The i-Graph and Other Variations on the γ-Graph

by

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BSc., Thompson Rivers University, 2010
MSc., University of Victoria, 2012

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We acknowledge with respect the Lekwungen peoples on whose traditional territory the university stands, and the Songhees, Esquimalt, and W̱SÁNEĆ peoples whose historical relationships with the land continue to this day.

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ABSTRACT

In graph theory, reconfiguration is concerned with relationships among solutions to a given problem. For a graph G, the *γ-graph of G*, $G(\gamma)$, is the graph whose vertices correspond to the minimum dominating sets of G, and where two vertices of $G(\gamma)$ are adjacent if and only if their corresponding dominating sets in G differ by exactly two adjacent vertices. We present several variations of the γ-graph including those using identifying codes, locating-domination, total-domination, paired-domination, and the upper domination number. For each, we show that for any graph H, there exist infinitely many graphs whose γ-graph variant is isomorphic to H.

The independent domination number $i(G)$ is the minimum cardinality of a maximal independent set of G. The i-graph of G, denoted $\mathscr{I}(G)$, is the graph whose vertices correspond to the i-sets of G, and where two i-sets are adjacent if and only if they differ by two adjacent vertices. In contrast to the parameters mentioned above, we show that not all graphs are i-graph realizable. We build a series of tools to show that known i-graphs can be used to construct new i-graphs and apply these results to build other classes of i-graphs, such as block graphs, hypercubes, forests, and unicyclic graphs. We determine the structure of the i-graphs of paths and cycles, and in the case of cycles, discuss the Hamiltonicity of their i-graphs. We also construct the i-graph seeds for certain classes of line graphs, a class of graphs known as theta graphs, and maximal planar graphs. In doing so, we characterize the line graphs and theta graphs that are i-graphs.

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Chapter 1

Introduction

1.1 Domination

Consider a labyrinthine castle with many interconnected rooms, filled with valuable treasures. We wish to protect these treasures by ensuring that guards are always nearby, no matter where a thief might strike. Here, we consider a room to be protected if there is either a guard stationed in that room, or in a room directly connected to it. How guards can be positioned to ensure this protection, and in particular, how it can be done with the fewest guards possible, is the problem of domination.

To model this building, we imagine that each room is the vertex of a graph G, and that two vertices are connected if and only if their corresponding rooms are joined. Then, each configuration of guards in rooms (assuming at most one guard occupies each room), becomes a matter of selecting vertex subsets of G. Formally, given a graph $G = (V, E)$, a set $S \subseteq V$ is a *dominating set* of G if for each $v \in V$, v is either in S or adjacent to a vertex in S. The minimum cardinality of a dominating set is the *domination number* $\gamma(G)$, and a set is a γ-*set* if it is dominating and has cardinality $\gamma(G)$. Thus, the minimum number of guards required to protect our castle modelled by the graph G is $\gamma(G)$.

Domination is a wide and well-studied field of graph theory. The domination number of a graph was first examined by Berge in 1958 [3] as the "coefficient of external stability". It was later in 1962 that Ore first used the term "domination number" [31], and then in their 1977 survey, Cockayne and Hedetniemi [10] introduced the $\gamma(G)$ notation. For basic notation, foundations, and definitions in this dissertation, we refer the reader to both volumes of Haynes, Hedetniemi, and Slater's text: *Fundamentals of Domination in Graphs* [19] and *Domination in Graphs* [18].

The *private neighbourhood* of a vertex v with respect to a vertex set S is the set $pn(v, S) = N[v] - N[S - \{v\}]$; therefore, a dominating set S is minimal dominating if for each $u \in S$, $pn(u, S)$ is nonempty. The *external private neighbourhood* of v with respect to S is the set $\text{epn}(v, S) = pn(v, S) - \{v\}$.

A set $S \subseteq V(G)$ is *irredundant* if $\text{pn}(v, S) \neq \varnothing$ for each $v \in S$, and *maximal irredundant* if S is irredundant but no proper superset of S is irredundant. The *irredundance number* $\text{ir}(G)$ is the minimum cardinality of a maximal irredundant set of G.

We denote the *independence number* of a graph G by $\alpha(G)$. The *independent domination number* of G, $i(G)$ is the minimum cardinality of a maximal independent set of G. A *well-covered* graph is a graph G for which $i(G) = \alpha(G)$. For other domination principles and terminology, see [19, 18].

In general, we follow the notation of [6]. In particular, the *disjoint union* of two graphs is denoted $G \cup H$, whereas the *join* of G and H, denoted $G \vee H$, is the graph obtained from $G \cup H$ by joining every vertex of G with every vertex of H.

1.2 Reconfiguration

Let us return to the problem of guarding our castle filled with treasures. Although we may have found a least-cost arrangement of guards to ensure that the castle is protected, what now happens when the guards move to different rooms? Say we have a different proposed morning and afternoon configuration of guards that both offer protection to the castle. Is it possible to transform the morning configuration into the afternoon configuration through a series of steps, such that one guard moves one room at a time, and so that the castle remains protected at all times? This brings us to the field of reconfiguration problems.

A *reconfiguration problem* asks whether it is possible to transform a given *source* (or *seed*) *solution* into a *target solution* through a series of incremental transformations (called *reconfiguration steps*) under some specified rule, such that each intermediate step is also a solution. The resulting chain of the source solution, intermediate solutions, and target solution is a *reconfiguration sequence*.

For example, when presented with two configurations of pieces on a chess board, one specified as the source and one as the target, a reachability problem might ask whether it is possible to begin with the source chess piece configuration, and then using only the standard allowable chess piece moves applied one at a time, eventually transform the board into the target chess piece configuration.

Variations to this might also ask whether a reconfiguration sequence of less than a specified number of steps exists, or whether a reconfiguration sequence is reversible when source and target are exchanged.

In graph theory, reconfiguration problems are often concerned with solutions that are vertex/edge subsets or labellings. In particular, when the solution is a vertex (or edge) subset, the reconfiguration problem can be viewed as a token manipulation problem, where a solution subset is represented by placing a token at each of the subset's vertices (edges). Then, the reconfiguration step for vertex subsets can be of one of three variants (edge subsets are handled analogously):

- ▷ **Token Slide (TS) Model**: A single token is slid along an edge between adjacent vertices.
- ▷ **Token Jump (TJ) Model**: A single token jumps from one vertex to another (without the vertices necessarily being adjacent).
- ▷ **Token Addition/Removal (TAR) Model**: A single token can either be added to a vertex or be removed from a vertex.

To represent the many possible solutions in a reconfiguration problem, each solution can be represented as a vertex of a separate graph, referred to as a *reconfiguration graph*, where adjacency between vertices follows one of the three token adjacency models, producing the *slide graph*, the *jump graph*, or the *TAR graph*, respectively.

More formally, given a graph G, the *slide graph* of G under some specified reconfiguration rule is the graph H, such that each vertex of H represents a solution of some problem on G, and two vertices u and v of H are adjacent if and only if the solution in G corresponding to u can be transformed into the solution corresponding to v by sliding a single token along an edge in G.

In this dissertation, we focus on two mirrored problems for the slide graph model of reconfiguration graphs:

(i) (**Structure**) Given a graph G (called a *seed graph*), what is the resulting reconfiguration graph of G?

(ii) (**Realizability**) Given a graph H (called a *target graph*), does there exist some seed graph G, such that the reconfiguration graph of G is isomorphic to H?

1.2.1 Gamma Graphs

First defined by Fricke, Hedetniemi, Hedetniemi, and Hutson in 2011 [13], the *γ-graph of a graph G* is the graph $G(\gamma) = (V(G(\gamma)), E(G(\gamma)))$, where each vertex $v \in V(G(\gamma))$ corresponds to a γ-set S_v of G. The vertices u and v in $G(\gamma)$ are adjacent if and only if there exist vertices u' and v' in G such that $u'v' \in E(G)$ and $S_v = (S_u - u') \cup \{v'\}$; this is a token-slide model of adjacency. For additional results on γ-graphs, see [4, 11, 12, 13].

An initial question of Fricke et al. was to determine exactly which graphs are γ-graphs [13]; they showed that every tree is the γ-graph of some graph, and further conjectured that every graph is the γ-graph of some graph. Later that year, Connelly, Hutson, and Hedetniemi [11] proved this conjecture to be true.

Theorem 1.1. [11] *For any graph H, there exists some graph G such that $G(\gamma) \cong H$. That is, every graph is the γ-graph of some graph.*

We give the explicit construction from Connelly et al. in Figure 2.1 of Chapter 2.

Subramanian and Sridharan [38] independently defined a different γ-graph of a graph G, denoted $\gamma \cdot G$. The vertex set of $\gamma \cdot G$ is the same as that of $G(\gamma)$; however, for $u, w \in V(\gamma \cdot G)$ with associated γ-sets S_u and S_w in G, u and w are adjacent in $\gamma \cdot G$ if and only if there exist some $v_u \in S_u$ and $v_w \in S_w$ such that $S_w = (S_u - \{v_u\}) \cup \{v_w\}$. This version of the γ-graph was dubbed the "single vertex replacement adjacency model" by Edwards [12], and is sometimes referred to as the "jump γ-graph" as it follows the TJ-Model for token reconfiguration. Further results concerning $\gamma \cdot G$ can be found in [25, 36, 35]. Notably, if G is a tree or a unicyclic graph, then there exists a graph H such that $\gamma \cdot H = G$ [36]. Conversely, if G is the (jump) γ-graph of some graph H, then G does not contain any induced $K_{3,2}$, $P_3 \vee K_2$, or $(K_2 \cup K_1) \vee 2K_1$ [25].

Also using a jump-adjacency model, Haas and Seyffarth [15] define the *k-dominating graph* of G, $D_k(G)$, as the graph with vertices corresponding to the k-dominating sets of G (i.e. the dominating sets of cardinality at most k). Two vertices in the k-dominating graph are adjacent if and only if the symmetric difference of their associated k-dominating sets contains exactly one element. Additional results can be found in [1, 16, 17, 39].

1.3 Outline of Dissertation

In this dissertation, we provide new results on the slide graph model of reconfiguration graphs.

In Chapter 2, we define several variations on the γ-graph using other domination parameters including the total domination number, the paired domination number, and the irredundance number, and solve the realizability problem in the affirmative for each; that is, we show that all graphs are π-graphs for each of their respective parameters π.

In Chapter 3, we define the i-graph: the reconfiguration graph of $i(G)$ (the independent domination number of a graph). We show that unlike the graphs in Chapter 2, not all i-graphs are realizable. In particular, all small non-i-graphs are part of the family of theta graphs (defined on page 28). Throughout the chapter, we build a series of tools to show how known i-graphs can be used to construct new i-graphs, and then apply these results to show that some famous classes of graphs, including all block graphs, are i-graphs.

Next, in Chapter 4, we move away from realizability of i-graphs, and focus on structural results; that is, given a graph G, what is the resulting i-graph of G? This question yields surprisingly complex answers for even the simplest of graph classes. We fully determine the structure of the i-graphs of paths and cycles, and discuss the Hamiltonicity of i-graphs of cycles.

Finally, in Chapter 5, we return to the topic of theta graphs and fully classify which theta graphs are i-graphs, and which are not. To do so, we apply a novel technique of examining the cliques in the complement of a graph G, rather than i-sets in G itself. Extending this technique, we also classify the family of line graphs for their i-graph realizability.

Chapter 2

Variations on the γ-Graph

The following chapter is taken from [30], except for Section 2.4, which contains new results regarding Roman and total-Roman dominating graphs.

Theorem 2.1. [11] *For any graph H, there exists some graph G such that $G(\gamma) \cong H$. That is, every graph is the γ-graph of some graph.*

To begin, we examine four domination-related parameters and their respective extensions of Theorem 2.1. A vertex set S is said to *totally-dominate* a graph G if it is dominating and for each $v \in S$ there exists $u \in S$ such that u and v are adjacent. The cardinality of a smallest total-dominating set is the *total-domination number* $\gamma_t(G)$, as first introduced by Cockayne et al. in [8]. Closely related, a *paired-dominating* set S is a total-dominating set with the additional requirement that the induced subgraph $G[S]$ has a perfect matching. The *paired-domination number* $\gamma_{pr}(G)$ was defined similarly by Haynes and Slater in [21]. Since every paired-dominating set is also a total-dominating set, for every graph G without isolated vertices, $\gamma(G) \leq \gamma_t(G) \leq \gamma_{pr}(G)$ [19]. A *connected-dominating* set S is a dominating set where $G[S]$ is connected, and the cardinality of a smallest connected-dominating set is the *connected-domination number* $\gamma_c(G)$, as defined by Sampathkumar and Walikar [32]. For all nontrivial connected graphs G, $\mathrm{ir}(G) \leq \gamma(G) \leq \gamma_t(G) \leq \gamma_c(G)$ and $\gamma(G) \leq 2\,\mathrm{ir}(G) - 1$ [19].

We say the ir-*graph*, the γ_t-*graph*, the γ_{pr}-*graph*, and the γ_c-*graph* of G are the graphs with vertices representing the minimum-cardinality maximal irredundant, total-dominating, paired-dominating, and connected dominating sets of G, respectively, and where adjacency is defined using the vertex-slide model. Using the same construction as Connelly et al. in [11] and restated below, we find a result analogous to Theorem 2.1 for the ir-graph, γ_t-graph, γ_{pr}-graph, and γ_c-graph.

Theorem 2.2. *Every graph H is the* ir*-graph, γ_{pr}-graph, γ_t-graph, and γ_c-graph of infinitely many graphs.*

For reference, we restate Connelly et al.'s construction for a graph G with $G(\gamma) \cong H$.

Construction: Given some graph H with $V(H) = \{v_1, v_2, \ldots, v_n\}$, to construct a graph G with $G(\gamma) \cong H$, begin with a copy of H and attach vertices a, b, c to every vertex of H. Then, add two (or more) pendant vertices to c, labelled as c_1 and c_2 (see Figure 2.1).

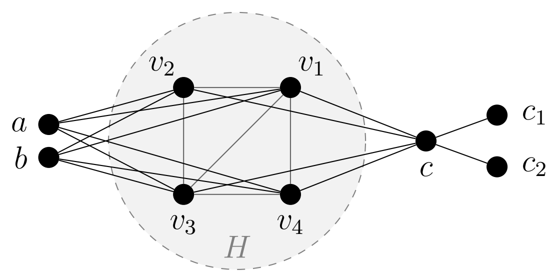


Figure 2.1: The graph G constructed from H from [11].

From the pendant vertices, c is in every γ-set of G; however, as a and b remain undominated, $\{c\}$ is not itself a γ-set. Thus, $\gamma_{pr}(G) \geq \gamma(G) \geq 2$ (likewise $\gamma_c(G) \geq \gamma_t(G) \geq \gamma(G) \geq 2$). It follows that for each $v_i \in V(H)$, the set $S_i = \{c, v_i\}$ is a γ-set. Since $\mathrm{ir}(G) \leq 2 = \gamma(G) \leq 2\,\mathrm{ir}(G) - 1$, it follows that S_i is an ir-set. Moreover, since $cv_i \in E(G)$, each S_i is also a γ_{pr}-set, a γ_t-set, and γ_c-set. Since neither $\{c, a\}$ nor $\{c, b\}$ is dominating, the collection $\{S_i : 1 \leq i \leq n\}$ consists of all the ir, γ, γ_t, γ_{pr}, γ_c-sets of G.

2.1 The γ^{ID}-Graph

A popular variation on domination is the topic of identifying codes. A vertex set S is an *identifying code* (ID-code) of a graph G if for each $v \in V$, the closed neighbourhood of v

and S have a unique, nonempty intersection. The *intersection set* of a vertex v with respect to a vertex subset S is the set $I_S(v) = N[v] \cap S$. Thus, S is an identifying code of G if all of its intersection sets are unique and nonempty. The cardinality of a minimum ID-code is denoted $\gamma^{ID}(G)$, and an ID-code with cardinality $\gamma^{ID}(G)$ is called an γ^{ID}*-set*. If $\gamma^{ID}(G)$ is finite, G is said to be *identifiable* (or *distinguishable*); otherwise, G is not identifiable, and $\gamma^{ID}(G)$ is defined to be $\gamma^{ID}(G) = \infty$.

Originally introduced by Karpovsky et al. in 1998 [24], ID-codes were proposed as a model for the positioning of fault-detection units on multiprocessor systems (for additional references, see Lobstein's extensive bibliography [26]). Consider now the problem of migrating the detecting units from one configuration to another, such that only one detecting unit can be moved at time to an adjacent processor, and at each step the configuration remains an ID-code. When given a certain starting configuration, what other configurations are reachable under these conditions? How many steps are required to move between them? Are there multiple routes from start to destination, or are we stuck with a single path? To aid in addressing this family of questions, we define the γ^{ID}*-graph of a graph* G, $G(\gamma^{ID}) = (V(\gamma^{ID}), E(\gamma^{ID}))$, similarly to the γ-graph, but where the vertices now correspond to the γ^{ID}-sets in G instead.

As a first result, we extend Theorem 2.1 to γ^{ID}-graphs. The construction and proof are similar; however, in consideration of the additional identification requirements, multiple copies of the graph $\mathcal{C} = C_4 \circleddash K_1$, the *depleted corona of* C_4, in Figure 2.2 are used to force certain vertices into the γ^{ID}-set. Given any graph G', we construct a new graph G by adding an edge between $x_1 \in V(\mathcal{C})$ and some $v \in V(G')$; we say $\mathcal{C}$ is *attached to* G' *at* v.

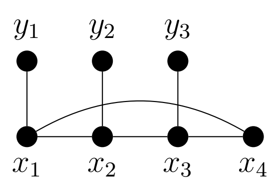


Figure 2.2: The graph $\mathcal{C}$ in Lemmas 2.3 and 2.4.

Lemma 2.3. *The vertex set* $X = \{x_1, x_2, x_3\}$ *is the unique* γ^{ID}*-set of* $\mathcal{C}$.

Proof. Suppose that $\mathcal{C}$ has a γ^{ID}-set S. Since the vertices in $Y = \{y_1, y_2, y_3\}$ are all pendant vertices in $\mathcal{C}$, for each $1 \le i \le 3$, either x_i or y_i is in S. Since X is identifying, it follows that $\gamma^{ID}(\mathcal{C}) = 3$ and that X is a γ^{ID}-set of $\mathcal{C}$.

Notice that since $|S| = 3$ and each y_i is pendant, $S \subseteq X \cup Y$ and $x_4 \notin S$. To dominate x_4, either x_1 or x_3 is in S. Without loss of generality, say $x_1 \in S$. To show uniqueness, we need only verify that the sets $S_2 = \{x_1, y_2, x_3\}$, $S_3 = \{x_1, x_2, y_3\}$ and $S_{2,3} = \{x_1, y_2, y_3\}$ are not identifying. For S_2, $I_{S_2}(x_3) = I_{S_2}(y_3) = \{x_3\}$ and is therefore not identifying. Likewise for $S_{2,3}$, $I_{S_{2,3}}(x_3) = I_{S_{2,3}}(y_3) = \{y_3\}$. Finally for S_3, $I_{S_3}(x_4) = I_{S_3}(y_1) = \{x_1\}$. It follows that $S = X = \{x_1, x_2, x_3\}$ is the unique γ^{ID}-set of $\mathcal{C}$. ■

Lemma 2.4. *Let G' be any graph, and construct G by attaching $\mathcal{C}$ to G' at some $v \in V(G')$. If S is any γ^{ID}-set of G, then $\{x_1, x_2, x_3\} \subseteq S$. Moreover, if S' is a γ^{ID}-set of G', then $S' \cup \{x_1, x_2, x_3\}$ is identifying in G.*

Proof. Let $V(G') = \{v_1, v_2, \ldots, v_n\}$, and suppose that G was constructed by attaching $\mathcal{C}$ to G' at v_n. Suppose that G has an γ^{ID}-set S. Regardless of whether v_n is in S or not, each vertex of $Y = \{y_1, y_2, y_3\}$ remains pendant and so for each $1 \le i \le 3$, either x_i or y_i is in S. Using the same arguments as in Lemma 2.3, $\{x_1, x_2, x_3\} \subseteq S$.

Now suppose that S' is a γ^{ID}-set of G' and consider $S = S' \cup \{x_1, x_2, x_3\}$ in G. Since S' is identifying, for each $v_i \in V(G')$, the intersection set $I_{S'}(v_i)$ in G' is unique. In G, all intersection sets of $v_1, v_2, \ldots, v_{n-1}$ remain the same. The intersection set of v_n becomes $I_S(v_n) = I_{S'}(v_n) \cup \{x_1\} \neq I_s(y_1)$. From the first portion of this lemma, the sets $I_S(x_i)$ for $1 \le i \le 4$ are also unique. It follows that S is identifying in G. ■

Notice that the converse of the second portion of Lemma 2.4 does not hold. For a counterexample, consider the graph G in Figure 2.3 constructed by attaching $\mathcal{C}$ to a copy of C_4 with $V(C_4) = \{v_1, v_2, v_3, v_4\}$. Although $\{v_1, v_3, x_1, x_2, x_3\}$ is a γ^{ID}-set of G, $\{v_1, v_3\}$ is not a γ^{ID}-set of C_4.

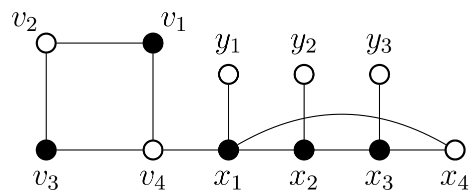


Figure 2.3: Counterexample to the converse of Lemma 2.4.

Theorem 2.5. *Every graph H is the γ^{ID}-graph of some graph.*

Proof. Let $H = (V(H), E(H))$ be any nonempty graph with $V(H) = \{v_1, v_2, \ldots, v_n\}$. We construct a new graph G such that $G(\gamma^{ID}) \cong H$.

Construction: Begin with a copy of H and for each $v_i \in V(H)$, attach two copies of the graph $\mathcal{C}$ from Figure 2.2 to H at v_i, labelled as $\mathcal{C}_i$ and $\mathcal{C}_i^*$ with

$$V(\mathcal{C}_i) = \{x_{i,1}, x_{i,2}, x_{i,3}, x_{i,4}, y_{i,1}, y_{i,2}, y_{i,3}\}, \text{ and } V(\mathcal{C}_i^*) = \{x_{i,1}^*, x_{i,2}^*, x_{i,3}^*, x_{i,4}^*, y_{i,1}^*, y_{i,2}^*, y_{i,3}^*\}.$$

Now, add vertices a and b so that $av_i \in E(G)$ and $bv_i \in E(G)$ for all $i = 1, 2, \ldots, n$. Finally, attach two more copies of $\mathcal{C}$, $\mathcal{C}_a$ and $\mathcal{C}_b$, at a and b, respectively (see Figure 2.4).

Consider the vertex set,

$$X = \left(\bigcup_{\substack{1 \le j \le n \\ 1 \le k \le 3}} \{x_{j,k}, x_{j,k}^*\} \right) \cup \left(\bigcup_{1 \le k \le 3} \{x_{a,k}, x_{b,k}\} \right).$$

In particular, notice that X consists of all the vertices within the various $\mathcal{C}$ graphs that Lemma 2.4 demonstrates are in every γ^{ID}-set of G.

We first show that for each $1 \le i \le n$, the vertex set $S_i = \{v_i\} \cup X$ is a γ^{ID}-set of G. From the construction of G, it is clear that S_i dominates G. Furthermore, from Lemma 2.4, we know that each vertex within a $\mathcal{C}$ subgraph is identified by S_i. For vertices within H, the identifying set $I_{S_i}(v_j)$ of the vertex v_j contains the unique pair $\{x_{j,1}, x_{j,1}^*\}$, ensuring that each v_j is identified in S_i. Finally, a and b also have the unique identifying sets $I_{S_i}(a) = \{x_{a,1}, v_i\}$ and $I_{S_i}(b) = \{x_{b,1}, v_i\}$, respectively. It follows that S_i is identifying and dominating in G.

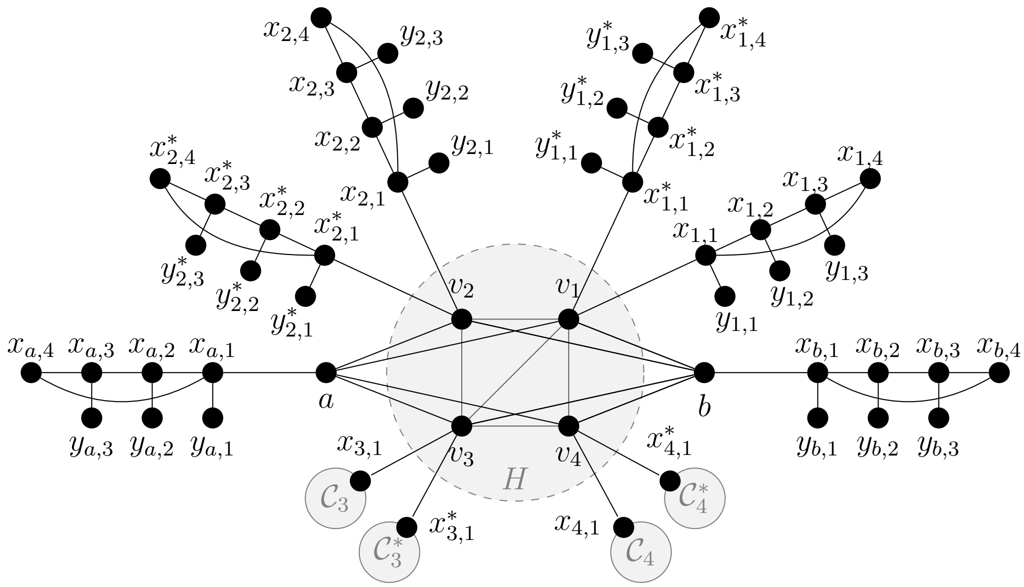


Figure 2.4: The graph G constructed from H in Theorem 2.5.

We now show that S_i is a γ^{ID}-set. Notice that there are $2n+6$ pendant vertices in G, and so $\gamma(G) \geq 2n+6$. Indeed, it is easy to see that $X = S_i - \{v_i\}$ is dominating in G, so $\gamma(G) = 2n+6$. Moreover, $\gamma^{ID}(G) \geq \gamma(G) = 2n+6$. Again, by Lemma 2.4, we know that any γ^{ID}-set of G contains all of X; however X is not identifying as $I_X(a) = \{x_{a,1}\} = I_X(y_{a,1})$, and so, $\gamma^{ID}(G) \geq (2n+6)+1$. Since $|S_i| = 2n+7$ and S_i is identifying, it follows that S_i is a γ^{ID}-set for all $1 \leq i \leq n$.

Let $\mathcal{S}_{\gamma^{ID}} = \{S_1, S_2, \ldots, S_n\}$. We claim that $\mathcal{S}_{\gamma^{ID}}$ is the collection of all γ^{ID}-sets of G. Since we have already established that $\gamma^{ID}(G) = 2n+7$ and that in every γ^{ID}-set, $2n+6$ of the vertices are from X, every γ^{ID}-set of G can viewed as "X-plus-one". However, there is no single vertex $w \in V(G) - V(H)$ such that $X \cup \{w\}$ identifies both pairs $\{a, y_{a,1}\}$ and $\{b, y_{b,1}\}$. The set $S_a = \{a\} \cup X$ with $|S_a| = 2n+7$ has $I_{S_a}(b) = \{x_{b,1}\} = I_{S_a}(y_{b,1})$ and is therefore not identifying. Similarly, $S_b = \{b\} \cup X$ is not identifying. Thus, $\mathcal{S}_{\gamma^{ID}}$ is the collection of all γ^{ID}-sets.

Consider now $G(\gamma^{ID}) = (V(G(\gamma^{ID})), E(G(\gamma^{ID})))$. By the above arguments, $V(G(\gamma^{ID}))$ $= \{v_1', v_2', \ldots, v_n'\}$, where the vertex v_i' corresponds to the set $S_i \in \mathcal{S}_{\gamma^{ID}}$ for each $1 \leq i \leq n$. Since v_i' and v_j' in $V(G(\gamma^{ID}))$ are adjacent in $G(\gamma^{ID})$ if and only if there exist $w_i \in S_i$ and $w_j \in S_j$ with $w_iw_j \in E(G)$ such that $S_i = (S_j - \{w_j\}) \cup \{w_i\}$ and S_i and S_j differ at exactly one vertex (that is, v_i versus v_j), it follows that $v_i'v_j' \in E(\gamma^{ID})$ if and only if $v_iv_j \in E(G)$. Therefore, $G(\gamma^{ID}) \cong H$ as required. ■

In the construction of the graph G in the proof of Theorem 2.5, if instead of attaching only two copies of the graph $\mathcal{C}$ to each vertex of the graph H, we attached three or more copies, the same γ^{ID}-graph H is obtained. This leads immediately to the following corollary.

Corollary 2.6. *Every graph H is the γ^{ID}-graph of infinitely many graphs.*

2.2 The γ_L and γ_t^L-Graphs

Introduced by Slater in 1988 [33, 34], a *locating-dominating set* S of graph $G = (V, E)$ is a dominating set such that for each $v \in V - S$, the set $N[v] \cap S$ is unique. In contrast to identifying codes, locating-dominating sets do not require that the vertices of the dominating set have unique neighbourhood intersection with the dominating set itself. The minimum cardinality of a locating-dominating set, denoted by $\gamma_L(G)$, is the *locating-domination number* of a graph G. A γ_L*-set* of a graph is a minimum locating-dominating vertex subset. Since all identifying codes are also locating-dominating sets, it follows that $\gamma_L \leq \gamma^{ID}$ for all graphs. We reuse the notation of the intersection set of a vertex v and set S from ID-codes; however, for locating-dominating sets, the sets $I_S(v)$ need only be unique for $v \notin S$.

We define the γ_L*-graph of a graph G*, $G(\gamma_L) = (V(\gamma_L), E(\gamma_L))$, to the be graph where the vertex set $V(\gamma_L)$ is the collection of γ_L-sets of G. As with the γ-graph, $u, w \in V(\gamma_L)$ associated with γ_L-sets S_u and S_w are adjacent in $G(\gamma_L)$ if and only if there exist $v_u \in S_u$ and $v_w \in S_w$ with $v_uv_w \in E(G)$, such that $S_u = (S_w - \{v_w\}) \cup \{v_u\}$.

Given the similarities between ID-codes and locating-dominating sets, it is not surprising that a similar result to Theorem 2.5 exists for γ_L-graphs.

Theorem 2.7. *Every graph H is the γ_L-graph of infinitely many graphs.*

Since $\mathcal{C}$ does not have a unique γ_L-set (for example, $\{x_1, x_2, x_3\}$ and $\{x_1, x_2, y_3\}$ are γ_L-sets), we cannot use it in the construction to prove Theorem 2.7. Instead, we use the very similar *Bull graph*, $\mathcal{B}$, as pictured in Figure 2.5. Notice that $S = \{x_1, x_2\}$ is a γ_L-set in $\mathcal{B}$. Moreover, since $S_1 = \{x_1, y_2\}$ and $S_2 = \{y_1, x_2\}$ give $I_{S_1}(y_1) = \{x_1\} = I_{S_1}(x_3)$, and $I_{S_2}(y_2) = \{x_2\} = I_{S_2}(x_3)$, S is the only γ_L-set of $\mathcal{B}$. The proof to Theorem 2.7 then proceeds identically to that of Theorem 2.5, substituting the use of $\mathcal{B}$ for $\mathcal{C}$.

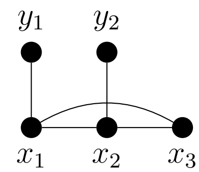


Figure 2.5: The Bull graph $\mathcal{B}$ used in the construction of Theorem 2.7.

A variant of locating-dominating sets, a vertex subset S is a *locating-total dominating set* (LTDS) of a graph G if S a locating-dominating set and if each vertex in $V(G)$ is adjacent to some vertex in S. The *locating-total domination number* $\gamma_t^L(G)$ is the minimum cardinality of a LTDS [20]. We define the γ_t^L-graph of a graph G analogously to the γ_L-graph. Since in the construction of Theorem 2.7, there were no independent vertices in the γ_L-sets, the following corollary is immediate.

Corollary 2.8. *Any graph H is the γ_t^L-graph of infinitely many graphs.*

2.3 The Γ-Graph

The next domination parameter we examine is $\Gamma(G)$, the *upper domination number* of a graph G, defined to be the cardinality of a largest minimal dominating set. The Γ-*graph of a graph G* and its associated parameters are defined analogously to $G(\gamma)$. Once again, this variation requires the use of a new gadget: the graph $\mathcal{Z}$ in Figure 2.6.

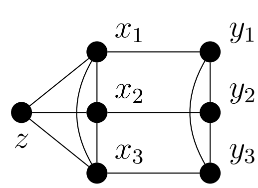


Figure 2.6: The graph $\mathcal{Z}$ used in Theorem 2.9.

Theorem 2.9. *Every graph H is the Γ-graph of infinitely many graphs.*

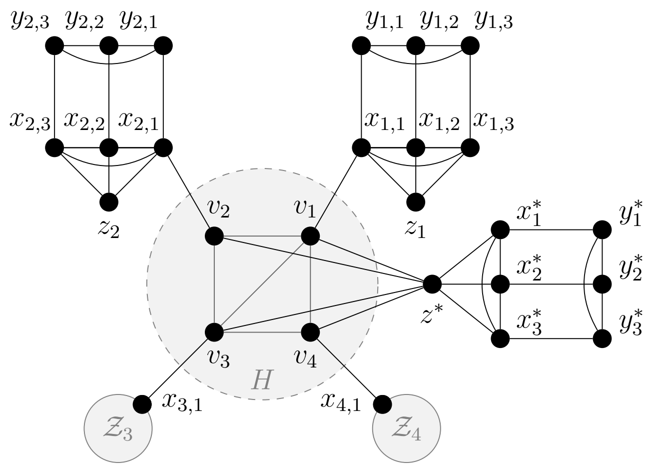


Figure 2.7: The graph G constructed from H in Theorem 2.9.

Proof. Construction: The construction of a graph G with $G(\Gamma) \cong H$ is similar to the previous results. Begin with a copy of the graph H with $V(H) = \{v_1, v_2, \ldots, v_n\}$, and to each v_i, attach a copy of the graph $\mathcal{Z}$ in Figure 2.6 labelled $\mathcal{Z}_i$ with $V(Z_i) = \{z_i, x_{i,1}, x_{i,2}, x_{i,3}, y_{i,1}, y_{i,2}, y_{i,3}\}$ at vertex $x_{i,1}$ to v_i. Attach a final copy of $\mathcal{Z}$ labelled $\mathcal{Z}^*$ $(V(\mathcal{Z}^*) = \{z^*, x_1^*, x_2^*, x_3^*, y_1^*, y_2^*, y_3^*\})$ by joining each v_i to z^*.

For reference, we define $X_i = \{x_{i,1}, x_{i,2}, x_{i,3}\}$, $Y_i = \{y_{i,1}, y_{i,2}, y_{i,3}\}$, $X^* = \{x_1^*, x_2^*, x_3^*\}$, and $Y^* = \{y_1^*, y_2^*, y_3^*\}$.

We claim that the Γ-sets of G are $S_1, S_2, \ldots, S_n$ where for each $1 \leq i \leq n$,

$$S_i = \{v_i\} \cup \left(\bigcup_{1 \leq j \leq n} X_j \right) \cup Y^*. \tag{2.1}$$

To begin, notice that S_i is minimal dominating with $|S_i| = 3(n+1) + 1 = 3n + 4$; for each $1 \leq j \leq 3$, $pn[x_{i,j}, S_i] = \{y_{i,j}\}$, $pn[y_j^*, S_i] = \{x_j^*\}$, and $pn[v_i, S_i] = \{z^*\}$. We proceed with a series of claims to demonstrate that the collection $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$ contains the only Γ-sets of G.

(i) *If S is a minimal dominating set, then for each $1 \leq i \leq n$ and $1 \leq j \leq 3$, $\{z_i, x_{i,j}\} \nsubseteq S$. Likewise, $\{z^*, x_j^*\} \nsubseteq S$*

Since $N[z_i] \subseteq N[x_{i,j}]$, either $X_i \cap S = \varnothing$, or one of the $x_{i,j}$ annihilates the private neighbourhood of z_i in S.

(ii) *If S is a minimal dominating set, and $x_{i,1} \notin S$, then $|S \cap V(\mathcal{Z}_i)| = 2$.*

If $z_i \in S$, then by (i), $X_i \cap S = \varnothing$. To dominate Y_i minimally, exactly one $y_{i,j} \in S$, and thus $|V(\mathcal{Z}_i) \cap S| = 2$. Suppose instead that $z_i \notin S$ and that z_i is externally dominated. Then $|X_i \cap S| \geq 1$, say without loss of generality, $x_{i,2} \in S$. To dominate $y_{i,1}$ minimally, some $y_{i,k}$ is also in S. Then $\mathcal{Z}_i$ is dominated by $\{x_{i,2}, y_{i,k}\}$ and again $|V(\mathcal{Z}_i) \cap S| = 2$.

(iii) *If S is a minimal dominating set, then for $1 \leq i \leq n$, $|S \cap V(\mathcal{Z}_i)| \leq 3$, and $|S \cap V(\mathcal{Z}^*)| \leq 3$.*

As in (ii), if $z_i \in S$ then $|S \cap V(\mathcal{Z}_i)| = 2$. For $1 \leq j \leq 3$, if $y_{i,j} \in S$, then to dominate z_i, either $z_i \in S$ (and $|S \cap V(\mathcal{Z}_i)| = 2$), or $x_{i,k} \in S$ for some $1 \leq k \leq 3$, in which case $\{x_{i,k}, y_{i,j}\}$ dominates $\mathcal{Z}_i$, and again, $|S \cap V(\mathcal{Z}_i)| = 2$. Otherwise, $z_i \notin S$, and $Y_i \cap S = \varnothing$, which implies $X_i \cap S \neq \varnothing$. Since only $x_{i,j}$ externally dominates $y_{i,j}$, it follows that $X_i \subseteq S$, and hence $|S \cap V(\mathcal{Z}_i)| = 3$.

(iv) *If S is a Γ-set of G, then $|V(H) \cap S| \leq 1$.*

Suppose to the contrary that $|V(H) \cap S| = m \geq 2$; say without loss of generality that $v_1, ..., v_m \in S$. For each $i = 1, ..., m$, $z^* \notin pn[v_i, S]$. Since z_i is dominated (by a vertex in $\{z_i, x_{i,1}, x_{i,2}, x_{i,3}\} \cap S$), $x_{i,1} \notin pn[v_i, S]$. Hence either $v_i \in pn[v_i, S]$ or $v_j \in pn[v_i, S]$ for some $j > m$. In the former case, $x_{i,1} \notin S$ and $|V(\mathcal{Z}_i) \cap S| = 2$, and in the latter case, $x_{j,1} \notin S$ and $|V(\mathcal{Z}_j) \cap S| = 2$. Thus, for each $i \in \{1, ..., m\}$ there exists a unique j such that $x_{j,1} \notin S$. By (ii), then, for each $i \in \{1, ..., m\}$ there exists a unique j such that $|S \cap V(\mathcal{Z}_j)| = 2$. Hence $|S \cap (V(H) \cup (\bigcup_{i=1}^{n} \mathcal{Z}_i))| \leq 3n$. By (iii), $|S \cap V(\mathcal{Z}^*)| \leq 3$. Hence $|S| \leq 3n + 3 < \Gamma(G)$, a contradiction.

From (i)-(iv), $\mathcal{S}$ consists of all the Γ-sets of G. The proof proceeds as in Theorem 2.5. To construct other graphs with a Γ-graph of H, attach additional copies of $\mathcal{Z}$ to any vertex of $V(H)$. ∎

2.4 Roman and Total Roman Domination

The final two variants of the γ-graph we examine are derived from the reconfiguration of Roman and Total Roman dominating functions. These are new results, not found in [30].

Roman domination was first presented by Ian Stewart in 1999 in the popular science magazine *Scientific American* [37]. Stewart asked how the 4th-century Emperor Constantine could distribute the fewest of his Roman legions about the empire so that each settlement either had a legion stationed within it or was adjacent to a city with two legions. Thus, in the event of an attack, a city was either already protected, or could request aid from a neighbouring settlement with a spare legion.

Unlike the previous domination variants we have examined, Roman domination is not a set-inclusion problem. Instead, a *Roman dominating function* of a graph G is a function $f : V(G) \to \{0, 1, 2\}$ such that each vertex v has either $f(v) \geq 1$ or there exists some $w \in N(v)$ such that $f(w) = 2$. The *cost* of a Roman dominating function f is $c(f) = \sum_{v \in V(G)} f(v)$, and the *Roman domination number* is the minimum cost of a Roman dominating function $\gamma_R = \min\{c(f) : f \text{ is a Roman dominating function}\}$. Going forward, we will assume all Roman dominating functions to be of minimum cost. See [9] for additional definitions and introductory results on Roman domination.

Since Roman domination is not a set-inclusion problem, we must alter our strategy for creating an analogous version of the γ-graph. Each vertex of the γ_R*-graph* corresponds to a (minimum cost) Roman dominating function f. Two vertices of the graph are adjacent if and only if they correspond to Roman dominating functions f and g such that for some pair of adjacent vertices $u, w \in V(G)$,

$$\begin{cases} f(v) = g(v), & \text{if } v \in V(G) - \{u, w\} \\ |f(v) - g(v)| = 1, & \text{if } v \in \{u, w\}. \end{cases}$$

The token-sliding model becomes particularly useful here. Since each vertex v has $f(v) \in \{0, 1, 2\}$, we now allow for the possibility of 0, 1, or 2 tokens (representing that number of legions) to occupy a single vertex. Each vertex of $G(\gamma_R)$ corresponds to a token configuration, where each vertex either has at least one token on it, or is adjacent to a vertex with two tokens on it. Two vertices of $G(\gamma_R)$ are adjacent if their corresponding token configurations can be transformed from one into the other by sliding a single token along an edge of G.

Before we consider the realizability of γ_R-graphs, we first note the following observation regarding minimum cost Roman dominating functions.

Observation 2.10. *Suppose that f is a Roman dominating function of a graph G of minimum cost and uw is an edge in G. If $f(u) = 2$, then $f(w) \neq 1$. Likewise, if $f(u) = 1$, then $f(w) \neq 2$.*

To see the observation, suppose otherwise. If $f(u) = 2$ and $f(w) = 1$, then each $v \in N[w] \backslash N[u]$, either has $f(v) = 1$ or is adjacent to some $z \in V(G)$ such that $f(z) = 2$. That is, no vertex in the neighbourhood of w is dominated by w alone. Thus, the function g where for $v \neq w$, $g(v) = f(v)$ and $g(w) = 0$, is also Roman dominating, and at lower cost, contradicting the minimality of f. The second statement follows similarly.

