ON THE ACHROMATIC NUMBER OF GRAPHS

by

Frederick John Hughes
B.Sc., University of Victoria, 1974

A Thesis Submitted in Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

in the Department of Mathematics and Statistics

We accept this thesis as conforming
to the required standard

[redacted]

Dr. G. MacGillivray, Supervisor
(Department of Mathematics and Statistics)

[redacted]

Dr. D. J. Miller, Department Member
(Department of Mathematics and Statistics)

[redacted]

Dr. D. Roelants van Baronaigien, Outside Member
(Department of Computer Science)

[redacted]

Dr. P. Hell, External Examiner
(School of Computing Science, Simon Fraser University)

University of Victoria

Supervisor: Dr. G. MacGillivray

Abstract

The achromatic number of a graph G is the largest number of colours that can be assigned to the vertices of G so that (i) adjacent vertices are assigned different colours, and (ii) any two different colours are assigned to some pair of adjacent vertices. This thesis contributes to the study of the achromatic number by outlining the research since 1967. The effects on the achromatic number of several different operations are surveyed, including that of the categorical product. The bounds on the achromatic number of paths, cycles, k-regular graphs and trees are then investigated, followed by an exploration of the relationship between clique number and achromatic number. Various results on n-minimal graphs are also reviewed. Finally, results concerning the computational complexity of the achromatic number problem for arbitrary and restricted graphs are presented. Included in this thesis are proofs of some theorems which illustrate an important technique or idea. Several original results are given, including n-minimal graph and Nordhaus-Gaddum type problems.

Examiners:

[redacted]

Dr. G. MacGillivray

[redacted]

Dr. D. J. Miller

[redacted]

Dr. D. Roelants van Baronaigien

[redacted]

Dr. P. Hell

Contents

List of Figures

Acknowledgements

I wish to acknowledge the generous guidance, assistance, editing skills, and sincere interest of Dr. Gary MacGillivray.

I also want to thank an old friend, Arne Arnason, who was the first to encourage me to pursue my Master's degree (before I lost the dream!).

Finally, I must thank my friend, Alana, for helping me get through these three years of study.

F.J.H.

Chapter 1

INTRODUCTION

The chromatic number of graphs is a well studied parameter which has been investigated for over a hundred years. The achromatic number, however, was first introduced by Harary, Hedetniemi and Prins in 1967 [32]. It is the purpose of this thesis to contribute to the ongoing study of the achromatic number. We use the definitions and terminology of Bondy and Murty [7].

A *graph* G is an ordered triple $(V(G), E(G), \iota_G)$ consisting of a nonempty set $V(G)$ of *vertices*, a set $E(G)$, disjoint from $V(G)$, of *edges*, and an *incidence function* ι_G that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G, called the *ends* of the edge. An edge with identical ends is called a *loop*. An edge with distinct ends is called a *link*. A graph is *finite* if both its vertex set and its edge set are finite. A *multigraph* is a graph with no loops. A *simple graph* has no loops and no two of its links join the same pair of vertices. The *underlying graph G of a multigraph G'* is the spanning subgraph of G' formed by deleting all but one edge between each pair of adjacent vertices. Unless otherwise noted, we assume that a graph is finite and simple. In this case, a graph can be regarded as an ordered pair $G = (V, E)$, where V is a finite non-empty set of vertices and E is a set of unordered pairs of distinct vertices.

A *k-colouring* of a graph G is a function $f : V(G) \rightarrow \{1, 2, \ldots, k\}$ such that if $uv \in E(G)$, then $f(u) \neq f(v)$. In other words, it is an assignment of k colours to the vertices of G so that no two adjacent vertices receive the same colour. A *complete k-colouring* of a graph G is a k-colouring in which, for every two different colours i and j, there are two adjacent vertices such that one is coloured i and the other is coloured j. Figure 1.1 shows several graphs with complete 3-colourings.

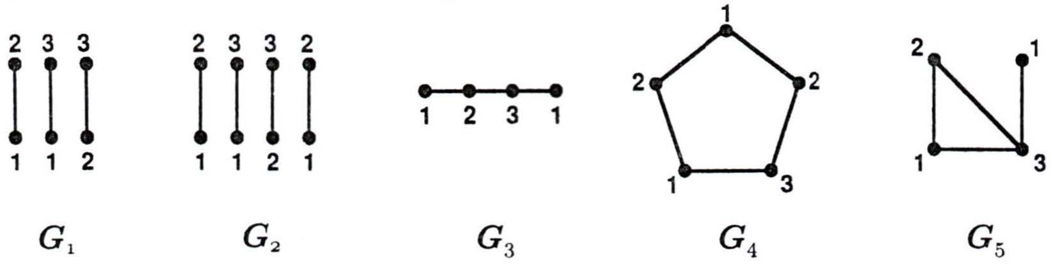


Figure 1.1 Complete 3-colourings of five different graphs.

Let G and H be graphs. A *homomorphism of G to H* is a function $f : V(G) \rightarrow V(H)$ such that if $uv \in E(G)$, then $f(u)f(v) \in E(H)$. That is, f is a mapping of the vertices of G to the vertices of H which preserves adjacency. If the vertices of K_n are regarded as colours, then it is easy to see by comparing the definitions, and noting that any two distinct vertices of K_n are adjacent, that a homomorphism of G to K_n is an n-colouring of

G. Note that if f is a homomorphism of G to H, then for all $h \in V(H)$, the set $f^{-1}(h)$ is an independent set in G.

A homomorphism f of G to H induces a function $f' : E(G) \to E(H)$ defined by $f'(uv) = f(u)f(v)$. If f is a homomorphism of G to H, then the *homomorphic image of G in H*, denoted by $f(G)$, is the subgraph of H with vertex set $f(V(G))$ and edge set $f'(E(G))$, where f' is defined as above. Figure 1.2 illustrates a homomorphism; the vertices of G are assigned the labels of the vertices to which they are mapped, and the edges of $f(G)$ are emphasized in bold.

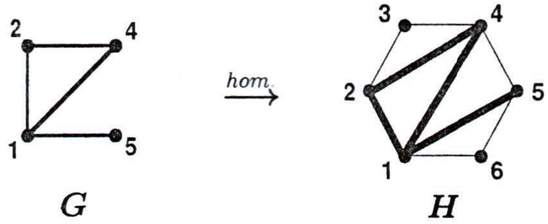


Figure 1.2. A homomorphism of G to H

A *full homomorphism* of G to H is a homomorphism f of G to H such that $f(G)$ is an induced (or full) subgraph of H (see Figure 1.3). Note that in Figure 1.2, $f(G)$ is not an induced subgraph of H.

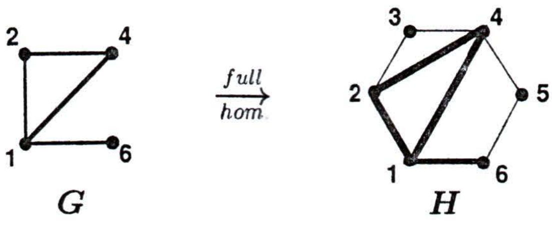


Figure 1.3. A full homomorphism of G to H

A *complete homomorphism* of G to H is a full homomorphism of G onto H, that is, a homomorphism f of G onto H such that $f(G) = H$. Figure 1.4 shows a homomorphism of G onto H and Figure 1.5 shows a complete homomorphism of G onto H. A complete homomorphism of G onto K_n can therefore be regarded as a complete n-colouring of G. If there is a complete homomorphism of G to H, (i.e., if $f(G) = H$), then we call H a *homomorphic image* of G.

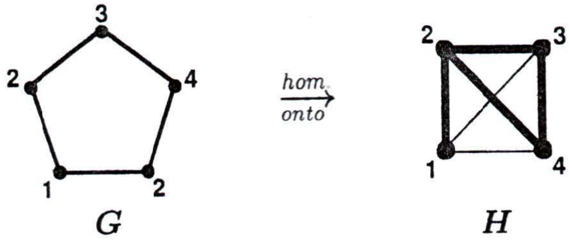


Figure 1.4: A homomorphism of G onto H.

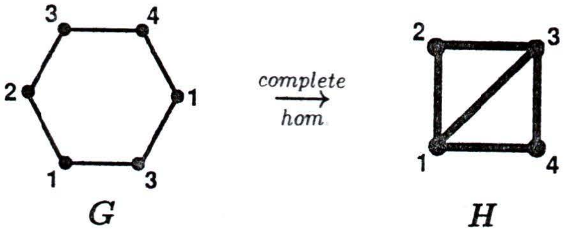


Figure 1.5: A complete homomorphism of G onto H.

Suppose ϵ is a complete homomorphism of G onto H, and $|V(H)| = |V(G)| - 1$. Then there exists a $w \in V(H)$ such that $|\epsilon^{-1}(w)| = 2$ and for all other $x \in V(H)$, $|\epsilon^{-1}(x)| = 1$. That is, H can be obtained from G by identifying a pair of independent vertices. In this case, we call H an *elementary homomorphic image of* G and refer to ϵ as an *elementary* *homomorphism of G to H* (see Figure 1.6).

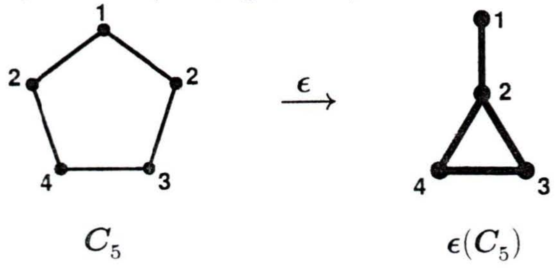


Figure 1.6. An elementary homomorphic image of C_5.

The *chromatic* (respectively *achromatic*) *number* of a graph G, denoted by $\chi(G)$ (resp. $achr(G)$), is the minimum (resp. maximum) number n for which G has a complete n-colouring. Equivalently, the chromatic (resp. achromatic) number of a graph G is the minimum (resp. maximum) number n such that K_n is a homomorphic image of G. A direct implication of this definition is that a graph G with achromatic number n must have at least $\binom{n}{2}$ edges.

A graph is *n-minimal* if $achr(G - e) < achr(G) = n$ for every edge e in G.

The graphs G and H are (vertex) *disjoint* if $V(G) \cap V(H) = \emptyset$.

A *clique* of G is a subset X of $V(G)$ such that $G[X]$ is a complete graph. The *clique number* of G, denoted by $\omega(G)$, is the number of vertices in a largest clique of G.

Note that if the context is clear, we abbreviate $\omega(G)$, $achr(G)$ and $\chi(G)$ to ω, $achr$ and χ, respectively.

We use P_n (resp. C_n) to denote a path (resp. cycle) with n vertices.

(Note that P_n has length $n-1$, while C_n has length n.)

A *k-partite graph* is a graph whose vertex set can be partitioned into subsets V_i, $i = 1, \ldots, k$, so that no edge has both ends in any one subset. A *complete k-partite* graph, denoted by $K_{n_1, n_2, \ldots, n_k}$, where V_i has size n_i for $i = 1, \ldots, k$, is one in which each vertex is joined to every vertex that is not in the same subset. The complete m-partite graph on n vertices in which each of the m parts has either $\left\lfloor \frac{n}{m} \right\rfloor$ or $\left\lceil \frac{n}{m} \right\rceil$ vertices is denoted by $T_{m,n}$ (see [7], page 6).

The *union* of two graphs G and H is the graph $G \cup H$ with $V(G \cup H) = V(G) \cup V(H)$, and $E(G \cup H) = E(G) \cup E(H)$. We denote the union of n disjoint copies G as nG.

The *join* of two disjoint graphs $G = (V, E)$ and $G' = (V', E')$ is the graph $G + G' = (V \cup V', X)$, where $X = E \cup E' \cup \{vv' | v \in V, \text{ and } v' \in V'\}$. That is, the graph obtained from $G \cup G'$ by adding all possible edges with one end in $V(G)$ and the other end in $V(G')$. Figure 1.7 shows the join of K_3 and K_2.

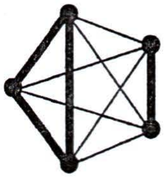


Figure 1.7. The join of K_3 and K_2 is K_5

The *wreath product* of disjoint graphs G and H is the graph $G \cdot H$ whose vertex set is $V(G) \times V(H)$ and whose edge set is $\{(u,x)(v,y) : (u = v$ and $xy \in E(H))$ or $(u \neq v$ and $uv \in E(G)\}$. Informally, it is the graph obtained from G by replacing each vertex $u \in V(G)$ by a copy H_u of H and for each edge uv of G adding all possible edges joining vertices in H_u and H_v. Figure 1.8 shows the wreath product of C_4 and K_2, the boldface edges showing the four K_2 graphs which replace the vertices of C_4.

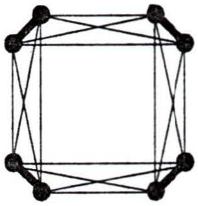


Figure 1.8: The wreath product of C_4 and K_2.

The *categorical product* of graphs G and H is the graph $G \times H$ whose vertex set is $V(G) \times V(H)$ and whose edge set is $\{(g,h)(g',h') : gg' \in E(G)$ and $hh' \in E(H)\}$. We give two examples of the categorical product: $P_n \times P_2$, and $C_n \times P_2$ (see Figures 1.9, and 1.10).

