SOME APPLICATIONS OF FRACTIONAL INTEGRAL OPERATORS AND RUSCHEWEYH DERIVATIVES

Y.C. KIM, K.S. LEE AND H.M. SRIVASTAVA

DMS-692-IR November 1994

SOME APPLICATIONS OF FRACTIONAL INTEGRAL OPERATORS AND RUSCHEWEYH DERIVATIVES

Yong Chan Kim, Kyung Suk Lee and H.M. Srivastava

Abstract

Recently, S. Owa *et al.* [13] introduced and studied a certain generalized fractional integral operator $J_{0,z}^{\alpha,\beta,\eta}$ involving the Gaussian hypergeometric function. The object of this paper is to investigate various properties and relationships involving $J_{0,z}^{\alpha,\beta,\eta}$, the Carlson-Shaffer operator $\mathcal{L}(a,c)$, and the Ruscheweyh derivative D^{λ}. A number of interesting subclasses of analytic and univalent functions are also considered in our investigation.

1991 *Mathematics Subject Classification.* Primary 30C45; Secondary 26A33, 33C20.

Key words and phrases. Fractional integral operator, Ruscheweyh derivative, Gaussian hypergeometric function, Carlson-Shaffer operator, analytic functions, univalent functions, starlike functions, convex functions, Gamma function, Hadamard products, Beta function, linear operator, prestarlike functions.

1. Introduction and Definitions

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = \sum_{n=0}^{\infty} a_{n+1}\, z^{n+1} \qquad (a_1 := 1), \tag{1.1}$$

which are analytic in the *open* unit disk

$$\mathcal{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Also let $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are univalent in the disk $\mathcal{U}$. Then a function $f(z)$ belonging to the class $\mathcal{S}$ is said to be *starlike of order* α $(0 \leq a < 1)$ in $\mathcal{U}$ if and only if

$$\Re\left(\frac{z\, f'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathcal{U};\ 0 \leq \alpha < 1). \tag{1.2}$$

We denote by $\mathcal{S}^*(\alpha)$ the class of all functions in $\mathcal{S}$ which are starlike of order α in $\mathcal{U}$.

A function $f(z)$ belonging to the class $\mathcal{S}$ is said to be *convex of order* α if and only if

$$\Re\left(1 + \frac{z\, f''(z)}{f'(z)}\right) > \alpha \qquad (z \in \mathcal{U};\ 0 \leq \alpha < 1). \tag{1.3}$$

We denote by $\mathcal{K}(\alpha)$ the class of all functions in $\mathcal{S}$ which are convex of order α in $\mathcal{U}$.

The classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ were first introduced by Roberston [15], and were studied subsequently by Schild [17], MacGregor [11], Pinchuk [14], Jack [6], and others (*cf.*, *e.g.*, [20]).

Let α_j $(j = 1, \cdots, p)$ and β_j $(j = 1, \cdots, q)$ be complex numbers with

$$\beta_j \neq 0, -1, -2, \cdots \qquad (j = 1, \cdots, q).$$

Then the generalized hypergeometric function ${}_pF_q(z)$ is defined by

$$\begin{aligned} {}_pF_q(z) &\equiv {}_pF_q\ (\alpha_1, \cdots, \alpha_p;\ \beta_1, \cdots, \beta_q;\ z) \\ &:= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} \qquad (p \leq q+1), \end{aligned} \tag{1.4}$$

where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_n := \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n = 0) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in \mathbb{N} := \{1, 2, 3, \cdots\}). \end{cases} \tag{1.5}$$

