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Complex Generalized Representation of Gamma Function Leading to the Distributional Solution of a Singular Fractional Integral Equation

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Abstract: Firstly, a basic question to find the Laplace transform using the classical representation of gamma function makes no sense because the singularity at the origin nurtures so rapidly that $\Gamma(z)e^{-sz}$ cannot be integrated over positive real numbers. Secondly, Dirac delta function is a linear functional under which every function f is mapped to $f(0)$. This article combines both functions to solve the problems that have remained unsolved for many years. For instance, it has been demonstrated that the power law feature is ubiquitous in theory but challenging to observe in practice. Since the fractional derivatives of the delta function are proportional to the power law, we express the gamma function as a complex series of fractional derivatives of the delta function. Therefore, a unified approach is used to obtain a large class of ordinary, fractional derivatives and integral transforms. All kinds of q-derivatives of these transforms are also computed. The most general form of the fractional kinetic integrodifferential equation available in the literature is solved using this particular representation. We extend the models that were valid only for a class of locally integrable functions to a class of singular (generalized) functions. Furthermore, we solve a singular fractional integral equation whose coefficients have infinite number of singularities, being the poles of gamma function. It is interesting to note that new solutions were obtained using generalized functions with complex coefficients.

Keywords: fractional Taylor series; H-function; singular integral equation; q-fractional derivatives

MSC: 33B15; 44A20; 33CXX

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1. Introduction and Motivation

Gelfand and Shilov remarked that "If the coefficients of equations have singularities, then new solutions may occur in generalized functions, while classical solutions may disappear" ([1], Vol. I, p. 42, Remark 1). Since the gamma function has an unlimited number of poles at negative integers containing 0. Therefore, it becomes more challenging to solve differential and integral equations containing the gamma function. In general, a function is typically thought of as being defined by a series, or an integral of a certain variable, or in terms of those functions that we consider to be "elementary". But the function must be regarded as a distinct entity, and one that can be represented by a series or an integral. When considering the theory of (special) functions, this point becomes more significant. All special functions have more than one representation, such as a series integral, an asymptotic representation, etc. For example, the integral representation ([2], Equation (1.1.1)) of the gamma function is defined in the positive half of the complex plane as

$$\Gamma(\omega) = \int_0^\infty t^{\omega-1}e^{-t}dt; (\omega \in \mathbb{C}; \Re(\omega) > 0), \tag{1}$$

and another representation ([2], Equation (1.1.4)) is as follows,

$$\Gamma(\omega) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\frac{1}{\omega+k} + \int_1^\infty t^{\omega-1}e^{-t}dt; (\omega \neq 0, -1, -2, \ldots\ldots), \tag{2}$$

and is defined in the whole complex plane except at negative integers including 0. Additionally, there may be multiple integral or series representations of the same function, which aid in differentiating the function in the various contexts. This is necessary in order to use the function for purposes other than those for which it was formerly defined. By doing so, one can also provide proofs of several known features that are simpler. Therefore, a generalized representation of the gamma function [3] in terms of complex delta functions is given by

$$\Gamma(x+\iota y) = 2\pi\sum_{l=0}^{\infty} \frac{(-1)^l}{l!}\,\delta(y - \iota(x+l)), \tag{3}$$

and was used to find its Laplace transform ([4], Equation (38)), which was not possible using its known (old) representations. A complex delta function can be represented as a Taylor series ([1], Vol. 1, p. 169, Equation (8)),

$$F\left[e^{xt}; y\right] = 2\pi\delta(y - \iota x) = \sum_{m=0}^{\infty} \frac{(-\iota x)^m}{m!}\delta^{(m)}(y). \tag{4}$$

More rigorously, a complex delta function is an element of a distribution space Z' so that for $\forall g \in Z'$ ([1], (pp. 159–160), Equation (4)) as well as ([5], p. 201, Equation (9)), we have the Taylor series expansion,

$$g(\omega+b) = \sum_{m=0}^{\infty} g^{(m)}(\omega)\frac{b^m}{m!} \quad (\omega, b \in \mathbb{C}), \tag{5}$$

and the corresponding fractional Taylor series is [6]

$$g(\omega+b) = \sum_{m=0}^{\infty} g^{(\nu m)}(\omega)\frac{b^{\nu m}}{\Gamma(\nu m+1)}; 0 < \nu < 1. \tag{6}$$

For further details of these distribution spaces, we suggest that an interested reader should refer to a more recent reference [4], subsection 2.2. Moreover, for $\nu \in \mathbb{R}^+$, a recently discussed definition of the fractional derivatives of delta function is given by [7]

$$\delta^{(\nu)}(\xi) * \chi(\xi) = \int_{0^-}^{u} \frac{d^\nu\delta(u-\xi)}{dt^\nu}\chi(\xi)d\xi = \frac{1}{\Gamma(-\nu)}\int_{0^-}^{u} \frac{\chi(\xi)}{(u-\xi)^{\nu+1}}d\xi = \frac{d^\nu\chi(u)}{du^\nu}, \tag{7}$$

which is basically the extension and modification of the classical works by Gelfand and Shilov [1]. Inspired by the study in [7], a non-integer-order distributional representation of the gamma function was discussed over a real domain [8]. In this article, we also extend these results over the complex domain. It is clear that the delta function's Riemann–Liouville (R-L) and Caputo derivatives [9–11] are identical. For a more thorough examination of such settings, the interested bibliophile can refer to [7–10] and their related literature.

This article is divided into the following sections: After a brief discussion of necessary definitions in Section 1.1, we find a novel depiction of the gamma function over its complex domain by making the use of non-integer derivatives of the delta function in Section 2. The complex portion contributes to the coefficients of this series representation, and fractional derivatives are used for the real part. Therefore, a new representation is also applicable to a class of functions whose fractional derivatives over the complex domain do not exist. With novel applications for the theory of distributions, the convergence of a new series is demonstrated. Highly nontrivial examples leading to the solution of a most general form of the fractional integrodifferential equation are also discussed. Using the Fourier transform of the gamma function, novel identities involving the gamma function are obtained using multiple Erdélyi–Kober (E-K) fractional derivatives in Section 3.1 and their fractional integral transforms in Section 3.2. In Section 3.3, the Laplace transform (LT) of the gamma function is utilized to obtain the distributional solution of the singular fractional integral problem in conjunction with the classical solution. By doing so, we obtain a large class of integrals and derivatives in a uniform way, i.e., q-fractional derivatives using the new representation are also computed in Section 3.4. Section 4 contains the conclusions of this article and lead to the future directions of this study.