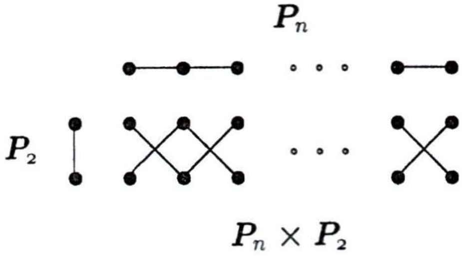


Figure 1.9: $P_n \times P_2 = 2P_n$.

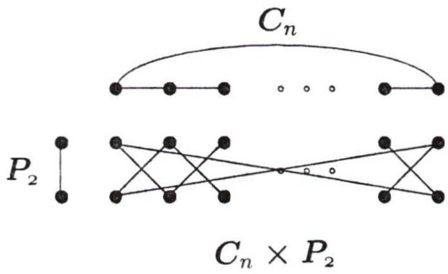


Figure 1.10: $C_n \times P_2 = 2C_n$ for even n, and C_{2n} for odd n

In this thesis we will next outline the general theory of the achromatic number. We will consider the effect on the achromatic number of several different operations: taking subgraphs, disjoint unions, joins, wreath products, categorical products, and the effect of homomorphisms. We then investigate the bounds on the achromatic number of paths, cycles, k-regular graphs and trees. The relationship between clique number and achromatic number is next explored, then we survey the results on n-minimal graphs. Subsequently we list a collection of inequalities relating the achromatic number and other graph parameters. A list of Nordhaus-Gaddum type inequalities is also given, along with examples of extremal graphs. We review results concerning the complexity of the achromatic number problem for arbitrary and restricted classes of graphs. Finally, we conclude by offering some open problems.

This is largely an expository thesis. For theorems which have previously appeared in the literature, proofs are included only when they illustrate an important point, technique, or idea. If no citation is given, the result is original.

Chapter 2

GENERAL THEORY

In this chapter we look at the achromatic number of subgraphs and certain product graphs, examine the effects of homomorphisms on the achromatic number, and finally investigate irreducible graphs.

2.1 Subgraphs

In this section, we study the effects of removing vertices or edges from a graph. The results, published in 1974, are due to Geller and Kronk [25].

Theorem 2.1.1 [25] *For any graph G and vertex $v \in V(G)$,*

$$achr(G) - 1 \leq achr(G - v) \leq achr(G).$$

Proof. We first establish the upper bound. Suppose $achr(G - v) = m$. If there exists a complete m-colouring of $G - v$ such that v is adjacent to a vertex of each colour, then assigning colour $m + 1$ to v results in a complete $m + 1$-colouring of G. Otherwise, assigning v a colour from $\{1, 2, \ldots, m\}$

not used on a neighbour of v gives a complete m-colouring of G. Therefore $achr(G-v) \leq achr(G)$.

We now establish the lower bound. Consider a complete n-colouring of G where, without loss of generality, v is assigned colour n. Notice that for any distinct colours i and j taken from $\{1, 2, \ldots, n-1\}$ there are adjacent vertices in $G-v$ such that one is coloured i and the other is coloured j. Hence $achr(G-v) \geq achr(G)-1$. □

The lower bound is achieved by K_n, $n \geq 1$. Equality in the upper bound is obtained by the graph G consisting of K_n, $n \geq 2$, with an additional vertex v connected to K_n by a single edge.

Corollary 2.1.2 [25] *If H is an induced subgraph of G, then*

$$achr(H) \leq achr(G).$$

Theorem 2.1.3 [25] *For any graph G and edge e of G,*

$$achr(G)-1 \leq achr(G-e) \leq achr(G)+1.$$

Proof. We prove the upper bound by contradiction. Let $e = uv$ and suppose that $achr(G) = n$ and $achr(G-e) = n+k > n+1$. Note that in any complete colouring of $G-e$ with more than n colours, u and v must be coloured the same colour, say colour 1. (Otherwise, the same colouring of $G-e$ is a complete colouring of G.) If one of the other $n+k-1$ colours, say i, is not used on a neighbour of v, then recolour v with colour i. This may not result in a complete $(n+k)$-colouring of G as some neighbour of v may have been the only vertex coloured j adjacent to a vertex of colour 1. Recolouring all vertices coloured 1 with colour j results in a complete $(n+k-1)$-colouring of $G-e$ with u and v assigned different colours. Replacing the edge uv, we obtain a complete colouring of G with $n+k-1 > n$ colours, a contradiction. On the other hand, if both u and v are adjacent to vertices of each of the $n+k-1$ other colours, then recolouring v with a new colour $n+k+1$ results in a complete $(n+k+1)$-colouring of $G-e$, except that colours 1 and $n+k+1$ are not adjacent. Replacing the edge uv, we obtain a complete $n+k+1$-colouring of G, again a contradiction. Hence $achr(G-e) \leq n+1$.

We now prove the lower bound. Consider a complete n-colouring of G where, without loss of generality, u is assigned colour $n-1$ and v is assigned colour n. Now if u and v are the only adjacent vertices in G which are coloured $n-1$ and n, then there exists a complete $(n-1)$-colouring of $G-e$, defined by recolouring all vertices of colour n with colour $n-1$. If u and v are not the only adjacent vertices in G which are coloured $n-1$ and n, then the same colouring is still a complete n-colouring of $G-e$. In either case, $achr(G-e) \geq n-1$. □

Equality in the lower bound occurs with K_n, $n \geq 2$. The upper bound is achieved by a graph G obtained from K_n, $n \geq 3$, by subdividing an edge.

Note that if a vertex is deleted the achromatic number either remains the same or decreases. If an edge is removed, however, the achromatic number may increase.

2.2 Product Graphs

This section surveys results on the achromatic number of the union, join, wreath product, and categorical product of graphs.

Theorem 2.2.1 [34] *Let G and H be (vertex) disjoint graphs. Then*

$$max\{achr(G), achr(H)\} \leq achr(G \cup H) \leq achr(G) \cdot achr(H).$$

The lower bound is shown to be best possible by considering the disjoint union of K_m and K_n. Hell and D. Miller [34] showed that the upper bound is best possible regardless of whether G and H are disjoint. Their example is the union of the graphs $G = T_{b,ab}$ and $H = T_{a,ab}$. Let $V(G) = V(H) = V(K_{ab})$ be the set $\{(x, y)|1 \leq x \leq a, 1 \leq y \leq b\}$. The edges of G join all pairs of vertices with different first coordinate and the edges of H join all pairs with different second coordinate. Hence $G \cup H = K_{ab}$ and $achr(G \cup H) = ab = achr(G) \cdot achr(H)$.

Theorem 2.2.2 [34] *Let $achr(G_i) = a$ for $i = 1, 2, \ldots, d$. Then,*

$$achr(dK_a) \leq achr(G_1 \cup \cdots \cup G_d) \leq a^d.$$

Further, for any fixed a and all sufficiently large d,
$achr(dK_a) \approx \sqrt{a^2 - a} \cdot \sqrt{d}$, *(where $\approx$ denotes "is approximately equal to").*

In particular, for large a (that is, for $1 \ll a \ll d$) we have

$$achr(dK_a) \approx a\sqrt{d}.$$

We now turn our attention to the join operation.

Theorem 2 2 3 [31] $achr(G_1 + G_2) = achr(G_1) + achr(G_2)$

From this, and the fact that $K_{n_1,n_2, \ ,n_r} = \overline{K}_{n_1} + \overline{K}_{n_2} + \quad + \overline{K}_{n_r}$, we get $achr(K_{n_1,n_2, \ ,n_r}) = r$ The achromatic number of a complete bipartite graph is therefore two

Recall that the wreath product of two graphs G and H is denoted by $G \ H$

Theorem 2 2 4 $achr(G \ H) \geq achr(G) \ achr(H)$

Proof Let g be a complete homomorphism of G onto K_n, and h be a complete homomorphism of H onto K_m Then the function $f \ V(G) \times V(H) \rightarrow \{(i \ j), 1 \leq i \leq n$ and $1 \leq j \leq m\}$ defined by $f(x, y) = (g(x), h(y))$ is a complete homomorphism of $G \ H$ onto $K_n \ K_m$, which is isomorphic to K_{mn} □

Strict inequality can occur For example, let $G = 5K_2$ and $H = K_2$ Then $achr(G) = 3$ and $achr(H) = 2$, so the theorem gives $achr(G \ H) \geq 6$ But $G \ H = 5K_4$ which has achromatic number 7 This difference can be made larger Consider $G = \left(\binom{n}{2} - 1\right) K_2$ and $H = K_2$ The theorem gives $achr(G \ H) \geq 2n - 2$, while $G \ H = \left(\binom{n}{2} - 1\right) K_4$, which by Theorem 2 2 2, has achromatic number approximately equal to 2 4n for large n

Recall that the categorical product of graphs G and H is the graph denoted by $G \times H$

Hell and D. Miller [34] showed that in almost all cases the achromatic number of the categorical product of two graphs is at least the sum of the achromatic numbers of the two graphs.

Theorem 2.2.5 [34] *For graphs G and H, with $achr(H) \leq achr(G)$,*

$$achr(G \times H) \geq achr(G) + achr(H),$$

unless

(i) $achr(H) = 3$ and $achr(G) \leq 5$, in which case

$achr(G \times H) \geq achr(G) + achr(H) - 1$, *or*

(ii) $achr(H) = 2$, in which case $achr(G \times H) \geq achr(G) + achr(H) - 2$.

Equality is achieved in the general case by K_m and K_n, in (i) by $2K_3$, and in (ii) by $2K_2$.

Hell and D. Miller [34] gave an example showing that the achromatic number of the categorical product can grow exponentially.
Let $b = \binom{a+1}{2} - 1$, $G = bK_2$, and $H = T_{a,2a(a+2)^{b-1}}$.
Then $achr(G \times H) \geq (a+2)^{\binom{a+1}{2}-1} \approx a^{a^2/2}$.

The achromatic number of the categorical product is bounded above, however.

Theorem 2.2.6 [34] *Let $achr(G) = achr(H) = a$. Then $achr(G \times H)$ is bounded above by a function which only depends on a.*

We conclude this section with a generalization of the previous theorem.

Corollary 2.2.7 [34] *For any positive integer* $k \geq 2$, *there exists a* k*-variable function* f_k *such that* $achr(G_1) = a_1$, $achr(G_2) = a_2$, $\ldots$, $achr(G_k) = a_k$ *implies that* $achr(G_1 \times \cdots \times G_k) \leq f_k(a_1, \ldots, a_k)$.

2.3 Homomorphisms

In this section, we will investigate the effects of homomorphisms on the achromatic number of a graph, then develop a homomorphism interpolation theorem.

The proof of the next theorem uses the fact that the composition of two homomorphisms is itself a homomorphism.

Theorem 2.3.1 [31] *If* H *is a homomorphic image of* G, *then*

$$achr(H) \leq achr(G).$$

Proof. Suppose $achr(H) = m$. Then there exists a complete homomorphism h of H to K_m. Since H is a homomorphic image of G, there exists a complete homomorphism g of G to H. Then $h \circ g$ is a complete homomorphism of G to K_m. Therefore $achr(G) \geq m$. □

Theorem 2.3.2 [31] *If* H *is an elementary homomorphic image of* G, *then*

$$achr(G) - 2 \leq achr(H) \leq achr(G).$$

The lower bound is achieved by the graphs constructed as follows. For integers $m, n \geq 3$, let uv and xy be a pair of independent edges of $K_{m,n}$, where u

and x belong to the same part of the bipartition. Let $G = K_{m,n} - \{uv, xy\}$, and H be the elementary homomorphic image of G obtained by identifying u and x. Then $achr(G) = 4$ and, since $H \cong K_{m-1,n}$, $achr(H) = 2$.

The upper bound is achieved by a graph consisting of K_n, $n \geq 2$, joined to an additional vertex u by a single edge. An elementary homomorphism which maps u to any non-neighbour in K_n will not effect the achromatic number.

Recall that the chromatic number, $\chi(G)$, of a graph G is the smallest n such that there is a complete homomorphism of G onto K_n, while the achromatic number, $achr(G)$, is the largest n such that there is a complete homomorphism of G onto K_n. We next show that, for any graph G, there exists a complete homomorphism of G onto K_k for each k, $\chi(G) \leq k \leq achr(G)$.

Lemma 2.3.3 [30] *If ϕ is a homomorphism of G to H, then there exist elementary homomorphisms, $\epsilon_1, \epsilon_2, \ldots, \epsilon_m$, such that*

$$\phi(G) = \epsilon_m \circ \epsilon_{m-1} \circ \cdots \circ \epsilon_1(G).$$

Lemma 2.3.4 [31] *If H is an elementary homomorphic image of G, then*

$$\chi(G) \leq \chi(H) \leq \chi(G) + 1.$$

Proof Let ϵ be an elementary homomorphism of G to H. If $\chi(H) = n$, then there exists a homomorphism h of H to K_n. Thus $h \circ \epsilon$ is a homomorphism of G to K_n, whence $\chi(G) \leq n$.