Returning to the γ_R-graph, we finally depart from the trend of the rest of Chapter 2, and encounter a parameter where not every graph is realizable as a γ_R-graph.

Proposition 2.11. *K_2 is not a γ_R-graph.*

Proof. Suppose to the contrary that there is some graph G such that $G(\gamma_R) \cong K_2$. Applying the token-sliding model, K_2 corresponds to two configurations of tokens on the vertices of G, say f and g, where a single token slides from a vertex $u \in V(G)$ (under configuration f), to the vertex $w \in V(G)$ (under configuration g). That is, $f(u) - g(u) = 1$ and $f(w) - g(w) = -1$.

If $f(u) = 1$, then by Observation 2.10, $f(w) \neq 2$; likewise, for each $v \in N(u)$, $f(v) \neq 2$. If $f(w) = 0$, then a token slide from u to w gives $g(u) = 0$, $g(w) = 1$, and for each $v \in N(u) - \{w\}$ $g(v) \neq 2$, thus leaving u not Roman-dominated by g. Hence, we conclude that if $f(u) = 1$, then $f(w) = 1$. After the token slide, this leaves $g(u) = 0$ and $g(w) = 2$.

Since we have supposed (for contradiction) that a token move between f and g does exist, then there is no $v \in N(w) - \{u\}$ such that v is Roman-dominated only by the two tokens at w under g; otherwise, f is not a Roman-dominating function. Therefore, instead of sliding the token at u to w in configuration f, nothing prevents the opposite: we slide the token from w to u, creating a third (Roman dominating) token configuration h, where $h(u) = 2$, $h(w) = 0$, and $h(v) = f(v)$ for all $v \in V(G) - \{u, w\}$. Thus, the γ_R-graph of G has at least three vertices, contradicting that $G(\gamma_R) \cong K_2$.

Similar arguments show that if $f(u) = 2$, then $f(w) = 0$, and so $g(u) = 1$, and $g(w) = 1$, and once again, there exists some third configuration h, where $h(u) = 0$, $h(w) = 2$ and $h(v) = f(v)$ for all $v \in V(G) - \{u, w\}$. Again, this is a contradiction. We conclude that there is no graph G such that $G(\gamma_R) \cong K_2$. ■

Although Proposition 2.11 shows that not every graph is a γ_R-graph, examining a more restrictive variation of Roman domination leads to a surprisingly different outcome.

A function $f : V(G) \to \{0, 1, 2\}$ is a *total Roman dominating function* of a graph G if it is a Roman dominating function and has the additional property that the subgraph of G induced by $V_f^+ := \{v \in V(G) : f(v) > 0\}$ has no isolated vertices. The *cost* of a total Roman dominating function f is similarly defined as $c(f) = \sum_{v \in V(G)} f(v)$, and the *total Roman domination number* is the minimum cost of a total Roman dominating function $\gamma_{tR} = \min\{c(f) : f \text{ is a total Roman dominating function}\}$. Going forward, we again assume all total Roman dominating functions to be of minimum cost.

Now that the token-bearing vertices induce a subgraph without isolated vertices, there is no result analogous to Observation 2.10 for total Roman dominating functions, and without that key observation, nothing prevents every graph from being a γ_{tR}-graph.

Theorem 2.12. *Every graph is the γ_{tR}-graph of some graph.*

Proof. Given a graph H, we construct a graph G such that $G(\gamma_{tR}) \cong H$.

Construction: The construction of a graph G with $G(\gamma_{tR}) \cong H$ is similar to those given previously in this chapter (and in [30]). Begin with a copy of the graph H with $V(H) = \{v_1, v_2, \ldots, v_n\}$. For each v_i, add two copies of the star $K_{1,3}$ with vertices labelled $\mathcal{Z}_i =$

$\{x_i, y_{i,1}, y_{i,2}, y_{i,3}\}$ and $\mathcal{Z}_i^* = \{x_i^*, y_{i,1}^*, y_{i,2}^*, y_{i,3}^*\}$, such that x_i and x_i^* are the centres of the stars with degree 3. Then, for each $i \in \{1, 2, \ldots, n\}$ connect each v_i to each x_i and x_i^* with an edge. Finally, add 2 more vertices labelled a and b, and connect every vertex in H to both. An example of this construction is given in Figure 2.8.

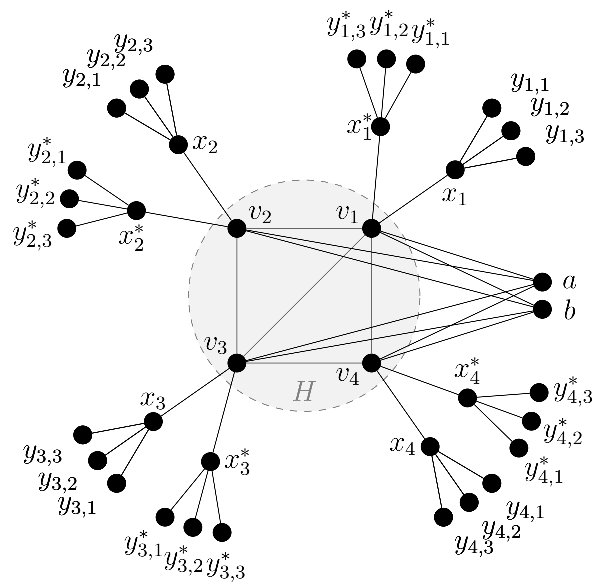


Figure 2.8: The graph G constructed from H in Theorem 2.12.

We define the family of functions $\mathcal{F} = \{f_1, f_2, \ldots, f_n\}$ for $1 \leq k \leq n$ by

$$f_k(v) = \begin{cases} 2 & \text{for } v \in \{x_i, x_i^*\}, i = 1, 2, \ldots, n, \\ 2 & \text{for } v = v_k, \\ 1 & \text{for } v = v_i, \text{ for } i \neq k, \\ 0 & \text{for } v \in \{a, b\}. \end{cases}$$

Then $c(f_k) = 5n + 1$, and f_k is total Roman dominating. We claim that the family $\mathcal{F}$ are all the least cost total Roman dominating functions of G.

Let g be a minimum total Roman dominating function, so that $c(g) \leq c(f_k)$. To begin, we determine $g(v_i)$. Suppose $g(v_i) = 0$ for some i. Then, to totally Roman dominate the star $\mathcal{Z}_i$, $\sum_{v \in \mathcal{Z}_i} g(v) \geq 3$, and likewise $\sum_{v \in \mathcal{Z}_i^*} g(v) \geq 3$. Define a function g^* by

$$g^*(v) = \begin{cases} f_k(v) & \text{for } v \in \mathcal{Z}_i \cup \mathcal{Z}_i^*, \\ 1 & \text{for } v = v_i, \\ g(v) & \text{otherwise.} \end{cases}$$

Then g^* is a total Roman dominating function of lower cost than g, contradicting that g is of minimum cost. Therefore, $g(v_i) \geq 1$ for each i.

Suppose $g(v_i) = 1$ for each v_i. Then, to totally Roman dominate a and b, $g(a), g(b) \geq 1$. Setting $g^*(a) = g^*(b) = 0$, and $g^*(v_i) = 2$ for any i, gives a total Roman dominating function of lower cost. Therefore, $g(v_i) = 2$ for at least one $i \in \{1, 2, \ldots, n\}$.

Next, we determine $g(x_i)$. Suppose now that $g(x_i) < 2$. Then, $\sum_{1 \leq j \leq 3} g(y_{i,j}) \geq 3$. Again, setting $g^*(x_i) = 2$ and $g^*(y_{i,j}) = 0$ for each $1 \leq j \leq 3$, gives a total Roman dominating function of lower cost. Hence $g(x_i) = 2$, and similarly, $g(x_i^*) = 2$ for each i.

Since $c(g) \leq c(f_k)$, it now follows easily that $g = f_k$ for some $k \in \{1, 2, \ldots, n\}$. Thus, $\gamma_{tR}(G) = 5n + 1$, and that $\mathcal{F}$ consists of all total Roman dominating functions of G. Moreover, the vertex corresponding to f_i in $\gamma_{tR}(G)$ is adjacent to the vertex corresponding to f_j if and only if v_i is adjacent to v_j in H. It follows that $G(\gamma_{tR}) \cong H$ as required. ■

In this chapter, we examined several variations on the γ-graph. In each case, with the exception of the γ_R-graph, we demonstrated that all graphs are π-graphs for their respective parameters π. Thus, the natural question to explore is what other parameters π are like γ_R, in that not every graph is a π-graph? For example, for the upper irredundance number IR, Mynhardt and Roux have shown that not all connected graphs are IR-graphs, but that all disconnected graphs are IR-graphs [28, 29]. For remainder of this dissertation we focus on a single such γ-graph variation, the i-graph, which like the γ_R-graph and the IR-graph, is not realizable for all graphs.

Chapter 3

Introduction to i-Graphs

Having examined in Chapter 2 several parameters π with the property that every graph is a π-graph of some graph, we turn our attention now to a much more challenging variant: the i-graph.

The *i-graph* of a graph G, denoted $\mathscr{I}(G) = (V(\mathscr{I}(G)), E(\mathscr{I}(G)))$, is the graph with vertices representing the minimum independent-dominating sets of G (that is, the *i-sets* of G). As with the graphs in Chapter 2, adjacency in $\mathscr{I}(G)$ follows a slide model where $u, v \in V(\mathscr{I}(G))$, corresponding to the i-sets S_u and S_v, respectively, of G, are adjacent in $\mathscr{I}(G)$ if and only if there exists $xy \in E(G)$ such that $S_u = (S_v - x) \cup \{y\}$. We say H is an i-graph (or is *i-graph realizable*) if there exists some graph G such that $\mathscr{I}(G) \cong H$. Moreover, we refer to G as the *seed graph* of the i-graph H. Going forward, we mildly abuse notation to denote both the i-set X of G and its corresponding vertex in H as X, so that $X \subseteq V(G)$ and $X \in V(H)$.

This version of a reconfiguration problem (and indeed, all of those discussed in Chapter 2) is also known as a *token-sliding model*, where each i-set of G can be imagined as a configuration of tokens lying on the vertices of a graph (see [17, 23, 27, 39]). Two configurations are adjacent in $\mathscr{I}(G)$ if and only if a single token can be slid along an edge of G to transform one configuration into another. Notice that the *token-jump model* of reconfiguration for independent domination is identical to the token-slide model. On a graph G a token may only "jump" from a vertex v in the i-set S_1 to another vertex w (to form the i-set S_2) if w is dominated only by v in S_1. Otherwise, if w is dominated by some other $u \neq v$

in S_1, then $(S_1 - u) \cup \{w\}$ is not an independent set as it contains the adjacent vertices u and w. A token is said to be *frozen* (in any reconfiguration model) if there are no available vertices to which it may be slid/jumped.

In acknowledgment of the slide-action in i-graphs, given i-sets $X = \{x_1, x_2, \ldots, x_k\}$ and $Y = \{y_1, x_2, \ldots x_k\}$ of G with $x_1y_1 \in E(G)$, we denote the adjacency of X and Y in $\mathscr{I}(G)$ as $X \overset{x_1y_1}{\sim} Y$, where we imagine transforming the i-set X into Y by sliding the token at x_1 along an edge to y_1. When discussing several graphs, we use the notation $X \overset{x_1y_1}{\sim}_G Y$ to specify that the relationship is on G. More generally, we use $x \sim y$ to denote the adjacency of vertices x and y (and $x \not\sim y$ to denote non-adjacency); this is used in the context of both the seed graph and the target graph.

Although the focus of this chapter is on the i-graph, for later comparison, we now also include the definition of the related *α-graph* of a graph G, denoted $\mathscr{A}(G)$. The α-graph is the slide reconfiguration graph with vertices representing the maximum independent sets of G (that is, the α-sets), and where adjacency is defined the same as in the i-graph. In Chapter 5, we present several constructions for i-graphs that are also constructions for α-graphs.

As the reader can likely surmise by noting that three chapters of this dissertation are dedicated to i-graphs, there is no such tidy theorem like those of Chapter 2 for i-graphs; as we see later in this chapter, not every graph is an i-graph and determining which classes are (or are not) i-graphs has proven to be an interesting challenge.

We begin our investigation into i-graph realizability in Section 3.1 by composing a series of observations and technical lemmas concerning the adjacency of vertices in an i-graph and the structure of their associated i-sets in the seed graph. In Section 3.2, we present the three smallest graphs which are not i-graphs, and then in Section 3.3, we show that several common graph classes, like trees and cycles, are i-graphs. Section 3.4 concludes this chapter by considering i-graphs from the opposing view: when given a graph G from a specific class, what is the resulting i-graph of G?

3.1 Observations

To begin, we propose several observations about the structure of i-sets within given i-graphs, which we then use to construct a series of useful lemmas.

Observation 3.1. *Let G be a graph and $H = \mathscr{I}(G)$. A vertex $X \in V(H)$ has $\deg_H(X) \geq 1$ if and only if for some $v \in X \subseteq V(G)$, there exists $u \in epn(v, X)$ such that u dominates $pn(v, X)$.*

From a token-sliding perspective, Observation 3.1 shows that a token on an i-set vertex v is frozen if and only if $epn(v) = \varnothing$ or $G[epn(v, X)]$ has no dominating vertex.

For some path $X_1, X_2, \ldots, X_k$ in H, at most one vertex of the i-set is changed at each step, and so X_1 and X_k differ on at most k vertices. This is yields the following immediate observation.

Observation 3.2. *Let G be a graph and $H = \mathscr{I}(G)$. Then for any i-sets X and Y of G, the distance $d_H(X, Y) \geq |X - Y|$.*

Lemma 3.3. *Let G be a graph with $H = \mathscr{I}(G)$. Suppose XY and YZ are edges in H with $X \overset{xy_1}{\sim} Y$ and $Y \overset{y_2z}{\sim} Z$, with $X \neq Z$. Then XZ is an edge of H if and only if $y_1 = y_2$.*

Proof. Let $X = \{x, v_2, v_3 \ldots, v_k\}$ and $Y = \{y_1, v_2, , v_3 \ldots, v_k\}$ so that $X \overset{xy_1}{\sim} Y$. To begin, suppose $y_1 = y_2$. Then $Y \overset{y_1z}{\sim} Z$ and $Z = \{z, v_2, , v_3 \ldots, v_k\}$, and $|X - Z| = 1$. Since X is dominating, z is adjacent to a vertex in $\{x, v_2, v_3 \ldots, v_k\}$; moreover, since Z is independent, z is not adjacent to any of $\{v_2, v_3, \ldots v_k\}$. Thus z is adjacent to x in G and $X \overset{xz}{\sim} Z$, so that $XZ \in E(H)$.

Conversely, suppose $y_1 \neq y_2$. Then, without loss of generality, say $y_2 = v_2$ and so $X = \{x, y_2, v_3 \ldots, v_k\}$, $Y = \{y_1, y_2, v_3 \ldots, v_k\}$, and $Z = \{y_1, z, v_3 \ldots, v_k\}$. Notice that $x \neq z$ since $x \sim y_1$ and $z \not\sim y_1$. Thus $|X - Z| = 2$, and it follows that $XZ \notin E(H)$. ■

Combining the results from Lemma 3.3 with Observation 3.2, yields the following observation for vertices of i-graphs at distance two.

Observation 3.4. *Let G be a graph and $H = \mathscr{I}(G)$. Then for any i-sets X and Y of G, if $d_H(X, Y) = 2$, then $|X - Y| = 2$.*

Lemma 3.5. *Let G be a graph and $H = \mathscr{I}(G)$. Suppose H contains an induced $K_{1,m}$ with vertex set $\{X, Y_1, Y_2, \ldots, Y_m\}$ and $\deg_H(X) = m$. Let $i \neq j$. Then in G:*

(i) $(X - Y_i) \neq (X - Y_j)$,

(ii) $|Y_i \cap Y_j| = i(G) - 2$, *and*

(iii) $m \leq i(G)$.

Proof. Suppose $X \overset{x_i y_i}{\sim} Y_i$ and $X \overset{x_j y_j}{\sim} Y_j$. Then, $(X - Y_i) = \{x_i\}$ and $(X - Y_j) = \{x_j\}$. From Lemma 3.3, since $Y_i \not\sim Y_j$, we have that $x_i \neq x_j$, which establishes Statement (i). Moreover, $Y_i \cap Y_j = X - \{x_i, x_j\}$, and so as these are i-sets, Statement (ii) also follows. Finally, for Statement (iii), again apply Lemma 3.3, $|\bigcap_{1 \leq i \leq m} Y_i| = |X| - m = i(G) - m \geq 0$. ■

Proposition 3.6. *Let G be a graph and $H = \mathscr{I}(G)$. Suppose H has an induced C_4 subgraph with vertices X, A, B, Y, where $XY, AB \notin E(H)$. Then, without loss of generality, the set composition of X, A, B, Y in G, and the edge labelling of the induced C_4 in H, are as in Figure 3.1.*

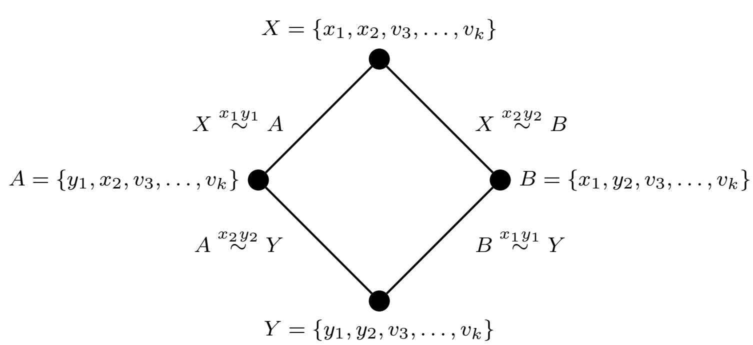


Figure 3.1: Reconfiguration structure of an induced C_4 subgraph from Proposition 3.6.

Proof. Suppose that the i-set X of G has $X = \{x_1, x_2, v_3, \ldots, v_k\}$. Then by Lemma 3.3, without loss of generality, the edge from X to A can be labelled as $X \overset{x_1 y_1}{\sim} A$ for some

$y_1 \in V(G) - X$, so that $A = \{y_1, x_2, v_3, \ldots, v_k\}$, while the edge from X to B can be labelled $X \overset{x_2y_2}{\sim} B$ for some y_2 and $B = \{x_1, y_2, v_3, \ldots, v_k\}$.

Consider now the edge $AY \in E(H)$ labelled $A \overset{ay^*}{\sim} Y$. From Lemma 3.3, since $XY \notin E(G)$, $a \neq y_1$. If, say, $a = v_3$, then $Y = \{y_1, x_2, y^*, \ldots, v_k\}$. However, neither y_1 nor x_2 is in B, so $|Y - B| \geq 2$, a contradiction of Observation 3.2. Thus, $a \neq v_i$ for any $3 \leq i \leq k$. This leaves $a = x_2$, and $Y = \{y_1, y^*, v_3 \ldots, v_k\}$. Since $|Y - B| = 1$, $y^* = y_2$ and $Y = \{y_1, y_2, v_3, \ldots, v_k\}$ as required. ∎

Finally, given an i-graph G with i-sets X and Y, and vertices $x_1 \in X$ and $y_1 \in Y$, we define a relation $\mathcal{R}_i$ on the ordered pairs (X, x_1) and (Y, y_1) so that $(X, x_1)\mathcal{R}_i(Y, y_1)$ if and only if $(X, x_1) = (Y, y_1)$, $X \overset{x_1y_1}{\sim} Y$. First notice that $\mathcal{R}_i$ is symmetric: if $X \overset{x_1y_1}{\sim} Y$, the slide of the token previously at x_1 to y_1 can be reversed, so that $Y \overset{y_1x_1}{\sim} X$. Moreover, Lemma 3.3 confirms that $\mathcal{R}_i$ is also transitive. Since reflexivity is embedded in the definition, the following observation is immediate.

Observation 3.7. *The relation $\mathcal{R}_i$ is an equivalence relation. The equivalence classes correspond to the components of the resulting i-graph.*

3.2 Realizability of i-Graphs

Having now established a series of observations and lemmas about the structures of i-graphs and the composition of their associate i-sets, we demonstrate that not all graphs are i-graphs by presenting three counterexamples: the diamond graph $\mathfrak{D}$, $K_{2,3}$ and κ, as pictured in Figure 3.2.

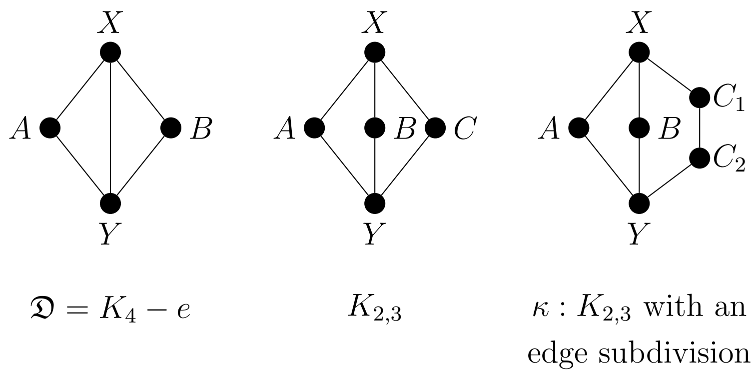


Figure 3.2: Three graphs not realizable as i-graphs.

Proposition 3.8. *The diamond graph $\mathfrak{D} = K_4 - e$ is not i-graph realizable.*

Proof. Suppose to the contrary there is some graph G with $\mathscr{I}(G) = \mathfrak{D}$. Let $V(\mathfrak{D}) = \{X, A, B, Y\}$ where $AB \notin E(\mathfrak{D})$. Say $X \overset{xy}{\sim} Y$. Then by Lemma 3.3, without loss of generality, the incident edges with A can be labelled as $X \overset{xa}{\sim} A$ and $A \overset{ay}{\sim} Y$. Likewise, $X \overset{xb}{\sim} B$ and $B \overset{by}{\sim} Y$ (see Figure 3.2). However, since $B \overset{bx}{\sim} X$ and $X \overset{xa}{\sim} A$, Lemma 3.3 implies that $AB \in E(\mathfrak{D})$, a contradiction. ■

Proposition 3.9. *The graph $K_{2,3}$ is not i-graph realizable.*

Proof. Suppose $K_{2,3} = \mathscr{I}(G)$ for some graph G. Let $\{\{X, Y\}, \{A, B, C\}\}$ be the bipartition of $K_{2,3}$. To begin, apply the exact labelling from Proposition 3.6 and Figure 3.1 to the i-sets and edges of X, A, B, and Y. We attempt to extend the labelling to C. By Lemma 3.3, since C is adjacent to X, but not A nor B, without loss of generality, $X \overset{v_3c}{\sim} C$ and $C = \{x_1, x_2, c, v_4, \dots, v_k\}$. As A is an i-set, $y_1v_3 \notin E(G)$. Since $v_3c \in E(G)$, $c \neq y_1$. Similarly, $c \neq y_2$. Now $|C - Y| = 3$ and $d(C, Y) = 1$, contradicting Observation 3.2. ■

Proposition 3.10. *The graph κ is not i-graph realizable.*

Proof. Suppose $\kappa = \mathscr{I}(G)$ for some graph G and let $V(\kappa) = \{X, A, B, C_1, C_2, Y\}$ as in Figure 3.2, and to the subgraph induced by X, A, B, Y, apply the labelling of Proposition 3.6 and Figure 3.1. Through additional applications of Proposition 3.6, we can as in the proof of Proposition 3.9, without loss of generality, assume $X \overset{x_3y_3}{\sim} C_1$. However, $d(C_1, Y) = 2$ but $|Y - C_1| = 3$, contradicting Observation 3.2. It follows that no such G exists and κ is not an i-graph. ■

The observant reader will have undoubtedly noticed the common structure between the graphs in the previous three propositions; they are all members of the class of *theta graphs* (see [5]), graphs that are the union of three internally disjoint paths with the same two distinct end vertices (not to be confused with the Theta graph, or Θ-graph, of computation geometry introduced by Clarkson [7]). The graph $\Theta\langle j, k, \ell\rangle$ with $j \leq k \leq \ell$, is the theta graph with paths of lengths j, k, and ℓ. In this notation, our three non i-graph realizable examples are $\mathfrak{D} \cong \Theta\langle 1, 2, 2\rangle$, $K_{2,3} \cong \Theta\langle 2, 2, 2\rangle$, and $\kappa \cong \Theta\langle 2, 2, 3\rangle$.

Further rumination on the similarity in structure suggests that additional subdivisions of the central path in κ could yield more graphs that are not i-graphs. However, the proof technique used for κ no longer applies when the central path between the starting and ending vertices is longer than four vertices. In Chapter 5, we explore an alternative method for determining the i-realizability of theta graphs.

3.3 Some Classes of i-Graphs

Having studied several graphs that are not i-graphs, we now examine the problem of i-graph realizability from the positive direction. To begin, it is easy to see that complete graphs are i-graphs; moreover, as with γ-graphs, complete graphs are their own i-graphs with $\mathscr{I}(K_n) \cong K_n$.

Proposition 3.11. *Complete graphs are i-graph realizable.*

Hypercubes Q_n (the Cartesian product of K_2 taken with itself n times) are also straightforward to construct as i-graphs, with $\mathscr{I}(nK_2) \cong Q_n$. Each K_2 pair can be viewed as a $0-1$ switch, with the vertex of the i-set in each component sliding between the two states.

Proposition 3.12. *Hypercubes are i-graph realizable.*

Hypercubes are a special case of the following result regarding Cartesian products of i-graphs.

Proposition 3.13. *If $\mathscr{I}(G_1) \cong H_1$ and $\mathscr{I}(G_2) \cong H_2$, then $\mathscr{I}(G_1 \cup G_2) \cong H_1 \,\square\, H_2$*

Proof. Let $\{X_1, X_2, \ldots, X_k\}$ be the i-sets of G_1 and let $\{Y_1, Y_2, \ldots, Y_\ell\}$ be the i-sets of G_2. Then, the i-sets of $G_1 \cup G_2$ are of the form $X_i \cup Y_j$. Clearly $X_i \cup Y_j \sim_{G_1 \cup G_2} X^*_{i^*} \cup Y^*_{j^*}$ if and only if $X_i \sim_G X^*_i$ and $Y_j = Y^*_j$, or $Y_j \sim_{G_2} Y^*_j$ and $X_i = X^*_i$. This gives a natural isomorphism to $H_1 \,\square\, H_2$, where $X_i \cup Y_j$ is the vertex (X_i, Y_j). ■

Moving to cycles, the constructions become markedly more difficult.

Proposition 3.14. *Cycles are i-graph realizable.*

Proof. The constructions for each cycle C_k for $k \geq 3$ are as described below.

(i) $\mathscr{I}(C_3) \cong C_3$

From Proposition 3.11.

(ii) $\mathscr{I}(2K_2) \cong C_4$

From Proposition 3.12.

(iii) $\mathscr{I}(C_5) \cong C_5$

Recall that $i(C_5) = 2$. A labelled C_5 and the resulting i-graph with $\mathscr{I}(C_5) \cong C_5$ are given in Figure 3.3 below.

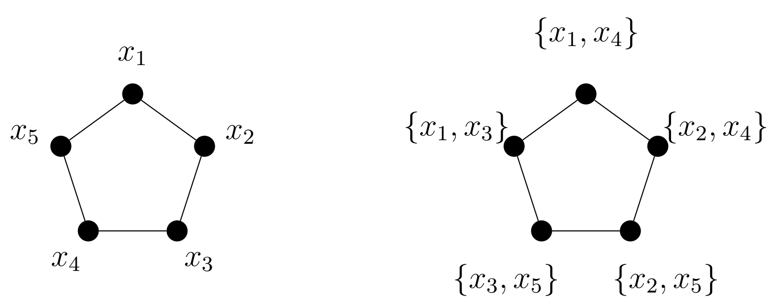


Figure 3.3: C_5 and $\mathscr{I}(C_5) \cong C_5$.

(iv) $\mathscr{I}(K_2 \,\square\, K_3) \cong C_6$

Let $V(K_2 \,\square\, K_3) = \{x_1, x_2, x_3, y_1, y_2, y_3\}$, where the x_i's and y_i's form the K_3's and $x_iy_i \in E(K_2 \,\square\, K_3)$ for each $1 \leq i \leq 3$. The set $\{x_i, y_j\}$ is an i-set of $K_2 \,\square\, K_3$ if and only if $i \neq j$, so that $|V(\mathscr{I}(K_2 \square K_3))| = 6$, and adjacencies are as in Figure 3.4 below.

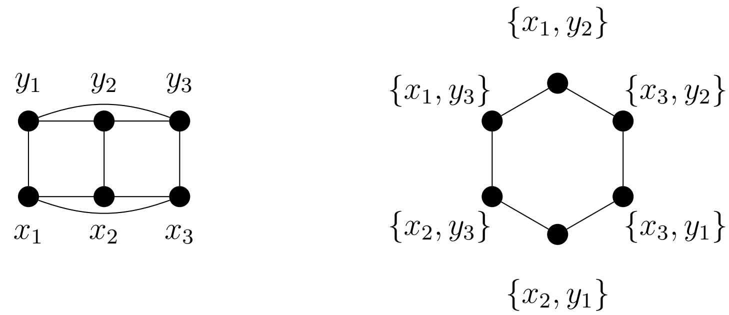


Figure 3.4: $K_2 \,\square\, K_3$ and $\mathscr{I}(K_2 \,\square\, K_3) \cong C_6$.

(v) *For any $k \geq 7$, construct the graph H with $V(H) = \{v_0, v_1, \ldots, v_{k-1}\}$, and $v_iv_j \in E(H)$ if and only if $j \not\equiv i-2, i-1, i, i+1, i+2$ (mod k). Then $\mathscr{I}(H) \cong C_k$.*

For convenience, we assume that all subscripts are given modulo k. Thus in H, for all $0 \leq i \leq k-1$, we have the following:

(I) $N[v_i] \backslash N[v_{i+1}] = \{v_i, v_{i+3}\}$

(II) $N[v_{i+1}] \backslash N[v_i] = \{v_{i-2}, v_{i+1}\}$.

Since H is vertex-transitive, suppose that v_i is in some i-set S. Then $v_{i-2}, v_{i-1}, v_{i+1}$, and v_{i+2} are not dominated by v_i. To dominate v_{i+1}, either v_{i+1} or v_{i-2} is in S, because all other vertices in $N[v_{i+1}]$ are also adjacent to v_i, as in (II). Begin by assuming that $v_{i+1} \in S$. Now since $\{v_i, v_{i+1}\}$ dominates all of H except v_{i+2} and v_{i-1}, and $N(v_{i+2}, v_{i-1}) \subseteq N(v_i, v_{i+1})$, either v_{i+2} or v_{i-1} is in S. Thus $S = \{v_i, v_{i+1}, v_{i+2}\}$ or $S = \{v_{i-1}, v_i, v_{i+1}\}$.

Suppose now instead that $v_{i-2} \in S$. Now, only v_{i-1} is not dominated by $\{v_i, v_{i-2}\}$; moreover, since $N(v_{i-1}) \subseteq N(\{v_{i-2}, v_i\})$, we have that $v_{i-1} \in S$, and so $S = \{v_{i-2}, v_{i-1}, v_i\}$. Combining the above two cases yields that $i(H) = 3$ and that all

i-sets of H have the form $S_i = \{v_i, v_{i+1}, v_{i+2}\}$, for each $0 \leq i \leq k-1$. Moreover, as there are k unique such sets, it follows that $|V(\mathscr{I}(H))| = k$.

We now consider the adjacencies of $\mathscr{I}(H)$. From our set definitions, $S_i \overset{v_{i-1}v_{i+2}}{\sim} S_{i+1}$, and $S_i \overset{v_{i+1}v_{i-2}}{\sim} S_{i-1}$. To see that S_i is not adjacent to any other i-set in H, notice that the token at v_i is frozen; $N(v_i)) \subseteq (N(v_{i-1}) \cup N(v_{i+1}))$. Moreover, by (II), the token at v_{i+1} can only slide to v_{i-2}, and likewise, the token at v_{i-1} can only slide to v_{i+2}. Thus $S_i \sim S_{i+1} \sim \cdots \sim S_{i-1} \sim S_i$, and so $\mathscr{I}(H) \cong C_k$ as required.

This completes the i-graph constructions for all cycles. ■

The constructions presented in Proposition 3.14 are, of course, not unique. Later in this chapter, we show that for $k \geq 5$ and $k \equiv 2 \pmod 3$, $\mathscr{I}(C_k) \cong C_k$. Moreover, in Chapter 5, we demonstrate how graph complements can be used to construct graphs with i-graphs that are cycles.

We now present three lemmas with the eventual goal of demonstrating that all forests are i-graphs. When considering the i-graph of some graph H, if a vertex v of some i-set S of H has no external private neighbours, then the token at v is frozen. In the first of the three lemmas, Lemma 3.15, we construct a new seed graph for a given target graph, where each vertex of the seed graph's i-set has a non-empty private neighbourhood. Ironically, although the addition of external private neighbours initially unfreezes tokens in the seed graph, we later use this lemma in a construction to force tokens into a certain configuration.

Lemma 3.15. *For any graph H, there exists a graph G ($G \ncong H$) such that $\mathscr{I}(G) \cong \mathscr{I}(H)$ and for any i-set S of G, all $v \in S$ have $epn(v, S) \neq \varnothing$.*

Proof. Suppose S is such an i-set of H having some $v \in S$ with $epn(v, S) = \varnothing$. Construct the graph G_1 from H by joining new vertices a and b to each vertex of $N[v]$.

To begin, we show that the i-sets of G_1 are exactly the i-sets of H. Let R be some i-set of H and say that v is dominated by $u \in R$. Then $u \in N[v]$, so u also dominates a and b in G_1; therefore, R is independent and dominating in G_1, and so $i(G_1) \leq i(H)$. Conversely, suppose that Q is an i-set of G_1. If neither a nor b is in Q, then Q is an independent dominating set of H, and so $i(H) \leq i(G_1)$. Hence, suppose instead that $a \in Q$. Notice that since Q is independent and a is adjacent to each vertex in $N(v)$, $N_H[v] \cap Q = \varnothing$. Some vertex in Q dominates b; however, since $N_H(a) = N_H(b)$ and Q is independent, it follows that b is self-dominating and so $b \in Q$. However, since $N_{G_1}[\{a,b\}] = N_{G_1}[v]$, the set $Q' = (Q - \{a,b\}) \cup \{v\}$ is an independent dominating set of G_1 such that $|Q'| < |Q|$, a contradiction. Thus, $i(G_1) = i(H)$ and the i-sets of H and G_1 are identical. In particular, S is an i-set of G_1, and moreover, $epn_{G_1}(v, S) = \{a, b\}$.

By repeating the above process for each i-set of G_i, $i \geq 1$, containing a vertex v with $epn(v, S) = \varnothing$, we eventually obtain a graph $G = G_k$ such that for each i-set of S of G and each vertex $v \in S$, $epn_G(v, S) \neq \varnothing$. Since the i-sets of H and G are identical and H is a subgraph of G, $\mathscr{I}(G) = \mathscr{I}(H)$ as required. ■

Next on our way to constructing forests, we demonstrate that given an i-graph, the resultant graph obtained by adding any number of isolated vertices is also an i-graph.

Lemma 3.16. *If H is the i-graph of some graph G, then there exists some graph G^* such that $\mathscr{I}(G^*) = H \cup \{v\}$.*

Proof. First, assume that $i(G) \geq 2$. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let W be an independent set of size $i(G) = k$ disjoint from $V(G)$, say $W = \{w_1, w_2, \ldots, w_k\}$. Construct a new graph G^* by taking the join of G with the vertices of W, so that $G^* = G \vee W$.

Notice that W is independent and dominating in G^*. Moreover, if an i-set S of G^* contains any vertex w_i of W, since W is independent and each vertex of W is adjacent to all of $\{v_1, v_2, \ldots, v_n\}$, it follows that S contains all of W, and so, $S = W$. That is, if an i-set of G^* contains any vertex of W, it contains all of W. Thus, $i(G) = i(G^*)$. Furthermore, any i-set of G is also an i-set of G^*, and so the i-sets of G^* comprise of W and the i-sets of G. That is, $V(\mathscr{I}(G^*)) = V(\mathscr{I}(G)) \cup \{W\} = V(H) \cup \{W\}$.

If S is an i-set of G, $S \cap W = \varnothing$. Thus, W is not adjacent to any other i-set $\mathscr{I}(G^*)$. Relabelling the vertex representing the i-set W in G^* as v in $\mathscr{I}(G^*)$ yields $\mathscr{I}(G^*) = H \cup \{v\}$ as required.

If $i(G) = 1$, then G has a dominating vertex; begin with $G \cup K_1$, which has $\mathscr{I}(G) = \mathscr{I}(G \cup K_1)$ and $i(G \cup K_1) = 2$, and then proceed as above. ■

As a final lemma before demonstrating the i-graph realizability of forests, we show that a pendant vertex can be added to any i-graph to create a new i-graph. The proof proceeds similarly to the construction for trees as γ-graphs by Fricke et al. in [13].

Lemma 3.17. *If H is the i-graph of some graph G, and H_u is the graph H with some pendant vertex u added, then there exists some graph G_u such that $\mathscr{I}(G_u) = H_u$.*

Proof. By Lemma 3.15 we may assume that for any i-set S of G, $epn(v, S) \neq \varnothing$ for all $v \in S$.

To construct G_u, begin with a copy of G. If w is the stem of u in H_u, then consider the i-set $W = \{v_1, v_2, \ldots, v_k\}$ in G corresponding to w. To each $v_i \in W$, attach a new vertex x_i for all $1 \leq i \leq k$. Then join each x_i to a new vertex y, and then to y, add a final pendant vertex z. Thus $V(G_u) = V(G) \cup \{x_1, x_2, \ldots, x_k, y, z\}$ as in Figure 3.5.

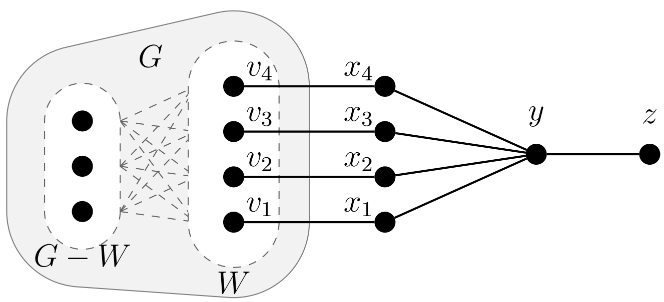


Figure 3.5: The construction of G_u from G in Lemma 3.17.

It is easy to see that if S is an i-set of G, then $S_y = S \cup \{y\}$ is an independent dominating set of G_u. The set $W_z = W \cup \{z\}$ is also an independent dominating set of G_u. Thus, $i(G_u) \leq i(G) + 1$. It remains only to show that these are i-sets and the only i-sets of G_u.

We claim that no x_i in $X = \{x_1, x_2, \ldots, x_k\}$ is in any i-set of G_u. To show this, suppose to the contrary that S^* is an i-set with $S^* \cap X = \{x_1, x_2, \ldots, x_\ell\}$ for some $1 \leq \ell \leq k$. Then, $y \notin S^*$; that is, $\{y, z\} \cap S^* = \{z\}$. To dominate the remaining $\{x_{\ell+1}, x_{\ell+2}, \ldots, x_k\}$, we have that $S^* = \{x_1, x_2, \ldots, x_\ell\} \cup \{v_{\ell+1}, v_{\ell+2}, \ldots, v_k\} \cup \{z\}$. Recall from our initial assumption on G that there exists some $v_1^* \in epn_G(v_1, W)$. Thus, $v_1^* \notin (N_{H_u}[\{v_{\ell+1}, v_{\ell+2}, \ldots, v_k\}] \cap V(G))$, and so v_1^* is undominated by S^*, which implies that S^* is not an i-set.

Thus in every i-set of G_u, y is dominated either by itself or by z. If y is not in a given i-set S (and so $z \in S$), then to dominate X, $W \subseteq S$, and so $S = W \cup \{z\}$. Conversely, if $y \in S$ (and $z \notin S$), then since the vertices of G can only be dominated internally, S is an i-set of G_u if and only if $S - \{y\}$ is an i-set of G, which completes the proof of our claim.

If u^* and w^* are the vertices in $\mathscr{I}(G_u)$ associated with W_z and $W_y = W \cup \{y\}$ respectively, then clearly $\mathscr{I}(G_u) - \{u^*\} \cong \mathscr{I}(G)$. Furthermore, since W_y is the only i-set with $|W_y - W_z| = 1$ and $yz \in E(G_u)$, it follows that $\deg(u^*) = 1$ and $u^*w^* \in E(\mathscr{I}(G_u))$, and we conclude that $\mathscr{I}(G_u) \cong H_u$. ■

Finally, we amalgamate the previous lemmas on adding isolated and pendant vertices to i-graphs to demonstrate that forests are i-graphs.

Theorem 3.18. *All forests are i-graph realizable.*

Proof. We show by induction on the number of vertices that if F is a forest with m components, then F is i-graph realizable. For a base, note that $\mathscr{I}(\overline{K_2}) = K_1$. Construct the graph $\overline{K_m}$ by repeatedly applying Lemma 3.16. Suppose that all forests on m components on at most n vertices are i-graph realizable. Let F be some forest with $|V(F)| = n+1$ and components $T_1, T_2, \ldots, T_m$. If all vertices of F are isolated, we are done, so assume there is some leaf v with stem w in component T_1. Let $F^* = F - \{v\}$. By induction there exists some graph G^* with $\mathscr{I}(G^*) \cong F^*$. Applying Lemma 3.17 to G^* at w constructs a graph G with $\mathscr{I}(G) \cong F$. ■

Moreover, by adding Proposition 3.14 to the previous results, we obtain the following immediate corollary.

Corollary 3.19. *Unicyclic graphs are i-graph realizable.*

With the completion of the constructions of forests and unicyclic graphs as i-graphs, we have now determined the i-graph realizability of many collections of small graphs. In particular, we draw the reader's attention to the following observation.

Observation 3.20. *Every graph on at most four vertices except $\mathfrak{D}$ is an i-graph.*

3.4 Building i-Graphs

In this section, we examine how new i-graphs can be constructed from known ones. To begin, we present three very useful tools for constructing new i-graphs: the **Max Clique Replacement Lemma**, the **Deletion Lemma**, and the **Inflation Lemma**. The first among these shows that maximal cliques in i-graphs can be replaced by arbitrarily larger maximal cliques.