The ${}_pF_q$ series in (1.4) converges absolutely for $|z| < \infty$ if $p < q+1$, and for $z \in \mathcal{U}$ if $p = q+1$. Furthermore, if we set

$$\omega = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j \,, \tag{1.6}$$

it is known that the ${}_pF_q$ series, with $p = q+1$, is absolutely convergent for

$$|z| = 1 \quad \text{if} \quad \Re(\omega) > 0,$$

and conditionally convergent for

$$|z| = 1 \quad (z \neq 1) \quad \text{if} \quad -1 < \Re(\omega) \leq 0.$$

For the functions $f_j(z)$ $(j = 1, 2)$ defined by

$$f_j(z) = \sum_{n=0}^{\infty} a_{j,n+1}\, z^{n+1} \qquad (a_{j,1} := 1;\;\; j = 1, 2), \tag{1.7}$$

let $(f_1 * f_2)(z)$ denote the *Hadamard product* or *convolution* of $f_1(z)$ and $f_2(z)$, defined by

$$(f_1 * f_2)(z) := \sum_{n=0}^{\infty} a_{1,n+1}\, a_{2,n+1}\, z^{n+1} \qquad (a_{j,1} := 1;\;\; j = 1, 2). \tag{1.8}$$

Now define the function $\phi(a, c; z)$ by

$$\phi(a, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n}\, z^{n+1} \qquad (c \neq 0, -1, -2, \cdots;\;\; z \in \mathcal{U}), \tag{1.9}$$

so that $\phi(a, c; z)$ is an *incomplete* Beta function with

$$\phi(a, c; z) = z \;\; {}_2F_1(1, a; c; z). \tag{1.10}$$

Corresponding to the function $\phi(a, c; z)$, Carlson and Shaffer [5] defined a linear operator $\mathcal{L}(a, c)$ on $\mathcal{A}$ by the convolution [5, p. 738, Equation (2.2)]:

$$\mathcal{L}(a, c)\, f(z) = \phi(a, c; z) * f(z) \qquad (f \in \mathcal{A}). \tag{1.11}$$

Clearly, $\mathcal{L}(a, c)$ maps $\mathcal{A}$ onto itself, and $\mathcal{L}(c, a)$ is the inverse of $\mathcal{L}(a, c)$, provided that

$$a \neq 0, -1, -2, \cdots.$$

Furthermore, $\mathcal{L}(a,a)$ is the identity operator, and

$$\mathcal{K}(\alpha) = \mathcal{L}(1,2)\,\mathcal{S}^*(\alpha) \qquad (0 \leq \alpha < 1) \tag{1.12}$$

or, inversely,

$$\mathcal{S}^*(\alpha) = \mathcal{L}(2,1)\,\mathcal{K}(\alpha) \qquad (0 \leq \alpha < 1). \tag{1.13}$$

Ruscheweyh [16] introduced an operator $D^\lambda : \mathcal{A} \to \mathcal{A}$ defined by the convolution:

$$D^\lambda f(z) = \frac{z}{(1-z)^{\lambda+1}} * f(z) \qquad (\lambda \geq -1;\ z \in \mathcal{U}), \tag{1.14}$$

which implies that

$$D^n f(z) = \frac{z(z^{n-1} f(z))^{(n)}}{n!} \qquad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}). \tag{1.15}$$

Ruscheweyh [16] also introduced the subclass

$$\mathcal{K}_\lambda := \left\{ f \in \mathcal{A} : \Re\left(\frac{D^{\lambda+1} f(z)}{D^\lambda f(z)}\right) > \frac{1}{2} \quad (z \in \mathcal{U}) \right\} \qquad (\lambda > -1) \tag{1.16}$$

of the class $\mathcal{S}^*\left(\frac{1}{2}\right)$.

With these notations, we have

$$\begin{aligned} f(z) \in \mathcal{S}^*(\alpha) &\Leftrightarrow \Re\left(\frac{D^1 f(z)}{D^0 f(z)}\right) > \alpha \qquad (z \in \mathcal{U};\ 0 \leq \alpha < 1) \\ f(z) \in \mathcal{K}(\alpha) &\Leftrightarrow \Re\left(\frac{D^2 f(z)}{D^1 f(z)}\right) > \frac{\alpha+1}{2} \qquad (z \in \mathcal{U};\ 0 \leq \alpha < 1). \end{aligned} \tag{1.17}$$