On the other hand, suppose $\chi(G) = m$. Then there exists a homomorphism g of G to K_m. Let u and v be the vertices of G such that $\epsilon(u) = \epsilon(v)$. If $g(u) = g(v)$, then colouring each vertex x of H with the (unique) colour in $g(\epsilon^{-1}(x))$ results in an m-colouring of H. If $g(u) \neq g(v)$, then colouring the vertices of $H - \epsilon(u)$ as above, and assigning colour $(m+1)$ to $\epsilon(u)$ results in an $(m+1)$-colouring of H. In either case, $\chi(H) \leq m+1$. □

Theorem 2.3.5 [32] *(The Homomorphism Interpolation Theorem)*
For any graph G and any integer k such that $\chi(G) \leq k \leq achr(G)$, *there is a complete homomorphism of G onto* K_k.

Proof Let $achr(G) = t$, and consider any complete homomorphism ϕ of G onto K_t. From Lemma 2.3.3, we know that ϕ can be expressed as a product of elementary homomorphisms, say $\phi(G) = \epsilon_m \circ \epsilon_{m-1} \circ \cdots \circ \epsilon_1(G) = K_t$. Since $\chi(G) \leq \chi(\epsilon(G)) \leq \chi(G) + 1$ from Lemma 2.3.4, we know that $\epsilon_{i+1} \circ \epsilon_i \circ \cdots \circ \epsilon_1(G)$ has chromatic number at most one greater than $\epsilon_i \circ \cdots \circ \epsilon_1(G)$. So for every k, $\chi(G) \leq k \leq t = achr(G)$, there exists at least one graph , say $\epsilon_i \circ \cdots \circ \epsilon_1(G)$, whose chromatic number is k. By definition, there is a complete homomorphism ϕ_i of this graph onto K_k. Hence $\phi_i \circ \epsilon_i \circ \cdots \circ \epsilon_2 \circ \epsilon_1$ is a complete homomorphism of G onto K_k. □

We conclude this section by pointing out a generalization of Theorem 2.3.5 by Cockayne, G.G. Miller, and Prins [18]. Let S be a finite set. If $\mathcal{P}$ is a collection of subsets of S, then $\mathcal{P}$ is called a *property* of S. We say a subset X of S *has* property $\mathcal{P}$ if $X \in \mathcal{P}$. A partition $\{S_1, S_2, \ldots, S_t\}$ of S is a *complete $\mathcal{P}$-partition of order t* if each S_i has property $\mathcal{P}$ but no $S_i \cup S_j$, $i \neq j$, has this property. $\mathcal{P}$ is *hereditary* if each subset of a set with property $\mathcal{P}$ has property $\mathcal{P}$.

Theorem 2.3.6 [18] *Let S be a set which has complete $\mathcal{P}$-partitions of orders r and t, where $\mathcal{P}$ is any hereditary property. Then for any integer s, such that $r \leq s \leq t$, S has a complete $\mathcal{P}$-partition of order s.*

2.4 Irreducible Graphs

Graphs with high achromatic number were investigated by Hell and D. Miller in 1975 and 1976 [36, 37]. A graph G of achromatic number n must contain at least $\binom{n}{2}$ edges, so a graph with high achromatic number must contain many edges. It is not necessarily true, however, that a graph with many edges has high achromatic number, for example $achr(K_{n,n}) = 2$ for any n. One method for measuring the freedom of a vertex to take on a colour is the reducing congruence R, defined as follows. Two vertices u, v of a graph G belong to the same congruence class of R if they have the same neighbourhood, (i.e., if $N(u) = N(v)$). The *reduced graph G/R* is the (quotient) graph whose vertices are the congruence classes of R, and where two classes are adjacent if there exists a vertex in one class adjacent to a vertex in the other. Since G/R is just the graph resulting from identification of vertices with the same neighbourhoods, it is a homomorphic image of G, so $achr(G/R) \leq achr(G)$.

Strict inequality can hold. For example, consider the graph G obtained by joining an end of P_3 to C_4 by a single edge. Then $achr(G) = 4$, whereas the graph $G/R \cong P_6$ has achromatic number 3. A graph is *irreducible* if distinct vertices have distinct neighbourhoods, that is, if $G/R \cong G$.

Lemma 2.4.1 [37] *If G is irreducible and $achr(G) \leq k$, then*
(i) G contains at most $\binom{k+1}{2}$ components, and
(ii) for each component G' of G, diam $G' \leq \left\lfloor \frac{k+3}{2} \right\rfloor \cdot (k-1)$.

Both of the above bounds are best possible. For (i), take G to be $K_1 \cup \left(\binom{k+1}{2} - 1 \right) K_2$; for (ii), take G to be the path of length $\left\lfloor \frac{k+3}{2} \right\rfloor \cdot (k-1)$.

Theorem 2.4.2 [37] *Let k be a given integer. There exists a constant K such that $|V(G)| \leq K$ for all irreducible graphs G with $achr(G) = k$.*

This important result implies that the number of non-isomorphic irreducible graphs of given achromatic number is finite. That is, up to duplication of vertices, the number of graphs of achromatic number k is finite. We can therefore define functions:

$$v(k) = max\ \{|V(G)| : G \text{ is irreducible}, achr(G) = k\},$$
$$e(k) = max\ \{|E(G)| : G \text{ is irreducible}, achr(G) = k\}.$$

Certain other results come out of the proof of Theorem 2.4.2.

Corollary 2.4.3 [37] *If G/R has more than $e(k)$ edges, then $achr(G) > k$.*

Corollary 2.4.4 [37] *If G is irreducible and $achr(G) = k$, then*

$$\Delta(G) \leq (v(k-1))^3, \text{ where } v(k) \text{ is defined as above.}$$

Theorem 2.4.5 [36] *Let G be any graph, and R be the reducing congruence on G. If n is the number of vertices of G/R (that is, the number of classes of R), then*

$$\sqrt{log\ log\ n} < achr(G) \leq \sqrt[3]{3^n} \qquad \text{if } n \text{ is large enough.}$$

This upper bound is reported to be sharp [36]. The lower bound implies that $achr(G/R) \rightarrow \infty$ as $n \rightarrow \infty$. Máté [43] improved this lower bound somewhat in 1981.

Theorem 2.4.6 [43] *Let G be an irreducible graph on n vertices. For all $\epsilon > 0$, there exists an N such that if $n \geq N$, then*

$$achr(G) \geq \frac{\left(\frac{1}{2}-\epsilon\right) log\ n}{log\ log\ n}$$

An example of an irreducible graph G with achr $(G) \leq \frac{log\ n}{log\ 2} + 2$ was attributed to Erdos in [43]. A slight variation on the Erdos example was given by Hell in [43]. He shows how to construct a graph with $k + 3 \cdot 2^k$ vertices and achromatic number at most $k + 2$.

Chapter 3

PATHS, CYCLES, K-REGULAR GRAPHS AND TREES

In this chapter, we examine the research concerning the bounds of the achromatic number of various classes of graphs.

3.1 Paths

Paths were logically the first type of graph to explore, and were researched by Geller and Kronk in 1974, and by Hell and D. Miller in 1976.

A complete homomorphism of a path to a graph G can be viewed as a walk in G in which every edge is traversed at least once. Equivalently, it can be viewed as an Eulerian trail in a multigraph for which G is the underlying graph. Figure 3.1 shows a complete homomorphism of P_8 onto the multigraph for which K_4 is the underlying graph. The arrows are added to show the Eulerian trail. Similar comments apply to a complete homomorphism of a cycle to G. This idea is formalized in Lemma 3.1.1 below. It was exploited in [36] to determine the achromatic number of P_n and C_n (see Theorems 3.1.2 and 3.2.2).

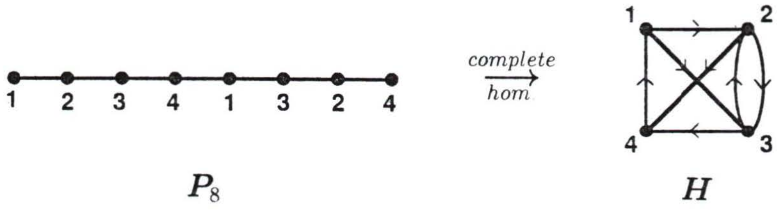


Figure 3.1. A complete homomorphism of P_8 onto a multigraph H with underlying graph K_4.

Lemma 3.1.1 [36] *Let G be a graph.*

(i) A complete homomorphism of P_n to G exists if and only if G is the underlying graph of some multigraph G' with $n-1$ edges such that G' admits an Eulerian trail.

(ii) A complete homomorphism of C_n to G exists if and only if G is the underlying graph of a multigraph G' with n edges such that G' admits an Eulerian tour.

Theorem 3.1.2 [36] $achr(P_n) = max\left\{k : (\left\lfloor \frac{k}{2} \right\rfloor + 1)(k-2) + 1 \leq n-1\right\}$

Proof. Let $f(k) - 1 = (\lfloor \frac{1}{2}k \rfloor + 1)(k-2) + 1$.

When k is odd, then $f(k) - 1 = \frac{1}{2}k(k-1) = \binom{k}{2}$ and each vertex of K_k has even degree. Thus K_k admits an Eulerian trail and $P_{f(k)}$ admits a complete k-colouring (Lemma 3 1 1) Since K_{k+1} has more edges than $P_{f(k)}$, $achr(P_{f(k)}) = k$

When k is even, each vertex of K_k has odd degree A multigraph that admits an Eulerian trail can be constructed by adding $\frac{1}{2}(k-2)$ new edges joining disjoint pairs of vertices in K_k This leaves exactly two vertices of odd degree, a necessary and sufficient condition for a connected multigraph to have an Eulerian trail [7] Since all vertices have odd degree, this is the minimum number of new edges necessary to obtain such a multigraph In this case, with k even, $f(k) - 1 = \frac{1}{2}k(k-1) + \frac{1}{2}(k-2)$, so the path $P_{f(k)}$ again admits a complete k-colouring, and $achr(P_{f(k)}) = k$ This argument also shows that, for all k, $achr(P_{f(k)-1}) < k$ □

Corollary 3 1 3 [25] *If $n > m$, then $achr(P_n) \geq achr(P_m)$*

The relationship between Euler tours and the achromatic number is further developed in Chapter 5

The minimum length of a path with given achromatic number can be determined from Theorem 3 1 2 This length is given explicitly in the following theorem

Theorem 3 1 4 [25] *For $k \geq 2$, let $n = k\left\lceil\frac{k-1}{2}\right\rceil$ Then if k is even, $achr(P_{n-1}) < achr(P_n) = k$, and if k is odd, $achr(P_n) < achr(P_{n+1}) = k$*

Later results (see Theorems 5.1.6 and 5.1.8.) give information regarding the achromatic number of a disjoint union of paths (of the same length).

3.2 Cycles

The achromatic number of cycles was also investigated in the papers by Geller and Kronk [25], and by Hell and D. Miller [36]. The results are similar to those for paths.

Theorem 3.2.1 [25] *If* $k\left\lceil\frac{k-1}{2}\right\rceil \leq n < (k+1)\left\lceil\frac{k}{2}\right\rceil$, *then*

$$achr(C_n) = \begin{cases} k & \text{if } k \geq 3 \text{ and } n = k\left\lceil\frac{k-1}{2}\right\rceil \\ k-1 & \text{if } n = k\left\lceil\frac{k-1}{2}\right\rceil + 1 \text{ and } k \text{ odd} \\ k & \text{otherwise} \end{cases}$$

We note that while $achr(P_n)$ is a monotone function of n, $achr(C_n)$ is not. For example, $achr(C_{10}) = 5$ while $achr(C_{11}) = 4$.

Theorem 3.2.2 [36] $achr(C_n) = max\left\{k : k\left\lfloor\frac{k}{2}\right\rfloor \leq n\right\} - s(n)$, *where s(n) is the number of positive integer solutions of the equation* $n = 2x^2 + x + 1$. *(Note that* $s(n) = 0$ *or 1.)*

The equation $n = 2x^2 + x + 1$ arises from $n = k\left(\frac{k-1}{2}\right) + 1$ when k is odd and $x = \frac{k-1}{2}$.

A subsequent result given in Chapter 5 states, in part, that the disjoint union of $2tm+1$ cycles of length m has achromatic number $2tm+1$, for $t \in \mathbf{Z}^+$.

3.3 k-Regular Graphs

In 1982, Z. Miller generalized Hell and D. Miller's work on cycles with a paper establishing upper bounds on the achromatic number of k-regular graphs on n vertices.

Theorem 3.3.1 [47] *Let $k \geq 2$ be an integer, and let M(n,k) be the maximum value of achr(G) over all k-regular graphs on n vertices. Then,*

(i) $M(n,k) \leq max\left\{\lambda \in \mathbf{Z}^+ : \lambda\lceil\frac{\lambda-1}{k}\rceil \leq n\right\}$ and,

(ii) There is a function Q(k) such that for all $n \geq Q(k)$, we have

$$M(n,k) = \begin{cases} max\left\{\lambda \in \mathbf{Z}^+ : \lambda\lceil\frac{\lambda-1}{k}\rceil \leq n\right\} & \text{if } n \notin \Omega \\ \\ max\left\{\lambda \in \mathbf{Z}^+ : \lambda\lceil\frac{\lambda-1}{k}\rceil \leq n\right\} - 1 & \text{if } n \in \Omega \end{cases}$$

where $\Omega = \{\, i(ik+1)+1 : i \in \mathbf{Z}^+ \,\}$.