1.1. Preliminaries Related to Multiple Erdélyi–Kober (E-K) Fractional Operators

The generalized fractional operators of order δ_k are defined as follows ([12], p. 8, Equation (18)),

$$I^{(\gamma_k),(\delta_k)}_{(\alpha_k),n} f(z) = \begin{cases} f(z); (\delta_k = 0) \\ \int_0^1 f(z\sigma) H^{n,0}_{n,n}\left[\sigma \middle| \begin{matrix} \left(\gamma_k + \delta_k - \frac{1}{\alpha_k} + 1, \frac{1}{\alpha_k}\right)_1^n \\ \left(\gamma_k - \frac{1}{\alpha_k} + 1, \frac{1}{\alpha_k}\right)_1^n \end{matrix}\right] d\sigma; \left(\sum_k \delta_k \geq 0\right) \\ = z^{-1} \int_0^z f(\xi) H^{n,0}_{n,n}\left[\frac{\xi}{z} \middle| \begin{matrix} \left(\gamma_k + \delta_k - \frac{1}{\alpha_k} + 1, \frac{1}{\alpha_k}\right)_1^n \\ \left(\gamma_k - \frac{1}{\alpha_k} + 1, \frac{1}{\alpha_k}\right)_1^n \end{matrix}\right] d\xi; \left(\sum_k \delta_k > 0\right) \end{cases} \tag{8}$$

where α_k's are arbitrary parameters, and γ_k's are used as weights. Then, $H^{n,0}_{n,n} = 0; \sigma > 1$ and the corresponding derivatives of different orders $(\delta_n \geq 0, \ldots, \delta_1 \geq 0) = \delta$ are as defined in ([12], p. 9), refer also [13,14]. These are named as multiple E–K fractional and derivative operators, respectively. One noteworthy aspect of these operators is their relationship with a few popular fractional integrals, as listed in the Table 1 below.

Table 1. Diverse kernels of multiple Erdélyi–Kober (E-K) fractional operators and integral transforms [14].

Cases of (8)	**Diverse Kernels of Non-Integer Transforms [12]**
$n = 3$ Marichev–Saigo–Maeda (M-S-M) $(1 = \alpha_1 = \alpha_2 = \alpha_3 = \alpha)$	$H^{3,0}_{3,3}\left(\frac{t}{x}\right) = G^{3,0}_{3,3}\left[\frac{t}{x} \middle| \begin{matrix} \gamma_1' + \gamma_2', \ \nu - \gamma_1, \nu - \gamma_2 \\ \gamma_1', \gamma_2', \ \nu - \gamma_1 - \gamma_2 \end{matrix}\right] = \frac{x^{-\gamma_1}}{\Gamma(\nu)}(x-t)^{\delta-1} t^{-\gamma_1'} F_3\left(\gamma_1, \gamma_1', \gamma_2, \gamma_2', \nu; 1 - \frac{t}{x}; 1 - \frac{x}{t}\right)$
$n = 2$ Saigo $\left(\alpha_1 = \alpha_2 = \alpha > 0; \sigma = \frac{t}{x} \wedge \sigma = \frac{x}{t}\right)$	$H^{2,0}_{2,2}\left[\sigma \middle| \begin{matrix} \left(\gamma_1 + \nu_1 + 1 - \frac{1}{\alpha}, \frac{1}{\alpha}\right), \left(\gamma_2 + \nu_2 + 1 - \frac{1}{\alpha}, \frac{1}{\alpha}\right) \\ \left(\gamma_1 + 1 - \frac{1}{\alpha}, \frac{1}{\alpha}\right), \left(\gamma_2 + 1 - \frac{1}{\alpha}, \frac{1}{\alpha}\right) \end{matrix}\right] = G^{2,0}_{2,2}\left[\sigma^\alpha \middle| \begin{matrix} \gamma_1 + \nu_1, \gamma_2 + \nu_2 \\ \gamma_1, \gamma_2 \end{matrix}\right] = \alpha \frac{\sigma^{\alpha\gamma_2}(1-\sigma^\alpha)^{\nu_1+\nu_2-1}}{\Gamma(\nu_1+\nu_2)} {}_2F_1(\nu_2 - \gamma_1 + \gamma_2, \nu_1; \nu_1 + \nu_2; 1 - \sigma^\alpha)$
$n = 1$ Erdélyi–Kober (E-K)	$H^{1,1}_{1,0}\left[\sigma \middle| \begin{matrix} \left(\gamma + \nu, \frac{1}{\alpha}\right) \\ \left(\gamma, \frac{1}{\alpha}\right) \end{matrix}\right] = \alpha\sigma^{\alpha-1} G^{1,1}_{1,0}\left[\sigma^\alpha \middle| \begin{matrix} \gamma + \nu \\ \gamma \end{matrix}\right] = \alpha \frac{\sigma^{\alpha(\gamma+1)-1}(1-\sigma^\alpha)^{\nu-1}}{\Gamma(\nu)}$
$n = 1\left(\alpha = 1; \frac{t}{x} = \sigma; \frac{x}{t} = \sigma\right)$ Riemann–Liouville (R-L)	$H^{1,1}_{1,0}\left[\sigma \middle| \begin{matrix} (\gamma + \nu, 1) \\ (\gamma, 1) \end{matrix}\right] = G^{1,1}_{1,0}\left[\frac{t}{x} \middle| \begin{matrix} \gamma + \nu \\ \gamma \end{matrix}\right] = \frac{(x-t)^{\nu-1} t^\gamma}{\Gamma(\nu)}$