Since

$$\frac{z}{(1-z)^{\lambda+1}} = z\ {}_2F_1(1, \lambda+1; 1; z),$$

we also have

$$D^\lambda f(z) = \mathcal{L}(\lambda+1, 1)\ f(z). \tag{1.18}$$

Ahuja ([1], [2]) defined the class $\mathcal{R}_\lambda(\alpha)$, which may be called the Ruscheweyh class of order α, satisfying the condition:

$$\Re\left(\frac{z\,(D^\lambda f(z))'}{D^\lambda f(z)}\right) > \alpha$$

$$(f \in \mathcal{A};\ \lambda > -1;\ \alpha < 1;\ z \in \mathcal{U}) \tag{1.19}$$

and proved that

$$\mathcal{R}_\lambda\left(\frac{1-\lambda}{2}\right) \equiv \mathcal{K}_\lambda \qquad (\lambda > -1).$$

For $\lambda = n \in \mathbb{N}_0$ and $0 \leq \alpha < 1$, the classes $\mathcal{R}_n(\alpha)$ and $\mathcal{R}_n(0)$ were, respectively, introduced in [3] and [4]. In particular, Ahuja and Silverman [4] showed that

$$\mathcal{R}_{n+1}(\alpha) \subset \mathcal{R}_n(\alpha) \quad \text{for each} \quad n \in \mathbb{N}_0 \quad \text{and for all} \quad \alpha.$$

These inclusion relations establish that

$$\mathcal{R}_n(\alpha) \subset \mathcal{S}^*(\alpha) \qquad (n \in \mathbb{N}_0; \quad 0 \leq \alpha < 1)$$

and

$$\mathcal{R}_n(\alpha) \subset \mathcal{K}(\alpha) \qquad (n \in \mathbb{N}; \quad 0 \leq \alpha < 1).$$

Furthermore, for α fixed and $n = n(\alpha)$ sufficiently large, we have (*cf.* [4])

$$\mathcal{R}_n := \mathcal{R}_n(0) \subset \mathcal{K}(\alpha) \qquad (0 \leq \alpha < 1).$$

The function

$$s_\alpha(z) := \frac{z}{(1-z)^{2(1-\alpha)}} \qquad (0 \leq \alpha < 1) \tag{1.20}$$

is the well-known extremal function for the class $\mathcal{S}^*(\alpha)$. A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{R}(\alpha, \beta)$ if

$$(f * s_\alpha)(z) \in \mathcal{S}^*(\beta) \qquad (0 \leq \alpha < 1; \; 0 \leq \beta < 1). \tag{1.21}$$

Note that $\mathcal{R}(\alpha, \alpha) \equiv \mathcal{R}(\alpha)$ is the subclass of $\mathcal{A}$ consisting of all functions which are *prestarlike of order* α. The family $\mathcal{R}(\alpha)$ was introduced in [17], where it was shown that

$$\mathcal{R}(\alpha) \subset \mathcal{R}(\beta) \quad \text{for} \quad \alpha < \beta < 1,$$

and

$$(f * g)(z) \in \mathcal{R}(\alpha) \quad \text{for} \quad f, g \in \mathcal{R}(\alpha).$$

Srivastava *et al.* [21] introduced a fractional integral operator $I_{0,z}^{\alpha,\beta,\eta}$ defined by (*cf.* [19])

$$I_{0.z}^{\alpha,\beta,\eta} f(z) = \frac{z^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^z (z-\zeta)^{\alpha-1} \ {}_2F_1\left(\alpha+\beta,\ -\eta; \alpha; 1-\frac{\zeta}{z}\right) f(\zeta)\, d\zeta \tag{1.22}$$

$$(\alpha > 0,\ \beta,\ \eta \in \mathbb{R};\ \ f(z) \in \mathcal{A})$$

and Owa *et al.* [13] studied the fractional integral operator $J_{0,z}^{\alpha,\beta,\eta}$ defined by (see also Kim *et al.* [10])

$$J_{0,z}^{\alpha,\beta,\eta} f(z) = \frac{\Gamma(2-\beta)\,\Gamma(2+\alpha+\eta)}{\Gamma(2-\beta+\eta)}\, z^\beta\ I_{0,z}^{\alpha,\beta,\eta} f(z) \qquad (f \in \mathcal{A}). \tag{1.23}$$