Lemma 3.21. ***(Max Clique Replacement Lemma)*** *Let H be an i-graph with a maximal m-vertex clique, $\mathcal{K}_m$. Then, the graph H_w formed by adding a new vertex w^* adjacent to all of $\mathcal{K}_m$ is also an i-graph.*

Proof. Suppose G is a graph such that $\mathscr{I}(G) = H$ and $i(G) = k + 1$ where $k \geq 1$, and let $\mathcal{K}_m = \{V_1, V_2, \ldots, V_m\}$ be a maximal clique in H. From Lemma 3.3, the corresponding i-sets $V_1, V_2, \ldots, V_m$ of G differ on exactly one vertex, so for each $1 \leq i \leq m$, let $V_i = \{v_i, z_1, z_2, \ldots, z_k\} \subseteq V(G)$, so that $Z = \{z_1, z_2, \ldots, z_k\} = \bigcap_{1 \leq i \leq m} V_i$. Notice also from Lemma 3.3, for each $1 \leq i < j \leq m$, $v_i v_j \in E(G)$, and so $Q_m = \{v_1, v_2, \ldots, v_m\}$ is a (not necessarily maximal) clique of size m in G.

In addition to Q_m and Z defined above, we further weakly partition (i.e. some of the sets of the partition may be empty) the vertices of G as:

$X = N(Q_m) \backslash N(Z)$, the vertices dominated by Q_m but not Z.

$Y = N(Q_m) \cap N(Z)$, the vertices dominated by both Q_m and Z.

$A = N(Z) \backslash N(Q_m)$, the vertices dominated by Z but not Q_m.

This partition (as well as the later defined construction of G_w) is illustrated in Figure 3.6. Before proceeding with the construction, we state the following series of claims regarding the set X:

Claim 1: *Each $x \in X$ is dominated by every vertex of Q_m.*
Otherwise, if some $x \in X$ is not adjacent to some $v_j \in Q_m$, then x is undominated in the the i-set $V_j = \{v_j\} \cup Z$.

Claim 2: $|X| \neq 1$.

If $X = 1$, say $X = \{x\}$, then $X^* = \{x\} \cup Z$ is independent, dominating, and has $|X^*| = i(G)$; that is, X^* is an i-set of G. However, since x is adjacent to all of Q_m in G, $X^* \overset{xv_j}{\sim} V_j$ for each $1 \leq j \leq m$, contradicting the maximality of the clique $\mathcal{K}_m$ in H.

Claim 3: *No $x \in X$ dominates all of X.*

If $x \in X$ dominates X, then $\{x\} \cup Z$ is an i-set of G. Following a similar argument of Claim 2, this contradicts the maximality of $\mathcal{K}_m$ in H.

Claim 4: *For any $v \in (X \cup Y \cup A)$, $\{v\} \cup Z$ is not an i-set.*

Combining Claims 2 and 3, if $v \in X$, then there exists some $x_i \in X$ such that $v \not\sim x_i$, and thus $\{v\} \cup Z$ does not dominate x_i. If $v \in (Y \cup A)$, then $v \in N(Z)$, and so $\{v\} \cup Z$ is not independent.

We construct a new graph G_w from G by joining a new vertex w to each vertex in $V(G) - Z$, as in Figure 3.6. We claim that $\mathscr{I}(G_w) \cong H_w$.

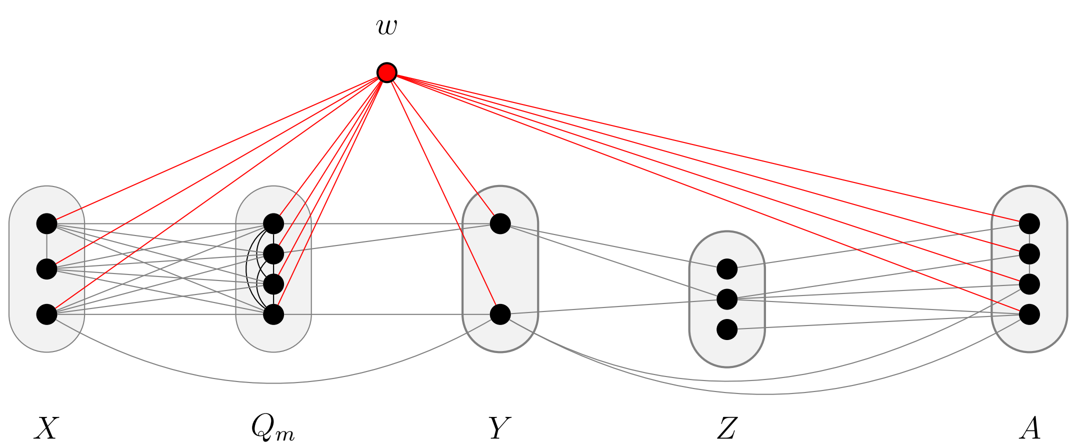


Figure 3.6: Construction of G_w from G in Lemma 3.21.

Let S be some i-set of G_w. If $w \notin S$, then $S \subseteq V(G)$ and so S is also independent dominating in G, implying $|S| = i(G) = k+1$. However, if $w \in S$, then since w is adjacent to all of $V(G) - Z$ and Z is independent, we have that $S = \{w\} \cup Z$. It follows that

$i(G_w) = i(G)$. Moreover, any i-set of G is also an i-set of G_w, and so $W := \{w\} \cup Z$ is the only new i-set generated in G_w. Thus $V(\mathscr{I}(G_w)) = V(\mathscr{I}(G)) \cup \{W\}$.

Consider now the edges of $\mathscr{I}(G_w)$. Since w is adjacent to all of Q_m in G_w, $W \overset{wv_j}{\sim} V_j$ for each $1 \leq j \leq m$, and thus $\mathcal{K}_m \cup W$ is a clique in $\mathscr{I}(G_w)$.

Finally, we demonstrate that W is adjacent only to the i-sets of $\mathcal{K}_m$. Consider some i-set $S \notin \mathcal{K}_m$, and suppose to the contrary that $W \sim S$. As W is the only i-set containing w, we have that $w \notin S$, and hence $W \overset{wu}{\sim} S$ for some vertex u. Since $w \sim u$, $u \notin Z$. Moreover, since W and S differ at exactly one vertex and $Z \subseteq W$, it follows that $Z \subseteq S$; that is, $S = \{u\} \cup Z$. If $u \in Q_m$ then $S \in \mathcal{K}_m$, a contradiction. If $u \in (X \cup Y \cup A)$, then by Claim 4, S is not an i-set, which is again a contraction. We conclude that $W \not\sim S$ for any i-set $S \notin \mathcal{K}_m$, and therefore $E(\mathscr{I}(G_w)) = E(\mathscr{I}(G)) \cup \left(\bigcup_{v_i \in Q_m} wv_i\right)$. If follows that $\mathscr{I}(G_w) \cong H_w$. ■

Our next result, the Deletion Lemma, shows that the class of i-graphs is closed under vertex deletion. It is unique among our other constructions; unlike most of our results which demonstrate how to build larger i-graphs from smaller ones, the Deletion Lemma instead shows that every induced subgraph of an i-graph is also an i-graph.

Lemma 3.22. ***(The Deletion Lemma)*** *If H is a nontrivial i-graph, then any induced subgraph of H is also an i-graph.*

Proof. Let G be a graph such that $H = \mathscr{I}(G)$ and $i(G) = k$. To prove this result, we show that for any $X \in V(H)$, there exists some graph G_X such that $\mathscr{I}(G_X) = H - X$.

To construct G_X, take a copy of G and add to it a vertex z so that z is adjacent to each vertex of $G - X$ (see Figure 3.7). Observe first that since H is nontrivial, there exists an i-set $S \neq X$ of G. Then, S is also an independent dominating set of G_X, and so $i(G_X) \leq k$. Consider now some i-set S_X of G_X. Clearly $S_X \neq X$ because X does not dominate z. If $z \in S_X$, then as S_X is independent, no vertex of $G - X$ is in S_X. Moreover, since X is also independent and its vertices have all of their neighbors in $G - X$, this leaves each vertex of X to dominate itself. That is, $X \subseteq S_X$, implying that $S_X = X \cup \{z\}$ and $|S_X| = k + 1$. This contradicts that $i(G_X) \leq k$, and thus we conclude that z is not in any

i-set of G_X. It follows that each i-set of G_X is composed only of vertices from G and so $i(G_X) = k$. Thus, $S_X \neq X$ is an i-set of G_X if and only if it is an i-set of G. Given that $V(\mathscr{I}(G_X)) = V(\mathscr{I}(G)) - \{X\} = V(H) - \{X\}$, we have that $\mathscr{I}(G_X) = H - X$ as required. ■

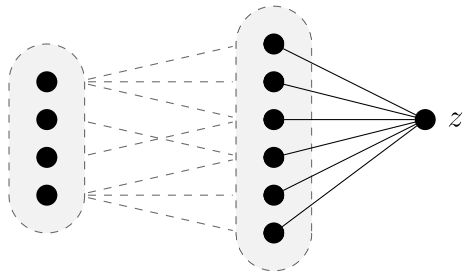


Figure 3.7: Construction of G_X in Lemma 3.22.

The following corollary is immediate as the contrapositive of Lemma 3.22.

Corollary 3.23. *If H is not an i-graph, then any graph containing an induced copy of H is also not an i-graph.*

This corollary, although simple in statement and proof, immediately removes many families of graphs from i-graph realizability. For example, all wheels, 2-trees, and maximal planar graphs on at least five vertices contain an induced copy of the Diamond graph $\mathfrak{D}$, which was shown in Proposition 3.8 to not be an i-graph. Moreover, given that i-graph realizability is an inherited property, this suggests that there may be a finite-family forbidden subgraph characterization for i-graph realizability.

We now alter course to examine how one may construct new i-graphs by combining several known i-graphs. Understandably, an immediate obstruction to combining the constructions of i-graphs of, say, $\mathscr{I}(G_1) = H_1$ and $\mathscr{I}(G_2) = H_2$ is that it is possible (and indeed, likely) that $i(G_1) \neq i(G_2)$.

Two solutions to this quandary are presented in the following lemmas. In the first, Lemma 3.24, given a graph G, we progressively construct an infinite family of seed graphs $\mathcal{G}$ with the same number of components as G, and such that $\mathscr{I}(G) = \mathscr{I}(G_i)$ for each $G_i \in \mathcal{G}$. The second, Lemma 3.25 or the **Inflation Lemma**, offers a more direct solution:

given an i-graph H, we demonstrate how to "inflate" a seed graph G to produce a new graph G^* such that $\mathscr{I}(G^*) = \mathscr{I}(G)$ and the i-sets of G^* are arbitrarily larger than the i-sets of G.

Lemma 3.24. *If G is a graph with $\mathscr{I}(G) \cong H$, then there exists an infinite family of graphs $\mathcal{G}$ such that $\mathscr{I}(G_i) \cong H$ for each $G_i \in \mathcal{G}$. Moreover, the number of components of $G_i \in \mathcal{G}$ is the same as G ($k(G) = k(G_i)$).*

Proof. Suppose $v \in V(G)$, and let G^* be the graph obtained by attaching a copy of the star $K_{1,3}$ with $V(K_{1,3}) = \{x, y_1, y_2, y_3\}$ ($\deg(x) = 3$) by joining v to y_1. As y_2 and y_3 are pendant vertices, $i(G^*) \geq i(G) + 1$. If S is an i-set of G, then $S^* = S \cup \{x\}$ is dominating and independent, and so $i(G^*) = i(G) + 1$. Thus, x is in every i-set of G^*, and we can conclude that S^* is an i-set of G^* if and only if $S^* - \{x\}$ is an i-set of G. It follows that $\mathscr{I}(G^*) \cong \mathscr{I}(G)$ as required. Attaching additional copies of $K_{1,3}$ as above at any vertex of H similarly creates the other graphs of $\mathcal{G}$. ■

Lemma 3.25. ***(The Inflation Lemma)*** *If H is the i-graph of some graph G, then for any $k \geq i(G)$ there exists a graph G^* such that $i(G^*) = k$ and $\mathscr{I}(G^*) \cong H$.*

Proof. Begin with a copy of G and add to it $\ell = k - i(G)$ isolated vertices, $S = \{v_1, v_2, \ldots, v_\ell\}$. Immediately, X is an i-set of G if and only if $X \cup S$ is an i-set of G^*. Moreover, if X and Y are i-sets of G such that $X \sim_G Y$ in H, then $(X \cup S) \sim_{G^*} (Y \cup S)$, and so $\mathscr{I}(G^*) \cong H$. ■

Now, when attempting to combine the constructions of $\mathscr{I}(G_1) = H_1$ and $\mathscr{I}(G_2) = H_2$ and $i(G_1) < i(G_2)$, we need only inflate G_1 until its i-sets are the same size as those in G_2. A powerful construction tool, the Inflation Lemma is used repeatedly in almost all of the following results of this section; in the first of these, we show how two i-graphs may be connected by an edge between any two vertices to produce a new i-graph.

Proposition 3.26. *Let H_1 and H_2 be disjoint i-graphs. Then the graph H, formed by connecting H_1 to H_2 by an edge between any $x \in V(H_1)$ and any $y \in V(H_2)$, is also an i-graph.*

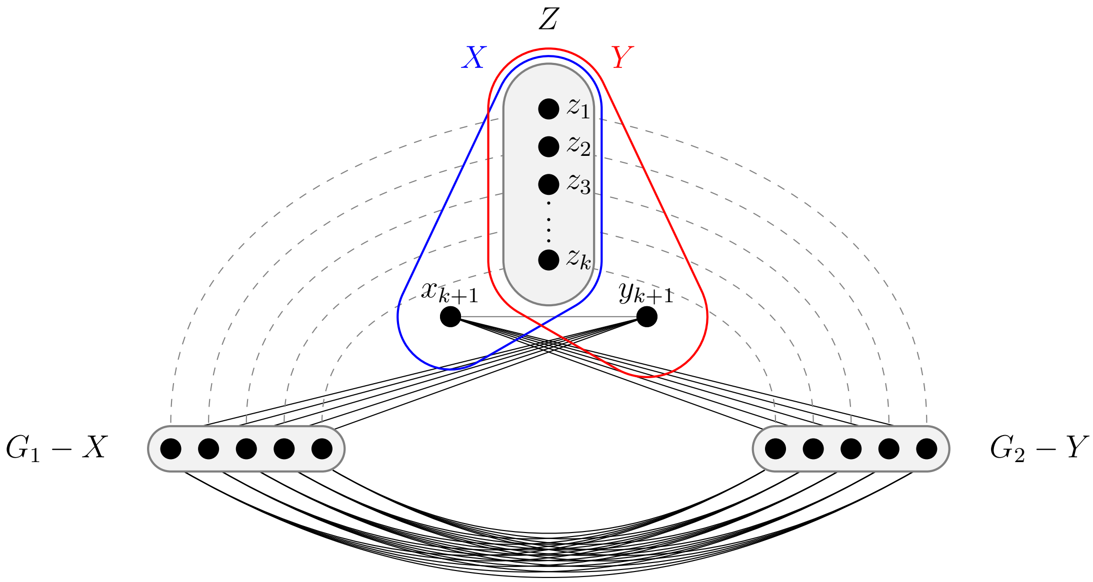


Figure 3.8: Construction of G in Proposition 3.26.

Proof. Suppose that G_1^* and G_2^* are graphs such that $\mathscr{I}(G_1^*) = H_1$ and $\mathscr{I}(G_2^*) = H_2$. Without loss of generality, let $k = i(G_1^*) \geq i(G_2^*)$. Applying the Inflation Lemma (Lemma 3.25), create graphs G_1 and G_2 with $\mathscr{I}(G_1) = H_1$, $\mathscr{I}(G_2) = H_2$, and $i(G_1) = i(G_2) = k+1$, by adding isolated vertices to G_1^* and G_2^*. In particular, let x_{k+1} be one of the added isolated vertices in G_1, and likewise y_{k+1} in G_2. Then x_{k+1} (respectively, y_{k+1}) is in every i-set of G_1 (respectively, G_2).

Now suppose that the i-set $X = \{x_1, x_2, \ldots, x_{k+1}\}$ of G_1 corresponds to the vertex $x \in V(H_1)$, and the i-set $Y = \{y_1, y_2, \ldots, y_{k+1}\}$ of G_2 corresponds to $y \in V(H_2)$. Construct the graph G by identifying $x_i \in V(G_1)$ with $y_i \in V(G_2)$ for all $1 \leq i \leq k$, and renaming the identified vertex z_i. Let $Z = \{z_1, z_2, \ldots, z_k\}$. Then, join each $x' \in V(G_1)\backslash Z$ to each $y' \in V(G_2)\backslash Z$. We claim that $\mathscr{I}(G) = H$. Through this proof (and those that follow), we mildly abuse notation by referring to both the seed graph used to build G, and the vertices of G corresponding to that graph as G_1 (and similarly with G_2).

Notice that Z does not dominate x_{k+1} nor y_{k+1}, and so Z is not itself an i-set of G. Suppose S^* is some i-set of G; clearly, S^* is not a subset of Z. Since all vertices of $G_1 - Z$ are adjacent to all vertices of $G_2 - Z$, either $S^* \subseteq G_1$ or $S^* \subseteq G_2$. Without loss of generality, say $S^* \subseteq G_1$. Then to dominate G_1, $|S^*| \geq k+1$. Now consider any i-set S of G_1. Since $x_{k+1} \in S$, S dominates G_2 and is therefore an independent dominating set of G. Thus $i(G) \leq k+1$, and so $i(G) = k+1$, and all i-sets of G_1 and G_2 are also i-sets of G. Moreover, since all i-sets of G are subsets of G_1 or G_2, we conclude that S is an i-set of G if and only if it is an i-set of G_1 or G_2.

If $X_1 \sim_{G_1} X_2$, then clearly $X_1 \sim_G X_2$, and likewise for some $Y_1 \sim_{G_2} Y_2$. Thus the subgraph of $\mathscr{I}(G)$ induced by the i-sets of G_1 (likewise, G_2) is H_1 (H_2). Moreover, from their definitions, $X \overset{x_{k+1}y_{k+1}}{\sim_G} Y$ in $\mathscr{I}(G)$.

Now suppose that $Y^* \neq Y$ is some i-set of G_2. Since every i-set of G_2 contains y_{k+1}, Y and Y^* differ at some other vertex: say, $y^* \in Y^* - Y$ and without loss of generality, $z_1 \in Y - Y^*$. Then $|X - Y^*| \geq 2$, and so X is not adjacent to Y^* in $\mathscr{I}(G)$, as required. Similar arguments show that there are no other edges between the i-sets of G_1 and G_2, except between X and Y. Thus the only edge between H_1 and H_2 in $\mathscr{I}(G)$ is xy. We concluded that $\mathscr{I}(G) = H$. ■

The following Corollary 3.27 provides a way to connect two i-graphs with a clique, rather than a bridge. Notice that the same result is obtainable by applying the Max Clique Replacement Lemma (Lemma 3.21) to Proposition 3.26, but we include the following alternative construction and proof below for completeness.

Corollary 3.27. *Let H_1 and H_2 be i-graphs, and let H be the graph formed from them as in Proposition 3.26 by creating a bridge xy between them. Then the graph H_m formed by replacing xy with a K_m for $m \geq 2$ is also an i-graph.*

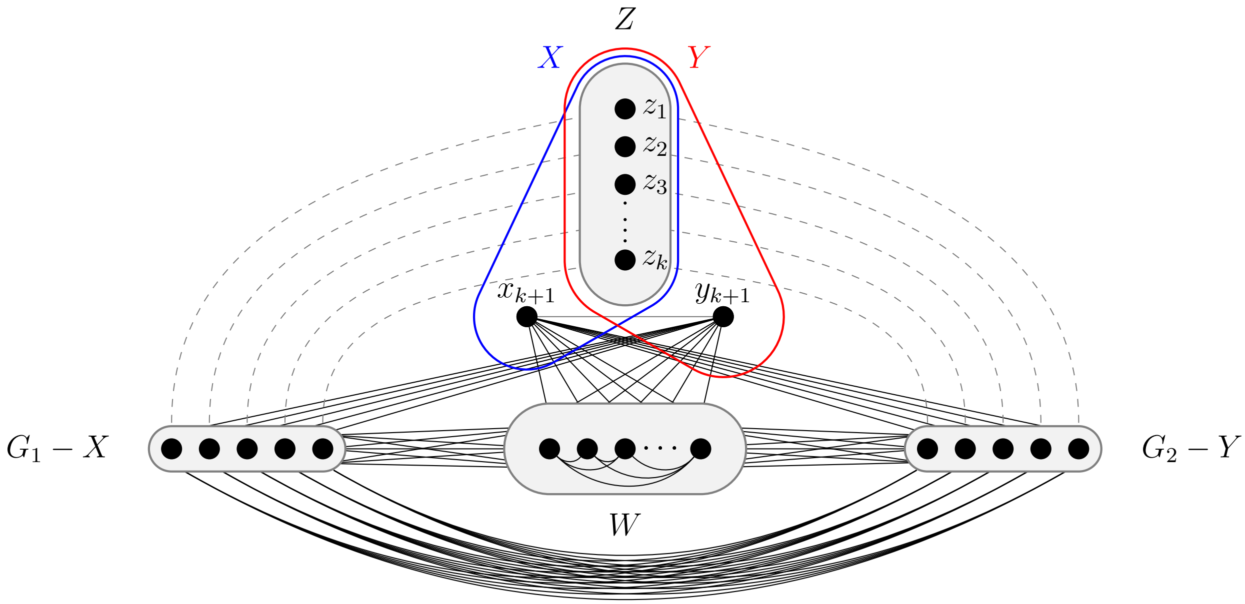


Figure 3.9: Construction of G in Corollary 3.27.

Proof. We proceed by fashioning a graph G_m and then demonstrating that $\mathscr{I}(G_m) = H_m$. Begin by constructing G from G_1 and G_2 as in the proof of Proposition 3.26; as before, identify k vertices of the i-set X, with k vertices of the i-set Y, and label them as $Z = \{z_1, z_2, \ldots, z_k\}$, so that $X = Z \cup \{x_{k+1}\}$ and $Y = Z \cup \{y_{k+1}\}$. In particular, recall that $i(G) = k+1$. Then, to construct G_m, add $m-2$ additional vertices $W = \{w_1, w_2, \ldots, w_{m-2}\}$ to G, making each w_i adjacent to every vertex in $G \backslash Z$ and to each other (i.e. $W = K_{m-2}$).

Suppose S is an i-set of G. Since the vertices of W form a clique, $|S \cap W| \leq 1$. Consider first the case where $|S \cap W| = 1$, so that, without loss of generality, $w_1 \in S$. The vertices of Z are not dominated by w_1. However, since w_1 is adjacent to everything in $G_m \backslash Z$ and Z is independent, the vertices of Z are therefore self-dominating in S. Thus, $W_1 = Z \cup \{w_1\} \subseteq S$, and since W_1 is dominating, we have that $W_1 = S$ with $|W_1| = k+1$. On the other hand, if $S \cap W = \varnothing$, then as in Proposition 3.26, S is a subset of either $V(G_1)$ or $V(G_2)$, and using similar arguments, $|S| = k+1$. Thus, we conclude that $i(G_m) = k+1$.

From the above, the i-sets of G_m are precisely the i-sets of G_1 and G_2, as well as the sets of the form $W_i = Z \cup \{w_i\}$ for each $i \in \{1, 2, \ldots, m-2\}$. Since $W \cup \{x_{k+1}, y_{k+1}\}$ is a clique, clearly $W_i \overset{w_i x_{k+1}}{\sim}_{G_m} X$, $W_i \overset{w_i y_{k+1}}{\sim}_{G_m} Y$, and $W_i \overset{w_i w_j}{\sim}_{G_m} W_j$, for each $1 \leq i < j \leq m-2$.

Therefore, in $\mathscr{I}(G_m)$, the vertex set $\{W_1, W_2, \ldots, W_{m-2}, X, Y\}$ is an induced clique.

Now consider an i-set $X^* \neq X$ of G_1 and the i-set W_i for some $1 \leq i \leq m-2$. Using the same arguments that were applied to X and Y in Proposition 3.26, we see that X^* and W_i are non-adjacent in $\mathscr{I}(G_m)$. Likewise for some i-set $Y^* \neq Y$ and W_i, and then again for X^* and Y^*. It follows that $\mathscr{I}(G_m) = H_m$. ∎

Next, Proposition 3.28 provides a method for combining two i-graphs without connecting them by an edge.

Proposition 3.28. *If H_1 and H_2 are i-graphs, then $H_1 \cup H_2$ is an i-graph.*

Proof. Suppose G_1 and G_2 are graphs such that $\mathscr{I}(G_1) = H_1$ and $\mathscr{I}(G_2) = H_2$. We assume that $i(G_1) = i(G_2) \geq 2$. Otherwise, apply the Inflation Lemma (Lemma 3.25) to obtain graphs with i-sets of equal size at least 2. Let $G = G_1 \vee G_2$, the join of G_1 and G_2. We claim that $\mathscr{I}(G) = H_1 \cup H_2$.

We proceed similarly to the proof of Proposition 3.26; namely, if S is an i-set of G_1, of G_2, then S is an independent dominating set of G. Likewise, we observe that any i-set of G is a subset of G_1 or G_2, and so, S is a i-set of G if and only if it is an i-set of G_1 or G_2.

Suppose $X \overset{xy}{\sim}_{G_1} Y$. Then in G, sets X and Y are still i-sets, and likewise, vertices X and Y are still adjacent, and so $X \overset{xy}{\sim}_{G} Y$. Now suppose instead that X is an i-set of G_1 and Y is an i-set of G_2. Within G, $X \cap Y = \varnothing$ and $|X| = |Y| \geq 2$, so X and Y are not adjacent in $\mathscr{I}(G)$. Therefore, $X \sim_G Y$ if and only if $X \sim_{G_1} Y$ or $X \sim_{G_2} Y$. It follows that $\mathscr{I}(G) = \mathscr{I}(G_1) \cup \mathscr{I}(G_2) = H_1 \cup H_2$ as required. ∎

Applying these new tools in combination yields some unexpected results. For example, the following corollary, which makes use of the previous Proposition 3.28 in partnership with the Deletion Lemma (a construction for combining i-graphs and a construction for vertex deletions) gives our first result on i-graph edge deletions.

Corollary 3.29. *Let H be an i-graph with a bridge e, such that the deletion of e separates H into components H_1 and H_2. Then,*

(i) H_1 and H_2 are i-graphs, and

(ii) the graph $H^* = H - e$ is an i-graph.

Proof. Result (i) follows immediately from Lemma 3.22. For (ii), by result (i), H_1 and H_2 are i-graphs. Then, from Proposition 3.28, the graph $H_1 \cup H_2 = H - e$ is also an i-graph. ∎

In the next result, we show that given i-graphs H_1 and H_2, a new i-graph can be formed by identifying any two vertices in H_1 and H_2. This result provides an alternative proof for Theorem 3.18.

Proposition 3.30. *Let H_1 and H_2 be i-graphs. Then the graph H, formed by identifying a vertex x of H_1 with a vertex y of H_2, is also an i-graph.*

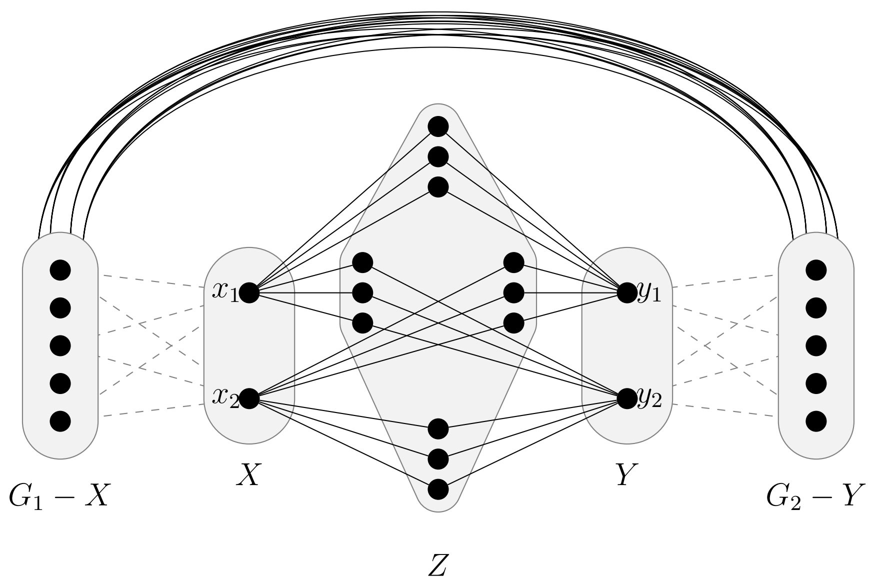


Figure 3.10: Construction of G in Proposition 3.30.

Proof. Suppose G_1 and G_2 are graphs such that $\mathscr{I}(G_1) = H_1$ and $\mathscr{I}(G_2) = H_2$. As before, we assume $i(G_1) = i(G_2) = k \geq 2$ via the Inflation Lemma. Let $X = \{x_1, x_2, \ldots, x_k\}$ and $Y = \{y_1, y_2, \ldots, y_k\}$ be the i-sets corresponding to x and y, respectively. Construct the graph G by adding $3k^2$ vertices to $G_1 \cup G_2$, labelled as $z_{i,j,\ell}$ for all $i, j \in \{1, 2, \ldots, k\}$ and $\ell \in \{1, 2, 3\}$, so that $z_{i,j,\ell}$ is adjacent to x_i and y_j for all $1 \leq \ell \leq 3$. Let Z be the set of these $z_{i,j,\ell}$ and note that Z is an independent set. Finally, add edges between all vertices of $G_1 \backslash X$ to all vertices of $G_2 \backslash Y$. We claim that $i(G) = 2k$ and $\mathscr{I}(G) = H$.

First, observe that $X \cup Y$ is an independent dominating set of G, and so $i(G) \leq 2k$. Let S be some i-set of G. To show that $S \cap Z = \varnothing$, suppose to the contrary that some vertex of Z is in S, say $z_{1,1,1} \in S$. Then, for independence, neither x_1 nor y_1 is in S, and so both $z_{1,1,2}$ and $z_{1,1,3}$ are self-dominating in S. Since $x_1 \notin S$, for any $1 \leq j \leq k$, to dominate $Z_{1,j} = \{z_{1,j,1}, z_{1,j,2}, z_{1,j,3}\}$ requires either $Z_{1,j} \subseteq S$ or $y_j \in S$. Likewise, since $y_1 \notin S$, for all $1 \leq i \leq k$, either $Z_{i,1} \subseteq S$ or $x_i \in S$. Thus, $|S| \geq 3 + 2(k-1) = 2k+1$. However, we have already established that $i(G) \leq 2k$, and so, we conclude that $S \cap Z = \varnothing$.

Now, suppose that some for some $1 \leq i, j \leq k$, $x_i \notin S$ and $y_j \notin S$. Then, since $Z \cap S = \varnothing$, the vertices of $\{z_{i,j,1}, z_{i,j,2}, z_{i,j,3}\}$ are undominated, contradicting that S is an i-set. Thus, any i-set of G has either X or Y as a subset. Suppose, without loss of generality, that $X \subseteq S$. Then, all of $G_1 \backslash X$ and Z are dominated by X (and none of G_2). Now, since $N_G[G_2] \cap X = \varnothing$, the vertices of G_2 are internally dominated; thus Y^* is an i-set of G_2 if and only if $X \cup Y^*$ is an i-set of G. Similarly, X^* is an i-set of G_1 if and only if $Y \cup X^*$ is an i-set of G, and so $i(G) = 2k$.

For adjacency, clearly $X_i \overset{x_i x_j}{\sim}_{G_1} X_j$ if and only if $(X_i \cup Y) \overset{x_i x_j}{\sim}_{G} (X_j \cup Y)$. Similarly, $Y_i \overset{y_i y_j}{\sim}_{G_2} Y_j$ if and only if $(X \cup Y_i) \overset{y_i y_j}{\sim}_{G} (X \cup Y_j)$. Consider some i-sets $X^* \neq X$ of G_1 and $Y^* \neq Y$ of G_2 and their corresponding i-sets in G: $(X^* \cup Y)$ and $(X \cup Y^*)$, respectively.

Since $|X \cap X^*| \geq 1$ and $|Y \cap Y^*| \geq 1$, $|(X^* \cup Y) \cap (X \cup Y^*)| \geq 2$, and so $(X^* \cup Y)$ and $(X \cup Y^*)$ are nonadjacent in $\mathscr{I}(G)$. Therefore, in $\mathscr{I}(G)$, there are no edges between the subgraphs induced by $\mathscr{I}(G_1)$ and $\mathscr{I}(G_2)$. Therefore, $(X \cup Y)$ is a cut vertex, and we conclude that $\mathscr{I}(G) = H$. ■

An alternative proof of Proposition 3.30 can be realized by utilizing Proposition 3.13. Consider i-graphs H_1 and H_2, with equal sized i-set seed graphs G_1 and G_2, respectively. From Proposition 3.13, there is some graph G such that $\mathscr{I}(G) = H_1 \,\square\, H_2$. Since the graph H described in the statement of Proposition 3.30 is an induced subgraph of $H_1 \,\square\, H_2$, applying Lemma 3.22, delete all other vertices of $H_1 \,\square\, H_2$ until only H remains.

Combining the results of Proposition 3.30 with Proposition 3.26 and Corollary 3.29, yields the following main result.

Theorem 3.31. *A graph G is an i-graph if and only if all of its blocks are i-graphs.*

As observed in Corollary 3.23, graphs with an induced $\mathfrak{D}$ subgraph are not i-realizable. If we consider the family of connected chordal graphs excluding those with an induced copy of $\mathfrak{D}$, we are left with the family of block graphs (also called clique trees): graphs where each block is a clique. As cliques are their own i-graph, the following is immediate.

Proposition 3.32. *Block graphs are i-graph realizable.*

Cacti are graphs whose blocks are cycles or edges. Thus, we have the following immediate corollary.

Corollary 3.33. *Cactus graphs are i-graph realizable.*

While the proof of Proposition 3.30 does provide a method for building block graphs, it is laborious to do so on a graph with many blocks, as the construction is iterative, with each block being appended one at a time; moreover, when the edges between $G_1 - X$ and $G_2 - Y$ (as in Figure 3.10) are added at each iteration, the resultant seed graph is often a confusing mess to behold. However, when we consider that the blocks of block graphs are complete graphs, and that complete graphs are their own i-graphs (and thus arguably the easiest to construct i-graphs), it is logical that there is a simpler construction. We offer one such construction below that uses similar gadgets to the proof of Proposition 3.30, but ultimately yields a cleaner and more illuminating seed graph. An example of this process is illustrated in Figure 3.11.

Construction 3.34. *Let H be a block graph with $V(H) = \{v_1, v_2, \ldots, v_n\}$ and let $\mathcal{B}_H = \{B_1, B_2, \ldots, B_m\}$ be the collection of maximal cliques of H. To construct a graph G such that $\mathscr{I}(G) = H$:*

(i) Begin with a copy of each of the maximal cliques of H, labelled $A_1, A_2, \ldots, A_m$ in G, where A_i of G corresponds to B_i of H for each $1 \leq i \leq m$, and the A_i are pairwise disjoint. Notice that each cut vertex of H has multiple corresponding vertices in G.

(ii) *Let $v \in V(H)$ be a cut vertex and $\mathcal{B}_v$ be the collection of blocks containing v in H; for notational ease, say $\mathcal{B}_v = \{B_1, B_2, \ldots, B_k\}$, and suppose that $W = \{w_1, w_2, \ldots, w_k\} \subseteq V(G)$ are the k vertices corresponding to v, where $w_i \in A_i$ for all $1 \leq i \leq k$.*

For each distinct pair w_i and w_j of W, add to G three internally disjoint paths of length two between w_i and w_j. Since v is in k blocks of H, $3\binom{k}{2}$ vertices are added in this process. These additions are represented as the green vertices in Figure 3.11.

(iii) *Repeat Step (ii) for each cut vertex of H.*

(a)

(b)

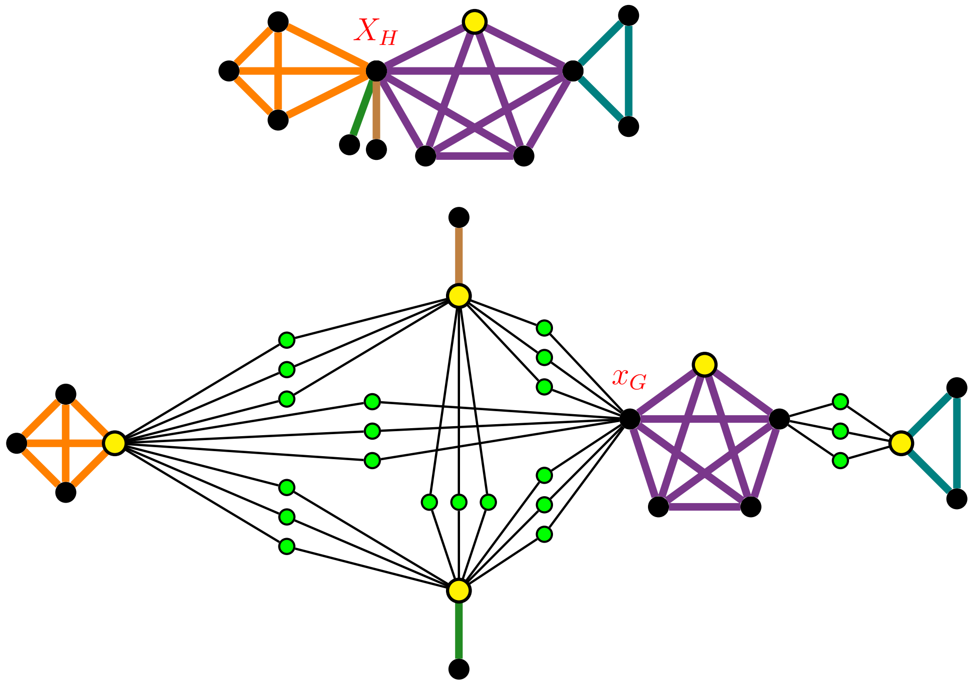


Figure 3.11: The construction of G from H in the proof of Proposition 3.32.

To see that the graph G from Construction 3.34 does indeed have $\mathscr{I}(G) = H$, notice that $i(G) = m$, where m is the number of blocks in H; if X is an i-set of G, then $|X \cap A_i|=1$ for each $A_i \in \{A_1, A_2, \ldots A_m\}$. Moreover, no i-set of G has vertices in the added green vertices, because, as with the proof of Proposition 3.30, the inclusion of any one of these green vertices in an independent dominating set necessitates the addition of them all.

In Figure 3.11(b), the five yellow vertices form the i-set corresponding to the yellow vertex of G in Figure 3.11(a). Only the token on the purple K_5 can move in G; the other four tokens remain frozen, thereby generating the corresponding purple K_5 of H. It is only when the token on the purple K_5 is moved to the vertex x_G that the tokens on the orange K_4, and the brown and green K_2's, unfreeze one clique at a time. This corresponds to the cut vertex i-set X_H of H. The freedom of movement now transfers from the purple K_5 to any of the three other cliques, allowing for the generation of their associated blocks in G as required.

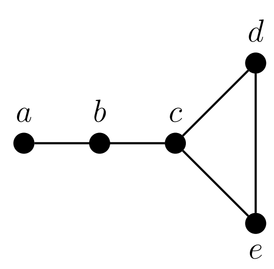
Finally, before we depart from block graphs, as chordal graphs are among the most well-studied families of graphs, we offer one additional reframing of this block graph result from the chordal graph perspective.

Corollary 3.35. *A chordal graph G is i-graph realizable if and only if G is $\mathfrak{D}$-free.*

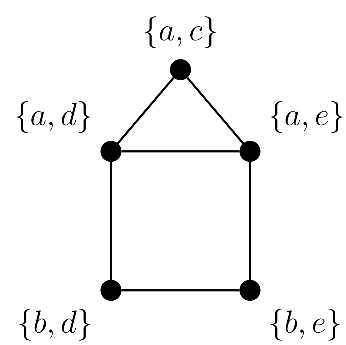
With the addition of Proposition 3.32 to the results used to build Observation 3.20, this leaves only the House graph (see Figure 3.12(b)) as unsettled with regard to its i-graph realizability among the 34 non-isomorphic graphs on five vertices. Although not strictly a result concerning the construction of larger i-graphs from known results, we include the following short proposition here for the sake of completeness.

Proposition 3.36. *The House graph $\mathcal{H}$ is an i-graph.*

To demonstrate Proposition 3.36, we provide an exact seed graph for the i-graph: the graph G in Figure 3.12(a) (K_3 with a P_3 tail) has $\mathscr{I}(G) = \mathcal{H}$. The i-sets of G and their adjacency are overlaid on $\mathcal{H}$ in Figure 3.12(b).



(a) A graph G such that $\mathscr{I}(G) = \mathcal{H}$.



(b) The House graph $\mathcal{H}$ with i-sets of G.

Figure 3.12: The graph G for Proposition 3.36 with $\mathscr{I}(G) = \mathcal{H}$.

The observant reader will have noted that $\mathcal{H}$ is a theta graph, namely $\mathcal{H} = \Theta\langle 1, 2, 3\rangle$. Thus, an alternative confirmation that $\mathcal{H}$ is an i-graph is given with the other theta graph results in Chapter 5 (see Lemma 5.9).

Having established the foundation for constructing new i-graphs from known i-graphs, in the following Chapter 4, we take a detour from i-graph realizability back to structural results to examine the properties of the i-graphs of paths and cycles. The realizability of i-graphs returns in Chapter 5, where we characterize which theta graphs are i-graphs.

Chapter 4

The i-Graphs of Paths and Cycles

In our previous chapters, we have explored whether a given graph H is i-graph realizable. That is, does there exist a graph G such that $\mathscr{I}(G) = H$? We now move to the opposing question: given a graph G, what is the structure of $\mathscr{I}(G)$? As we have already observed, although not every graph is i-graph realizable, every graph does have an i-graph. The exact structure of the resulting i-graph can vary among families of graphs from the simplest isolated vertex to surprisingly complex structures.

In this chapter, we examine the i-graphs of two of the most famous classes of graphs: paths and cycles. To begin, we use generating functions to count the distinct i-sets of the path P_n and the cycle C_n. That is, we determine $|V(\mathscr{I}(P_n))|$ for $n \geq 1$ and $|V(\mathscr{I}(C_n))|$ for $n \geq 3$.

