi}(\mathcal{A}_T)} + ||\epsilon\nabla_\xi f_\epsilon(t,\cdot,\cdot)||_{L^\infty_{x,\xi}(\mathcal{A}_T)} \\ \leq C(T, R^T, ||\mathbf{B}_e||_{W^{1,\infty}}, ||(\bar{f}^{in}, E^{in}, B^{in})||_{W^{1\infty}_{x,\xi}(\mathcal{A}_T)}) < \infty.\end{aligned} \tag{2.38}$$

Furthermore, if $\bar{f}^{in}$ *is prepared in the sense of Definition 1, then*

$$\begin{aligned}|\partial_t(f_\epsilon, E_\epsilon, B_\epsilon)(t,\cdot,\cdot)| + |\nabla_x(f_\epsilon, E_\epsilon, B_\epsilon)(t,x,\xi)| + |\nabla_\xi f_\epsilon(t,x,\xi)| \\ \leq C(T, R^T, ||\mathbf{B}_e||_{W^{1,\infty}}, ||(\bar{f}^{in}, E^{in}, B^{in})||_{W^{1\infty}_{x,\xi}(\mathcal{A}_T)}) < \infty.\end{aligned} \tag{2.39}$$

The conjecture above can be interpreted as saying the stationary solution $\mathbf{U}^s = (M_\epsilon(|\xi|), 0, 0)$ is a stable solution to (2.1)-(2.3) in the sup-norm.

Chapter 3

Fundamental Solutions

This Chapter is devoted to constructing solutions of the HMRVM system. We start Section 3.1 by recalling a classical energy result of the Vlasov Maxwell system. The function $\mathbf{f}$ is a positive density, so the associated energy is positive. This can be used to achieve weak solutions to the RVM system such as in [8] . However, the perturbation $\bar{f}$ can be negative, so the the associated energy can also be negative. Next, Section 3.2 derives representation formulas for the electromagnetic fields (E, B). Unlike in [7], we take a more direct approach by avoiding the usage of vector and scalar potentials. Instead, we show directly that the fields (E, B) solve a linear wave equation. The solutions to the fields (E, B) are then represented using the fundamental solutions of the wave equation and Kirchhoff's formula. In doing so, this introduces a source term depending on the derivatives $\partial_t \bar{f}$ and $\nabla_x \bar{f}$. Section 3.3 then uses a classical division lemma of [4] to pass the time and spatial derivatives on this source term to the transport operator $\partial_t + v(\xi) \cdot \nabla_x$. This allows us to substitute the Vlasov equation. Since the current and charge densities posses an integration in the momentum, we can then integrate by parts to remove these derivatives from $\bar{f}$ and estimate the fields (E, B) in term of $\bar{f}$. This allows us to arrive at a similar expression presented in [4] now in the presence of an applied magnetic field. However, unlike [7], we are not able to uniformly estimate the fields with respect to ϵ. This is due to the fact, that we no longer have a cold plasma, and so cannot recover the factor of ϵ using the small momentum assumption. Section 3.4 states precisely this difficulty and why the methods of [7] fail in the hot plasma regime. Next in Section 3.5, we solve the Vlasov equation using the method of characteristics and Duhamel's principle. Then with an addition weight of ϵ on the fields, we prove the estimate (2.34) in Theorem 1. We finally conclude with Section 3.6 which reformulates the Vlasov equation in new canonical coordinates. This has the advantage demonstrating the penalization term $v(\xi) \times \epsilon^{-1} \mathbf{B}_e(x)$. The remainder of Theorem 1 is addressed in Chapter 4.

3.1 Classical Energy

The following is a classic energy result for the Vlasov Maxwell system (2.1)-(2.3).

Lemma 1. *Suppose that* $(\mathbf{E}, \mathbf{B}) \in [L^2 \cap C^1([0,T] \times \mathbb{R}^3; \mathbb{R}^3)]^2$ *and* $\mathbf{f} \in C^1([0,T] \times \mathbb{R}^3 \times \mathbb{R}^3; \mathbb{R})$ *are compactly supported in* x *and* ξ *and are solution to (2.1)-(2.3) with initial data satisfying (2.30). Then the energy*

$$\mathcal{E}(t) := \int \int \langle \xi \rangle \, \mathbf{f} d\xi dx + \frac{1}{2} \int |\mathbf{E}|^2 + |\mathbf{B}|^2 dx \tag{3.1}$$

is constant in time, meaning that

$$\forall t \in [0,T], \qquad \mathcal{E}(t) = \mathcal{E}(0). \tag{3.2}$$

Assume that the perturbed solution $(\bar{f}, E, B)$ *of (2.21)-(2.25) is compactly supported in* x *and* ξ*. Then the modified energy*

$$\bar{\mathcal{E}}(t) = \int \int \langle \xi \rangle \, \bar{f} dx d\xi + \frac{1}{2} \int \epsilon |E|^2 + \epsilon |B|^2 dx \tag{3.3}$$

is also constant in time, meaning that

$$\forall t \in [0,T], \qquad \bar{\mathcal{E}}(t) = \bar{\mathcal{E}}(0). \tag{3.4}$$

Furthermore if

$$|\int\int \langle \xi \rangle \bar{f}^{in} dx d\xi| \leq \epsilon C, \tag{3.5}$$

then the perturbed energy is small

$$|\bar{\mathcal{E}}(t)| \leq \epsilon C + \epsilon ||(E^{in}, B^{in})||_{L^2}^2 \tag{3.6}$$

As already mentioned, both $\mathbf{f}$ and $\bar{f}$ cannot be compactly supported in x simultaneously, therefore it will not suffice to simply substitute $\mathbf{f} = \bar{f} + M_\epsilon(|\xi|)$ into (3.1) to obtain (3.3) as this would introduce a divergence integral in x. Because of this we treat each case separately. Furthermore, the perturbed energy $\bar{\mathcal{E}}$ can be negative since, unlike $\mathbf{f}$ and as a consequence of (2.27), the expression $\bar{f}$ cannot have a specific sign.

Proof. First we differentiate the integrated Vlasov Equation

$$\begin{aligned}\frac{d}{dt}\int\int \langle \xi \rangle \mathbf{f} d\xi dx = &\int\int -\nabla_x \cdot (\xi \mathbf{f}) + \nabla_\xi \cdot ([\xi \times (\mathbf{B} + \epsilon^{-1}\mathbf{B}_e)]\mathbf{f}) d\xi dx \\ &+ \int\int \langle \xi \rangle \mathbf{E} \cdot \nabla_\xi \mathbf{f} d\xi dx.\end{aligned}$$

Integrating by parts gives

$$\frac{d}{dt}\int\int \langle \xi \rangle \mathbf{f} d\xi dx = -\int \mathbf{E} \cdot \int v(\xi)\mathbf{f} d\xi dx = -\int \mathbf{E} \cdot \mathbf{J}(\mathbf{f}) dx. \tag{3.7}$$

Taking the dot product of Maxwell's equations with $\mathbf{E}$ and $\mathbf{B}$ we find

$$\frac{1}{2}\frac{d}{dt}|\mathbf{E}|^2 - \mathbf{E} \cdot \nabla_x \times \mathbf{B} = \mathbf{E} \cdot \mathbf{J}(f), \tag{3.8}$$

$$\frac{1}{2}\frac{d}{dt}|\mathbf{B}|^2 + \mathbf{B} \cdot \nabla_x \times \mathbf{E} = 0. \tag{3.9}$$

Then using the identity

$$\mathbf{v} \cdot (\nabla_x \times \mathbf{w}) = -\nabla_x \cdot (\mathbf{v} \times \mathbf{w}) + \mathbf{w} \cdot (\nabla_x \times \mathbf{v}), \tag{3.10}$$

and adding (3.8)-(3.9) gives

$$\frac{1}{2}\frac{d}{dt}(|\mathbf{E}|^2 + |\mathbf{B}|^2) + \nabla_x \cdot (\mathbf{E} \times \mathbf{B}) = \mathbf{E} \cdot \mathbf{J}(\mathbf{f}). \tag{3.11}$$

Then integrating with respect to x and adding to (3.7) we obtain

$$\frac{d}{dt}\mathcal{E}(t) = 0.$$

This proves the first part. Next, compute

$$\begin{aligned}\frac{d}{dt}\int\int \langle\xi\rangle\,\bar f dx d\xi &= \epsilon\int\int \langle\xi\rangle\, E\cdot\nabla_\xi \bar f d\xi dx + \int\int \langle\xi\rangle\, M'(|\xi|)\frac{\xi}{|\xi|}\cdot E dx d\xi\\ &= -\epsilon\int E\cdot\Big(\int \bar f\,\nabla_\xi \langle\xi\rangle\, d\xi\Big)dx + \int E\cdot\Big(\int \langle\xi\rangle\, M'(|\xi|)\frac{\xi}{|\xi|}d\xi\Big)dx.\end{aligned}$$

Since $\nabla_\xi \langle\xi\rangle = v(\xi)$ and because the integral of an odd function is zero, there remains

$$\frac{d}{dt}\int\int \langle\xi\rangle\,\bar f dx d\xi = -\epsilon\int E\cdot J(\bar f)dx, \tag{3.12}$$

and similarly as before

$$\frac{1}{2}\frac{d}{dt}\int |E|^2+|B|^2dx = \int E\cdot J(\bar f)dx. \tag{3.13}$$

Adding (3.12) to (3.13) with an additional factor of ϵ it then follows that

$$\frac{d}{dt}\Big(\int\int \langle\xi\rangle\,\bar f dx d\xi + \frac{1}{2}\int \epsilon|E|^2+\epsilon|B|^2dx\Big) = 0. \tag{3.14}$$

□

3.2 Fundamental Solution of the Wave Equation

One approach to obtain representation formulas of the electromagnetic fields (E, B) is though a wave equation. We define the $3D$ d'Alembertian as follows.

$$\Box_{t,x} := \partial_t^2 - \Delta_x = \partial_t^2 - \sum_{i=1}^{3}\partial_{x_i}^2$$

Then the following lemma gives the precise relation between (E, B) and the operator $\Box_{t,x}$.

Lemma 2. *Let $\bar f \in C^1([0,T]\times\mathbb{R}^6;\mathbb{R})$ be a solution of (2.21). Then the self-consistent electromagnetic field (E, B) is in $C^0([0,T]\times\mathbb{R}^3;\mathbb{R}^3)$ and it solves*

$$\begin{aligned}\Box_{t,x}E &= \int v(\xi)\partial_t\bar f + \nabla_x\bar f d\xi,\\ \Box_{t,x}B &= -\int \nabla_x\times(v(\xi)\bar f)d\xi.\end{aligned} \tag{3.15}$$

Proof. Consider differentiating (2.22)-(2.23) with respect to t. This gives

$$\begin{aligned}\partial_t^2E - \nabla_x\times\partial_tB &= \int v(\xi)\partial_t\bar f d\xi,\\ \partial_t^2B + \nabla_x\times\partial_tE &= 0.\end{aligned} \tag{3.16}$$

We substitute $(\partial_tE, \partial_tB)$ from Maxwell's equations into (3.16) and use the vector identity

$$\nabla\times(\nabla\times A) = \nabla(\nabla\cdot A) - \Delta A,$$

to obtain

$$\begin{aligned}\int v(\xi)\bar{f}d\xi &= \partial_t^2 E - \nabla_x \times (-\nabla_x \times E) = \partial_t^2 E - \Delta_x E + \nabla_x(\nabla_x \cdot E)\\ &= \partial_t^2 E - \Delta_x E - \int \nabla_x \bar{f} d\xi.\end{aligned}$$

Similarly,

$$\begin{aligned}0 &= \partial_t^2 B + \nabla_x \times \big(\nabla_x \times B + \int v(\xi)\bar{f}d\xi\big)\\ &= \partial_t^2 B - \Delta_x B + \nabla_x(\nabla_x \cdot B) + \int \nabla_x \times (v(\xi)\bar{f})d\xi\\ &= \partial_t^2 B - \Delta_x B + \int \nabla_x \times (v(\xi)\bar{f})d\xi.\end{aligned}$$

This is the desired result. □

Next, we introduce the fundamental solution of the wave equation. This will allow us to write a solution of (3.15) in terms of the derivatives $\partial_t \bar{f}$ and $\nabla_x \bar{f}$. We first define a space of distributions and well known functional transforms.

Definition 2. *We define the space $\mathcal{D}'(\mathbb{R}^n;\mathbb{R})$ to be the set of continuous linear functionals on $C_c^\infty(\mathbb{R}^n;\mathbb{R})$. For $\phi \in C_c^\infty(\mathbb{R}^n;\mathbb{R})$, and $S \in \mathcal{D}'(\mathbb{R}^n;\mathbb{R})$ we use the notation*

$$S(\phi) = \langle S, \phi \rangle = \int_{\mathbb{R}^n} S(x)\phi(x)dx, \tag{3.17}$$

where the rightmost term is imprecise, but will be used for formal computations.

One important example of such a distribution is the classical Dirac mass.

Definition 3. *Define the distribution $\delta \in D'(\mathbb{R};\mathbb{R})$ such that for all $\phi \in C_c^\infty(\mathbb{R};\mathbb{R})$*

$$\langle \delta, \phi \rangle = \phi(0). \tag{3.18}$$

Next we introduce the following well known transform's and notation we will consider.

Definition 4. *We define the Laplace transform for a function $\phi \in L^1(\mathbb{R};\mathbb{R})$ as*

$$\mathcal{L}(\phi)(s) := \lim_{\alpha \to 0^-} \int_\alpha^\infty e^{-st}\phi(t)dt \tag{3.19}$$

and the Fourier transform of $\psi \in L^1(\mathbb{R}^n;\mathbb{R})$ as

$$\begin{aligned}\hat{\psi}(k) &:= \mathcal{F}_x(\psi)(k) := \frac{1}{(2\pi)^{n/2}} \lim_{r\to\infty} \int_{|x|\le r} e^{-ik\cdot x}\psi(x)dx,\\ \psi(x) &= \mathcal{F}_x^{-1}(\hat{\psi})(x) := \frac{1}{(2\pi)^{n/2}} \lim_{r\to\infty} \int_{|k|\le r} e^{ik\cdot x}\hat{\psi}(k)dk.\end{aligned} \tag{3.20}$$

We will not concern ourselves here with technical details involved in distribution theory, but remark we have the formal computations for the 1 dimensional Dirac mass which is valid from our definitions of $\mathcal{F}_x$ and $\mathcal{L}$

$$\begin{aligned}\mathcal{F}_x(\delta) &= \frac{1}{\sqrt{2\pi}} \lim_{r\to\infty} \int_{|x|\le r} \delta(x) e^{-ikx} dx = \frac{1}{\sqrt{2\pi}},\\ \mathcal{L}(\delta) &= \lim_{\alpha\to 0^-} \int_{\alpha}^{\infty} e^{-st}\delta(t)dt = 1.\end{aligned} \tag{3.21}$$

As a consequence of (3.21) we have

$$\delta(x) = \mathcal{F}_x^{-1}(\frac{1}{\sqrt{2\pi}}) = \frac{1}{2\pi} \lim_{|r|\to\infty} \int_{|k|\le r} e^{ikx} dk. \tag{3.22}$$

In the sense that, for $\phi \in L^1(\mathbb{R};\mathbb{R})$ we define the Fourier transform of a distribution as

$$\int \frac{1}{\sqrt{2\pi}} \phi(x)dx = \langle \mathcal{F}_x(\delta), \phi\rangle := \langle \delta, \mathcal{F}(\phi)\rangle = \hat{\phi}(0) \tag{3.23}$$

and

$$\hat{\phi}(0) = \left\langle \delta, \hat{\phi} \right\rangle = \left\langle \mathcal{F}_x^{-1}(\frac{1}{\sqrt{2\pi}}), \hat{\phi} \right\rangle := \left\langle \frac{1}{\sqrt{2\pi}}, \phi \right\rangle = \frac{1}{\sqrt{2\pi}} \int \phi(x)dx \tag{3.24}$$

We are now are now equipped to obtain representation formulas for the wave equation using fundamental solution.

Lemma 3. *Consider the Cauchy problem,*

$$\Box_{t,x} u = g(t,x),\ u(0,x) = u_0(x),\ \partial_t u|_{t=0} = u_1(x),\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \tag{3.25}$$

with $g \in C_c^1(\mathbb{R}_+ \times \mathbb{R};\mathbb{R}^3)$ and $u_0, u_1 \in C^1(\mathbb{R}^3;\mathbb{R}^3)$. Then for $t > 0$, $u(t,\cdot)$ has the representation formula

$$u(t,x) = \frac{1}{4\pi t^2} \int_{|y-x|=t} \Bigg[t u_1(y) + u_0(y) + [(x-y)\cdot\nabla_y] u_0 \Bigg] dS(y) + Y *_{t,x} \mathbb{1}_{t>0} g(t,x). \tag{3.26}$$

Here $*_{t,x}$ denotes the convolution in t and x and $Y \in \mathcal{D}'(\mathbb{R}^4;\mathbb{R})$, the space of distributions on $C_c^\infty(\mathbb{R}^4;\mathbb{R})$, is the fundamental solution of

$$\Box_{t,x} Y = \delta(t,x), \quad Y|_{t=0} = 0, \quad \partial_t Y(0,x) = 0. \tag{3.27}$$

Given by the distributional representation

$$Y(t,x) := \frac{\delta(t-|x|)}{4\pi t} \mathbb{1}_{t>0}. \tag{3.28}$$

Proof. First consider the problem (3.27). Let $\hat{Y}$ denote the spatial Fourier transform of Y. Then it follows that $\hat{Y}$ solves the ODE

$$\partial_t^2 \hat{Y} + |k|^2 \hat{Y} = \delta(t) \frac{1}{(2\pi)^{3/2}}, \quad \hat{Y}|_{t=0} = 0, \quad \partial_t \hat{Y}|_{t=0} = 0.$$

Then denote $\hat{Y}_L$ as the Laplace transform of $\hat{Y}$. Remark that we are only considering forward solutions of $Y(t,x)$, so that $t \geq 0$, so we compute

$$s^2\hat{Y}_L + |k|^2\hat{Y}_L = \int_0^\infty e^{-st}\frac{\delta(t)}{(2\pi)^{3/2}}dt = \frac{1}{(2\pi)^{3/2}}.$$

Solving the algebraic problem yields

$$\hat{Y}_L = \frac{1}{|k|(2\pi)^{3/2}}\bigg(\frac{|k|}{s^2+|k|^2}\bigg).$$

Thus the inverse Laplace transform of $\hat{Y}_L$ is given by

$$\hat{Y} = \frac{1}{|k|(2\pi)^{3/2}}\sin(|k|t)\mathbb{1}_{t>0}.$$

Next we compute the inverse Fourier transform by converting to polar coordinates with $k = r\omega$, where $r \in [0,\infty)$ and $\omega := \frac{k}{|k|} \in \mathbb{S}^2$:

$$\begin{aligned} Y(t,x) &= \frac{1}{(2\pi)^3}\mathbb{1}_{t>0}\int_{\mathbb{R}^3} e^{ik\cdot x}\frac{\sin(t|k|)}{|k|}dk \\ &= \frac{1}{(2\pi)^3}\mathbb{1}_{t>0}\int_0^\infty \sin(rt)r\int_{\mathbb{S}^2} e^{irx\cdot\omega}d\omega dr. \end{aligned}$$

Consider now the well known identity

$$\frac{1}{4\pi}\int_{\mathbb{S}^2} e^{ir\omega\cdot x}d\omega = \frac{\sin(r|x|)}{r|x|}, \tag{3.29}$$

along with the representation (3.22). Thus Y is computed as follows (remark that these integrals should be interpreted in the sence of (3.23) and (3.24))

$$\begin{aligned} Y(t,x) &= \frac{4\pi}{(2\pi)^3|x|}\mathbb{1}_{t>0}\int_0^\infty \sin(rt)\sin(|x|r)dr \\ &= \frac{1}{(2\pi)^2|x|}\mathbb{1}_{t>0}\int_{-\infty}^\infty \sin(rt)\sin(|x|r)dr \\ &= \frac{1}{(2\pi)^2|x|}\mathbb{1}_{t>0}\frac{1}{(2i)^2}\int_{-\infty}^\infty (e^{irt}-e^{-irt})(e^{i|x|r}-e^{-i|x|r})dr \\ &= \frac{-1}{16\pi^2|x|}\mathbb{1}_{t>0}(2\pi)\int_{-\infty}^\infty \bigg(\frac{e^{irt}e^{ir|x|}}{2\pi} - \frac{e^{irt}e^{-ir|x|}}{2\pi} - \frac{e^{-irt}e^{ir|x|}}{2\pi} + \frac{e^{-irt}e^{-ir|x|}}{2\pi}\bigg)dr \\ &= \frac{-1}{16\pi^2|x|}\mathbb{1}_{t>0}(2\pi)\bigg(\delta(t+|x|)-\delta(t-|x|)-\delta(-t+|x|)+\delta(-t-|x|)\bigg) \\ &= \frac{1}{16\pi^2|x|}\mathbb{1}_{t>0}(2\pi)(2\delta(t-|x|)) = \frac{1}{4\pi|x|}\mathbb{1}_{t>0}\delta(t-|x|), \end{aligned}$$

where $\delta(-t-|x|) = 0$ in the sense of distributions for $t > 0$. Furthermore since $\operatorname{supp}(\delta(|x|-t) = \{(t,x)\ ||x| = t\}$, we have $\frac{\delta(t-|x|)}{|x|} = \frac{\delta(t-|x|)}{t}$ in the sense of distributions. Next, consider the Cauchy problem

$$\Box_{t,x}u_p = g(t,x), \quad u_p(0,x) = 0, \quad \partial_t u_p(0,x) = 0. \tag{3.30}$$

Then, if $*_{t,x}$ denotes the convolution in x and t, we have for all $t > 0$, and $u_p(t, \cdot)$ defined by

$$u_p(t, x) := (Y *_{t,x} g\mathbb{1}_{t>0})(t, x), \tag{3.31}$$

solves (3.30) since by construction $\Box_{t,x} Y = \delta(t, x)$. Next, let u_c solve

$$\Box_{t,x} u_c = 0, \quad u_c(0, x) = u_0(x), \quad \partial_t u_c(0, x) = u_1(x), \tag{3.32}$$

so that $u(t, x) = u_p(t, x) + u_c(t, x)$. Finally u_c is a standard result given by Kirchhoff's formula (22) in the graduate text [9] giving the desired result. □

Using Lemmas 2 and 3 we obtain the representation formula for (E, B).

Corollary 1. *Let $\bar{f} \in C^1([0, T] \times \mathbb{R}^3 \times \mathbb{R}^3; \mathbb{R})$ be a solution of (2.21), then $(E, B)(t, \cdot)$ solving (2.22)-(2.23) has the solution*

$$E(t, x) = K_1(E^{in}) + Y *_{t,x} \mathbb{1}_{t>0} \int [v(\xi)\partial_t \bar{f} + \nabla_x \bar{f}] d\xi, \tag{3.33}$$

$$B(t, x) = K_2(B^{in}) - Y *_{t,x} \mathbb{1}_{t>0} \int [\nabla_x \times (v(\xi)\bar{f})] d\xi, \tag{3.34}$$

where K_i depends on the initial data

$$K_1(E^{in})(t, x) := \frac{1}{4\pi t^2} \int_{|y-x|=t} \big[t\partial_t E|_{t=0}(y) + E^{in}(y) + [(y - x) \cdot \nabla_y] E^{in}(y)\big] dS(y), \tag{3.35}$$

$$K_2(B^{in})(t, x) := \frac{1}{4\pi t^2} \int_{|y-x|=t} \big[t\partial_t B|_{t=0}(y) + B^{in}(y) + [(y - x) \cdot \nabla_y] B^{in}\big] dS(y), \tag{3.36}$$

with

$$\partial_t E|_{t=0} = J(\bar{f}^{in}) + \nabla \times B^{in}, \quad \partial_t B|_{t=0} = -\nabla_x \times E^{in}. \tag{3.37}$$

3.3 Transfer of Derivatives

The idea is to pass the derivatives $\partial_t \bar{f}$ and $\nabla_x \bar{f}$ in (3.33) to a derivative with respect to ξ and integrate by parts in order to apply a Gronwall lemma to estimate the fields (E, B) in terms of $\bar{f}$ only. This is done by using a corollary of the *division lemma* from [4] to obtain a transport operator on $\bar{f}$ for which the Vlasov equation can be substituted in (3.33). First define the spaces of smooth homogeneous functions $\mathcal{M}_k$ on $\mathbb{R}^n - \{0\}$,

$$\mathcal{M}_k(\mathbb{R}^n - \{0\}) := \Big\{\phi \in C^\infty(\mathbb{R}^n - \{0\}) \mid \phi(\alpha x) = \alpha^k \phi(x), \ \forall \alpha > 0\Big\}. \tag{3.38}$$

Then set $\mathfrak{M}_k(\mathbb{R}^n - \{0\})$ to be the space of homogeneous distributions on $\mathbb{R}^n - \{0\}$ of degree k. This means $S \in \mathfrak{M}_k(\mathbb{R}^n - \{0\})$ if for all $\lambda > 0$ and $\phi \in C_c^\infty(\mathbb{R}^n - \{0\})$ we have

$$\langle S, M_\lambda \phi \rangle = \lambda^{k+n} \langle S, \phi \rangle, \tag{3.39}$$

where $M_\lambda \phi(x) := \phi(\lambda^{-1} x)$. Remark that we will not make the distinction here between homogeneous distributions on $\mathbb{R}^n$ and $\mathbb{R}^n - \{0\}$, since we will only consider distributions of degree $k > -n$. By a result in [15] any homogeneous distribution on $\mathbb{R}^n - \{0\}$ of degree $k > -n$ has a unique homogeneous extension to $\mathbb{R}^n$. Thus we simply identify the distributions on $\mathbb{R}^n - \{0\}$ with those on $\mathbb{R}^n$

$$\mathcal{M}_k(\mathbb{R}^n) \sim \mathcal{M}_k(\mathbb{R}^n - \{0\}) \text{ and } \mathfrak{M}_k(\mathbb{R}^n) \sim \mathfrak{M}_k(\mathbb{R}^n - \{0\}), \text{ for } k > -n.$$

Remark that $Y \in \mathfrak{M}_{-2}(\mathbb{R}^4)$. Next we define the transport operator (also known as the convective derivative) T as

$$T := T(\xi) = \partial_t + v(\xi) \cdot \nabla_x. \tag{3.40}$$

The goal is to exchange $[v(\xi)\partial_t + \nabla_x]\bar{f}$ and $\nabla_x \times (v(\xi)\bar{f})$ in (3.33) by commuting the derivatives onto Y through the convolution, and express $\partial_i Y$ in terms of T. This is given precisely in the following lemma.

