A CERTAIN SUBCLASS OF STARLIKE FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract

The authors present a systematic investigation of various interesting properties of a certain subclass of starlike functions with negative coefficients. In particular, they give numerous sharp results including coefficient estimates, distortion theorem, and radius of convexity for functions belonging to this subclass. They also show that the class studied in this paper is closed under arithmetic mean and convex linear combinations.

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1. Introduction and Definitions

Let $\mathcal{S}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic and univalent in the *open* unit disk

$$\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

A function $f(z) \in \mathcal{S}$ is said to be *starlike of order* α $(0 \leq \alpha < 1)$ in $\mathcal{U}$ if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathcal{U}; \quad 0 \leq \alpha < 1).$$

Also, a function $f(z) \in \mathcal{S}$ is said to be *convex of order* α $(0 \leq \alpha < 1)$ in $\mathcal{U}$ if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (z \in \mathcal{U}; \quad 0 \leq \alpha < 1).$$

We denote by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ the classes of all functions in $\mathcal{S}$ which are, respectively, starlike and convex of order α in $\mathcal{U}$.

Let $\mathcal{T}$ be the subclass of $\mathcal{S}$ consisting of functions $f(z)$ of the form (1.1) whose nonzero coefficients a_n $(n \geq 2)$ are negative. Thus an analytic and univalent function $f(z)$ in $\mathcal{U}$ is in the class $\mathcal{T}$ if and only if it can be expressed in the form:

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \tag{1.2}$$

for all z in $\mathcal{U}$.

We begin by setting

$$F_\lambda(z) = (1-\lambda)f(z) + \lambda z f'(z) \qquad (\lambda \geq 0; \quad f \in \mathcal{T}), \tag{1.3}$$

so that, obviously,

$$F_\lambda(z) = z - \sum_{n=2}^{\infty} [1 + (n-1)\lambda] |a_n| z^n, \tag{1.4}$$

since $f \in \mathcal{T}$ is given by (1.2).

When $\lambda = \frac{1}{2}$ in (1.3), we get

$$2F_{\frac{1}{2}}(z) = \left(z\,f(z)\right)'. \tag{1.5}$$

Sarangi and Uralegaddi ([8], [9]) studied the basic properties of functions $F_\lambda(z)$ defined by (1.3).

Definition. A function $f \in \mathcal{T}$ is said to be in the class $\mathcal{T}^*_\lambda(A, B, \alpha, \beta, \gamma)$ if the function $F_\lambda(z)$, defined by (1.3), satisfies the inequality:

$$\left| \frac{z\{F'_\lambda(z)/F_\lambda(z)\} - 1}{(B-A)\,\gamma\,[z\{F'_\lambda(z)/F_\lambda(z)\} - \alpha] - B\,[z\{F'_\lambda(z)/F_\lambda(z)\} - 1]} \right| < \beta \qquad (z \in \mathcal{U}),$$

where (and *throughout this paper*) the parameters α, β, γ, A, and B are constrained as follows:

$$\begin{gathered} 0 \leq \alpha < 1; \quad 0 < \beta \leq 1; \quad -1 \leq A < B \leq 1; \quad 0 < B \leq 1; \\ \frac{B}{B-A} < \gamma \leq \begin{cases} B/\{(B-A)\alpha\} & (\alpha \neq 0) \\ 1 & (\alpha = 0). \end{cases} \end{gathered} \tag{1.6}$$

We note that

1. The subclass $\mathcal{T}^*_0\ (-1, 1, \alpha, \beta, 1)$ was studied by Gupta and Jain ([4], [5]).
2. The class $\mathcal{T}^*(\alpha) = \mathcal{T}^*_0\ (-1, 1, \alpha, 1, 1)$ was studied by Silverman [10].

The subclass $\mathcal{T}^*(\alpha)$ and various other subclasses of $\mathcal{T}$ have been studied rather extensively by Bhoosnurmath and Swamy [3], Srivastava and Owa [11], Owa and Aouf [7], Nunokawa and Aouf [6], and others (see also [1], [2], and [12]).