4.1 The Number of i-Sets of Paths

For the remainder of this chapter, we assume that the vertices of the path P_n are labelled as $P_n = (v_1, v_2, \ldots, v_n)$. Given that we are discussing i-sets, which are both independent and dominating, if X is an i-set of P_n, then two consecutive vertices of X are separated by one or two vertices of $P_n - X$; the different interval lengths between these consecutive vertices of X therefore correspond to the different i-sets of P_n. This provides our method for counting the distinct i-sets of P_n.

To begin, recall the following well-known result regarding the independent domination number for both paths and cycles.

Lemma 4.1. *[14] For the path and cycle,* $i(P_n) = i(C_n) = \lceil n/3 \rceil$.

Thus, deleting the vertices of an arbitrary i-set X of P_n divides $V(P_n) - X$ into $t = i(P_n) + 1$ subsets $X_1, X_2, \ldots, X_t$, where X_1 and X_t could be empty or consist of a single vertex, and where $1 \le |X_j| \le 2$ for $j \ne \{1, t\}$. An example for P_{10} with sets $X_1, X_2, \ldots, X_5$ is given below in Figure 4.1. In particular, notice that $X_1 = \varnothing$.

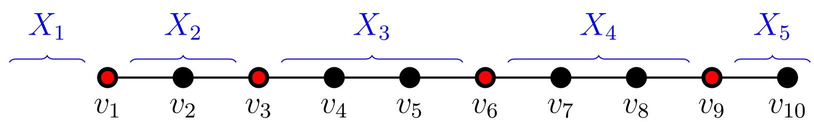


Figure 4.1: The sets $X_1, X_2, \ldots, X_5$ of P_{10}.

The number of i-sets, therefore, equals the number of integer solutions to the following equation, where $r = |V(P) - X| = n - i(P_n) = n - \lceil n/3 \rceil$, and where $x_j = |X_j|$:

$$x_1 + \cdots + x_t = r, \text{ where } 0 \le x_1, x_t \le 1, \text{ and } 1 \le x_j \le 2 \text{ for } j \in \{2, \ldots, t-1\}. \quad (4.1)$$

The corresponding generating function is

$$(1 + x)^2 (x + x^2)^{t-2} = x^{t-2}(1 + x)^t. \quad (4.2)$$

- If $n = 3k + 1$ then $t = i(P_n) + 1 = k + 2$ and $r = 2k$. Thus we want to the coefficient of $2k$ in (4.2), that is, the coefficient of x^k in $(1 + x)^{k+2}$, which is $\binom{k+2}{k}$.
- If $n = 3k + 2$, then $t = k + 2$ and $r = 2k + 1$. Here we want the coefficient of x^{k+1} in $(1 + x)^{k+2}$, which is $\binom{k+2}{k+1} = k + 2$.
- If $n = 3k + 3$, then $t = k + 2$ and $r = 2k + 2$. Here we want the coefficient of x^{k+2} in $(1 + x)^{k+2}$, which is 1.

We summarize the above results in the following lemma.

Lemma 4.2. *For $n \geq 1$, the order of $\mathscr{I}(P_n)$ is*

$$|V(\mathscr{I}(P_n))| = \begin{cases} 1 & \textit{if} \ \ n = 3k \\ \binom{k+2}{k} & \textit{if} \ \ n = 3k+1 \\ k+2 & \textit{if} \ \ n = 3k+2. \end{cases}$$

4.2 The i-Graph of P_n

We remind the reader that the vertices of the path P_n are labelled as $P_n = (v_1, v_2, \ldots, v_n)$. As we return to our token-sliding model for i-graphs, notice that the i-set tokens on P_n that are free to slide are very limited.

For example, in Figure 4.2 below, we have two different i-sets on P_{10}. In the first case, the token at v_6 which is surrounded on each side by two consecutive non-i-set vertices is frozen; a slide in either direction would leave either v_5 or v_7 undominated. That is, in this first configuration $|X_3| = |X_4| = 2$. Similarly, near the end of the path, the token at v_9 is frozen. Indeed, the only token that can slide is v_3. Thus the i-graph vertex associated with the i-set in the first figure would have only degree 1. On the other hand, in the second i-set of Figure 4.2, we see that there are 4 possible token slides, corresponding to an i-graph vertex of degree four. Indeed, we see that due to the limited availability of slack between i-set vertices, the maximum degree of an i-graph vertex of P_n is 4, as we now explain. Moving the token between X_i and X_{i+1} to the right, we increase $|X_i|$ by one, and decrease $|X_{i+1}|$ by one.

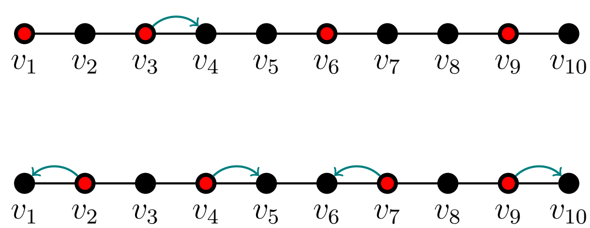


Figure 4.2: Two i-sets of P_{10}.

Thus, a token between X_i and X_{i+1} can slide to the right if and only if

$$|X_i| = \begin{cases} 0 & \text{if } i = 0 \\ 1 & \text{if } i \neq 0 \end{cases} \quad \text{and} \quad |X_{i+1}| = \begin{cases} 1 & \text{if } i = t \\ 2 & \text{if } i \neq t. \end{cases}$$

Similarly, a token between X_i and X_{i+1} can slide to the left if and only if

$$|X_i| = \begin{cases} 1 & \text{if } i = 0 \\ 2 & \text{if } i \neq 0 \end{cases} \quad \text{and} \quad |X_{i+1}| = \begin{cases} 0 & \text{if } i = t \\ 1 & \text{if } i \neq t. \end{cases}$$

With these results in mind, we can now consider the actual structure of $\mathscr{I}(P_n)$.

From Lemma 4.2, the i-graphs of P_n for $n = 3k$ and $n = 3k+2$ ($k \geq 0$) are immediate. In particular, a token can move only if it is adjacent to some X_i when $x_i = |X_i|$ is at its lower bound in (4.1). For $n = 3k$, there are no such X_i, so all tokens are frozen, and $\mathscr{I}(P_n) = K_1$. If $n = 3k+2$, then exactly one X_i is at its lower bound. If $i = 0$ or t, then one token can move. Otherwise, $1 \leq i \leq t-1$ and X_i has both a left and right token that can move. Hence $\Delta(\mathscr{I}(P_{3k+2})) \leq 2$. It is easy to confirm that $\mathscr{I}(P_{3k+2}) = P_{k+2}$.

Lemma 4.3. *For $n \geq 1$, $k \geq 0$,*

$$\mathscr{I}(P_n) = \begin{cases} K_1 & \textit{if } n = 3k \\ P_{k+2} & \textit{if } n = 3k+2. \end{cases}$$

It is only the third case where $n = 3k+1$ that remains to be described. Using the fact that two X_i are at their lower bound, we get $\Delta(\mathscr{I}(P_n)) \leq 4$. In order to eventually determine the structure of $\mathscr{I}(P_{3k+1})$, we first detour slightly to define a new class of graphs.

We define the *worn k-lattice graph* $\mathfrak{L}_k$ to be the graph with $1+2+\cdots+k+(k+1) = \binom{k+2}{2}$ vertices, such that $V(\mathfrak{L}_k) = \{w_{i,j} : 0 \leq j \leq i \leq k\}$, and where the vertex $w_{a,b}$ is adjacent to vertex $w_{c,d}$ if and only if one of the following holds:

(i) $a = c-1$ and $b = d$ or $d-1$, or

(ii) $a = c + 1$ and $b = d$ or $d + 1$.

The graph $\mathfrak{L}_3$ is given as an example below in Figure 4.3.

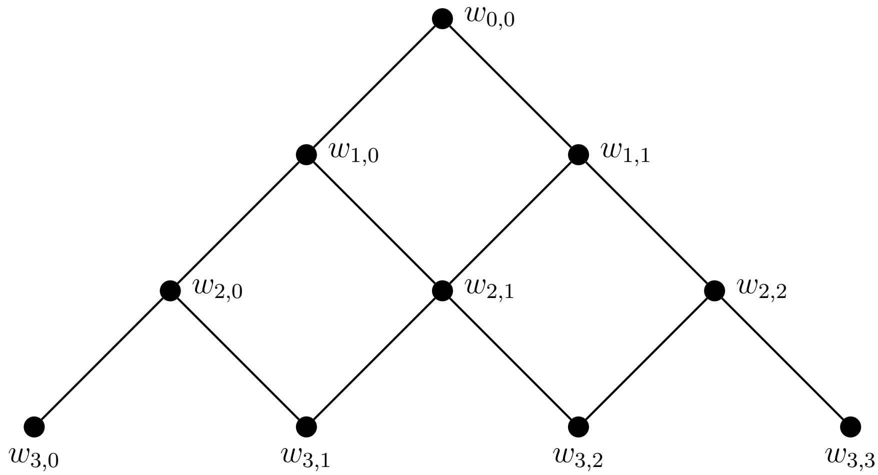


Figure 4.3: The worn lattice graph $\mathfrak{L}_3$.

The observant reader will have already noted several similarities to $\mathfrak{L}_k$ and the structure of $\mathscr{I}(P_{3k+1})$. Notably, recall from Lemma 4.2 that $\mathscr{I}(P_{3k+1})$ also has $\binom{k+2}{2}$ vertices. We have also noted that for each $v \in V(\mathscr{I}(P_k))$, $\deg(v) \leq 4$. In Lemma 4.4 below, we show exactly this connection, with a key observation being that just as one can imagine growing P_{3k+1} into $P_{3(k+1)+1}$ by adding a *tail* of three additional vertices to the end of P_{3k+1}, so can we grow $\mathfrak{L}_k$ into $\mathfrak{L}_{k+1}$ by adding an additional path of $(k+1)+1$ vertices down one outer-diagonal. We call these vertices the *path tail* vertices of $\mathfrak{L}_{k+1}$.

The cases for $k = 1$ and $k = 2$ (i.e. with P_4 and P_7) are illustrated in Figures 4.4 and 4.5 below. Here the vertices of $\mathfrak{L}_k$ are labelled with the subscripts of the vertices of an i-set of P_{3k+1}. So, for example, the vertex labelled "1, 4" at the top of $\mathfrak{L}_1$ corresponds to the i-set $\{v_1, v_4\}$ of P_4, while the vertex labelled "2, 4" corresponds to the i-set $\{v_2, v_4\}$. Cases $k = 3$ (P_{10}) and $k = 4$ (P_{13}) are given in Figures 4.6 and 4.7, respectively, and the added path tail vertices highlighted in green and blue.

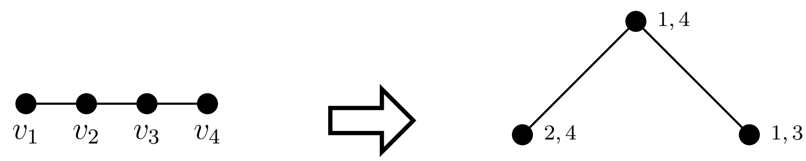


Figure 4.4: $\mathscr{I}(P_4) \cong \mathfrak{L}_1$.

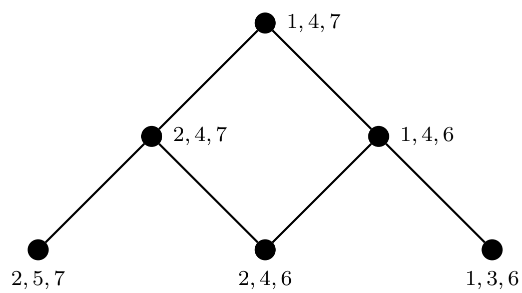


Figure 4.5: $\mathscr{I}(P_7) \cong \mathfrak{L}_2$.

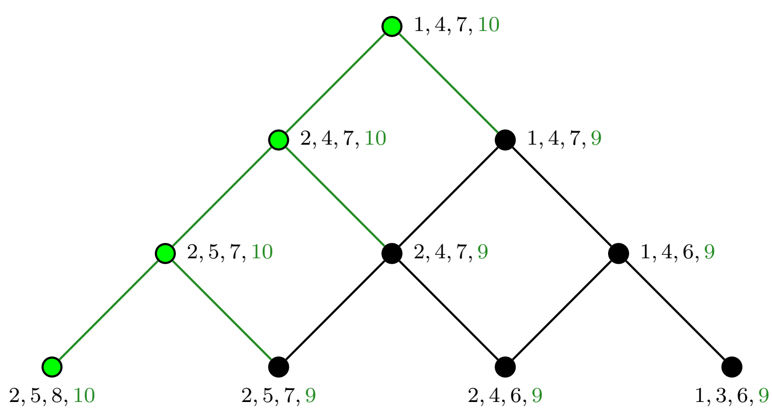


Figure 4.6: $\mathscr{I}(P_{10}) \cong \mathfrak{L}_3$.

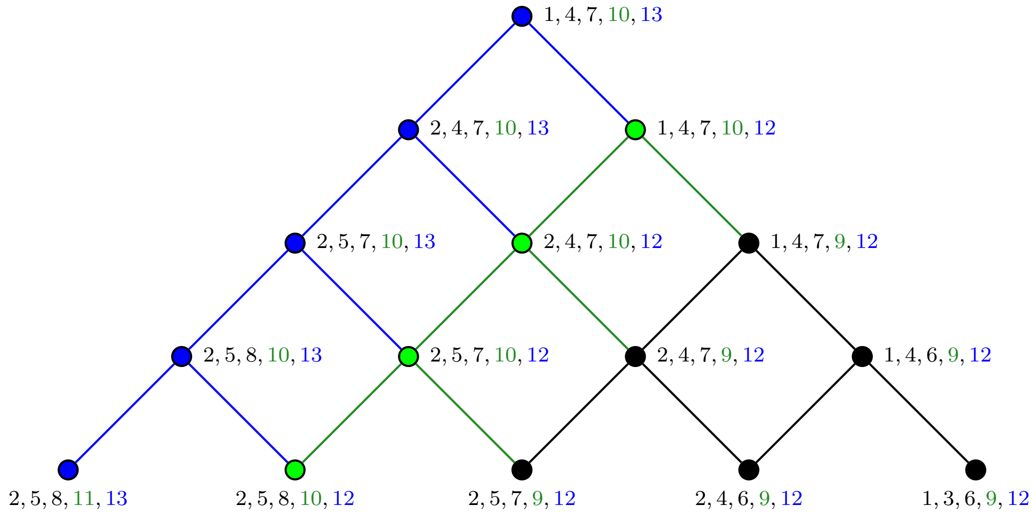


Figure 4.7: $\mathscr{I}(P_{13}) \cong \mathfrak{L}_4$.

Lemma 4.4. *For $k \geq 1$, $\mathscr{I}(P_{3k+1}) \cong \mathfrak{L}_k$.*

Proof. We proceed by induction on k. The base cases with $k \in \{1, 2, 3, 4\}$ are given explicitly above in Figures 4.4 - 4.7. Assume that $\mathscr{I}(P_{3k+1}) \cong \mathfrak{L}_k$. We now consider $\mathscr{I}(P_{3k+4})$. Observe that every i-set of P_{3k+4} contains v_{3k+3} or v_{3k+4}.

Let S be some i-set of P_{3k+1}, and define in P_{3k+4} the set $S' = S \cup \{v_{3k+3}\}$. Then S' is both independent and dominating in P_{3k+4}, and since $i(P_{3k+4}) = i(P_{3k+1}) + 1 = k + 2$, S' is an i-set of P_{3k+4} (and corresponds to a vertex of $\mathscr{I}(P_{3k+4})$). Conversely, given an i-set of P_{3k+4} containing v_{3k+3}, say S', the set $S = S' - \{v_{3k+3}\}$ is an i-set of P_{3k+1}. This follows since $N[v_{3k+3}] = \{v_{3k+2}, v_{3k+3}, v_{3k+4}\}$.

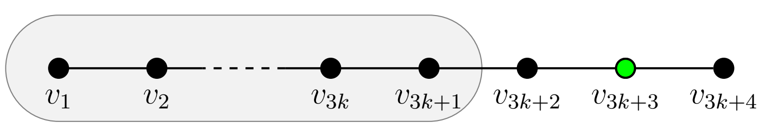


Figure 4.8: P_{3k+4}.

Moreover, if S_1 and S_2 are i-sets of P_{3k+1}, we have that $S_1 \sim S_2$ if and only if $S_1' \sim S_2'$. It follows that $\mathscr{I}(P_{3k+4})$ contains an induced copy of $\mathscr{I}(P_{3k+1}) = \mathfrak{L}_k$. This accounts for $\binom{k+2}{2}$ of the $\binom{k+3}{2}$ vertices in $\mathscr{I}(P_{3k+4})$. These are precisely the i-sets of P_{3k+4} containing v_{3k+3} (these correspond to the black and green vertices in Figure 4.7).

Now, consider some i-set of P_{3k+4} containing v_{3k+4}, say S'. We observe S' must contain either v_{3k+1} or v_{3k+2}. We consider the former first. Let $S = S' - \{v_{3k+4}\}$. Then S dominates $\{v_1, v_2, \ldots, v_{3k+1}\}$, is independent, and has size k. So, S is an i-set of P_{3k+1}.

Conversely, given an i-set of P_{3k+1} containing the vertex v_{3k+1}, say S, the set $S' = S \cup \{v_{3k+4}\}$ is an i-set of P_{3k+4}. Finally, given two such sets S_1 and S_2 and the corresponding S_1' and S_2', we have that $S_1 \sim S_2$ if and only if $S_1' \sim S_2'$. This accounts for $k+1$ i-sets of P_{3k+4}. We emphasize that these sets are all different from the $\binom{k+2}{2}$ i-sets mentioned above.

Let S_1' be an i-set of P_{3k+4} containing v_{3k+4} and not v_{3k+2}. Then, let $S_2' = \{S_1' \cup \{v_{3k+3}\}\} - \{v_{3k+4}\}$. Now S_2' is an independent dominating set of size $k+1$ and clearly $S_1' \sim S_2'$. Further, $S_2 = S_2' - \{v_{3k+3}\}$ is a dominating set of P_{3k+1} containing v_{3k+1}, and so S_2 is a path-tail vertex of $\mathfrak{L}_k$. Thus, S_2 is a path-tail vertex the left-hand side of the copy of $\mathfrak{L}_k$ inside of $\mathfrak{L}_{k+1}$ constructed above. These correspond to the blue vertices matched to the green vertices in Figure 4.7.

The only remaining i-set of P_{3k+4} has both v_{3k+2} and v_{3k+4}. This set is precisely $\{v_2, v_5, \ldots, v_{3k+2}, v_{3k+4}\}$. This is the lower left blue vertex in Figure 4.7, and is adjacent only to $\{v_2, v_5, \ldots, v_{3k+1}, v_{3k+4}\}$, thus forming the (unique) path-tail vertex of $\mathfrak{L}_{k+1}$ of degree 1.

Note there are $\binom{k+2}{2} + (k+1) + 1 = \binom{k+3}{2}$ i-sets, which accounts for all the i-sets of P_{3k+4}. We emphasize that the i-sets of P_{3k+4} containing v_{3k+4} are precisely the path-tail vertices of $\mathfrak{L}_{k+1}$ (down the left-hand side), verifying the invariant for $k+1$. The result follows by induction. ■

Bringing Lemmas 4.3 and 4.4 together yields the following theorem.

Theorem 4.5. *For $n \geq 1$, $k \geq 0$, and $\mathfrak{L}_k$ as defined in Section 4.2,*

$$\mathscr{I}(P_n) = \begin{cases} K_1 & \text{if } n = 3k, \\ \mathfrak{L}_k & \text{if } n = 3k+1, \\ P_{k+2} & \text{if } n = 3k+2. \end{cases}$$

4.3 The Number of i-Sets of Cycles

Our methods for counting the i-sets of cycles are similar to paths, but with an added complication. Although one might be tempted to use the equation in Equation (4.1) (restated here for reference) with the constraints $1 \leq x_j \leq 2$ for $j \in \{1, \ldots, t\}$, this method does not take the rotational symmetries of the i-sets of C_n into account:

$$x_1 + \cdots + x_t = r, \text{ where } 0 \leq x_1, x_t \leq 1, \text{ and } 1 \leq x_j \leq 2 \text{ for } j \in \{2, \ldots, t-1\}. \quad (4.3)$$

For example, it will count only one i-set of C_n, $n \equiv 0 \pmod 3$, whereas C_n has three i-sets in this case. Moreover, different i-sets have different types of rotational symmetries, and so we cannot simply proceed as above and then multiply the answer by a constant.

Let $e = v_0 v_{n-1}$ be an edge of C_n and consider $P_n = C_n - e = (v_0, \ldots, v_{n-1})$. Any i-set X of C_n corresponds to either an i-set of P_n or to a "near i-set of P_n", in which

(i) v_2 is the first vertex and v_{n-1} is the last vertex of X on P_n, or

(ii) v_0 is the first vertex and v_{n-3} is the last vertex of X on P_n.

These three varieties of i-sets are pairwise disjoint; therefore, we count the number of i-sets by using Equation (4.2) with different conditions on the integer variables. The number of i-sets of C_n equals the sum of the number of integer solutions to the following equations, where r, t and x_j are defined as in Section 4.1:

$$x_1 + \cdots + x_t = r, \text{ where } x_1 = 0 \text{ and } 1 \leq x_j \leq 2 \text{ for } j \in \{2, \ldots, t\}, \quad (4.4)$$

$$x_1 + \cdots + x_t = r, \text{ where } x_t = 0 \text{ and } 1 \leq x_j \leq 2 \text{ for } j \in \{1, \ldots, t-1\}, \quad (4.5)$$

$$x_1 + \cdots + x_t = r, \text{ where } x_1 = x_t = 1 \text{ and } 1 \leq x_j \leq 2 \text{ for } j \in \{2, \ldots, t-1\}. \quad (4.6)$$

The corresponding generating function is

$$(x + x^2)^{t-1} + (x + x^2)^{t-1} + x^2(x + x^2)^{t-2} = 2x^{t-1}(1 + x)^{t-1} + x^t(1 + x)^{t-2}. \quad (4.7)$$

- If $n = 3k+1$, then $t = i(P_n)+1 = k+2$ and $r = 2k$. Thus, we require the coefficient of $2k$ in (4.7), that is, in $2x^{k+1}(1+x)^{k+1} + x^{k+2}(1+x)^k$, which is $2\binom{k+1}{k-1} + \binom{k}{k-2} = k(3k+1)/2$.

- If $n = 3k+2$, then $t = k+2$ and $r = 2k+1$. Thus, we require the coefficient of $2k+1$ in $2x^{k+1}(1+x)^{k+1} + x^{k+2}(1+x)^k$, which is $2\binom{k+1}{k} + \binom{k}{k-1} = 3k+2 = n$.

- If $n = 3k+3$, then $t = k+2$ and $r = 2k+2$. Thus, we require the coefficient of $2k+2$ in $2x^{k+1}(1+x)^{k+1} + x^{k+2}(1+x)^k$, which is 3.

We summarize this result in the lemma below.

Lemma 4.6. *For $n \geq 3$, the order of $\mathscr{I}(C_n)$ is*

$$|V(\mathscr{I}(C_n))| = \begin{cases} 3 & \text{if } n = 3k \\ k(3k+1)/2 & \text{if } n = 3k+1 \\ n & \text{if } 3k+2. \end{cases}$$

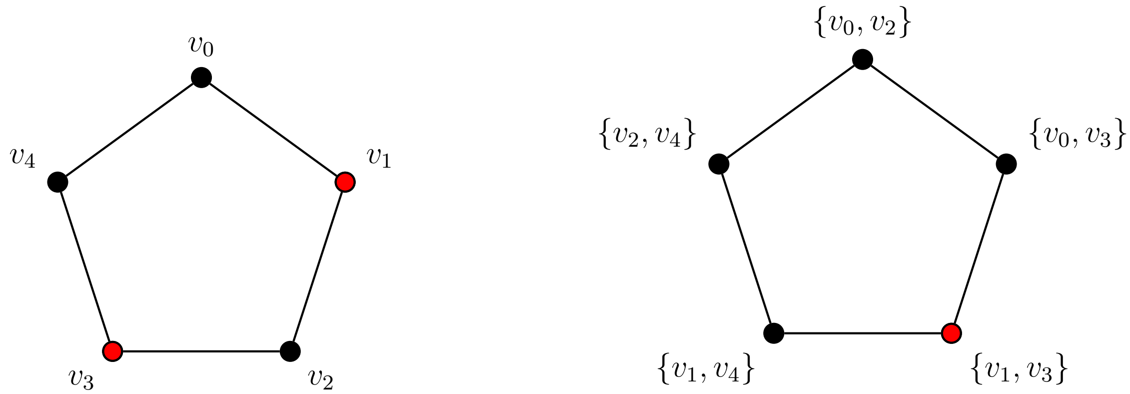
4.4 The i-Graphs of Cycles

Contrary to our conventions in the previous sections on paths, we assume that all cycles have labelled vertices $V(C_n) = (v_0, v_1, \ldots, v_{n-1})$.

Immediately, we discover that some of the i-graphs for C_n are fairly straight-forward. For a C_3, this is a complete graph, and hence $\mathscr{I}(C_3) = \mathscr{I}(K_3) = C_3$. For C_{3k} with $3k \geq 6$, $|V(C_{3k})|$ is divisible by 3, and hence there are three distinct i-sets of C_{3k}, in which each i-set vertex has two non-i-set vertices between it and the next i-set vertex; thus each i-set vertex is frozen, and hence $\mathscr{I}(C_n)$ consists of three singletons.

For C_n with $n \equiv 2 \pmod 3$, say $n = 3k+2$, each i-set of C_n contains exactly one pair of vertices v_{j-1} and v_{j+1} that are separated by exactly one vertex, v_j, not in the i-set (the common neighbour of these two vertices), while all other pairs of consecutive i-set vertices are separated by exactly two vertices not in the i-set. Hence, the i-set has exactly two vertices, namely v_{j-1} and v_{j+1}, that are not frozen, and each of them can slide in only one direction. The vertex v_{j-1} can move to v_{j-2}, and v_{j+1} can move to v_{j+2}. In the former case, the vertex that has two neighbours in the i-set is now v_{j-3}, while in the latter, it is v_{j+3}. As a result, $\mathscr{I}(C_n)$ is 2-regular and since 3 is coprime to n, the sequence v_j, v_{j+3}, v_{j+6}, $\ldots$, of the vertex dominated by two vertices in the i-set will visit each vertex in C_n after n-slides. Thus, each i-set is generated; that is, $\mathscr{I}(C_n)$ is connected. As it is 2-regular and connected, we conclude it is a cycle.

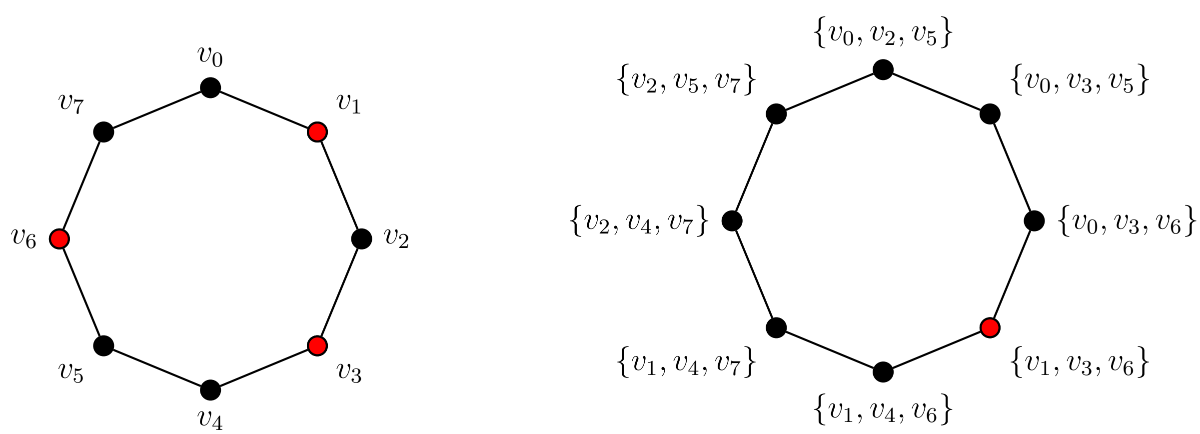
We provide examples of C_5 and C_8 in Figures 4.9 and 4.10 below.



(a) C_5 with the i-set $\{v_1, v_3\}$ in red.

(b) $\mathscr{I}(C_5)$ with the vertex corresponding to the i-set $\{v_1, v_3\}$ in red.

Figure 4.9: C_5 and its i-graph.



(a) C_8 with the i-set $\{v_1, v_3, v_6\}$ in red.

(b) $\mathscr{I}(C_8)$ with the vertex corresponding to the i-set $\{v_1, v_3, v_6\}$ in red.

Figure 4.10: C_8 and its i-graph.

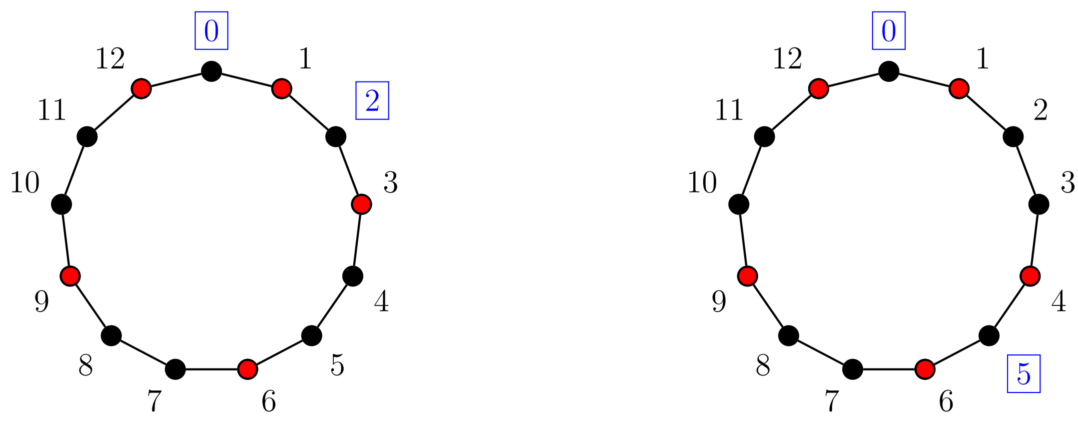
We summarize these results in the lemma below.

Lemma 4.7. *For $n \geq 3$, $k \geq 0$,*

$$\mathscr{I}(C_n) \cong \begin{cases} K_3 & \text{if } n = 3 \\ 3K_1 & \text{if } n = 3k \geq 6 \\ C_n & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Once again, the case that requires deeper analysis is $n = 3k + 1$ for $k \geq 1$. To help us tackle this final case, we first introduce a new notation for referencing the i-sets of cycles.

For cycles C_{3k+1}, given any i-set X, there are exactly two vertices in $C_{3k+1} - X$ that are doubly dominated (that is, are adjacent to two different vertices of X). Rather than referring to the i-set by its elements, which given a large k, could be numerous, we instead refer to the i-sets by these two unique vertices. For clarity, we use a wide-angled bracketed notation when using this convention. Figure 4.11 below illustrates this system: for (a), rather than calling the i-set $X = \{v_1, v_3, v_6, v_9, v_{12}\}$, we refer to it as $\langle 0, 2\rangle = \langle 2, 0\rangle$. Similarly in (b), instead of $Y = \{v_1, v_4, v_6, v_9, v_{12}\}$, we denote the i-set as $\langle 0, 5\rangle$.



(a) $X = \{v_1, v_3, v_6, v_9, v_{12}\}$, denoted as $\langle 0, 2\rangle$.

(b) $Y = \{v_1, v_4, v_6, v_9, v_{12}\}$, denoted as $\langle 0, 5\rangle$.

Figure 4.11: Two i-sets of C_{13}.

With this new labelling in place, we move now to a proposed family of graphs that we have dubbed *bracelet graphs*, $\mathfrak{B}_k$. The vertex set of $\mathfrak{B}_k$ consists of all distinct 2-subsets $\{j, \ell\}$

of $\{0, 1, \ldots, 3k\}$ such that $0 \le j \le 3k$ and $\ell \equiv j + 3s + 2 (\text{mod } 3k+1)$, $s \in \{0, 1, \ldots, k-1\}$. For example, the subsets containing 0 are $\{0, 2\}, \{0, 5\}, \ldots, \{0, 3k-4\}, \{0, 3k-1\}$.

To simplify notation, assume the vertices of C_{3k+1} are labelled $0, 1, \ldots, 3k$ in clockwise order, as illustrated in Figure 4.11. In $\mathfrak{B}_k$, the neighbours of the vertex $\{j, \ell\}$ are described below.

1. Suppose $j - \ell \equiv 2$ or -2 (mod $3k+1$). Assume without loss of generality that j precedes ℓ in clockwise order around C_{3k+1}. (Thus, for the subsets $\{0, 2\}$ and $\{11, 0\}$ of C_{13}, for example, 0 precedes 2, while 11 precedes 0.) With arithmetic performed modulo $3k+1$, the neighbours of $\{j, \ell\}$ are $\{j, \ell+3\}$ and $\{j-3, \ell\}$, and $\{j, \ell\}$ has degree 2 in $\mathfrak{B}_k$. (Thus, $\{0, 2\}$ is adjacent to $\{0, 5\}$ and $\{10, 2\}$ in $\mathfrak{B}_4$, while $\{11, 0\}$ is adjacent to $\{11, 3\}$ and $\{8, 0\}$.)

2. Suppose $j - \ell \not\equiv 2$ or -2 (mod $3k+1$). Then the neighbours of $\{j, \ell\}$ in $\mathfrak{B}_k$ are $\{j-3, \ell\}, \{j+3, \ell\}, \{j, \ell-3\}$ and $\{j, \ell+3\}$ (arithmetic modulo $3k+1$), and $\{j, \ell\}$ has degree 4 in $\mathfrak{B}_k$. (For example, the neighbours, in $\mathfrak{B}_4$, of $\{0, 5\}$ are $\{10, 5\}, \{3, 5\}, \{0, 2\}$ and $\{0, 8\}$).

We emphasize that the sets $\{j, \ell\}$ are unordered. For example, the neighbours of $\{2, 7\}$ in $\mathfrak{B}_3$ are $\{2, 0\} = \{0, 2\}, \{2, 4\}, \{9, 7\} = \{7, 9\}$ and $\{5, 7\}$. Examples of these graphs are given in Figures 4.12, 4.13, 4.14, and 4.15 below, but where the set braces are removed to reduce visual clutter.

The two-number identifiers on the vertices of a bracelet graph and the i-sets of a cycle C_{3k+1} are no coincidence; in the following series of lemmas and observations, we show that they are one and the same. That is, we show that the vertex $\{i, j\}$ in the bracelet graph $\mathfrak{B}_k$ corresponds to the i-set of C_{3k+1} represented by $\langle i, j \rangle$, and hence that $\mathscr{I}(C_{3k+1}) \cong \mathfrak{B}_k$.

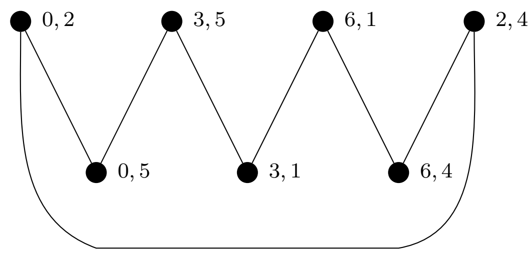


Figure 4.12: $\mathfrak{B}_2 \cong \mathscr{I}(C_7)$.

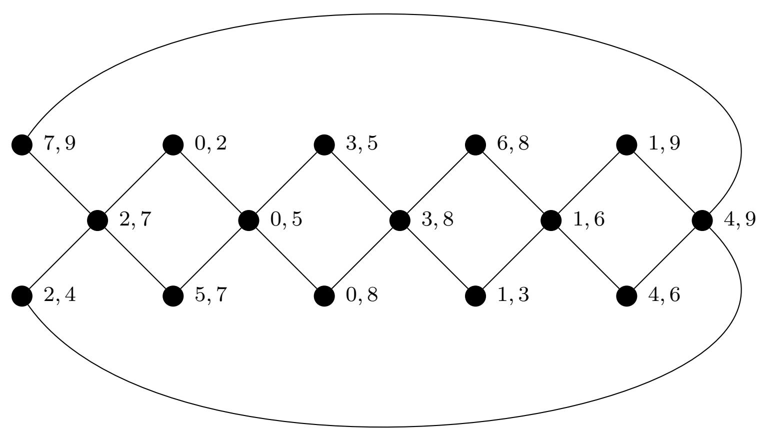


Figure 4.13: $\mathfrak{B}_3 \cong \mathscr{I}(C_{10})$.

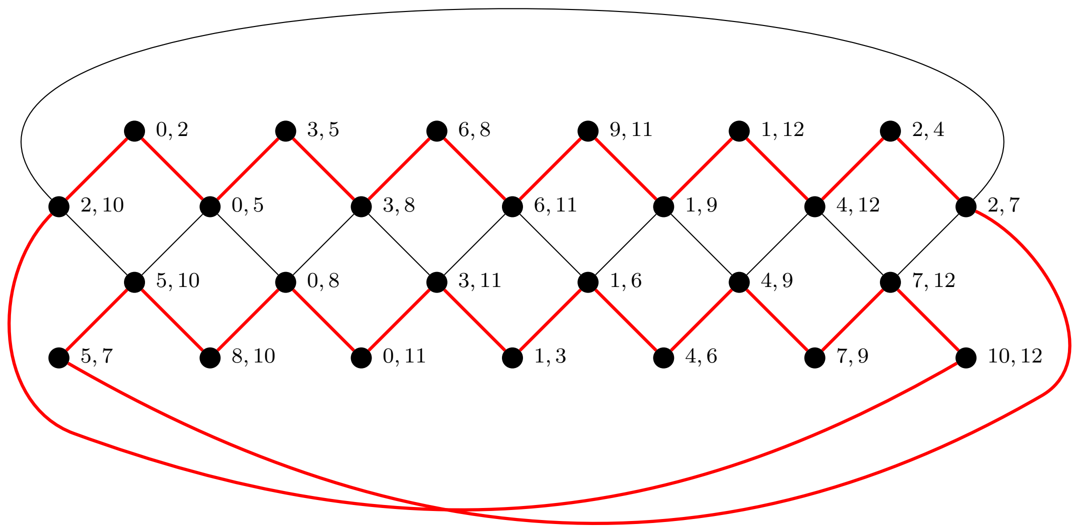


Figure 4.14: $\mathfrak{B}_4 \cong \mathscr{I}(C_{13})$ with Hamiltonian cycle in red.

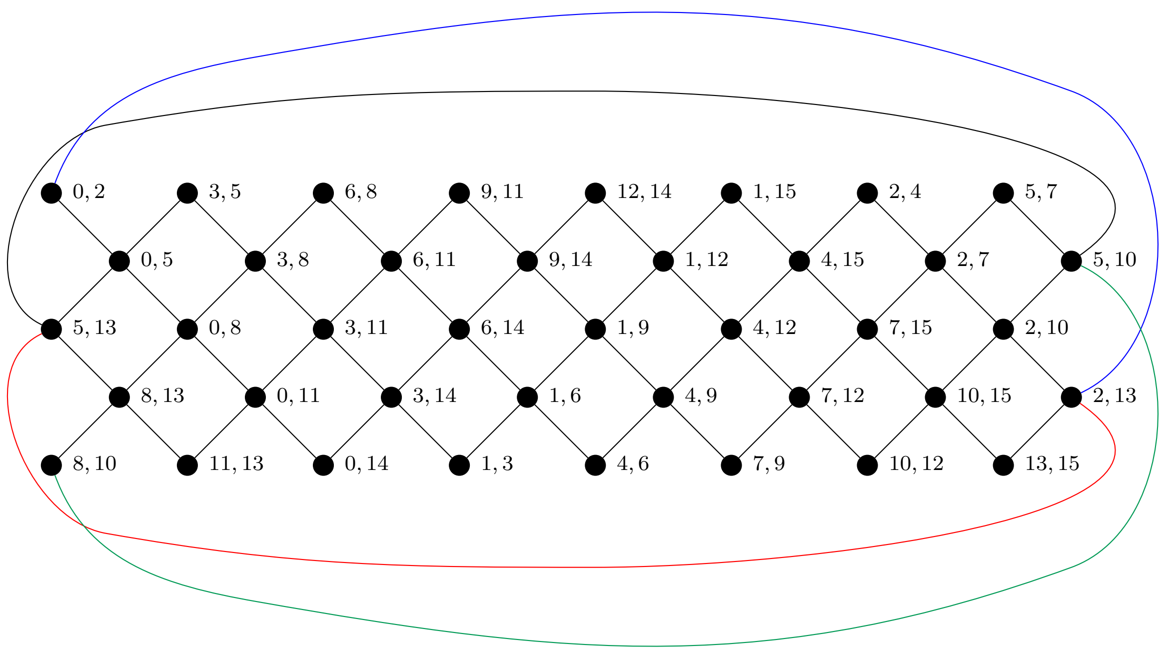


Figure 4.15: $\mathfrak{B}_5 \cong \mathscr{I}(C_{16})$.

Lemma 4.8. *For $k \geq 1$, $\mathscr{I}(C_{3k+1}) \cong \mathfrak{B}_k$.*

To prove Lemma 4.8, we first establish several lemmas, beginning with a formulation of the observation from page 59, for the case $n = 3k+1$. Going forward, we assume that all arithmetic in the i-set notation is performed modulo $3k+1$.