Lemma 4. *Let Y be the fundamental solution of the wave equation given by (3.28). Then there exits homogeneous functions $p, a^0 \in \mathcal{M}_0(\mathbb{R}^4)$ and $a^1, q \in \mathcal{M}_{-1}(\mathbb{R}^4)$ such that*

$$\begin{aligned} [v(\xi)\partial_t + \nabla_x]Y &= -T(\xi)(pY) + qY \in \mathfrak{M}_{-3}(\mathbb{R}^4), \\ \nabla_x \times (v(\xi)Y) &= T(\xi)(a^0 Y) + a^1 Y \in \mathfrak{M}_{-3}(\mathbb{R}^4). \end{aligned} \tag{3.41}$$

In fact, we have the precise expressions

$$p(t, x, \xi) := \frac{v(\xi)t - x}{v(\xi) \cdot x - t}, \quad q(t, x, \xi) := \frac{1}{\langle \xi \rangle^2} \frac{v(\xi)t - x}{(v(\xi) \cdot x - t)^2}, \tag{3.42}$$

with similar expression for a^0 and a^1 given in appendix A.

The proof of Lemma 4 is in Appendix A. The reason is that Lemma 4 is rather technical and involves some deep results in homogeneous distribution theory which hides much of the physics of the problem. However lemma 4 can be physically interpreted as follows. The Vlasov equation has a speed of propagation in the spatial variable of $v(\Xi)$, where Ξ is a partial solution (the ladder 3 components of the 6 dimensional vector field) of the characteristic curves. The main remark is that for compactly support momentum, we have the control $|v(\Xi)| < 1$. Physically this means that individual particle velocities are uniformly bounded away from the speed of light. On the other hand, the electromagnetic waves (E, B) travel at a speed $c = 1$, ahead of the transport. This feature that transport speed never surpasses the wave speed is crucial which allows for The distributions p and q (as well as a^0 and a^1) to be well defined away from the light cone $\{|x| = t\}$.

The next two lemmas enable us to write (3.33) in a way that allows both the use of Lemma 4 and a way to estimate (E, B).

Lemma 5. *Let $p \in \mathcal{M}_m(\mathbb{R}^4)$ with $m \geq -1$ and $\bar{f} \in L^\infty(\mathbb{R} \times \mathbb{R}^3; \mathbb{R})$. Then the following expression can be written*

$$\begin{aligned} \bar{u}(t, x) &:= (pY) * (\bar{f}\mathbb{1}_{t>0}) \\ &= \int_0^t \int_{\mathbb{S}^2} \frac{p(1, \omega)}{4\pi} \bar{f}(t - s, x - s\omega) s^{1+m} d\omega ds, \end{aligned} \tag{3.43}$$

where $\omega = \frac{y}{|y|} \in \mathbb{S}^2$. Furthermore, from this we obtain the estimate

$$|\bar{u}(t, x)| \leq \frac{t^{1+m}}{3} ||p(1, \cdot)||_{L^\infty(\mathbb{S}^2)} \int_0^t ||\bar{f}(s, \cdot)||_{L^\infty(\mathbb{R}^3_x)} ds. \tag{3.44}$$

Proof. By direct formal computation, upon converting to polar coordinates, we have

$$\begin{aligned}\bar{u}(t,x) &= \int_{\mathbb{R}^4} p(s,y)\frac{\delta(s-|y|)}{4\pi s}\mathbb{1}_{s>0}\bar{f}(t-s,x-y)\mathbb{1}_{t-s>0}dsdy \\ &= \int_0^t \int_{\mathbb{S}^2}\int_0^\infty p(s,\omega r)\frac{\delta(s-r)}{4\pi s}\bar{f}(t-s,x-r\omega)r^2drd\omega ds \\ &= \int_0^t \int_{\mathbb{S}^2} p(s,\omega s)\frac{1}{4\pi}\bar{f}(t-s,x-r\omega)sd\omega ds \\ &= \int_0^t \int_{\mathbb{S}^2} \frac{p(1,\omega)}{4\pi}\bar{f}(t-s,x-s\omega)s^{1+m}d\omega ds.\end{aligned}$$

Then it is easy to conclude

$$|\bar{u}(t,x)| \leq \frac{t^{1+m}}{4\pi}|\mathbb{S}^2|||p(1,\cdot)||_{L^\infty(\mathbb{S}^2)}\int_0^t ||\bar{f}(s,\cdot)||_{L^\infty(\mathbb{R}^3)}ds.$$

Given $|\mathbb{S}^2| = \frac{4\pi}{3}$, we are done. □

The next lemma allows us to commute the time derivative in (3.33) onto the distribution Y. Remark the challenge is to pass ∂_t through the characteristic function $\mathbb{1}_{t>0}$.

Lemma 6. *For* $\bar{f} \in W^{1,\infty}(\mathbb{R}^4;\mathbb{R})$ *we have the identity*

$$\partial_t(Y *_{t,x} \mathbb{1}_{t>0}\bar{f}) = Y * \mathbb{1}_{t>0}\partial_t\bar{f} + \frac{t}{4\pi}\int_{\mathbb{S}^2}\bar{f}(0,x-t\omega)d\omega. \tag{3.45}$$

Proof. First note, from Lemma 5 with $p \equiv 1 \in \mathcal{M}_0(\mathbb{R}^4)$, we have

$$Y *_{t,x} \mathbb{1}_{t>0}\bar{f} = \int_{\mathbb{S}^2}\int_0^t \frac{s}{4\pi}\bar{f}(t-s,x-s\omega)dsd\omega,$$

therefore

$$\begin{aligned}\partial_t(Y *_{t,x} \mathbb{1}_{t>0}\bar{f}) &= \int_{\mathbb{S}^2}\int_0^t \frac{s}{4\pi}\partial_t\bar{f}(t-s,x-s\omega)dsd\omega + \int_{\mathbb{S}^2}\frac{t}{4\pi}\bar{f}(0,x-t\omega)d\omega \\ &= Y * \mathbb{1}_{t>0}\partial_t\bar{f} + \frac{t}{4\pi}\int_{\mathbb{S}^2}\bar{f}(0,x-t\omega)d\omega.\end{aligned}$$

□

Lemmas 4, 5 and 6 then allow us to then manipulate (3.33) as follows

$$\begin{aligned}&Y *_{t,x} \mathbb{1}_{t>0}\int[v(\xi)\partial_t\bar{f} + \nabla_x\bar{f}]d\xi \\ &\qquad= \int(-T(pY) + qY) *_{t,x} \mathbb{1}_{t>0}\bar{f}d\xi - \frac{t}{4\pi}\int\int_{\mathbb{S}^2} v(\xi)\bar{f}(0,x-t\omega,\xi)d\omega d\xi \\ &\qquad= -\int pY *_{t,x} \mathbb{1}_{t>0}T(\bar{f})d\xi - \frac{t}{4\pi}\int\int_{\mathbb{S}^2} p(1,\omega,\xi)\bar{f}(0,x-t\omega,\xi)d\omega d\xi \\ &\qquad\quad+ \int qY *_{t,x} \mathbb{1}_{t>0}\bar{f}d\xi - \frac{t}{4\pi}\int\int_{\mathbb{S}^2} v(\xi)\bar{f}(0,x-t\omega,\xi)d\omega d\xi.\end{aligned} \tag{3.46}$$

Remark that the term in (3.35) involving $J(\bar{f}^{in})$ can be written

$$\begin{aligned}\frac{1}{4\pi t^2}\int_{|y-x|=t} tJ(\bar{f}^{in})dS(y) &= \frac{1}{4\pi t}\int\int_{|x-y|=t} v(\xi)\bar{f}^{in}(y,\xi)dS(y)\xi \\ &= \frac{t}{4\pi}\int\int_{\mathbb{S}^2} v(\xi)\bar{f}^{in}(x-t\omega,\xi)d\omega d\xi\end{aligned}$$

This cancels with the last term in (3.46). A similar computation for B, leads to a wonderful representation formula for the fields (E, B):

$$\begin{aligned}E(t,x) = &-\int p(t,x,\xi)Y(t,x) *_{t,x} (\mathbb{1}_{t>0}T(\bar{f}))d\xi + \int q(t,x,\xi)Y(t,x) *_{t,x} (\mathbb{1}_{t>0}\bar{f})d\xi \\ &+ \frac{1}{4\pi t^2}\int_{|x-y|=t}\Big[t\nabla_x \times B^{in}(y) + E^{in}(y) + [(y-x)\cdot\nabla_y]E^{in}(y)\Big]dS(y) \\ &- \frac{t}{4\pi}\int\int_{\mathbb{S}^2} p(1,\omega,\xi)\bar{f}^{in}(x-t\omega,\xi)d\omega d\xi, \qquad (3.47)\\ B(t,x) = &\int a^0(t,x,\xi)Y(t,x) *_{t,x} (\mathbb{1}_{t>0}T(\bar{f}))d\xi + \int a^1(t,x,\xi)Y(t,x) *_{t,x} (\mathbb{1}_{t>0}\bar{f})d\xi \\ &+ \frac{1}{4\pi t^2}\int_{|x-y|=t}\Big[-t\nabla_x \times E^{in}(y) + B^{in}(y) + [(y-x)\cdot\nabla_y]B^{in}(y)\Big]dS(y) \\ &+ \frac{t}{4\pi}\int\int_{\mathbb{S}^2} a^0(1,\omega,\xi)\bar{f}^{in}(x-t\omega,\xi)d\omega d\xi. \qquad (3.48)\end{aligned}$$

3.4 Obstruction for Uniform Estimates

The difficulty for obtaining uniform in ϵ estimates of the fields comes from the first terms in (3.47). That is when we replace the $T(\bar{f})$ using the vlasov equation, this introduces the term of order ϵ, coming from the applied field. We will only consider computations for E and simply state the final results for B as they are similar. Using lemma 5, the estimate shown in [7] for p and q with $|\xi| \le R^T_\xi$ are given by

$$||p(1,\cdot,\xi)||_{L^\infty(\mathbb{S}^2)} \le 2\frac{\sqrt{1+(R^T_\xi)^2}}{\sqrt{1+(R^T_\xi)^2}-(R^T_\xi)} = 2\big(1+(R^T_\xi)^2+(R^T_\xi)\sqrt{1+(R^T_\xi)^2}\big) < \infty, \qquad (3.49)$$

and

$$||q(1,\cdot,\xi)||_{L^\infty(\mathbb{S}^2)} \le 2\frac{1+(R^T_\xi)^2}{(\sqrt{1+(R^T_\xi)^2}-(R^T_\xi)^2} = 2\big(1+(R^T_\xi)^2+(R^T_\xi)\sqrt{1+(R^T_\xi)^2}\big)^2 < \infty. \qquad (3.50)$$

We then immediately obtain the estimate for the field E

$$\begin{aligned}|E(t,x)| \le C(t,R_x,R_\xi,E^{in},B^{in},f^{in}) &+ \int_{\mathrm{supp}(f(t,x,\cdot))} ||q(1,\cdot,\xi)||_{L^\infty(\mathbb{S}^2)}d\xi\int_0^t ||f(s,\cdot,\cdot)||_{L^\infty}ds \\ &+ \left|\int p(t,x,\xi)Y(t,x) *_{t,x} (\mathbb{1}_{t>0}T(\bar{f}))d\xi\right|. \qquad (3.51)\end{aligned}$$

The idea to estimate this remaining term and apply Gronwall's lemma is to pass the derivative $T(\bar{f})$ to the Vlasov equation and integrate by parts in ξ as follows

$$\begin{aligned}
&\int p(t,x,\xi)Y(t,x) *_{t,x} (\mathbb{1}_{t>0}T(\bar{f}))d\xi \\
&\quad = \int p(t,x,\xi)Y(t,x) *_{t,x} \left(\mathbb{1}_{t>0}\{\nabla_\xi \cdot ([\epsilon E + v(\xi) \times (\epsilon B + \epsilon^{-1}\mathbf{B}_e)]\bar{f}) + M'_\epsilon(|\xi|)\frac{\xi}{|\xi|} \cdot E\}\right)d\xi \\
&\quad = -\epsilon \int \nabla_\xi p(t,x,\xi)Y(t,x) *_{t,x} (\mathbb{1}_{t>0}[E + v(\xi) \times B]\bar{f})d\xi \\
&\qquad - \epsilon^{-1} \int \nabla_\xi p(t,x,\xi)Y(t,x) *_{t,x} (\mathbb{1}_{t>0}[v(\xi) \times \mathbf{B}_e]\bar{f})d\xi \\
&\qquad + \int p(t,x,\xi)Y(t,x) *_{t,x} (\mathbb{1}_{t>0}M'_\epsilon(|\xi|)\frac{\xi}{|\xi|} \cdot E)d\xi. \qquad (3.52)
\end{aligned}$$

Similarly for B. This is now in a suitable form to apply Gronwall's estimates (after applying Lemma 5 one more time of course), provided the solution $\bar{f}$ has compact support in ξ which allows the use of the estimates (3.49)-(3.50). More specifically we apply a non-linear Gronwall estimate known as the Bihari-LaSalle inequality due to the quadratic term $[E + v \times B]\bar{f}$. This is given in appendix A. Assuming $\bar{f}$ remains bounded in L^∞, we do have the fields (E, B) are uniformly bounded in L^∞ with respect to ϵ, but only on a time interval $T_\epsilon > 0$, which may shrink to zero as ϵ tends to zero. Unlike the cold case in [7], at this stage, it is not apparent that we can achieve uniform estimates on a times interval $t = \mathcal{O}(1)$ due to the penalization $\epsilon^{-1}\mathbf{B}_e(x)$ (see Remark 1 below). Therefore we pay special attention to the term

$$-\epsilon^{-1} \int \nabla_\xi p(t,x,\xi)Y(t,x) *_{t,x} (\mathbb{1}_{t>0}[v(\xi) \times \mathbf{B}_e]\bar{f})d\xi.$$

We can however generalize the results of [7] by considering various scaling of solutions in terms of the small parameter ϵ. This is done in Section 5.5. Chapter 4 is devoted to overcoming this difficulty of achiving a uniform lifetime $0 < T < T_\epsilon$ in the hot regime. To accomplish this, we also require representation formulas for the Vlasov equation. This is done in the next section. Finally we conclude Section 3.6 by constructing a canonical set of coordinates which simplifies the analysis of Chapter 4.

Remark 1. *In [7], the cold assumption leads to the replacement of* $p(\cdot,\cdot,\xi)$ *with* $p_\epsilon(\cdot,\cdot,\xi)$ *given by the relationship* $p_\epsilon(\cdot,\cdot,\xi) := p(\cdot,\cdot,\epsilon\xi)$ *and hence* $\nabla_\xi p$ *is replaced with* $\epsilon\nabla_\xi p_\epsilon$ *which compensates the term* $\epsilon^{-1}\mathbf{B}_e(x)$ *allowing for uniform bounds in* L^∞ *of* (E, B).

Remark 2. *One does however have the estimate*

$$\begin{aligned}
|\epsilon E(t,x)| \leq \epsilon C + C\int_0^t (1+\epsilon)||\bar{f}(s,\cdot,\cdot)||_{L^\infty_{x,\xi}}ds + C||M'_\epsilon||_{L^\infty}\int_0^t ||\epsilon(E,B)(s,\cdot)||_{L^\infty_x}ds \\
+ C\int_0^t ||\epsilon(E,B)(s,\cdot)||_{L^\infty_{x,\xi}}||\epsilon\bar{f}(s,\cdot,\cdot)||_{L^\infty_{x,\xi}}ds, \qquad (3.53)
\end{aligned}$$

where the constant C *depends on the momentum support* $\{|\xi| \leq R^T_\xi\}$ *of* $\bar{f}$, $||(\nabla_\xi p, q)(1,\cdot,\cdot)||_{L^\infty(\mathbb{S}^2\times\{|\xi|\leq R^T_\xi\})}$, $||\mathbf{B}_e||_{L^\infty}$ *and initial data.*

In the next section we will derive representation formulas for the Vlasov equation.

3.5 Vlasov Representation Formula

The approach to solving the Vlasov Equation, a transport equation, is through the method of characteristics. Consider the ODE system, depending on given fields (E, B), defined as solutions of

$$\dot{X} = v(\Xi), \qquad X(0,x,\xi) = x, \tag{3.54}$$
$$\dot{\Xi} = -\epsilon^{-1}v(\Xi) \times \mathbf{B}_e(X) - \epsilon[E(t,X) + v(\Xi) \times B(t,X)], \qquad \Xi(0,x,\xi) = \xi. \tag{3.55}$$

Then for as long as the solution $(X,\Xi)(t) := (X,\Xi)(t,x,\xi)$ exists (here we omit the dependence on (x,ξ) in our notation), it follows that

$$\frac{d}{dt}\bar{f}(t,X(t),\Xi(t)) = M_\epsilon'(|\Xi(t)|)\frac{\Xi}{|\Xi|} \cdot E(t,X). \tag{3.56}$$

Thus we must justify the flow map defined by

$$\begin{aligned} \mathcal{F}_t : \mathbb{R}^3 \times \mathbb{R}^3 &\mapsto \mathbb{R}^3 \times \mathbb{R}^3 \\ (x,\xi) &\mapsto (X(t,x,\xi), \Xi(t,x,\xi)) \end{aligned}$$

is invertible up to some time t. First remark that $|\dot{X}| < 1$ and therefore

$$|X(t) - x| < t. \tag{3.57}$$

Next we compute

$$\frac{d}{dt}|\Xi|^2 = \Xi \cdot \dot{\Xi} = \epsilon\Xi \cdot E(t,X(t)).$$

Therefore the Bahari-LaSalle inequality implies

$$|\Xi|(t,x,\xi) \leq |\xi| + \epsilon C\int_0^t ||E(s,\cdot)||_{L^\infty(|x-y|\leq t)}ds. \tag{3.58}$$

Therefore as long as $||\epsilon E(s,\cdot)||_{L^\infty(|x-y|\leq t)} < \infty$ it follows that $|\Xi(t)| < \infty$. Therefore the characteristics $(X,\Xi)(t)$ remain in a compact set. Furthermore, we have the right hand side of the vector field (3.54)-(3.55) is divergence free $\nabla_{X,\Xi} \cdot (\dot{X}, \dot{\Xi}) \equiv 0$, and therefore the flow $\mathcal{F}_t$ is a volume preserving diffeomorphism. Thus the Duhamel Principal on (3.56) implies

$$\bar{f}(t,x,\xi) = \bar{f}^{in}(X(-t),\Xi(-t)) + \int_0^t \left[M_\epsilon'(|\xi|)\frac{\xi}{|\xi|} \cdot E\right](s, X(t-s), \Xi(t-s))ds.$$

Note that (3.57) implies $|X(t-s)-x| \leq t-s \leq t$ for $s \in [0,t]$. This then gives the immediate estimate

$$|\bar{f}(t,x,\xi)| \leq ||\bar{f}^{in}||_{L^\infty_{x,\xi}} + ||M_\epsilon'||_{L^\infty_\xi}\int_0^t ||E(s,\cdot)||_{L^\infty(|x-y|\leq t)}ds. \tag{3.59}$$

Lemma 7. *The estimate (2.34) in Theorem 1 holds as long as there exists $C > 0$ such that for all $\epsilon > 0$ such that (2.33) holds*

Proof. We simply add (3.53) to (3.59) and apply Gronwall's (Bihari-LaSalle) Lemma. This gives gives for all $t \in [0, T]$

$$||(\bar{f}, \epsilon E, \epsilon B)(t, \cdot, \cdot)||_{L^{\infty}_{x,\xi}(\mathcal{A}_T)} \leq C(T, R^T_{\xi}, ||\mathbf{B}_e||_{L^{\infty}}). \tag{3.60}$$

Note the quadratic term in (3.53) has a factor of ϵ. Therefore we can extend the lifetime T for smaller values of ϵ. Note the remaining terms of (3.52) that do not involve ϵ^{-1} are controlled the same as in [7]. $\square$

At this stage it is not apparent how uniform estimates should be obtained when the system is not dilute. Moreover, these estimates do not show how one could remove the weight of ϵ to achieve uniform Sup-norm estimates of the fields. Before we address these issues, we will consider a canonical set of coordinates though a field straightening procedure. This will involve a rotation of the the applied magnetic field to align with the x_3-axis. The advantage is to introduce a single oscillatory variable θ in cylindrical coordinates as the characteristic curve trajectories wrap around the x_3-axis.