In the present paper, sharp results involving the coefficients a_n, distortion theorem, and radius of convexity for the class $\mathcal{T}^*_\lambda\ (A, B, \alpha, \beta, \gamma)$ are investigated. In the last section we also show that the class $\mathcal{T}^*_\lambda\ (A, B, \alpha, \beta, \gamma)$ is closed under *arithmetic mean* and *convex linear combinations.*

2. A Coefficient Theorem

Theorem 1. *Let $f \in \mathcal{T}$ be defined by (1.2). Then $f(z)$ is in the class $\mathcal{T}^*_\lambda\ (A, B, \alpha, \beta, \gamma)$ if and only if*

$$\begin{aligned} &\sum_{n=2}^{\infty} |a_n|[1 + (n-1)\lambda]\,\{(n-1) + \beta\,[(B-A)\gamma(n-\alpha) - B(n-1)]\} \\ &\qquad \leq (B-A)\gamma\beta(1-\alpha), \end{aligned} \tag{2.1}$$

where the parameters are constrained as in (1.6).

Proof. Suppose that the function $f(z)$ defined by (1.2) is in the class

$$\mathcal{T}_\lambda^*(A, B, \alpha, \beta, \gamma).$$

Then, in view of the Definition and the series expansion (1.4) with $f \in \mathcal{T}$, we have

$$\left| \frac{z\{F_\lambda'(z)/F_\lambda(z)\} - 1}{(B-A)\gamma\{z\{F_\lambda'(z)/F_\lambda(z)\} - \alpha\} - B\{z\{F_\lambda'(z)/F_\lambda(z)\} - 1\}} \right|$$

$$= \left| \frac{-\sum_{n=2}^{\infty}(n-1)[1+(n-1)\lambda]\,|a_n|\,z^{n-1}}{(B-A)\gamma(1-\alpha) - \{(B-A)\gamma(n-\alpha) - B(n-1)\} \cdot \sum_{n=2}^{\infty}[1+(n-1)\lambda]\,a_n|\,z^{n-1}} \right| < \beta, \tag{2.2}$$

for $z \in \mathcal{U}$.

Since $\Re(z) \leq |z|$ for all z, it follows from (2.2) that

$$\Re\left\{ \frac{\sum_{n=2}^{\infty}(n-1)[1+(n-1)\lambda]\,|a_n|\,z^{n-1}}{(B-A)\gamma(1-\alpha) - \{(B-A)\gamma(n-\alpha) - B(n-1)\} \cdot \sum_{n=2}^{\infty}[1+(n-l)\lambda]\,|a_n|\,z^{n-1}} \right\} < \beta. \tag{2.3}$$

Choose the values of z on the real axis so that $z\{F_\lambda'(z)/F_\lambda(z)\}$ is real. Upon clearing the denominator in (2.3) and letting $z \to 1-$ through real values, we obtain

$$\sum_{n=2}^{\infty}(n-1)[1+(n-1)\lambda]\,|a_n|$$

$$\leq \beta\left\{(B-A)\gamma(1-\alpha) - [(B-A)\gamma(n-\alpha) - B(n-1)]\sum_{n=2}^{\infty}[1+(n-1)\lambda]\,|a_n|\right\}$$

or

$$\sum_{n=2}^{\infty}[1+(n-1)\lambda]\,|a_n|\,\{(n-1) + \beta\,[(B-A)\gamma(n-\alpha) - B(n-1)]\}$$

$$\leq (B-A)\gamma\,\beta(1-\alpha),$$

which proves that the condition (2.1) is necessary for $f \in \mathcal{T}$ to be in the class

$$\mathcal{T}_\lambda^*(A, B, \alpha, \beta, \gamma).$$

In order to prove that the condition (2.1) is also sufficient for $f \in \mathcal{T}$ to be in the class