Observation 4.9. *Each i-set X of C_{3k+1} contains exactly two pairs of vertices, say v_{j-1}, v_{j+1} and $v_{\ell-1}, v_{\ell+1}$, such that each pair has a common neighbour in $C_{3k+1} - X$; all other pairs of consecutive vertices of X on C_{3k+1} are separated by two vertices of $C_{3k+1} - X$.*

Lemma 4.10. *Each vertex of $\mathscr{I}(C_{3k+1})$ has degree 2 or 4.*

Proof. For each i-set X of C_{3k+1}, we deduce from Observation 4.9 that X is one of two types (with notation as in Observation 4.9):

> **Type 1:** $\{v_{j-1}, v_{j+1}\} \cap \{v_{\ell-1}, v_{\ell+1}\} \neq \varnothing$; in this case $|\{v_{j-1}, v_{j+1}\} \cap \{v_{\ell-1}, v_{\ell+1}\}| = 1$. Say $j+1 = \ell - 1$. In our wide-angled notation, Type 1 i-sets are of the form $X = \langle j, j+2 \rangle$. Then the subpath $(v_{j-1}, v_j, v_{j+1}, v_{j+2}, v_{j+3})$ of C_{3k+1} has tokens on v_{j-1}, v_{j+1}, and v_{j+3}. A token on v_{j-1} can only slide counterclockwise to v_{j-2}, a token on v_{j+3} can only slide clockwise to v_{j+4}, and a token on v_{j+1} is frozen. Hence X has degree 2 in $\mathscr{I}(C_{3k+1})$ (all other tokens are frozen).
>
> **Type 2:** $\{v_{j-1}, v_{j+1}\} \cap \{v_{\ell-1}, v_{\ell+1}\} = \varnothing$. When X is a Type 2 i-set, and $v_{j-1}, v_{j+1}, v_{\ell-1}$, and $v_{\ell+1}$ occur in this order in a clockwise direction on C_{3k+1}, then each of v_{j-1} and $v_{\ell-1}$ is immediately preceded (counterclockwise) by two vertices of $C_{3k+1} - X$, and each of v_{j+1} and $v_{\ell+1}$ is immediately followed (clockwise) by two vertices of $C_{3k+1} - X$. Hence tokens on v_{j-1} and $v_{\ell-1}$ can slide counterclockwise to v_{j-2} and $v_{\ell-1}$, respectively, while tokens on v_{j+1} and $v_{\ell+1}$ can slide clockwise to v_{j+2} and $v_{\ell+2}$, respectively. Hence X has degree 4 in $\mathscr{I}(C_{3k+1})$.

■

The following is straightforward from the above proof, but we include a proof here for completeness.

Lemma 4.11. *Let X be an i-set of C_{3k+1}.*

(i) If $X = \langle j, j+2 \rangle$ for some $j \in \{0, ..., 3k\}$, then the neighbours of X in $\mathscr{I}(C_{3k+1})$ are $\langle j, j+5 \rangle$ and $\langle j-3, j+2 \rangle$.

(ii) If $X = \langle j, \ell \rangle$ for some $j \in \{0, ..., 3k\}$ and some ℓ, where $j - \ell \not\equiv 2$ or $-2 \pmod{3k+1}$, then $\ell = j + 2 + 3s$, where $s \in \{1, ..., k-2\}$. The neighbours of X in $\mathscr{I}(C_{3k+1})$ are $\langle j, \ell+3 \rangle, \langle j, \ell-3 \rangle, \langle j+3, \ell \rangle$, and $\langle j-3, \ell \rangle$.

Proof. We prove the statement for $j = 0$; the general cases follow by symmetry.

(i) Consider the i-set $X = \langle 0, 2 \rangle$. Then X is a Type 1 i-set (see Lemma 4.10) of C_{3k+1}. Then $\{v_{-1}, v_1, v_3\} \subseteq X$. By Lemma 4.10, the token on v_1 is frozen, the token on v_{-1} can slide to v_{-2}, creating the i-set $\langle -3, 2 \rangle$, and the token on v_3 can slide to v_4, creating the i-set $\langle 0, 5 \rangle$.

(ii) Consider the i-set $X = \langle 0, \ell \rangle$, where $\ell \not\equiv 2$ or $-2 \pmod{3k+1}$. Then X is a Type 2 i-set of C_{3k+1}, and there are tokens on $v_{-1}, v_1, v_{\ell-1}$, and $v_{\ell+1}$. Moreover, since v_0 and v_ℓ are the only vertices that have two neighbours in X, the other vertices, in a clockwise direction from v_1, contained in X are $v_4, v_7, ..., v_{\ell-1}$, which shows that $\ell = 3s + 2$ for some $s \geq 1$. Since X is a Type 2 i-set, $\ell \neq 3k - 1$, that is, $s \neq k - 1$. Hence $\ell = 3s + 2$, where $s \in \{1, ..., k-2\}$.

Similarly to the above, the token on v_{-1} can slide to v_{-2}, forming the i-set $\langle -3, \ell \rangle$, the token on v_1 can slide to v_2, forming the i-set $\langle 3, \ell \rangle$, the token on $v_{\ell-1}$ can slide to $v_{\ell-2}$, forming the i-set $\langle 0, \ell-3 \rangle$, and the token on $v_{\ell+1}$ can slide to $v_{\ell+2}$, forming the i-set $\langle 0, \ell+3 \rangle$.

■

This completes the proof of Lemma 4.8. Finally, combining Lemma 4.7 and Lemma 4.8 reveals the full result for i-graphs of cycles.

Theorem 4.12. *For $n \geq 3$, $k \geq 0$,*

$$\mathscr{I}(C_n) = \begin{cases} K_3 & \text{if } n = 3 \\ 3K_1 & \text{if } n = 3k \geq 6 \\ \mathfrak{B}_k & \text{if } n = 3k+1 \\ C_n & \text{if } n \equiv 2 \ (mod\ 3). \end{cases}$$

4.5 Hamiltonicity of $\mathscr{I}(C_n)$

In some of the figures presented in Section 4.4, a Hamiltonian cycle or path is easily found; in others, it is not. This leads to the problem of determining for which values of n $\mathscr{I}(C_n)$ is Hamiltonian or Hamilton traceable (i.e. has a Hamiltonian path). In most cases, this is not too difficult to determine, as we show next.

Theorem 4.13. *For $n \geq 3$,*

- *If $n \equiv 0 \ (mod\ 3)$, $\mathscr{I}(C_n)$ is disconnected.*
- *If $n \equiv 2 \ (mod\ 3)$, $\mathscr{I}(C_n)$ is trivially Hamiltonian.*
- *If $n \equiv 4 \ (mod\ 6)$, $\mathscr{I}(C_n)$ is neither Hamiltonian nor Hamilton traceable.*

Proof. The first two cases are trivial, and so assume that $n \equiv 4 \pmod 6$. Since $\mathscr{I}(C_4) = \overline{K_2}$, it is non-Hamiltonian; hence, assume $n > 4$, say $n = 6k+4$, $k \geq 1$. With notation as above, we first count the number of vertices $\langle j, \ell \rangle$ of $\mathscr{I}(C_n)$ such that $j \equiv \ell \pmod 2$. For $j = 0$, the set of these vertices is

$$\mathscr{X}_0 = \{\langle 0, 2\rangle, \langle 0, 8\rangle, \ldots, \langle 0, 6k+2\rangle\}$$

and $|\mathscr{X}_0| = k+1$. Similarly,

$$\mathscr{X}_j = \{\langle j, j+2\rangle, \langle j, j+8\rangle, \ldots, \langle j, j+6k+2\rangle\}$$

and

$$\sum_{j=0}^{6k+3} |\mathscr{X}_j| = (k+1)(6k+4).$$

But each vertex $\langle j, \ell \rangle$, $j \equiv \ell \pmod 2$ occurs in exactly two sets, namely $\mathscr{X}_j$ and $\mathscr{X}_\ell$. Hence,

$$\left| \bigcup_{j=0}^{6k+3} \mathscr{X}_j \right| = (k+1)(3k+2).$$

Since

$$V(\mathscr{I}(C_n)) = \frac{n(n-1)}{6} = (3k+2)(2k+1),$$

$\mathscr{I}(C_n)$ has $k(3k+2)$ vertices $\langle j, \ell \rangle$ such that $j \not\equiv \ell \pmod 2$.

We show next that each vertex $\langle j, \ell \rangle$ such that $j \equiv \ell \pmod 2$ is adjacent only to vertices $\langle j', \ell' \rangle$ such that $j' \not\equiv \ell' \pmod 2$, and vice versa.

Let $\langle j, \ell \rangle$ be any vertex of $\mathscr{I}(C_n)$ such that $j \equiv \ell \pmod 2$. Then, with arithmetic in the subscripts performed modulo n,

$$N(\langle j, \ell \rangle) = \begin{cases} \{\langle j, \ell+3 \rangle, \langle j, \ell-3 \rangle, \langle j+3, \ell \rangle, \langle j-3, \ell \rangle\} & \text{if } \ell - j \not\equiv \pm 2 \pmod n \\ \{\langle j, \ell+3 \rangle, \langle j-3, \ell \rangle\} & \text{if } \ell - j \equiv 2 \pmod n \\ \{\langle j, \ell-3 \rangle, \langle j+3, \ell \rangle\} & \text{if } \ell - j \equiv -2 \pmod n, \end{cases}$$

where, since n is even, $j \not\equiv \ell \pm 3 \pmod 2$ and $\ell \not\equiv j \pm 3 \pmod 2$. Hence each vertex $\langle j, \ell \rangle$ of $\mathscr{I}(C_n)$ such that $j \equiv \ell \pmod 2$ is adjacent only to vertices $\langle j', \ell' \rangle$ such that $j' \not\equiv \ell' \pmod 2$.

Therefore, $\mathscr{I}(C_n)$ is bipartite with $(k+1)(3k+2)$ vertices in one partite set, and $k(3k+2)$ in the other. Since the cardinalities of the partite sets differ by more than one, the result follows. ■

The case for $n \equiv 1 \pmod 6$ is more complicated. From Figures 4.12 and 4.14 given above, $\mathscr{I}(C_7)$ and $\mathscr{I}(C_{13})$ are Hamiltonian: $\mathscr{I}(C_7)$ trivially so, and for $\mathscr{I}(C_{13})$, illustrated in red. For $n \equiv 1 \pmod 6$ with $n \geq 19$, we claim that $\mathscr{I}(C_n)$ is not Hamiltonian. Consider $\mathscr{I}(C_{19})$ given in Figure 4.16 below. Any Hamiltonian cycle on $\mathscr{I}(C_{19})$ would include all of the vertices of degree 2 and their degree 4 neighbours, as highlighted in red in Figure 4.16. However, this (proper) subset of vertices induces a cycle in $\mathscr{I}(C_{19})$, and so $\mathscr{I}(C_{19})$ is not Hamiltonian. A similar argument follows for larger n with $n \equiv 1 \pmod 6$, as we show next.

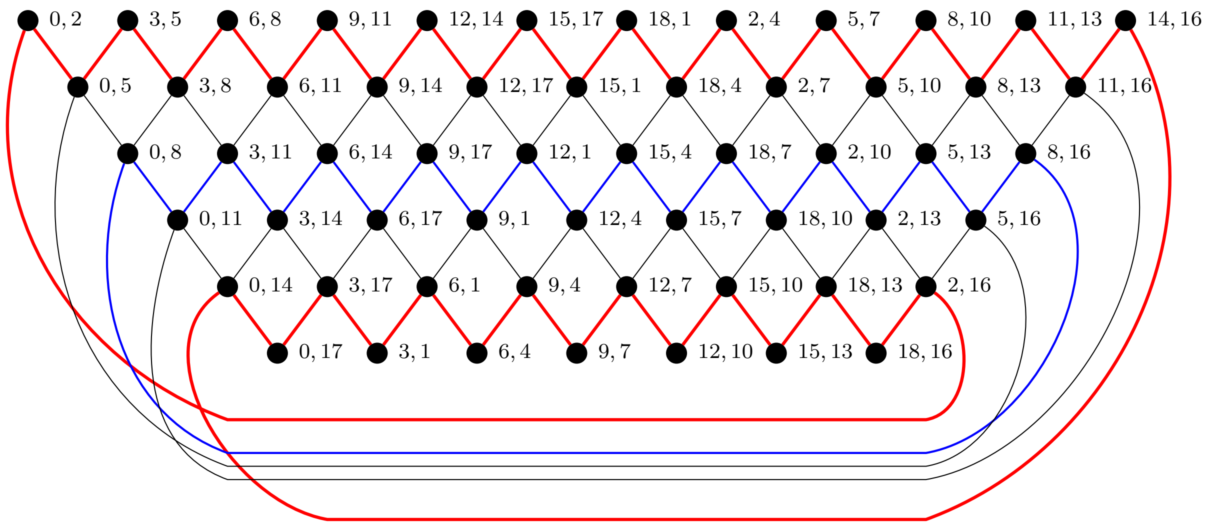


Figure 4.16: $\mathscr{I}(C_{19})$.

Theorem 4.14. *If $n \geq 19$ and $n \equiv 1 \pmod 6$, then $\mathscr{I}(C_n)$ is not Hamiltonian.*

Proof. As shown in Lemma 4.11 (i), the neighbours of the i-set $X = \langle 0, 2 \rangle$ in $\mathscr{I}(C_{6k+1})$ are $\langle -3, 2 \rangle$ and $\langle 0, 5 \rangle$. On the other hand, the neighbours of $\langle 0, 5 \rangle$ are $\langle 0, 2 \rangle$, $\langle 0, 8 \rangle$, $\langle 3, 5 \rangle$ and $\langle -3, 5 \rangle$. Note that $\langle 0, 2 \rangle$ and $\langle 3, 5 \rangle$ are Type 1 vertices while $\langle 0, 8 \rangle$ and $\langle -3, 5 \rangle$ are Type 2. Similarly, $\langle -3, 2 \rangle$ has Type 1 neighbours $\langle 0, 2 \rangle$ and $\langle -3, -1 \rangle$, and Type 2 neighbours $\langle -3, 5 \rangle$ and $\langle -6, 2 \rangle$. A similar remark holds for any i-set $\langle j, j+5 \rangle$, $j \in \{0, ..., 3k\}$. We call these i-sets *Type 2a* i-sets.

Consider the i-set $X = \langle 0, 2 + 3s \rangle$, where $s \in \{2, ..., 2k - 3\}$ (hence $k \geq 3$). As proved in Lemma 4.11 (ii), the neighbours of X in $\mathscr{I}(C_{6k+1})$ are the Type 2 i-sets

$$\langle 0, 2 + 3(s+1) \rangle, \langle 0, 2 + 3(s-1) \rangle, \langle 3, 2 + 3s \rangle, \langle -3, 2 + 3s \rangle.$$

Observe that for the given range of s, these are all Type 2 i-sets. A similar remark holds for any i-set $\langle j, j+2+3s\rangle$, where $s \in \{2, ..., k-3\}$. We call these i-sets *Type 2b* i-sets.

Therefore there are exactly $6k+1$ Type 1 i-sets of C_{6k+1} and the same number of Type 2a i-sets, and each Type 1 i-set has exactly two neighbours in $\mathscr{I}(C_{6k+1})$, both of which are Type 2a i-sets, and, conversely, each Type 2a i-set has two Type 1 neighbours in $\mathscr{I}(C_{6k+1})$. We deduce that the subgraph of $\mathscr{I}(C_{6k+1})$ induced by its Type 1 vertices and their neighbours consists of only Type 1 and Type 2a vertices, and is 2-regular. Hence if $\mathscr{I}(C_{6k+1})$ has Type 2b vertices, that is, if $k \geq 3$, then $\mathscr{I}(C_{6k+1})$ is non-Hamiltonian. ■

Traceability of $\mathscr{I}(C_{6k+1})$, where $k \geq 3$.

When $k \geq 3$ and $n = 6k+1$, we have previously established that $\mathscr{I}(C_n)$ has no Hamiltonian cycle. We now instead prove that it has a Hamilton path.

Theorem 4.15. *For $n = 6k+1$, $k \geq 3$, $\mathscr{I}(C_n)$ is Hamilton traceable.*

Proof. Say $k \geq 3$, and consider C_{6k+1} and an i-set $\langle j, \ell\rangle$. The distance from j to ℓ on C_{6k+1} is the length of the shorter path, thus $d(j,\ell) \in \{2, 5, 8, \ldots, 3k-1\}$.

- If k is even, say $k = 2k'$, then, in $C_{12k'+1}$, we see that $d(j,\ell) \in \{2, 5, 8, \ldots, 6k'-1\}$. This set contains an equal number of even and odd integers.
- If k is odd, say $k = 2k'+1$, then, in $C_{12k'+7}$, we see that $d(j,\ell) \in \{2, 5, 8, \ldots, 6k'+2\}$. This set contains more even than odd integers.

Consider, again, the subgraph of $\mathscr{I}(C_{6k+1})$ induced by its Type 1 vertices and their neighbours, which is 2-regular (as above) and has order $2(6k+1)$. Denote this graph by $\mathcal{H}_{2,5}$. In Figure 4.16, $\mathcal{H}_{2,5}$ is the subgraph induced by the vertices:

$$\{\langle 0,2\rangle, \langle 0,5\rangle, \langle 3,5\rangle, \langle 3,8\rangle, \langle 6,8\rangle \ldots, \langle 2,16\rangle\}.$$

We argue below that $\mathcal{H}_{2,5}$ is in fact connected; that is, $\mathcal{H}_{2,5}$ is a cycle.

Note that, with arithmetic modulo $6k+1$,

$$\mathcal{W}_{2,5} = \langle 0,2\rangle, \langle 0,2+3\rangle, \langle 3,2+3\rangle, \langle 3,2+2\cdot 3\rangle, \langle 2\cdot 3,2+2\cdot 3\rangle, \\ \ldots, \langle 3x,2+3x\rangle, \langle 3x,2+3(x+1)\rangle \ldots$$

is a walk in $\mathcal{H}_{2,5}$. When does $\langle 0,2\rangle$ recur? There are two cases to consider, each having two subcases.

Case 1: When $\langle 0,2\rangle$ recurs for the first time, an even cycle is formed. Then $\langle 0,2\rangle = \langle 3x,2+3x\rangle$ for some integer x. Since $\langle 0,2\rangle = \langle 2,0\rangle$, there are two subcases.

Case 1.1: $3x \equiv 2 \pmod{6k+1}$ and $2+3x \equiv 0 \pmod{6k+1}$, that is, $3x \equiv -2 \pmod{6k+1}$. But then $2 \equiv -2 \pmod{6k+1}$, which is impossible because $k > 0$.

Case 1.2: $3x \equiv 0 \pmod{6k+1}$ and $2+3x \equiv 2 \pmod{6k+1}$. Since $\gcd(3, 6k+1) = 1$, $x \equiv 0 \pmod{6k+1}$. Then the first time $\langle 0,2\rangle$ recurs on $\mathcal{W}_{2,5}$ is therefore when $x = 6k+1$. It follows that $\mathcal{W}_{2,5}$ contains the cycle

$$\langle 0,2\rangle, \langle 0,2+3\rangle, \langle 3,2+3\rangle, \langle 3,2+2\cdot 3\rangle, \langle 2\cdot 3,2+2\cdot 3\rangle, \ldots, \langle 0,2\rangle$$

of length $2(6k+1) = |V(\mathcal{H}_{2,5})|$. Therefore $\mathcal{H}_{2,5} \cong C_{12k+2}$.

Case 2: When $\langle 0,2\rangle$ recurs for the first time, an odd cycle is formed. Then $\langle 0,2\rangle = \langle 3x,2+3(x+1)\rangle$ for some integer x. Again there are two subcases.

Case 2.1: $3x \equiv 2 \pmod{6k+1}$ and $2+3(x+1) \equiv 0 \pmod{6k+1}$. Then $7 \equiv 0 \pmod{6k+1}$, which is impossible because $k > 1$.

Case 2.2: $3x \equiv 0 \pmod{6k+1}$ and $2+3(x+1) \equiv 2 \pmod{6k+1}$. This is likewise impossible.

Therefore, we conclude in all cases that $\mathcal{H}_{2,5} \cong C_{12k+2}$.

In general, for fixed $\ell \equiv 2 \pmod 6$ and $2 \le \ell \le 3k-1$, denote the subgraph of $\mathscr{I}(C_{6k+1})$ induced by the i-sets of the form $\langle j, j+\ell\rangle$ and $\langle j, j+\ell+3\rangle$, where $j \in \{0,...,6k\}$, by $\mathcal{H}_{\ell,\ell+3}$.

Then

$$\mathcal{W}_{\ell,\ell+3} = \langle 0,\ell\rangle, \langle 0,\ell+3\rangle, \langle 3,\ell+3\rangle, \langle 3,\ell+2\cdot 3\rangle, \langle 2\cdot 3,\ell+2\cdot 3\rangle,$$
$$\ldots, \langle 3x,\ell+3x\rangle, \langle 3x,\ell+3(x+1)\rangle \ldots$$

is a walk in $\mathcal{H}_{\ell,\ell+3}$. When does $\langle 0,\ell\rangle$ recur? Again there are two cases to consider.

Case 3: When $\langle 0,\ell\rangle$ recurs for the first time, an even cycle is formed. Then $\langle 0,\ell\rangle = \langle 3x,\ell+3x\rangle$ for some integer x. Since $\langle 0,\ell\rangle = \langle \ell,0\rangle$, there are two subcases.

Case 3.1: $3x \equiv \ell \pmod{6k+1}$ and $\ell+3x \equiv 0 \pmod{6k+1}$, that is, $3x \equiv -\ell \pmod{6k+1}$. Then $2\ell \equiv 0 \pmod{6k+1}$ and, since $\gcd(2,6k+1) = 1$, $\ell \equiv 0 \pmod{6k+1}$. Since $2 \le \ell \le 3k-1$, this is impossible.

Case 3.2: $3x \equiv 0 \pmod{6k+1}$ and $\ell+3x \equiv \ell \pmod{6k+1}$, i.e., $x \equiv 0 \pmod{6k+1}$. Therefore the first time $\langle 0,\ell\rangle$ recurs on $\mathcal{W}_{\ell,\ell+3}$ is when $x = 6k+1$. It follows that $\mathcal{W}_{\ell,\ell+3}$ contains the cycle

$$\langle 0,\ell\rangle, \langle 0,\ell+3\rangle, \langle 3,\ell+3\rangle, ..., \langle 3(x-1),\ell+3x\rangle, \langle 0,\ell\rangle$$

of length $2(6k+1) = |V(\mathcal{H}_{\ell,\ell+3})|$. Therefore $\mathcal{H}_{\ell,\ell+3} \cong C_{12k+2}$.

Case 4: When $\langle 0,\ell\rangle$ recurs for the first time, an odd cycle is formed. Then $\langle 0,\ell\rangle = \langle 3x,\ell+3(x+1)\rangle$ for some integer x.

Case 4.1: $3x \equiv \ell \pmod{6k+1}$ and $\ell+3(x+1) \equiv 0 \pmod{6k+1}$. Then $2\ell+3 \equiv 0 \pmod{6k+1}$, or $2\ell \equiv -3 \equiv 6k-2 \pmod{6k+1}$. This implies that $\ell \equiv 3k-1 \pmod{6k+1}$ and the restrictions on ℓ show that $\ell = 3k-1$. That is, there is exactly one value of ℓ for which these congruences hold. Moreover, $6x+3 \equiv 0 \pmod{6k+1}$, i.e., $2x \equiv -1 \equiv 6k \pmod{6k+1}$. Hence $x = 3k$.

Therefore $\mathcal{W}_{\ell,\ell+3}$ contains the cycle

$$\langle 0,\ell\rangle, \langle 0,\ell+3\rangle, \langle 3,\ell+3\rangle, ..., \langle 3x,\ell+3x\rangle, \langle \ell,0\rangle$$

of length $2x+1 = 6k+1$. Consider the distances $d(0, 3k-1)$ and $d(0, 3k+2)$ on C_{6k+1}. Observe that $d(0,\ell) = d(0, 3k-1) = 3k-1$ and $d(0, \ell+3) = d(0, 3k+2) = 6k+1-(3k+2) = 3k-1$. It follows that $\mathcal{H}_{\ell,\ell+3}$ consists of all i-sets $\langle p, q\rangle$ such that, on C_{6k+1}, $d(p,q) = 3k-1$, and there are exactly $6k+1$ such i-sets. Hence $|V(\mathcal{H}_{\ell,\ell+3})| = 6k+1$, that is, $\mathcal{H}_{\ell,\ell+3}$ is exactly the cycle $\langle 0,\ell\rangle, \langle 0, \ell+3\rangle, \langle 3, \ell+3\rangle, ..., \langle 3x, \ell+3x\rangle, \langle \ell, 0\rangle$.

Since $\ell = 3k-1$ and $\ell \equiv 2 \pmod 6$, we deduce that k is odd; say $k = 2k'+1$, where $k' \geq 1$. Then $\ell = 6k'+2$. The smallest cycle C_{6k+1} where $k \geq 3$ for which this case occurs is C_{19}, in which case $\ell = 8$ (see Figure 4.16).

Case 4.2: $3x \equiv 0 \pmod{6k+1}$ and $\ell + 3(x+1) \equiv \ell \pmod{6k+1}$. From the first congruence, $x \equiv 0 \pmod{6k+1}$, and so, from the second congruence, $3 \equiv 0 \pmod{6k+1}$. This is impossible.

To summarize: Fix $\ell \in \{2, 5, 8, ..., 3k-1\}$.

- If $k = 2k'$, then by (1) and Cases 3 and 4, the subgraphs $\mathcal{H}_{2,5}, ..., \mathcal{H}_{6k'-4,6k'-1}$ of $\mathscr{I}(C_{12k'+1})$ all have order $24k'+2$, and $\mathcal{H}_{2,5} \cong \cdots \cong \mathcal{H}_{6k'-4,6k'-1} \cong C_{24k'+2}$.
- If $k = 2k'+1$, then by (2) and Cases 3 and 4, the subgraphs $\mathcal{H}_{2,5}, ..., \mathcal{H}_{6k'-4,6k'-1}$ of $\mathscr{I}(C_{12k'+7})$ all have order $24k'+14$, and $\mathcal{H}_{2,5} \cong \cdots \cong \mathcal{H}_{6k'-4,6k'-1} \cong C_{24k'+14}$. However, the subgraph $\mathcal{H}_{6k'+2,6k'+5}$ of $\mathscr{I}(C_{12k'+7})$ has order $12k'+7$ and, as for the subgraph $\mathcal{H}_{8,11}$ of $\mathscr{I}(C_{19})$, is a cycle, that is, $\mathcal{H}_{6k'+2,6k'+5} \cong C_{12k'+7}$.

In either case, each vertex of $\mathscr{I}(C_{6k+1})$ belongs to $H_{\ell,\ell+3}$ for some $\ell \equiv 2 \pmod 6$.

Connecting the Subgraphs $\mathcal{H}_{\ell,\ell+3}$ to Form a Hamilton Path of $\mathscr{I}(C_{6k+1})$

Denote the subgraph of $\mathscr{I}(C_{6k+1})$ that consists of the union of the cycles $\mathcal{H}_{\ell,\ell+3}$ by $\mathcal{H}$, and the set of edges of $\mathscr{I}(C_{6k+1})$ that do not belong to $\mathcal{H}$ by $\mathcal{E}$. Since each vertex of $\mathscr{I}(C_{6k+1})$ belongs to $H_{\ell,\ell+3}$ for some $\ell \equiv 2 \pmod 6$, $\mathcal{H}$ is a spanning subgraph of $\mathscr{I}(C_{6k+1})$. We consider two cases, depending on whether k is even or odd.

Case 1: $k = 2k'$. Then

$$\mathcal{P} : \langle 0,2\rangle , \langle 0,5\rangle , \langle 0,8\rangle , ..., \langle 0,6k'-4\rangle , \langle 0,6k'-1\rangle$$

is a path in $\mathscr{I}(C_{12k'+1})$ whose edges belong alternately to $\mathcal{H}$ and to $\mathcal{E}$, beginning with the edge $(\langle 0,2\rangle , \langle 0,5\rangle)$ in $\mathcal{H}_{2,5}$ and ending with the edge $(\langle 0,6k'-4\rangle , \langle 0,6k'-1\rangle)$ in $\mathcal{H}_{6k'-4,6k'-1}$. Moreover, $\mathcal{P}$ contains at least one vertex of each $\mathcal{H}_{\ell,\ell+3}$. Let $\mathcal{T}$ be the subgraph of $\mathscr{I}(C_{12k'+1})$ obtained by deleting all edges of $\mathcal{P}$ from $\mathcal{H}$, then adding the edges of $E(\mathcal{P}) \cap \mathcal{E}$. Observe that $\mathcal{T}$ is a spanning subgraph of $\mathscr{I}(C_{12k'+1})$. Since the edges of $\mathcal{P}$ were alternately deleted and added, all vertices of $\mathcal{T}$ have degree 2, except for $\langle 0,2\rangle$ and $\langle 0,6k'-1\rangle$, which have degree 1. Also, by construction, $\mathcal{T}$ is connected. Therefore, $\mathcal{T}$ is a Hamiltonian path of $\mathscr{I}(C_{12k'+1})$.

Case 2: $k = 2k'+1$. The argument is similar.

Therefore, in all cases $\mathscr{I}(C_{6k+1}), k \geq 3$ is Hamilton traceable. ■

This completes the characterization of which cycles have Hamiltonian or Hamiltonian traceable i-graphs.

Chapter 5

Constructions of i-Graphs Using Complements

When visualizing the connections between the i-sets of a graph G, it is sometimes advantageous to consider its complement, $\overline{G}$, instead. From a human perspective, it is curiously easier see to which vertices a vertex v is adjacent, rather than to which vertices v is non-adjacent. This is especially true when $i(G) = 2$ or 3, when we may interpret the adjacency of i-sets of G as the adjacencies of edges and triangles (i.e. K_3), respectively, in $\overline{G}$.

In the following sections, we examine how the use of graph complements can be exploited to construct the i-graph seeds for certain classes of line graphs, theta graphs, and maximal planar graphs. In Sections 5.2 and 5.3 we further confirm that these theta graph constructions complete the full characterization of theta graphs that are i-graphs, by proving that each of the remaining theta graphs are not realizable as i-graphs.

5.1 Line Graphs and Claw-free Graphs

To begin, consider a graph G with $i(G) = 2$ and where $X = \{u, v\}$ is an i-set of G. In $\overline{G}$, u and v are adjacent, so X is represented as the edge uv. Moreover, no other vertex w is adjacent to both vertices of X in $\overline{G}$; otherwise, $\{u, v, w\}$ is independent in G, contrary to X being an i-set.

Consider now the line graph of $\overline{G}$, $L(\overline{G})$. If $X = \{u, v\}$ is an i-set of G, then $e = uv$ is an edge of $\overline{G}$ and hence e is a vertex of $L(\overline{G})$. Thus, the i-sets of G correspond to a subset of the vertices of $L(\overline{G})$. In the case where $\overline{G}$ is triangle-free (that is, G has no independent sets of cardinality three or more, i.e. G is well-covered), these i-sets of G are exactly the vertices of $L(\overline{G})$. Now suppose Y is an i-set of G adjacent to X; say, $X \overset{uw}{\sim} Y$, so that $Y = \{v, w\}$. Then, in $\overline{G}$, $f = vw$ is an edge, and so in $L(\overline{G})$, $f \in V(L(\overline{G}))$. Since e and f are both incident with v in $\overline{G}$, $ef \in E(L(\overline{G}))$. That is, for i-sets X and Y of a well-covered graph G, $X \sim Y$ if and only if X and Y correspond to adjacent vertices in $L(\overline{G})$. Thus $\mathscr{I}(G) \cong L(\overline{G})$ for well-covered graphs G with $i(G) = \alpha(G) = 2$.

In the example illustrated in Figure 5.1 below, $\mathcal{H}$ is the house graph, where $X = \{a, c\}$ and $Y = \{c, e\}$ are i-sets with $X \overset{ae}{\sim} Y$. In $L(\overline{\mathcal{H}})$, the two vertices in each of these i-sets are likewise adjacent.

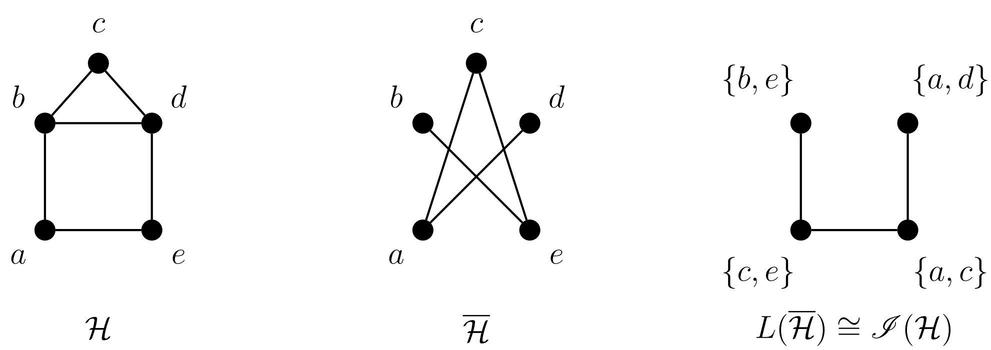


Figure 5.1: The complement and line graphs complement of the well-covered house graph $\mathcal{H}$.

Before continuing with the theme of i-graph realizability, we make the following observation on the order of $\mathscr{I}(G)$ when $i(G) = 2$.

Proposition 5.1. *Let G be a graph of order n with $i(G) = 2$. Then G has at most $\frac{1}{2}n(n-1) - \frac{1}{2}\sum_{v \in V(G)} deg(v)$ distinct i-sets. That is, $|V(\mathscr{I}(G))| \leq \frac{1}{2}n(n-1) - \frac{1}{2}\sum_{v \in V(G)} \deg(v)$.*

Proof. Since $i(G) = 2$, each i-set of G corresponds to an edge of $\overline{G}$. So, $|V(\mathscr{I}(G))| \leq |E(\overline{G})| = \binom{n}{2} - |E(G)| = \frac{1}{2}n(n-1) - \frac{1}{2}\sum_{v \in V(G)} \deg(v)$. ■

This connection between graphs with $i(G) = 2$ and line graphs helps us not only understand the structure of $\mathscr{I}(G)$, but also lends itself towards some interesting realizability results. We follow this thread for the remainder of this section, and build towards determining the i-graph realizability of line graphs and claw-free graphs.

Lemma 5.2. *The line graph of a connected graph G of order at least four contains $\mathfrak{D}$ as an induced subgraph if and only if G contains a triangle.*

Proof. Suppose G contains a triangle with vertices a, b, and c, and let $d \notin \{a, b, c\}$ be a vertex adjacent to (say) a. Then $\{ab, ac, ad\}$ and $\{ab, ac, bc\}$ are triangles in $L(G)$. Since $\{b, c\} \cap \{a, d\} = \varnothing$ in G, bc is nonadjacent to ad in $L(G)$. That is, $\mathfrak{D}$ is an induced subgraph of $L(G)$.

Conversely, suppose $L(G)$ contains $\mathfrak{D}$ as an induced subgraph. Then, G has at least four edges, and hence $G \ncong K_3$. Since the graph H in Figure 5.2 (i.e. "the paw" or the "3-pan") has $L(H) \cong \mathfrak{D}$, it follows from the uniqueness of line graphs (see [42]) that G contains H as a subgraph. ■

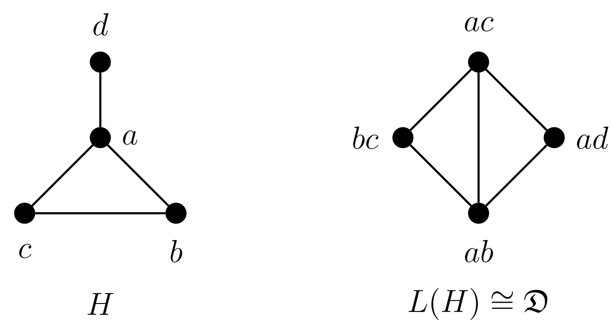


Figure 5.2: The "paw" H with $L(H) \cong \mathfrak{D}$.

Theorem 5.3. *Let H be a connected line graph. Then H is an i-graph if and only if H is $\mathfrak{D}$-free.*

Proof. Suppose H is an i-graph. By Proposition 3.8 and Corollary 3.23, H is $\mathfrak{D}$-free.

Conversely, suppose that H is $\mathfrak{D}$-free. If H is complete, then H is the i-graph of itself. So, assume that H is not complete. Say H is the line graph of some graph F, where we may assume F has no isolated vertices (as isolated vertices do not affect line graphs). Since H is $\mathfrak{D}$-free and connected, F has no triangles by Lemma 5.2. Since F has edges (which it does since H exists), $\alpha(\overline{F}) \leq 2$. Moreover, as F is connected, $\overline{F}$ has no universal vertices, and so $i(\overline{F}) \geq 2$. Thus, $i(\overline{F}) = \alpha(\overline{F})$ and $\overline{F}$ is well-covered. It follows that every edge of F corresponds to an i-set of $\overline{F}$. Since H is the line graph of F, it is the i-graph of $\overline{F}$. ■

Finally, if we examine Beineke's forbidden subgraph characterization for line graphs (see [2]), it is interesting to note that eight of the nine minimal non-line graphs contain an induced $\mathfrak{D}$, and are therefore not i-graphs. The ninth minimal non-line graph is the claw, $K_{1,3}$. Thus, claw-free graphs without an induced $\mathfrak{D}$ are $\mathfrak{D}$-free line graphs, and therefore are i-graphs.

Corollary 5.4. *Let H be a connected claw-free graph. Then H is an i-graph if and only if H is $\mathfrak{D}$-free.*

Proof. If H is i-graph realizable, it is $\mathfrak{D}$-free. Conversely, suppose that H is $\mathfrak{D}$-free. Then by the forbidden subgraph characterization of line graphs [2], H is a line graph. From Theorem 5.3, H is a $\mathfrak{D}$-free line graphs and thus an i-graph. ■

While Theorem 5.3 and Corollary 5.4 reveal the i-graph realizability of many famous graph families (including another construction for cycles, which are connected, claw-free, and $\mathfrak{D}$-free), the realizability problem for graphs containing claws remains unresolved. Moreover, among clawed graphs are the theta graphs which we first alluded to in Section 3.2 as containing three of the small known non-i-graphs. In the next section we apply similar techniques with graph complements to construct all theta graphs that are realizable as i-graphs.

5.2 Theta Graphs From Graph Complements

Consider now a graph G with $i(G) = 3$. Each i-set of G is represented as a triangle (i.e., an induced K_3) in $\overline{G}$. If X and Y are two i-sets of G with $X \overset{uv}{\sim} Y$, then $\overline{G}[X]$ and $\overline{G}[Y]$ are triangles in $\overline{G}$, and have $|X \cap Y| = 2$. Although it is technically the induced subgraphs $\overline{G}[X]$ and $\overline{G}[X]$ that are the triangles of $\overline{G}$, for notational simplicity going forward, we refer to X and Y as triangles. In $\overline{G}$, the triangle X can be transformed into the triangle Y by removing the vertex u and adding in the vertex v (where $u \not\sim_{\overline{G}} v$). Thus, we say that two triangles are *adjacent* if they share exactly one edge (two vertices). Moreover, since two i-sets of a graph G with $i(G) = 3$ are adjacent if and only if their associated triangles in $\overline{G}$ are adjacent, we use the same notation for i-set adjacency in G as triangle adjacency in $\overline{G}$; that is, the notation $X \sim Y$ represents both i-sets X and Y of G being adjacent, and triangles X and Y of $\overline{G}$ being adjacent.

In the following section, we use triangle adjacency to construct *complement seed graphs* for the i-graphs of theta graphs; that is, a graph $\overline{G}$ such that $\mathscr{I}(G)$ is isomorphic to some desired theta graph. Before proceeding with these constructions, we note some observations which will help us with this process.

Observation 5.5. *If a graph $\overline{G}$ has a bridge, then $i(G) \leq 2$.*

Observation 5.6. *A graph G has $i(G) = 2$ if and only if $\overline{G}$ is nonempty and has an edge that does not lie on a triangle.*

If $\overline{G}$ has an edge uv that does not lie on a triangle, then every other vertex is adjacent to at least one of u or v in G. That is, $\{u, v\}$ is independent and dominating in G, and so $i(G) \leq 2$. When building our various seed graphs with $i(G) = 3$, it is therefore necessary to ensure that every edge of the complement $\overline{G}$ belongs to a triangle.

Observation 5.7. *Let G be a graph with $i(G) = 3$. If S with $|S| \geq 4$ is a (possibly non-maximal) clique in $\overline{G}$, then no three-vertex subset of S is an i-set of G.*

Suppose that $S = \{u, v, w, x\}$ is such a clique of $\overline{G}$. Then, for example, x is undominated by $\{u, v, w\}$ in G, and so $\{u, v, w\}$ is not an i-set of G. Conversely, suppose that $\{u, v, w\}$ is a triangle in a graph $\overline{G}$ with $i(G) = 3$. By attaching a new vertex x to all of $\{u, v, w\}$ in $\overline{G}$, we remove $\{u, v, w\}$ as an i-set of G, while keeping all other i-sets of G. This observation proves to be very useful in the constructions in the following section: we now have a technique to eliminate any unwanted triangles in $\overline{G}$ (and hence i-sets of G) that may arise. Notice that this technique is an application of the Deletion Lemma (Lemma 3.22) in $\overline{G}$, instead of the usual G.

The use of triangle adjacency in a graph $\overline{G}$ to determine i-set adjacency in G provides a key technique to finally resolve a question posed in Chapter 3: which theta graphs are

i-graphs? In the following Theorem 5.8, we show that all theta graphs except the seven listed exceptions are i-graphs. The proofs of the lemmas for the affirmative cases make up most of the remainder of Section 5.2, and almost all use this method of complement triangles. The proofs of the lemmas for the seven negative cases are given in Section 5.3.

Theorem 5.8. *A theta graph is an i-graph if and only if it is not one of the seven exceptions listed below:*

$$\Theta\langle 1,2,2\rangle,$$
$$\Theta\langle 2,2,2\rangle, \quad \Theta\langle 2,2,3\rangle, \quad \Theta\langle 2,2,4\rangle, \quad \Theta\langle 2,3,3\rangle, \quad \Theta\langle 2,3,4\rangle,$$
$$\Theta\langle 3,3,3\rangle.$$

Table 5.1 summarizes the cases used to establish Theorem 5.8 and their associated results.