3.6 Field Straightening

As mentioned it will be convenient to work with a single, singular variable. To do this, we will rotate our system in the following way. Let $O : \mathbb{R}^3 \mapsto SO(3)$ be a map defined by the relation

$$O^t(x)\mathbf{B}_e(x) = b_e(x)(0, 0, 1)^t.$$

Remark the superscript t is used to denote a matrix transpose and should not be confused with time. Thus, O^t is a rotation by angle $\vartheta(x) \in [0, 2\pi)$ defined by $\cos(\vartheta(x)) := \frac{\mathbf{B}^3_e(x)}{b_e(x)}$ about the axis $\mathbf{B}^{\perp}_e := (\mathbf{B}^2_e(x), -\mathbf{B}^1_e(x), 0)^t = \mathbf{B}_e \times e_3$. Clearly when $\mathbf{B}^{\perp}_e(x) \equiv 0$, we take $\vartheta(x) = 0$. Recall our assumption that $b_e > 0$. So more precisely, $O^t(\cdot)$ is determined by Euler-Rodrigues' formula

$$O^t(x) := \frac{\mathbf{B}^3_e(x)}{b_e(x)} I_3 + \frac{|\mathbf{B}^{\perp}_e(x)|}{b^2_e(x)} [\mathbf{B}_e \times] + \big(1 - \frac{\mathbf{B}^3_e}{b_e(x)}\big) \frac{\mathbf{B}^{\perp}_e \otimes \mathbf{B}^{\perp}_e}{b^2_e(x)}, \tag{3.61}$$

with the cross product matrix and usual Euclidean outer product

$$[\mathbf{B}_e \times] := \begin{bmatrix} 0 & -\mathbf{B}^3_e & \mathbf{B}^2_e \\ \mathbf{B}^3_e & 0 & -\mathbf{B}^1_e \\ -\mathbf{B}^2_e & \mathbf{B}^1_e & 0 \end{bmatrix}, \quad \mathbf{B}^{\perp}_e \otimes \mathbf{B}^{\perp}_e := \mathbf{B}^{\perp}_e (\mathbf{B}^{\perp}_e)^t = \begin{bmatrix} (\mathbf{B}^2_e)^2 & -\mathbf{B}^1_e \mathbf{B}^2_e & 0 \\ -\mathbf{B}^1_e \mathbf{B}^2_e & (\mathbf{B}^1_e)^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The precise construction of $O(x)$ is not of high importance, but retain that it is a smooth, rational function of the components of $\mathbf{B}_e$ with matrix norm $||O^t||_{L^{\infty}} = 1$. Next define a new function f according to the following variable change

$$f(t, x, \xi) := \bar{f}(t, x, O(x)\xi). \tag{3.62}$$

As implied by our notation, we will focus on the function f. It follows that f is a solution of

$$\partial_t f + v(O(x)\xi) \cdot \nabla_x f - O^t(x)\nabla_x(O(x)\xi)v(O(x)\xi) \cdot \nabla_\xi f - \epsilon^{-1}\frac{b_e(x)}{\langle\xi\rangle}\partial_\theta f$$

$$= \epsilon[O^t(x)E + v(\xi) \times O^t(x)B] \cdot \nabla_\xi f + \frac{M'(|\xi|)}{|\xi|}O(x)\xi \cdot E, \tag{3.63}$$

$$f(0,x,\xi) = \bar{f}^{in}(x, O(x)\xi) := f^{in}(x,\xi), \tag{3.64}$$

where

$$\partial_\theta := \xi_2\partial_{\xi_1} - \xi_1\partial_{\xi_2} = [\xi \times O^t(x)\frac{\mathbf{B}_e(x)}{b_e(x)}] \cdot \nabla_\xi = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xi \cdot \nabla_\xi. \tag{3.65}$$

For now, we may think of ∂_θ defined above to be given in a Cartesian coordinate system as in the far right expression of (3.65). Later we will convert our new characteristic curves to a cylindrical coordinate system and the notation will become clear. Furthermore, to be unambiguous, the components of the matrix $\nabla_x(O(x)\xi)$ are defined by

$$[\nabla_x(O(x)\xi)]_{i,j} := \sum_{k=1}^{3} \partial_{x_j} O_{ik}\xi_k, \quad (i,j) \in \{1,2,3\}^2.$$

This convention will be used whenever we write the gradient of a vector valued function. Note that because $\det(O(x)) = 1$ for all $x \in \mathbb{R}^3$, it follows that the charge and current density become

$$\rho(\bar{f})(t,x) = \rho(f)(t,x) = \int f(t,x,\xi)d\xi,$$

$$J(\bar{f})(t,x) = \int v(O(x)\xi)f(t,x,\xi)d\xi. \tag{3.66}$$

So the compatibility conditions (2.26) - (2.30) are satisfied. The characteristic curves of (3.63) are defined by solutions of

$$\dot{X} = v(O(X)\Xi) \qquad X(0) = x, \tag{3.67}$$

$$\dot{\Xi} = \frac{Q(X,\Xi)}{\langle\Xi\rangle} - \epsilon^{-1}v(\Xi) \times O^t(X)\mathbf{B}_e(X)$$

$$- \epsilon[O^t(x)E(t,X) + v(\Xi) \times O^t(x)B(t,X)] \qquad \Xi(0) = \xi, \tag{3.68}$$

where for more compact notation we have set the quadratic in ξ term Q to be given by

$$Q(x,\xi) := -O^t(x)\nabla_x(O(x)\xi)O(x)\xi. \tag{3.69}$$

See remark 3 below for the derivation of Q. Note that the transformation $(t,x,\xi) \mapsto (t,x,O(x)\xi)$ is volume preserving with respect to $dxd\xi$ for all t. So it is expected that the flow,

$$\mathcal{F}_t(x,\xi) := (X(t,x,\xi), \Xi(t,x,\xi)), \tag{3.70}$$

should also preserve volume. The following lemma guarantees that this will be the case for any transformation η with non-zero constant Jacobian. I.e. $|D\eta|(x,\xi) = C \in \mathbb{R}^*$, $\forall (x,\xi) \in \mathbb{R}^3 \times \mathbb{R}^3$.

Lemma 8. *Suppose that we have a PDE with a given function* $F : \mathbb{R}^n \mapsto \mathbb{R}^n$ *and* C^1 *solution* $f : \mathbb{R}^n \mapsto \mathbb{R}$ *satisfying*

$$F(x) \cdot \nabla f(x) = 0, \tag{3.71}$$

with

$$\nabla \cdot F \equiv 0.$$

Suppose we consider a variable change

$$y := \eta(x), \ \textit{with } |D_x \eta| = 1 \tag{3.72}$$

and define

$$\bar{f}(y) := \bar{f}(\eta(x)) = f(x).$$

Then it follows that

$$\tilde{F}(y) := [(D_x\eta)^t F \circ (\eta^{-1}(y))], \quad \tilde{F}(y) \cdot \nabla_y \tilde{f}(y) = 0, \tag{3.73}$$

with the convention that

$$(D_x\eta)_{i,j} := J_{i,j} = \frac{\partial \eta_i}{\partial x_j}, \quad J^{-1}_{i,j} = \frac{\partial x_i}{\partial \eta_j}. \tag{3.74}$$

Moreover the following divergence free property is preserved for any constant Jacobian transform

$$\nabla_y \cdot \tilde{F}(y(x)) = \nabla_x \cdot F(x) + F(x) \cdot \nabla_x \ln(|D_x\eta|(x)) = 0. \tag{3.75}$$

Proof. Using index notation (while not distinguishing between upper and lower indices) we have by the chain rule

$$\begin{aligned} 0 = F(x) \cdot \nabla f(x) = F_i(x) \frac{\partial f}{\partial x_i} &= F_i(x(y)) \frac{\partial y_j}{\partial_{x_i}} \frac{\partial \tilde{f}}{\partial y_j} \\ &= \big[\frac{\partial y_j}{\partial x_i} F_i\big](x(y)) \frac{\partial \tilde{f}}{\partial y_j}(y), \end{aligned}$$

which is exactly (3.73). Consider next the y-divergence of $\tilde{F}$

$$\nabla_y \cdot \tilde{F}(y) = \partial_{y_j} \big[\frac{\partial y_j}{\partial_{x_i}} F_i\big](x(y)) = \frac{\partial F_i}{\partial x_k} \frac{\partial x_k}{\partial y_j} \frac{\partial y_j}{\partial x_i} + F_i \frac{\partial}{\partial y_j} \big[\frac{\partial y_j}{\partial x_i}\big].$$

Then using (3.74) it follows that

$$\begin{aligned} \nabla_y \cdot \tilde{F}(y) &= \frac{\partial F_i}{\partial x_k} \delta_i^k + F_i \frac{\partial x_k}{\partial y_j} \frac{\partial^2 y_j}{\partial x_k \partial x_i} \\ &= \frac{\partial F_i}{\partial x_i} + F_i \big[\frac{\partial y_j}{\partial x_k}\big]^{-1} \frac{\partial}{\partial x_i} \big[\frac{\partial y_j}{\partial x_k}\big] \\ &= \nabla \cdot F + F_i \mathrm{Tr}\big(J^{-1} \partial_{x_i} J\big). \end{aligned}$$

Then Jacobi's Formula gives that for any invertible matrix $A(t)$ we have

$$\partial_t \ln(|A|) = \frac{\partial_t |A|}{|A|} = \mathrm{Tr}\big(A^{-1}\partial_t A\big).$$

Thus we finally arrive at

$$\nabla_y \cdot \tilde{F}(y) = \nabla_x \cdot F(x(y)) + F(x(y)) \cdot \nabla_x \ln(|D\eta|)(x(y)).$$

□

Remark 3. *Note that in our case we use the transformation* $(t, x, \tilde{\xi}) := (t, x, O^t(x)\xi)$, *so that* $f(t, x, \tilde{\xi}) = \bar{f}(t, x, \xi)$ *and the term* Q *comes from*

$$\begin{aligned}
\frac{Q_j}{\langle \xi \rangle} &= \frac{\partial \tilde{\xi}_j}{\partial x_i} v_i(\xi) = \partial_{x_i}(O^t_{j,k}\xi_k)\frac{\xi_i}{\langle \xi \rangle} \\
&= \partial_{x_i}(O^t_{j,k})O_{k,\ell}\tilde{\xi}_\ell \frac{O_{i,m}\tilde{\xi}_m}{\left\langle \tilde{\xi} \right\rangle} \\
&= \underbrace{\partial_{x_i}(O^t_{j,k}O_{k,\ell}\tilde{\xi}_\ell)}_{=0} \frac{O_{i,m}\tilde{\xi}_m}{\left\langle \tilde{\xi} \right\rangle} - O^t_{j,k}\partial_{x_i}(O_{k,\ell}\tilde{\xi}_\ell)\frac{O_{i,m}\tilde{\xi}_m}{\left\langle \tilde{\xi} \right\rangle} \\
&= -\big[O^t(x)\nabla_x(O(x)\tilde{\xi})v(O(x)\tilde{\xi})\big]_j,
\end{aligned}$$

where it is clear that $\langle \xi \rangle = \left\langle \tilde{\xi} \right\rangle$. *This is exactly the equation given by (3.69). Furthermore* Q *is orthogonal to* ξ

$$\begin{aligned}
-\langle \xi \rangle \xi \cdot Q &= \xi_i O^t_{i,j}\partial_{x_k}(O_{j,\ell}\xi_\ell)O_{k,m}\xi_m \\
&= \partial_k(\xi_i O^t_{i,j}O_{j,\ell}\xi_\ell)O_{k,m}\xi_m - \partial_k(\xi_i O^t_{i,j})O_{j,\ell}\xi_\ell O_{k,m}\xi_m \\
&= \underbrace{\nabla_x(O(x)\xi \cdot O(x)\xi)}_{=\nabla_x(\xi\cdot\xi)=0} \cdot O(x)\xi - \xi_\ell O^t_{\ell,j}\partial_k(O_{j,i}\xi_i)O_{k,m}\xi_m \\
&= \langle \xi \rangle \xi \cdot Q,
\end{aligned}$$

where we have relabeled ℓ *and* j *in the last lines implying that* $\xi \cdot Q \equiv 0$.

This leads to the immediate corollary:

Corollary 2. *The rotated flow of (3.67)-(3.68) preserves volume with respect to the Liouville measure* $dxd\xi$ *as long as the characteristics do not cross.*

Similarly to the non-rotated flow, the solution of (3.63) will exist up to time $T > 0$ provided the characteristics remain in a compact set up to time T. Suppose a priori that $(E, B) \in C^1([0,T] \times \mathbb{R}^3 \times \mathbb{R}^3)$ is a classical solution. It follows that for $t \leq T$

$$|\dot{X}| \leq 1 \implies |X(t, x, \xi)| \leq R^0_x + T.$$

Next consider the pointwise estimate of $|\Xi|$ using remark 3 that $\xi \cdot Q = 0$,

$$\partial_t |\Xi|^2 = 2\Xi \cdot \partial_t \Xi = \Xi \cdot \epsilon O^t E \leq \epsilon |\Xi| ||E||_{L^\infty}. \tag{3.76}$$

Then integrating gives

$$|\Xi|^2 \leq (R_\xi^0)^2 + \epsilon \int_0^t ||E(s,\cdot)||_{L^\infty} |\Xi(s)| ds. \tag{3.77}$$

If $\epsilon E \in L^\infty([0,T] \times \{|x| \leq R_x^0\})$, then in fact , $|\Xi|$ can be controlled by the Bahari LaSalle inequality which gives us the estimate

$$|\Xi(t)|^2 \leq (R_\xi^0)^2 (1 + Cte^{Ct}).$$

This means that the characteristics (3.67) - (3.68) remain in a bounded set for any finite $T > 0$ and are thus globally defined.

Chapter 4

Proof of Main Theorem

This chapter is devoted to the proof of Theorem 1. Section 4.1 begins by considering an external inhomogeneous magnetic field orientated along a fixed direction. Furthermore, we study a linearized version of the Vlasov Maxwell system and derive an asymptotic approximation of the associated characteristics in terms of ϵ. This approximation is given by Lemma 9 and is accomplished using a strategy similar to the methods of [10], involving a non-stationary phase argument for the rapidly oscillating characteristics. In Section 4.2 we prove the well posedness of the linear system and derive estimates in the Sup and Lipschitz-norms. The Sup-norm is uniform in ϵ, while a weight ϵ is necessary for a uniform Lipschitz norm unless the data is well prepared in the sense of Definition 1. Furthermore, although the computation is not done explicitly, the well posedness results of the linear system hold for applied fields with variable direction. This remark is resolved by Lemma 10. Finally, Section 4.3 uses the linear system described in 4.1 to prove Theorem 1. The linear solution serves as a good $\mathcal{O}(\epsilon)$-approximation of f, while only an $\mathcal{O}(1)$ approximation of the fields (E, B), which is still enough to deduce Theorem 1. In other words, we establish well posedness on a uniform times interval $[0, T]$ of the HMRVM system for dilute equilibrium and well prepared data.

4.1 Asymptotics of Linear Characteristic Curves

For simplicity, we first consider the case of an inhomogeneous, magnetic field with constant direction aligned along the x_3-axis, that is

$$\mathbf{B}_e(x) = b_e(x)\,{}^t(0, 0, 1).$$

This implies $Q \equiv 0$ and $O(x) = Id_{3\times 3}$. The goal will be to first study the dilute, linearized system in an inhomogeneous magnetic field with fixed direction. We define the linear system by dropping the non-linear term of order ϵ from (2.21), namely:

$$\begin{cases} \partial_t f_\ell + v(\xi)\cdot\nabla_x f_\ell - \epsilon^{-1}\,\langle\xi\rangle^{-1}\,b_e(x)\,\partial_\theta f_\ell = \epsilon M'(|\xi|)\,|\xi|^{-1}\,\xi\,\cdot E_\ell \\ \partial_t E_\ell - \nabla_x\times B_\ell = J(f_\ell), \quad \nabla_x\cdot E_\ell = -\rho(f_\ell) \\ \partial_t B_\ell + \nabla_x\times E_\ell = 0, \qquad\quad \nabla_x\cdot B_\ell = 0 \end{cases} \tag{4.1}$$

together with

$$(f_\ell, E_\ell, B_\ell)|_{t=0} = (f^{in}, E^{in}, B^{in}). \tag{4.2}$$

Furthermore, consider the characteristic curves $(X_\ell, \Xi_\ell)(t, x, \xi) = (X_\ell, \Xi_\ell)(t)$ of the linearized system solving

$$\begin{aligned} \dot{X}_\ell &= \frac{\Xi_\ell}{\langle\Xi_\ell\rangle}, & X_\ell(0) &= x, \\ \dot{\Xi}_\ell &= -\frac{b_e(X_\ell)}{\epsilon\,\langle\Xi_\ell\rangle}\,{}^t(\Xi_{\ell 2}, -\Xi_{\ell 1}, 0), & \Xi_\ell(0) &= \xi. \end{aligned} \tag{4.3}$$

Then the solution f_ℓ can be expressed using Duhamel's principal in terms of these characteristic curves as

$$f_\ell(t,x,\xi) = f^{in}(X_\ell(-t),\Xi_\ell(-t)) + \epsilon \int_0^t \left[M'(|\xi|)\frac{\xi}{|\xi|}\cdot E_\ell \right](s, X_\ell(t-s), \Xi_\ell(t-s))ds.$$

Remark that system (4.3) is divergence free and the flow is therefore volume preserving for all times. Define the horizontal and perpendicular momentum variables as

$$\bar{\xi} := {}^t(\xi_1,\xi_2,0), \quad \xi^\perp := {}^t(\xi_2,-\xi_1,0),$$

as well as the following phase Φ and remainder functions R_ϵ as follows.

$$\Phi(t,x,\xi) := b_e(x)t - \epsilon\nabla_x b_e(x)\cdot\left(t\frac{1}{b_e(x)}\xi^\perp - t^2\left(\frac{\nabla_x b_e(x)\cdot\xi^\perp}{4\langle\xi\rangle b_e(x)^2}\right)\bar{\xi} + t^2\left(\frac{\nabla_x b_e(x)\cdot\bar{\xi}}{4\langle\xi\rangle b_e(x)^2}\right)\xi^\perp \right), \tag{4.4}$$

$$\begin{aligned} R_\epsilon(t,x,\xi) := {} & \frac{1}{b_e(x)}\left(\sin\left(\frac{\Phi(t,x,\xi)}{\epsilon\langle\xi\rangle}\right)\bar{\xi} + \cos\left(\frac{\Phi(t,x,\xi)}{\epsilon\langle\xi\rangle}\right)\xi^\perp - \xi^\perp \right) \\ & + t\left(\frac{\nabla_x b_e(x)\cdot\xi^\perp}{2\langle\xi\rangle b_e(x)^2}\right)\bar{\xi} - t\left(\frac{\nabla_x b_e(x)\cdot\bar{\xi}}{2\langle\xi\rangle b_e(x)^2}\right)\xi^\perp. \end{aligned} \tag{4.5}$$

Then we have the following approximation for the linear, inhomogeneous characteristics.

Lemma 9. *Consider the ODE system (4.3). For any $T>0$ and $R^0_\xi > 0$, there exists $C = C(T, ||b_e||_{W^{2,\infty}}, R^0_\xi) \geq 0$ such that for all $t\in[0,T]$ and $|\xi| \leq R^0_\xi$,*

$$\left| X_\ell(t,x,\xi) - x - \frac{t\xi_3}{\langle\xi\rangle}e_3 - \epsilon R_\epsilon(t,x,\xi) \right| \leq \epsilon^2 C, \tag{4.6}$$

$$\left| \Xi_\ell(t,x,\xi) - \cos\left(\frac{\Phi(t,x,\xi)}{\epsilon\langle\xi\rangle}\right)\bar{\xi} + \sin\left(\frac{\Phi(t,x,\xi)}{\epsilon\langle\xi\rangle}\right)\xi^\perp - \xi_3 e_3 \right| \leq \epsilon C. \tag{4.7}$$

Proof. For neatness, we will omit the subscript ℓ, but note that (X,Ξ) should not be confused with (3.67)-(3.68). Remark that $\nabla_x\cdot\mathbf{B}_e = 0$ and $\mathbf{B}_e(x) = b_e(x)e_3$ imply that b_e depends only on the horizontal spatial components, $b_e(x) = b_e(x^1,x^2)$. Furthermore $\frac{d}{dt}|\Xi|^2 = 0$ and $|X| \leq x + t$ so the solution $(X,\Xi)(t)$ is globally defined. Moreover, the solution $\Xi(t)$ in (4.3) can be expressed as

$$\Xi(t,x,\xi) = \cos\left(\frac{\theta_\epsilon(t,x,\xi)}{\epsilon\langle\xi\rangle}\right)\bar{\xi} - \sin\left(\frac{\theta_\epsilon(t,x,\xi)}{\epsilon\langle\xi\rangle}\right)\xi^\perp + e_3\xi_3, \tag{4.8}$$

where

$$\theta_\epsilon(t,x,\xi) := \int_0^t b_e(X(s,x,\xi))ds.$$

Retain that, due to (2.6), we have

$$\frac{d}{dt}\theta_\epsilon(t,x,\xi) = b_e(X(t,x,\xi)) \geq c(K) > 0. \tag{4.9}$$

This part is similar to the setting of [10]. Note that since $|\Xi| = |\xi|$ we also have $\langle\Xi\rangle = \langle\xi\rangle$. Hence we can integrate to obtain an expression for $X(t)$

$$X(t,x,\xi) = x + \frac{t\xi_3}{\langle\xi\rangle}e_3 + \frac{1}{\langle\xi\rangle}\int_0^t \Big(\cos(\frac{\theta_\epsilon(s,x,\xi)}{\epsilon\langle\xi\rangle})\bar{\xi} - \sin(\frac{\theta_\epsilon(s,x,\xi)}{\epsilon\langle\xi\rangle})\xi^\perp\Big)\, ds.$$

The time integral is rapidly oscillating, so an integration by parts gives

$$\begin{aligned} X - x - \frac{t\xi_3}{\langle\xi\rangle}e_3 &= \epsilon\int_0^t \frac{1}{b_e(X(s))}\partial_s\big(\sin(\frac{\theta_\epsilon(s,x,\xi)}{\epsilon\langle\xi\rangle})\bar{\xi} + \cos(\frac{\theta_\epsilon(s,x,\xi)}{\epsilon\langle\xi\rangle})\xi^\perp\big)ds \\ &= \epsilon\frac{1}{b_e(X(t))}\big(\sin(\frac{\theta_\epsilon(t,x,\xi)}{\epsilon\langle\xi\rangle})\bar{\xi} + \cos(\frac{\theta_\epsilon(t,x,\xi)}{\epsilon\langle\xi\rangle})\xi^\perp\big) - \epsilon\frac{1}{b_e(x)}\xi^\perp \\ &\quad + \epsilon\int_0^t \frac{\nabla_x b_e(X)\cdot\dot{X}}{b_e(X(s))^2}\big(\sin(\frac{\theta_\epsilon(s,x,\xi)}{\epsilon\langle\xi\rangle})\bar{\xi} + \cos(\frac{\theta_\epsilon(s,x,\xi)}{\epsilon\langle\xi\rangle})\xi^\perp\big)ds. \end{aligned} \tag{4.10}$$

Therefore we have the estimate

$$|X(t) - x - \frac{t\xi_3}{\langle\xi\rangle}e_3| \le \epsilon|\xi|\big(\frac{3}{b_-} + 2t||\nabla_x(\frac{1}{b_e})||_{L^\infty}\big),$$

where

$$0 < c(K) \le b_- = b_-(t,x) := \min_{|x-y|\le t} b_e(y). \tag{4.11}$$

We can then Taylor expand $b_e(X)^{-2}\,\nabla_x b_e(X)$ in the last line of (4.10) with respect to X about the point $x + t\langle\xi\rangle^{-1}\xi_3 e_3$. When doing this, since $b_e(x) = b_e(x^1,x^2)$ does not depend on x_3, the shift $t\langle\xi\rangle^{-1}\xi_3 e_3$ does not appear, so that:

$$b_e(X)^{-2}\,\nabla_x b_e(X) = b_e(x)^{-2}\,\nabla_x b_e(x) + \mathcal{O}(\epsilon).$$

and integrate by parts once more after substituting $\dot{X} = v(\Xi)$. The only terms of size ϵ which remain are the ‘slow terms’ with non-zero mean. For instance, using standard trig identities and substituting (4.8) we have

$$\begin{aligned} \dot{X}\sin(\frac{\theta_\epsilon(t,x,\xi)}{\epsilon\langle\xi\rangle}) &= \frac{\Xi}{\langle\xi\rangle}\sin(\frac{\theta_\epsilon(t,x,\xi)}{\epsilon\langle\xi\rangle}) \\ &= \frac{1}{\langle\xi\rangle}\Bigg[\cos(\frac{\theta_\epsilon(t,x,\xi)}{\epsilon\langle\xi\rangle})\bar{\xi} - \sin(\frac{\theta_\epsilon(t,x,\xi)}{\epsilon\langle\xi\rangle})\xi^\perp + e_3\xi_3\Bigg]\sin(\frac{\theta_\epsilon(t,x,\xi)}{\epsilon\langle\xi\rangle}) \\ &= \frac{1}{2\langle\xi\rangle}\Bigg[\sin(\frac{2\theta_\epsilon(t,x,\xi)}{\epsilon\langle\xi\rangle})\bar{\xi} + \cos(\frac{2\theta_\epsilon(t,x,\xi)}{\epsilon\langle\xi\rangle})\xi^\perp + 2e_3\xi_3\sin(\frac{\theta_\epsilon(t,x,\xi)}{\epsilon\langle\xi\rangle})\Bigg] - \frac{1}{2\langle\xi\rangle}\xi^\perp. \end{aligned}$$

Similarly,

$$\begin{aligned} \dot{X}\cos(\frac{\theta_\epsilon(t,x,\xi)}{\epsilon\langle\xi\rangle}) &= \frac{\Xi}{\langle\xi\rangle}\cos(\frac{\theta_\epsilon(t,x,\xi)}{\epsilon\langle\xi\rangle}) \\ &= \frac{1}{2\langle\xi\rangle}\Bigg[\cos(\frac{2\theta_\epsilon(t,x,\xi)}{\epsilon\langle\xi\rangle})\bar{\xi} - \sin(\frac{2\theta_\epsilon(t,x,\xi)}{\epsilon\langle\xi\rangle})\xi^\perp + 2e_3\xi_3\cos(\frac{\theta_\epsilon(t,x,\xi)}{\epsilon\langle\xi\rangle})\Bigg] + \frac{1}{2\langle\xi\rangle}\bar{\xi}. \end{aligned}$$