For $k = 2$ in the above theorem, the result is precisely that of Theorem 3.2.2. Examples of graphs with maximum achromatic number for some other values of k and n are shown in Figure 3.2. G_1 is the 4-regular graph on 6 vertices with $M(6,4) = achr(G_1) = 3$, and G_2, the Petersen graph, is a 3-regular graph on 10 vertices with $M(10,3) = achr(G_2) = 5$.

G_1 G_2

Figure 3.2 Two k-regular graphs on n vertices which attain $M(n, k)$.

Since the proof of the previous theorem was constructive, it is possible to determine the minimum n for which there exists a k-regular graph on n vertices having achromatic number λ. This minimum n is denoted by $n(\lambda, k)$.

Corollary 3.3.2 [47] *For any $k \geq 2$ and λ sufficiently large (depending on k), we have $n(\lambda, k) = min\left\{n : n \geq \lambda\lceil\frac{\lambda-1}{k}\rceil \text{ and } n \text{ even if } k \text{ odd}\right\}$.*

3.4 Trees

Most parameters of trees are easy to compute. The achromatic number is an exception. Even the complexity of determining the achromatic number of a given tree is still an open question, although polynomial algorithms have been found for several classes of trees (see Chapter 8).

The achromatic number of trees has been studied by Farber, Hahn, Hell, and D. Miller who arrived at the following bounds in 1986 [22].

Lemma 3.4.1 [22] *If T is a tree with max. degree Δ and $achr(T) \leq k$, then*

$$|E(T)| \leq \begin{cases} (k-1)\Delta + \binom{k-1}{2} & \text{if } k \leq \Delta \\ (k-1)k + \binom{\Delta-1}{2} & \text{if } k \geq \Delta \end{cases}$$

Theorem 3.4.2 [22] *Let* T *be a tree with* m *edges and*
$t = max\left\{0, m - \binom{\Delta-1}{2} - \frac{3}{4}\right\}$.

$$\text{Then } \left\lfloor \tfrac{3}{2} + t^{\frac{1}{2}} \right\rfloor \leq achr(T) \leq \left\lfloor \tfrac{1}{2} + \left(2m + \tfrac{1}{4}\right)^{\frac{1}{2}} \right\rfloor$$

Proof. The upper bound holds since any graph with achromatic number k has at least $\binom{k}{2}$ edges:

$$\left\lfloor \tfrac{1}{2} + \left(2m + \tfrac{1}{4}\right)^{\frac{1}{2}} \right\rfloor \geq \left\lfloor \tfrac{1}{2} + \left(2\binom{k}{2} + \tfrac{1}{4}\right)^{\frac{1}{2}} \right\rfloor = k.$$

The lower bound follows from Lemma 3.4.1 by solving for k in the equation $m = (k-2)(k-1) + \binom{\Delta-1}{2} + 1$. This bound holds for all Δ since $(k-2)(k-1) + \binom{\Delta-1}{2} \geq (k-2)\Delta + \binom{k-2}{2}$ for all $\Delta, k \in \mathbf{Z}^+$, and hence the left hand side of this inequality is always an upper bound for the number of edges in a tree T with achr$(T) \leq k-1$. □

Simpler bounds can be obtained if a relationship between the number of edges and the maximum degree is known.

Corollary 3.4.3 [22] *If* T *is a tree with* m *edges and* $\Delta \leq (4m)^{\frac{1}{4}}$, *then*

$$achr(T) \geq \sqrt{m}.$$

Corollary 3.4.4 [22] *If* T *is a tree with* $\Delta < n$ *and at least* $3\binom{n}{2}$ *edges, then* $achr(T) > n$.

The case of generalized stars (i.e., subdivisions of $K_{1,d}$) has been considered by Lopez-Bracho [42]. Some of these results are presented on page 39.

Considering the difficulty of ascertaining the achromatic number of a tree, a result of a later chapter (see Theorem 5.1.7) is interesting in that it states, in part, that if the tree T has m edges, then for $t \in \mathbf{Z}^+$, the forest $t(2tm+1)T$ has achromatic number $2tm + 1$.

Chapter 4

CLIQUE NUMBER AND ACHROMATIC NUMBER

We now focus on the restricted class of graphs for which the clique number equals the achromatic number. We begin by giving an original characterization of these graphs.

4.1 Graphs With Equal Achromatic And Clique Numbers

We say that a graph G has *Property* $\mathcal{H}$ if for every homomorphic image H of G there exists a homomorphism of H (back) to G. Figure 4.1 shows three graphs each having this property. Note that in each example, the achromatic number equals the clique number. The families of graphs described in Sections 4.2 and 4.3 also have Property $\mathcal{H}$, constructions for these graphs are given in the relevant section.

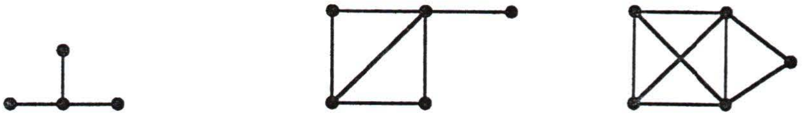

G_1 G_2 G_3

Figure 4.1. Three graphs with Property $\mathcal{H}$.

Theorem 4.1.1 *A graph G has Property $\mathcal{H}$ if and only if $achr(G) = \omega(G)$.*

Proof. Suppose G has Property $\mathcal{H}$, that is, there exists a homomorphism of H to G whenever there exists a complete homomorphism of G onto H. If $achr(G) = n$, there exists a complete homomorphism of G onto K_n. Thus there also exists a homomorphism f of K_n to G. Since $f(K_n)$ is a complete subgraph of G, $\omega(G) \geq achr(G)$. Hence $\omega(G) = achr(G)$.

Now suppose $achr(G) = \omega(G) = n$. Let H be a homomorphic image of G. By Theorem 2.3.1, $achr(H) \leq achr(G) = n$, so there exists a homomorphism h of H to K_n. Since K_n is a subgraph of G, there is a homomorphism $\imath$ of K_n to G ($\imath$ can be taken to be the inclusion map). The function $\imath \circ h$ is then a homomorphism of H to G. □

It remains an open problem to find a structural characterization of these graphs. In the remainder of this chapter we examine several subclasses of these graphs for which such a characterization has been determined.

4.2 Perfect Graph Theorems

In this section we review perfect graph colourings, which were investigated by Christen and Selkow [15] in 1979.

A *Grundy n-colouring* of a graph is an n-colouring such that for each colour i, each vertex coloured i is adjacent to at least one vertex coloured j for each $j < i$. These have also been called Canonical colourings by Bondy and Murty [7], and Ordered colourings by Cockayne and Thomason [19], and are the colourings produced by a greedy algorithm. The *Grundy number* γ of a graph is the maximum number n for which the graph has a Grundy n-colouring.

It is clear that for any graph, $\omega \leq \chi \leq \gamma \leq achr$.

For α and β distinct elements of the set $\{\omega, \chi, \gamma, achr\}$, a graph G is called *(α, β)-perfect* if for each induced subgraph H of G, $\alpha(H) = \beta(H)$. The (χ, ω)-perfect graphs are more commonly referred to as perfect graphs and have been extensively studied in the literature (e.g., see [26]). Christen and Selkow characterized $(achr, \omega)$-perfect graphs and $(achr, \chi)$-perfect graphs.

Theorem 4.2.1 [15] *The following are equivalent.*

(i) G is $(achr, \omega)$-perfect.

(ii) G is $(achr, \chi)$-perfect.

(iii) G does not contain an induced subgraph isomorphic to one of the graphs P_4, $P_3 \cup P_2$, or $P_2 \cup P_2 \cup P_2$.

(iv) no homomorphic image of G contains an induced subgraph isomorphic

to P_4

They next provided an inductive characterization of the class of $(achr, \omega)$-perfect graphs (and therefore also of the $(achr, \chi)$ -perfect graphs)

Theorem 4 2 2 [15] *A finite graph is* $(achr, \chi)$*-perfect (equivalently* $(achr, \omega)$*-perfect) if and only if it can be generated from* K_1 *or from* $K_m \cup K_n$ *, for some* $m, n \in \mathbf{Z}^+$*, by finitely many applications of the addition of isolated vertices and of the join operation*

Proof We use induction on $|V(G)|$ Suppose G is $(achr, \chi)$-perfect If $|V(G)| = 1$, then G is isomorphic to K_1

Suppose the theorem is true for all graphs G_k for which $|V(G_k)| = k$ We show that it is also true for graphs G_{k+1} with $k + 1$ vertices If G_{k+1} has a component which is a single vertex, it can be generated by adding an isolated vertex to some G_k, which by the induction hypothesis is generable from K_1 or $K_m \cup K_n$ in the appropriate way

If all components of G_{k+1} are non-trivial, then by Theorem 4 2 1, G_{k+1} does not contain an induced subgraph isomorphic to $P_2 \cup P_2 \cup P_2$, and thus has at most two components If G_{k+1} has two components, then since it does not contain an induced subgraph isomorphic to $P_3 \cup P_2$, both components are complete and we are finished If G_{k+1} has only one component, this, together with a result of Seinsche [51], that if a graph H is $(achr, \chi)$-perfect then either H or $\overline{H}$ is disconnected, implies that G_{k+1} is the join of two smaller graphs The assertion follows from the induction hypothesis

Now suppose G can be generated from K_1 or $K_m \cup K_n$ in the appropriate way. We prove this part by induction on the number of operations necessary to generate the graph G. If this number is 0, then G is isomorphic to K_1 or $K_m \cup K_n$, which do not contain an induced graph isomorphic to one of the forbidden graphs listed in Theorem 4.2.1, and G is therefore $(achr, \chi)$-perfect.

Suppose the theorem is true for all graphs G_t for which the number of operations equals t. We show that it is also true for G_{t+1}. If G_{t+1} is generated by adding isolated vertices to G_t, then it clearly contains no induced subgraph isomorphic to a forbidden graph, and thus is $(achr, \chi)$-perfect.

If G_{t+1} is the join of two graphs G_i and G_j, both of which can be generated in less than $t+1$ operations, then by the induction hypothesis both G_i and G_j contain no induced subgraph isomorphic to a forbidden graph. However, if p and q are non-adjacent vertices in G_{t+1} they must both be in G_i or both in G_j. By the definition of join, any induced subgraph of G_{t+1} isomorphic to one of the three forbidden graphs must be contained entirely in G_i or entirely in G_j, contrary to the induction hypothesis. Thus, G_{t+1} is $(achr, \chi)$-perfect, and the result follows. □

In the same paper, Christen and Selkow gave a family of forbidden induced subgraphs of a $(achr, \gamma)$-perfect graph. The characterization of $(achr, \gamma)$-perfect graphs remains open, however. Further results concerning the Grundy number can be found in [15, 17, 21].

4.3 Homomorphically Full Graphs

In this section we investigate another class of graphs, which turn out to be a subset of the class of perfect graphs. A graph G for which every homomorphic image of G is a subgraph of G is called a *homomorphically full graph*.

Theorem 4.3.1 [14] *If G is homomorphically full, then $\omega(G) = \chi(G) = achr(G)$, where ω denotes the clique number of G.*

Proof. For any graph G, $\omega(G) \leq \chi(G) \leq achr(G)$, so it suffices to show that $\omega(G) \geq achr(G)$. By definition, $achr(G)$ is the largest n for which K_n is a homomorphic image of G. Suppose $achr(G) = m$. Then K_m is a homomorphic image of G. Since G is homomorphically full, this means K_m is a subgraph of G. Therefore $\omega(G) \geq m$. □

The graph $2K_2$ shows that the converse is false.

Brewster and MacGillivray [14] characterized homomorphically full graphs.

Theorem 4.3.2 [14] *Let G be a graph. The following statements are equivalent.*

(i) G is homomorphically full.

(ii) For every pair u, v of non-adjacent vertices of G, either $N(u) \subseteq N(v)$ or $N(v) \subseteq N(u)$.

(iii) Every homomorphic image of G is an induced subgraph of G.

(iv) G contains neither $2K_2$ nor P_4 as an induced subgraph.

In analogy with the recursive construction of $(achr, \omega)$-perfect graphs in the previous section, the homomorphically full graphs may be constructed using the following rules [14]:

(i) K_1 is homomorphically full;

(ii) If G is homomorphically full, then so is $G \cup K_1$;

(iii) If G and H are homomorphically full, then so is $G + H$.

Other characterizations and constructions of homomorphically full graphs can also be found in [14].