A variety of such significant transforms can also be found in [15,16]. The above definitions involve the H-function [17,18] given by

$$H^{m,n}_{p,q}(z) = H^{m,n}_{p,q}\left[z\middle|\begin{matrix}(a_i, A_i)\\(b_j, B_j)\end{matrix}\right] = \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\prod_{j=1}^{m}\Gamma\left(b_j+B_j\mathfrak{s}\right)\prod_{i=1}^{n}\Gamma\left(1-a_i-A_i\mathfrak{s}\right)}{\prod_{j=m+1}^{q}\Gamma\left(1-b_j-B_j\mathfrak{s}\right)\prod_{i=n+1}^{p}\Gamma\left(a_i+A_i\mathfrak{s}\right)} z^{-\mathfrak{s}} d\mathfrak{s}, \quad (9)$$
$$\left(i = 1, \cdots, p;\ 0 \le n \le p; j = 1, \cdots, q;\ 1 \le m \le q;\ A_i > 0;\ B_j > 0;\ a_i \in \mathbb{C};\ b_j \in \mathbb{C}\right).$$

In this definition of the H-function, the singularities of $\left\{\Gamma\left(b_j + B_j\mathfrak{s}\right)\right\}_{j=1}^{m}$ and $\left\{\Gamma\left(1 - a_i - A_i\mathfrak{s}\right)\right\}_{i=1}^{n}$ are kept separated using the contour $\mathcal{L}$. If $A_i = 1 = B_j$, then, we have the following relationship between H and G functions [17,18]:

$$H^{m,n}_{p,q}\left[z\middle|\begin{matrix}(a_i, 1)\\(b_j, 1)\end{matrix}\right] = G^{m,n}_{p,q}\left[z\middle|\begin{matrix}a_i\\b_j\end{matrix}\right]. \quad (10)$$

Furthermore, for $A_i \epsilon \mathbb{R}^+ (i = 1, \ldots, p)$; $B_j \epsilon \mathbb{R}^+ (j = 1, \ldots, q)$; and $1 + \sum_{j=1}^{q} B_j - \sum_{i=1}^{p} A_i > 0$, the Fox–Wright function [18] is defined as follows:

$${}_p\Psi_q\left[\begin{matrix}(a_i, A_i)\\(b_j, B_j)\end{matrix}; z\right] = \sum_{m=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(a_i + A_i m) z^m}{\prod_{j=1}^{q}\Gamma\left(\left(b_j + B_j m\right) m!\right.}. \quad (11)$$

It has the following relationship with the hypergeometric function [2,19]:

$${}_p\Psi_q\left[\begin{matrix}(a_i, 1)\\(b_j, 1)\end{matrix}; z\right] = {}_pF_q\left(a_i; b_j; z\right).\frac{\Gamma(a_i)}{\Gamma(b_j)}; \left(a_i > 0;\ i = 1, \ldots. p; b_j \notin \mathbb{Z}_0^-;\ j = 1, \ldots. q\right). \quad (12)$$

The generalized Mittag–Leffler function [19,20] is given by

$$E^{\gamma}_{\alpha,\beta}(z) = \frac{1}{\Gamma(\gamma)} {}_1\Psi_1\left[\begin{matrix}(\gamma, 1)\\(\beta, \alpha)\end{matrix}; z\right] = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{k!\Gamma(\alpha k + \beta)}; \alpha, \beta, \gamma \epsilon \mathbb{C}, \Re(\alpha) > 0, \quad (13)$$

where $(\gamma)_k$ are Pochhammer symbols given by

$$(\gamma)_k = \frac{\Gamma(\gamma + k)}{\Gamma(\gamma)} = \begin{cases} 1\ (k = 0,) \\ \gamma(\gamma + 1) \ldots (\gamma + n - 1)\ (k \in \mathbb{C} \setminus \{0\}; k = n \in \mathbb{N}; \gamma \in \mathbb{C}). \end{cases} \quad (14)$$

Further special cases of (13) are as follows:

$$E^1_{\alpha,\beta}(z) = E_{\alpha,\beta}(z) = {}_1\Psi_1\left[\begin{matrix}(1, 1)\\(\beta, \alpha)\end{matrix}; z\right] = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}; E^1_{\alpha,1}(z) = E_{\alpha}(z) = {}_1\Psi_1\left[\begin{matrix}(1, 1)\\(1, \alpha)\end{matrix}; z\right] = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}. \quad (15)$$

Unless otherwise mentioned during this research, the conditions of the parameters shall be deemed normal as specified in Section 1.

2. Complex Generalized Representation of the Gamma Function and Its Convergence

This section presents a complex generalized Taylor series of the gamma function denoted by $\Gamma_\nu(x + \imath y)$, where ν is the fractional parameter as defined in Equation (6) in the form of non-integer derivatives of delta function. The major objective is to use the fractional operator to present a basic special function in the form of a generalized function. To model and solve the most general form of the fractional kinetic equation, a fractional generalization of the differential operators was necessary, which can assist to explain why this conclusion can only be accomplished by using fractional derivatives.

Proposition 1. *A subsequent complex generalized representation is computed for the gamma function.*

$$\Gamma_{\nu}(x+\iota y)=2\pi\sum_{l,m=0}^{\infty}\frac{(-1)^{l}(-\iota(x+l))^{\nu m}}{l!\Gamma(m\nu+1)}\delta^{(\nu m)}(y). \quad (16)$$

Proof. This is obtained by using Equation (6) in Equation (3) for the complex delta function being an element of Z'. □

Remark 1. *The delta function is used in an infinite series in this gamma function exemplification, which is only correct when it is specified in the sense of distributions (generalized functions). A subsequent theorem implies the interesting fact that the representation (16) acts as a distribution over a specific set of functions with fractional derivatives.*

Theorem 1. *Let $\mathfrak{I}$ denote a set of functions that have well-defined non-integer derivatives, then prove that the series representation (16) acts as a distribution over this set of functions.*

Proof. First, one may consider the subsequent inner product of the complex generalized representation of the gamma function (16) as well as a suitable liner combination of testing functions, $\chi_1(y),\chi_2(y)\epsilon\mathfrak{I}$, and $c_1,c_2>0$,

$$\begin{aligned}\langle\Gamma_{\nu}(x+\iota y),c_1\chi_1(y)+c_2\chi_2(y)\rangle&=2\pi\sum_{l,m=0}^{\infty}\frac{(-1)^{l}(-\iota(x+l))^{\nu m}}{l!\Gamma(\nu m+1)}\delta^{(\nu m)}(y)*(c_1\chi_1(y)+c_2\chi_2(y))\\&=c_1\Gamma_{\nu}(x+\iota y)*\chi_1(y)+c_2\Gamma_{\nu}(x+\iota y)*\chi_2(y).\end{aligned} \quad (17)$$