Therefore, Taylor expanding the first term in the integrand of (4.10) gives

$$
\begin{aligned}
&\frac{\nabla_x b_e(X)\cdot \dot{X}}{b_e(X)^2}\sin\Big(\frac{\theta_\epsilon(t,x,\xi)}{\epsilon\left\langle \xi\right\rangle}\Big)\bar{\xi} = \frac{\nabla_x b_e(x)\cdot \dot{X}}{b_e(x)^2}\sin\Big(\frac{\theta_\epsilon(t,x,\xi)}{\epsilon\left\langle \xi\right\rangle}\Big)\bar{\xi} + \mathcal{O}(\epsilon)\\
&= \Bigg[\frac{\nabla_x b_e(x)}{2\left\langle \xi\right\rangle b_e(x)^2}\cdot\Bigg(\sin\Big(\frac{2\theta_\epsilon(t,x,\xi)}{\epsilon\left\langle \xi\right\rangle}\Big)\bar{\xi} + \cos\Big(\frac{2\theta_\epsilon(t,x,\xi)}{\epsilon\left\langle \xi\right\rangle}\Big)\xi^\perp + 2e_3\xi_3\sin\Big(\frac{\theta_\epsilon(t,x,\xi)}{\epsilon\left\langle \xi\right\rangle}\Big)\Bigg)\Bigg]\bar{\xi}\\
&\quad - \Bigg(\frac{\nabla_x b_e(x)\cdot \xi^\perp}{2\left\langle \xi\right\rangle b_e(x)^2}\Bigg)\bar{\xi} + \mathcal{O}(\epsilon),
\end{aligned}
\tag{4.12}
$$

and the other term in (4.10) becomes

$$
\begin{aligned}
&\frac{\nabla_x b_e(X)\cdot \dot{X}}{b_e(X)^2}\cos\Big(\frac{\theta_\epsilon(t,x,\xi)}{\epsilon\left\langle \xi\right\rangle}\Big)\xi^\perp = \frac{\nabla_x b_e(x)\cdot \dot{X}}{b_e(x)^2}\cos\Big(\frac{\theta_\epsilon(t,x,\xi)}{\epsilon\left\langle \xi\right\rangle}\Big)\xi^\perp + \mathcal{O}(\epsilon)\\
&= \Bigg[\frac{\nabla_x b_e(x)}{2\left\langle \xi\right\rangle b_e(x)^2}\cdot\Bigg(\cos\Big(\frac{2\theta_\epsilon(t,x,\xi)}{\epsilon\left\langle \xi\right\rangle}\Big)\bar{\xi} - \sin\Big(\frac{2\theta_\epsilon(t,x,\xi)}{\epsilon\left\langle \xi\right\rangle}\Big)\xi^\perp + 2e_3\xi_3\cos\Big(\frac{\theta_\epsilon(t,x,\xi)}{\epsilon\left\langle \xi\right\rangle}\Big)\Bigg)\Bigg]\xi^\perp\\
&\quad + \Bigg(\frac{\nabla_x b_e(x)\cdot \bar{\xi}}{2\left\langle \xi\right\rangle b_e(x)^2}\Bigg)\xi^\perp + \mathcal{O}(\epsilon).
\end{aligned}
\tag{4.13}
$$

Therefore after substituting (4.12) and (4.13) into (4.10) and integrating the oscillating terms by parts, up to order ϵ^2, we have

$$
\begin{aligned}
X - x - \frac{t\xi_3}{\left\langle \xi\right\rangle}e_3 &= \epsilon\frac{1}{b_e(X(t))}\Big(\sin\Big(\frac{\theta_\epsilon(t,x,\xi)}{\epsilon\left\langle \xi\right\rangle}\Big)\bar{\xi} + \cos\Big(\frac{\theta_\epsilon(t,x,\xi)}{\epsilon\left\langle \xi\right\rangle}\Big)\xi^\perp\Big) - \epsilon\frac{1}{b_e(x)}\xi^\perp\\
&\quad - \epsilon\int_0^t\Bigg[-\Bigg(\frac{\nabla_x b_e(x)\cdot \xi^\perp}{2\left\langle \xi\right\rangle b_e(x)^2}\Bigg)\bar{\xi} + \Bigg(\frac{\nabla_x b_e(x)\cdot \bar{\xi}}{2\left\langle \xi\right\rangle b_e(x)^2}\Bigg)\xi^\perp\Bigg]ds + \mathcal{O}(\epsilon^2)\\
&= \epsilon\frac{1}{b_e(x)}\Big(\sin\Big(\frac{\theta_\epsilon(t,x,\xi)}{\epsilon\left\langle \xi\right\rangle}\Big)\bar{\xi} + \cos\Big(\frac{\theta_\epsilon(t,x,\xi)}{\epsilon\left\langle \xi\right\rangle}\Big)\xi^\perp\Big) - \epsilon\frac{1}{b_e(x)}\xi^\perp\\
&\quad + \epsilon t\Bigg(\frac{\nabla_x b_e(x)\cdot \xi^\perp}{2\left\langle \xi\right\rangle b_e(x)^2}\Bigg)\bar{\xi} - \epsilon t\Bigg(\frac{\nabla_x b_e(x)\cdot \bar{\xi}}{2\left\langle \xi\right\rangle b_e(x)^2}\Bigg)\xi^\perp + \mathcal{O}(\epsilon^2).
\end{aligned}
\tag{4.14}
$$

Similarly we can Taylor expand θ_ϵ and integrate the oscillating terms by parts

$$
\begin{aligned}
&\theta_\epsilon(t,x,\xi) = \int_0^t b_e(x) + \nabla_x b_e(x)\cdot\Big[X - x - \frac{t\xi_3}{\left\langle \xi\right\rangle}e_3\Big]\,ds + \mathcal{O}(\epsilon^2)\\
&= b_e(x)t + \nabla_x b_e(x)\cdot\Bigg(-\epsilon t\frac{1}{b_e(x)}\xi^\perp + \epsilon t^2\Bigg(\frac{\nabla_x b_e(x)\cdot \xi^\perp}{4\left\langle \xi\right\rangle b_e(x)^2}\Bigg)\bar{\xi} - \epsilon t^2\Bigg(\frac{\nabla_x b_e(x)\cdot \bar{\xi}}{4\left\langle \xi\right\rangle b_e(x)^2}\Bigg)\xi^\perp\Bigg) + \mathcal{O}(\epsilon^2)\\
&= \Phi(t,x,\xi) + \mathcal{O}(\epsilon^2).
\end{aligned}
\tag{4.15}
$$

After replacing this inside (4.8), we get (4.7). Finally, we can replace θ_ϵ inside (4.14) as indicated in (4.15) to recover (4.6). □

Lemma 9 gives the immediate corollary which follows.

Corollary 3. *There exists ϵ_0 and $T > 0$ independent of ϵ_0, such that for all $\epsilon \in (0, \epsilon_0]$, $t \in [0, T]$ and $|\xi| \leq R^0_\xi$ the solution maps $x \mapsto X(t,x,\xi)$ of (4.3) is a diffeomorpshism.*

Proof. The proof easily follows by computing $D_x X$ and taking the operator sup-norm

$$||(D_x X - Id_{3\times 3})(t,\cdot)||_{L^\infty_x(|x-y|\le t)} \le Ct + \mathcal{O}(\epsilon),$$

where C depends only on R^0_ξ and $||b_e||_{W^{2,\infty}}$. so that for t and ϵ small enough one has

$$||(D_x X - Id_{3\times 3})(t,\cdot)||_{L^\infty_x(|x-y|\le t)} < 1.$$

The map $X(t,\cdot,\xi)$ is therefore a local diffeomorphism. It is injective (uniqueness part of Cauchy-Lipschitz Theorem) and it is surjective (it suffices to integrate the flow in the opposite direction, from t to 0). It is bijective, and thereby it is a global diffeomorphism. □

Remark 4. *A similar approximation to (4.6) and (4.7) holds when we include the quadratic term Q coming from the linearized version of the characteristics (3.67)-(3.68). That is to say, the normal form procedure of Lemma 9 holds when we allow the direction of the applied field $\mathbf{B}_e$ to vary. However, the procedure and approximation is much less explicit as it depends on the matrix $O(x)$. Furthermore, we no longer have $b_e(x)$ independent of x_3, so the 3rd component of the approximation (4.6) is less trivial.*

4.2 Uniform Bounds of Dilute Linear Model in Inhomogeneous Magnetic Field

In this section we derive uniform estimates for a linear model of a dilute plasma in an inhomogeneous magnetic field given by the following Cauchy problem:

$$\begin{cases} \partial_t f_\ell + v(O(x)\xi)\cdot\nabla_x f_\ell - \epsilon^{-1}\langle\xi\rangle^{-1} b_e(x)\partial_\theta f_\ell \\ \qquad + Q(x,\xi)\cdot\nabla_\xi f_\ell = \epsilon M'(|\xi|)|\xi|^{-1}O(x)\xi\cdot E_\ell, \\ \partial_t E_\ell - \nabla_x\times B_\ell = \displaystyle\int v(O(x)\xi) f_\ell d\xi, & \nabla_x\cdot E = -\rho(f_\ell) \\ \partial_t B_\ell + \nabla_x\times E_\ell = 0, & \nabla_x\cdot B_\ell = 0 \end{cases} \tag{4.16}$$

together with

$$\nabla_x\cdot E^{in} = -\rho(f^{in}), \quad \nabla_x\cdot B^{in} = 0. \tag{4.17}$$

Again we assume the compatibility conditions on the initial data (f^{in},E^{in},B^{in}):

$$\nabla_x\cdot E^{in} = -\rho(f^{in}), \quad \nabla_x\cdot B^{in} = 0. \tag{4.18}$$

Note that (4.16) is the linearized version of the straightened system (3.63). In this section we prove the following proposition.

Proposition 1. *(Uniform estimates for the solutions of the inhomogeneous linear model) Let $(f^{in},E^{in},B^{in})\in C^3_c(\mathbb{R}^6;\mathbb{R})\times[C^1_c(\mathbb{R}^3;\mathbb{R}^3)]^2$ satisfying the compatibility conditions (4.18). Suppose there is $R^0_\xi>0$ such that supp$(f^{in}(x,\cdot))\subset\{|\xi|\le R^0_\xi\}$ and denote by $(f_\epsilon,E_\epsilon,B_\epsilon)$ the solution in $C^1(\mathbb{R}_+;L^\infty_{x,\xi})$ of the linear Cauchy problem (4.16)-(4.17). Then, there exists $T>0$ and $\epsilon_0(T)\in(0,1]$ such that for all $\epsilon\in(0,\epsilon_0]$ we can find a constant C_T depending on T, R^0_ξ and $||(f^{in},E^{in},B^{in})||_{W^{1,\infty}_{x,\xi}}$ such that for all $t\in[0,T]$, we have*

$$\begin{aligned} ||(f_\epsilon,E_\epsilon,B_\epsilon)(t)||_{L^\infty_{x,\xi}} + ||\epsilon\partial_t(f_\epsilon,E_\epsilon,B_\epsilon)(t)||_{L^\infty_{x,\xi}} + ||\partial_{x_3}(f_\epsilon,E_\epsilon,B_\epsilon)(t)||_{L^\infty_{x,\xi}} \\ + ||\epsilon\bar\nabla_x(f_\epsilon,E_\epsilon,B_\epsilon)(t)||_{L^\infty_{x,\xi}} + ||\epsilon\nabla_\xi f_\epsilon(t)||_{L^\infty_{x,\xi}} \le C_T, \end{aligned} \tag{4.19}$$

where $\bar{\nabla}_x := (\partial_{x_1}, \partial_{x_2}, 0)$*. Furthermore, in the case of prepared data, when*

$$||\partial_\theta f^{in}||_{L^\infty_{x,\xi}} \leq \epsilon C, \tag{4.20}$$

the preceding Lipschitz norm control becomes

$$\begin{aligned}||(f_\epsilon, E_\epsilon, B_\epsilon)(t)||_{L^\infty_{x,\xi}} + ||\partial_t(f_\epsilon, E_\epsilon, B_\epsilon)(t)||_{L^\infty_{x,\xi}} + ||\partial_{x_3}(f, E, B)(t)||_{L^\infty_{x,\xi}} \\ + ||\bar{\nabla}_x(f_\epsilon, \epsilon E_\epsilon, \epsilon B_\epsilon)(t)||_{L^\infty_{x,\xi}} + ||\nabla_\xi f_\epsilon(t)||_{L^\infty_{x,\xi}} \leq C_T.\end{aligned} \tag{4.21}$$

Before proving Proposition 1, we would like to illustrate the optimally of its estimates. Indeed, the prepared data assumption is necessary for uniform Lipschitz estimates in both the linear and non-linear system. The underlying mechanism is local (we can forget the condition on the support), and it does not involve the spatial variable x. Thus, we can explain it below by looking at functions f depending only on t and ξ.

Example 1. *Consider the initial data given by*

$$f^{in}(\xi(r,\theta,z)) = \chi(r,z)\cos(2\theta), \quad (E^{in}, B^{in}) = (0,0),$$

where (r,θ,z) *are the cylindrical coordinates for* ξ*, and take* $\chi \in C^1_c(\mathbb{R}_+ \times \mathbb{R})$*. Then*

$$\begin{aligned} &f(t,x,\xi) = \chi(r,z)\cos(2(\theta + \frac{t}{\epsilon\langle\xi\rangle})), \\ &(E,B)(t,x) \equiv (0,0), \end{aligned}$$

solves the nonlinear problem with $M \equiv 0$*:*

$$\begin{cases} \partial_t f + v(\xi)\cdot\nabla_x f - \frac{1}{\epsilon\langle\xi\rangle}\partial_\theta f - \epsilon[E + v(\xi)\times B]\cdot\nabla_\xi f = 0 \\ \partial_t E - \nabla_x \times B = J(f), \quad \nabla_x \cdot E = -\rho(f) \\ \partial_t B + \nabla_x \times E = 0, \qquad\quad \nabla_x \cdot B = 0 \\ (f,E,B)|_{t=0} = (f^{in}, 0, 0) \end{cases}$$

This follows by construction, as we have $J(f) = 0$ *and* $\rho(f) = 0$*. When* $\chi \not\equiv 0$*, we do not have (4.20), and we see that the control (4.21) is not satisfied since both* $|\partial_t f|$ *and* $|\nabla_\xi f|$ *are of order* ϵ^{-1}*.*

Proof of Proposition 1. The non singular terms inside (3.52) can be handled as in [7] or as briefly explained in Subsection 3.4. Thus, we can focus on the more problematic term implied by (3.52), the one which is of order ϵ^{-1}. This involves $\bar{f}$ and not the expression f obtained through the change of variables (3.62). The change of variable $\xi = O(x)\eta$ allows to remedy this, yielding

$$-\epsilon^{-1}\int \nabla_\xi p\big(t,x,O(x)\eta\big)Y(t,x) *_{t,x} \big(\mathbb{1}_{t>0}[v\big(O(x)\eta\big)\times \mathbf{B}_e]f_\ell(t,x,\eta)\big)\,d\eta. \tag{4.22}$$

The aim of Lemma 10 is to reformulate (4.22). Thanks to (4.22), we can handle the solution f_ℓ of (4.16). The interest is that we can solve f_ℓ in (4.16) using Duhamel's Principle

$$f_\ell(t,x,\xi) = f^{in}(X(-t),\Xi(-t)) + \epsilon\int_0^t \big(M'(|\xi|)\frac{O(X)\Xi}{|\xi|}\cdot E_\ell\big)(s,X(t-s),\Xi(t-s))ds, \tag{4.23}$$

where $(X,\Xi)(t)$ solve the characteristics of the linear Vlasov equation in the system (4.16). Our goal is to absorb the singular factor ϵ^{-1} inside (4.22). To accomplish this, the strategy is to substitute (4.23) into (4.22). This yields a sum of two terms. The first is

$$-\epsilon^{-1}\int \nabla_\xi p\big(t,x,O(x)\eta\big)Y(t,x) *_{t,x} \big(\mathbb{1}_{t>0}[v\big(O(x)\eta\big)\times \mathbf{B}_e]f^{in}(X(-t),\Xi(-t))\big)\,d\eta. \tag{4.24}$$

The second is

$$-\int \nabla_\xi p\big(t,x,O(x)\eta\big) *_{t,x} \Big(\mathbb{1}_{t>0}[v\big(O(x)\eta\big)\times \mathbf{B}_e] \\ \times \int_0^t \big(M'(|\eta|)\frac{O(X)\Xi}{|\eta|}\cdot E_\ell\big)(s,X(t-s),\Xi(t-s))ds\Big)\,d\eta. \tag{4.25}$$

Lemma 11 is devoted to estimate (4.24). We accept to lose derivatives of the data f^{in}, and therefore we can apply non-stationary phase arguments. The idea is to take advantage of the rapid oscillations implied by the characteristics.

Lemma 12 deals with (4.25). We are saved by the dilute equilibrium condition $M'_\epsilon(|\xi|)=\mathcal{O}(\epsilon)$ which explains why the singular factor ϵ^{-1} has disappeared from (4.25).

Briefly, Proposition 1 is proved in four stages made of three lemmas 10, 11 and 12, followed by a closing paragraph "*End of proof or Proposition 1*". $\square$

A preliminary step is to reformulate (4.24) using f instead of $\bar f$, with f as in (3.62).

Lemma 10. *[Transfer of derivatives after straightening of the field lines] Let $\bar g(t,\cdot)\in W^{1,\infty}_{x,\xi}$. Then, under the change of variables*

$$g(t,x,\xi):=\bar g(t,x,O(x)\xi),\quad \bar g(t,x,\xi)=g(t,x,O^t(x)\xi),$$

we have the following identity

$$\int p(t,x,\xi)Y(t,x) *_{t,x} \mathbb{1}_{t>0}\nabla_\xi\cdot[(v(\xi)\times\frac{1}{\epsilon}\mathbf{B}_e(x))\bar g(t,x,\xi)]d\xi \\ = -\frac{1}{4\pi\epsilon}\int_0^t\int_{\mathbb{S}^2}\int s\big(\frac{d}{d\theta}[p(1,\omega,O(s\omega)\eta)]\big)\frac{b_e(x-s\omega)}{\langle\eta\rangle}g(t-s,x-s\omega,\eta)d\eta dS(\omega)ds, \tag{4.26}$$

where $d/d\theta$ is the total derivative with respect to the new variable $\eta:=O^t(x)\xi$,

$$\frac{d}{d\theta}:=\eta_2\partial_{\eta_1}-\eta_1\partial_{\eta_2}=\eta^\perp\cdot\nabla_\eta.$$

Proof. First consider the variable change, with x as a parameter

$$\xi:=O(x)\eta,\quad \eta:=O^t(x)\xi,\quad d\xi=|\det(O(x))|d\eta=d\eta. \tag{4.27}$$

Then using the formula (3.75), remarking that $|\det(O(x))|=1$, and again that $O^t\mathbf{B}_e(x)=b_e(x)e_3$ we have the divergence in the new variable becomes

$$\begin{aligned}\forall x\in\mathbb{R}^3,\ \nabla_\xi\cdot\big[\big(\frac{\xi}{\langle\xi\rangle}\times\mathbf{B}_e(x)\big)\bar g(t,x,\xi)\big] &= \nabla_\eta\cdot\big[O^t(x)\big(\frac{O(x)\eta}{\langle\eta\rangle}\times\mathbf{B}_e(x)\big)\bar g(t,x,O(x)\eta)\big]\\ &= \nabla_\eta\cdot\big[O^t(x)O(x)\big(\frac{\eta}{\langle\eta\rangle}\times O^t\mathbf{B}_e(x)\big)g(t,x,\eta)\big]\\ &= \nabla_\eta\cdot\big[b_e(x)\frac{\eta^\perp}{\langle\eta\rangle}g(t,x,\eta)\big]\end{aligned}$$

Therefore we can change variables, apply Lemma 5, then integrate by parts in the new momentum variable η to arrive at the conclusion

$$\begin{aligned}
&\int p(t,x,\xi)Y(t,x) *_{t,x} \mathbb{1}_{t>0}\nabla_\xi \cdot [(v(\xi) \times \frac{1}{\epsilon}\mathbf{B}_e(x))\bar{g}(t,x,\xi)]d\xi \\
&= \int_0^t \int\int p(t-s, x-y, \xi)Y(t-s,x-y)\nabla_\xi \cdot [(v(\xi) \times \frac{1}{\epsilon}\mathbf{B}_e(y))\bar{g}(s,y,\xi)]d\xi dy ds \\
&= \epsilon^{-1}\int_0^t \int\int p(t-s,x-y,O(y)\eta)Y(t-s,x-y)\nabla_\eta \cdot \big(b_e(y)\frac{\eta^\perp}{\langle\eta\rangle}g(s,y,\eta)\big)d\eta dy ds \\
&= \frac{1}{4\pi\epsilon}\int_0^t \int_{\mathbb{S}^2}\int sp(1,\omega,O(s\omega)\eta)\nabla_\eta \cdot \big(b_e(x-s\omega)\frac{\eta^\perp}{\langle\eta\rangle}g(t-s,x-s\omega,\eta)\big)d\eta dS(\omega)ds \\
&= -\frac{1}{4\pi\epsilon}\int_0^t \int_{\mathbb{S}^2}\int s\eta^\perp \cdot \nabla_\eta\Big[p(1,\omega,O(s\omega)\eta)\Big]\frac{b_e(x-s\omega)}{\langle\eta\rangle}g(t-s,x-s\omega,\eta)d\eta dS(\omega)ds \\
&= -\frac{1}{4\pi\epsilon}\int_0^t \int_{\mathbb{S}^2}\int s\frac{d}{d\theta}\Big[p(1,\omega,O(s\omega)\eta)\Big]\frac{b_e(x-s\omega)}{\langle\eta\rangle}g(t-s,x-s\omega,\eta)d\eta dS(\omega)ds.
\end{aligned}$$

Observe that there are no more derivatives on g, but instead derivatives on the symbol $p(\cdot)$ given by (3.42). We can check that this derivative on $p(\cdot)$ is non zero. □

Next we will state lemma 11, which allows us to regain a factor of ϵ, by taking advantage of the time averaged rapid oscillations coming from the characteristics. Given Lemma 10, the method will hold when $O(x) \not\equiv Id_{3\times3}$. However, as stated in Remark 4, the approximations of Lemma 9 are less explicit. Therefore from now on we assume

$$\mathbf{B}_e(x) = b_e(x)^t(0,0,1),\ O(x) \equiv Id_{3\times3},\ Q(x,\xi) \equiv 0.$$

Then we have the following, where ξ comes to replace η to fit with the presentation of (4.1).