We observe that

$$J_{0,z}^{\alpha,\beta,\eta} f(z) = \mathcal{L}(2, 2-\beta)\, \mathcal{L}(2-\beta+\eta,\ 2+\alpha+\eta)\, f(z). \tag{1.24}$$

Various families of integral and other operators (including, for example, some of those mentioned above) were studied recently in connection with numerous subclasses of analytic functions (*cf.*, *e.g.*, [7] to [10]; see also [19] and [20]). In this paper, we first derive some interesting properties of the fractional integral operator $J_{0,z}^{\alpha,\beta,\eta}$ and the linear operator $\mathcal{L}(a, c)$. In Section 3, we shall investigate the value of δ_α, where

$$\Re\left(\frac{D^{\alpha+1} f(z)}{D^\alpha f(z)}\right) > \delta_\alpha.$$

Also we shall consider an application for the Ruscheweyh class $\mathcal{R}_\lambda(\alpha)$ defined above.

2. Results Involving Carlson-Shaffer and Fractional Integral Operators

Theorem 1. *If* $0 \leq \alpha < 1$ *and* $0 \leq \beta < 1$, *then*

$$\mathcal{L}(3-2\alpha, 2)\, \mathcal{R}(\alpha, \beta) \subset \frac{1-2\alpha}{2-2\alpha}\, \mathcal{K}(\beta) + \frac{1}{2-2\alpha}\, \mathcal{S}^*(\beta).$$

Proof. Since

$$\begin{aligned} s_\alpha(z) = \frac{z}{(1-z)^{2(1-\alpha)}} &= z\ {}_2F_1(1, 2-2\alpha; 1; z) \\ &= \phi(2-2\alpha, 1; z), \end{aligned} \tag{2.1}$$

it follows from (1.11) that

$$f(z) * \frac{s_\alpha(z)}{1-z} = \mathcal{L}(3-2\alpha, 1)\, f(z). \tag{2.2}$$

From the identity:

$$\begin{aligned} f(z) * \frac{s_\alpha(z)}{1-z} &= \frac{1-2\alpha}{2-2\alpha}\ (s_\alpha * f)(z) \\ &\qquad + \frac{1}{2-2\alpha}\, z\, \frac{d}{dz} \{(s_\alpha * f)(z)\} \\ &= \frac{1-2\alpha}{2-2\alpha} (s_\alpha * f)(z) \\ &\qquad + \frac{1}{2-2\alpha} \{s_\alpha(z) * z\, f'(z)\}, \end{aligned} \tag{2.3}$$

we have

$$\begin{aligned} \mathcal{L}(3-2\alpha, 1)\, f(z) &= \frac{1-2\alpha}{2-2\alpha} \mathcal{L}(2-2\alpha, 1)\, f(z) \\ &\qquad + \frac{1}{2-2\alpha} \mathcal{L}(2-2\alpha, 1)\, \mathcal{L}(2,1)\, f(z). \end{aligned} \tag{2.4}$$

It follows from (2.4) that

$$\begin{aligned} \mathcal{L}(3-2\alpha, 1)\, \mathcal{R}(\alpha, \beta) &\subset \frac{1-2\alpha}{2-2\alpha} \mathcal{L}(2-2\alpha, 1)\, \mathcal{R}(\alpha, \beta) \\ &+ \frac{1}{2-2\alpha} \mathcal{L}(2-2\alpha, 1)\, \mathcal{L}(2,1)\, \mathcal{R}(\alpha, \beta). \end{aligned} \tag{2.5}$$

Noting that [*cf.* Equation (1.21)]

$$\begin{aligned} \mathcal{R}(\alpha, \beta) &= \{f(z) \in \mathcal{A} : f(z) * s_\alpha(z) \in \mathcal{S}^*(\beta)\} \\ &= \mathcal{L}(1, 2-2\alpha)\, \mathcal{S}^*(\beta), \end{aligned} \tag{2.6}$$

we can easily rewrite the inclusion relation (2.5) in the form:

$$\begin{aligned} &\mathcal{L}(3-2\alpha, 1)\, \mathcal{R}(\alpha, \beta) \\ &\quad \subset \frac{1-2\alpha}{2-2\alpha} \mathcal{L}(2-2\alpha, 1)\, \mathcal{L}(1, 2-2\alpha)\, \mathcal{S}^*(\beta) \\ &\qquad + \frac{1}{2-2\alpha} \mathcal{L}(2-2\alpha, 1)\, \mathcal{L}(2,1)\, \mathcal{L}(1, 2-2\alpha)\, \mathcal{S}^*(\beta) \\ &\quad = \frac{1-2\alpha}{2-2\alpha} \mathcal{S}^*(\beta) + \frac{1}{2-2\alpha} \mathcal{L}(2,1)\, \mathcal{S}^*(\beta). \end{aligned} \tag{2.7}$$

Therefore

$$\begin{aligned}
&\mathcal{L}(3-2\alpha,1)\,\mathcal{L}(1,2)\,\mathcal{R}(\alpha,\beta)\\
&\quad\subset \frac{1-2\alpha}{2-2\alpha}\,\mathcal{L}(1,2)\,\mathcal{S}^*(\beta)+\frac{1}{2-2\alpha}\,\mathcal{L}(2,1)\,\mathcal{L}(1,2)\,\mathcal{S}^*(\beta)\\
&\quad=\frac{1-2\alpha}{2-2\alpha}\,\mathcal{K}(\beta)+\frac{1}{2-2\alpha}\,\mathcal{S}^*(\beta).
\end{aligned}\tag{2.8}$$

Since

$$\mathcal{L}(3-2\alpha,2)\,\mathcal{R}(\alpha,\beta)=\mathcal{L}(3-2\alpha,1)\,\mathcal{L}(1,2)\,\mathcal{R}(\alpha,\beta,)$$

the proof of Theorem 1 is completed.

Theorem 2. *If* $\alpha>0$, $0\le\beta\le1$, *and* $\eta\in\mathbb{R}$, *then*

$$\mathcal{L}(2+\alpha+\eta,\ 2-\beta+\eta)\,J_{0,z}^{\alpha,\beta,\eta}\,\mathcal{K}(\alpha)=\mathcal{R}\left(\frac{\beta}{2},\alpha\right).\tag{2.9}$$

Proof. From the equations (1.12) and (1.24), we readily have

$$\begin{aligned}
&\mathcal{L}(2+\alpha+\eta,\ 2-\beta+\eta)\,J_{0,z}^{\alpha,\beta,\eta}\,\mathcal{K}(\alpha)\\
&=\mathcal{L}(2,\,2-\beta)\,\mathcal{K}(\alpha)\\
&=\mathcal{L}(2,\,2-\beta)\,\mathcal{L}(1,2)\,\mathcal{S}^*(\alpha)\\
&=\mathcal{L}(1,\,2-\beta)\,\mathcal{S}^*(\alpha)\\
&=\mathcal{R}\left(\frac{\beta}{2},\alpha\right).
\end{aligned}$$

Letting $\beta=1$ in Theorem 2, we have

Corollary 1. *If* $0\le\alpha<1$ *and* $\eta\in\mathbb{R}$, *then*

$$\mathcal{L}(2+\alpha+\eta,\ 1+\eta)\,J_{0,z}^{\alpha,1,\eta}\,\mathcal{K}(\alpha)=\mathcal{S}^*(\alpha).$$

3. Applications of the Ruscheweyh Derivatives

The following known result will be required in our proof of Theorem 3 below:

Lemma 1 (*Cf.*, *e.g.*, Miller and Mocanu [12, p. 298, Theorem 5]). *Let $w(u,v)$ be a complex-valued function, that is,*

$$w : \mathbb{D} \to \mathbb{C} \qquad (\mathbb{D} \subset \mathbb{C} \times \mathbb{C})$$

and let

$$u = u_1 + iu_2 \quad \textit{and} \quad v = v_1 + iv_2.$$

Suppose that the function $w(u,v)$ satisfies the following conditions:

(i) $w(u,v)$ *is continuous in* $\mathbb{D}$;

(ii) $(1,0) \in \mathbb{D}$ *and* $\Re\{w(1,0)\} > 0$;