It demonstrates that this novel representation of the gamma function behaves as a linear operator over $\mathfrak{I}$. Next, one may consider a random sequence $\mathfrak{I}\supset\left\{\chi_{\mu}{}^{(\nu m)}(y)\right\}_{\mu=1}^{\infty}$ converging to zero and then consider the following convolution product,

$$\begin{aligned}\Rightarrow\{\langle\Gamma_{\nu}(x+\iota y)*\chi_{\mu}(y)\rangle\}_{\mu=1}^{\infty}&=2\pi\sum_{l,m=0}^{\infty}\frac{(-1)^{l}(-\iota(x+l))^{\nu m}}{l!\Gamma(\nu m+1)}\{\delta^{(\nu m)}(y)*\chi_{\mu}(y)\}_{\mu=1}^{\infty}\\&=2\pi\sum_{l,m=0}^{\infty}\frac{(-1)^{l}(-\iota(x+l))^{\nu m}}{l!\Gamma(\nu m+1)}\chi_{\mu}{}^{(\nu m)}(y),\end{aligned} \quad (18)$$

and we can show that $\{\langle\delta^{(\nu m)}(y)*\chi_{\mu}(y)\rangle\}_{\mu=1}^{\infty}$ converges to zero. This implies that the complex generalized representation of the gamma function $\Gamma_{\nu}(x+\iota y)$ acts as a linear as well as a continuous operator over $\mathfrak{I}$. Therefore, we consider the following equation,

$$\text{sum over the coefficients}=\sum_{l=0}^{\infty}\frac{(-1)^{l}}{l!}E_{\nu}\left((-\iota(x+l))^{\nu}\right)=\sum_{l=0}^{\infty}\frac{(-1)^{l}}{l!}{}_1\Psi_1\left[\begin{matrix}(1,\,1)\\(1,\nu)\end{matrix};(-\iota(x+l))^{\nu}\right] \quad (19)$$

where $\chi_{\mu}{}^{(\nu m)}(y)\in\mathfrak{I}$ exists and is well defined, showing that $\Gamma_{\nu}(x+\iota y)*\chi_{\mu}(y)$ converges for $\forall\chi(y)\epsilon\mathfrak{I}$ using the famous Abel theorem. □

One should note that, by making the use of (7) and (16) for any function $\chi(y)\epsilon\mathfrak{I}$, we obtain the following:

$$\begin{aligned}\langle\Gamma_{\nu}(x+\iota y),\chi(y)\rangle&=2\pi\sum_{l,m=0}^{\infty}\frac{(-1)^{l}(-\iota(x+l))^{\nu m}}{l!\Gamma(\nu m+1)}\delta^{(\nu m)}(y)*\chi(y)\\&=2\pi\sum_{l,m=0}^{\infty}\frac{(-1)^{l}(-\iota(x+l))^{\nu m}}{l!\Gamma(\nu m+1)}\chi^{(\nu m)}(y).\end{aligned} \quad (20)$$

The existing literature contains a wide range of non-integer-order operators, and even for the computation of very fundamental functions, a thorough comprehension of the selected operator is necessary (refer [21] and the bibliography cited therein).

Example 1. *For example, using the non-integer-order derivative in the Caputo sense* [21] *of* e^{cy}*, we obtain*

$$\Gamma_{\nu}(x+\iota y)*e^{cy} = 2\pi\sum_{l,m=0}^{\infty}\frac{(-1)^{l}(-\iota(x+l))^{\nu m}}{l!\Gamma(\nu m+1)}\frac{d^{\nu m}}{dy^{\nu m}}(e^{cy}) = \frac{2\pi}{y^{\nu}}\sum_{l,m=0}^{\infty}\frac{(-1)^{l}((-\iota(x+l))^{\nu}cy)^{m}}{l!\Gamma(\nu m+1)}E_{1,m-\nu+1}(cy), \tag{21}$$

and using the Grünwald–Letnikov derivative [21], ${}^{\nu m}_{GL}D\ e^{cy} = c^{\nu m}\ e^{cy}$ *in (20), the new identity obtained is as follows:*

$$\Gamma_{\nu}(x+\iota y)*e^{cy} = 2\pi\sum_{l,m=0}^{\infty}\frac{(-1)^{l}(-\iota(x+l))^{\nu m}}{l!\Gamma(\nu m+1)}c^{\nu m}e^{cy} = 2\pi e^{cy}\sum_{l=0}^{\infty}\frac{(-1)^{l}}{l!}E_{\nu}(-\iota c(x+l))^{\nu}. \tag{22}$$

Next, we consider the following examples for the Riemann–Liouville (R-L) fractional derivative of the Hurwitz–Lerch zeta function [22] and the Mittag–Leffler function [23].

Example 2. *Consider* $\chi(y) = y^{\nu m}\Phi(y,s,a)$*, then* ${}^{RL}_{0}D^{\nu m}_{y}\ \chi(y) = \Gamma(\nu m+1)\Phi^{*}_{\nu m+1}(y,s,a)$*, and* Φ, Φ^{*} *denote the Hurwitz–Lerch zeta and the generalized Hurwitz–Lerch zeta functions* [22]*. Using (7) and (16) for these functions, we obtain the following:*

$$\begin{aligned} y^{\nu m}\Phi(y,s,a)*\Gamma_{\nu}(x+\iota y) &= 2\pi\sum_{l,m=0}^{\infty}\frac{(-1)^{l}(-\iota(x+l))^{\nu m}}{l!\Gamma(\nu m+1)}\frac{d^{\nu m}}{dy^{\nu m}}(y^{\nu m}\Phi(y,s,a)) \\ &= 2\pi\sum_{l,m=0}^{\infty}\frac{(-1)^{l}(-\iota(x+l))^{\nu m}\Phi^{*}_{\nu m+1}(y,s,a)}{l!}. \end{aligned} \tag{23}$$