Lemma 11. *[Impact of the oscillating flow] For any $T > 0$, there exists $\epsilon_0(T) \in (0,1]$, and constant $C_T := C_T(R^0_\xi, ||f^{in}||_{W^{1,\infty}_{x,\xi}}, ||b_e||_{W^{2,\infty}})$ such that for all $\epsilon \in (0,\epsilon_0]$, and all $t \in [0,T]$, the following estimate holds*

$$\begin{aligned}
\Big|\int\int_{\mathbb{S}^2}\int_0^t &\frac{s\partial_\theta p(1,\omega,\xi)}{4\pi}\frac{b_e(x-s\omega)}{\langle\eta\rangle} \\
&\times f^{in}\circ\big[(X_\ell,\Xi_\ell)(-(t-s),x-s\omega,\xi)\big]dsdS(\omega)d\xi\Big| \le \epsilon C_T. \qquad (4.28)
\end{aligned}$$

Proof. We can first simplify our analysis by Taylor expanding f^{in} composed with the flow with respect to X using Lemma 9

$$f^{in}(X(t),\Xi(t)) = f^{in}(x + \frac{t\xi_3}{\langle\xi\rangle}e_3, \Xi(t)) + \mathcal{O}(\epsilon||\nabla_x f^{in}||_{L^\infty}).$$

This remaining term of order $\mathcal{O}(\epsilon||\nabla_x f^{in}||_{L^\infty})$, when substituted into (4.28), is controlled using Lemma 5 by the constant

$$C_1(t) := \frac{t^2}{3}||p(1,\cdot,\cdot)||_{L^\infty(\mathbb{S}^2\times\{|\xi|\le R^0_\xi\})}||b_e||_{L^\infty(|x|\le R^0_x+t)}||\nabla_x f^{in}||_{L^\infty}||R_\epsilon||_{L^\infty(|x|\le R^0_x+t,\ |\xi|\le R^0_\xi)}. \qquad (4.29)$$

Next remark that the momentum component $\Xi(t)$ given by (4.8) can be viewed as a rotation as follows

$$\Xi(t,x,\xi)=\mathcal{R}(\frac{\Phi(t,x,\xi)}{\epsilon\left\langle\xi\right\rangle})\xi+\mathcal{O}(\epsilon),$$

where $\mathcal{R}$ is the rotation matrix about the ξ_3-axis

$$\mathcal{R}(\theta)=\begin{bmatrix}\cos(\theta) & -\sin(\theta) & 0\\ \sin(\theta) & \cos(\theta) & 0\\ 0 & 0 & 1\end{bmatrix}.$$

Therefore we may convert $\Xi(t)$ to cylindrical coordinates

$$\Xi(t,x,\xi(r,\theta,z))=\begin{bmatrix}r\cos(\theta+\frac{\Phi(t,x,\xi)}{\epsilon\langle\xi\rangle})\\ r\sin(\theta+\frac{\Phi(t,x,\xi)}{\epsilon\langle\xi\rangle})\\ z\end{bmatrix}+\mathcal{O}(\epsilon).$$

where $z=\Xi_3=\xi_3$ and $r=\sqrt{\Xi_1^2+\Xi_2^2}=\sqrt{\xi_1^2+\xi_2^2}$ are independent of time. Then with a slight abuse of notation on the dependence of r and z we consider the Fourier series

$$f^{in}(X(t),\Xi(t))=\sum_{n\in\mathbb{Z}}f_n^{in}(x+\frac{t\xi_3}{\langle\xi\rangle}e_3,r,z)e^{in(\theta+\frac{\Phi(t,x,\xi)}{\epsilon\langle\xi\rangle})}+\mathcal{O}\big(\epsilon||\nabla_{\xi,x}f^{in}||\big).\tag{4.30}$$

Remark a similar estimate to (4.29) holds for the order $\epsilon||\nabla_\xi f^{in}||$ term, now including a momentum derivative. Then substituting the order 1 term of (4.30), which must be evaluated at the position $(-(t-s),x-s\omega,\xi)$, into (4.28), it remains to consider

$$\int\int_{\mathbb{S}^2}\int_0^t\frac{s\partial_\theta p(1,\omega,\xi)}{4\pi}\frac{b_e(x-s\omega)}{\epsilon\left\langle\xi\right\rangle}\times\\ \sum_{n\in\mathbb{Z}^*}f_n^{in}(x+\frac{(s-t)z}{\langle\xi\rangle}e_3-s\omega,r,z)e^{in(\theta+\frac{\Phi(s-t,x-s\omega,\xi)}{\epsilon\langle\xi\rangle})}dsdS(\omega)d\xi.\tag{4.31}$$

In the above sum, the integer $n=0$ does not appear because the integration with respect to the variable θ of the derivative $\partial_\theta p$ is simply zero.The next step is to gain back the factor of ϵ by a time integration along the lines of the proof of lemma 9. First note

$$e^{in\frac{\Phi(s-t,x-s\omega,\xi)}{\epsilon\langle\xi\rangle}}=\frac{\epsilon\left\langle\xi\right\rangle}{in[\partial_t\Phi-\omega\cdot\nabla_x\Phi](s-t,x-s\omega,\xi)}\partial_s(e^{in\frac{\Phi(s-t,x-s\omega,\xi)}{\epsilon\langle\xi\rangle}}),\ n\neq 0.\tag{4.32}$$

Therefore we can integrate by parts in time s, as long as the denominator of (4.32) does not vanish. Recalling (4.4) we have

$$(\partial_t\Phi-\omega\cdot\nabla_x\Phi)(t,x,\xi)=b_e(x)-\omega\cdot\nabla_x b_e(x)t\\ -\epsilon\big[\partial_t-\omega\cdot\nabla_x\big]\bigg(\nabla_x b_e(x)\cdot\big(t\frac{1}{b_e(x)}\xi^\perp-t^2(\frac{\nabla_x b_e(x)\cdot\xi^\perp}{4\left\langle\xi\right\rangle b_e(x)^2})\bar{\xi}+t^2(\frac{\nabla_x b_e(x)\cdot\bar{\xi}}{4\left\langle\xi\right\rangle b_e(x)^2})\xi^\perp\big)\bigg).$$

Therefore choose $T > 0$ small enough such that

$$b_- - T||\nabla_x b_e||_{L^\infty} > \frac{3}{4}b_- > 0,$$

where b_- is defined by (4.11). Furthermore, choose $\epsilon_0(T)$ small enough, such that for $|\xi| \le R_\xi^T$ fixed, we can bound the reciprocal according to

$$\forall t \in [0,T],\ \Big|\frac{1}{\partial_t \Phi - \omega \cdot \nabla_x \Phi}\Big| \le \frac{2}{b_-}.$$

Thus integrating (4.31) by parts in time using (4.32), we find the term (4.31) remains uniformly bounded, depending only on initial data as long as the Fourier series is absolutely convergent in the sense that

$$\sum_{n \in \mathbb{Z}-\{0\}} \frac{||f_n^{in}||_{W_x^{1,\infty}, L_{r,z}^\infty}}{|n|} \le C,$$

which is guaranteed by the C^3-smoothness of f^{in}. □

Next, we consider (4.25).

Lemma 12. *There exists $C_T = C_T(||b_e||_{L^\infty}, ||M'||_{L^\infty}, R_\xi^0)$ such that the following estimate holds*

$$\int \partial_\theta p Y *_{t,x} (H \mathbb{1}_{t\ge 0}) d\xi \le C_T \int_0^t \sup_{s' \in [0,s]} ||E(s',\cdot)||_{L_x^\infty} ds, \tag{4.33}$$

where

$$H(t,x,\xi) := \frac{b_e(x)}{\langle \xi \rangle} \int_0^t \left(M'(|\xi|)\frac{\Xi}{|\xi|} \cdot E \right)(s, X(t-s), \Xi(t-s)) ds. \tag{4.34}$$

Proof. Once again Lemma 5 implies

$$\begin{aligned}
&\int \partial_\theta p Y *_{t,x} (H \mathbb{1}_{t\ge 0}) d\xi = \\
&\int \int_{\mathbb{S}^2} \int_0^t \frac{s \partial_\theta p(1,\omega,\xi)}{4\pi} \frac{b_e(x - s\omega)}{\langle \xi \rangle} \times \\
&\qquad \left[\int_0^{t-s} \left(M'(|\xi|)\frac{\Xi}{|\xi|} \cdot E \right)(s', X(t-s-s'), \Xi(t-s-s')) ds' \right] ds dS(\omega) d\xi \\
&\qquad \lesssim \frac{||b_e||_{L^\infty(B(x,t))} t^2}{3} \int \frac{1}{\langle \xi \rangle} |M'(|\xi|)| ||\partial_\theta p(1,\cdot,\xi)||_{L^\infty(\mathbb{S}^2)} d\xi \int_0^t \sup_{s' \in [0,s]} ||E(s',\cdot)||_{L_x^\infty} ds.
\end{aligned} \tag{4.35}$$

□

As an immediate corollary to lemma 11 and 12 we obtain the uniform Sup norm estimate in Proposition 1. We now prove the remainder of Proposition 1, which is related to the information involving derivatives .

End of proof or Proposition 1. There is nothing left to be done to estimate the Sup-norm. To estimate the Lipschitz norm we use a trick from geometric optics. We first compute ∂_i with $i = t, x_1, x_2$ or x_3, by first dividing the Vlasov equation by $b_e(x)$, apply ∂_i, then re-multiply by $b_e(x)$. By this way there is no term of size ϵ^{-1} acting as a source:

$$\begin{cases} \partial_t(\partial_i f) + v(\xi)\cdot\nabla_x(\partial_i f) - \frac{b_e(x)}{\epsilon\langle\xi\rangle}\partial_\theta(\partial_i f) = \epsilon M'(|\xi|)\frac{\xi}{|\xi|}\cdot(\partial_i E) \\ \qquad\qquad + \partial_i \ln(b_e)[\partial_t f + v(\xi)\cdot\nabla_x f - \epsilon M'(|\xi|)\frac{\xi}{|\xi|}\cdot E] \\ \partial_t(\partial_i E) - \nabla_x\times(\partial_i B) = J(\partial_i f), \quad \nabla_x\cdot(\partial_i E) = -\rho(\partial_i f) \\ \partial_t(\partial_i B) + \nabla_x\times(\partial_i E) = 0, \qquad \nabla_x\cdot(\partial_i B) = 0 \end{cases} \tag{4.36}$$

Remark that if $\partial_{x_i}\ln(b_e) = \mathcal{O}(\epsilon)$ (such as the constant case when $\partial_{x_i} b_e \equiv 0$), the sup-norm proof would follow once more since the new source term introduced into the first equation in (4.36) would be of size ϵ. Therefore, in the case when $i = t$ or x_3 we have $\partial_i \ln(b_e) \equiv 0$ and therefore, we can once again integrate along the flow

$$\partial_{x_3} f(t,x,\xi) = \partial_{x_3} f^{in}(X(-t),\Xi(-t)) + \epsilon\int_0^t \left(M'(|\xi|)\frac{\Xi}{|\xi|}\cdot\partial_{x_3}E\right)(s, X(t-s), \Xi(t-s))ds$$

and

$$\partial_t f(t,x,\xi) = \partial_t f|_{t=0}(X(-t),\Xi(-t)) + \epsilon\int_0^t \left(M'(|\xi|)\frac{\Xi}{|\xi|}\cdot\partial_t E\right)(s, X(t-s), \Xi(t-s))ds. \tag{4.37}$$

For general (non prepared) data, the time derivative $\partial_t f$ in the right hand side of (4.36) and in (4.37) are of size ϵ^{-1}. The weight ϵ must be put in factor of $\partial_t f$, $\partial_1 f$ and $\partial_2 f$ to compensate this. Then, by the previous argument, we have

$$||\epsilon\partial_t(f,E,B)(t)||_{L^\infty_{x,\xi}} + ||\partial_{x_3}(f,E,B)(t)||_{L^\infty_{x,\xi}} \leq C_T, \tag{4.38}$$

and if the data is prepared to ensure $||\partial_t f|_{t=0}||_{L^\infty_{x,\xi}} \leq \epsilon C$, then we have uniform estimates

$$||\partial_t(f,E,B)(t)||_{L^\infty_{x,\xi}} + ||\partial_{x_3}(f,E,B)(t)||_{L^\infty_{x,\xi}} \leq C_T. \tag{4.39}$$

When we consider $i = x_1$ and x_2 we can integrate along the flow to estimate $\partial_i f$ as follows

$$\begin{aligned} |\partial_i f(t,x,\xi)| \leq{}& ||\partial_i f^{in}(\cdot)||_{L^\infty_{x,\xi}} + C\int_0^t ||\epsilon E(s,\cdot)||_{L^\infty_x} + ||\epsilon\partial_i E(s,\cdot)||_{L^\infty_x} ds \\ &+ C\int_0^t ||\partial_t f(s,\cdot)||_{L^\infty_{x,\xi}} + ||\nabla_x f(s,\cdot)||_{L^\infty_{x,\xi}} ds. \end{aligned} \tag{4.40}$$

For the fields, similar to (3.47), we again have

$$\Box\partial_i E = \int v(\xi)\partial_t(\partial_i f) + \nabla_x(\partial_i f) d\xi.$$

Thus

$$\begin{aligned} \partial_i E ={}& K_1(\partial_i E^{in}) - \frac{t}{4\pi}\int\int_{\mathbb{S}^2} p(1,\omega,\xi)\partial_i f^{in}(x - t\omega, \xi)d\omega d\xi \\ &- \int p(t,x,\xi)Y(t,x) *_{t,x} (\mathbb{1}_{t>0} T(\partial_i f))d\xi + \int q(t,x,\xi)Y(t,x) *_{t,x} (\mathbb{1}_{t>0}\partial_i f)d\xi, \end{aligned}$$

and it follows, after multiplying by ϵ and replacing $T(\partial_i f)$, that $\partial_i E$ can be estimated as follows

$$
\begin{aligned}
|\epsilon\partial_i E(t,x)| \leq{}& C_T(R^0_\xi)||\epsilon\partial_i(f^{in},E^{in},B^{in})||_{L^\infty_{x,\xi}} \\
&+ C_T(R^0_\xi)\int_0^t (1+\epsilon)||\partial_i f(s,\cdot)||_{L^\infty_{x,\xi}} + ||\epsilon\partial_t f(s,\cdot)||_{L^\infty_{x,\xi}} + ||\epsilon\nabla_x f(s,\cdot)||_{L^\infty_{x,\xi}} ds \\
&+ C_T(R^0_\xi)\int_0^t ||\epsilon\partial_i E(s,\cdot)||_{L^\infty_x} + ||\epsilon^2 E(s,\cdot)||_{L^\infty_x} ds \qquad (4.41)
\end{aligned}
$$

and similarly for B_i. Thus adding (4.40) and (4.41) (and the similar expression for $\partial_i B$) and applying Gronwall's lemma we have for prepared data

$$
||\partial_{x_1}(f,\epsilon E,\epsilon B)(s,\cdot)||_{L^\infty_{x,\xi}} + ||\partial_{x_2}(f,\epsilon E,\epsilon B)(s,\cdot)||_{L^\infty_{x,\xi}} \leq C_T \qquad (4.42)
$$

For the momentum derivatives we again use a method from geometric optics. We can estimate $\nabla_\xi f$ using the cylindrical operator

$$
\nabla_\xi = e_\theta\frac{1}{r}\partial_\theta + e_r\partial_r + e_z\partial_z.
$$

Remark that although the commutator $[\frac{1}{r}\partial_\theta,\partial_r] = \frac{1}{r^2}\partial_\theta \neq 0$, we do in fact have $[\partial_\theta,\partial_r]=0$. Therefore, we first multiply (4.16) by $\langle\xi\rangle$, then apply ∂_j with $j\in\{r,z\}$ and then divide once again by $\langle\xi\rangle$ leading to the expression

$$
\begin{aligned}
\partial_t(\partial_j f) + v(\xi)\cdot\nabla_x(\partial_j f) - \frac{b_e(x)}{\epsilon\langle\xi\rangle}\partial_\theta(\partial_j f) ={}& \epsilon\partial_j\big(M'(|\xi|)\frac{\xi}{|\xi|}\big)\cdot E \\
&- \partial_j(v(\xi))\cdot\nabla_x f + \partial_j(\ln(\langle\xi\rangle))\big[\partial_t f + v(\xi)\cdot\nabla_x f - \epsilon M'(|\xi|)\frac{\xi}{|\xi|}\cdot E\big]. \qquad (4.43)
\end{aligned}
$$

Assume that $M'(0)=0$ and $M'(|\xi|)=\mathcal{O}(|\xi|)$ at $\xi=0$ to ensure $|\frac{M'(|\xi|)}{|\xi|}|$ remains bounded at $|\xi|=0$. This is satisfied as long as there is a differentiable extension of $M(\cdot)$ from $\mathbb{R}_+$ to $\mathbb{R}$. Thus integrating along the flow gives the estimate

$$
|\partial_j f(t,x,\xi)| \leq ||\partial_j f^{in}||_{L^\infty_{x,\xi}} + C(R^0_\xi)\int_0^t ||\partial_t f(s,\cdot)||_{L^\infty_{x,\xi}} + ||\nabla_x f(s,\cdot)||_{L^\infty_{x,\xi}} + ||\epsilon E(s,\cdot)||_{L^\infty_x} ds. \qquad (4.44)
$$

For the θ derivative we simply apply the operator $\frac{1}{r}\partial_\theta$ directly since $\partial_\theta(\frac{1}{\langle\xi\rangle})=0$. This gives

$$
\partial_t(\frac{1}{r}\partial_\theta f) + v(\xi)\cdot\nabla_x(\frac{1}{r}\partial_\theta f) - \frac{1}{\epsilon\langle\xi\rangle}\partial_\theta(\frac{1}{r}\partial_\theta f) = -\epsilon M'(|\xi|)\frac{\xi^\perp}{r|\xi|}\cdot E - \frac{\xi^\perp}{r\langle\xi\rangle}\cdot\nabla_x f.
$$

With the same assumptions on M, we then arrive at

$$
|\frac{1}{r}\partial_\theta f(t,x,\xi)| \leq ||\frac{1}{r}\partial_\theta f^{in}||_{L^\infty_{x,\xi}} + ||\frac{M'(|\xi|)}{|\xi|}||_{L^\infty_\xi}\int_0^t ||\epsilon E(s,\cdot)||_{L^\infty_x} ds + \int_0^t ||\nabla_x f(s,\cdot)||_{L^\infty_{x,\xi}}. \qquad (4.45)
$$

Thus adding (4.44) and (4.45) and using the established estimates (4.42) and (4.39) we can conclude for prepared data

$$||\nabla_\xi f(t,\cdot)||_{L^\infty_{x,\xi}} \leq C_T$$

□

Remark 5. *The previous proof relies heavily on the dilute assumption to ensure no loss of derivatives when estimating the fields. In order to control the initial data for f along the flow a non-stationary phase argument was used. This implied derivatives on the initial data of f. However, the singular term coming from the source term of the vlasov equation was required to be small (coming from the dilute assumption) in order to avoid this integration by parts step which would give a loss of derivatives when estimating the fields (E, B).*

4.3 Approximation of Dilute HMRVM System

In this section we prove Theorem 1. The goal of this section is use the results for the linear models in the previous section to deduce results for complete non-linear problem under the dilute assumption. In this section we will denote (f, E, B) as a solution to the Cauchy problem

$$\begin{cases} \partial_t f + v(\xi)\cdot\nabla_x f - \frac{b_e(x)}{\epsilon\langle\xi\rangle}\partial_\theta f - \epsilon[E + v(\xi)\times B]\cdot\nabla_\xi f = \epsilon M'(|\xi|)\frac{\xi}{|\xi|}\cdot E \\ \partial_t E - \nabla_x \times B = J(f), \quad \nabla_x\cdot E = -\rho(f) \\ \partial_t B + \nabla_x\times E = 0, \qquad \nabla_x\cdot B = 0 \\ (f,E,B)|_{t=0} = (f^{in}, E^{in}, B^{in}) \end{cases} \tag{4.46}$$

Then let (f_ℓ, E_ℓ, B_ℓ) denote the solution to the associated linear system

$$\begin{cases} \partial_t f_\ell + v(\xi)\cdot\nabla_x f_\ell - \frac{b_e(x)}{\epsilon\langle\xi\rangle}\partial_\theta f_\ell = \epsilon M'(|\xi|)\frac{\xi}{|\xi|}\cdot E_\ell \\ \partial_t E_\ell - \nabla_x \times B_\ell = J(f_\ell), \quad \nabla_x\cdot E_\ell = -\rho(f_\ell) \\ \partial_t B_\ell + \nabla_x\times E_\ell = 0, \qquad \nabla_x\cdot B_\ell = 0 \\ (f,E,B)|_{t=0} = (f^{in}, E^{in}, B^{in}) \end{cases} \tag{4.47}$$

In the dilute case, the ability to approximate the non-linear system with the linear version is due to the fact the first equation in (4.46) only $(\epsilon E, \epsilon B)$ appear instead of (E, B). This allows for a linearization in the variable f when the data is prepared. This guarantees the non linear term $\epsilon[E + v\times B]\cdot\nabla_\xi f$ remains small. Theorem 2 gives a precise relationship between (4.46) and (4.47). Furthermore, Theorem 1 follows as a corollary of Theorem 2.

Theorem 2. *(Local in time solution of non-linear system for prepared data) Let (f_ℓ, E_ℓ, B_ℓ) be a solution of (4.47) with initial data (f^{in}, E^{in}, B^{in}) satisfying the compatibility (4.18) and be prepared in the sense of (4.20). Then there exits $T > 0$ and $\epsilon_0 \in (0, 1]$ such that for all $\epsilon \in (0, \epsilon_0]$ there is a unique solution $(f_\epsilon, E_\epsilon, B_\epsilon) \in C^1([0,T]; L^\infty_{x,\xi})$ to (4.46) such that $(f_\epsilon, E_\epsilon, B_\epsilon)|_{t=0} = (f^{in}, E^{in}, B^{in})$ and a constant C_T depending on $||(f^{in}, E^{in}, B^{in})||_{W^{1,\infty}_{x,\xi}}$ such that for all $t \in [0,T]$,*

$$\begin{aligned} ||(f_\epsilon - f_\ell)(t)||_{L^\infty_{x,\xi}} &\leq \epsilon C_T, \\ ||(E_\epsilon - E_\ell, B_\epsilon - B_\ell)(t)||_{L^\infty_{x,\xi}} &\leq C_T. \end{aligned} \tag{4.48}$$

Proof. First consider the anzatz

$$\begin{aligned} f_\epsilon &= f_\ell + \epsilon f^\delta, \\ (E_\epsilon, B_\epsilon) &= (E_\ell, B_\ell) + (E^\delta, B^\delta). \end{aligned} \tag{4.49}$$

Here, the δ in $(f^\delta, E^\delta, B^\delta)$ is not a parameter, but instead symbolizes a difference of solutions. Therefore $(f^\delta, E^\delta, B^\delta)$ satisfies

$$\begin{cases} \partial_t f^\delta + v(\xi)\cdot\nabla_x f^\delta - \frac{b_e(x)}{\epsilon\langle\xi\rangle}\partial_\theta f^\delta - \epsilon[E^\delta + E_\ell + v(\xi)\times(B^\delta + B_\ell)]\cdot\nabla_\xi f^\delta \\ \qquad = M'(|\xi|)\frac{\xi}{|\xi|}\cdot E^\delta + [E_\ell + E^\delta + v(\xi)\times(B_\ell + B^\delta)]\cdot\nabla_\xi f_\ell \\ \partial_t E^\delta - \nabla_x \times B^\delta = \epsilon J(f^\delta), \quad \nabla_x\cdot E^\delta = -\epsilon\rho(f^\delta) \\ \partial_t B^\delta + \nabla_x\times E^\delta = 0, \qquad \nabla_x\cdot B^\delta = 0 \end{cases} \tag{4.50}$$

Remark that the dilute assumption is key here, otherwise the right hand side of the equation on f^δ would involve $\epsilon^{-1}M'(|\xi|)\frac{\xi}{|\xi|}\cdot E^\delta$. Furthermore, the difference of scaling between f^δ and (E^δ, B^δ), introduces ϵJ and $\epsilon\rho$ in the current and charge density. The methods of [7] can then be repeated to obtain uniform estimates without the difficulty involved in passing the transport operator $T(f^\delta)$ to a ξ derivative which can be integrated by parts. First note that f^δ can be easily integrated along the full, non-linear flow $\mathcal{F}$ associated with the characteristics of (4.50). Recall that $f^\delta|_{t=0}\equiv 0$, then it follows by the Duhamel Principle

$$f^\delta(t,x,\xi) = \int_0^t \left[M'(|\xi|)\frac{\xi}{|\xi|}\cdot E^\delta + [E_\ell + E^\delta + v\times(B_\ell + B^\delta)]\cdot\nabla_\xi f_\ell\right]\circ\big(s, \mathcal{F}(t-s,x,\xi)\big)ds.$$

This gives the estimate

$$\begin{aligned} |f^\delta(t,x,\xi)| \le{}& ||M'||_{L^\infty}\int_0^t ||E^\delta(s,\cdot)||_{L^\infty_x} ds \\ &+ \int_0^t ||\nabla_\xi f_\ell(s,\cdot)||_{L^\infty_{x,\xi}}\big(||(E^\delta,B^\delta)(s,\cdot)||_{L^\infty_x} + ||(E_\ell,B_\ell)(s,\cdot)||_{L^\infty_x}\big)ds. \end{aligned} \tag{4.51}$$

This is precisely were the prepared data assumption is needed. It is to ensure uniform control on $||\nabla_\xi f_\ell(s,\cdot)||_{L^\infty_{x,\xi}}$. Next, due to the compensation of ϵ on the current and charge density, the fields (E^δ, B^δ) satisfy the wave equation

$$\begin{aligned} \Box E^\delta &= \epsilon\int v(\xi)\partial_t f^\delta + \nabla_x f^\delta d\xi, \\ \Box B^\delta &= \epsilon\int \nabla_x\times(v(\xi)f^\delta)d\xi. \end{aligned}$$

Again recalling $(f^\delta, E^\delta, B^\delta)\equiv 0$, the solution to the fields is given by

$$\begin{aligned} E^\delta(t,x) &= -\epsilon\int p(t,x,\xi)Y(t,x) *_{t,x} (\mathbb{1}_{t>0}T(f^\delta))d\xi + \epsilon\int q(t,x,\xi)Y(t,x) *_{t,x} (\mathbb{1}_{t>0}f^\delta)d\xi, \\ B^\delta(t,x) &= \epsilon\int a^0(t,x,\xi)Y(t,x) *_{t,x} (\mathbb{1}_{t>0}T(f^\delta))d\xi + \epsilon\int a^1(t,x,\xi)Y(t,x) *_{t,x} (\mathbb{1}_{t>0}f^\delta)d\xi. \end{aligned}$$

Replacing $T(f)$ with the Vlasov equation and integrating by parts in ξ we can estimate E by

$$\begin{aligned}
|E^\delta(t,x)| \le\ & C(R_\xi^T, ||b_e||_{L^\infty}) \int_0^t ||f^\delta(s,\cdot,\cdot)||_{L^\infty_{x,\xi}} ds \\
& + \epsilon C(R_\xi^T) \int_0^t ||f^\delta(s,\cdot,\cdot)||_{L^\infty_{x,\xi}} \big(1 + \epsilon||(E^\delta, B^\delta)(s,\cdot)||_{L^\infty_x} + \epsilon||(E_\ell, B_\ell)(s,\cdot)||_{L^\infty_x}\big) ds \\
& + \epsilon C(R_\xi^T) \int_0^t ||f_\ell(s,\cdot,\cdot)||_{L^\infty_{x,\xi}} \big(1 + \epsilon||(E^\delta, B^\delta)(s,\cdot)||_{L^\infty_x} + \epsilon||(E_\ell, B_\ell)(s,\cdot)||_{L^\infty_x}\big) ds \\
& + \epsilon C(R_\xi^0)||M'||_{L^\infty} \int_0^t ||E^\delta(s,\cdot)||_{L^\infty} ds. \qquad (4.52)
\end{aligned}$$

And similarly of B^δ. Therefore adding (4.52) and (4.51) and applying Gronwall's lemma gives the result (4.48). As an immediate consequence we also have Theorem 1.