Chapter 5

N-MINIMAL GRAPHS

This chapter will explore those graphs which have the minimum number of edges necessary in order to have a given achromatic number. A graph G is *n-minimal* if $achr(G - e) < achr(G) = n$ for every edge e in G. This definition is due to Bhave [5] in 1979, and Kelly [42] in 1978 (where such graphs were called *maximally achromatic*). They have also been called *ψ-critical* in [43]. An equivalent statement is that a graph G is n-minimal if $achr(H) < achr(G)$ for all $H \subsetneq G$ (up to deleting isolated vertices).

Theorem 5.1.1 [5] *A graph G with achromatic number n is n-minimal if and only if it has exactly $\binom{n}{2}$ edges.*

Proof. Suppose G is n-minimal. Then there exists a complete homomorphism ϕ which maps G onto K_n. Thus G has at least $\binom{n}{2}$ edges. Suppose G has more than $\binom{n}{2}$ edges. Then there exists edges uv and xy of G such that $\phi'(uv) = \phi'(xy)$, where ϕ' is the induced edge mapping function of $E(G)$ to $E(K_n)$ defined by $\phi'(ab) = \phi(a)\phi(b)$. This implies that G is not n-minimal, since the deletion of uv, say, leaves a graph which still admits a complete homomorphism onto K_n. Hence G must have exactly $\binom{n}{2}$ edges.

Now suppose $achr(G) = n$ and G has $\binom{n}{2}$ edges. Then, for any edge $e \in E(G)$, $achr(G - e) \leq n - 1$ by Theorem 2.1.3. Hence G is n-minimal. $\square$

We note that a path with $\binom{n}{2}$ edges is n-minimal when n is odd.

Gao and Hahn [24] give the following construction of a family of n-minimal trees for n even or odd. Let $V(T) = \{u^i_j : i = 1, \ldots, n-1, j = i, \ldots, n-1\} \cup \{u^0_0\}$, and $E(T) = \{u^i_i u^{i+1}_j : i = 0, 1, \ldots, n-2, j = i+1, \ldots, n-1\}$. A complete homomorphism h of T onto K_n with $V(K_n) = \{v_0, \ldots, v_{n-1}\}$ is defined by $h(u^i_j) = v_j$.

We now describe a method, proposed by Bhave [5], for constructing the set of all n-minimal graphs from the set of all $(n-1)$-minimal graphs. We first need a few definitions and lemmas.

Let G be a graph, and $\mathcal{P} = \{V_1, V_2, \ldots, V_n\}$ be a partition of $V(G)$ into non-empty sets. The *partition graph, $\mathcal{P}(G)$, of G* is the graph with vertex set $\mathcal{P}$ where V_i and V_j are adjacent if and only if there exist $v_i \in V_i$ and $v_j \in V_j$ such that v_iv_j is an edge in G. A partition $\mathcal{P}$ of $V(G)$ is *complete* if $\mathcal{P}(G)$ is a complete graph. A graph H is *partition realizable* from a graph G if $\mathcal{P}(G) = H$ for some partition $\mathcal{P}$ of $V(G)$.

Let H be a subgraph of G. The subgraph of G obtained by deleting all edges of H and the resulting isolated vertices in G will be denoted by $G \dot{-} H$. For example, $K_n \dot{-} K_{n-1} = K_{1,n-1}$.

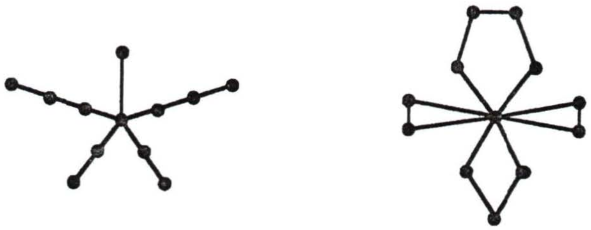
Lemma 5.1.2 [5] *Let a graph H be partition realizable from G, and let H_1 be an induced subgraph of H, then*
(i) H_1 is partition realizable from an induced subgraph G_1 of G, and
(ii) $H \dot{-} H_1$ is partition realizable from $G \dot{-} G_1$.

Corollary 5.1.3 [5] *If K_n is partition realizable from G, then there exists an induced subgraph G_1 of G such that*
(i) K_{n-1} is partition realizable from G_1, and
(ii) $K_{1,n-1}$ is partition realizable from $G \dot{-} G_1$.

Bhave's method [5] of constructing the set of all n-minimal graphs from the set of all $(n-1)$-minimal graphs is as follows (only graphs with no isolated vertices are considered).

Let $\{G_i\}$ be the set of all $(n-1)$-minimal graphs and $\{H_j\}$ be the set of all graphs with $n-1$ edges, such that $K_{1,n-1}$ is partition realizable from any element of $\{H_j\}$. Since K_{n-1} is partition realizable from any element of $\{G_i\}$ and $K_{1,n-1}$ is partition realizable from any element of $\{H_j\}$, K_n is partition realizable from each of the graphs formed below, each of which has $\binom{n}{2}$ edges and hence is n-minimal. Consider a graph $G_i \in \{G_i\}$. Let $\mathcal{P}= \{V_1, V_2, \ldots, V_{n-1}\}$ be a complete partition of $V(G_i)$. It is easy to see that each $H_j \in \{H_j\}$ has at least $n-1$ vertices of degree one, say $u_1, u_2, \ldots, u_{n-1}$. Let G be a graph obtained from G_i and H_j by identifying some, all, or none of these vertices with the vertices of G_i such that no two of $u_1, u_2, \ldots, u_{n-1}$ are identified with vertices belonging to the same part $V_i \in \mathcal{P}$. The following argument shows that any n-minimal graph is isomorphic to a graph obtained above. Let G be n-minimal and $\mathcal{P}(G) = K_n$. Then as K_{n-1} is an induced subgraph of K_n, there exists an induced subgraph G'_i of G such that K_{n-1} is partition realizable from G'_i and $K_{1,n-1}$ is partition realizable from $G \dot{-} G'_i$ by Corollary 5.1.3. Therefore G'_i has $\binom{n-1}{2}$ edges and hence is $(n-1)$-minimal and $G \dot{-} G'_i$ has $n-1$ edges. Therefore G is isomorphic to one of the graphs obtained above.

The n-minimal trees with a single vertex of degree at least 3 (that is, generalized stars), and "flowers" were investigated by Lopez-Bracho [42] in 1983. If v is a vertex of a tree T and the components of the subgraph $T - v$ are $V_1, V_2, \ldots, V_d$, then the subgraphs $T_i = T[V_i \cup \{v\}]$, $i = 1, 2, \ldots, d$, are called the *branches* of T with respect to v. (The branches of a generalized star with respect to the vertex of degree at least 3 are paths.) A *flower* is a connected graph F composed of (induced) cycles which have exactly one vertex in common. The cycles themselves are called *petals*. Figure 5.1 shows a generalized star S and a flower F.



S **F**

Figure 5.1 A generalized star S and a flower F

Theorem 5.1.4 [42] *Let $T_m(u_1, u_2, \ldots, u_d)$ be a tree with just one vertex v with degree $d > 2$, where m is the number of edges of T, d is the number of branches of T with respect to v and $u_1, u_2, \ldots, u_d$ are the lengths of these branches. Then*

(i) If n is even, T is n-minimal if and only if $m = \binom{n}{2}$ and $d = n - 1$.

(ii) If $n \geq 3$ is odd, T is n-minimal if and only if $m = \binom{n}{2}$, $d \leq n - 1$, and at least $\frac{d}{2}$ branches of T have length > 1.

Theorem 5.1.5 [42] *Let F be a flower, then for odd $n \geq 5$, F is n-minimal if and only if it has $\binom{n}{2}$ edges and at most $\frac{(n-1)}{2}$ petals.*

Various types of n-minimal graphs which are disjoint unions of identical graphs were first constructed by Kelly [41] in 1981.

Theorem 5.1.6 [41] *Let F be a forest with m edges. Let $t \in \mathbf{Z}^+$. Then $t(2tm + 1)F$ is $(2tm + 1)$-minimal.*

In other words, K_{2tm+1} is the union of $t(2tm + 1)$ edge disjoint copies of F.

Theorem 5.1.7 [41] *Let C_m be a cycle with m edges. Let $t \in \mathbf{Z}^+$. Then $t(2tm + 1)C_m$ is $(2tm + 1)$-minimal.*

In other words, K_{2tm+1} is the union of $t(2tm + 1)$ edge disjoint copies of C_m.

Theorem 5.1.8 [41] *For any integer n such that* $m|\binom{n}{2}$, $\frac{1}{m}\binom{n}{2}P_{m+1}$ *is n-minimal.*

These results have been recently improved. Let G be a graph. We note that the statement that kG is n-minimal is equivalent to the statement that K_n can be factored into k edge-disjoint copies of G. The next two theorems give conditions under which kP_m and kC_m, respectively, are n-minimal. In each case, the given conditions are necessary to assure the graph in question has exactly $\binom{n}{2}$ edges.

Theorem 5.1.9 [2] *For any odd integers* n, m, $n \geq m \geq 3$, *if* $m|\binom{n}{2}$, *then* $\frac{1}{m}\binom{n}{2}C_m$ *is n-minimal.*

Theorem 5.1.10 [54] *For any integer n such that* $n \equiv 0 (mod\ m)$ *and* $n \equiv 1(mod\ m-1)$ *when* $m \neq 3, 5$ *and* $(m-1)|\binom{n}{2}$, $\frac{1}{m-1}\binom{n}{2}P_m$ *is n-minimal.*

Theorem 5.1.11 [55] *If m is an even integer, then* $\frac{1}{m}\binom{n}{2}K_{1,m}$ *is n-minimal if and only if* $n \equiv 0(mod\ m+1)$ *and* $n \equiv 1(mod\ 2m)$.

Other similar results for classes of trees are found in [55].

The above three results are samples only; a comprehensive reference to decompositions of graphs is the text by Bosak [10] in 1990.

In 1986, Farber, Hahn, Hell and D. Miller established a Hall-like necessary condition for any graph G to be n-minimal. We first need some definitions. An *edge-monomorphism of G to H* is a homomorphism f of G to H such that the induced edge mapping f' is injective. If G has exactly $\binom{n}{2}$ edges then a complete n-colouring corresponds to an edge-monomorphism of G to K_n. A *μ-image of G* is a homomorphic image of G under some edge-monomorphism. Thus, a graph is Eulerian if and only if it is a μ-image of a cycle.

Theorem 5.1.12 [22] *Suppose $G = (V, E)$ is n-minimal. For $W \subseteq V$, let $\left|E(G[W])\right|$ be the number of edges of G with both ends in W. Then*

$$\sum_{v \in W} \deg(v) \leq \left|E(G[W])\right| + \left|W\right|(n-1) - \binom{|W|}{2}$$

for each set $W \subseteq V$ of cardinality at most n. In particular, if W is an independent set with at most n vertices, then the average degree in G of the vertices in W is at most $n - \frac{|W|+1}{2}$.

Proof. Let $\left|[V(G) - W, W]\right|$ be the number of edges of G with exactly one end in W. Note that $\sum_{v \in W} \deg(v) = 2\left|E(G[W])\right| + \left|[V(G) - W, W]\right|$.

Consider an edge-monomorphism f of G onto K_n. Since f is an edge-monomorphism,

$\left|E(G[W])\right| + \left|[V(G) - W, W]\right| \leq \left|E(K_n[f(W)])\right| + \left|[V(K_n) - f(W), f(W)]\right|$. Therefore,

$$\begin{aligned}
\sum_{v \in W} deg(v) &\leq \left|E(G[W])\right| + \left|E(K_n[f(W)])\right| + \left|[V(K_n) - f(W), f(W)]\right| \\
&= \left|E(G[W])\right| + \left(\sum_{u \in f(W)} \deg_{K_n}(u)\right) - \left|E(K_n[f(W)])\right| \\
&= \left|E(G[W])\right| + \left|f(W)\right|(n-1) - \binom{|f(W)|}{2} \\
&\leq \left|E(G[W])\right| + \left|W\right|(n-1) - \binom{|W|}{2},
\end{aligned}$$

since $\left|f(W)\right| \leq \left|W\right| \leq n$. □

Corollary 5.1.13 [22] *If G is an Eulerian graph with $\binom{n}{2}$ edges, where n is even, then $achr(G) < n$.*

Proof. Suppose G has $\binom{n}{2}$ edges and $achr(G) = n$. Then K_n is a μ-image of G. Clearly any μ-image of an Eulerian graph is Eulerian. Since K_n is Eulerian if and only if n is odd, then either n is odd or G is not Eulerian.

□

The following theorem can be thought of as a generalization of Theorem 5.1.5. It uses a more general definition of petal. Here a *petal* of G is a component of $G \backslash B$, where B is the set of vertices of degree at least 3 (that is, a component of the subgraph induced by the vertices of degree at most 2).

Theorem 5.1.14 [22] *There is a function $g : \mathbf{Z}^+ \times \mathbf{Z}^+ \rightarrow \mathbf{Z}^+$ with the property that $achr(G) = n$ for each Eulerian graph G with at most k vertices of degree at least 3, at most l petals, and exactly $\binom{n}{2}$ edges, as long as n is odd and $n \geq g(k, l)$.*

Theorem 5.1.15 [22] *Suppose T is a tree with at most k leaves and exactly $\binom{n}{2}$ edges, where n is odd and large with respect to k (say $n \geq g(k, 2k)$, where g is the function in the previous theorem). Then $achr(T) = n$ if and only if T has an Eulerian μ-image.*

We conclude this chapter with an original result on n-minimal graphs. We begin with a result by Bhave [5].