Example 3. *Consider* $\chi(y) = y^{\nu m}E_{\alpha,\beta}(y)$*, then* ${}^{RL}_{0}D^{\nu m}_{y}\ \chi(y) = \Gamma(\nu m+1)E^{\rho}_{\alpha,\beta}(y)$*, where* $E^{\rho}_{\alpha,\beta}(y)$ *denotes the Mittag–Leffler function* ([23]*, Equation (12)). Using (7) and (16) for this function, we obtain the following:*

$$\begin{aligned} y^{\nu m}E_{\alpha,\beta}(y)*\Gamma_{\nu}(x+\iota y) &= 2\pi\sum_{l,m=0}^{\infty}\frac{(-1)^{l}(-\iota(x+l))^{\nu m}}{l!\Gamma(\nu m+1)}\frac{d^{\nu m}}{dy^{\nu m}}(y^{\nu m}E_{\alpha,\beta}(y)) \\ &= 2\pi\sum_{l,m=0}^{\infty}\frac{(-1)^{l}(-\iota(x+l))^{\nu m}E^{\nu m+1}_{\alpha,\beta}(y)}{l!}. \end{aligned} \tag{24}$$

In the same way, more novel and innovative results can be calculated that are valuable in solving the several problems arising in fractional calculus that are also proposed in [24]. For instance, many different types of fractional-order kinetic equations have been employed recently, particularly in the modelling and analysis of several significant physics and astrophysics problems [25,26]. Such kinetic equations combined by the notion of Continuous-Time Random Walk (CTRW) has led to an apparent increase in the popularity of kinetic equations of the fractional order [27,28]. These equations are currently being utilized to decipher the diffusion in porous media, the relaxation and response in complex systems, anomalous diffusion, so on and so forth. This section is concluded considering the subsequent general non-integer kinetic integrodifferential equation discussed in [24].

If c, d, and η are constants, and f is a locally integrable function, then we have

$$c(D^{a,b}_{0+}X)(t) - X_0 f(t) = d\left(I^{\lambda}_{0+}X\right)(t), \tag{25}$$

with an initial condition of

$$\left(I^{(1-b)(1-a)}_{0+}X\right)(0+) = \eta. \tag{26}$$

Here, we apply this general form of a new fractional representation (16) to obtain the solution ([24], Equation (5.9)–(5.10)), by extending the general result from locally integrable functions to a class of singular distributions,

$$\begin{aligned} X(t) = \sum_{k=0}^{\infty} \frac{\left(\frac{d}{c}\right)^k}{\Gamma(c+k(\lambda+c))} y^{c+k(\lambda+c)-1} * \sum_{l,m=0}^{\infty} \frac{(-1)^l(-\iota(x+l))^{\nu m}}{l!\Gamma(\nu m+1)} \delta^{(\nu m)}(y) \\ +\eta \sum_{k=0}^{\infty} \frac{\left(\frac{d}{c}\right)^k}{\Gamma(c-d(1-c)+k(\lambda+c))} y^{c-d(1-c)+k(\lambda+c)-1}, \end{aligned} \tag{27}$$

and from this equation, we obtain

$$\begin{aligned} X(t) = \frac{X_0}{c} \sum_{k=0}^{\infty} \frac{\left(\frac{d}{c}\right)^k}{\Gamma(a+k(\lambda+a))} \sum_{l,m=0}^{\infty} \frac{(-1)^l(-\iota(x+l))^{\nu m}}{l!\Gamma(\nu m+1)} y^{a+k(\lambda+a)-1} * \delta^{(\nu m)}(y) \\ +\eta \sum_{k=0}^{\infty} \frac{\left(\frac{d}{c}\right)^k}{\Gamma(a-b(1-a)+k(\lambda+a))} y^{a-b(1-a)+k(\lambda+a)-1}; c \neq 0, \end{aligned} \tag{28}$$

and using (7), the required solution is

$$\begin{aligned} X(t) = \frac{X_0}{c} \sum_{k,l,m=0}^{\infty} \frac{\left(\frac{d}{c}\right)^k \Gamma(\nu m-(c+k(\nu+c)-1)) y^{c+k(\lambda+c)-1-\nu m} (-1)^l (-\iota(x+l))^{\nu m}}{\Gamma(a+k(\lambda+a))\Gamma(-(a+k(\lambda+a)-1)) l! \Gamma(\nu m+1)} \\ +\eta \sum_{k=0}^{\infty} \frac{\left(\frac{d}{c}\right)^k}{\Gamma(a-d(1-a)+k(\lambda+a))} y^{a-b(1-a)+k(\lambda+a)-1}; c \neq 0. \end{aligned} \tag{29}$$

3. Fourier Transform of Gamma Function Using New Representation and Multiple Erdélyi–Kober (E-K) Fractional Operators with Application in Recently Popular Transforms

This section contains new non-integer formulae involving the gamma function by distributing it into two subsections.

Fourier transformations are significant to solve many physical problems. Here, we compute them for the new fractional representation (16) using the following equation explored in [7]

$$\mathcal{F}\{\delta^{(\nu)}(t); \omega\} = (\iota\omega)^{\nu}. \tag{30}$$

Hence, we obtain

$$\begin{aligned} \mathcal{F}(\Gamma_\nu(x+\iota y); \omega) = \mathcal{F}\left(2\pi \sum_{l,m=0}^{\infty} \frac{(-1)^l(-\iota(x+l))^{\nu m}}{l!\Gamma(\nu m+1)} \delta^{(\nu m)}(y); \omega\right) \\ = 2\pi \sum_{l,m=0}^{\infty} \frac{(-1)^l(-\iota(x+l))^{\nu m}}{l!\Gamma(\nu m+1)} \mathcal{F}\left(\delta^{(\nu m)}(y); \omega\right) \\ = 2\pi \sum_{l,m=0}^{\infty} \frac{(-1)^l(-\iota(x+l))^{\nu m}}{l!\Gamma(\nu m+1)} (\iota\omega)^{\nu m} = 2\pi \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} E_\nu((x+l)\omega)^{\nu}, \end{aligned} \tag{31}$$

where E_ν denotes the Mittag–Leffler function. Considering $\nu = 1$ in (31), the equation obtained is as follows:

$$\mathcal{F}(\Gamma(x+\iota y); \omega) = 2\pi \sum_{l,m=0}^{\infty} \frac{(-1)^l(-\iota(x+l))^m}{l!m!} (\iota\omega)^m = 2\pi \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} e^{x\omega+l\omega} = 2\pi e^{x\omega} \exp(-e^{\omega}). \tag{32}$$