(iii) $\Re\{w(iu_2, v_1)\} \leq 0$ *for all* $(iu_2, v_1) \in \mathbb{D}$ *and such that*

$$v_1 \leq -\frac{(1+u_2^2)}{2}.$$

Let

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots$$

be regular in $\mathcal{U}$ such that

$$(p(z),\, zp'(z)) \in \mathbb{D} \quad \textit{for all} \quad z \in \mathcal{U}.$$

If

$$\Re\{w(p(z),\, zp'(z))\} > 0 \qquad (z \in \mathcal{U}),$$

then

$$\Re\{p(z)\} > 0.$$

Theorem 3. *If $f(z) \in \mathcal{A}$ satisfies*

$$\Re\left(\frac{D^{\alpha+2} f(z)}{D^{\alpha+1} f(z)}\right) > \beta \qquad (z \in \mathcal{U};\ \alpha > -1) \tag{3.1}$$

for some β $\left(\frac{1}{2} \leq \beta < 1\right)$, *then*

$$\Re\left(\frac{D^{\alpha+1} f(z)}{D^{\alpha} f(z)}\right) > \gamma \qquad (z \in \mathcal{U};\ \alpha > -1), \tag{3.2}$$

where

$$\gamma := \frac{\{2\beta(\alpha+2)-3\} + \sqrt{\{2\beta(\alpha+2)-3\}^2 + 8(\alpha+1)}}{4(\alpha+1)}. \tag{3.3}$$

Proof. It is known that [3, p. 250, Equation (1.15)]

$$z\big(D^{\alpha} f(z)\big)' = (\alpha+1)\, D^{\alpha+1} f(z) - \alpha\, D^{\alpha} f(z) \qquad (\alpha > -1), \tag{3.4}$$

which immediately yields

$$\frac{z\big(D^{\alpha} f(z)\big)'}{D^{\alpha} f(z)} = (\alpha+1)\,\frac{D^{\alpha+1} f(z)}{D^{\alpha} f(z)} - \alpha. \tag{3.5}$$

If we define the function $p(z)$ by

$$\frac{D^{\alpha+1} f(z)}{D^{\alpha} f(z)} = \gamma + (1-\gamma)\, p(z) \tag{3.6}$$

with γ defined, as before, by (3.3), then

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots$$

is analytic in $\mathcal{U}$. Differentiating both sides of (3.6) logarithmically, we have

$$\frac{z\big(D^{\alpha+1} f(z)\big)'}{D^{\alpha+1} f(z)} = \frac{z\big(D^{\alpha} f(z)\big)'}{D^{\alpha} f(z)} + \frac{(1-\gamma)\, z p'(z)}{\gamma + (1-\gamma)\, p(z)}. \tag{3.7}$$

Combining the equations (3.5) and (3.7), we obtain

$$\frac{D^{\alpha+2} f(z)}{D^{\alpha+1} f(z)} = \frac{\alpha+1}{\alpha+2}\,\frac{D^{\alpha+1} f(z)}{D^{\alpha} f(z)} + \frac{1}{\alpha+2} + \frac{(1-\gamma)\, z p'(z)}{(\alpha+2)\{\gamma + (1-\gamma)\, p(z)\}}, \tag{3.8}$$

which readily yields

$$\Re\left(\frac{D^{\alpha+2} f(z)}{D^{\alpha+1} f(z)} - \beta\right)$$

$$= \frac{1}{\alpha+2}\,\Re\left\{(\alpha+1)\gamma + 1 - \beta(\alpha+2) + (\alpha+1)(1-\gamma)\, p(z) + \frac{(1-\gamma)\, z p'(z)}{\gamma + (1-\gamma)\, p(z)}\right\} \tag{3.9}$$

$$> 0.$$

Therefore, if we define the function $w(u,v)$ by

$$w(u,v) = (\alpha+1)\gamma + 1 - \beta(\alpha+2) + (\alpha+1)(1-\gamma)\,u(z) + \frac{(1-\gamma)\,v(z)}{\gamma + (1-\gamma)\,u(z)}, \tag{3.10}$$

then we see that

(i) $w(u,v)$ is continuous in $\mathbb{D} = \left(\mathbb{C} - \left\{\frac{\gamma}{\gamma-1}\right\}\right) \times \mathbb{C}$;