$\Theta\langle j,k,\ell\rangle$	Realizability	Result
$\Theta\langle 1,2,2\rangle$	non-i-graph	$\mathfrak{D}$. Proposition 3.8
$\Theta\langle 1,2,\ell\rangle$, $\ell \geq 3$	i-graph	Lemma 3.21
$\Theta\langle 1,k,\ell\rangle$, $3 \leq k \leq \ell$	i-graph	Lemma 5.11
$\Theta\langle 2,2,2\rangle$	non-i-graph	$K_{2,3}$. Proposition 3.9
$\Theta\langle 2,2,3\rangle$	non-i-graph	κ. Proposition 3.10
$\Theta\langle 2,2,4\rangle$	non-i-graph	Proposition 5.41
$\Theta\langle 2,2,\ell\rangle$, $\ell \geq 5$	i-graph	Lemma 5.15
$\Theta\langle 2,3,3\rangle$	non-i-graph	Proposition 5.42
$\Theta\langle 2,3,4\rangle$	non-i-graph	Proposition 5.43
$\Theta\langle 2,3,\ell\rangle$, $\ell \geq 5$	i-graph	Lemma 5.17
$\Theta\langle 2,4,4\rangle$	i-graph	Lemma 5.19
$\Theta\langle 2,k,5\rangle$, $4 \leq k \leq 5$	i-graph	Lemma 5.21
$\Theta\langle 2,k,\ell\rangle$, $k \geq 4$, $\ell \geq 6$, $\ell \geq k$	i-graph	Lemma 5.23
$\Theta\langle 3,3,3\rangle$	non-i-graph	Proposition 5.44
$\Theta\langle 3,3,4\rangle$	i-graph	Lemma 5.24
$\Theta\langle 3,3,5\rangle$	i-graph	Lemma 5.26
$\Theta\langle 3,3,\ell\rangle$, $\ell \geq 6$	i-graph	Lemma 5.28
$\Theta\langle 3,4,4\rangle$	i-graph	Lemma 5.30
$\Theta\langle 3,4,\ell\rangle$, $\ell \geq 5$	i-graph	Lemma 5.32
$\Theta\langle 3,5,5\rangle$	i-graph	Lemma 5.34
$\Theta\langle 4,4,4\rangle$	i-graph	Lemma 5.36
$\Theta\langle j,k,5\rangle$, $4 \leq j \leq k \leq 5$.	i-graph	Lemma 5.38
$\Theta\langle j,k,\ell\rangle$, $3 \leq j \leq k \leq \ell$, and $\ell \geq 6$.	i-graph	Lemma 5.40

Table 5.1: i-graph realizability of theta graphs.

5.2.1 $\Theta\langle 1, k, \ell\rangle$

We have already seen that the diamond graph $\mathfrak{D} = \Theta\langle 1, 2, 2\rangle$ is not an i-graph (Proposition 3.8) and that the House graph $\mathcal{H} = \Theta\langle 1, 2, 3\rangle$ is an i-graph (Proposition 3.36). Indeed, we can further exploit previous results to see that all graphs $\Theta\langle 1, 2, \ell\rangle$ for $\ell \geq 3$ are i-graphs by taking a cycle C_n with $n \geq 4$, and replacing one of its maximal cliques (i.e. an edge) with a K_3. By the Max Clique Replacement Lemma (Lemma 3.21), the resultant $\Theta\langle 1, 2, n-1\rangle$ is also i-graph. Notice that when we likewise attempt to replace an edge of a C_3 with a K_3, the result no longer holds; a single edge in a C_3 is not a maximal clique. For reference, we explicitly state this result as a lemma below.

Lemma 5.9. *For $\ell \geq 3$, the theta graphs $\Theta\langle 1, 2, \ell\rangle$ are i-graphs.*

Construction of $\overline{G}$ for $\Theta\langle 1, k, \ell\rangle$, $3 \leq k \leq \ell$

In Figure 5.3 below, we provide a first example of the technique we employ repeatedly throughout this section to construct our theta graphs. To the left is a graph $\overline{G}$, where each of its nine triangles corresponds to an i-set of its complement G. The resultant i-graph of G, $\mathscr{I}(G) = \Theta\langle 1, 4, 5\rangle$, is presented on the right. For consistency, we use X and Y to denote the triangles corresponding to the degree 3 vertices in the theta graphs in this example, as well as all constructions to follow.

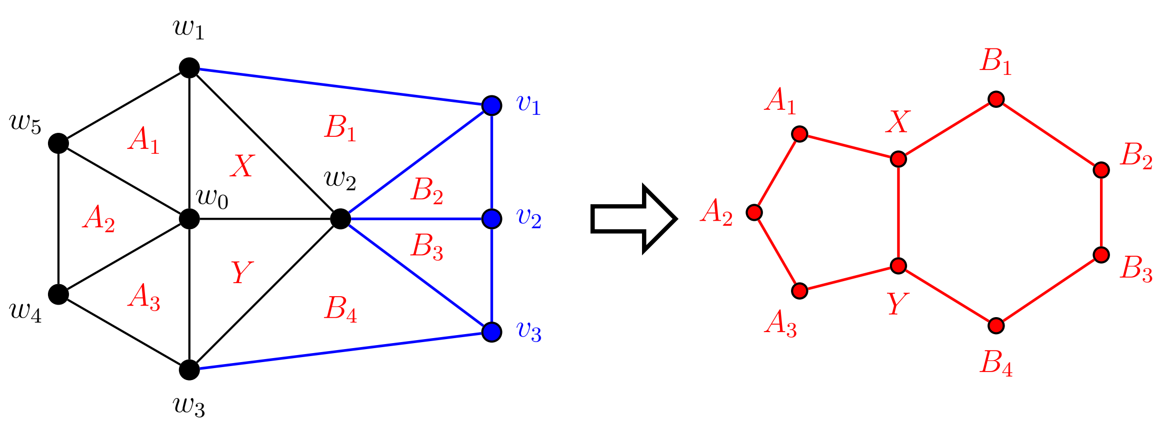


Figure 5.3: A graph $\overline{G}$ (left) such that $\mathscr{I}(G) = \Theta\langle 1, 4, 5\rangle$ (right).

We proceed now to the general construction of a graph $\overline{G}$ with $\mathscr{I}(G) = \Theta\langle 1, k, \ell\rangle$ for

$3 \leq k \leq \ell$. As it is our first construction using this triangle technique, we provide the construction and proof for Lemma 5.11 with an abundance of detail.

Construction 5.10. *(See Figure 5.4) Let* $\overline{H} \cong W_{k+2} = C_{k+1} \vee K_1$, *where* $C_{k+1} = (w_1, ..., w_{k+1}, w_1)$ *and* w_0 *is the central hub, that is, the vertex with degree* $k+1$. *Add a path* $P_{\ell-2} : (v_1, ..., v_{\ell-2})$, *joining each* v_i, $i = 1, ..., \ell-2$, *to* w_2. *Also join* v_1 *to* w_1 *and* $v_{\ell-2}$ *to* w_3. *(If* $\ell = 3$, *then* $v_1 = v_{\ell-2}$, *hence* v_1 *is adjacent to* w_1, w_2 *and* w_3.*) This is the (planar) graph* $\overline{G}$.

Lemma 5.11. *If* $\overline{G}$ *is the graph constructed by Construction 5.10, then* $\mathscr{I}(G) = \mathscr{A}(G) = \Theta\langle 1, k, \ell\rangle$, $3 \leq k \leq \ell$.

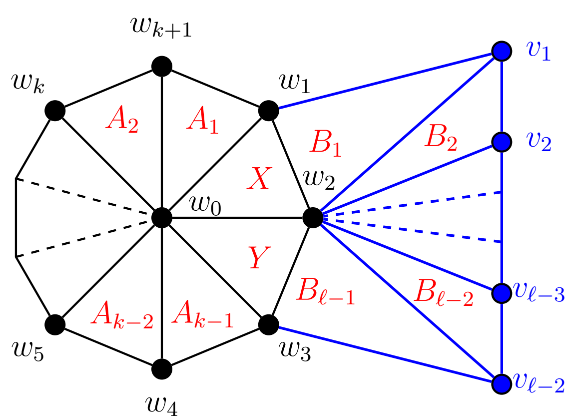


Figure 5.4: The graph $\overline{G}$ from Construction 5.10 such that $\mathscr{I}(G) = \Theta\langle 1, k, \ell\rangle$ for $3 \leq k \leq \ell$.

Proof. To begin, notice that since in $\overline{G}$, the vertices $\mathcal{W} = \{w_0, w_1, \ldots, w_k, w_{k+1}\}$ form a wheel on at least five vertices, $\overline{H} \not\cong K_4$. Likewise, the graph induced by $\{w_0, w_1, w_2, w_3, v_1, v_2, \ldots, v_{\ell-2}\}$ in $\overline{G}$ is also a wheel on $\ell + 2$ vertices, where w_2 is the central hub, and so it too contains no K_4. Therefore, $\overline{G}$ is K_4-free. Moreover, since the triangles of $\overline{G}$ are its smallest maximal cliques, these triangles are precisely the maximal cliques of $\overline{G}$, and so $\omega(\overline{G}) = i(G) = \alpha(G) = 3$. Since the i-sets of G are identical to its α-sets, $\mathscr{I}(G) = \mathscr{A}(G)$, and so for ease of notation, we will refer only to $\mathscr{I}(G)$ throughout the remainder of this proof.

We label the triangles as in Figure 5.4 by dividing them into two collections. The first are the triangles composed only of the vertices from $\mathcal{W}$ and each containing w_0: let

$X = \{w_0, w_1, w_2\}$, $Y = \{w_0, w_2, w_3\}$, $A_1 = \{w_0, w_{k+1}, w_1\}$, $A_2 = \{w_0, w_k, w_{k+1}\}, \ldots$ $A_{k-2} = \{w_0, w_4, w_5\}$, $A_{k-1} = \{w_0, w_3, w_4\}$. The remainder are the triangles with vertex sets not fully contained in $\overline{H}$: $B_1 = \{w_2, w_1, v_1\}$, $B_2 = \{w_2, v_1, v_2\}$, $B_3 = \{w_2, v_2, v_3\}$, $\ldots$, $B_{\ell-2} = \{w_2, v_{\ell-3}, v_{\ell-2}\}$, and $B_{\ell-1} = \{w_2, v_{\ell-2}, w_3\}$. We refer to these collections as $\mathcal{S} = \{X, Y\}$, $\mathcal{A} = \{A_1, A_2, \ldots, A_{k-1}\}$, and $\mathcal{B} = \{B_1, B_2, \ldots, B_{\ell-1}\}$, and claim that these are the only triangles of $\overline{G}$.

To demonstrate this, we show that each vertex v_i on the path $P_{\ell-2}$ is incident with exactly two of the triangles defined above.

Consider first a vertex v_i for $2 \leq i \leq \ell - 3$ on $P_{\ell-1}$ (in $\overline{G}$). From the above definitions, v_i is already on the triangles B_i and B_{i+1}. Since $\overline{G}$ is a planar graph, and $\deg(v_i) = 3$, there is at most one more triangle incident with v_i. The triangle B_i is incident with edges $v_{i-1}v_i$ and v_iw_2, and B_{i+1} is incident with edges v_iv_{i+1} and v_iw_2. Thus, a third triangle incident with v_i would make use of edges v_iv_{i-1} and v_iv_{i+1}. However, since $v_{i-1}v_{i+1} \notin E(\overline{G})$, no such third triangle exists; v_i is incident only with B_{i-1} and B_i. In the special case where $i = 1$, the vertex v_1 is incident with triangles B_1 and B_2. Since $w_1v_2 \notin E(\overline{G})$, these are the only two such triangles. Similarly, for $i = \ell - 3$, $v_{\ell-3}$ is incident only to $B_{\ell-2}$ and $B_{\ell-1}$, since $w_3v_{\ell-3} \notin E(\overline{G})$. Thus, we conclude that for $1 \leq i \leq \ell - 2$, the vertex v_i is incident to the triangles B_i and B_{i+1}, and no others.

Now, since all triangles incident with v_i have already been considered, any unaccounted for triangles would be composed only of vertices from $\mathcal{W}$. However, clearly the vertices of $\mathcal{W}$ form a wheel, for which we have already accounted for all its $2 + (k-1) = k + 1$ triangles. We conclude that the only triangles of $\overline{G}$ are the $2 + (k-1) + (\ell - 1) = k + \ell$ such ones defined above. It follows that these $k + \ell$ triangles represent all the i-sets of G, and so $V(\mathscr{I}(G)) = \{X, Y, A_1, A_2, \ldots, A_{k-1}, B_1, B_2, \ldots, B_{\ell-1}\}$.

Having determined all triangles of $\overline{G}$ (and thus $V(\mathscr{I}(G))$), we now show that the required adjacencies hold. From the original construction of $\overline{G}$, the following are immediate for $\mathscr{I}(G)$:

(i) $X \overset{w_1 w_3}{\sim} Y$,

(ii) $X \overset{w_0 v_1}{\sim} B_1 \overset{w_1 v_2}{\sim} B_2 \overset{v_1 v_3}{\sim} B_3 \ldots B_{\ell-2} \overset{v_{\ell-3} w_3}{\sim} B_{\ell-1} \overset{v_{\ell-2} w_0}{\sim} Y$,

(iii) $X \overset{w_2 w_{k+1}}{\sim} A_1 \overset{w_1 w_k}{\sim} A_2 \ldots A_{k-2} \overset{w_5 w_3}{\sim} A_{k-1} \overset{w_4 w_2}{\sim} Y$.

Hence, we need only show that there are no additional unwanted edges generated in the construction of $\mathscr{I}(G)$.

Since $\overline{G}$ is a planar graph and all of its triangles are facial (that is, the edges of the K_3 form a face in the planar embedding of the graph shown in Figure 5.4), each triangle is adjacent to at most three others. From (i)-(iii) above, triangles X and Y are both adjacent to the maximum three (and hence $\deg_{\mathscr{I}(G)}(X) = \deg_{\mathscr{I}(G)}(Y) = 3$).

Recall that to be adjacent, two triangles share exactly two vertices. Notice that the triangles of $\mathcal{A}$ are composed entirely of vertices from $\mathcal{W} - \{w_2\}$, and that for $2 \leq i \leq \ell - 2$, $B_i \cap \mathcal{W} = \{w_2\}$; furthermore, $B_1 \cap \mathcal{W} = \{w_1, w_2\}$ and $B_{\ell-2} \cap \mathcal{W} = \{w_2, w_3\}$. We conclude that no triangle of $\mathcal{B}$ is adjacent to any triangle of $\mathcal{A}$.

It remains only to show that there are no additional unwanted adjacencies between two triangles of $\mathcal{A}$ or two triangles of $\mathcal{B}$. Clearly from the definitions, there is no triangle of $\mathcal{A}$ aside from A_1 that contains both the vertices w_1 and w_{k+1}. For $2 \leq i \leq k-2$, consider a triangle A_i. From (iii), we have already shown that A_i is adjacent to two other triangles: one sharing the vertex pair $\{w_0, w_{(k+2)-i}\}$ and the other sharing the vertices $\{w_0, w_{(k+3)-i}\}$. From their definitions, we see that there is no triangle A_j with $i \neq j$ sharing the third vertex pair with A_i. That is, there is no A_j such that $\{w_{(k+2)-i}, w_{(k+3)-i}\} \subseteq A_j$. Hence, for all $A_i \in \mathcal{A}$, A_i is adjacent to exactly two other triangles, and so $\deg_{\mathscr{I}(G)}(A_i) = 2$.

Applying arguments very similar to those from $\mathcal{A}$ to the triangles of $\mathcal{B}$ shows that no additional unwanted edges are generated in the construction of the i-graph. This completes the exhaustive examination of the triangles of $\overline{G}$. We conclude that the graph $\overline{G}$ generated by Construction 5.10 yields $\mathscr{I}(G) = \mathscr{A}(G) = \Theta\langle 1, k, \ell\rangle$. ■

5.2.2 The i-graphs of the Complements of Wheels Are Cycles

Before we proceed with the remainder of the theta graph constructions, let us return to Figure 5.4 to notice the prominence of the wheel subgraph in the complement seed graph $\overline{G}$. In the constructions throughout this chapter, this wheel subgraph will appear repeatedly; indeed, all of the complement seed graphs for the i-graphs of theta graphs have a similar basic form: begin with a wheel, add a path of some length, and then add some collection of edges between them. Lemma 5.12 below demonstrates that a wheel in a triangle-based complement seed graph $\overline{G}$ corresponds to a cycle in the i-graph of G. Using this result, in our later constructions with a wheel subgraph, we already have two of the three paths of a theta graph formed. Hence, we need only confirm that whatever unique additions are present in a given construction form the third path in the i-graph.

Notice that when $k = 3$, the wheel W_4 is isomorphic to the complete graph K_4, the complement of which has only a single i-set, namely, the set including all vertices. Hence, $\mathscr{I}(\overline{W_4}) = \mathscr{I}(\overline{K_4}) = K_1$.

Lemma 5.12. *For $k \geq 4$, let H_k be the wheel $W_{k+1} = C_k \vee K_1$. Then $\mathscr{I}(\overline{H_k}) \cong \mathscr{A}(\overline{H_k}) \cong C_k$.*

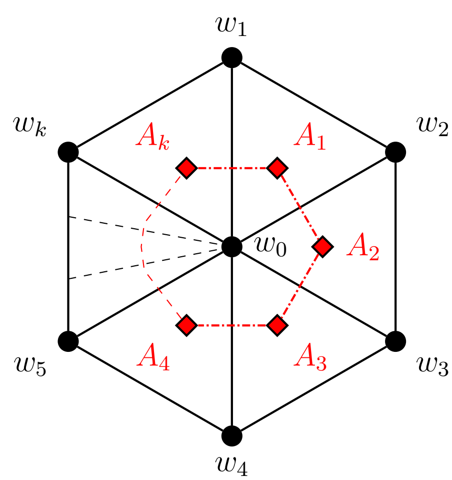


Figure 5.5: The wheel $H_k = W_{k+1}$, with the i-graph of its complement, $\mathscr{I}(\overline{H_k}) \cong \mathscr{A}(\overline{H_k}) \cong C_k$, embedded in red.

Proof. Let H_k be the wheel graph W_{k+1} for $k \geq 4$. We apply the standard vertex labelling used throughout this section, where the degree 3 vertices are labelled $w_1, w_2, \ldots, w_k$, and the central degree k vertex is w_0.

Since H_k is K_4-free and each vertex of H_k lies on a triangle, $i(\overline{H_k}) = 3$; moreover, each triangle of H_k represents an i-set of $\overline{H_k}$. For the same reasons, we have that $\alpha(\overline{H_k}) = 3$ and the triangles represent also the α-sets of $\overline{H_k}$.

We label these triangles as $A_i = \{w_0, w_i, w_{i+1}\}$ for $1 \leq i \leq k-1$ and $A_k = \{w_0, w_k, w_1\}$.

To check that there are no other unaccounted for triangles in H_k, consider one of the exterior wheel vertices of H_k, say w_i with, for ease of notation, $2 \leq i \leq k-1$ (similar arguments with worse notation hold for $i = 1$ and $i = k$). Since $deg(w_i) = 3$, w_i is on at most $\binom{3}{2} = 3$ triangles, of which we have already accounted for two: $A_{i-1} = \{w_0, w_{i-1}, w_i\}$ (using edges w_0w_i and w_0w_{i-1}) and $A_i = \{w_0, w_i, w_{i+1}\}$ (using edges w_0w_i and w_0w_{i+1}). The final potential triangle would make use of the edges w_iw_{i+1} and $w_{i-1}w_i$; however, since $w_{i-1}w_{i+1} \notin E(H_k)$, this does not form a triangle. Thus, w_i (and hence all of the exterior wheel vertices) is (are) on exactly two triangles. We have accounted for all triangles of H_k and therefore all i-sets of $\overline{H_k}$, and so $V(\mathscr{I}(\overline{H_k})) = \{A_1, A_2, \ldots, A_k\}$.

Now, since H_k is planar, and each triangle $A_1, A_2, \ldots, A_k$ is a facial triangle, each triangle is adjacent to at most three other triangles. Clearly, for $2 \leq i \leq k-1$, $A_{i-1} \sim A_i \sim A_{i+1}$, which makes use of the shared edges w_0w_i and w_0w_{i+1} of A_i. As observed before, the third edge, w_iw_{i+1}, is only on A_i, and so A_i is adjacent only to A_{i-1} and A_{i+1}. Similar arguments follow for A_1 and A_k. Thus, $\deg_{\mathscr{I}(\overline{H_k})}(A_i) = 2$ for all $1 \leq i \leq k$ and since $A_1 \sim A_2 \sim \cdots \sim A_k \sim A_1$, it follows that $\mathscr{I}(\overline{H_k}) \cong \mathscr{A}(\overline{H_k}) \cong C_k$. ∎

We note the following analogous - although less frequently applied - result for fans of the form $K_1 + P_k$. The proof proceeds similarly to the previous result for wheels, and is omitted.

Lemma 5.13. *For $k \geq 2$, let H_k be the k-fan $K_1 \vee P_k$. Then $\mathscr{I}(\overline{H_k}) \cong \mathscr{A}(\overline{H_k}) \cong P_{k-1}$.*

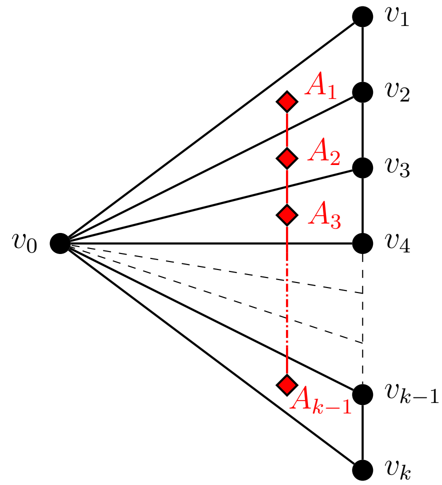


Figure 5.6: The fan $H_k = K_1 \vee P_k$, with the i-graph of its complement, $\mathscr{I}(\overline{H_k}) \cong \mathscr{A}(\overline{H_k}) \cong P_{k-1}$, embedded in red.

5.2.3 $\Theta\langle 2, k, \ell\rangle$ for $2 \leq k \leq \ell$

Construction of $\overline{G}$ for $\Theta\langle 2, 2, \ell\rangle$, $\ell \geq 5$

In Chapter 3, we saw that $K_{2,3} \cong \Theta\langle 2, 2, 2\rangle$ (Proposition 3.9) and $\kappa \cong \Theta\langle 2, 2, 3\rangle$ (Proposition 3.10) are not i-graphs. Extending these results, we find that the length of the third path in $\Theta\langle 2, 2, \ell\rangle$ has a transition point between $\ell = 4$ and $\ell = 5$; while $\ell = 4$ is still too short to form an i-graph (see Lemma 5.41), for $\ell \geq 5$, $\Theta\langle 2, 2, \ell\rangle$ is i-graph realizable.

Construction 5.14. *Refer to Figures 5.7 and 5.8. Begin with a copy of the graph $\overline{H} \cong W_5 = C_4 \vee K_1$, labelling the degree 3 vertices as w_1, w_2, w_3, w_4 and the central degree 4 vertex as w_0.*

- *(a) If $\ell \geq 6$ (as in Figure 5.7), attach to $\overline{H}$ a path $P_{\ell-3} : (v_1, v_2, \ldots, v_{\ell-3})$ by joining w_1 to $v_1, v_2, \ldots, v_{\ell-4}$. Join $v_{\ell-5}$ to $v_{\ell-3}$. Next, join w_2 to v_1, and w_3 to $v_{\ell-3}$. Then, join w_4 to $v_{\ell-4}$ and $v_{\ell-3}$. Add a new vertex z, joined to w_1, w_4, and $v_{\ell-4}$.*
- *(b) If $\ell = 5$ (as in Figure 5.8), attach to $\overline{H}$ a path of $P_2 : (v_1, v_2)$, by joining w_1 to v_1, and w_2 to v_1 and v_2. Then join w_3 to v_2, and w_4 to both v_1 and v_2. Add two new vertices, z_1 and z_2, joining z_1 to v_1, w_1, and w_4, and z_2 to v_2, w_2 and w_3.*

We label the resultant (planar) graph $\overline{G}$.

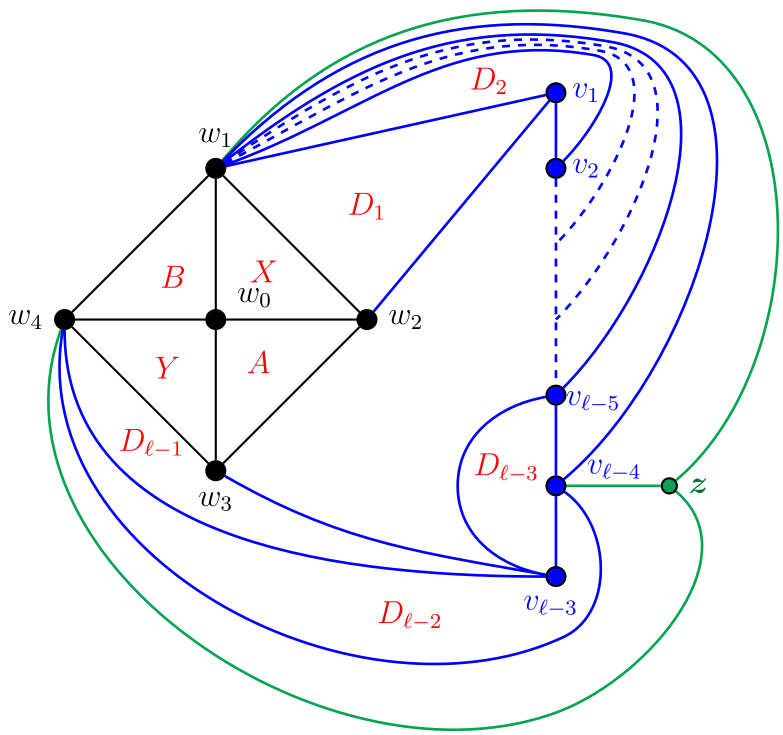


Figure 5.7: The graph $\overline{G}$ from Construction 5.14 such that $\mathscr{I}(G) = \Theta\langle 2, 2, \ell\rangle$ for $\ell \geq 6$.

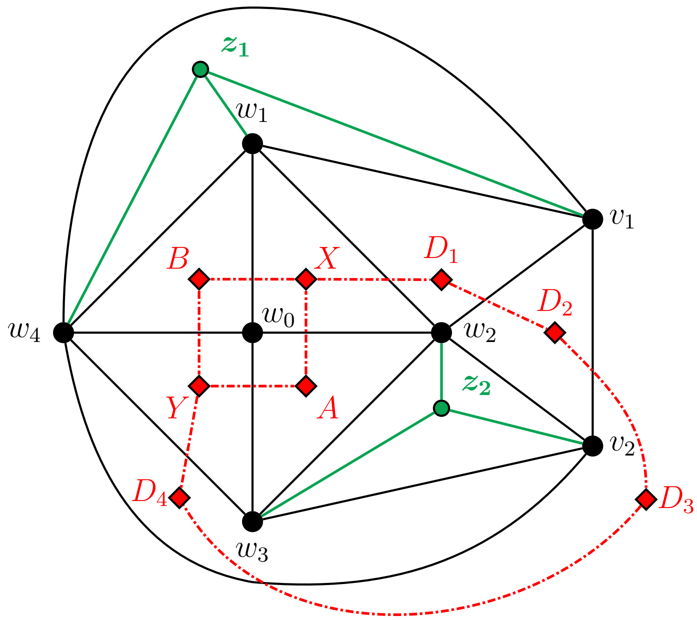


Figure 5.8: The graph $\overline{G}$ from Construction 5.14 such that $\mathscr{I}(G) = \Theta\langle 2,2,5\rangle$, with $\Theta\langle 2,2,5\rangle$ overlaid in red.

As with our other constructions, the triangles of $\overline{G}$ are its smallest maximal cliques, and so $i(G) = 3$. As before, we use these triangle faces to visualize the movements of tokens between the i-set reconfigurations; however, we now employ a technique of adding vertices to create K_4's through $\overline{G}$ and eliminate any "unwanted" triangles that might arise in our construction. In (a), the addition of z prevents $\{w_1, w_4, v_{\ell-3}\}$ from being a maximal clique of $\overline{G}$ and hence an i-set of G. Similarly, in (b), z_1 and z_2 eliminate triangles $\{w_1, w_4, v_1\}$ and $\{w_2, w_3, v_2\}$, respectively. The unfortunate trade-off in this triangle-elimination technique is that the remaining triangles are no longer α-sets; the constructions work only for i-graphs, not α-graphs.

Lemma 5.15. *If $\overline{G}$ is the graph constructed by Construction 5.14, then $\mathscr{I}(G) = \Theta\langle 2,2,\ell\rangle$, for $\ell \geq 5$.*

Proof. We prove the two constructions separately, beginning with the more general case.

(a) $\mathbf{\Theta}\left\langle \mathbf{2}, \mathbf{2}, \ell \right\rangle, \ell \geq \mathbf{6}$.

As in the previous constructions, since each edge of $\overline{G}$ belongs to a triangle and some triangles are not contained in K_4's, these maximal triangles of $\overline{G}$ form the smallest maximal cliques of $\overline{G}$ and, hence, the i-sets of G. We label these triangles as in Figure 5.7; in particular,

$$\begin{aligned} X &= \{w_0, w_1, w_2\}, & D_1 &= \{w_1, w_2, v_1\}, \\ Y &= \{w_0, w_3, w_4\}, & D_i &= \{w_1, v_{i-1}, v_i\} \text{ for } 2 \leq i \leq \ell - 4, \\ A &= \{w_0, w_2, w_3\}, & D_{\ell-3} &= \{v_{\ell-5}, v_{\ell-4}, v_{\ell-3}\}, \\ B &= \{w_0, w_1, w_4\}, & D_{\ell-2} &= \{w_4, v_{\ell-4}, v_{\ell-3}\}, \\ & & D_{\ell-1} &= \{w_3, w_4, v_{\ell-3}\}. \end{aligned}$$

We further partition the vertices of $\overline{G}$ into two sets: the wheel vertices $W = \{w_0, w_1, \ldots, w_4\}$ and the path vertices $Q = \{v_1, v_2, \ldots, v_{\ell-3}\}$.

We first show that the only induced K_4 in $\overline{G}$ is $Z = \{z, w_1, w_4, v_{\ell-4}\}$, and hence each of the above listed $\ell + 3$ sets is an i-set of G. The vertices of Q induce a path of length $\ell - 2$, with an extra edge between $v_{\ell-3}$ and $v_{\ell-5}$, so no K_4 is contained entirely in Q. There is likewise no K_4 with three vertices in Q, since the only set of three mutually adjacent vertices from Q is $\{v_{\ell-5}, v_{\ell-4}, v_{\ell-3}\}$, but $v_{\ell-5}$ and $v_{\ell-3}$ have no common neighbours outside of Q in $\overline{G}$. Moreover, there is also no K_4 using exactly two vertices of Q; notice that w_2 and w_3 both have only a single edge to Q, and so any K_4 with two vertices on Q would necessarily contain w_1 and w_4. Since $N_{\overline{G}}(\{w_1, w_4\}) \cap Q = \{v_{\ell-4}\}$, the only K_4 containing two vertices from $\mathcal{W}$ is Z, as required.

Next, we show that there are no additional i-sets in G beyond the $\ell + 3$ labelled maximal clique triangle faces of $\overline{G}$ listed above. As all of the neighbours of z lie on a K_4, we can divide any potential unaccounted for i-sets into two categories: those with at least two vertices on the wheel W, and those with at least two vertices on the path Q.

In the first case, $N_{\overline{G}}(w_0) = \{w_1, w_2, w_3, w_4\}$, so every triangle containing w_0 has already been accounted for (on the faces of the wheel), including those with three vertices on W. As noted above, the open neighbourhoods of w_3 and w_4 share only $v_{\ell-3}$ on Q, and we have already labelled $D_{\ell-1} = \{w_3, w_4, v_{\ell-3}\}$. The only neighbour of w_2 on Q is v_1, which is not adjacent to w_3, so there are no triangles containing both w_2 and w_3, aside from A. The only other wheel vertex adjacent to v_1 is w_1, but we have again already accounted for $D_1 = \{w_1, w_2, v_1\}$. Finally, $(N_{\overline{G}}(w_1) \cap N_{\overline{G}}(w_4)) \cap Q = \{v_{\ell-4}\}$, and since $\{w_1, w_4, v_{\ell-4}\}$ is a part of the clique Z, it is not an i-set of G.

We now show that there are no additional triangles of $\overline{G}$ with at least two vertices on Q that are i-sets of G. For a vertex $v_i \in Q$ with $2 \leq i \leq \ell - 6$, we have that $N(v_i) \cap Q = \{v_{i-1}, v_{i+1}\}$ and, furthermore, $(N_{\overline{G}}(v_i) \cap N_{\overline{G}}(v_{i+1})) \cap W = \{w_1\}$. Each such triangle $D_{i+1} = \{w_i, v_i, v_{i+1}\}$ has already been accounted for. With the vertices of Q along the middle of the path now checked, we consider special cases present in the end vertices. The only triangle entirely contained in Q is $D_{\ell-3} = \{v_{\ell-5}, v_{\ell-4}, v_{\ell-3}\}$, which has already been considered. For v_1, $N_{\overline{G}}(v_1) \cap Q = \{v_2\}$, and as before, since $N_{\overline{G}}(v_1) \cap N_{\overline{G}}(v_2) = \{w_1\}$, we have already listed the triangle $D_2 = \{v_1, v_2, w_2\}$ as an i-set of G. Once again, $N_{\overline{G}}(v_{\ell-4}) \cap N_{\overline{G}}(v_{\ell-5}) = \{w_1, v_{\ell-3}\}$, so $v_{\ell-4}$ and $v_{\ell-5}$ are on only the previously considered triangles $D_{\ell-3}$ and $D_{\ell-4}$. Finally, $N_{\overline{G}}(v_{\ell-3}) \cap N_{\overline{G}}(v_{\ell-4}) = \{w_4\}$, which is once more associated only with the previously labelled triangle $D_{\ell-1} = \{w_3, w_4, v_{\ell-3}\}$. This concludes all possible cases where a triangle of $\overline{G}$ could be an i-set of G; the i-sets of G, and hence the vertices of $\mathscr{I}(G)$, are exactly the $\ell + 3$ sets listed above.

Moving to the edges of $\mathscr{I}(G)$, the following adjacencies are clear from Figure 5.7.

(i) $X \overset{w_1 w_3}{\sim} A \overset{w_2 w_4}{\sim} Y$,

(ii) $X \overset{w_2 w_4}{\sim} B \overset{w_1 w_3}{\sim} Y$,

(iii) $X \overset{w_0 v_1}{\sim} D_1 \overset{w_2 v_2}{\sim} D_2 \overset{v_1 v_3}{\sim} D_3 \ldots D_{\ell-3} \overset{w_1 v_{\ell-2}}{\sim} D_{\ell-2} \overset{v_{\ell-4} w_4}{\sim} D_{\ell-1} \overset{v_{\ell-2} w_0}{\sim} Y$.

We need only show that there are no additional adjacencies in $\mathscr{I}(G)$ not listed above. Since $\overline{G}$ is planar, and an i-set triangle is adjacent to another i-set triangle if and only if they share exactly one edge if $\overline{G}$, each i-set of G is adjacent to at most three others. In our lists above, X and Y are each adjacent to three other i-sets, and thus no others. The i-set A is adjacent to both X and Y in the list, and from the above, no other triangles of $\overline{G}$ contain both w_2 and w_3, and so there is no triangle adjacent to A sharing the edge w_2w_3. Similarly, the i-set B is listed above as being adjacent to both X and Y. While $\{w_1, w_4, z\}$ is a triangle adjacent to B, it is part of the K_4, Z, and so not an i-set.

All that remains is to show that each $D_i \in \mathscr{I}(V(G))$ has $\deg_{\mathscr{I}(G)}(D_i) = 2$ for all $i = 1, 2, \ldots \ell - 1$. Suppose that some D_i had $\deg_{\mathscr{I}(G)}(D_i) \geq 3$. We have already established that D_i is not adjacent to A nor to B, and unless $i = 1$ or $\ell - 1$, D_i is not adjacent to any of X or Y either. If $1 \leq i \leq \ell - 5$, then because $\deg_{\overline{G}}(v_i) = 3$, the triangle D_i is adjacent only to D_{i-1} and D_{i+1} as in the lists above. The triangle $D_{\ell-4}$ is adjacent to both $D_{\ell-3}$ and $D_{\ell-5}$, and while it does share the edge w_1v_4 with the triangle $\{w_1, v_4, z\} \subseteq Z$, the triangle is not an i-set of G. The triangle $D_{\ell-3}$ is adjacent to both $D_{\ell-2}$ and $D_{\ell-4}$, and no other triangles contain the edge $v_{\ell-5}v_{\ell-3}$. Similarly to $D_{\ell-4}$, the triangle $D_{\ell-2}$ does share the edge w_4v_4 with the triangle $\{w_4, v_4, z\} \subseteq Z$, which is not an i-set of G. The triangle $D_{\ell-1}$ is adjacent to both Y and $D_{\ell-2}$, and no other triangles contain the edge $w_3v_{\ell-3}$. This completes the exhaustive verification that $\deg_{\mathscr{I}(G)}(D_i) = 2$ for all $1 \leq i \leq \ell - 1$. Moreover, we have confirmed that no additional edges are present in the i-graph of G beyond those given in $(i) - (iii)$. We conclude that $\mathscr{I}(G) = \Theta\langle 2, 2, \ell\rangle$.

(b) $\boldsymbol{\Theta\langle 2, 2, 5\rangle}$

As in part (a), we begin by listing the triangle sets:

$$\begin{aligned} X &= \{w_0, w_1, w_2\}, & D_1 &= \{w_1, w_2, v_1\}, \\ Y &= \{w_0, w_3, w_4\}, & D_2 &= \{w_2, v_1, v_2\}, \\ A &= \{w_0, w_2, w_3\}, & D_3 &= \{w_4, v_1, v_2\}, \\ B &= \{w_0, w_1, w_4\}, & D_4 &= \{w_3, w_4, v_2\}. \end{aligned}$$

Using this labelling of the vertices and triangles of $\overline{G}$ (and as in Figure 5.8), we see that many of the arguments for this case proceed similarly to those in (a). In particular, the triangles of $\overline{G}$ correspond to the i-sets of G, and so the triangles A, B, X and Y, which form a wheel in $\overline{G}$, are an induced C_4 in $\mathscr{I}(G)$. We need only confirm that a third disjoint path in $\mathscr{I}(G)$ is formed between X and Y (and nothing else). Explicitly, the three pathways are:

(i) $X \overset{w_1w_3}{\sim} A \overset{w_2w_4}{\sim} Y$,

(ii) $X \overset{w_2w_4}{\sim} B \overset{w_1w_3}{\sim} Y$,

(iii) $X \overset{w_0v_1}{\sim} D_1 \overset{w_1v_2}{\sim} D_2 \overset{w_2w_4}{\sim} D_3 \overset{v_1w_3}{\sim} D_4 \overset{v_2w_0}{\sim} Y$.

To see that there are no additional i-sets of G beyond those corresponding to the eight labelled triangles of Figure 5.8, notice first that neither z_1 nor z_2 are in any i-sets as each of their closed neighbourhoods forms a K_4. The vertices of the wheel $\{w_0, w_1, \ldots, w_4\}$ form a C_4 in $\mathscr{I}(G)$ by Lemma 5.12, so any potential additional triangles contain at least one of v_1 or v_2. Since $\overline{G}$ is planar, each vertex v_i is in at most $\deg_{\overline{G}}(v_i)$ triangles of $\overline{G}$ and hence at $\deg_{\overline{G}}(v)$ i-sets of G. For v_1, since $z_1 \in N_{\overline{G}}(v_1)$ and z_1 is not any i-sets, v_1 is in at most $\deg_{\overline{G}}(v_1) - 2 = 5 - 2 = 3$ i-sets, of which we have already accounted for all three (D_1, D_2, and D_3). Similar arguments for v_2 follow. Thus, the eight i-sets listed above are the only i-sets of G.

Again, since $\overline{G}$ is planar and all triangles are facial, each i-set of G is adjacent to at most three others in $\mathscr{I}(G)$. For D_1, notice that the edge w_1v_1 in $\overline{G}$ is part of a K_4 with z_1, and hence not adjacent to another triangle over that shared edge, and so $\deg_{\mathscr{I}(G)}(D_1) = 2$. Similarly, each D_i for $i \in \{2, 3, 4\}$ also has at least one shared edge with a K_4 in $\overline{G}$, containing either z_1 or z_2, so that $\deg_{\mathscr{I}(G)}(D_i) = 2$ for all $1 \leq i \leq 4$; there are no additional edges in $\mathscr{I}(G)$ than those listed above. We can therefore conclude that X, D_1, D_2, D_3, D_4, Y forms the third disjoint path of length five of a theta graph in $\mathscr{I}(G)$, as required.

■

Construction of $\overline{G}$ for $\Theta\langle 2, 3, \ell\rangle$ for $\ell \geq 5$

For many of the results going forward, we apply small modifications to previous constructions. In the first of these, we begin with the graphs from Construction 5.14, which were used to find i-graphs for $\Theta\langle 2,2,5\rangle$ and $\Theta\langle 2,2,\ell\rangle$ for $\ell \geq 6$, and expand the central wheel used in there to build i-graphs for $\Theta\langle 2,3,5\rangle$ and $\Theta\langle 2,3,\ell\rangle$ for $\ell \geq 6$.

Construction 5.16. *Refer to Figures 5.9 and 5.10.*

(a) If $\ell \geq 6$, begin with a copy of the graph $\overline{G}$ from Construction 5.14. Subdivide the edge w_1w_4, adding the new vertex w_5. Join w_5 to w_0, so that $w_0, w_1, \ldots, w_5$ forms a wheel. Delete the vertex z.

(b) If $\ell = 5$, begin with a copy of the graph $\overline{G}$ from Construction 5.14. Subdivide the edge w_1w_4, adding the new vertex w_5. Join w_5 to w_0, so that $w_0, w_1, \ldots, w_5$ forms a wheel. Delete the vertex z_1.

We rename the resultant (planar) graph $\overline{G_{2,3,\ell}}$ for $\ell \geq 5$.

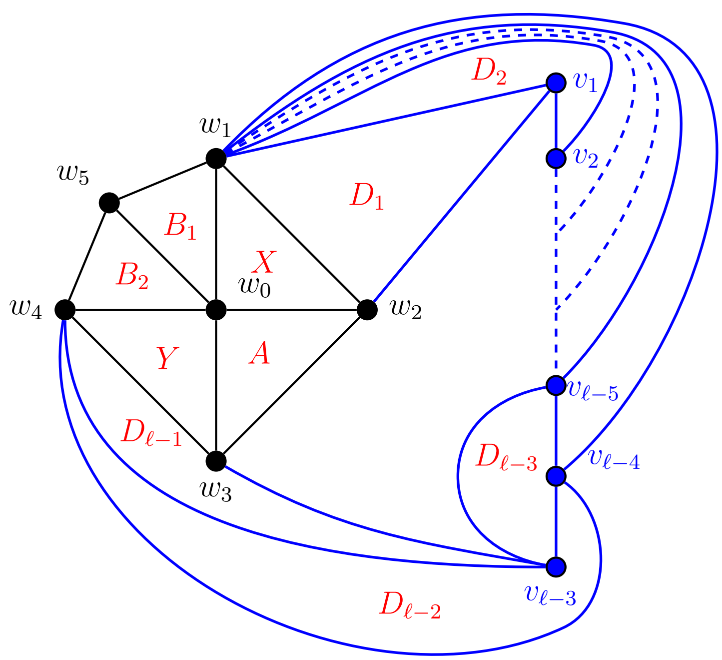


Figure 5.9: The graph $\overline{G_{2,3,\ell}}$ from Construction 5.16 (a) such that $\mathscr{I}(G_{2,3,\ell}) = \mathscr{A}(G_{2,3,\ell}) = \Theta\langle 2,3,\ell\rangle$ for $\ell \geq 6$.