$$\mathcal{T}_\lambda^*(A, B, \alpha, \beta, \gamma),$$

we first suppose that the condition (2.1) holds true for all admissible values of A, B, α, β, and γ. In view of the Definition, we then consider the expression:

$$\mathcal{M}(f) := |z\,F_\lambda'(z) - F_\lambda(z)| - \beta|(B-A)\gamma\,\{z\,F_\lambda'(z) - \alpha\,F_\lambda(z)\}$$
$$- B\{z\,F_\lambda'(z) - F_\lambda(z)\}| \qquad (z \in \mathcal{U}),$$

which, upon substituting from (1.4) and putting

$$|z| = r < 1,$$

yields

$$\begin{aligned}
\mathcal{M}(f) = &\left| -\sum_{n=2}^{\infty}(n-1)[1+(n-1)\lambda]\,|a_n|\,z^{n-1}\right| \\
&- \beta\Bigg|(B-A)\gamma(1-\alpha) - \{(B-A)\gamma(n-\alpha) - B(n-1)\} \\
&\qquad \cdot \sum_{n=2}^{\infty}[1+(n-1)\lambda]\,|a_n|\,z^{n-1}\Bigg| \\
\leqq &\sum_{n=2}^{\infty}(n-1)[1+(n-1)\lambda]\,|a_n|\,r^{n-1} \\
&- (B-A)\gamma\beta(1-\alpha) + \beta\{(B-A)\gamma(n-\alpha) - B(n-1)\} \\
&\qquad \cdot \sum_{n=2}^{\infty}[1+(n-1)\lambda]\,|a_n|\,r^{n-1} \\
= &\sum_{n=2}^{\infty}[1+(n-1)\lambda]\Big\{(n-1) + \beta[(B-A)\gamma(n-\alpha) - B(n-1)]\Big\} \\
&\cdot |a_n|\,r^{n-1} - (B-A)\gamma\beta(1-\alpha).
\end{aligned} \tag{2.4}$$

Finally, letting $r \to 1-$ in (2.4), and making use of the condition (2.1), we have

$$\mathcal{M}(f) \leq \sum_{n=2}^{\infty}[1+(n-1)\lambda]\Big\{(n-1)+\beta[(B-A)\gamma(n-\alpha)-B(n-1)]\Big\} \cdot |a_n| - (B-A)\gamma\beta(1-\alpha)$$

$$\leq 0,$$

which evidently completes the proof of Theorem 1.

Corollary. *If $f(z)$, defined by (1.2), is in the class $\mathcal{T}_\lambda^*(A,B,\alpha,\beta,\gamma)$, then*

$$|a_n| \leq \frac{(B-A)\gamma\,\beta(1-\alpha)}{[1+(n-1)\lambda]\,\{(n-1)+\beta\,[(B-A)\gamma(n-\alpha)-B(n-1)]\}} \quad (n=2,3,4,\cdots) \tag{2.5}$$

with equality for each n for functions in the form:

$$f_n(z) = z - \frac{(B-A)\gamma\beta(1-\alpha)}{[1+(n-1)\lambda]\,\{(n-1)+\beta\,[(B-A)\gamma(n-\alpha)-B(n-1)]\}}\, z^n \quad (n=2,3,4,\cdots). \tag{2.6}$$

3. A Distortion Theorem and Its Consequence

With the help of Theorem 1, we first derive

Theorem 2. *If $f(z) \in \mathcal{T}_\lambda^*(A,B,\alpha,\beta,\gamma)$, then*

$$\begin{aligned} r - \frac{(B-A)\gamma\beta(1-\alpha)\,r^2}{(1+\lambda)\,\{1+\beta\,[(B-A)\gamma(2-\alpha)-B]\}} &\leq |f(z)| \\ &\leq r + \frac{(B-A)\gamma\beta(1-\alpha)\,r^2}{(1+\lambda)\,\{1+\beta\,[(B-A)\gamma(2-\alpha)-B]\}} \\ &(|z| = r < 1). \end{aligned} \tag{3.1}$$