Theorem 5.1.16 *If m and n are any two integers with $1 < m \leq n$, then there exists a graph G with $\chi(G) = m$ and $achr(G) = n$.*

One family of n-minimal graphs with chromatic number m is $K_m \cup \left(\binom{n-m}{2} + m(n-m)\right) K_2$. These graphs, however, contain a clique of size m. In what follows, we investigate the connection between the clique number ω, the chromatic number χ, and the achromatic number. Recall that $\omega \leq \chi \leq achr$.

Let $f(m, n)$ denote the minimum number of vertices in a graph G with $\chi(G) = n$ and $\omega(G) = m$. Such graphs exist for all m and n with $1 < m \leq n$. For example, $f(3, 2) = 5$, $f(4, 2) = 11$, and $f(5, 3) = 11$ [29].

Proposition 5.1.17 *For any integers m and n with $1 < m \leq n$, there exists a graph G with $\omega(G) = m$, $\chi(G) = n$, and $achr(G) \geq f(m, n)$.*

Proof. To construct the graph G with $achr(G) = f(m, n)$, we colour the $f(m, n)$ vertices of G with different colours, then add sufficient disjoint edges to give $achr(G) = f(m, n)$. □

More generally, if we let $g(m, n)$ denote the minimum value of $achr(G)$, taken over all graphs G with $\omega(G) = m$ and $\chi(G) = n$, then for all integers m and n, $1 < m \leq n$, there exists a graph H with $\omega(H) = m$, $\chi(H) = n$, and $achr(H) \geq g(m, n)$.

It is an open problem to determine the integers $1 < w \leq c \leq a$ for which there exists a graph with $\omega = w$, $\chi = c$, and $achr = a$. (Note: this is equivalent to determining $g(m, n)$.) In what follows, we seek a-minimal graphs with this property. We show that $\omega = n$, $\chi = n + 1$, $achr = t$ is realizable by a t-minimal graph if and only if $t > n+2$. We use a result from Gallai [23], 1963.

Theorem 5.1.18 [23] *If G is t-(colour)-critical and $|V(G)| \leq 2t - 2$ then $\overline{G}$ is disconnected (i.e., G has a spanning complete bipartite graph, or equivalently there are vertex disjoint graphs G_1 and G_2 such that $G = G_1 + G_2$).*

Lemma 5.1.19 *For $n \geq 2$, the minimum number of edges in a graph G with $\omega(G) = n$ and $\chi(G) = n+1$ is $\binom{n-2}{2} + 5(n-2) + 5$. Furthermore, $K_{n-2} + C_5$ is the unique graph which achieves equality.*

Proof. We first show that such a graph G has at least $n + 3$ vertices. Since K_{n+1} is the only $(n + 1)$-vertex graph with chromatic number $n + 1$, $|V(G)| \geq n + 2$. Suppose $|V(G)| = n + 2$. Then $V(G) = W \cup \{u, v\}$, where W is a clique of size n. Since G has no complete subgraph of size $n+1$, there exist vertices $x, y \in W$ such that $ux, vy \notin E(G)$. If $x \neq y$ then assigning each vertex of W a different colour , and assigning u, v the same colours as x, y, respectively, yields an n-colouring of G, a contradiction. Therefore $x = y$, and both u and v are adjacent to every vertex in $W - \{x\}$. Furthermore, $uv \in E(G)$, otherwise the above still describes an n-colouring of G. But now $(W - \{x\}) \cup \{u, v\}$ is a clique of size $n + 1$ in G, a contradiction. Therefore $|V(G)| \geq n + 3$.

We now show that there is no graph G with $\omega(G) = n$, $\chi(G) = n + 1$ and fewer than $\binom{n-2}{2} + 5(n-2) + 5$ edges. The result is clear if $n = 2$. Suppose it is true for $n = 2, 3, \ldots, k-1$, and let G be a graph with $\omega(G) = k$, $\chi(G) = k + 1$ and the minimum number of edges. Thus G is $(k+1)$-critical, and so $\delta(G) \geq k$. Since $|V(G)| \cdot k \leq 2|E(G)|$, we have $\frac{1}{2}|V(G)| \cdot k \leq \binom{k-2}{2} + 5(k-2) + 5 = \frac{k^2+5k-4}{2}$. Therefore $|V(G)| \leq k + 5 - \frac{4}{k}$. If $k = 3$, then $|V(G)| \leq 6 \leq 2(k+1) - 2$, and if $k \geq 4$ then $|V(G)| \leq k + 4 \leq 2(k+1) - 2$. In either case, Theorem 5.1.18 applies. Hence $G = G_1 + G_2$. If we now let $\chi(G_1) = \omega(G_1) = x$, then $\chi(G_2) = k - x + 1$, and $\omega(G_2) = k - x$. Since G has the minimum number of edges, $G_1 = K_x$. It now follows from the definition of join that we can assume $x = 1$.

Since $\omega(G_2) = k - 1$ and $\chi(G_2) = k$, it follows from the induction hypothesis that $|E(G_2)| \geq \binom{k-3}{2} + 5(k-3) + 5$. Thus

$$\begin{aligned}
|E(G)| &\geq (|V(G)| - 1) + \binom{k-3}{2} + 5(k-3) + 5 \\
&\geq k + 3 - 1 + \tfrac{(k-3)(k-4)}{2} + 5(k-2) \\
&= \tfrac{2((k-3)+5)}{2} + \tfrac{(k-3)(k-4)}{2} + 5(k-2) \\
&= \tfrac{2\cdot 5}{2} + \tfrac{(k-2)(k-3)}{2} + 5(k-2) \\
&= \binom{k-2}{2} + 5(k-2) + 5, \text{ as required.}
\end{aligned}$$

It follows from the above argument that if G has $\omega(G) = n$, $\chi(G) = n+1$ and $|E(G)| = \binom{n-2}{2} + 5(n-2) + 5$, then G has $n + 3$ vertices and, in fact, $G = K_1 + H$, where $|E(H)| = \binom{n-3}{2} + 5(n-3) + 5$, $\omega(H) = n - 1$, and $\chi(H) = n$. Since C_5 is the unique extremal graph when $n = 2$, an easy induction argument shows that $C_5 + K_{n-2}$ is the unique graph for which equality is achieved $\square$

Theorem 5.1.20 *For $n \geq 2$, there exists a t-minimal graph G with $\omega(G) = n$, $\chi(G) = n + 1$ and $achr(G) = t$ if and only if $t > n + 2$*

Proof. Consider a graph G with $\omega(G) = n$ and $\chi(G) = n + 1$. By Lemma 5.1.19, G must have at least $\binom{n-2}{2} + 5(n-2) + 5 = \frac{n^2+5n-4}{2}$ edges.

Since $\frac{n^2+5n-4}{2} > \frac{n^2+3n+2}{2} = \binom{n+2}{2}$ for all $n \geq 4$, G is not $(n+2)$-minimal. It remains to consider $n = 3$ and $n = 2$. The only graph with $\chi = 4$, $\omega = 3$ and 10 edges is $C_5 + K_1$, which has achromatic number 4. The only graphs with $\chi = 3$, $\omega = 2$ and 6 edges are $C_5 \cup K_2$, and C_5 connected to K_1 by a single edge. Both of these graphs have achromatic number 3. Thus for $n \geq 2$ there exists no t-minimal graph with $\chi = n + 1$ and $\omega = n$ such that $t \leq n + 2$.

We establish the existence of t-minimal graphs with $\omega = n$ and $\chi = n+1$ $(n \geq 2)$ for all $t > n + 2$ by construction. Observe that if G is a k-minimal graph with $\omega(G) = n$ and $\chi(G) = n+1$, then $G \cup k \cdot K_2$ is a $(k+1)$-minimal graph with the same clique and chromatic numbers as G. Thus it suffices to find, for all $n \geq 2$, an $(n + 3)$-minimal graph with $\omega = n$ and $\chi = n + 1$. But this is easy since $(C_5 + K_{n-2}) \cup 5K_2$ is such a graph. This completes the proof $\square$

We note in closing that if we relax the minimality condition, then all parameter sets of the form n, $n+1$, t, where $t \geq n+1$, are realizable. By the above theorem, we need only consider $t = n+1$ and $t = n+2$. The required graphs are $(C_5 + K_{n-2})$ and $(C_5 + K_{n-2}) \cup 2K_2$, respectively.

Chapter 6

OTHER PARAMETERS

In this chapter we list relations between the achromatic number and other graph parameters, as recorded by Xu [52] in 1991. We introduce each parameter with its notation and an informal definition. We denote the maximum degree by Δ, minimum degree by δ, connectivity (minimum number of vertices of G which must be deleted to disconnect G) by κ, edge connectivity (minimum number of edges which must be deleted to disconnect G) by κ', domination number (minimum size of a subset D of $V(G)$ such that all vertices not in D are adjacent to at least one vertex in D) by γ, independence number (size of a maximum independent set of vertices) by α, covering number (minimum size of a subset C of $V(G)$ so that all edges in G are incident to at least one vertex in C) by β, circumference (length of longest cycle) by c, length of longest path by l, girth (length of shortest cycle) by g, diameter (maximum of the distances between all pairs of vertices) by d, edge independence number (size of a maximum matching) by α', and edge covering number (minimum size of a subset C of $E(G)$ so that all vertices in G are incident to at least one edge in C) by β'. Each inequality below is followed by graphs which attain the lower and upper bounds. We have supplied the extremal graphs which are not directly cited.

Theorem 6.1.1 [52] *Let G be a finite graph with n vertices. We assume that G is connected in the case of diameter, and we further assume that a cycle exists in the case of girth and circumference.*

(1) $1 - n \le \delta(G) - achr(G) \le n - 2\sqrt{n}$ ($K_{n-1} \cup K_1$ [52] ; C_4)

(2) $1 - n \le \kappa(G) - achr(G) \le n - 2\sqrt{n}$ ($K_{n-1} \cup K_1$ [52] , C_4)

(3) $1 - n \le \kappa'(G) - achr(G) \le n - 2\sqrt{n}$ ($K_{n-1} \cup K_1$ [52] , C_4)

(4) $-\frac{n}{4} - 1 \le \Delta(G) - achr(G) \le n - 3$ (*inexact* [52] , $K_{1,n}$ [52])

(5) $achr(G)(achr(G) - 1) \le n\Delta(G)$ (K_n)

(6) $0 \le achr(G) - \chi(G) \le \frac{n}{2} - 1$ (K_n , K_2)

(7) $2\, achr(G) - \chi(G) \le n$ (K_n)

(8) $l(G) - achr(G) \le n - 3$ ($K_{n/2,n/2}$ [52])

(9) $2\, achr(G) - l(G) \le n + 1$ (K_n)

(10) $c(G) - achr(G) \le n - 2$ ($K_{n/2,n/2}$ [52])

(11) $2\, achr(G) - c(G) \le n$ (K_n)

(12) $g(G) \le \left\lceil \frac{achr(G)}{2} \right\rceil (achr(G) + 1) + 1$ (C_n, *such that* $n = 2a^2 + a + 1,\ a \in \mathbf{Z}^+$)

(13) $6 \le g(G) + achr(G) \le n + \sqrt{2n} + 1$ ($T_{n,3n}$, C_n, *such that*
$n = 2a^2 + a,\ a \in \mathbf{Z}^+$)

(14) $g(G) + 2\,achr(G) \le 2n + 3$ (K_n)

(15) $d(G) < \frac{(achr(G)+1)^2}{2}$

(16) $d(G) + 2\,achr(G) \le 2n + 1$ (K_n)

(17) $4 \le d(G) + achr(G) \le n + \sqrt{2n}$ ($K_{n,n}$, C_4)

(18) $2\sqrt{n} \le achr(G) + \alpha(G) \le n + 1$ ($T_{\sqrt{n},n}$, $\sqrt{n} \in \mathbf{Z}^+$, $K_{1,n}$)

(19) $n \le achr(G)\alpha(G) \le \frac{(n+1)^2}{4}$ (K_n ; $K_{\frac{n+1}{2}} \cup \frac{n+1}{2}K_1$,
$n = 1, 3, 5, \ldots$ [52])

(20) $-1 \le \beta(G) - achr(G) \le n - 2\sqrt{n}$ ($K_{1,m}$, C_4)

(21) $-1 \le 2\alpha'(G) - achr(G) \le n - 2$ (K_{2n+1} [52] ; $K_{n/2,n/2}$)

(22) $n + 2 \le achr(G) + 2\beta'(G) \le 2n + 1$ ($K_{n/2,n/2}$ [52] , K_{2n+1})

(23) $3 \le achr(G) + \gamma(G) \le n + 1$ (K_2 , K_n)

(24) $2 \le achr(G)\gamma(G) \le \frac{(n+1)^2}{4}$ (K_2 ; $K_{\frac{n+1}{2}} \cup \frac{n+1}{2}K_1$,
$n = 1, 3, 5, \ldots$ [52])

Chapter 7

NORDHAUS-GADDUM TYPE RESULTS

In this chapter we study results which stem from research by Nordhaus and Gaddum [49]. Let G, $\overline{G}$ denote a graph and its complement and η be a graph parameter. The calculations of extremum values of $\eta(G) + \eta(\overline{G})$ and $\eta(G)\eta(\overline{G})$ taken over all n-vertex graphs G are known as Nordhaus-Gaddum type problems, due to the results where $\eta(G)$ is the chromatic number of G [49].