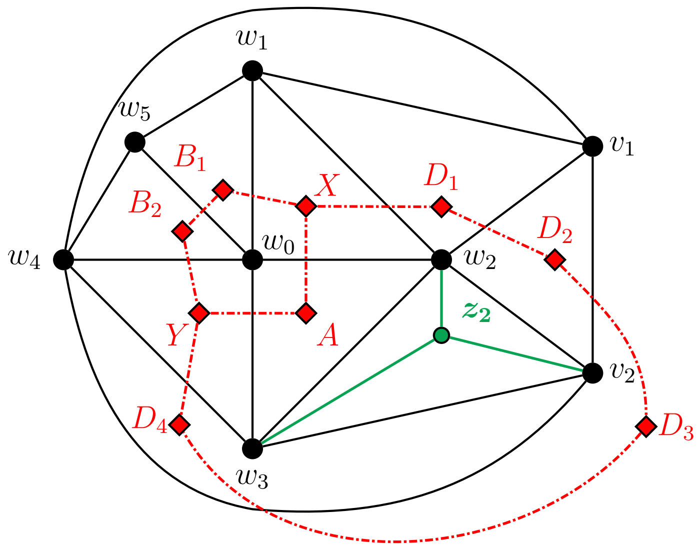


Figure 5.10: The graph $\overline{G_{2,3,5}}$ from Construction 5.16 (b) such that $\mathscr{I}(G_{2,3,5}) = \Theta\langle 2,3,5\rangle$, with $\mathscr{I}(G_{2,3,5})$ overlaid in red.

In Construction 5.16 (a), notice that the vertex z is deleted from $\overline{G}$. In the original Construction 5.14 for a graph $\overline{G}$ with $\mathscr{I}(G) = \Theta\langle 2,2,\ell\rangle$ for $\ell \geq 6$, z served to eliminate the unwanted triangle formed by $\{w_1, w_4, v_{\ell-4}\}$. Now with the expanded wheel including w_5, $\{w_1, w_4, v_{\ell-4}\}$ is not a triangle in $\overline{G_{2,3,\ell}}$ ($\ell \geq 6$), and z is not needed. Indeed, as $\overline{G_{2,3,\ell}}$ now has $\alpha(G_{2,3,\ell}) = 3$, its triangles are also α-sets in G, and so we can immediately extend the construction from i-graphs to α-graphs.

The extension, however, does not apply to Construction 5.16 (b) for the graph $\overline{G_{2,3,5}}$. Here, we no longer require z_1 (which served to eliminate the unwanted triangle formed by $\{w_1, w_4, v_1\}$), but z_2 remains and forms the clique $\{w_2, w_3, z_2, v_2\}$; thus, $\alpha(G_{2,3,5}) = 4$.

The proof for the following Lemma 5.17 is otherwise very similar to the proof of Lemma 5.15, and so is omitted.

Lemma 5.17. *If $\overline{G_{2,3,5}}$ is the graph constructed in Construction 5.16 (b), then $\mathscr{I}(G_{2,3,5}) = \Theta\langle 2,3,5\rangle$. For $\ell \geq 6$, if $\overline{G_{2,3,\ell}}$ is the graph constructed in Construction 5.16 (a), then $\mathscr{I}(G_{2,3,\ell}) = \mathscr{A}(G_{2,3,\ell}) = \Theta\langle 2,3,\ell\rangle$ for $\ell \geq 6$.*

Construction of $\overline{G}$ for $\Theta\langle 2,4,4\rangle$

In the following construction for a graph $\overline{G}$ with $\mathscr{I}(G) = \Theta\langle 2,4,4\rangle$, we again apply the technique of adding vertices to eliminate unwanted triangles.

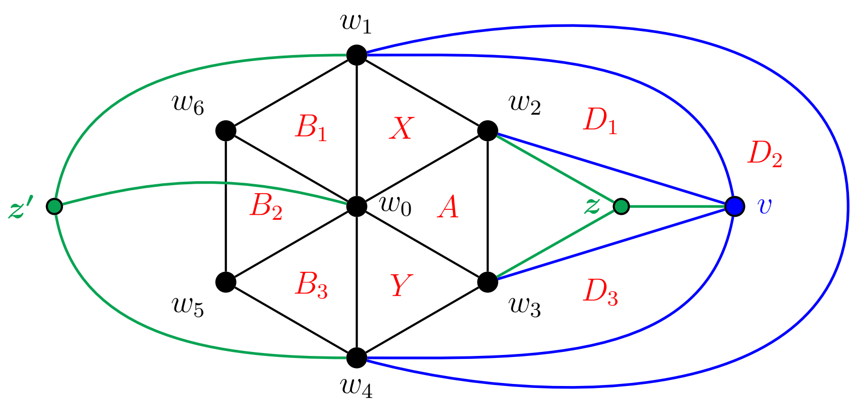


Figure 5.11: A graph $\overline{G}$ such that $\mathscr{I}(G) = \Theta\langle 2,4,4\rangle$.

Construction 5.18. *Refer to Figure 5.11. Begin with a copy of the graph $\overline{H} \cong W_7 = C_6 \vee K_1$, labelling the degree 3 vertices as $w_1, w_2, \ldots, w_6$ and the central degree 6 vertex as w_0. Join w_1 to w_4. Add a new vertex v to $\overline{H}$, joining v to $w_1, \ldots, w_4$. Then, add the new vertex z, joined to v, w_2 and w_3, and the new vertex z', joined to w_0, w_1, and w_4. We label the resultant (non-planar) graph $\overline{G}$.*

Lemma 5.19. *If $\overline{G}$ is the graph constructed by Construction 5.18, then $\mathscr{I}(G) = \Theta\langle 2,4,4\rangle$.*

Proof. Once again, the triangles of $\overline{G}$ form the smallest maximal cliques of $\overline{G}$, and are therefore the i-sets of G. As in Figure 5.11, we label them as follows:

$$\begin{aligned}
X &= \{w_0, w_1, w_2\}, & B_1 &= \{w_0, w_1, w_6\}, & D_1 &= \{w_1, w_2, v\},\\
Y &= \{w_0, w_3, w_4\}, & B_2 &= \{w_0, w_5, w_6\}, & D_2 &= \{w_1, w_4, v\},\\
A &= \{w_0, w_2, w_3\}, & B_3 &= \{w_0, w_4, w_5\}, & D_3 &= \{w_3, w_4, v\}.
\end{aligned}$$

Similarly to the construction for $\Theta\langle 2,2,5\rangle$ in the proof of Lemma 5.15, the vertices z and z' are added to ensure that $\{v, w_2, w_3\}$ and $\{w_0, w_1, w_4\}$, respectively, are not maximal cliques in $\overline{G}$, and hence are not i-sets of G.

To see that there are no other triangles in $\overline{G}$ beyond the nine listed above (and thus, no unaccounted for i-sets in G), consider how one such unaccounted for triangle might be constructed. Since $N[z]$ forms a maximal clique, z (and likewise z') is not on any such triangle. Moreover, all triangles comprised entirely of vertices from the wheel $W_7 = \{w_0, w_1, \ldots, w_6\}$ have already been considered. This leaves only triangles potentially formed by two adjacent wheel vertices and v. However, $\deg(v) = 4$, so it is incident with at most $\binom{4}{2} = 6$ triangles, three of which have already been accounted for in D_1, D_2, D_3, and one that has been eliminated by the clique $\{v, w_2, w_3, z\}$. For the remaining two edge combinations, vw_3 and vw_1 are not on a triangle as $w_1w_2 \notin E(\overline{G})$, and likewise, vw_2 and vw_4 are not on a triangle as $w_2w_4 \notin E(\overline{G})$. We therefore conclude that the collection of triangles of G are exactly the nine sets listed above; moreover, these nine sets are the i-sets of G and hence form $V(\mathscr{I}(G))$.

From Figure 5.11, the following triangle adjacencies are immediate:

(i) $X \overset{w_1w_3}{\sim} A \overset{w_2w_4}{\sim} Y$,

(ii) $X \overset{w_2w_6}{\sim} B_1 \overset{w_1w_5}{\sim} B_2 \overset{w_6w_4}{\sim} B_3 \overset{w_5w_3}{\sim} Y$,

(iii) $X \overset{w_0v}{\sim} D_1 \overset{w_2w_4}{\sim} D_2 \overset{w_1w_3}{\sim} D_3 \overset{vw_0}{\sim} Y$.

Lemma 5.12 ensures that there are no additional adjacencies between two triangles on the wheel $\overline{H}$. Thus, we need only check that there are no unaccounted for edges between the triangles of the collection $\mathcal{D} = \{D_1, D_2, D_3\}$, either among themselves, or with triangles from the wheel. The former is easily removed by observing from Figure 5.11 that D_1 and D_3 share no edges and are therefore not adjacent. For the latter, notice that each wheel triangle contains the vertex w_0, which is not a part of any of $\mathcal{D}$. Thus only triangles of $\mathcal{D}$ containing two consecutively numbered wheel vertices (i.e. w_iw_{i+1} or w_6w_1) are adjacent to wheel triangles. From the figure, this occurs exactly twice: w_1w_2 in D_1 and w_3w_4 in D_3, and has already been accounted for in the adjacencies $X \overset{w_0v}{\sim} D_1$ and $Y \overset{w_0v}{\sim} D_3$. This ensures that there are no additional edges in $\mathscr{I}(G)$ than the ten outlined in the list above, and so we conclude that $\mathscr{I}(G) = \Theta\langle 2, 4, 4\rangle$ as required. ∎

Notice that Construction 5.18 is our first theta graph construction that is not planar. In particular, the constructed graph $\overline{G}$ has a $K_{3,3}$ minor, as illustrated in Figure 5.12. Indeed, with the exception of the constructions that are based upon Construction 5.18, all of our i-graph constructions use planar complement seed graphs.

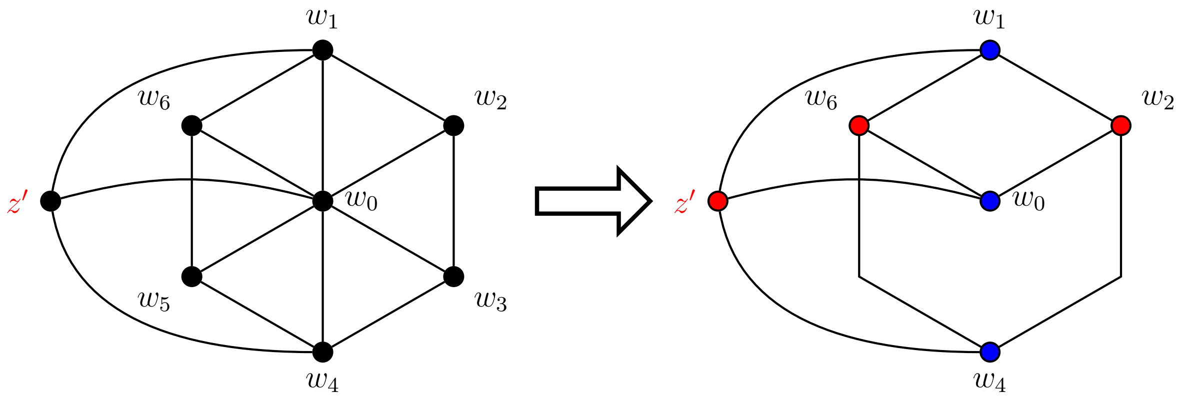


Figure 5.12: A $K_{3,3}$ minor in a subgraph of $\overline{G}$ from Figure 5.11.

Problem 1. *(i) Find a planar graph-complement construction for* $\Theta\langle 2,4,4\rangle$.

(ii) Do all i-graphs with largest induced stars of $K_{1,3}$*, always have a planar graph-complement construction?*

A large target graph requires a large seed graph in order to generate a sufficient number of unique i-sets. Can a target graph become too dense to allow for a planar graph-complement construction?

Moving forward, we will no longer explicitly check that there are no additional unaccounted for triangles in our constructions. Should the construction indeed result in triangles of $\overline{G}$ that produce extraneous vertices in $\mathscr{I}(G)$, we can easily remove them using the Deletion Lemma (Lemma 3.22). In fact, in the previous results like in Construction 5.18, the addition of the vertices z and z' were exactly such applications. Recall that in the Deletion Lemma, when removing a vertex X from the i-graph, a new vertex z is added to the seed graph and joined to every vertex *except* those of X. In the complement seed graph $\overline{G}$, this amounts to adding z and joining to everything in X (and nothing else), exactly as we did in Construction 5.18.

Although we are quite certain that the graphs presented in the following constructions are the exact ones that build the given theta graphs, it is of no actual consequence to the existence of the theta graph as an i-graph if they are not; by the Deletion Lemma, it is enough to present the graph $\overline{G}$ with the knowledge that its complement produces an i-graph that contains the desired theta graph as an induced subgraph.

Constructions of $\overline{G}$ for $\Theta\langle 2,4,5\rangle$ and $\Theta\langle 2,5,5\rangle$

Construction 5.20. *Refer to Figure 5.13.*

(a) Begin with a copy of the graph $\overline{G_{2,3,5}}$ from Construction 5.16 (b) for $\Theta\langle 2,3,5\rangle$. Subdivide the edge w_1w_5, adding the new vertex w_6. Join w_6 to w_0, so that $w_0, w_1, \ldots, w_6$ forms a wheel. Call this graph $\overline{G_{2,4,5}}$.

(b) Begin with a copy of the graph $\overline{G_{2,4,5}}$ from Construction 5.20 (a). Subdivide the edge w_1w_6, adding the new vertex w_7. Join w_7 to w_0, so that $w_0, w_1, \ldots, w_7$ forms a wheel. Call this graph $\overline{G_{2,5,5}}$.

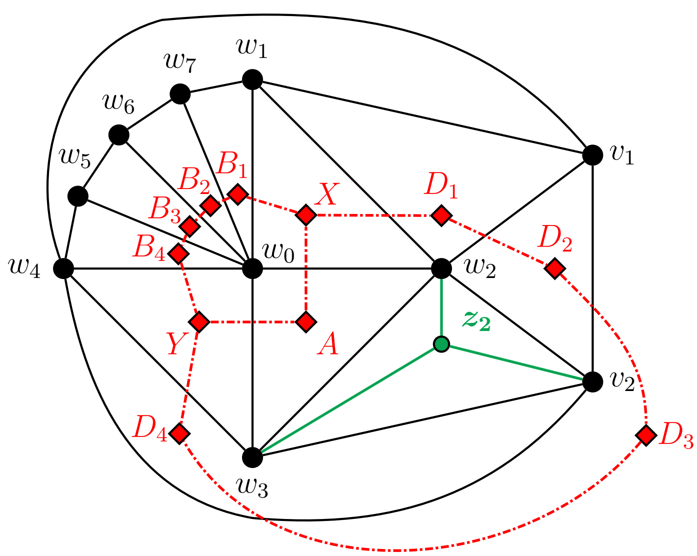


Figure 5.13: The graph $\overline{G_{2,5,5}}$ from Construction 5.20 such that $\mathscr{I}(G) = \Theta\langle 2,5,5\rangle$, with $\mathscr{I}(G)$ overlaid in red.

Lemma 5.21. *If $\overline{G_{2,k,5}}$ is the graph constructed by Construction 5.20, then $\mathscr{I}(G_{2,k,5}) = \Theta\langle 2,k,5\rangle$, for $4 \le k \le 5$.*

The proof for Lemma 5.21 is similar to that of Lemma 5.15(b) and is omitted.

Construction of $\overline{G}$ for $\Theta\langle 2,k,\ell\rangle$ for $\ell \ge k \ge 4$ and $\ell \ge 6$

Construction 5.22. *See Figure 5.14. Begin with a copy of the graph $\overline{G_{2,3,\ell}}$ from Construction 5.16 (a) for $\ell \ge 5$. Subdivide the edge w_1w_5 $k-3$ times (for $k \le \ell$), adding the new vertices $w_6, w_7, \ldots, w_{k+2}$. Join $w_6, w_7, \ldots, w_{k+2}$ to w_0, so that $w_0, w_1, \ldots, w_{k+2}$ forms a wheel. Call this graph $\overline{G_{2,k,\ell}}$.*

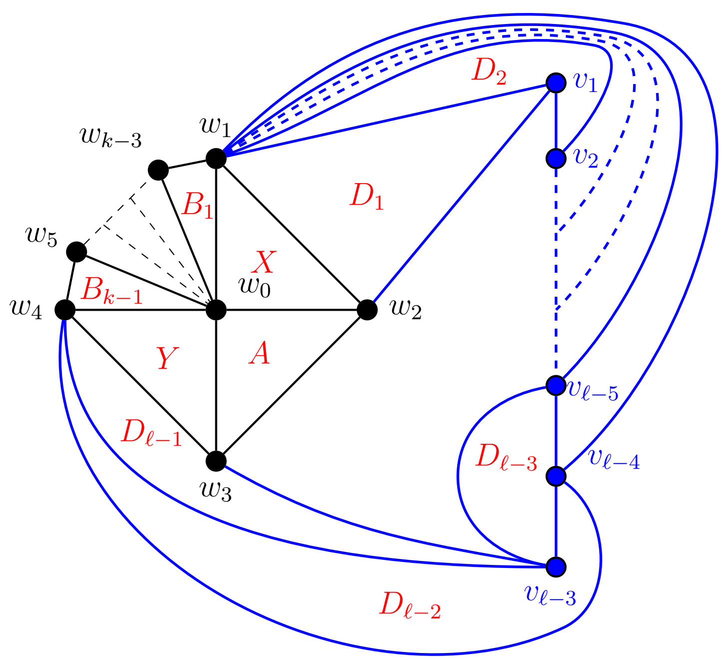


Figure 5.14: The graph $\overline{G_{2,k,\ell}}$ from Construction 5.22 such that $\mathscr{I}(\overline{G_{2,k,\ell}}) = \Theta\langle 2,k,\ell\rangle$ for $\ell \ge k \ge 4$ and $\ell \ge 6$.

The proof for Lemma 5.23 again proceed similarly to the proof of Lemma 5.15(a) and is also omitted.

Lemma 5.23. *If $\overline{G_{2,k,\ell}}$ is the graph constructed by Construction 5.22, then $\mathscr{I}(G_{2,k,\ell}) = \mathscr{A}(G_{2,k,\ell}) = \Theta\langle 2,k,\ell\rangle$, for $\ell \ge k \ge 4$ and $\ell \ge 6$.*

5.2.4 $\Theta\langle 3,k,\ell\rangle$

Construction of $\overline{G}$ for $\Theta\langle 3,3,4\rangle$

When initially exploring the i-graph realizability of $\Theta\langle 3,3,4\rangle$, we encountered difficulties in trying to use a similar method as above utilizing graph complements to create a graph $\overline{G}$ such that $\mathscr{I}(G)=\Theta\langle 3,3,4\rangle$. Through trial and error, we instead found the graph G in Figure 5.15 which has $\mathscr{I}(G)=\Theta\langle 3,3,4\rangle$. Although the following Lemma 5.24 does not use graph complements, we include it here for the completeness of determining the i-graph realizability of theta graphs.

Lemma 5.24. $\Theta\langle 3,3,4\rangle$ *is an i-graph.*

Proof. Consider the graph G in Figure 5.15. We claim that $\mathscr{I}(G)=\Theta\langle 3,3,4\rangle$, with corresponding i-sets as illustrated in Figure 5.16.

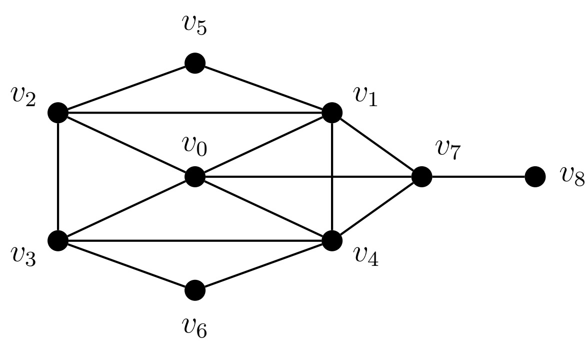


Figure 5.15: A graph G such that $\mathscr{I}(G)=\Theta\langle 3,3,4\rangle$.

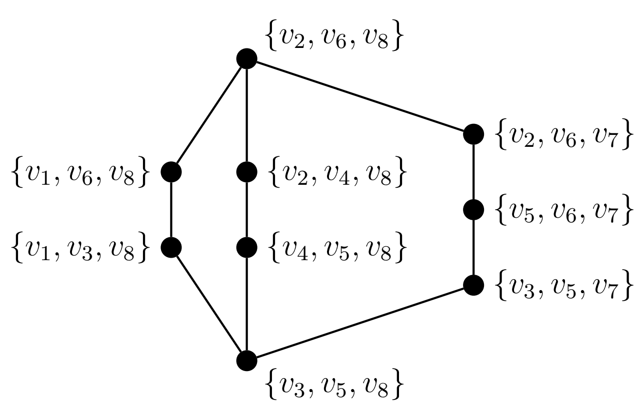


Figure 5.16: The i-graph of G from Figure 5.15.

Notice that at least one vertex of $T_1 = \{v_1, v_2, v_5\}$, $T_2 = \{v_3, v_4, v_6\}$ and $\{v_7, v_8\}$ is in each i-set of G, and hence $i(G) = 3$. It is straight-forward to confirm that each set illustrated in Figure 5.15 is an i-set of G, and that the i-set adjacencies (and edges of $\mathscr{I}(G)$) are as drawn.

Thus, we need only confirm that G has only the nine i-sets listed there, and that none are unaccounted for in Figure 5.15. To do so, we proceed with a counting argument. If $v_8 \in S$, then there are three options for each of T_1 and T_2, except for the cases where the selected vertices of each triangle are adjacent to each other: $\{v_2, v_3\}$, $\{v_1, v_4\}$, and the case where v_0 is left undominated, $\{v_5, v_6\}$. Thus there are $(3)(3) - 3 = 6$ i-sets containing v_8. If an i-set contains v_7, then v_1 and v_4 are not in the i-set, so exactly one each of $T_1 - \{v_1\} = \{v_2, v_5\}$ and $T_2 - \{v_4\} = \{v_3, v_6\}$ are in the i-set. We again must exclude the case where both adjacent v_2 and v_3 are present; since v_7 dominates v_0, we no longer exclude the case where both v_5 and v_6 are present. Thus, there are $(2)(2) - 1 = 3$ i-sets containing v_7. This accounts for all nine of the i-sets required to construct $\Theta\langle 3, 3, 4\rangle$. ■

For completeness, in Figure 5.17 we include a sketch of the complement of the graph G given in Figure 5.15, using a layout more familiar to the format we saw in the previous constructions, but with the same vertex labellings as in Figure 5.15. Although very similar, it is one of only two the constructions (along with Construction 5.18) that require an additional edge between two of the wheel vertices (i.e. v_5 and v_6). Moreover, this representation of $\overline{G}$ - and the set D_2 in particular - highlights how easily triangles of $\overline{G}$ can be overlooked when they are non-facial triangles. We also note that this construction joins $\Theta\langle 2, 4, 4\rangle$ among those currently lacking a planar graph complement construction.

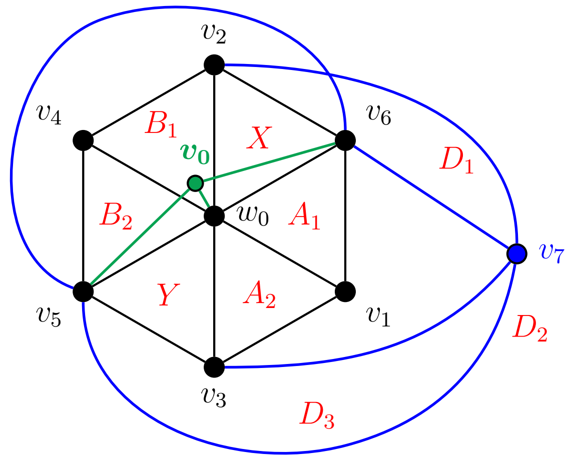


Figure 5.17: The complement of the graph G from Figure 5.15: a graph $\overline{G}$ such that $\mathscr{I}(G) = \Theta\langle 3, 3, 4\rangle$.

Construction of $\overline{G}$ for $\Theta\langle 3, 3, 5\rangle$

Construction 5.25. *Refer to Figure 5.18. Begin with a copy of the graph* $\overline{H} \cong W_7 = C_6 \vee K_1$*, labelling the degree 3 vertices as* $w_1, w_2, \ldots, w_6$ *and the central degree 6 vertex as* w_0*. Add new vertices* v_1 *and* v_2 *to* $\overline{H}$*, joined to each other. Then, join* v_1 *to each of* $\{w_1, w_2, w_5\}$*, and* v_2 *to each of* $\{w_2, w_4, w_5\}$*. Call this graph* $\overline{G_{3,3,5}}$*.*

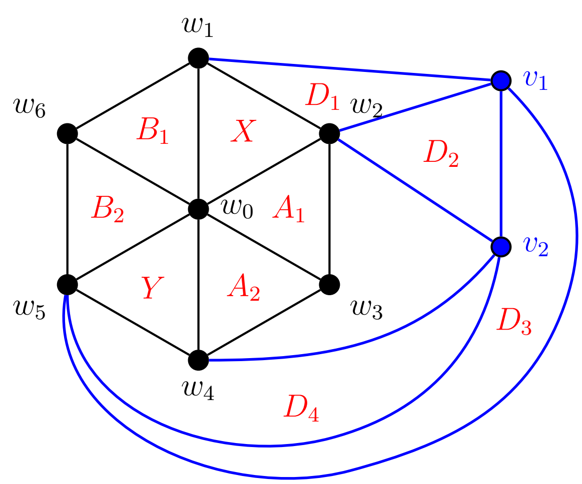


Figure 5.18: A graph $\overline{G_{3,3,5}}$ from Construction 5.25 such that $\mathscr{I}(G_{3,3,5}) = \Theta\langle 3, 3, 5\rangle$.

Lemma 5.26. *If $\overline{G_{3,3,5}}$ is the graph constructed by Construction 5.25, then $\mathscr{I}(G_{3,3,5}) = \mathscr{A}(G_{3,3,5}) = \Theta\langle 3,3,5\rangle$.*

Proof. From Lemma 5.12, the wheel $\mathcal{W} = \{w_0, w_1, \ldots, w_6\}$ in $\overline{G}$ forms a cycle C_6 with vertices $\{X, A_1, A_2, Y, B_1, B_2\}$ in $\mathscr{I}(G)$. We need only check that in $\mathscr{I}(G)$, $\mathcal{D} = \{D_1, D_2, D_3, D_4\}$ (where each D_i is defined as in Figure 5.18) forms the necessary third disjoint path between X and Y. As discussed above, we need not concern ourselves over whether there are additional triangles of $\overline{G}$ (and hence vertices of $\mathscr{I}(G)$) present beyond the ten labelled in red in Figure 5.18: they can easily be removed via the Deletion Lemma if needed.

From Figure 5.18, we have that

$$X \overset{w_0v_1}{\sim} D_1 \overset{w_1v_2}{\sim} D_2 \overset{w_2w_5}{\sim} D_3 \overset{v_1w_4}{\sim} D_4 \overset{v_2w_0}{\sim} Y,$$

which forms the additional path between X and Y in $\mathscr{I}(G)$. Since D_2 and D_3 both contain v_1 and v_2, of the four triangles in $\mathcal{D}$, only D_1 and D_4 contain at least two of the wheel vertices from $\mathcal{W}$. Thus, only D_1 and D_2 are adjacent to wheel triangles, namely X and Y, respectively. It follows that there are no additional edges between the vertices of $\mathcal{D}$ in $\mathscr{I}(G)$. We conclude that $\mathscr{I}(\overline{G}) = \Theta\langle 3,3,5\rangle$. ■

Construction of $\overline{G}$ for $\Theta\langle 3,3,\ell\rangle$ for $\ell \geq 6$

Construction 5.27. *Refer to Figure 5.20. Begin with a copy of the graph $\overline{H} \cong W_7 = C_6 \vee K_1$, labelling the degree 3 vertices as $w_1, w_2, \ldots, w_6$ and the central degree 4 vertex as w_0. For $\ell \geq 6$, add to $\overline{H}$ a new path of $\ell - 3$ vertices labelled as $v_1, v_2, \ldots, v_{\ell-3}$. Then, join w_1 to each of $\{v_1, v_2, \ldots, v_{\ell-4}\}$, w_4 to $v_{\ell-3}$, and w_5 to $v_{\ell-4}$ and $v_{\ell-3}$. Finally, join $v_{\ell-5}$ to $v_{\ell-3}$, so that $\{v_{\ell-5}, v_{\ell-4}, v_{\ell-3}\}$ form a K_3. Call this graph $\overline{G_{3,3,\ell}}$ for $\ell \geq 6$.*

Although the general construction still applies for the case when $\ell = 6$, we include a separate figure for the construction of $\Theta\langle 3,3,6\rangle$ for reference below, because of the additional complication that now $v_1 = v_{\ell-5}$, and so the single vertex now has dual roles in the construction.

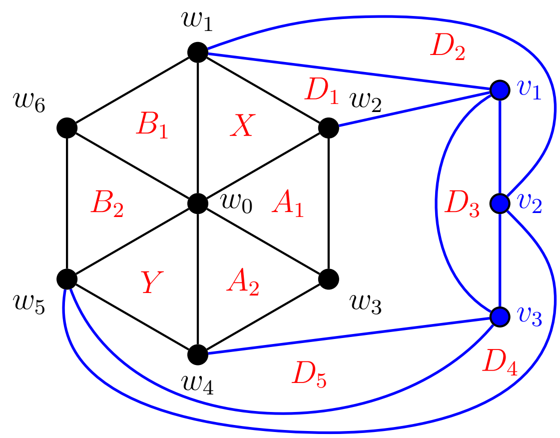


Figure 5.19: The graph $\overline{G_{3,3,6}}$ from Construction 5.27 such that $\mathscr{I}(G_{3,3,6}) = \mathscr{A}(G_{3,3,6}) = \Theta\langle 3, 3, 6\rangle$.

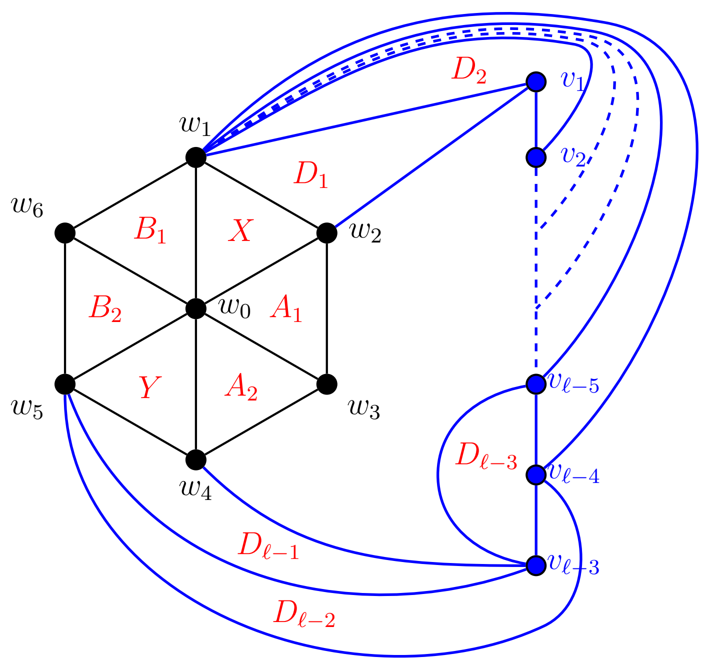


Figure 5.20: The graph $\overline{G_{3,3,\ell}}$ from Construction 5.27 such that $\mathscr{I}(G) = \mathscr{A}(G) = \Theta\langle 3, 3, \ell\rangle$ for $\ell \geq 6$.

Lemma 5.28. *If $\overline{G_{3,3,\ell}}$ is the graph constructed by Construction 5.27, then $\mathscr{I}(G_{3,3,\ell}) = \mathscr{A}(G_{3,3,\ell}) = \Theta\langle 3, 3, \ell\rangle$ for $\ell \geq 6$.*

Proof. We proceed as in the proof of Lemma 5.26 for $\Theta\langle 3, 3, 5\rangle$, labelling the triangles as in Figure 5.20, with the amendment that we now ensure that $\mathcal{D} = \{D_1, D_2, \ldots, D_{\ell-1}\}$ forms the necessary third disjoint path between X and Y in $\mathscr{I}(\overline{G})$.

Clearly from Figure 5.20,

$$X \overset{w_0v_1}{\sim} D_1 \overset{w_2v_2}{\sim} D_2 \overset{v_1v_3}{\sim} D_3 \overset{v_2v_4}{\sim} \ldots \overset{v_{\ell-6}v_{\ell-4}}{\sim} D_{\ell-4} \overset{w_1v_{\ell-3}}{\sim} D_{\ell-3} \overset{v_{\ell-5}w_5}{\sim} D_{\ell-2} \overset{v_{\ell-4}w_4}{\sim} D_{\ell-1} \overset{v_{\ell-3}w_0}{\sim} Y,$$

where $D_i \overset{v_{i-1}v_{i+1}}{\sim} D_{i+1}$ for $3 \leq i \leq \ell - 5$.

Similarly to before, of the triangles in $\mathcal{D}$, only D_1 and $D_{\ell-1}$ contain at least two of the wheel vertices from $\mathcal{W}$, and so only D_1 and $D_{\ell-1}$ are adjacent to wheel triangles. We may assume that each i-set of $\overline{G}$ corresponds to a facial triangle of G; if this is not the case, then as previously observed, we can use the Deletion Lemma (Lemma 3.22) to remove any non-facial triangle i-sets from the construction. Thus, a vertex of $\mathscr{I}(G)$ corresponding to a triangle D_i of $\overline{G}$ is adjacent to at most three others.

For $i = 2, 3, \ldots, \ell - 5$, we have already seen that $D_{i-1} \overset{v_{i-2}v_i}{\sim} D_i \overset{v_{i-1}v_{i+1}}{\sim} D_{i+1}$. If D_i were adjacent to a third facial triangle, it would be to a triangle sharing the edge $v_{i-1}v_i \in E(\overline{G})$; however, $N_{\overline{G}}(\{v_{i-1}, v_i\}) = \{w_1\}$ and since $D_i = \{w_1, v_{i-1}, v_i\}$, no such third triangle exists. Thus, we need only check that for $i \in \{1, \ell-4, \ell-3, \ell-2, \ell-1\}$, $\deg_{\overline{G}}(D_i) = 2$. For these cases, it is not overly arduous to exhaustively check that the edges v_1w_2, $w_1v_{\ell-4}$, $v_{\ell-5}v_{\ell-3}$, $v_{\ell-4}w_5$, and $v_{\ell-3}w_4$, respectively, are incident with exactly one triangle in $\overline{G}$. It follows that in $\mathscr{I}(G)$, $X, D_1, D_2, \ldots, D_{\ell-2}, D_{\ell-1}, Y$ forms a path of length ℓ from X to Y, and hence that $\mathscr{I}(G) = \Theta\langle 3, 3, \ell\rangle$. ■

The proofs of the remaining lemmas in Section 5.2 are all similar to proofs of previous lemmas and are therefore omitted without further comments.

Construction of $\overline{G}$ for $\Theta\langle 3, 4, 4\rangle$

Construction 5.29. *See Figure 5.21. Begin with a copy of the graph $\overline{G}$ from Construction 5.18 for $\Theta\langle 2, 4, 4\rangle$, which we rename here as $\overline{G_{2,4,4}}$. Subdivide the edge w_2w_3, adding a new vertex u_1, and joining u_1 to w_0. Delete the vertex z. Call this graph $\overline{G_{3,4,4}}$.*

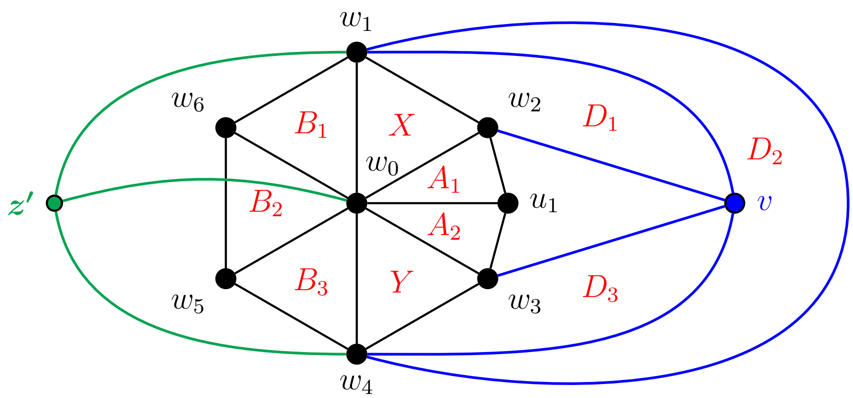


Figure 5.21: The graph $\overline{G_{3,4,4}}$ from Construction 5.29 such that $\mathscr{I}(G_{3,4,4}) = \Theta\langle 3, 4, 4\rangle$.

Lemma 5.30. *If $\overline{G_{3,4,4}}$ is the graph constructed by Construction 5.29, then $\mathscr{I}(G_{3,4,4}) = \Theta\langle 3, 4, 4\rangle$.*

Construction of $\overline{G}$ for $\Theta\langle 3, 4, \ell\rangle$, $\ell \geq 5$

Construction 5.31. *Begin with a copy of the graph $\overline{G_{3,4,4}}$ from Construction 5.29 for $\Theta\langle 3, 4, 4\rangle$. Subdivide the edge w_1w_6 $\ell - 4$ times (for $\ell \geq 5$), adding the new vertices w_7, $w_8, \ldots, w_{\ell+2}$. Join $w_7, w_8, \ldots, w_{\ell+2}$ to w_0, so that $w_0, w_1, \ldots, w_{\ell+2}$ forms a wheel. Call this graph $\overline{G_{3,4,\ell}}$.*

Lemma 5.32. *If $\overline{G_{3,4,\ell}}$ is the graph constructed by Construction 5.31, then $\mathscr{I}(G_{3,4,\ell}) = \Theta\langle 3, 4, \ell\rangle$ for $\ell \geq 5$.*

Construction of $\overline{G}$ for $\Theta\langle 3, 5, 5\rangle$

Construction 5.33. *Refer to Figure 5.22. Begin with a copy of the graph $\overline{G_{3,3,5}}$ from Construction 5.25. Subdivide the edge w_1w_6 twice, adding the new vertices w_7 and w_8. Join w_7 and w_8 to w_0, so that $w_0, w_1, \ldots, w_8$ forms a wheel. Call this graph $\overline{G_{3,5,5}}$.*

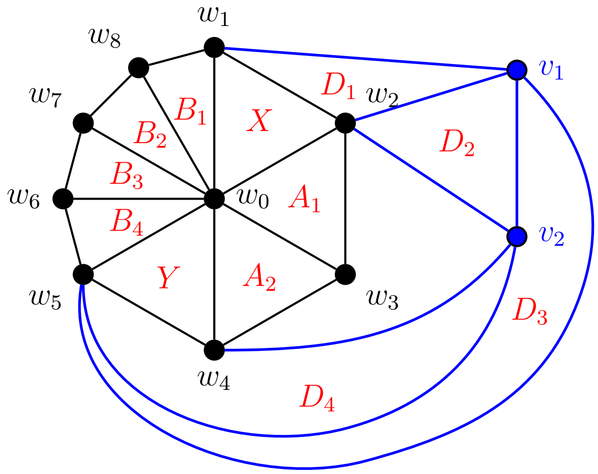


Figure 5.22: A graph $\overline{G_{3,5,5}}$ from Construction 5.33 such that $\mathscr{I}(G_{3,5,5}) = \Theta\langle 3,5,5\rangle$.

Lemma 5.34. *If $\overline{G_{3,5,5}}$ is the graph constructed by Construction 5.33, then $\mathscr{I}(G_{3,5,5}) = \mathscr{A}(G_{3,5,5}) = \Theta\langle 3,5,5\rangle$.*

Notice that subdividing only once in Construction 5.33 (adding only w_7 and not w_8) gives an alternative (planar) construction for $\Theta\langle 3,4,5\rangle$.

5.2.5 $\Theta\langle j,k,\ell\rangle$ for $4 \le j \le k \le \ell$ and $3 \le j \le k \le \ell$, $\ell \ge 6$

Construction of $\overline{G}$ for $\Theta\langle j,k,\ell\rangle$ for $4 \le j \le k \le \ell \le 5$

Construction 5.35. *Refer to Figure 5.23. Begin with a copy of the graph $\overline{G_{3,4,4}}$ from Construction 5.29. Subdivide the edge u_1w_3, adding the new vertex u_2. Join u_2 to w_0, so that*

$$w_0, w_1, w_2, u_1, u_2, w_3, \ldots, w_6$$

forms a wheel. Call this graph $\overline{G_{4,4,4}}$.

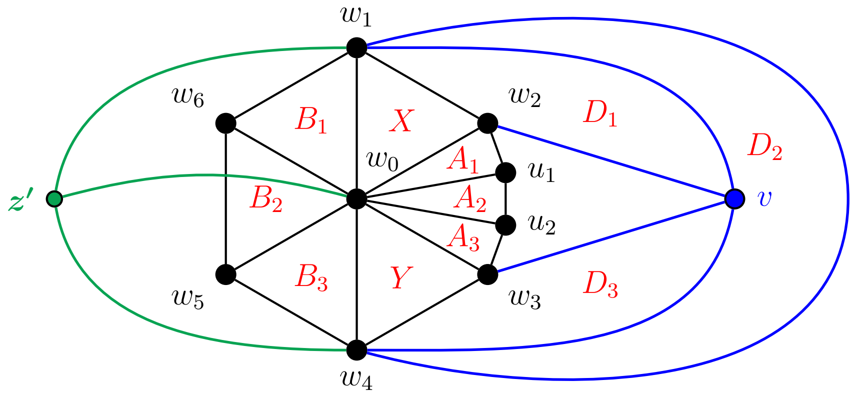


Figure 5.23: The graph $\overline{G_{4,4,4}}$ from Construction 5.35 such that $\mathscr{I}(G_{4,4,4}) = \Theta\langle 4,4,4\rangle$.