The bounds in (3.1) are sharp for the function $f(z)$ given by

$$f(z) = z - \frac{(B-A)\gamma\beta(1-\alpha)\,z^2}{(1+\lambda)\,\{1+\beta[(B-A)\gamma(2-\alpha)-B]\}} \qquad (z = \pm r). \tag{3.2}$$

Proof. Since $f \in \mathcal{T}_\lambda^*(A,B,\alpha,\beta,\gamma)$, Theorem 1 readily yields

$$\sum_{n=2}^{\infty} |a_n| \leq \frac{(B-A)\gamma\beta(1-\alpha)}{(1+\lambda)\{1+\beta[(B-A)\gamma(2-\alpha)-B]\}}. \tag{3.3}$$

Hence (3.1) follows from the following obvious consequence of the definition (1.2):

$$r - r^2 \sum_{n=2}^{\infty} |a_n| \leq |f(z)| \leq r + r^2 \sum_{n=2}^{\infty} |a_n|$$

$$(|z| = r < 1).$$

Next we prove

Theorem 3. *If $f(z) \in \mathcal{T}_\lambda^*(A,B,\alpha,\beta,\gamma)$, then $f(z)$ is convex in the disk*

$$|z| < \rho_\lambda(A,B,\alpha,\beta,\gamma),$$

where

$$\rho_\lambda(A,B,\alpha,\beta,\gamma)$$
$$= \inf_{n\geq 2}\left\{\frac{[1+(n-1)\lambda]\{(n-1)+\beta[(B-A)\gamma(n-\alpha)-B(n-1)]\}}{n^2(B-A)\gamma\beta(1-\alpha)}\right\}^{1/(n-1)}. \tag{3.4}$$

The result is sharp for the functions $f_n(z)$ $(n = 2,3,4,\cdots)$ given by (2.6).

Proof. It suffices to show that $|zf''(z)/f'(z)| \leq 1$ for all z constrained as in Theorem 3. Indeed we first find from (1.2) that

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{\sum_{n=2}^{\infty} n(n-1)\,|a_n|\;|z|^{n-1}}{1-\sum_{n=2}^{\infty} n\,|a_n|\;|z|^{n-1}}.$$

Thus the desired result follows if

$$\sum_{n=2}^{\infty} n(n-1)\,|a_n|\;|z|^{n-1} \leq 1 - \sum_{n=2}^{\infty} n|a_n|\;|z|^{n-1}, \tag{3.5}$$

or, equivalently, if

$$\sum_{n=2}^{\infty} n^2|a_n|\;|z|^{n-1} \leq 1.$$

This last inequality (3.5), in conjunction with Theorem 1, shows that $f(z)$ is convex if

$$n^2\,|z|^{n-1} \leq \frac{[1+(n-1)\lambda]\,\{(n-1)+\beta\,[(B-A)\gamma(n-\alpha)-B(n-1)]\}}{(B-A)\gamma\beta(1-\alpha)},$$

that is, if

$$|z| \leq \left\{\frac{[1+(n-1)\lambda]\,\{(n-1)+\beta\,[(B-A)\gamma(n-\alpha)-B(n-1)]\}}{n^2\,(B-A)\gamma\,\beta(1-\alpha)}\right\}^{1/(n-1)} \tag{3.6}$$

$$(n=2,3,4,\cdots).$$

Writing $\rho_\lambda(A,B,\alpha,\beta,\gamma)$ for $|z|$ in (3.6), we immediately obtain the radius of convexity asserted in Theorem 3.

4. A Set of Closure Theorems

Theorem 4. *Let each of the functions $f_j(z)$ $(j=1,2)$ defined by*

$$f_j(z) = z - \sum_{n=2}^{\infty} |a_{n,j}|\,z^n \qquad (j=1,2) \tag{4.1}$$

be in the class $\mathcal{T}_\lambda^*(A,B,\alpha,\beta,\gamma)$. Then the function $h(z)$ defined by

$$h(z) = z - \frac{1}{2}\sum_{n=2}^{\infty} |a_{n,1}+a_{n,2}|\,z^n \tag{4.2}$$

is also in the class $\mathcal{T}_\lambda^*(A,B,\alpha,\beta,\gamma)$.