The following result is due to Gupta [27] in 1969:

Theorem 7.1.1 [27] *For any graph G with n vertices,*

$$achr(G) + achr(\overline{G}) \leq \left\lceil \tfrac{4}{3}n \right\rceil, \text{ and this bound is best possible.}$$

To show that it is best possible, Gupta [27] considered inductive graph constructions of the form $n = 3r + 1$, $r = 1, 2, \ldots$ with $(2r+1)$-colourings f_r and $\overline{f}_r$ of G_r and $\overline{G}_r$ respectively. For $r = 1$, they define G_1, f_1, $\overline{f}_1$ as shown

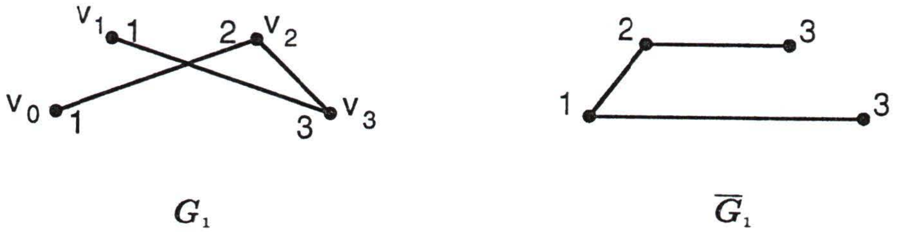
$V(G_1) = \{v_0, v_1, v_2, v_3\},$

$E(G_1) = \{[v_0, v_2], [v_1, v_3], [v_2, v_3]\},$

$f_1(v_0) = f_1(v_1) = 1, \quad f_1(v_2) = 2, \quad f_1(v_3) = 3\ ,$

$\overline{f}_1(v_0) = 1, \quad \overline{f}_1(v_1) = 2, \quad \overline{f}_1(v_2) = \overline{f}_1(v_3) = 3$

Figure 7.1 shows the basis graphs G_1 and $\overline{G}_1$ with their complete 3-colourings.



G_1 $\overline{G}_1$

Figure 7.1: Gupta's graphs G_1 and $\overline{G}_1$ on 4 vertices.

Now suppose G_{r-1}, f_{r-1}, and $\overline{f}_{r-1}$ have been defined for some $r \geq 2$. They define G_r, f_r, $\overline{f}_r$ as follows.

$V(G_r) = V(G_{r-1}) \cup \{v_{3r-2}, v_{3r-1}, v_{3r}\},$

$E(G_r) = E(G_{r-1}) \cup \{[v_{3r-2}, v_{3i-1}] | i = 1, 2, \ldots, r-1\} \cup \{[v_{3r-1}, v_{3i}] | i = 0, 1, \ldots, r\}$

$\cup \{[v_{3r}, v_{3i-1}], [v_{3r}, v_{3i}] | i = 1, 2, \ldots, r-1\} \cup \{[v_{3r}, v_1]\},$

$f_r(v_{3r-2}) = f_r(v_{3r-1}) = 2r,$

$f_r(v_{3r}) = 2r + 1,$ and

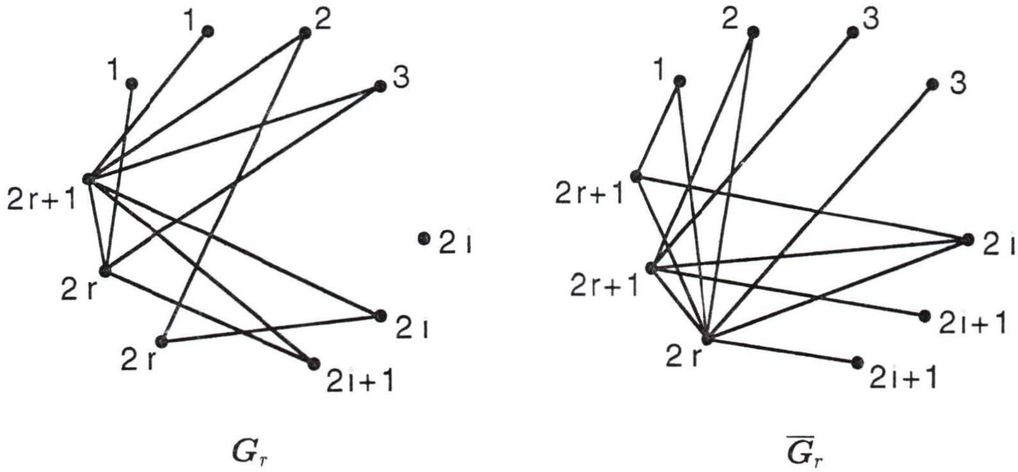
$f_r(v_i) = f_{r-1}(v_i)$ for $i = 0, 1, \ldots, 3r-3\ ;$

$\overline{f}_r(v_{3r-2}) = 2r,$

$\overline{f}_r(v_{3r-1}) = \overline{f}_r(v_{3r}) = 2r + 1,$ and

$\overline{f}_r(v_i) = \overline{f}_{r-1}(v_i)$ for $i = 0, 1, \ldots, 3r-3$.

This is demonstrated by Figure 7.2.



G_r $\overline{G}_r$

Figure 7.2: Gupta's graphs G_r and $\overline{G}_r$ on $3r+1$ vertices with $achr(G_r) = achr(\overline{G}_r) = 2r+1$.

It is easy to check that f_r and $\overline{f}_r$ are complete $(2r+1)$-colourings of G_r and $\overline{G}_r$, respectively. Thus $achr(G)+achr(\overline{G}) \geq 2(2r+1) = \left\lceil \frac{4}{3}(3r+1) \right\rceil$. So, from the theorem, the equality must hold and the bound is the best possible.

We offer the following corollary:

Corollary 7.1.2 *For any graph G with n vertices,*

$$achr(G) \cdot achr(\overline{G}) \leq \begin{cases} \frac{4n^2}{9} & if\ n \equiv 0 (mod\, 3) \\ \frac{4n^2+4n+1}{9} & if\ n \equiv 1 (mod\, 3) \\ \frac{4n^2+2n-2}{9} & if\ n \equiv 2 (mod\, 3) \end{cases}$$

and these bounds are best possible.

Proof. From Theorem 7.1.1 we have the best possible bound $achr(G) + achr(\overline{G}) \le \left\lceil \frac{4}{3}n \right\rceil$. So $achr(\overline{G}) \le \left\lceil \frac{4}{3}n \right\rceil - achr(G)$, and $achr(G) \cdot achr(\overline{G}) \le achr(G) \cdot \left\lceil \frac{4}{3}n \right\rceil - achr(G)^2$.

To maximize this upper bound, we take the overlying continuous function $f(x) = x \cdot \left\lceil \frac{4}{3}n \right\rceil - x^2$. We have $\frac{d}{dx}\left(x \cdot \left\lceil \frac{4}{3}n \right\rceil - x^2\right) = \left\lceil \frac{4}{3}n \right\rceil - 2x$, and the maximum occurs at $x = \frac{1}{2}\left\lceil \frac{4}{3}n \right\rceil$.

If $n \equiv 0 (mod\, 3)$, then the maximum is at $x = \frac{2n}{3}$, which is an integer, so the maximum $achr(G) = \frac{2n}{3}$, and $achr(G) \cdot achr(\overline{G}) \le \frac{2n}{3} \cdot \frac{4n}{3} - \left(\frac{2n}{3}\right)^2 = \frac{4n^2}{9}$.

If $n \equiv 1 (mod\, 3)$, then the maximum is at $x = \frac{2n+1}{3}$, which is also an integer, therefore the maximum $achr(G) = \frac{2n+1}{3}$, and $achr(G) \cdot achr(\overline{G}) \le \frac{2n+1}{3} \cdot \frac{4n+2}{3} - \left(\frac{2n+1}{3}\right)^2 = \left(\frac{2n+1}{3}\right)^2$.

If $n \equiv 2 (mod\, 3)$, then the maximum is at $x = \frac{4n+1}{6}$, which is halfway between two integers on a quadratic function, so the maximum $achr(G) = \frac{2n-1}{3}$ or $\frac{2n+2}{3}$. Both values yield the same bound: $achr(G) \cdot achr(\overline{G}) \le \left(\frac{2n-1}{3}\right)\left(\frac{2n+2}{3}\right)$.

That this is best possible for $n \equiv 1 (mod\, 3)$ is shown by Gupta's graphs, G_r and $\overline{G}_r$, each on $3r+1$ vertices with achromatic number $2r+1$: $achr(G) \cdot achr(\overline{G}) \ge (2r+1)^2 = \left(\frac{6r+3}{3}\right)^2 = \left(\frac{2(3r+1)+1}{3}\right)^2$.

For $n \equiv 2 (mod\, 3)$ and $n \equiv 0 (mod\, 3)$, the bounds are attained by $G_r + K_1$ and $(G_r + K_1) \cup K_1$, respectively. □

Many people have studied generalized Nordhaus-Gaddum type problems, that is, the problem of finding the extrema of $\eta_1(G) + \eta_2(\overline{G})$ for two graph parameters η_1 and η_2. It frequently turns out that if both η_1 and η_2 attain their maxima for complete graphs, then we can obtain a bound for $\eta_1(G) + \eta_2(\overline{G})$, since complementary graphs cannot both be complete.

S. Xu [52] listed the following generalized Nordhaus-Gaddum type results. In the following theorem we use the notation and restrictions found in Theorem 6.1.1, again following each relation with graphs of the lower and upper bounds. As before, we have provided the extremal graphs which are not directly cited.

Theorem 7.1.3 [52] *For any finite graph G with n vertices,*

(1) $2\sqrt{n} \le \chi(G) + achr(\overline{G}) \le n+1$ ($\sqrt{n} \cdot K_{\sqrt{n}}$, K_n)

(2) $2\sqrt{n} - 1 \le achr(G) + \Delta(\overline{G}) \le 2n-2$ ($T_{\sqrt{n},n}$, $\sqrt{n} \in \mathbf{Z}^+$, $K_{n-1} \cup K_1$)

(3) $n \le achr(G)(\Delta(\overline{G})+1) \le n(n-1)$ (K_n , $K_{n-1} \cup K_1$)

(4) $2 \le achr(G) + \delta(\overline{G})$ (K_2)

(5) $2 \le achr(G) + \kappa(\overline{G})$ (K_2)

(6) $2 \le achr(G) + \kappa'(\overline{G})$ (K_2)

(7) $0 \le achr(G) - \gamma(\overline{G}) \le n-2$ (K_n , $2K_n$)

(8) $0 \le achr(G) - \alpha(\overline{G})$ (K_n)

(9) $n \le \beta(G) + achr(\overline{G})$ (nK_1)

(10) $2\sqrt{n} \le c(G) + achr(\overline{G})$ ($\sqrt{n} \cdot K_{\sqrt{n}}$)

(11) $2\sqrt{n} - 1 \le l(G) + achr(\overline{G})$ ($\sqrt{n} \cdot K_{\sqrt{n}}$)

(12) $achr(G) - g(\overline{G}) \leq n - 5$ ($K_{n-2} \cup 2K_1$)

(13) $\Delta(G) > 0 \Rightarrow g(\overline{G}) \leq 2\,achr(G)$ ($2K_2$)

(14) $d(\overline{G}) \leq 2\,achr(G) - 2$ ($2K_2$)

(15) $n \leq 2\alpha'(G) + achr(\overline{G})$ (K_n *for odd* n [52])

(16) $2\beta'(G) - achr(\overline{G}) \leq n$ (K_n, *for odd* n [52])

In 1983, Akiyama, Harary and Ostrand [1] specified the graphs for which the achromatic number of G and $\overline{G}$ are both 2, and both 3. The graphs G such that both G and $\overline{G}$ have achromatic number 2 are C_4, $2K_2$, $K_{1,2}$ and $K_2 \cup K_1$. There are exactly 41 graphs G such that both G and $\overline{G}$ have achromatic number 3: six with 7 vertices, twenty with 6 vertices, fourteen with 5 vertices and just one with 4 vertices.

Chapter 8

COMPLEXITY

This chapter discusses the computational complexity of the achromatic number problem as it relates to different classes of graphs.

The achromatic number problem is: Given a graph G and a positive integer k, is $achr(G) \geq k$? Yannakakis and Gavril [53] explored this problem in 1980, and Bodlaender revisited it in 1989 [6].

Theorem 8.1.1 [53] *The achromatic number problem is NP-complete even for complements of bipartite graphs.*

A *cograph* is a graph which does not have P_4 as an induced subgraph [20]. A graph $G = (V, E)$ is an *interval graph* if one can associate to each vertex $v \in V$ an interval $[\, a_v, b_v \,] \subseteq \mathbf{R}$, such that $vw \in E \Leftrightarrow [\, a_v, b_v \,] \cap [\, a_w, b_w \,] \neq \emptyset$ [26].