□

Remark that a there is a difficulty with the Lipschitz estimate when we continue with this argument. Although for prepared data one does have $|\nabla_\xi f_\ell(t)| \lesssim C$, it is not true in general that $|\partial_{\xi_i}\partial_{\xi_j} f_\ell(t)|$ will also remain uniformly bounded. In fact we require $|\partial_\theta f^{in}| \lesssim \epsilon^2 C$ to obtain such estimates. Furthermore, the prepared data assumption is not a necessary condition for uniform estimates as shown in Example 1.

Chapter 5

Discussion

This chapter outlines some original results about the linearized HMRVM system in a constant magnetic field as well as difficulties associated with improving Theorem 1 in order to make progress towards Conjecture 1. Section 5.1 establishes well possedness of the linearized HMRVM system in a constant magnetic field for non-dilute equilibrium using the Fourier transform to deduce Fourier L^1 estimates. A simple embedding theorem then implies Sup-norm estimates. Although this is an improvement of Proposition 1, the bootstrap argument of Theorem 2 still requires the dilute assumption. Next, Section 5.2 explains why the Fourier transform method of Section 5.1 does not work for inhomogeneous fields. Furthermore, it establishes a possible route to studying stability of the non-dilute linear system in an inhomogeneous magnetic field using a non-linear Fourier transform coming from the flow of the characteristics. The Beurling Helson Theorem in [1] may pose an answer to this question of stability. This is left for future work however. Section 5.3 then explains why the proof of Proposition 1 does not apply for non-dilute equilibrium, but also possible way to overcome the difficulty. Furthermore, Section 5.3.1 uses a Picard scheme to give insight into why the prepared data assumption is necessary for for the bootstrap argument of Theorem 2. We finish by addressing the choice of scaling used in the anzatz (2.16). This generalizes the results of [7] for a larger class of perturbations.

5.1 Constant Magnetic Field

As alluded to the proof of Proposition 1, which required the dilute assumption $M_\epsilon = \epsilon M$, may not be optimal. In the section we assume the applied magnetic field is a constant vector along the e_3 direction

$$\mathbf{B}_e(x) = {}^t(0,0,1).$$

This implies that $O(x) = id_{3\times 3}$ and $Q \equiv 0$. Furthermore, we will derive uniform estimates of the following simplified linear Cauchy problem in a constant magnetic field

$$\begin{cases} \partial_t f + v(\xi)\cdot\nabla_x f - \frac{1}{\epsilon\langle\xi\rangle}\partial_\theta f = M'(|\xi|)\frac{\xi}{|\xi|}\cdot E \\ \partial_t E - \nabla_x \times B = J(f), \quad \nabla_x\cdot E = -\rho(f) \\ \partial_t B + \nabla_x \times E = 0, \qquad\quad \nabla_x \cdot B = 0 \\ (f,E,B)|_{t=0} = (f^{in}, E^{in}, B^{in}) \end{cases} \tag{5.1}$$

We emphasize that are considering the non-dilute case using equilibrium profile M. Again we assume the compatibility (4.18). In order to deduce uniform estimates of the system (5.1) we will consider the Fourier transformed equations in the variable x only. Therefore we will consider the space L^1_k (sometimes denoted $\mathcal{F}L^1$) which denotes the space of functions whose Fourier transform (in x) is in L^1. More precisely

$$L^1_k(\mathbb{R}^n;\mathbb{R}^m) := \{f : \mathbb{R}^n \mapsto \mathbb{R}^m \text{ measurable } \mid ||\mathcal{F}_x(f)||_{L^1}(\mathbb{R}^n) < \infty\}.$$

Similarly we set $W_k^{1,1}$ to be the corresponding sobolev space

$$W_k^{1,1}(\mathbb{R}^n;\mathbb{R}^m) := \{f : \mathbb{R}^n \mapsto \mathbb{R}^m \text{ measurable } \mid ||\mathcal{F}_x(f)||_{L^1(\mathbb{R}^n)} + ||\mathcal{F}_x(\nabla_x f)||_{L^1(\mathbb{R}^n)} < \infty\}.$$

The idea is to Fourier transform the system (5.1) with respect to the variable x and use the inequality of the bounded linear operator $\mathcal{F}_x : L^1 \mapsto L^\infty$

$$||\phi||_{L^\infty_x} \leq ||\hat{\phi}||_{L^1_k} \tag{5.2}$$

to deduce uniform estimates in L^∞_x. More precisely we have the following lemma.

Lemma 13. *(Global in time solution of linear homogeneous model) Let* $(f^{in}, E^{in}, B^{in}) \in C^1(\mathbb{R}^6;\mathbb{R}) \times [C^1(\mathbb{R}^3;\mathbb{R}^3)]^2$ *with* $R^0_\xi > 0$ *such that* $supp(f^{in}(x,\cdot)) \subset \{|\xi| \leq R^0_\xi\}$ *and* $(f^{in}, E^{in}, B^{in})(\cdot,\xi) \in W_k^{1,1}(\mathbb{R}^3;\mathbb{R}^7)$ *and satisfies the compatibility condition (4.18). Let* $T > 0$*. Then for each* $\epsilon \in (0,1]$ *there exists a unique solution* $(f_\epsilon, E_\epsilon, B_\epsilon) \in C^1([0,T];L^\infty_{x,\xi})$ *to system (5.1) with* $(f_\epsilon, E_\epsilon, B_\epsilon)|_{t=0} = (f^{in}, E^{in}, B^{in})$*. Furthermore, there is a constant* C_T *depending only on* T *and* $||(\hat{f}^{in}, \hat{E}^{in}, \hat{B}^{in})||_{W_k^{1,1},L^\infty_\xi}$ *such that for all* $t \in [0,T]$,

$$\begin{aligned} ||(f_\epsilon, E_\epsilon, B_\epsilon)(t)||_{L^\infty_{x,\xi}} &+ ||\partial_t(\epsilon f_\epsilon, E_\epsilon, B_\epsilon)(t)||_{L^\infty_{x,\xi}} \\ &+ ||\nabla_x(f_\epsilon, E_\epsilon, B_\epsilon)(t)||_{L^\infty_{x,\xi}} + ||\epsilon\nabla_\xi f_\epsilon(t)||_{L^\infty_{x,\xi}} \leq C_T. \end{aligned} \tag{5.3}$$

Furthermore if f^{in} *is prepared in the sense of definition 1, or equivalently*

$$||\partial_\theta f^{in}||_{L^\infty_{x,\xi}} \leq \epsilon C, \tag{5.4}$$

then the Lipschitz norm is also uniform in ϵ

$$||(f_\epsilon, E_\epsilon, B_\epsilon)(t)||_{L^\infty_{x,\xi}} + ||\partial_t(f_\epsilon, E_\epsilon, B_\epsilon)(t)||_{L^\infty_{x,\xi}} + ||\nabla_x(f_\epsilon, E_\epsilon, B_\epsilon)(t)||_{L^\infty_{x,\xi}} + ||\nabla_\xi f_\epsilon(t)||_{L^\infty_{x,\xi}} \leq C_T. \tag{5.5}$$

Proof. For neatness, we will omit the subscripts ϵ on the solution $(f_\epsilon, E_\epsilon, B_\epsilon)$. Consider taking the Fourier Transform of the system (5.1). Since the divergence conditions on E and B are fixed by the compatibility conditions, the evolutionary part of the transformed system then reads:

$$\begin{cases} \partial_t \hat{f} - v(\xi)\cdot ik\hat{f} - \frac{1}{\epsilon\langle\xi\rangle}\partial_\theta \hat{f} = M'(|\xi|)\frac{\xi}{|\xi|}\cdot\hat{E} \\ \partial_t \hat{E} + ik \times \hat{B} = J(\hat{f}) \\ \partial_t \hat{B} - ik \times \hat{E} = 0 \\ (\hat{f}, \hat{E}, \hat{B})|_{t=0} = (\hat{f}^{in}, \hat{E}^{in}, \hat{B}^{in}) \end{cases} \tag{5.6}$$

The first remark is that the transport term $v(\xi)\cdot ik\hat{f}$ does not generate any growth of the L^1_k norm. This can be seen in two ways. One approach is change variables form $\theta \mapsto \theta - \frac{t}{\epsilon\langle\xi\rangle}$ in the spirit of an integrating factor. Furthermore, this change of variables introduces no penalization of size ϵ^{-1} in the Jacobian (which is identically 1) introduced in the charge and current density. One then can show the semigroup associated with (5.1) containing $ik\cdot v(\xi)$, after this change of variables is uniformly bounded in k. The second approach is similar which introduces a PDE on $|\hat{f}|$. This is the approach we take here. First we multiplying the first equation on $\hat{f}$ by the complex conjugate $\hat{f}^*$, which implies

$$\partial_t |\hat{f}|^2 - v(\xi)\cdot ik|\hat{f}|^2 - \frac{1}{\epsilon\langle\xi\rangle}\partial_\theta|\hat{f}|^2 = M'(|\xi|)\frac{\xi}{|\xi|}\cdot\hat{E}\hat{f}^*.$$

Similarly, taking the complex conjugate and multiplying by $\hat{f}$, and adding the two expressions (i.e. taking the real part) gives

$$\partial_t |\hat{f}|^2 - \frac{1}{\epsilon\langle\xi\rangle}\partial_\theta|\hat{f}|^2 = M'(|\xi|)\frac{\xi}{|\xi|}\cdot(\hat{E}\hat{f}^* + \hat{E}^*\hat{f}).$$

We can solve this equation through method of characteristics in cylindrical coordinates. If we set $\xi(r,\theta,z) = (r\cos(\theta), r\sin(\theta), z)$, then the Duhamel principle gives

$$\begin{aligned}|\hat{f}(t,k,\xi)|^2 &= |\hat{f}^{in}(k,\xi(r,\theta+\frac{t}{\epsilon\langle\xi\rangle},z))|^2\\ &\quad+\int_0^t\left(\frac{1}{2}M'(|\xi|)\frac{\xi}{|\xi|}\cdot(\hat{E}\hat{f}^*+\hat{E}^*\hat{f})\right)\circ\big(s,k,\xi(r,\theta+\frac{t-s}{\epsilon\langle\xi\rangle},z)\big)ds, \qquad (5.7)\end{aligned}$$

implying the point-wise estimate

$$|\hat{f}(t,x,\xi)|^2 \le |\hat{f}^{in}(k,\xi(r,\theta+\frac{t}{\epsilon\langle\xi\rangle},z))|^2 + C(R_M)||M'||_{L^\infty}\int_0^t |\hat{E}(s,k)|\;||f(s,k,\cdot)||_{L^\infty_\xi}ds. \qquad (5.8)$$

Remark that (5.7) implies $\hat{f}$ remains compactly supported in the variable ξ for all times $t \ge 0$ since

$$|\hat{f}(t,k,\xi)| = 0 \text{ for } |\xi| \notin \operatorname{supp}(f^{in}(k,\cdot)) \cup \operatorname{supp}(M(|\cdot|)) \subset B(0,R^0_\xi).$$

To estimate the fields we consider the ODE

$$\partial_t\begin{bmatrix}\hat{E}\\ \hat{B}\end{bmatrix} = \begin{bmatrix}0 & -i[k]_\times\\ i[k]_\times & 0\end{bmatrix}\begin{bmatrix}\hat{E}\\ \hat{B}\end{bmatrix} + \begin{bmatrix}\int v(\xi)\hat{f}d\xi\\ 0\end{bmatrix} := A_k\begin{bmatrix}\hat{E}\\ \hat{B}\end{bmatrix} + \begin{bmatrix}\int v(\xi)\hat{f}d\xi\\ 0\end{bmatrix}, \qquad (5.9)$$

where

$$[k]_\times := \begin{bmatrix}0 & -k^3 & k^2\\ k^3 & 0 & -k^1\\ -k^2 & k^1 & 0\end{bmatrix}.$$

The matrix A_k has eigenvalues

$$\lambda_1 = 0, \quad \lambda_\pm = \pm i\sqrt{|k|},$$

implying that the operator e^{tA_k} is uniformly bounded in k

$$\sup_{k\in\mathbb{R}^3}||e^{tA_k}||_{\mathbb{R}^6\mapsto\mathbb{R}^6} \le C. \qquad (5.10)$$

Therefore the variation of a constant formula on (5.9) gives

$$\begin{bmatrix}\hat{E}\\ \hat{B}\end{bmatrix} = e^{tA_k}\begin{bmatrix}\hat{E}^{in}\\ \hat{B}^{in}\end{bmatrix} + \int_0^t e^{(t-s)A_k}\begin{bmatrix}\int v(\xi)\hat{f}d\xi\\ 0\end{bmatrix} ds.$$

Then since $\hat{f}$ has compact support in ξ and using the estimate (5.10) we obtain

$$|(\hat{E},\hat{B})(t,k)| \le C|(\hat{E}^{in},\hat{B}^{in})(k)| + \frac{4\pi}{3}(R_\xi^0)^3\int_0^t ||\hat{f}(t,k,\cdot)||_{L_\xi^\infty} ds. \tag{5.11}$$

Therefore adding (5.8) and (5.11) and applying Gronwall's inequality imply

$$||(\hat{f},\hat{E},\hat{B})(t,k,\cdot)||_{L_\xi^\infty} \le C(R_\xi^0,T)||(\hat{f}^{in},\hat{E}^{in},\hat{B}^{in})(k,\cdot)||_{L_\xi^\infty}.$$

Integrating with respect to k implies the L_k^1 estimate

$$||(\hat{f},\hat{E},\hat{B})|(t,\cdot,\cdot)||_{L_\xi^\infty,L_k^1} \le C(R_\xi^0,T)||(\hat{f}^{in},\hat{E}^{in},\hat{B}^{in})(\cdot,\cdot)||_{L_\xi^\infty,L_k^1}.$$

Finally we use that $\mathcal{F}: L^1 \mapsto L^\infty$ is a bounded linear operator

$$||(f,E,B)(t,\cdot,\cdot)||_{L_\xi^\infty,L_x^\infty} \le C(R_\xi^0,T)||(\hat{f}^{in},\hat{E}^{in},\hat{B}^{in})(\cdot,\cdot)||_{L_\xi^\infty,L_k^1}. \tag{5.12}$$

Lipschitz estimates in time and space are similar. Differentiated the system (5.1) with respect to ∂_i with $i = x_1, x_2, x_3$ or t we have $\partial_i(f,E,B)$ solves

$$\begin{cases} \partial_t\partial_i f + v(\xi)\cdot\nabla_x\partial_i f - \frac{1}{\epsilon\langle\xi\rangle}\partial_\theta\partial_i f = M'(|\xi|)\frac{\xi}{|\xi|}\cdot\partial_i E \\ \partial_t\partial_i E - \nabla_x\times\partial_i B = J(\partial_i f) \\ \partial_t\partial_i B + \nabla_x\times\partial_i E = 0 \end{cases} \tag{5.13}$$

Therefore $\partial_i(f,E,B)$ is also a solution of PDE (5.1). Thus repeating the previous proof for the L^∞ estimates gives

$$||\partial_i(f,E,B)(t,\cdot,\cdot)||_{L_\xi^\infty,L_x^\infty} \le C(R_\xi^0,T)||\partial_i(\hat{f},\hat{E},\hat{B})|_{t=0}(\cdot,\cdot)||_{L_\xi^\infty,L_k^1}. \tag{5.14}$$

In the case of $i = x_1, x_2, x_3$, since $b_e(x) \equiv 1$, imposing more regularity on $(\hat{f}^{in},\hat{E}^{in},\hat{B}^{in})$ to be in $W_x^{k,1}$ gives uniform estimates on $\partial_i^k(f,E,B)$ for all $k\in\mathbb{N}$. Furthermore we can bound $\partial_t(E,B)$ since

$$\begin{aligned} |\partial_t E| &\le |\nabla_x\times B| + |\int v(\xi)fd\xi|, \\ |\partial_t B| &\le |\nabla_x\times E|. \end{aligned}$$

For $\partial_t f$ however, note that

$$\partial_t|\hat{f}|^2\big|_{t=0}(k,\xi) = \frac{1}{\epsilon\langle\xi\rangle}\partial_\theta|\hat{f}^{in}|^2(k,\xi) + M'(|\xi|)\frac{\xi}{|\xi|}\cdot\left(\hat{E}^{in}(\hat{f}^{in})^* + (\hat{E}^{in})^*\hat{f}^{in}\right).$$

Therefore for $i = t$, the right hand side of (5.14) is uniformly bounded in ϵ as long as (5.4) holds. For the momentum derivatives, we could follow along the lines of (4.43), but instead we take a different approach to emphasize the presence of rapid oscillations occurring in the characteristics. Remark that the Vlasov equation can be solved with an exact solution along the flow using the Duhamel principle as follows

$$\begin{aligned} f(t,x,\xi(r,\theta,z)) &= f^{in}\big(x - \Xi^m(t,\xi), \xi(r, \theta + \frac{t}{\epsilon\langle\xi\rangle}, z)\big) \\ &\quad + \frac{M'(|\xi|)}{|\xi|}\int_0^t \xi(r,\theta + \frac{t-s}{\epsilon\langle\xi\rangle}, z)\cdot E(s, x - \Xi^m(t-s,\xi))ds, \end{aligned} \tag{5.15}$$

where

$$\Xi^m(t,\xi) := \begin{bmatrix} \epsilon r\sin(\theta + \frac{t}{\epsilon\langle\xi\rangle}) - \epsilon r\sin(\theta) \\ -\epsilon r\cos(\theta + \frac{t}{\epsilon\langle\xi\rangle}) + \epsilon r\cos(\theta) \\ \frac{zt}{\langle\xi\rangle} \end{bmatrix}.$$

We can then estimate $\nabla_\xi f$ in cylindrical coordinates using the gradient operator

$$\nabla_\xi = e_\theta \frac{1}{r}\partial_\theta + e_r\partial_r + e_z\partial_z.$$

Firstly, we can estimate $\frac{1}{r}\partial_\theta f$, along the same lines as before with ∂_t and ∂_{x_i} since $\frac{1}{r}\partial_\theta f$ satisfies

$$\partial_t(\frac{1}{r}\partial_\theta f) + v(\xi)\cdot\nabla_x(\frac{1}{r}\partial_\theta f) - \frac{1}{\epsilon\langle\xi\rangle}\partial_\theta(\frac{1}{r}\partial_\theta f) = -M'(|\xi|)\frac{\xi^\perp}{r|\xi|}\cdot E - \frac{\xi^\perp}{r\langle\xi\rangle}\cdot\nabla_x f.$$

Integrating along the flow then gives

$$\begin{aligned} \frac{1}{r}\partial_\theta f(t,x,\xi) &= \frac{1}{r}\partial_\theta f^{in}\big(x - \Xi^m(t,\xi), \xi(r,\theta + \frac{t}{\epsilon\langle\xi\rangle}, z)\big) \\ &\quad - \frac{M'(|\xi|)}{r|\xi|}\int_0^t \xi^\perp(r,\theta + \frac{t-s}{\epsilon\langle\xi\rangle}, z)\cdot E(s, x - \Xi^m(t-s,\xi))ds \\ &\quad - \frac{1}{r\langle\xi\rangle}\int_0^t \xi^\perp(r,\theta + \frac{t-s}{\epsilon\langle\xi\rangle}, z)\cdot\nabla_x f(s, x - \Xi^m(t-s,\xi))ds \end{aligned} \tag{5.16}$$

Again assume that $M'(0) = 0$ and $M'(|\xi|) = \mathcal{O}(|\xi|)$ at $\xi = 0$ to ensure $\frac{M'(|\xi|)}{|\xi|}$ remains bounded at $|\xi| = 0$. Therefore

$$|\frac{1}{r}\partial_\theta f(t,x,\xi)| \le ||\frac{1}{r}\partial_\theta f^{in}||_{L^\infty_{x,\xi}} + ||\frac{M'(|\xi|)}{|\xi|}||_{L^\infty_\xi}\int_0^t ||E(s,\cdot)||_{L^\infty_x}ds + \int_0^t ||\nabla_x f(s,\cdot)||_{L^\infty_{x,\xi}}. \tag{5.17}$$

From the definition of Ξ^m, the norm L^∞_x is in fact taken over the bounded set

$$\mathcal{A} = \mathcal{A}(x,T,\epsilon) := \{y \in \mathbb{R}^3 \mid |y^1 - x^1| + |y^2 - x^2| \le 2\epsilon, |y^3 - x^3| \le T\}.$$

This suffices for uniform estimates of $|\frac{1}{r}\partial_\theta f|$. However, as noted we have $|\partial_t \partial_{x_i}^k f|$ remains bounded for all $k \in \mathbb{N}$ for prepared data. Therefore integrating (5.16) by parts in time and Taylor expanding the initial data term with respect to ϵ for Ξ^m implies

$$\begin{aligned}
&|\frac{1}{r}\partial_\theta f(t,x,\xi) - \frac{1}{r}\partial_\theta f^{in}\big(x - \frac{zt}{\langle\xi\rangle}e_3, \xi(r,\theta+\frac{t}{\epsilon\langle\xi\rangle},z)\big)| \\
&\qquad \le \epsilon C T \sup_{t\in[0,T]} \big(||\partial_t E(t,\cdot)||_{L^\infty_x} + ||\partial_t \nabla_x f(t,\cdot)||_{L^\infty_{x,\xi}}\big) + 2\epsilon ||\partial_\theta \nabla_x f^{in}||_{L^\infty_{x,\xi}} \\
&\qquad \lesssim \epsilon C(T+1).
\end{aligned}$$

Next we consider $\partial_j f$ with $j = r, z$ by differentiating (5.15). But remark that the derivatives of Ξ^m are uniformly bounded in the sense that

$$||\nabla_\xi \Xi^m(t,\xi)|| + |\partial_t \Xi^m(t,\xi)| \le C(t,|\xi|) \le C(T, R^0_\xi). \tag{5.18}$$

Thus we have

$$\begin{aligned}
&\partial_j f(t,x,\xi(r,\theta,z)) = -\partial_j \Xi^m \cdot \nabla_x f^{in}\big(x - \Xi^m(t,\xi), \xi(r,\theta+\frac{t}{\epsilon\langle\xi\rangle},z)\big) \\
&- \frac{tj}{\epsilon\langle\xi\rangle^3}\partial_\theta f^{in}\big(x - \Xi^m(t,\xi), \xi(r,\theta+\frac{t}{\epsilon\langle\xi\rangle},z)\big) \\
&+ \int_0^t \Big\{ \partial_j\Big(\frac{rM'(|\xi|)}{|\xi|}\Big) \begin{bmatrix} \cos(\theta+\frac{t-s}{\epsilon\langle\xi\rangle}) \\ \sin(\theta+\frac{t-s}{\epsilon\langle\xi\rangle}) \\ 0 \end{bmatrix} \cdot E(s, x - \Xi^m(t-s,\xi)) \\
&\qquad\qquad + \partial_j\big(\frac{M'(|\xi|)z}{|\xi|}\big) E^3(s, x - \Xi^m(t-s,\xi)) \Big\} ds \\
&- \int_0^t \frac{M'(|\xi|)}{|\xi|} \xi(r,\theta+\frac{t-s}{\epsilon\langle\xi\rangle},z) \cdot \nabla_x E(s, x - \Xi^m(t-s,\xi)) \partial_j \Xi^m(t-s,\xi) ds \\
&- \int_0^t \frac{M'(|\xi|)}{|\xi|} \frac{(t-s)rj}{\epsilon\langle\xi\rangle^3} \begin{bmatrix} -\sin(\theta+\frac{t-s}{\epsilon\langle\xi\rangle}) \\ \cos(\theta+\frac{t-s}{\epsilon\langle\xi\rangle}) \\ 0 \end{bmatrix} \cdot E(s, x - \Xi^m(t-s,\xi)) ds.
\end{aligned} \tag{5.19}$$

For prepared data each term in the above is uniformly bounded in ϵ except the last integral. We recover this loss of ϵ^{-1} by integrating by parts

$$
\begin{aligned}
&\int_0^t \frac{(t-s)rj}{\epsilon\left\langle\xi\right\rangle^3}\begin{bmatrix}-\sin(\theta+\frac{t-s}{\epsilon\langle\xi\rangle})\\ \cos(\theta+\frac{t-s}{\epsilon\langle\xi\rangle})\\ 0\end{bmatrix}\cdot E(s,x-\Xi^m(t-s,\xi))ds\\
&\qquad=\int_0^t \frac{(t-s)rj}{\left\langle\xi\right\rangle^2}\partial_s\begin{bmatrix}\cos(\theta+\frac{t-s}{\epsilon\langle\xi\rangle})\\ \sin(\theta+\frac{t-s}{\epsilon\langle\xi\rangle})\\ 0\end{bmatrix}\cdot E(s,x-\Xi^m(t-s,\xi))ds\\
&=\frac{trj}{\left\langle\xi\right\rangle^2}\begin{bmatrix}\cos(\theta+\frac{t}{\epsilon\langle\xi\rangle})\\ \sin(\theta+\frac{t}{\epsilon\langle\xi\rangle})\\ 0\end{bmatrix}\cdot E^{in}(x-\Xi^m(t,\xi))\\
&-\int_0^t \frac{rj}{\left\langle\xi\right\rangle^2}\begin{bmatrix}\cos(\theta+\frac{t-s}{\epsilon\langle\xi\rangle})\\ \sin(\theta+\frac{t-s}{\epsilon\langle\xi\rangle})\\ 0\end{bmatrix}\cdot\\
&\qquad\qquad\Big(-E+(t-s)\partial_t E+(\nabla_x E)\partial_t\Xi^m(t-s,\xi)\Big)(s,x-\Xi^m(t-s,\xi))ds.
\end{aligned}
\tag{5.20}
$$

We can combine (5.19) and (5.20) and again assume that $M'(0)=0$ and $M'(|\xi|)=\mathcal{O}(|\xi|)$ in order to ensure $\partial_j(\frac{rM'(|\xi|)}{|\xi|})$ remains bounded at $|\xi|=0$, to obtain an estimate for $|\partial_j f|$:

$$
\begin{aligned}
|\partial_j f(t,x,\xi)| \le C(T,R^0_\xi)||\nabla_x f^{in}||_{L^\infty_{x,\xi}} + T||\frac{1}{\epsilon}\partial_\theta f^{in}||_{L^\infty_{x,\xi}} + T||E^{in}||_{L^\infty_x}\\
+ C(T,R^0_\xi)\int_0^t ||E(s,\cdot)||_{L^\infty_x} + ||\nabla_x E(s,\cdot)||_{L^\infty_x} + ||\partial_t E(s,\cdot)||_{L^\infty_x} ds.
\end{aligned}
$$

This completes the proof of existence. Uniqueness follows immediately from consider the difference of two solutions, and using the linearity of the system. The inequalities (5.12) and (5.14) then imply the difference must be zero implying uniqueness.