3.1. Multiple Erdélyi–Kober (E-K) Fractional Derivatives with Application in Recently Popular Transforms

The following result stated in [12], Theorem 4, is significant to simplify the complicated forms involving ratios of the gamma function:

$$D_{(\beta_k),n}^{(\gamma_k)_1^n,(\delta_k)}\left\{z^c{}_p\Psi_q\left[\begin{matrix}(a_i,\nu_i)_1^p\\(b_j,\beta_j)_1^q\end{matrix};\mu z^\mu\right]\right\}=z^c\left\{{}_{p+n}\Psi_{q+n}\left[\begin{matrix}(a_i,\nu_i)_1^p,\left(\gamma_k+\delta_k+1+\frac{c}{\beta_k},\frac{1}{\beta_k}\right)_1^n\\(b_j,\beta_j)_1^q,\left(\gamma_k+1+\frac{c}{\beta_k},\frac{1}{\beta_k}\right)_1^n\end{matrix};\mu z^\mu\right]\right\}. \tag{33}$$

We use it to obtain the closed form of multiple Erdélyi–Kober derivatives by the means of the complex generalized representation of the gamma function that is formulated as follows:

$$D_{(\beta_i),n}^{(\gamma_i)_1^n,(\delta_i)}\left(\omega^{\mu-1}\mathcal{F}(\Gamma_\nu(x+\iota y);\omega)\right)=2\pi\omega^{\mu-1}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}{}_n\Psi_{n+1}\left[\begin{matrix}- & \left(\gamma_i+\delta_i+1+\frac{\mu}{\beta_i},\frac{\nu}{\beta_i}\right)_1^n\\(1,\nu) & \left(\gamma_i+1+\frac{\mu}{\beta_i},\frac{\nu}{\beta_i}\right)_1^n\end{matrix}\Bigg\vert(-(x+l)\omega)^\nu\right]. \tag{34}$$

Further important cases of this result are listed in the following Table 2.

Table 2. Non-integer-order derivatives containing the Fourier transformation of the gamma function.

$n=3$	*M-S-M non-integer-order derivatives*
$D_{0+}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\delta}\left(\omega^{\mu-1}\mathcal{F}(\Gamma_\nu(x+\iota y);\omega)\right)=2\pi\omega^{\delta+\mu-\gamma_1-\gamma_1'-1}$ $\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}{}_2\Psi_3\left[\begin{matrix}- & (\mu,\nu) & (\mu-\gamma_2+\gamma_1,\nu) & (\mu+\gamma_{1+}\gamma_1'+\gamma_2'-\delta,\nu)\\(1,\nu) & (\mu-\gamma_2,\nu) & (\mu-\delta+\gamma_1++\gamma_2',\nu) & (\mu-\delta+\gamma_1'+\gamma_1,\nu)\end{matrix}\Big\vert(-(x+l)\omega)^\nu\right]$	
$D_{-}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\delta}\left(\omega^{\mu-1}\mathcal{F}(\Gamma_\nu(x+\iota y);\omega)\right)=2\pi\omega^{\delta+\mu-\gamma_1-\gamma_1'-1}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}$ ${}_2\Psi_3\left[\begin{matrix}- & (1-\mu+\gamma_2',-\nu) & (1+\gamma_2'-\mu-\gamma_2+\gamma_1,-\nu) & (1-\mu-\gamma_{1-}\gamma_1'+\delta,-\nu)\\(1,-\nu) & (1-\mu,-\nu) & (1-\mu-\gamma_1'+\gamma_2',-\nu) & (1-\mu+\delta-\gamma_1'-\gamma_1-\gamma_2,-\nu)\end{matrix}\Big\vert(-(x+l)\omega)^\nu\right]$	
$n=2$	*Saigo fractional-order derivatives*
	$D_{0+}^{\gamma_1,\gamma_2,\delta}\left(\omega^{\mu-1}\mathcal{F}(\Gamma_\nu(x+\iota y);\omega)\right)=2\pi\omega^{\mu-\gamma_1-1}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}$ ${}_1\Psi_2\left[\begin{matrix}- & (\mu,\nu) & (\mu+\delta+\gamma_2+\gamma_1,\nu)\\(1,\nu) & (\mu+\gamma_2,\nu) & (\mu+\delta,\nu)\end{matrix}\Big\vert(-(x+l)\omega)^\nu\right]$
	$D_{-}^{\gamma_1,\gamma_2,\delta}\left(\omega^{\mu-1}\mathcal{F}(\Gamma_\nu(x+\iota y);\omega)\right)=2\pi\omega^{\mu-\gamma_1-1}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}$ ${}_1\Psi_2\left[\begin{matrix}- & (1-\mu-\gamma_2,-\nu) & (1-\mu+\delta+\gamma_1,-\nu)\\(1,-\nu) & (1-\mu+\delta-\gamma_2,-\nu) & (1-\mu,-\nu)\end{matrix}\Big\vert(-(x+l)\omega)^\nu\right]$
$n=1$	*E-K fractional-order derivatives*
	$D_{0+}^{\gamma,\delta}\left(\omega^{\mu-1}\mathcal{F}(\Gamma_\nu(x+\iota y);\omega)\right)=$ $2\pi\omega^{\mu-\gamma_1-1}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}{}_1\Psi_2\left[\begin{matrix}- & (\gamma+\delta+\mu,1)\\(1,\nu) & (\gamma+\mu,1)\end{matrix}\Big\vert(-(x+l)\omega)^\nu\right]$
	$D_{-}^{\gamma,\delta}\left(\omega^{\mu-1}\mathcal{F}(\Gamma_\nu(x+\iota y);\omega)\right)=$ $2\pi\omega^{\mu-1}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}{}_1\Psi_2\left[\begin{matrix}- & (1-\mu+\gamma+\delta,-\nu)\\(1,-\nu) & (1-\mu+\gamma,-\nu)\end{matrix}\Big\vert(-(x+l)\omega)^\nu\right]$
$n=1$	*R-L fractional derivatives*
	$D_{0+}^{\delta}\left(\omega^{\mu-1}\mathcal{F}(\Gamma_\nu(x+\iota y);\omega)\right)=2\pi\omega^{\mu-1}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}{}_1\Psi_2\left[\begin{matrix}- & (\mu,\nu)\\(1,\nu) & (\mu-\delta,\nu)\end{matrix}\Big\vert(-(x+l)\omega)^\nu\right]$
	$D_{-}^{\delta}\left(\omega^{\mu-1}\mathcal{F}(\Gamma_\nu(x+\iota y);\omega)\right)=$ $2\pi\omega^{\mu-1}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}{}_1\Psi_2\left[\begin{matrix}- & (\delta-\lambda+1,-\nu)\\(1,-\nu) & (1-\lambda,-\nu)\end{matrix}\Big\vert(-\omega(x+l))^\nu\right]$