Lemma 5.36. *If $\overline{G_{4,4,4}}$ is the graph constructed by Construction 5.35, then* $\mathscr{I}(G_{4,4,4}) = \Theta\langle 4,4,4\rangle$.

Construction 5.37. *Begin with a copy of the graph $\overline{G_{3,3,5}}$ from Construction 5.25. For $k = 4$ subdivide the edge w_1w_6 once, adding the vertex w_7; for $k = 5$, subdivide a second time, adding the vertex w_8. For $j = 4$, subdivide the edge w_2w_3, adding the vertex u_1; for $j = 5$ ($j \leq k$), subdivide a second time, adding the vertex u_2. Connect all new vertices to w_0 to form a wheel. Call this graph $\overline{G_{j,k,5}}$ for $4 \leq j \leq k \leq 5$.*

An example of the construction of $\overline{G_{5,5,5}}$ is given in Figure 5.24 below.

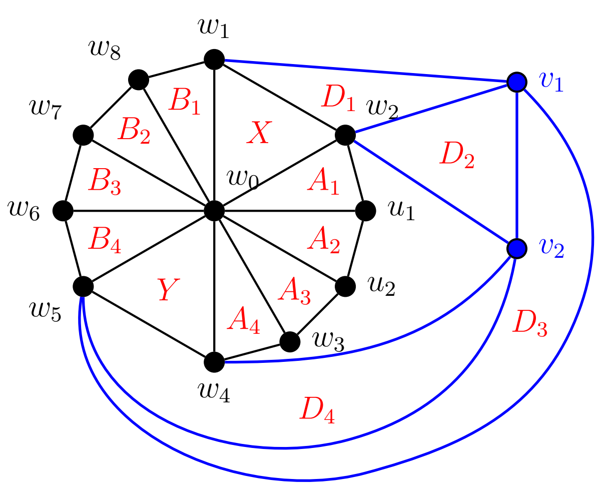


Figure 5.24: A graph $\overline{G_{5,5,5}}$ from Construction 5.33 such that $\mathscr{I}(G_{5,5,5}) = \Theta\langle 5,5,5\rangle$.

Lemma 5.38. *If $\overline{G_{j,k,5}}$ is the graph constructed by Construction 5.37, then $\mathscr{I}(G_{j,k,5}) = \mathscr{A}(G_{j,k,5}) = \Theta\langle j,k,5\rangle$ for $4 \le j \le k \le 5$.*

Construction of $\overline{G}$ for $\Theta\langle j,k,\ell\rangle$ for $3 \le j \le k \le \ell$, $\ell \ge 6$.

Construction 5.39. *Begin with a copy of the graph $\overline{G_{3,3,\ell}}$ from Construction 5.27 for $\Theta\langle 3,5,\ell\rangle$ for $\ell \ge 6$. For $3 \le k \le \ell$, subdivide the edge w_1w_6 $k-3$ times, adding the new vertices $w_7, w_8, \ldots, w_{k+3}$. Join each of $w_7, w_8, \ldots, w_{k+3}$ to w_0, so that $w_0, w_1, \ldots, w_{k+3}$ forms a wheel. Then, for $3 \le j \le k$, subdivide the edge w_2w_3 $j-3$ times, adding the new vertices $u_1, u_2, \ldots, u_{j-3}$. Again, join each of $u_1, u_2, \ldots, u_{j-3}$ to w_0 to form a wheel. Call this graph $\overline{G_{j,k,\ell}}$ for $3 \le j \le k \le \ell$ and $\ell \ge 6$.*

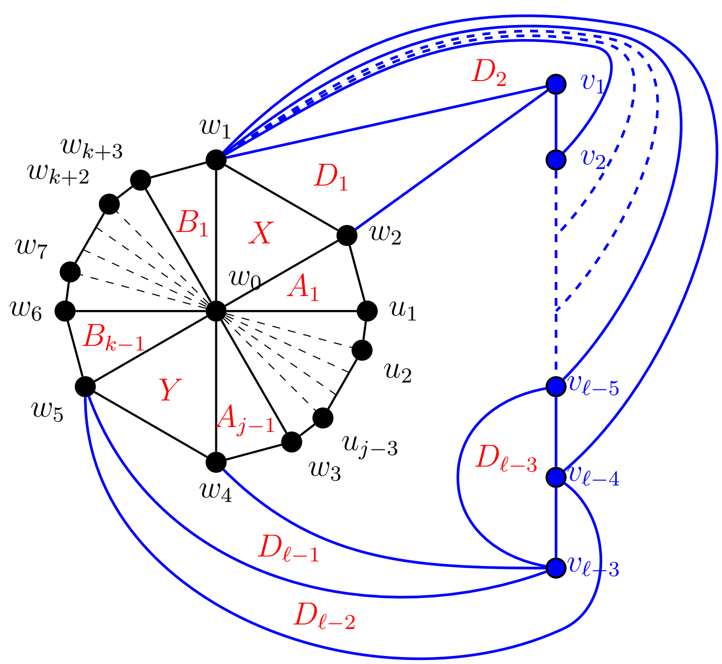


Figure 5.25: The graph $\overline{G_{j,k,\ell}}$ from Construction 5.39 such that $\mathscr{I}(G_{j,k,\ell}) = \Theta\langle j,k,\ell\rangle$ for $3 \le j \le k \le \ell$ and $\ell \ge 6$.

Lemma 5.40. *If $\overline{G_{j,k,\ell}}$ for $3 \le j \le k \le \ell$ and $\ell \ge 6$ is the graph constructed by Construction 5.39, then $\mathscr{I}(G_{j,k,\ell}) = \mathscr{A}(G_{j,k,\ell}) = \Theta\langle j,k,\ell\rangle$.*

The lemmas above imply the sufficiency of Theorem 5.8: if a theta graph is not one of seven exceptions listed, then it is an i-graph. In the next section, we complete the proof by examining the exception cases.

5.3 Theta Graphs That Are Not i-Graphs

In this section we show that $\Theta\langle 2,2,4\rangle$, $\Theta\langle 2,3,3\rangle$, $\Theta\langle 2,3,4\rangle$, and $\Theta\langle 3,3,3\rangle$ are not i-graphs; together with Propositions 3.8, 3.9, and 3.10, this completes the proof of Theorem 5.8.

$\Theta\langle 2,2,4\rangle$ Is Not an i-Graph

Proposition 5.41. *The graph $\Theta\langle 2,2,4\rangle$ is not i-graph realizable.*

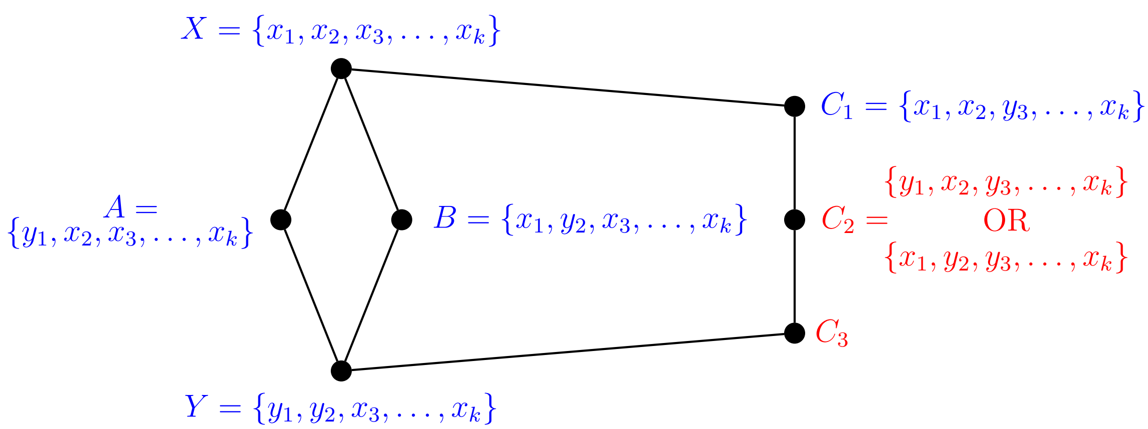


Figure 5.26: $H = \Theta\langle 2,2,4\rangle$ non-construction.

Proof. Suppose to the contrary that $\Theta\langle 2,2,4\rangle$ is realizable as an i-graph, and that $H = \Theta\langle 2,2,4\rangle \cong \mathscr{I}(G)$ for some graph G. Label the vertices of H as in Figure 5.26.

From Proposition 3.1, and similarly to the proofs of Propositions 3.9 and 3.10, the composition of the following i-sets of G are immediate:

$$X = \{x_1, x_2, x_3, \ldots, x_k\}, \qquad Y = \{y_1, y_2, x_3, \ldots, x_k\},$$
$$A = \{y_1, x_2, x_3, \ldots, x_k\}, \qquad B = \{x_1, y_2, x_3, \ldots, x_k\},$$
$$C_1 = \{x_1, x_2, y_3, \ldots, x_k\}$$

where $k \geq 3$ and y_1, y_2, y_3 are three distinct vertices in $G - X$. These sets are illustrated in blue in Figure 5.26. This leaves only the composition of C_2 and C_3 (in red) to be determined. As we construct C_2, notice first that $y_3 \in C_2$; otherwise, if say some other $z \in C_2$ so that $C_1 \overset{y_3z}{\sim} C_2$, then $X \overset{x_3z}{\sim} C_2$, and $XC_2 \in E(H)$. Thus, a token on one of $\{x_1, x_2, x_4, \ldots, x_k\}$ moves in the transition from C_1 to C_2. We consider three cases.

Case 1: The token on x_1 moves. If $C_1 \overset{x_1z}{\sim} C_2$ for some $z \notin \{y_1, y_2\}$, then $|C_2 \cap Y| = 3$, contradicting the distance requirement between i-sets from Observation 3.2. Moreover, from the composition of B, $x_1 \not\sim y_2$, and so $C_1 \overset{x_1y_1}{\sim} C_2$, so that $C_2 = \{y_1, x_2, y_3, \ldots, x_k\}$. However, since $x_3 \sim y_3$, we have that $A \overset{x_3y_3}{\sim} C_2$, so that $AC_2 \in E(G)$, a contradiction.

Case 2: The token on x_2 moves. An argument similar to Case 1 constructs $C_2 = \{x_1, y_2, y_3, \ldots, x_k\}$, with $B \overset{x_3y_3}{\sim} C_2$, resulting in the contradiction $BC_2 \in E(G)$.

Case 3: The token on x_i for some $i \in \{4, 5, \ldots, k\}$ moves. From the compositions of X and Y, x_i is not adjacent to any of $\{x_3, y_1, y_2\}$, so the token at x_i moves to some other vertex, say z, so that $C_1 \overset{x_iz}{\sim} C_2$ and $\{x_1, x_2, y_3, z\} \subseteq C_2$. This again contradicts the distance requirement of Observation 3.2 as $|C_2 \cap Y| = 4$.

In all cases, we fail to construct a graph G with $\mathscr{I}(G) \cong \Theta\langle 2, 2, 4\rangle$ and so conclude that no such graph exists. ■

$\Theta\langle 2,3,3\rangle$ Is Not an i-Graph

Proposition 5.42. *The graph $\Theta\langle 2,3,3\rangle$ is not i-graph realizable.*

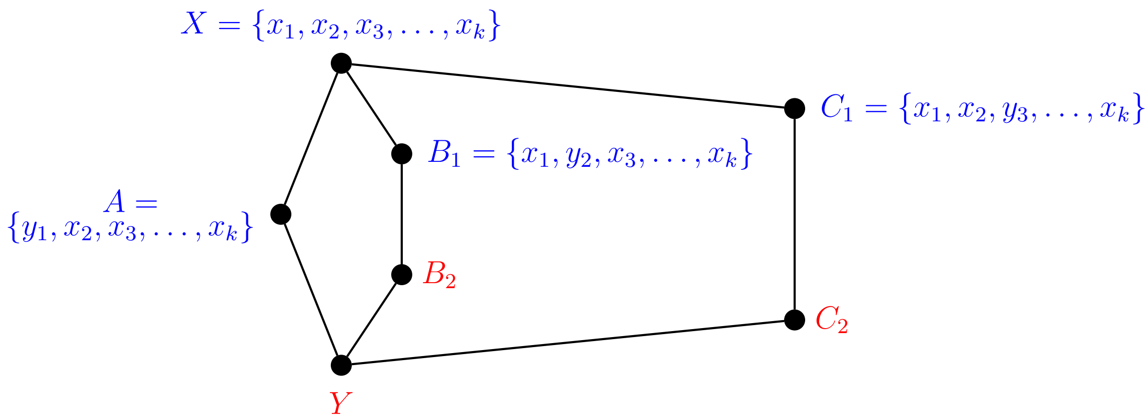


Figure 5.27: $H = \Theta\langle 2,3,3\rangle$ non-construction.

Proof. To begin, we proceed similarly to the proof of Proposition 5.41: suppose to the contrary that $\Theta\langle 2,3,3\rangle$ is realizable as an i-graph, and that $H = \Theta\langle 2,3,3\rangle \cong \mathscr{I}(G)$ for some graph G. Label the vertices of H as in Figure 5.27. As before, the corresponding i-sets in blue are established from previous results, and those in red are yet to be determined. Moreover, from the composition of these four blue i-sets, we observe that for each $i \in \{1,2,3\}$, $x_i \sim y_j$ if and only if $i = j$.

Unlike the construction for $\Theta\langle 2,2,4\rangle$, we no longer start with knowledge of the exact composition of Y. We proceed with a series of observations on the contents of the various i-sets:

(i) $y_1 \in Y$, $y_2 \in B_2$, and $y_3 \in C_2$ by three applications of Proposition 3.6.

(ii) $y_1 \notin B_2$ and $y_1 \notin C_2$. If $y_1 \in B_2$, then $B_1 \overset{x_1y_1}{\sim} B_2$ (because A shows that y_1 is not adjacent to $x_3, \dots, x_k$) so that $B_2 = \{y_1, y_2, x_3, \dots, x_k\}$, and therefore $A \overset{x_2y_2}{\sim} B_2$, which is impossible. Similarly, if $y_1 \in C_2$, then $A \overset{x_3y_3}{\sim} C_2$, which is also impossible.

(iii) $y_3 \notin B_2$. Otherwise, $B_2 = \{x_1, y_2, y_3, x_4, \dots, x_k\}$ and so $C_1 \overset{x_2y_2}{\sim} B_2$.

(iv) $y_2 \notin Y$ and $y_3 \notin Y$. If $y_2 \in Y$, then $Y = \{y_1, y_2, x_3, \dots, x_k\}$ and so $B_1 \overset{x_1y_1}{\sim} Y$. Likewise, if $y_3 \in Y$ then $C_1 \overset{x_1y_1}{\sim} Y$.

From (i) and (ii), $y_1 \in Y$ but $y_1 \notin B_2$, and similarly from (iv) $y_2 \in B_2$ but $y_2 \notin Y$; therefore, $B_2 \overset{y_2 y_1}{\sim} Y$. Now, since $x_1 \sim y_1$, and $y_1 \in Y$, we have that $x_1 \notin Y$. Thus, if x_1 were in B_2, its token would move in the transition from B_2 to Y. However, have already established that it is the token at y_2 that moves, and so $x_1 \notin B_2$. We conclude that $B_1 \overset{x_1 z}{\sim} B_2$ for some $z \notin \{y_1, y_3\}$, so that $B_2 = \{z, y_2, x_3, \ldots, x_k\}$. Notice that since B_2 is independent, and $x_2 \sim y_2$, it follows that $z \neq x_2$.

Using similar arguments, we determine that $C_2 \overset{y_3 y_1}{\sim} Y$, and that $C_1 \overset{x_1 w}{\sim} C_2$ for some $w \notin \{y_1, y_2, x_1\}$. Moreover, $C_2 = \{w, x_2, y_3, x_4, \ldots, x_k\}$. Again, note that since $x_3 \sim y_3$, $w \neq x_3$.

From $B_2 \overset{y_2 y_1}{\sim} Y$, we have that $Y = \{y_1, z, x_3, x_4 \ldots, x_k\}$. However, from $C_2 \overset{y_3 y_1}{\sim} Y$, we also have that $Y = \{y_1, x_2, w, x_4, \ldots, x_k\}$. As we have already established that $z \neq x_2$ and $w \neq x_3$, we arrive at two contradicting compositions of Y. Thus, no such graph G exists, and we conclude that $\Theta \langle 2, 3, 3 \rangle$ is not an i-graph. ■

$\Theta \langle 2, 3, 4 \rangle$ Is Not an i-Graph

Proposition 5.43. *The graph $\Theta \langle 2, 3, 4 \rangle$ is not i-graph realizable.*

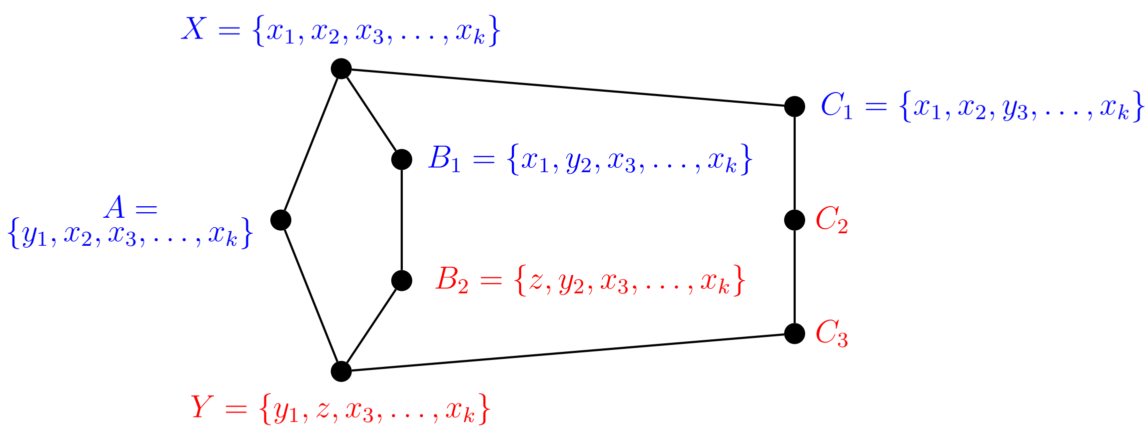


Figure 5.28: $H = \Theta \langle 2, 3, 4 \rangle$ non-construction.

Proof. The construction for our contradiction begins similarly to that of $\Theta \langle 2, 3, 3 \rangle$ in Proposition 5.42. As before, we illustrate the graph in Figure 5.28, labelling the known sets in blue, and those yet to be determined in red. Given the similarity of $\Theta\langle 2,3,3\rangle$ and $\Theta\langle 2,3,4\rangle$, many of the observations from Proposition 5.42 carry through to our current proof. In particular, all of (i) - (iv) hold here, including that $y_1 \notin C_2$ from (ii). Moreover, the compositions of Y and B_2 also hold, where z is some vertex with $z \notin \{y_1, x_2, y_2\}$.

We now attempt to build C_2. From Proposition 3.6, since $X \not\sim C_2$, $y_3 \in C_2$ (and $x_3 \notin C_2$). From the distance requirement of Observation 3.2, $|X - C_2| \leq 2$, and so at least one of x_1 or x_2 is in C_2. Recall from the construction for Proposition 5.42 that $A \overset{x_2 z}{\sim} Y$ and $B_1 \overset{x_1 z}{\sim} B_2$, and so z is adjacent to both x_1 and x_2. Hence, $z \notin C_2$.

Gathering these results shows that none of $\{x_3, z, y_1\}$ are in C_2, and thus, $d(C_2, Y) \geq 3$, contradicting the distance requirement of Observation 3.2. We conclude that no graph G exists such that $\mathscr{I}(G) = \Theta\langle 2,3,4\rangle$. ■

$\Theta\langle 3,3,3\rangle$ Is Not an i-Graph

Proposition 5.44. *The graph $\Theta\langle 3,3,3\rangle$ is not i-graph realizable.*

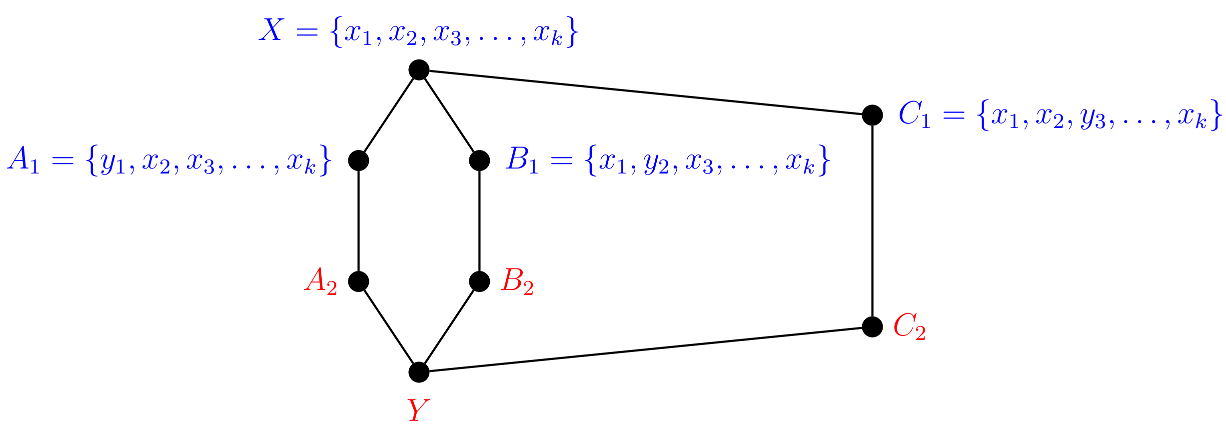


Figure 5.29: $H = \Theta\langle 3,3,3\rangle$ non-construction.

Proof. Let H be the theta graph $\Theta\langle 3,3,3\rangle$, with vertices labelled as in Figure 5.29. Suppose to the contrary that there exists some graph G such that H is the i-graph of G; that is, $\mathscr{I}(G) = \Theta\langle 3,3,3\rangle$.

Since $d_H(A_1, Y) = 2$, by Observation 3.4, $|A_1 - Y| = 2$. Similarly, $|B_1 - Y| = 2$ and $|C_1 - Y| = 2$. Suppose that, say, $x_4 \notin Y$. Hence, by Observation 3.2 $|\{x_1, x_2, x_3\} \cap Y| \geq 1$.

Without loss of generality, say $x_1 \in Y$. Then since $y_1 \notin Y$ and $x_4 \notin Y$, both x_2 and $x_3 \in Y$ to satisfy $|A_1 - Y| = 2$. However, then $Y = \{x_1, x_2, x_3, z, x_5, \ldots, x_k\}$ for some vertex $z \sim x_4$, and so $X \overset{x_4 z}{\sim} Y$, which is not so. We therefore conclude that $x_4 \in Y$, and likewise $x_i \in Y$ for $i \geq 4$. Thus, $\{x_1, x_2, x_3\} \cap Y = \varnothing$.

Returning to A_1, since $d(A_1, Y) = 2$ and $x_2, x_3 \notin Y$, we have that $y_1 \in Y$. Similarly, $y_2, y_3 \in Y$. Thus, $Y = \{y_1, y_2, y_3, x_3, \ldots, x_k\}$. Moreover, A_2 is obtained from A_1 by replacing one of x_2 or x_3, by y_2 or y_3, respectively. Say, $A_1 \overset{x_2 y_2}{\sim} A_2$ so that $A_2 = \{y_1, y_2, x_3, \ldots, x_k\}$. Now, however, we have that $B_1 \overset{x_1 y_1}{\sim} A_2$, but clearly $B_1 \not\sim A_2$. It follows that $\Theta \langle 3, 3, 3 \rangle$ is not an i-graph. ■

This completes the proof of Theorem 5.8.

5.4 Other Results

In the final section of this chapter we first display a graph that is neither a theta graph nor an i-graph, and then use the method of graph complements to show that every cubic 3-connected bipartite planar graph is an i-graph.

5.4.1 A Non-Theta Non-i-Graph

So far, every non-i-graph we have observed is either one of the seven theta graphs from Theorem 5.8, or contains one of those seven as an induced subgraph (as per Corollary 3.23). This leads naturally to the question of whether theta graphs provide a forbidden subgraph characterization for i-graphs. Unfortunately, this is not the case.

Consider the graph $\mathfrak{T}$ in Figure 5.30: it is not a theta graph, and although it contains several theta graphs as induced subgraphs, none of those induced subgraphs are among the seven non-i-graph theta graphs. In Proposition 5.45 we confirm that $\mathfrak{T}$ is not an i-graph.

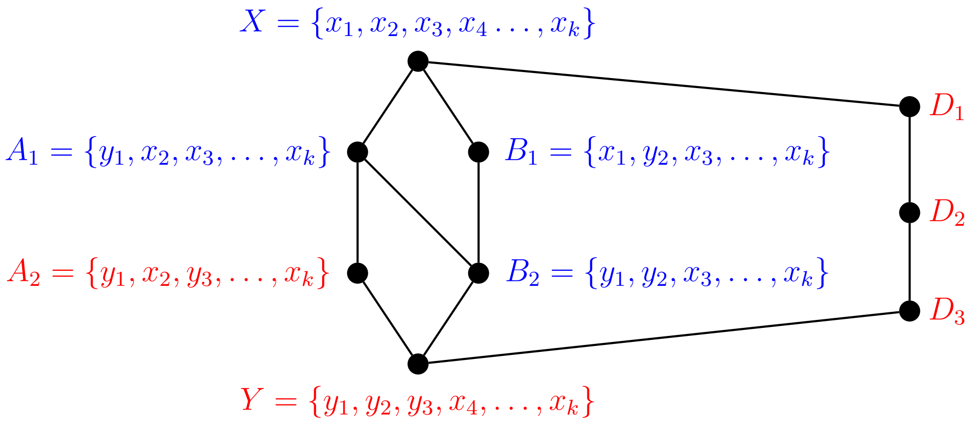


Figure 5.30: A non-theta non-i-graph $\mathfrak{T}$.

Proposition 5.45. *The graph $\mathfrak{T}$ in Figure 5.30 is not an i-graph.*

Proof. We proceed similarly to the proofs for the theta non-i-graphs in Section 5.3. Let $\mathfrak{T}$ be the graph in Figure 5.30, with vertices as labelled, and suppose to the contrary that there is some graph G such that $\mathscr{I}(G) \cong \mathfrak{T}$.

To begin, we determine the vertices of the two induced C_4's of $\mathfrak{T}$. Immediately from Proposition 3.6, the vertices of X, A_1, B_1, and B_2 are as labelled in Figure 5.30. Using a second application of Proposition 3.6, A_2 differs from A_1 in exactly one position, different from B_2. Without loss of generality, say, $A_1 \overset{x_3y_3}{\sim} A_2$. Then again by the proposition, $Y = \{y_1, y_2, y_3, x_4 \ldots, x_k\}$ as in Figure 5.30.

We now construct the vertices of D_1 through a series of claims:

(i) x_3 is not in D_1.

From Lemma 3.3, D_1 differs from X in one vertex, different from both A_1 and B_1; moreover, x_1 and x_2 are in D_1. Notice that $\{x_4, x_5, \ldots, x_k\} \subseteq D_1$: suppose to the contrary that, say, $x_4 \notin D_1$, so that $X \overset{x_4z}{\sim} D_1$. Then $z \neq y_1$, since $y_1 \sim_G x_1$ and D_1 is independent. Likewise $z \neq y_2$ and $z \neq y_3$. Thus $D_1 = \{x_1, x_2, x_3, z, x_5, \ldots, x_k\}$ has $|Y \cap D_1| = 4$, but $d(D_1, Y) = 3$, contradicting Observation 3.4.

(ii) y_1, y_2, and y_3 are not in D_1.

As above, y_1 and y_2 are not in D_1, as D_1 is an independent set containing x_1 and x_2. Suppose to the contrary that $X \overset{x_3y_3}{\sim} D_1$, so that $D_1 = \{x_1, x_2, y_3, \ldots, x_k\}$. Then, since $x_1 \sim_G y_1$, we have that $D_1 \overset{x_1y_1}{\sim} A_2$, a contradiction.

(iii) $D_1 = \{x_1, x_2, z, x_4, \ldots, x_k\}$, where $z \notin \{y_1, y_2, y_3\}$.

Immediate from (i) and (ii).

A contradiction for the existence of G arises as we construct D_2. Since $d_{\mathfrak{T}}(D_1, Y) = 3$ and $|D_1 \cap Y| = 3$, at each step along the path through D_2, D_3, and Y, exactly one token departs from a vertex of D_1 and moves to one of $Y - D_1 = \{y_1, y_2, y_3\}$. However, $D_2 \neq \{y_1, x_2, z, x_4, \ldots, x_k\}$, since otherwise, $A_1 \overset{x_3z}{\sim} D_2$. Likewise, $D_2 \neq \{x_1, y_2, z, x_4, \ldots, x_k\}$. Finally $D_2 \neq \{x_1, x_2, y_3, x_4, \ldots, x_k\}$ as again we have $A_2 \overset{y_1x_1}{\sim} D_2$. Thus, we cannot build a set D_2 such that $|D_2 \cap Y| \leq 2$ as required. We so conclude that no such graph G exists, and that $\mathfrak{T}$ is not an i-graph. ■

Like the seven non-i-graph realizable theta graphs of Theorem 5.8, $\mathfrak{T}$ is a minimal obstruction to a graph being an i-graph; every induced subgraph of $\mathfrak{T}$ is a theta graph.

5.4.2 Maximal Planar Graphs

We conclude this chapter with a result to demonstrate that certain planar graphs are i-graphs and α-graphs. Our proof uses the following three known results.

Theorem 5.46. *[6, Theorem 4.6] A cubic graph is 3-connected if and only if it is 3-edge connected.*

Theorem 5.47. *[41] A connected planar graph G is bipartite if and only if its dual $\widetilde{G}$ is Eulerian.*

Theorem 5.48. *[22, 40] A maximal planar graph G of order at least 3 has $\chi(G) = 3$ if and only if G is Eulerian.*

In our proof of the following theorem, we consider a graph G that is cubic, 3-connected, bipartite, and planar. We then examine its dual $\widetilde{G}$, and how those specific properties of

G translate to $\widetilde{G}$. Then, we construct the complement of $\widetilde{G}$, which we refer to as H. We claim that H is a seed graph of G; that is, $\mathscr{I}(H)$ contains an induced copy of G.

Theorem 5.49. *Every cubic 3-connected bipartite planar graph is an i-graph and an α-graph.*

Proof. Let G be a cubic 3-connected bipartite planar graph and consider the dual $\widetilde{G}$ of G. Since G is bridgeless, $\widetilde{G}$ has no loops. Moreover, since G is 3-connected, Theorem 5.46 implies that no two edges separate G; hence, $\widetilde{G}$ has no multiple edges. Therefore, $\widetilde{G}$ is a (simple) graph. Further, since G is cubic, each face of $\widetilde{G}$ is a triangle, and so $\widetilde{G}$ is a maximal planar graph.

We note the following two key observations. First, since each face of $\widetilde{G}$ is a triangle,

(1) each edge of $\widetilde{G}$ belongs to a triangle.

For the second, note that since G is bipartite, $\widetilde{G}$ is Eulerian (by Theorem 5.47). Then, by Theorem 5.48, $\chi(\widetilde{G}) = 3$, so $\widetilde{G}$ does not contain a copy of K_4. Thus,

(2) every triangle of $\widetilde{G}$ is a maximal clique.

Now, by the duality of $\widetilde{G}$ and G, there is a one-to-one correspondence between the facial triangles of $\widetilde{G}$ and the vertices of G. Let H be the complement of $\widetilde{G}$. By (1) and (2), $i(H) = \alpha(H) = 3$, and every maximal independent set of G corresponds to a triangle of $\widetilde{G}$. But again, by duality, the facial triangles of $\widetilde{G}$ and their adjacencies correspond to the vertices of G and their adjacencies. Therefore, $\mathscr{I}(H)$ contains G as an induced subgraph. Any additional unwanted vertices of $\mathscr{I}(H)$ can be removed by applying the Deletion Lemma (Lemma 3.22). ■

Chapter 6

Conclusion

In this dissertation, we provided new results on the slide graph model of reconfiguration graphs for domination parameters.

In Chapter 2, we defined several variations on the γ-graph using other parameters such as total domination number, the paired domination number, the irredundance number, and the identification number, and showed that in each case all graphs are π-graphs for their respective parameters π. The singular exception was the Roman dominating graph, where we demonstrated that K_2 is not a γ_R-graph.

In Chapter 3, we showed that unlike the graphs of Chapter 2, not all graphs are i-graph realizable. We built a series of tools to show that known i-graphs can be used to construct new i-graphs and applied these results to build other classes of i-graphs, such as block graphs, hypercubes, forests, and unicyclic graphs. We also showed that a chordal graph is i-graph realizable if and only if it is diamond-free.

In Chapter 4, we determined the structure of the i-graphs of paths and cycles, and in the case of cycles, determined exactly which cycles have Hamiltonian or Hamiltonian traceable i-graphs.

In Chapter 5 we used graph complements to construct the i-graph seeds for certain classes of line graphs, theta graphs, and maximal planar graphs. In doings so, we characterized the line graphs and theta graphs that are i-graphs. The results concerning the i-graph realizability of theta graph are listed in Table 5.1.

We conclude this dissertation with a list of open problems, some of which were previously mentioned in the text.

Open Problems

Of the open problems presented in this dissertation, we believe the following to be the most important:

Problem 2. *Does there exist a finite forbidden subgraph characterization of i-graph realizable graphs?*

We now list the remaining open problems in the order in which their associated topic was discussed in the dissertation.

Problem 3. *Determine conditions on the graph G under which each γ-graph variation is connected/disconnected.*

Problem 4. *Let π be any of the above-mentioned domination-related parameters for which every graph is a π-graph. Is it true that every* ***bipartite*** *graph is the π-graph of a* ***bipartite*** *graph?*

Problem 5. *If H is i-graph realizable, does there exist a well-covered seed graph G such that $\mathscr{I}(G) \cong H$?*

Problem 6. *We have already seen that classes of graphs trees, cycles, and, more generally, block graphs, are i-graphs. As we build new graphs from these families of i-graphs, using tree structures, which of those are also i-graphs? For example, are cycle-trees i-graphs? Path-trees? For what families of graphs H are H-trees i-graphs?*

Problem 7. *Determine the structure of i-graphs of various families of trees. For example, consider caterpillars in which every vertex has degree 1 or 3.*

Problem 8. *Find more classes of i-graphs that are Hamiltonian, or Hamiltonian traceable.*

Problem 9. *Let π be any of the previously mentioned domination-related parameters. Suppose $G_1, G_2, \ldots$ are graphs such that $\pi(G_1) \cong G_2$, $\pi(G_2) \cong G_3$, $\pi(G_3) \cong G_4, \ldots$ Under which conditions does there exist an integer k such that $\pi(G_k) \cong G_1$?*

As a special case of Problem 9, we saw in Chapter 3 that for any $n \geq 1$, $\mathscr{I}(K_n) \cong K_n$, and that for $k \equiv 2 \pmod 3$, $\mathscr{I}(C_k) \cong C_k$.

Problem 10. *Determine for which graphs G, $\mathscr{I}(G) \cong G$.*

Bibliography

[1] S. Alikhani, D. Fatehi, and S. Klavžar. On the structure of dominating graphs. *Graphs Combin.*, 33(4):665–672, 2017.

[2] L. W. Beineke. Characterizations of derived graphs. *J. Combinatorial Theory*, 9:129–135, 1970.

[3] C. Berge. *The theory of graphs and its applications.* Methuen & Co., Ltd., London; John Wiley & Sons, Inc., New York, 1962. Translated by Alison Doig.

[4] A. Bień. Gamma graphs of some special classes of trees. *Ann. Math. Sil.*, (29):25–34, 2015.

[5] J. A. Bondy. The "graph theory" of the Greek alphabet. In *Graph theory and applications (Proc. Conf., Western Michigan Univ., Kalamazoo, Mich., 1972; dedicated to the memory of J. W. T. Youngs)*, Lecture Notes in Math., Vol. 303, pages 43–54. Springer, Berlin, 1972.

[6] G. Chartrand, L. Lesniak, and P. Zhang. *Graphs and Digraphs.* Chapman & Hall, London, 6th edition, 2015.

[7] K. Clarkson. Approximation algorithms for shortest path motion planning. In *Pro 19th ann. ACM symp. Theory computing (STOC '87)*, pages 56–65. ACM, New York, NY, USA, 1987.

[8] E. J. Cockayne, R. M. Dawes, and S. T. Hedetniemi. Total domination in graphs. *Networks*, 10(3):211–219, 1980.

[9] E. J. Cockayne, P. A. Dreyer, Jr., S. M. Hedetniemi, and S. T. Hedetniemi. Roman domination in graphs. *Discrete Math.*, 278(1-3):11–22, 2004.

[10] E. J. Cockayne and S. T. Hedetniemi. Towards a theory of domination in graphs. *Networks*, 7(3):247–261, 1977.

[11] E. Connelly, K. R. Hutson, and S. T. Hedetniemi. A note on γ-graphs. *AKCE Int. J. Graphs Comb.*, 8(1):23–31, 2011.

[12] M. Edwards. *Vertex-critically and bicritically for independent domination and total domination in graphs.* PhD thesis, University of Victoria, 2015.

[13] G. H. Fricke, S. M. Hedetniemi, S. T. Hedetniemi, and K. R. Hutson. γ-graphs of graphs. *Discuss. Math. Graph Theory*, 31(3):517–531, 2011.

[14] W. Goddard and M. A. Henning. Independent domination in graphs: a survey and recent results. *Discrete Math.*, 313(7):839–854, 2013.

[15] R. Haas and K. Seyffarth. The k-dominating graph. *Graphs Combin.*, 30(3):609–617, 2014.

[16] R. Haas and K. Seyffarth. Reconfiguring dominating sets in some well-covered and other classes of graphs. *Discrete Math.*, 340(8):1802–1817, 2017.

[17] A. Haddadan, T. Ito, A. E. Mouawad, N. Nishimura, H. Ono, A. Suzuki, and Y. Tebbal. The complexity of dominating set reconfiguration. *Theoret. Comput. Sci.*, 651:37–49, 2016.

[18] T. Haynes, S. Hedetniemi, and P. Slater. *Domination in Graphs.* Marcel Dekker, New York, 1998.

[19] T. Haynes, S. Hedetniemi, and P. Slater. *Fundamentals of Domination in Graphs.* Marcel Dekker, New York, 1998.

[20] T. W. Haynes, M. A. Henning, and J. Howard. Locating and total dominating sets in trees. *Discrete Appl. Math.*, 154(8):1293–1300, 2006.

[21] T. W. Haynes and P. J. Slater. Paired-domination in graphs. *Networks*, 32(3):199–206, 1998.

[22] P. Heawood. On the four-color map theorem. *Quart. J. Pure Math.*, 629:270–285, 1898.

[23] T. Ito, E. D. Demaine, N. J. A. Harvey, C. H. Papadimitriou, M. Sideri, R. Uehara, and Y. Uno. On the complexity of reconfiguration problems. *Theoret. Comput. Sci.*, 412(12-14):1054–1065, 2011.

[24] M. G. Karpovsky, K. Chakrabarty, and L. B. Levitin. On a new class of codes for identifying vertices in graphs. *IEEE Trans. Inform. Theory*, 44(2):599–611, 1998.

[25] S. A. Lakshmanan and A. Vijayakumar. The gamma graph of a graph. *AKCE Int. J. Graphs Comb.*, 7(1):53–59, 2010.

[26] A. Lobstein. Watching systems, identifying, locating-dominating and discriminating codes in graphs. Online bibliography (`https://www.lri.fr/~lobstein/debutBIBidetlocdom.pdf`).

[27] C. M. Mynhardt and S. Nasserasr. Reconfiguration of colourings and dominating sets in graphs. In *50 years of combinatorics, graph theory, and computing*, Discrete Math. Appl. (Boca Raton), pages 171–191. CRC Press, Boca Raton, FL, 2020.

[28] C. M. Mynhardt and A. Roux. Irredundance graphs. *Discrete Appl. Math.*, 322:36–48, 2022.

[29] C. M. Mynhardt and A. Roux. Irredundance trees of diameter 3. *Discrete Math.*, 345(12):Paper No. 113079, 10, 2022.

[30] C. M. Mynhardt and L. E. Teshima. A note on some variations of the γ-graph. *J. Combin. Math. Combin. Comput.*, 104:217–230, 2018.

[31] O. Ore. *Theory of graphs*. American Mathematical Society Colloquium Publications, Vol. 38. American Mathematical Society, Providence, R.I., 1962.

[32] E. Sampathkumar and H. B. Walikar. The connected domination number of a graph. *J. Math. Phys. Sci.*, 13(6):607–613, 1979.

[33] P. J. Slater. Domination and location in acyclic graphs. *Networks*, 17(1):55–64, 1987.

[34] P. J. Slater. Dominating and reference sets in a graph. *J. Math. Phys. Sci.*, 22(4):445–455, 1988.

[35] N. Sridharan, S. Amutha, and S. B. Rao. Induced subgraphs of gamma graphs. *Discrete Math. Algorithms Appl.*, 5(3):1350012, 5, 2013.

[36] N. Sridharan and K. Subramanian. Trees and unicyclic graphs are γ-graphs. *J. Combin. Math. Combin. Comput.*, 69:231–236, 2009.

[37] I. Stewart. Defend the roman empire! *Sci. Amer.*, 281(6):136–139, 1999.

[38] K. Subramanian and N. Sridharan. γ-graph of a graph. *Bull. Kerala Math. Assoc.*, 5(1):17–34, 2008.

[39] A. Suzuki, A. E. Mouawad, and N. Nishimura. Reconfiguration of dominating sets. *J. Comb. Optim.*, 32(4):1182–1195, 2016.

[40] M.-T. Tsai and D. B. West. A new proof of 3-colorability of Eulerian triangulations. *Ars Math. Contemp.*, 4(1):73–77, 2011.

[41] D. J. A. Welsh. Euler and bipartite matroids. *J. Combinatorial Theory*, 6:375–377, 1969.

[42] H. Whitney. Congruent graphs and the connectivity of graphs. *Amer. J. Math.*, 54(1):150–168, 1932.