□

5.2 Fourier Analysis for Inhomogeneous Fields

When we consider the case of an inhomogeneous, magnetic field with constant direction a difficulty in the previous proof arises when we attempt to take the Fourier transform of the Vlasov equation. In this case, the Vlasov equation becomes

$$
\partial_t f + v(\xi)\cdot\nabla_x f - \frac{b_e(x)}{\epsilon\left\langle\xi\right\rangle}\partial_\theta f = M'_\epsilon(|\xi|)\frac{\xi}{|\xi|}\cdot E.
$$

But then the transformed equation reads

$$
\partial_t\hat{f} - v(\xi)\cdot ik\hat{f} - \frac{\hat{b}_e(k)}{\epsilon\left\langle\xi\right\rangle} *_k \partial_\theta\hat{f} = M'_\epsilon(|\xi|)\frac{\xi}{|\xi|}\cdot\hat{E}.
$$

Thus, due to the convolution $\frac{\hat{b}_e}{\epsilon\langle\xi\rangle} *_k \partial_\theta\hat{f}$, we no longer can solve for $|\hat{f}|$ along the flow. But remark that after taking a Fourier transform of the fields (E,B), the variation of a constant formula given by (5.9) still remains. We can attempt to avoid this difficulty be solving the non-transformed Vlasov equation first

$$f(t,x,\xi) = f^{in}(X(-t),\Xi(-t)) + \int_0^t \big(M'(|\Xi|)\frac{\Xi}{|\Xi|}\cdot E\big)(s,X(t-s),\Xi(t-s))ds, \tag{5.21}$$

where (X,Ξ) are solutions to the characteristic curves of the inhomogeneous linear system.

$$\begin{aligned} \dot{X} &= \frac{\Xi}{\langle \Xi \rangle}, & X(0) &= x, \\ \dot{\Xi} &= \frac{-b_e(X)}{\epsilon \langle \Xi \rangle}\Xi^{\perp}, & \Xi(0) &= \xi. \end{aligned} \tag{5.22}$$

Remark once again, the system (5.22) is divergence free and is therefore the flow is volume preserving for all times. For simplicity we write $(X,\Xi)(t,x,\xi) = (X,\Xi)(t)$ and therefore the transformed current density coming from (5.9) after substituting (5.21) and changing variables is

$$\begin{aligned} \int v(\xi)\hat{f}d\xi &= \frac{1}{(2\pi)^{3/2}}\int\int e^{-ik\cdot x}v(\xi)f(t,x,\xi)dxd\xi \\ &= \frac{1}{(2\pi)^{3/2}}\int\int e^{-ik\cdot X(t)}v(\Xi(t))f^{in}(x,\xi)dxd\xi \\ &\quad + \frac{1}{(2\pi)^{3/2}}\int\int\int_0^t e^{-ik\cdot X(t-s)}v(\Xi(t-s))M'(|\xi|)\frac{\xi}{|\xi|}\cdot E(s,x)dsdxd\xi. \end{aligned} \tag{5.23}$$

In the constant case, where $b_e(x) \equiv 1$, the map $x \mapsto X(t,x,\xi) = x + \Xi^m(t,\xi)$ was linear in x and $\Xi(t) = \xi(r, \theta + \frac{t}{\epsilon\langle\xi\rangle}, z)$ so (5.23) reduces to

$$\begin{aligned} \int v(\xi)\hat{f}d\xi &= \frac{1}{(2\pi)^{3/2}}\int e^{-ik\cdot\Xi^m}\frac{\xi(r,\theta+\frac{t}{\epsilon\langle\xi\rangle},z)}{\langle\xi\rangle}\int e^{-ik\cdot x}f^{in}(x,\xi)dxd\xi \\ &\quad + \frac{1}{(2\pi)^{3/2}}\int_0^t\int e^{-ik\cdot\Xi^m(t-s,\xi)}M'(|\xi|)\frac{\xi(r,\theta+\frac{(t-s)}{\epsilon\langle\xi\rangle},z)}{\langle\xi\rangle}\Big(\xi\cdot\int e^{ik\cdot x}E(s,x)dx\Big)d\xi ds \end{aligned}$$

$$\begin{aligned} &= \int e^{-ik\cdot\Xi^m}\frac{\xi(r,\theta+\frac{t}{\epsilon\langle\xi\rangle},z)}{\langle\xi\rangle}\hat{f}^{in}d\xi \\ &\quad + \int_0^t\int e^{-ik\cdot\Xi^m(t-s,\xi)}M'(|\xi|)\frac{\xi(r,\theta+\frac{(t-s)}{\epsilon\langle\xi\rangle},z)}{\langle\xi\rangle}\big(\frac{\xi}{|\xi|}\cdot\hat{E}(s,k)\big)d\xi ds, \end{aligned}$$

which allowed for uniform estimates in L^1_k since the transformed current density could be bounded by

$$|\int v(\xi)\hat{f}d\xi| \leq ||\hat{f}^{in}(k,\cdot)||_{L^\infty_\xi} + ||M'(|\cdot|)||_{L^1_\xi}\int_0^t |\hat{E}(s,k)|ds.$$

When b_e does depend on x, it may be possible to deduce uniform estimates of the fields by remarking that the inhomogeneous flow is a perturbation of identity as explained precisely in Lemma 9. Using this in (5.23), we have up to order ϵ^2,

$$\int v(\xi)\hat{f}d\xi = \frac{1}{(2\pi)^{3/2}}\int\int e^{-ik\cdot X(t)}v(\Xi(t))f^{in}(x,\xi)dxd\xi$$
$$+\frac{1}{(2\pi)^{3/2}}\int_0^t\int e^{-ik^3\frac{(t-s)\xi_3}{\langle\xi\rangle}}\int e^{-ik\cdot(x+\epsilon R_\epsilon(t-s,x,\xi)}\frac{\Xi(t-s,x,\xi)}{\langle\xi\rangle}M'(|\xi|)\frac{\xi}{|\xi|}\cdot E(s,x)dxd\xi ds.$$

If we consider the Beurling-Helson theorem from [1], the non-linear exponent, $e^{ik\cdot(x+\epsilon R_\epsilon(t-s,x,\xi))}$, may imply ill-posedness in the L^1_k norm when we attempt to obtain an estimate of the form

$$\left|\int_0^t\int\int e^{ik\cdot(x+\epsilon R_\epsilon(t-s,x,\xi))}\phi(s,x,\xi)dxd\xi ds\right| \overset{?}{\leq} C(R^\infty_\xi)\int_0^t ||\hat{\phi}(s,k,\cdot)||_{L^\infty_\xi}ds.$$

However due to the imprecision of (5.2) this would not immediately imply ill-posedness in L^∞_x. Moreover, the map $x \mapsto x+\epsilon R(t,x,\xi)$ is an approximation of identity. In other words we have the point wise limit,

$$\mathcal{F}_x(\lim_{\epsilon\to 0^+} e^{\epsilon R_\epsilon(t-s,\cdot,\xi)})(k) = \mathcal{F}_x(1) = \sqrt{2\pi}\delta(k),$$

in the sense of distributions. Therefore, an argument may exists to deduce uniform estimates in L^1_k for very small ϵ. This is not immediately clear and will be further explored in more details in forthcoming work.

5.3 Difficulty For Unprepared Initial Data

There is a difficulty which occurs when we attempt to extend the uniform results for the non-dilute, linear model to the non-linear system even in the case of a homogeneous magnetic field

$$\begin{cases} \partial_t f + v(\xi)\cdot\nabla_x f - \frac{1}{\epsilon\langle\xi\rangle}\partial_\theta f - \epsilon[E+v(\xi)\times B]\cdot\nabla_\xi f = M'(|\xi|)\frac{\xi}{|\xi|}\cdot E \\ \partial_t E - \nabla_x\times B = J(f), \quad \nabla_x\cdot E = -\rho(f) \\ \partial_t B + \nabla_x\times E = 0, \qquad \nabla_x\cdot B = 0 \\ (f,E,B)|_{t=0} = (f^{in},E^{in},B^{in}) \end{cases} \tag{5.24}$$

and the associated linear system

$$\begin{cases} \partial_t f_\ell + v(\xi)\cdot\nabla_x f_\ell - \frac{1}{\epsilon\langle\xi\rangle}\partial_\theta f_\ell = M'(|\xi|)\frac{\xi}{|\xi|}\cdot E_\ell \\ \partial_t E_\ell - \nabla_x\times B_\ell = J(f_\ell), \quad \nabla_x\cdot E_\ell = -\rho(f_\ell) \\ \partial_t B_\ell + \nabla_x\times E_\ell = 0, \qquad \nabla_x\cdot B_\ell = 0 \\ (f,E,B)|_{t=0} = (f^{in},E^{in},B^{in}) \end{cases} \tag{5.25}$$

First the proof given for Lemma 13 fails due to the non-linear term $\epsilon[E+v\times B]\cdot\nabla_\xi f$ when one attempts to compute the Fourier transforms in x. Since the space L^1_k is an algebra, a non-linearity such as Ef would be fine, but the derivative ∇_ξ does not allow for uniform L^1_k estimates by following the same lines. Furthermore, as remarked (remark 5), Proposition 1 fails in the non-linear case since the source term in the Vlasov equation is no longer small.

To illustrate this, if we Taylor expand the linear solution f_ℓ for a non-dilute plasma in with respect to ϵ in the variable $x - \Xi^m$:

$$\begin{aligned} f_\ell(t,x,\xi(r,\theta,z)) &= f^{in}\big(x - \Xi^m(t,\xi), \xi(r, \theta + \frac{t}{\epsilon\langle\xi\rangle}, z)\big) \\ &\qquad + \frac{M'(|\xi|)}{|\xi|}\int_0^t \xi(r,\theta+\frac{t-s}{\epsilon\langle\xi\rangle},z)\cdot E_\ell(s, x - \Xi^m(t-s,\xi))ds \\ &= f^{in}(x - \frac{tz}{\langle\xi\rangle}e_3, \xi(r,\theta+\frac{t}{\epsilon\langle\xi\rangle},z) \\ &\qquad + \frac{zM'(|\xi|)}{|\xi|}\int_0^t E_\ell(s, x - \frac{z(t-s)}{\langle\xi\rangle}e_3)ds + \mathcal{O}\big(\epsilon|\bar{\nabla}_x f^{in}| + \epsilon|\bar{\nabla}_x E_\ell|\big), \end{aligned} \tag{5.26}$$

where $\bar{\nabla}_x = (\partial_{x_1}, \partial_{x_2}, 0)$. Then when we consider the difficult term like (3.52), the term above involving the initial data f^{in} is rapidly oscillating, so the proof of Lemma 11 can be used as on this term. Similarly, if we assume a priori control over $|\epsilon\bar{\nabla}_x E_\ell|$, the remaining term in the above could be handled without the dilute assumption since

$$\partial_\theta\Big(\frac{zM'(|\xi|)}{|\xi|}\int_0^t E_\ell(s, x - \frac{z(t-s)}{\langle\xi\rangle}e_3)ds\Big) \equiv 0. \tag{5.27}$$

Thus a similar argument could used for the non-linear case as long as there was a priori control on $|\epsilon\bar{\nabla}_x E|$. To summarize, there are two issues with the preceding arguments for extending the linear case to the non-linear case in a dilute setting

- The space L^1_k does not behave well with quasi-linear terms like $\epsilon E\cdot\nabla_\xi f$.
- The wave equation introduces a loss of derivatives in order to regain the factor ϵ from passing $T(f)$ to a momentum derivative by means of $\epsilon\nabla_x E$ for sup-norm estimates on E unless the plasma is dilute and the non-stationary phase argument is not needed on the source term of the Vlasov equation.

5.3.1 Picard Scheme

Now suppose we attempt a Picard iterative scheme to try and avoid the prepared initial data assumption. We will show the linearization procedure will not yield a good approximation for ill-prepared data. For simplicity, in this discussion we assume here that $M \equiv 0$ and $b_e \equiv 1$. First consider the first approximation (f_0, E_0, B_0) given by the following system

$$\begin{cases} \partial_t f_0 + v(\xi)\cdot\nabla_x f_0 - \frac{1}{\epsilon\langle\xi\rangle}\partial_\theta f_0 = 0 \\ \partial_t E_0 - \nabla_x \times B_0 = J(f_0), \quad \nabla_x\cdot E_0 = -\rho(f_0) \\ \partial_t B_0 + \nabla_x\times E_0 = 0, \qquad \nabla_x\cdot B_0 = 0 \\ (f_0, E_0, B_0)|_{t=0} = (f^{in}, E^{in}, B^{in}) \end{cases} \tag{5.28}$$

Then let f_1 be a solution to the linear system,

$$\begin{cases} \partial_t f_1 + v(\xi)\cdot\nabla_\xi f_1 - \frac{1}{\epsilon\langle\xi\rangle}\partial_\theta f_1 = \epsilon[E_0 + v(\xi)\times B_0]\cdot\nabla_\xi f_0, \\ f_1|_{t=0} = f^{in}. \end{cases} \tag{5.29}$$

The we have the following relation

Lemma 14. *Let (f_0, E_0, B_0) solve (5.28) and f_1 solve (5.29). These are global solutions C^1 in time. Then for all $T > 0$, there exists $C_T > 0$ such that for all $t \in [0,T]$ we have*

$$\left| f_1(t,x,\xi) - f_0(t,x,\xi) + \frac{tz}{\langle \xi \rangle} \partial_\theta f^{in}(X(-t), \Xi(-t)) \int_0^t E_0^3(s, x + \frac{(t-s)z}{\langle \xi \rangle} e_3) ds \right| \leq \epsilon C_T, \tag{5.30}$$

where $(X, \Xi)(t)$ solve the linear homogeneous characteristics of (5.28) given by

$$X(t,x,\xi) = x - \Xi^m(-t,\xi), \quad \Xi(t,x,\xi) = \xi(t, \theta - \frac{t}{\epsilon \langle \xi \rangle}, z)$$

Proof. Again remark that the most singular part of $\nabla_\xi f_0$ is given by

$$\epsilon \nabla_\xi f_0(t,x,\xi) = -\frac{t\xi}{\langle \xi \rangle^3} \partial_\theta f^{in}(X(-t), \Xi(-t)) + \mathcal{O}(\epsilon).$$

Where $(X, \Xi)(t)$ is the linear characteristic curve solution of f_0. Therefore, up to order ϵ we have

$$\epsilon[E_0 + v(\xi) \times B_0] \cdot \nabla_\xi f_0 = -E_0 \cdot \frac{t\xi}{\langle \xi \rangle^3} \partial_\theta f^{in}(X(-t), \Xi(-t)) + \mathcal{O}(\epsilon).$$

Solving for f_1 yields

$$\frac{d}{dt}[f_1(t, X(t), \Xi(t))] = -E_0(t, X(t)) \cdot \frac{t\Xi(t)}{\langle \xi \rangle^3} \partial_\theta f^{in}(x,\xi) + \mathcal{O}(\epsilon).$$

Thus

$$f_1(t, X(t), \Xi(t)) = f^{in}(x,\xi) - \frac{t}{\langle \xi \rangle^3} \partial_\theta f^{in}(x,\xi) \int_0^t \big[E_0(s, X(s)) \cdot \Xi(s) \big] ds + \mathcal{O}(\epsilon).$$

We then write

$$\Xi(s) = \xi(r, \theta - \frac{s}{\epsilon \langle \xi \rangle}, z) = -\epsilon \langle \xi \rangle \frac{d}{ds}[\xi^\perp(r, \theta - \frac{s}{\epsilon \langle \xi \rangle}, z)] + z e_3.$$

Therefore integrating by parts yields

$$\begin{aligned} -\int_0^t \big[E_0(s, X(s)) \cdot \Xi(s)\big] ds &= \epsilon \langle \xi \rangle \big[E_0(t, X(t)) \cdot \xi^\perp(r, \theta - \frac{t}{\epsilon \langle \xi \rangle}, z) - E^{in}(x) \cdot \xi^\perp \big] \\ &- \epsilon \langle \xi \rangle \int_0^t [\partial_s E_0(s, X(s)) + \nabla_x E_0(s, X(s)) X(s)] \cdot \xi^\perp(r, \theta - \frac{s}{\epsilon \langle \xi \rangle}, z) ds \\ &- z \int_0^t E_0^3(s, X(s)) ds. \end{aligned}$$

Next can approximate the linear flow X via

$$X(s) = x + \frac{sz}{\langle \xi \rangle} e_3 + \mathcal{O}(\epsilon).$$

Recalling the estimates from theorem 13

$$|\partial_t E_0| + |\nabla_x E_0| \lesssim C(t),$$

we can then solve f_1 along the flow

$$f_1(t, X(t), \Xi(t)) = f^{in}(x,\xi) - \frac{tz}{\langle\xi\rangle^3}\partial_\theta f^{in}(x,\xi)\int_0^t E_0^3(s, x + \frac{sz}{\langle\xi\rangle}e_3)ds + \mathcal{O}(\epsilon).$$

and finally

$$\begin{aligned} f_1(t,x,\xi) &= f^{in}(X(-t),\Xi(-t)) \\ &\qquad - \frac{tz}{\langle\xi\rangle^3}\partial_\theta f^{in}(X(-t),\Xi(-t))\int_0^t E_0^3(s, x + \frac{(s-t)z}{\langle\xi\rangle}e_3)ds + \mathcal{O}(\epsilon) \\ &= f_0(t,x,\xi) - \frac{tz}{\langle\xi\rangle^3}\partial_\theta f^{in}(X(-t),\Xi(-t))\int_0^t E_0^3(s, x + \frac{(s-t)z}{\langle\xi\rangle}e_3)ds + \mathcal{O}(\epsilon). \end{aligned}$$

□

This presentation shows that the nonlinear term $\epsilon[E+v\times B]\cdot\nabla_\xi f$ may lead to non-trivial growth in the sup-norm unless initial data is prepared.

5.4 Loss of Derivatives in L^1_k for Non-Linear Problem

In this section we describe another potential problem by attempting to compute uniform L^1_k estimates on the fields in for the non-linear system. First consider computing the time derivative of the nonlinear f along the linear flow in the inhomogeneous setting ($b_e(x) \equiv 1$):

$$\begin{aligned} X(t) &= x - \Xi^m(-t,\xi), \\ \Xi(t) &= \xi(r, \theta - \frac{t}{\epsilon\langle\xi\rangle}, z). \end{aligned}$$

Then we have that

$$\begin{aligned} &\frac{d}{dt}[f((X,\Xi)(t))] = \epsilon[\nabla_\xi\cdot([E+v\times B]f)]\circ(X,\Xi) + M'(|\xi|)\frac{\Xi}{|\xi|}\cdot E(t,X) \\ &= \nabla_\xi\cdot[\epsilon([E+v\times B]f)\circ(X,\Xi)] + \partial_\theta[([E+v\times B]f)\circ(X,\Xi)]\cdot\frac{\xi t}{\langle\xi\rangle^3} + M'(|\xi|)\frac{\Xi}{|\xi|}\cdot E(t,X) \end{aligned}$$

Next we consider the variable change as follows

$$g(t,x,\xi) = f((X,\Xi)(t,x,\xi))$$

So that g satisfies

$$\partial_t g = \nabla_\xi\cdot[\epsilon[E+v\times B]\circ(X,\Xi)g] + \partial_\theta[([E+v\times B]\circ(X,\Xi)g]\cdot\frac{\xi t}{\langle\xi\rangle^3} + M'(|\xi|)\frac{\Xi}{|\xi|}\cdot E(t,X)$$

Suppose we seek to transform Maxwell's equations using a Fourier transform in x. Then the transformed current density, after changing variables along the the volume preserving diffeomorphism $(x,\xi)\mapsto(X,\Xi)(t)$ is given by

$$\begin{aligned}\int v(\xi)\hat{f}d\xi &= \int\int v(\xi)e^{ik\cdot x}f(t,x,\xi)dxd\xi\\ &= \int\int v(\Xi(t))e^{ik\cdot(x-\Xi^m(-t,\xi))}gdxd\xi\\ &= \int\int v(\Xi)e^{ik\cdot(x+\Xi^m(t,\xi))}\bigg\{\int_0^t \nabla_\xi\cdot[\epsilon[E+v\times B]\circ(X,\Xi)(s)g(s)]\\ &\qquad \partial_\theta[([E+v\times B]\circ(X,\Xi)(s)g(s)]\cdot\frac{\xi s}{\langle\xi\rangle^3}+M'(|\xi|)\frac{\Xi}{|\xi|}\cdot E(s,X)\bigg\}dsdxd\xi\end{aligned}$$

Consider for example the first term in the above. We can first integrate by parts and the flow can be inverted as follows

$$\begin{aligned}&\int\int\epsilon\nabla_\xi\Big(v(\Xi(t,\xi))e^{ik\cdot(x+\Xi^m(t,\xi))}\Big)\circ(X^{-1},\Xi^{-1})\int_0^t[E+v\times B]fdsdxd\xi\\ &=\int\int\epsilon\Big(\nabla_\xi[v(\Xi)]+v(\Xi)\otimes\nabla_\xi[ik\cdot\Xi^m]\Big)\circ\Xi^{-1}\int_0^t e^{ik\cdot x}[E+v\times B]fdsdxd\xi\end{aligned}$$

remark that we have the estimates

$$||\epsilon\nabla_\xi[v(\Xi)]\circ\Xi^{-1}||_{L^\infty_\xi}\leq C$$

whereas

$$||\epsilon v(\Xi)\otimes\nabla_\xi[ik\cdot\Xi^m]\circ\Xi^{-1}||_{L^\infty_\xi}\leq\epsilon kC$$

Therefore this term can be controlled by

$$\begin{aligned}|\int v(\xi)\hat{f}d\xi| &\leq C\int_0^t||(\hat{E},\hat{B})(s,\cdot)||_{L^1}||\hat{f}(s,\cdot,\cdot)||_{L^1_k,L^\infty_\xi}ds\\ &\qquad + C\int_0^t||\epsilon k(\hat{E},\hat{B})(s,k)||_{L^1_k}||\hat{f}(s,\cdot,\cdot)||_{L^1_k,L^\infty_\xi}ds\end{aligned}$$

But this is not suitable for uniform L^1_k estimates due to the multiplicative factor k.