If $\nu=1$, then the above table yields further novel results.

3.2. Multiple Erdélyi–Kober (E-K) Fractional Integrals with Application in Recently Popular Transforms

The following action of the Erdélyi–Kober (E-K) fractional transform ([12], p. 9, Equation (27)) is significant for this research to obtain various results that are depicted in Table 3:

$$I_{(\beta_i),n}^{(\gamma_i),(\delta_i)}\{z^p\} = \prod_{i=1}^{n} \frac{\Gamma\left(\gamma_i + 1 + \frac{p}{\beta_i}\right)}{\Gamma\left(\gamma_i + \delta_i + 1 + \frac{p}{\beta_i}\right)} z^p; \ ([-\beta_i(1+\gamma_i)] < p; \delta_i \geq 0; i = 1, \ldots, n). \quad (35)$$

Table 3. Non-integer-order integrals containing the Fourier transformation of the gamma function.

$n = 3$	*M-S-M fractional integrals*
	$\mathrm{I}_{0+}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\delta}(\omega^{\mu-1}\mathcal{F}(\Gamma_\nu(x+\imath y);\omega)) = 2\pi s^{\delta-\gamma_1-\gamma_1'} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!}\, {}_3\Psi_4\left[\begin{matrix} - & (\mu,\nu) & (\mu+\delta-\gamma_1-\gamma_1'-\gamma_2,\nu) & (\mu+\gamma_2'-\gamma_1',\nu) \\ (1,\nu) & (\mu+\gamma_2',\nu) & (\mu+\delta-\gamma_1-\gamma_1',\nu) & (\mu+\delta-\gamma_1'-\gamma_2,\nu) \end{matrix} \,\vert\, (-(x+l)\omega)^\nu\right]$
	$\mathrm{I}_{0-}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\delta}(\omega^{\mu-1}\mathcal{F}(\Gamma_\nu(x+\imath y);\omega)) = 2\pi\omega^{\delta+\mu-\gamma_1-\gamma_1'-1} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!}\, {}_3\Psi_4 \left[\begin{matrix} - & (1-\nu-\delta+\gamma_1+\gamma_1',-\nu) & (1-\mu+\gamma_1+\gamma_2'-\delta,-\nu) & (1-\mu-\gamma_1,-\nu) \\ (1,-\nu) & (1-\mu,-\nu) & (1-\mu+\gamma_1+\gamma_1'+\gamma_2+\gamma_2'-\delta,-\nu) & (1-\mu+\gamma_1-\gamma_2,-\nu) \end{matrix} \,\vert\, (-(x+l)\omega)^\nu\right]$
$n = 2$	*Saigo fractional integrals*
	$I_{0+}^{\gamma_1,\gamma_2,\delta}(\omega^{\mu-1}\mathcal{F}(\Gamma_\nu(x+\imath y);\omega)) = 2\pi\omega^{\lambda-\gamma_1-1} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!}\, {}_2\Psi_3\left[\begin{matrix} - & (\lambda,\nu) & (\lambda+\gamma_2-\gamma_1,\nu) \\ (1,\nu) & (\lambda-\gamma_2,\nu) & (\lambda+\delta+\gamma_2,\nu) \end{matrix} \,\vert\, (-(x+l)\omega)^\nu\right]$
	$I_{-}^{\gamma_1,\gamma_2,\delta}(\omega^{\mu-1}\mathcal{F}(\Gamma_\nu(x+\imath y);\omega)) = 2\pi\omega^{\lambda-\gamma_1-1} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!}\, {}_2\Psi_3\left[\begin{matrix} - & (\gamma_1-\mu+1,-\nu) & (\gamma_2-\mu+1,-\alpha) \\ (1,-\nu) & (1-\mu,-\nu) & (\gamma_1+\gamma_2+\delta-\mu+1,-\nu) \end{matrix} \,\vert\, (-(x+l)\omega)^\nu\right]$
$n = 1$	*E-K fractional integrals*
	$I_{0+}^{\gamma,\delta}(\omega^{\mu-1}\mathcal{F}(\Gamma_\nu(x+\imath y);\omega)) = 2\pi\omega^{\mu-1} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!}\, {}_1\Psi_2\left[\begin{matrix} - & (\mu+\gamma,\nu) \\ (1,\nu) & (\mu+\gamma+\delta,\nu) \end{matrix} \,\vert\, (-(x+l)\omega)^\nu\right]$
	$I_{0-}^{\gamma,\delta}(\omega^{\mu-1}\mathcal{F}(\Gamma_\nu(x+\imath y);\omega)) = 2\pi\omega^{\mu-1} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!}\, {}_1\Psi_2\left[\begin{matrix} - & (\gamma-\mu+1,-\nu) \\ (1,-\nu) & (\gamma+\delta-\mu+1,-\alpha) \end{matrix} \,\vert\, (-(x+l)\omega)^\nu\right]$
$n = 1$	*R-L fractional integrals*
	$\mathrm{I}_{0+}^{\delta}(\omega^{\mu-1}\mathcal{F}(\Gamma_\nu(x+\imath y);\omega)) = 2\pi\omega^{\mu+\delta-1} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!}\, {}_1\Psi_2\left[\begin{matrix} - & (\mu,\nu) \\ (1,\nu) & (\delta+\mu,\nu) \end{matrix} \,\vert\, (-(x+l)\omega)^\nu\right]$ $\mathrm{I}_{0+}^{\delta}(\omega^{\mu-1}\mathcal{F}(\Gamma_\nu(x+\imath y);s)) = 2\pi\omega^{\mu+\delta-1} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!}\, {}_1\Psi_2\left[\begin{matrix} - & (\mu,\nu) \\ (1,\nu) & (\delta+\mu,\nu) \end{matrix} \,\vert\, (-(x+l)\omega)^\nu\right]$
	$\mathrm{I}_{-}^{\delta}(\omega^{\mu-1}\mathcal{F}(\Gamma_\nu(x+\imath y);\omega)) = 2\pi\omega^{\mu+\delta-1} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!}\, {}_1\Psi_2\left[\begin{matrix} - & (1-\delta-\mu,-\nu) \\ (1,-\nu) & (1-\mu,-\nu) \end{matrix} \,\vert\, (-(x+l)\omega)^\nu\right]$