(ii) $(1,0) \in \mathbb{D}$ and $\Re\{w(1,0)\} = (\alpha+2)(1-\beta) > 0$;

(iii) for all $(iu_2, v_1) \in \mathbb{D}$ and such that

$$v_1 \leq -\frac{(1+u_2^2)}{2},$$

$$\begin{aligned} \Re\{w(iu_2,v_1)\} &= (\alpha+1)\gamma + 1 - \beta(\alpha+2) + \frac{\gamma(1-\gamma)\,v_1}{\gamma^2 + (1-\gamma)^2\,u_2^2} \\ &\leq (\alpha+1)\gamma + 1 - \beta(\alpha+2) - \frac{\gamma(1-\gamma)\,(1+u_2^2)}{2\,\{\gamma^2 + (1-\gamma)^2\,u_2^2\}} \\ &= \frac{\gamma[2(\alpha+1)\,\gamma^2 - \{2\beta(\alpha+2)-3\}\,\gamma - 1]}{2\,\{\gamma^2 + (1-\gamma)^2\,u_2^2\}} \\ &\quad - \frac{(1-\gamma)\,[\gamma + (1-\gamma)\{2\beta(\alpha+2) - 2\gamma(\alpha+1) - 2\}]}{2\,\{\gamma^2 + (1-\gamma)^2\,u_2^2\}}\,u_2^2. \end{aligned} \tag{3.11}$$

From Equation (3.3), we have

$$2(\alpha+1)\,\gamma^2 - \{2\beta(\alpha+2) - 3\}\,\gamma - 1 = 0. \tag{3.12}$$

For $\beta \geq \gamma$ and $\beta \geq \frac{1}{2}$, we find from (3.11) and (3.12) that

$$\Re\,\{w(iu_2,\,v_1)\} \leq 0.$$

This implies that the function $w(u,v)$ satisfies the hypotheses of Lemma 1. Thus we conclude that

$$\Re\left(\frac{D^{\alpha+1}\,f(z)}{D^{\alpha}\,f(z)}\right) > \gamma = \frac{\{2\beta(\alpha+2)-3\} + \sqrt{\{2\beta(\alpha+2)-3\}^2 + 8(\alpha+1)}}{4(\alpha+1)},$$

which evidently completes the proof of Theorem 3.

Corollary 2. *If* $\alpha \geq 0$ *and* $0 \leq \beta < 1$, *then*

$$\mathcal{R}_{\alpha+1}(\beta) \subset \mathcal{R}_{\alpha}\big(\gamma(\alpha+1)-\alpha\big)$$

where

$$\gamma := \frac{\{2(\alpha+\beta)-1\} + \sqrt{\{2(\alpha+\beta)-1\}^2 + 8(\alpha+1)}}{4(\alpha+1)}. \tag{3.13}$$

Moreover

$$\gamma(\alpha+1)-\alpha \geq \beta.$$

Proof. Let $f \in \mathcal{R}_{\alpha+1}(\beta)$. Then, from the definition (1.19), we have

$$\Re\left(\frac{z\big(D^{\alpha+1}f(z)\big)'}{D^{\alpha+1}f(z)}\right) > \beta \qquad (z \in \mathcal{U}). \tag{3.14}$$

By a simple computation, using (3.5) and (3.14), we obtain

$$\Re\left(\frac{D^{\alpha+2}f(z)}{D^{\alpha+1}f(z)}\right) > \frac{\alpha+\beta+1}{\alpha+2} \qquad (z \in \mathcal{U}). \tag{3.15}$$

Applying Theorem 3, we have

$$\Re\left(\frac{D^{\alpha+1}f(z)}{D^{\alpha}f(z)}\right) > \gamma \qquad (z \in \mathcal{U}),$$

where γ is defined by (3.13). Using (3.5), we have

$$\Re\left(\frac{z\big(D^{\alpha}f(z)\big)'}{D^{\alpha}f(z)}\right) > \gamma(\alpha+1)-\alpha \qquad (z \in \mathcal{U}).$$

Hence

$$f \in \mathcal{R}_{\alpha}\big(\gamma(\alpha+1)-\alpha\big),$$

which obviously proves the main assertion of Corollary 2.