Theorem 8.1.2 [6] *The achromatic number problem is NP-complete, even when restricted to connected graphs that are simultaneously a cograph and an interval graph.*

The exact bipartite achromatic number problem is: If G is a bipartite graph with exactly $\binom{k}{2}$ edges, is $achr(G) = k$? (Equivalently, given a bipartite graph G with exactly $\binom{k}{2}$ edges, is G k-minimal?) The complexity of this problem was established by Farber, Hahn, Hell and D. Miller [22] in 1986.

Theorem 8.1.3 [22] *The exact bipartite achromatic number problem is NP-complete.*

In contrast to these conclusions Farber et al. give the following results:

Lemma 8.1.4 [22] *There is a function* $f : \mathbf{Z}^+ \rightarrow \mathbf{Z}^+$ *such that each graph with at least* $f(k)$ *reducing congruence classes has achromatic number at least* k.

Theorem 8.1.5 [22] *For each fixed integer k, there is an algorithm which, for an arbitrary graph* $G = (V, E)$, *determines whether* $achr(G) \geq k$ *in time* $O(|E|)$.

Outline of Proof. Let G_k be the graph obtained from G by deleting $m - k$ vertices from each reducing congruence (r.c.) class with m vertices, whenever $m > k$. The authors first show that $achr(G) \geq k$ if and only if $achr(G_k) \geq k$. Two lists are then constructed: one of the r.c. classes of G, the other of the neighbourhoods of each of these r.c. classes. These lists can be made in linear time. If during the process G has at least $f(k)$ r.c. classes (where

$f(k)$ is the function of Lemma 8.1.4), then $achr(G) \geq k$. Otherwise, given complete lists of the r.c. classes and their neighbourhoods, we can construct G_k in linear time. We can check whether $achr(G_k) \geq k$ in constant time. If $achr(G_k) \geq k$ then $achr(G) \geq k$, otherwise $achr(G) < k$. □

Unfortunately the proof of the above theorem is non-constructive, and while the algorithms exist, only those for $k \leq 4$ have been found because the list of all reduced graphs has only been completed for $k \leq 4$.

A related problem is the complexity of the achromatic number problem when restricted to trees or forests. The two following results establish the existence of algorithms which determine whether certain trees are n-minimal. The first pertains to trees with few leaves, the second to trees with few non-leaves.

Theorem 8.1.6 [22] *For each fixed k, there is a polynomial algorithm which, given a tree T with* $\binom{n}{2}$ *edges and at most k leaves, will determine whether* $achr(T) = n$.

Theorem 8.1.7 [22] *There is a polynomial algorithm which, given any tree T with* $\binom{n}{2}$ *edges and at least* $\binom{n-1}{2} + 1$ *leaves, will determine whether* $achr(T) = n$.

Five algorithms which compute approximate solutions to the achromatic number problem have been compared in terms of running time and accuracy

These heuristics may be found in Brewster [13]. We will briefly describe three of his algorithms. The first method colours the vertices in a depth first search order by colouring a new vertex with a new colour n only once a complete homomorphism onto K_{n-1} has been constructed with the previously coloured vertices. A more accurate approach is to begin by assigning each vertex a different colour and then iteratively merging pairs, X, Y, of colour classes for which there is no edge with one end in X and the other end in Y. A variation is to begin instead with a k-colouring of G, where $k = max\{t : \binom{t}{2} \leq |E(G)|\}$. Brewster reports that the second method generally gives better approximations than the first, while the third is much faster than the second and is comparable in accuracy. He also compares these algorithms when they are augmented by a re-colouring step.

Chapter 9

SUMMARY OF OPEN PROBLEMS

We conclude with a list of six problems for further study. This list is not meant to be comprehensive, rather it is a sample of the topics which have been discussed in this thesis.

1. Determine bounds for the achromatic number of the Cartesian Product of graphs.

2. Determine the list of irreducible graphs with $achr = k$, for $k \geq 5$.

3. Exactly determine the achromatic number of the $m \times n$ grid (the Cartesian Product of P_n and P_m) and the n-dimensional cube (the n-fold Cartesian Product of K_2 with itself).

4. Find a structural characterization of the graphs with $achr = \omega$.

5. Determine those integers $1 \leq w \leq c \leq a$ such that there exists a (minimal) graph with $\omega = w$, $\chi = c$ and $achr = a$.

6. Complete the solution to the exact bipartite achromatic number problem restricted to trees.

Bibliography

[1] J. Akiyama, F. Harary and P. Ostrand, A graph and its complement with specified properties vi: chromatic and achromatic numbers, *Pac. J. Math.* 104 (1983) 15-27.

[2] B. Alspach, P. Schellenberg, D. Stinson and D. Wagner, The Oberwolfach problem and factors of uniform odd length cycles, *J. Comb. Theory Series* A 52 (1989) 20-43.

[3] B. Auerbach and R. Laskar, Some coloring numbers for complete r-partite graphs, *J. Comb. Theory Series* B 21 (1976) 169-170.

[4] C. Berge and P. Duchet, Strongly perfect graphs, *Ann. Discrete Math.* 21 (1984) 57-61.

[5] V. N. Bhave, On the pseudoachromatic number of a graph, *Fund. Math.* 102 (1979) 159-164.

[6] H. Bodlaender, Achromatic number is N-P complete for cographs and interval graphs, *Inform. Processing Letters* 31 (1989) 135-138.

[7] J. Bondy and U. S. R. Murty, *Graph Theory with Applications*, North Holland, New York, N. Y. ,1976.

[8] F. Bories, Etude du nombre achromatique de certains graphes, *Cahiers Centre d'Etudes et Recherche Operationnelle* 17 (1975) 155-171.

[9] F. Bories and J. Jolivet, On complete colorings of graphs, Recent Advances in Graph Theory, *Proc. Symp. Prague* (1974) 75-87.

[10] J. Bosak, *Decompositions of Graphs, First English Edition,* Kluwer Academic Publications, Dordrecht, Boston, 1990.

[11] A. Bouchet, Indice achromatique des graphes multiparti complets et reguliers, *Cahiers Centre d'Etudes et Recherche Operationnelle* 20 (1978) 331-340.

[12] A. Bouchet and R. Lopez-Bracho, Decomposition of a complete graph into trails of given lengths, *Discrete Math.* 42 (1982) 145-152.

[13] R. Brewster, Heuristics for computing the achromatic number of a graph. Manuscript, 1986.

[14] R. Brewster and G. MacGillivray, Homomorphically full graphs, *Discrete Applied Math.* (to appear).

[15] C. Christen and S. Selkow, Some perfect coloring properties of graphs, *J. Comb. Theory Series* B 27 (1979) 49-59.

[16] V. Chvátal, Perfectly ordered graphs, in: Topics on Perfect Graphs, (C. Berge and V. Chvatal, eds.), *Ann. Discrete Math.* 21 (1984) 63-65.

[17] E. Cockayne and S. Hedetniemi, Interpolation systems, *Proc. 3rd S-E Conf. Comb., Graph Theory and Computing, Congr. Numer.* (1972) 117-129.

[18] E. Cockayne, G. Miller and G. Prins, An interpolation theorem for partitions which are complete with respect to hereditary properties, *J. Comb. Theory* 13 (1972) 290-297.

[19] E. Cockayne and A. Thomason, Ordered colourings of graphs, *J. Comb. Theory Series* B 32 (1982) 286-292.

[20] D. G. Corneil, H. Lerchs and L. Stewart Burlingham, Complement reducible graphs. *Discrete Applied Math.* 3 (1981) 163-174.

[21] P. Erdös, W. Hare, S. Hedetniemi and R. Lasker, On the equality of the grundy and ochromatic numbers of a graph, *J. Graph Theory* 11 (1987) 157-159.

[22] M. Farber, G. Hahn, P. Hell and D. Miller, Concerning the achromatic number of graphs, *J. Comb. Theory Series* B 40 (1986) 21-39.

[23] T. Gallai, Kritische graphen II, *Publ. Math. Inst. Hung. Acad. Sci.* 8 (1963) 373-395.

[24] G. Gao and G. Hahn, Minimal graphs that fold into K_n, Manuscript, 1993.

[25] D. Geller and H. Kronk, Further results on the achromatic number, *Fund. Math.* 85 (1974) 285-290.

[26] M. C. Golumbic, *Algorithmic Graph Theory and Perfect Graphs*, Academic Press, New York, 1980.

[27] R P Gupta, Bounds on the chromatic and achromatic numbers of complementary graphs, *Recent Progress in Combinatorics* (W T Tutte, ed) Academic Press, New York, 1969

[28] R Haggkvist, P Hell, D J Miller and V Neumann Lara, On multiplicative graphs and the product conjecture, *Combinatorica* 8 (1988) 71-81

[29] D Hanson, G MacGillivray, and D Young, The size of a minimum five-chromatic K_4-free graph, *Discrete Math* 122 (1993) 353-355

[30] F Harary, *Graph Theory*, Addison-Wesley, Reading, Mass , 1969

[31] F Harary and S Hedetniemi, Achromatic number of a graph, *J Comb Theory* 8 (1970) 154-161

[32] F Harary, S Hedetniemi and G Prins, An interpolation theorem for graphical homomorphisms, *Portugal Math* 26 (1967) 453-462

[33] F Harary, D Hsu and Z Miller, The bichromaticity of a tree, *Lecture Notes in Math* 642 (1978) 235-246

[34] P Hell and D Miller, Achromatic numbers and graph operations, *Discrete Math* 108 (1992) 297-305

[35] P Hell and D Miller, Graphs with forbidden homomorphic images, *Ann N Y Acad Sci* 319 (1979) 270-280

[36] P Hell and D J Miller, Graphs with given achromatic number, *Discrete Math* 16 (1976) 195-207

[37] P Hell and D J Miller, On forbidden quotients and the achromatic number, in *Proc 5th British Combinatorics Conference, Congr Numer XV, Utilitas Math* (1976) 283-292

[38] L Hsu, Generalized homomorphism graph functions, *Discrete Math* 84 (1990) 31-37

[39] R Jamison, On the edge achromatic numbers of complete graphs, *Discrete Math* 74 (1989) 99-115

[40] R Jamison, On the achromatic index of K_{12} , *Congr Numerantium* 81 (1991) 143-148

[41] J Kelly, Difference systems, graph designs, and coloring problems, *J Comb Theory Series* B 30 (1981) 144-165

[42] R Lopez-Bracho, Le nombre achromatique d'une étoile, *Ars Combinatoria* 18 (1983) 187-194

[43] A Máté, A lower estimate for the achromatic number of irreducible graphs, *Discrete Math* 33 (1981) 171-183

[44] C McDiarmid, Colouring random graphs badly, in *Graph Theory and Combinatorics* (R J Wilson, ed), Pitman Research Notes in Math , No 34, Pitman, New York, 1979

[45] M Middendorf and F Pfeiffer, On the complexity of recognizing perfectly orderable graphs, *Discrete Math* 80 (1990) 327-333

[46] D J Miller, The automorphism group of a product of graphs, *Proc Am Math Soc* 25 (1970) 24-28

[47] Z Miller, Extremal regular graphs for the achromatic number, *Discrete Math* 40 (1982) 235-253

[48] J W Moon and L Moser, On cliques in graphs, *Israel J Math* 3 (1965) 23-28

[49] E Nordhaus and J Gaddum, On complementary graphs, *Amer Math Monthly* 63 (1956) 175-177

[50] E Sampathkumar and V Bhave, Partition graphs and coloring numbers of a graph, *Discrete Math* 16 (1976) 57-60

[51] D Seinsche, On a property of the class of n-colorable graphs, *J Comb Theory Series* B 16 (1974) 191-193

[52] S Xu, Relations between parameters of a graph, *Discrete Math* 89 (1991) 65-88

[53] M Yannakakis and F Gavril, Edge dominating sets in graphs, *SIAM J of Applied Math* 38 (1980) 364-372

[54] M -L Yu, Resolvable tree designs, *J Comb Theory Series* A 61 (1992) 302-308

[55] M -L Yu, On tree factorizations of K_n, *J Graph Theory* 17 (1993) 713-726

[56] H Zhou, Chromatic difference sequences and homomorphisms, *Discrete Math* 113 (1993) 285-292

[57] H Zhou, Inequalities with respect to graph homomorphism, *Discrete Math* 112 (1993) 295-298

VITA

HUGHES, Frederick John

Place of Birth: Victoria, B.C., Canada	Date of Birth: 51-03-06

Educational Institutions Attended:

University of Victoria	1969-1974
Simon Fraser University	1977-1978
University of Victoria	1991-1994

Degrees Awarded:

B.Sc. University of Victoria	1974
B.C. Teaching Certificate-Simon Fraser University	1978

Honours and Awards

University of Victoria Fellowship	1991-1994

PARTIAL COPYRIGHT LICENSE

Title of Thesis: **On The Achromatic Number of Graphs**

Author: [redacted]

Frederick John Hughes

September 25, 1994