5.5 Scaling

Coming back for a moment to the system (2.1)-(2.3), and considering the expression (3.47) there is a clear challenge in deducing uniform in ϵ estimates on (E,B) in terms of $\bar{f}$ using the approach of [7]. However, theses methods can be generalized for a wider variety of perturbations. It is apparent there is a strong dependence on 5 parameters, that are fixed at time $t=0$. These are: 1, the size of the in initial data $\bar{f}$ in the sense of L^∞, 2, the size of the initial fields (E,B), 3, the size of the momentum support of $\bar{f}$, 4, the size of the momentum size support of the equilibrium profile M, and finally 5, the size of M in the sense of L^∞. These can be summarized by consider a general anzatz as follows:

$$\mathbf{f}(t,x,\xi) = \epsilon^{\beta} M(\epsilon^{\alpha}|\xi|) + \epsilon^{\gamma} f(t,x,\epsilon^{\mu}\xi), \qquad \mathbf{f}^{in}(x,\xi) = \epsilon^{\beta} M(\epsilon^{\alpha}|\xi|) + \epsilon^{\gamma} f^{in}(x,\epsilon^{\mu}\xi), \tag{5.31}$$

$$(\mathbf{E},\mathbf{B})(t,x) = \epsilon^{\nu}(E,B)(t,x), \qquad (\mathbf{E}^{in},\mathbf{B}^{in}) = \epsilon^{\nu}(E^{in},B^{in})(x). \tag{5.32}$$

In a completely general situation we simply have $\beta,\alpha,\gamma,\mu,\nu \in \mathbb{R}$ fixed constants. Here we do not scale the spatial variable x as it is already well adjusted. This is physically relevant as we assume the characteristic length scale $|x| \lesssim R_x^0$ is fixed, such as confinement to a tokamak or confined to Earth in the Van Allen Belts. The momentum variable however, can vary since it will depend on the plasma temperature (which is independent of the characteristic length scale). In the new momentum variable $\eta := \epsilon^{\mu}\xi$ this perturbed system becomes:

$$\begin{cases} \partial_t f + v(\epsilon^{-\mu}\eta)\cdot\nabla_x f - \epsilon^{-1}[\frac{\eta}{\langle \epsilon^{-\mu}\eta\rangle} \times \mathbf{B}_e(x)]\cdot\nabla_\eta f \\ \qquad -\epsilon^{\mu+\nu}[E + v(\epsilon^{-\mu}\eta)\times B]\cdot\nabla_\eta f = \epsilon^{\beta+\alpha+\nu-\gamma} M'(\epsilon^{\alpha-\mu}|\eta|)\frac{\eta}{|\eta|}\cdot E \\ \partial_t E - \nabla_x \times B = \epsilon^{\gamma-3\mu-\nu}\int v(\epsilon^{-\mu}\eta) f(t,x,\eta)d\eta \\ \nabla_x\cdot E = -\epsilon^{\gamma-3\mu-\nu}\int f(t,x,\eta)d\eta - \epsilon^{\beta-3\alpha-\nu}\int M(|\eta|)d\eta + \rho_i \\ \partial_t B + \nabla_x\times E = 0, \quad \nabla_x\cdot B = 0 \end{cases} \tag{5.33}$$

We will consider this system under the following conditions:

$$\text{for } \mu > 0: \qquad \begin{cases} r_1' := \gamma - 8\mu - \nu - 1 \ge 0, \\ r_2' := \gamma - 8\mu \ge 0, \\ r_3' := \beta - 2\alpha - 2\mu \ge 0 \end{cases} \tag{5.34}$$

$$\text{for } -1 \le \mu \le 0: \qquad \begin{cases} r_1 := \gamma - 4\mu - \nu - 1 \ge 0, \\ r_2 := \gamma - 4\mu \ge 0 \\ r_3 := \beta - 2\alpha \ge 0 \end{cases} \tag{5.35}$$

$$\text{for } \mu < -1: \qquad \begin{cases} r_1'' := \gamma - 3\mu - \nu \ge 0, \\ r_2'' := \gamma - 4\mu \ge 0 \\ r_3'' := \beta - 2\alpha \ge 0 \end{cases} \tag{5.36}$$

and

$$\rho_i = \epsilon^{\beta-3\alpha-\nu}\int M(|\eta|)d\eta, \quad \mu+\nu \ge 0. \tag{5.37}$$

Under these conditions we have the following result:

Theorem 3. *(Sufficient condition for uniform sup-norm following following the strategies of [7]) For initial data $(f^{in},E^{in},B^{in})(\cdot)$ satisfying (2.26)-(2.30), there exists $\epsilon_0 \in (0,1]$ and some $T>0$, independent of ϵ_0, such that for all $\epsilon \in (0,1]$, there is a unique C^1 solution $(f_\epsilon,E_\epsilon,B_\epsilon)$ on $[0,T]$, with $(f_\epsilon,E_\epsilon,B_\epsilon)|_{t=0} = (f^{in},E^{in},B^{in})$ to (5.33) such that*

$$\sup_{(x,\xi)\in\mathbb{R}^6} ||(f_\epsilon,E_\epsilon,B_\epsilon)(t,x,\xi)|| \le C(T,R_0^\infty,||(f^{in},E^{in},B^{in}||_{L^\infty_{x,\xi}}) < \infty, \tag{5.38}$$

if the scaling parameter criteria (5.34) - (5.37) are satisfied.

Remark 6. *Physically we should assume that $\mu \leq 0$ to ensure that we have uniform control of the initial support, supp*$(\mathbf{f}^{in}(x, \epsilon^{\mu}\xi)) \subset B(0, R_x^0) \times B(0, R_\xi^0)$.

The example given in [7] consider the critical case of $\mu = -1$ and $r_1 = r_2 = 0$. This meant to consider solutions of the form

$$\mathbf{f}(t,x,\xi) = \epsilon^{-2}M(\epsilon^{-1}|\xi|) + \epsilon^{-2}f(t,x,\epsilon^{-1}\xi), \tag{5.39}$$

$$(\mathbf{E},\mathbf{B})(t,x) = \epsilon(E,B)(t,x), \tag{5.40}$$

$$\rho_i = \int M(|\eta|)d\eta. \tag{5.41}$$

Another result which follows from repeating the procedure of [7], which was not considered, would be to take

$$\mathbf{f}(t,x,\xi) = \epsilon M(|\xi|) + \epsilon^2 f(t,x,\xi), \tag{5.42}$$

$$(\mathbf{E},\mathbf{B})(t,x) = \epsilon(E,B)(t,x), \tag{5.43}$$

$$\rho_i = \epsilon\int M(|\eta|)d\eta. \tag{5.44}$$

Remark 7. *The proposed anzatz (2.16) with equilibrium profile (2.14) does not satisfy (5.35) since $r_2 = -1$, although $r_1 = 0$ and $r_3 = 0$. Similarly for the case of equilibrium profile (2.15) these condition are not satisfied as $r_2 = -1$ whereas $r_1 = 0$ and $r_3 = 1$. The question becomes whether (5.34)-(5.34) are necessary conditions or not. It turns out that in the case of equilibrium profile (2.15), we can obtain uniform estimates provided the initial data f^{in} is prepared in the sense of definition 1.*

Proof. (Outline) Following the lines of [7], the proof of theorem 3 can be sketched as follows explaining each of the conditions (5.34)-(5.35). First, equation (3.47) under this scaling reads

$$\begin{aligned} E(t,x) = & -\epsilon^{\gamma-3\mu-\nu}\int p(t,x,\epsilon^{-\mu}\xi)Y(t,x) *_{t,x} (\mathbb{1}_{t>0}T(\epsilon^{-\mu}\xi)f(t,x,\xi))d\xi \\ & + \epsilon^{\gamma-3\mu-\nu}\int q(t,x,\epsilon^{-\mu}\xi)Y(t,x) *_{t,x} (\mathbb{1}_{t>0}f(t,x,\xi))d\xi \\ & + \frac{1}{4\pi t^2}\int_{|x-y|=t}\Big[t\nabla_x \times B^{in}(y) + E^{in}(y) + [(y-x)\cdot\nabla_y]E^{in}(y)\Big]dS(y) \\ & - \epsilon^{\gamma-3\mu-\nu}\frac{t}{4\pi}\int\int_{\mathbb{S}^2} p(1,\omega,\epsilon^{-\mu}\xi)f^{in}(x-t\omega,\xi)d\omega d\xi \\ := & \, I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{5.45}$$

After passing the transport operator to the Vlasov equation the first term becomes

$$\begin{aligned} I_1 = & -\epsilon^{\gamma-3\mu}\int \nabla_\xi p(t,x,\epsilon^{-\mu}\xi)Y(t,x) *_{t,x} (\mathbb{1}_{t>0}[E + v(\epsilon^{-\mu}\xi)\times B]f)d\xi \\ & - \epsilon^{\gamma-4\mu-\nu-1}\int \nabla_\xi p(t,x,\epsilon^{-\mu}\xi)Y(t,x) *_{t,x} (\mathbb{1}_{t>0}[\frac{\xi}{\langle\epsilon^{-\mu}\xi\rangle}\times\mathbf{B}_e]f)d\xi \\ & + \epsilon^{\beta+\alpha-3\mu}\int p(t,x,\epsilon^{-\mu}\xi)Y(t,x) *_{t,x} (\mathbb{1}_{t>0}M'(\epsilon^{\alpha-\mu}|\xi|)\frac{\xi}{|\xi|}\cdot E)d\xi \\ := & \, I_1^1 + I_1^2 + I_1^3. \end{aligned} \tag{5.46}$$

Owing to the estimates (3.49) and (3.50), the 2nd and 4th terms in (5.45) are estimated respectively with

$$\begin{aligned} I_2 &\leq \epsilon^{\gamma-3\mu-\nu}(1+\epsilon^{-4\mu})C(R_\xi^T)\int_0^t ||f(s,\cdot,\cdot)||_{L^\infty_{x,\xi}}ds, \\ I_4 &\leq \epsilon^{\gamma-3\mu-\nu}(1+\epsilon^{-2\mu})tC(||f^{in}||_{L^\infty_{x,\xi}}, R_\xi^0). \end{aligned}$$

Term I_3 is easily controlled by initial data and is independent of ϵ. Furthermore, one has

$$||\nabla_\xi p(1,\omega,\cdot)||_{L^\infty(\{|\xi|\leq R_\xi^T\})} \leq 8\big(1+(R_\xi^T)^2+(R_\xi^T)\sqrt{1+(R_\xi^T)^2}\big)^2$$

Therefore the first two terms of (5.46) are similarly controlled by

$$\begin{aligned} I_1^1 &\leq \epsilon^{\gamma-4\mu}(1+\epsilon^{-4\mu})C(R_\xi^T)\int_0^t ||(E,B)(s,\cdot)||_{L^\infty_x}||f(s,\cdot,\cdot)||_{L^\infty_{x,\xi}}ds, \\ I_1^2 &\leq \epsilon^{\gamma-4\mu-\nu-1}(1+\epsilon^{-4\mu})C(R_\xi^T, ||\mathbf{B}_e||_{L^\infty_x})\int_0^t ||f(s,\cdot,\cdot)||_{L^\infty_{x,\xi}}ds, \end{aligned}$$

and finally the last term in (5.46) is bounded by

$$\begin{aligned} I_1^3 &\leq C\epsilon^{\beta+\alpha-3\mu}(1+(\epsilon^{-\mu}R_\xi^0)^2)\int |M'(\epsilon^{\alpha-\mu}|\xi|)|d\xi\int_0^t ||E(s,\cdot)||_{L^\infty_x}ds \\ &\leq C(R_\xi^0)\epsilon^{\beta-2\alpha}(1+(\epsilon^{-\mu}R_\xi^0)^2)||M'||_{L^1}\int_0^t ||E(s,\cdot)||_{L^\infty_x}ds. \end{aligned}$$

This justifies (5.34)-(5.35). Therefore in the worst case scenario when $\mu > 0$ we require:

$$\mu > 0: \quad \begin{cases} r_1' := \gamma - 8\mu - \nu - 1 \geq 0, \\ r_2' := \gamma - 8\mu \geq 0, \\ r_3' := \beta - 2\alpha - 2\mu \geq 0 \end{cases}$$

For $\mu \leq 0$ we must have:

$$\begin{cases} \gamma - 4\mu - \nu - 1 \geq 0, \\ \gamma - 3\mu - \nu \geq 0, \\ \gamma - 4\mu \geq 0 \\ \beta - 2\alpha \geq 0 \end{cases}$$

Therefore if we consider two regimes $\mu \in [-1,0]$ and $\mu \in (-\infty, 1)$, we have either $-(1+\mu) \leq 0$ or $-(1+\mu) > 0$ respectively. Therefore this reduces to

$$-1 \leq \mu \leq 0: \quad \begin{cases} r_1 := \gamma - 4\mu - \nu - 1 \geq 0, \\ r_2 := \gamma - 4\mu \geq 0 \\ r_3 := \beta - 2\alpha \geq 0 \end{cases}$$

and

$$\mu < -1 : \quad \begin{cases} r_1'' := \gamma - 3\mu - \nu \geq 0, \\ r_2'' := \gamma - 4\mu \geq 0 \\ r_3'' := \beta - 2\alpha \geq 0 \end{cases}$$

Remark, the -1 comes from the size of the applied field $\mathcal{O}(\epsilon^{-1})$. Finally, the estimate (3.58) becomes

$$|\Xi(t)| \leq |\xi| + \epsilon^{\mu+\nu} C \int_0^t ||E(s, \cdot)||_{L_x^\infty} ds \tag{5.47}$$

Therefore, with the a priori assumption that $||E(t, \cdot)||_{L_x^\infty}$ remains uniformly bounded in ϵ on some time interval $[0, T]$, we must have $\mu + \nu \geq 0$ to ensure the characteristics curves $(X, \Xi)(t)$ remain in a compact set on a uniform time interval $[0, T]$. $\square$

Appendix A

A.1 Homogeneous Distribution Theory

In this section we introduce some results homogeneous distributions in order to prove lemma 4. We first start with a result given in [16], about commuting vector fields

Lemma 15. *(Commuting Vector Fields) Define*

$$L_i := x_i\partial_t + t\partial_{x_i}. \tag{5.48}$$

Then in the sense of distributions we have the identities

$$\begin{aligned}
[L_i, \Box_{t,x}] = L_i\Box_{t,x} - \Box_{t,x}L_i = 0, && (5.49)\\
L_iY = 0, && (5.50)\\
L_i\delta(t,x) = 0. && (5.51)
\end{aligned}$$

Proof. Consider the triplet $\{i,j,k\} = \{1,2,3\}$ with possible rearrangement of order. For neatness, we will identify $\partial_{x_i} = \partial_i$. Then by direct computation we have

$$\begin{aligned}
\Box_{t,x}L_i &= (\partial_t^2 - \Delta_x)(x_i\partial_t + t\partial_{x_i})\\
&= [x_i\partial_t^3] + [2\partial_{ti}^2 + t\partial_{itt}] - [x_i(\partial_j^2 + \partial_k^2)\partial_t + \partial_i^2(x_i\partial_t)] - [t\Delta\partial_i]\\
&= [x_i\partial_t^3] + [2\partial_{ti}^2 + t\partial_{itt}] - [x_i\Delta\partial_t + 2\partial_{it}^2] - [t\Delta\partial_i]\\
&= x_i\partial_t^3 + t\partial_{itt}^3 - x_i\Delta\partial_t - t\Delta\partial_i\\
&= (x_i\partial_t + t\partial_i)(\partial_t^2 - \Delta) = L_i\Box_{t,x}. && (5.52)
\end{aligned}$$

Next we take $\phi(t,x) \in C_c^\infty(\mathbb{R}^4)$ and consider

$$\begin{aligned}
\langle L_i\delta, \phi\rangle &= \int([x_i\partial_t + t\partial_i]\delta(t,x))\phi(t,x)dxdt\\
&= -\int \delta(t,x)[x_i\partial_t + t\partial_i]\phi(t,x)dxdt\\
&= -[x_i\partial_t\phi(x,t) + t\partial_i\phi(t,x)]|_{(t,x)=(0,0)} = 0. && (5.53)
\end{aligned}$$

Finally we compute

$$\begin{aligned}
\langle L_iY, \phi\rangle &= -\langle Y, (t\partial_i + x_i\partial_t)\phi\rangle\\
&= -\int_{\mathbb{R}^4} \frac{\delta(t-|x|)}{4\pi|x|}(t\partial_i + x_i\partial_t)\phi(t,x)dtdx\\
&= -\frac{1}{4\pi}\int_{\mathbb{R}^3} \partial_i\phi(|x|,x) + \frac{x_i}{|x|}\partial_t\phi(|x|,x)dx\\
&= -\frac{1}{4\pi}\int_{\mathbb{R}^3} \partial_i[\phi(|x|,x)]dx = 0. && (5.54)
\end{aligned}$$

□

Before we can use these to proof lemma 4, we recall a result from [15]. Recall that $S \in \mathfrak{M}_\alpha(\mathbb{R}^n)$ if for all $\phi \in C_c^\infty(\mathbb{R}^n - \{0\})$ and $\lambda > 0$,

$$\langle S, M_\lambda \phi \rangle = \lambda^{\alpha+n} \langle S, \phi \rangle , \tag{5.55}$$

where $M_\lambda \phi(x) := \phi(\lambda^{-1}x)$.

Lemma 16. *Let $S \in \mathfrak{M}_\alpha(\mathbb{R}^n)$. Then if $\alpha > -n$, there exists a unique homogeneous extension $\dot{S} \in \mathcal{D}'(\mathbb{R}^n)$ of degree α.*

Proof. (Proof of lemma 4) The following proof holds for any $v \in \mathbb{R}^3$ satisfying $|v| < 1$. Let

$$L := (L_1, L_2, L_3) = x\partial_t + t\nabla_x, \tag{5.56}$$

then

$$\begin{aligned} v \cdot L &= (v \cdot x)\partial_t + tv \cdot \nabla_x \\ &= (x \cdot v)\partial_t + t[\partial_t + v \cdot \nabla_x] - t\partial_t \\ &= (x \cdot v - t)\partial_t + tT(v). \end{aligned} \tag{5.57}$$

By a straight forward computation we have

$$(t - x \cdot v)L + x(v \cdot L) = t[(t - x \cdot v)\nabla_x + xT(v)]. \tag{5.58}$$

Next using (5.50) we obtain

$$0 = (v \cdot L)Y = tT(v)Y + (x \cdot v - t)\partial_t Y, \tag{5.59}$$

$$0 = [(t - x \cdot v)L + x(v \cdot L)]Y = t[(t - x \cdot v)\nabla_x + xT(v)]Y. \tag{5.60}$$

Therefore away from $x \cdot v = t \neq 0$ it follows that

$$\begin{aligned} \partial_t Y &= \frac{t}{t - x \cdot v} T(v)Y := a_0(t,x)T(v)Y, \\ \partial_i Y &= \frac{x_i}{x \cdot v - t} T(v)Y := a_i(t,x)T(v)Y. \end{aligned} \tag{5.61}$$

Remark that since $|v| < 1$ the singular set $x \cdot v = t$ of a_i intersects the support of Y only at the origin

$$\{x \cdot v = t, \text{ or } t = 0\} \cap \{|x| \le t\} = \{(0,0)\} \tag{5.62}$$

Denoting $\partial_0 = \partial_t$ it then follows for $i \in \{0,1,2,3\}$

$$\operatorname{supp}(\partial_i Y - a_i T(v)Y) = \{(0,0)\}. \tag{5.63}$$

Therefore $a_i T(v)Y \in \mathfrak{M}_{-3}(\mathbb{R}^4)$ (as $a_i \in \mathcal{M}_0(\mathbb{R}^4)$) and therefore, by lemma 16, has a unique homogeneous extension to all of $\mathbb{R}^4$, i.e.

$$\partial_i Y - a_i T(v)Y \in \mathcal{D}(\mathbb{R}^4). \tag{5.64}$$

Introduce an auxiliary cut-off function

$$\chi := \chi_v \in C_c^\infty(\mathbb{R}^+), \quad 0 \le \chi \le 1, \quad \chi|_{[0,\frac{1}{2}+\frac{1}{2|v|}]} = 1, \quad \operatorname{supp}(\chi) \subset [0, \frac{1}{|v|}). \tag{5.65}$$

Therefore, on the light cone $\mathcal{LC} := \{(t,x) \mid |x| = t\}$, we have $\chi(\frac{|x|}{t})|_{\mathcal{LC}} \equiv 1$. This implies

$$\partial_i Y = \chi(\frac{|x|}{t})\partial_i Y = \chi(\frac{|x|}{t}) a_i(t,x)T(v)Y(t,x) := a_i^0(t,x)T(v)Y(t,x). \tag{5.66}$$

Thus

$$\partial_i Y = T(v)[a_i^0 Y] - [T(v)a_i^0]Y. \tag{5.67}$$

Therefore we set

$$\begin{aligned} a^0(t,x,\xi) &:= (a_1^0, a_2^0, a_3^0), \\ a^1(t,x,\xi) &:= T(v)a^0, \\ p(t,x,\xi) &:= -v(\xi)a_0^0 - a^0, \\ q(t,x,\xi) &:= v(\xi)T(v)a_0^0 + a^1. \end{aligned} \tag{5.68}$$

Giving the desired result

$$[v(\xi)\partial_t + \nabla_x]Y = -T(v)(pY) + qY. \tag{5.69}$$

Similarly

$$\nabla_x \times (v(\xi)Y) = [v(\xi) \times \nabla_x]Y = v(\xi) \times (T(v)[a^0 Y] + a^1 Y). \tag{5.70}$$

□

A.2 Bihari-LaSalle Inequality

We first introduce the classic Gronwall's lemma.

Lemma 17. *(Gronwall: Integral form) Let u and β be continuous, real valued function defined on an interval $I = [0,b)$ (where $b > 0$ may be $+\infty$). Let α be an integral function on I and suppose that u satisfies*

$$u(t) \leq \alpha(t) + \int_0^t \beta(s)u(s)ds, \ \forall t \in I, \tag{5.71}$$

then

$$u(t) \leq \alpha(t) + \int_0^t \alpha(s)\beta(s)e^{\int_s^t \beta(\tau)d\tau}ds, \forall t \in I. \tag{5.72}$$

If in addition α is non-decreasing then

$$u(t) \leq \alpha(t)e^{\int_0^t \beta(s)ds} \tag{5.73}$$

We omit the proof this well known result. However will prove a non-linear version of Gronwall's lemma known as the Bihari-LaSalle inequality. This result was originally given in [2]. For our purposes, the non-linear function $w(u)$ in the following lemma will simply be a quadratic polynomial such as in the expression (3.53) for instance.

Lemma 18. *(Bihari-LaSalle) Let u and F be non-negative, continuous functions on an interval $I = [0, b)$, $b > 0$. Furthermore, let w be a non-negative, non-decreasing, continuous function on $[0, \infty)$ and let $\alpha > 0$. Suppose that u satisfies*

$$u(t) \leq \alpha + \int_0^t F(s) w(u(s)) ds. \tag{5.74}$$

Then

$$u(t) \leq G^{-1}\Big(G(\alpha) + \int_0^t F(s) ds\Big), \forall t \in [0, T], \tag{5.75}$$

where

$$G(x) := \int_{x_0}^x \frac{dy}{w(y)},\ \ x \geq 0,\ \ x_0 > 0. \tag{5.76}$$

and $b \geq T > 0$ is chosen so that the inverse function G^{-1} evaluated above is well defined

$$G(\alpha) + \int_0^t F(s) ds \in Dom(G^{-1}), \forall t \in [0, T]. \tag{5.77}$$

Proof. First define the right hand side of (5.74) as

$$v(t) := \alpha + \int_0^t F(s) w(u(s)) ds. \tag{5.78}$$

Then since w is non-decreasing, and the inequality (5.74) implies

$$w(u(t)) \leq w(v(t)), \tag{5.79}$$

and since $\alpha > 0$ we have $w(v(t)) > 0$, and therefore dividing by $w(v(t))$ and multiplying by $F(t)$ gives

$$\frac{w(u(t))}{w(v(t))} F(t) \leq F(t). \tag{5.80}$$

Furthermore,

$$v'(t) = F(t) w(u(t)), \tag{5.81}$$

so that

$$\frac{v'(t)}{w(v(t))} \leq F(t) \tag{5.82}$$

Moreover we compute with Leibniz's integral formula

$$\frac{d}{dt}[G(v(t))] = \frac{v'(t)}{w(v(t))} \leq F(t) \tag{5.83}$$

Therefore integrating both side gives

$$G(v(t)) - G(v(0)) = G(v(t)) - G(\alpha) \leq \int_0^t F(s)ds. \tag{5.84}$$

Equivalently we have

$$G(v(t)) \leq G(\alpha) + \int_0^t F(s)ds \tag{5.85}$$

Finally, since w is non-decreasing, G^{-1} is increasing and we finally arrive at

$$u(t) \leq v(t) \leq G^{-1}\left(G(\alpha) + \int_0^t F(s)ds\right), \forall t \in [0, T], \tag{5.86}$$

where T is chosen so that the above argument is well defined.

□

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