Acknowledgments

The authors were supported by Yeungnam University and KOSEF (Project No. 94-1400-02-01-3). This work was supported, in part, also by the Natural Sciences and Engineering Research Council of Canada under Grant OGP0007353.

References

1. O.P. Ahuja, Integral operators of certain univalent functions, *Internat. J. Math. and Math. Sci.* **8**(1985), 653-662.
2. O.P. Ahuja, On the radius problem of certain analytic functions, *Bull. Korean Math. Soc.* **22**(1985), 31-36.
3. O.P. Ahuja, On the generalized Ruscheweyh class of analytic functions of complex order, *Bull. Austral. Math. Soc.* **47**(1993), 247-257.
4. O.P. Ahuja and H. Silverman, Function classes related to Ruscheweyh derivatives, *J. Austral. Math. Soc. Ser. A* **47**(1989), 438-444.
5. B.C. Carlson and D.B. Shaffer, Starlike and prestarlike hypergeometric functions, *SIAM J. Math. Anal.* **15**(1984), 737-745.
6. I.S. Jack, Functions starlike and convex of order α, *J. London Math. Soc.* (2) **3**(1971), 469-474.
7. Y.C. Kim and C.K. Bae, Some applications of a certain integral operator associated with the Hardy space, *Math. Japon.* **39**(1994), 261-265.
8. Y.C. Kim, K.S. Lee, and H.M. Srivastava, Certain classes of integral operators associated with the Hardy space of analytic functions, *Complex Variables Theory Appl.* **20**(1992), 1-12.
9. Y.C. Kim, S.H. Lee, and H.M. Srivastava, Some properties of convolution operators in the class $\mathcal{P}_\alpha(\beta)$, *J. Math. Anal. Appl.* **187**(1994), 498-512.
10. Y.C. Kim, Y.S. Park, and H.M. Srivastava, A class of inclusion theorems associated with some fractional integral operators, *Proc. Japan Acad. Ser. A Math. Sci.* **67**(1991), 313-318.
11. T.H. MacGregor, The radius of convexity for starlike functions of order $\frac{1}{2}$, *Proc. Amer. Math. Soc.* **14**(1963), 71-76.
12. S.S. Miller and P.T. Mocanu, Second order differential inequalities in the complex plane, *J. Math. Anal. Appl.* **65**(1978), 289-305.
13. S. Owa, M. Saigo, and H.M. Srivastava, Some characterization theorems for starlike and convex functions involving a certain fractional integral operator, *J. Math. Anal.*

Appl. **140**(1989), 419-426.

14. B. Pinchuk, On starlike and convex functions of order α, *Duke Math. J.* **35**(1968), 721-734.
15. M.S. Robertson, On the theory of univalent functions, *Ann. of Math.* **37**(1936), 374-408.
16. S. Ruscheweyh, New criteria for univalent functions, *Proc. Amer. Math. Soc.* **49**(1975), 109-115.
17. A. Schild, On starlike functions of order α, *Amer. J. Math.* **87**(1965), 65-70.
18. T. Sheil-Small, H. Silverman, and E. Silvia, Convolution multipliers and starlike functions, *J. Analyse Math.* **41**(1982), 181-192.
19. H.M. Srivastava and S. Owa (Editors), *Univalent Functions, Fractional Calculus, and Their Applications*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane, and Toronto, 1989.
20. H.M. Srivastava and S. Owa (Editors), *Current Topics in Analytic Function Theory*, World Scientific Publishing Company, Singapore, New Jersey, London, and Hong Kong, 1992.
21. H.M. Srivastava, M. Saigo, and S. Owa, A class of distortion theorems involving certain operators of fractional calculus, *J. Math. Anal. Appl.* **131**(1988), 412-420.

YONG CHAN KIM and KYUNG SUK LEE
Department of Mathematics
College of Education
Yeungnam University
Gyongsan 713-749
Korea

H.M. SRIVASTAVA
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3P4
Canada