Proof. Using Theorem 1, we easily see that

$$\frac{1}{2}\sum_{n=2}^{\infty}[1+(n-1)\lambda]\,\{(n-1)+\beta[(B-A)\gamma(n-\alpha)-B(n-1)]\}\,|a_{n,1}+a_{n,2}|$$

$$\leq \frac{1}{2}\sum_{n=2}^{\infty}[1+(n-1)\lambda]\,\{(n-1)+\beta[(B-A)\gamma(n-\alpha)-B(n-1)]\}\,(|a_{n,1}|+|a_{n,2}|)$$

$$\leq (B-A)\gamma\beta(1-\alpha),$$

which implies that $h(z)\in\mathcal{T}_\lambda^*(A,B,\alpha,\beta,\gamma)$.

Theorem 5. *Let* $f_1(z) = z$ *and*

$$f_n(z) = z - \frac{(B-A)\gamma\beta(1-\alpha)}{[1+(n-1)\lambda]\{(n-1)+\beta[(B-A)\gamma(n-\alpha)-B(n-1)]\}} z^n \qquad (n = 2, 3, 4, \cdots). \tag{4.3}$$

Then $f(z)$ *is in the class* $\mathcal{T}^*_\lambda(A, B, \alpha, \beta, \gamma)$ *if and only if it can be expressed in the form:*

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z), \tag{4.4}$$

where

$$\lambda_n \geq 0 \quad and \quad \sum_{n=1}^{\infty} \lambda_n = 1. \tag{4.5}$$

Proof. Let

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)$$

$$= z - \sum_{n=2}^{\infty} \frac{\lambda_n (B-A)\gamma\beta(1-\alpha)}{[1+(n-1)\lambda]\{(n-1)+\beta[(B-A)\gamma(n-\alpha)-B(n-1)]\}} z^n,$$

which implies that

$$\sum_{n=1}^{\infty} \left\{ \frac{[1+(n-1)\lambda]\{(n-1)+\beta[(B-A)\gamma(n-\alpha)-B(n-1)]\}}{(B-A)\gamma\beta(1-\alpha)} \cdot \frac{(B-A)\gamma\beta(1-\alpha)\,\lambda_n}{[1+(n-1)\lambda]\{(n-1)+\beta[(B-A)\gamma(n-\alpha)-B(n-1)]\}} \right\}$$

$$= \sum_{n=2}^{\infty} \lambda_n = 1 - \lambda_1 \leq 1.$$

By Theorem 1, we thus have $f(z) \in \mathcal{T}^*_\lambda(A, B, \alpha, \beta, \gamma)$.

Conversely, let us suppose that

$$f(z) \in \mathcal{T}^*_\lambda(A, B, \alpha, \beta, \gamma).$$

Then Theorem 1 (or its Corollary) gives us

$$|a_n| \leq \frac{(B-A)\gamma\beta(1-\alpha)}{[1+(n-1)\lambda]\{(n-1)+\beta[(B-A)\gamma(n-\alpha)-B(n-1)]\}} \qquad (n = 2, 3, 4, \cdots). \tag{4.6}$$

Setting

$$\lambda_n = \frac{[1+(n-1)\lambda]\,\{(n-1)+\beta[(B-A)\gamma(n-\alpha)-B(n-1)]\}}{(B-A)\gamma\beta(1-\alpha)}\,|a_n|$$

$$(n = 2, 3, 4, \cdots).$$

and

$$\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n,$$

we find from (4.3) that [*cf.* Equation (4.4)]

$$f(z) = \sum_{n=1}^{\infty} \lambda_n\, f_n(z),$$

which completes the proof of Theorem 5.

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