Theorem 2. *The closed form of multiple Erdélyi–Kober (E-K) fractional integral transforms involving the gamma function is given by*

$$\mathrm{I}_{(\beta_i),n}^{(\gamma_i),(\delta_i)}(\omega^{\mu-1}\mathcal{F}(\Gamma_\nu(x+\imath y);\omega)) = 2\pi\omega^{\mu-1} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!}\, {}_n\Psi_{n+1}\left[\begin{matrix} - & \left(\gamma_i + 1 - \frac{1-\mu}{\beta_i}, \frac{\nu}{\beta_i}\right)_1^n \\ (1,\nu) & \left(\gamma_i + \delta_i + 1 - \frac{1-\mu}{\beta_i}, \frac{\nu}{\beta_i}\right)_1^n \end{matrix} \middle| ((x+l)\omega)^\nu\right] \quad (36)$$
$$([-\beta_i(1+\gamma_i)] < \nu n + \mu - 1; \delta_i \geq 0; i = 1, \ldots, n).$$

Proof. Let us first consider

$$\mathrm{I}_{(\beta_i),n}^{(\gamma_i),(\delta_i)}\left(\omega^{\mu-1}\mathcal{F}(\Gamma_\nu(x+\imath y);\omega)\right) = \mathrm{I}_{(\beta_i),n}^{(\gamma_i),(\delta_i)}\left(\sum_{l,m=0}^{\infty} \frac{(-1)^l (x+l)^{\nu m}}{l!\Gamma(\nu m+1)} \omega^{\nu m+\mu-1}\right), \quad (37)$$

and then, by interchanging the summation and integration, we obtain

$$I^{(\gamma_i),(\delta_i)}_{(\beta_i),n}(\mathcal{F}(\Gamma_\nu(x+\iota y);\omega)) = 2\pi\sum_{l,m=0}^{\infty}\frac{(-1)^l(-\iota(x+l))^{\nu m}}{l!\Gamma(\nu m+1)}I^{(\gamma_i),(\delta_i)}_{(\beta_i),n}\left(\omega^{\nu m+\mu-1}\right), \tag{38}$$

and then, we obtain to the following equation using (35),

$$I^{(\gamma_i),(\delta_i)}_{(\beta_i),n}(\mathcal{F}(\Gamma_\nu(x+\iota y);\omega)) = 2\pi\sum_{l,m=0}^{\infty}\frac{(-1)^l(-\iota(x+l))^{\nu m}}{l!\Gamma(\nu m+1)}\prod_{i=1}^{n}\frac{\Gamma\left(\gamma_i+1+\frac{\nu m+\mu-1}{\beta_i}\right)}{\Gamma\left(\gamma_i+\delta_i+1+\frac{\nu m+\mu-1}{\beta_i}\right)}\omega^{\nu m}, \tag{39}$$

which after replacing the summation index m by n and using Equation (11), leads to the essential compact form. □

Further important special cases are listed in the following Table 3.

If $\nu = 1$, then the above table yields further novel results. Consequently, it validates that the outcomes of the new fractional representation are consistent with the current findings. Based on this, we move forward and extract the distributional solution of the singular fractional integral equation, which is only possible because of this fractional distributional representation.

3.3. Solution of a Singular Fractional Integral Equation including the Fractional Derivatives of the Delta Function

The Laplace transformation of the fractional derivatives of the Dirac delta function is given below, as computed in [7],

$$L\left\{\delta^{(\nu)}(t);s\right\} = s^\nu \tag{40}$$

and therefore, by using the above equation in (16), we obtain

$$L(\Gamma_\nu(x+\iota y);s) = 2\pi\sum_{l,m=0}^{\infty}\frac{(-1)^l(-\iota(x+l))^m}{l!m!}s^{\nu m} = 2\pi\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}E_\nu\left((-\iota(x+l)s)^\nu\right). \tag{41}$$

Considering $\nu = 1$ in (41), we obtain

$$L(\Gamma(x+\iota y);s) = 2\pi e^{-\iota x s}\exp\left(-e^{-\iota s}\right). \tag{42}$$

The basic kinetic equation, $p(X_t) - d(X_t) = \frac{dX(t)}{dt}; X_t(t^*) = X(t-t^*); t^* > 0$, to analyse the reaction rate X(t) in terms of production p(X) and destruction d(X) was formulated in [25] by Haubold and Mathai. Then, disregarding the variation in species and position, we obtain $\frac{dX_k}{dt} = -a_k X_k(t), X_k(t=0) = X_0$, where $X_k = X_k(t)$ is the count of the density of species k. At that point, we integrate it by dropping the subscript k, to obtain $X(t) - X_0 = -aI^{-1}_{0+}X(t)$. Hence, the corresponding fractional kinetic equation formulated by the means of the R-L fractional integral is $X(t) - X_0 = -a^\lambda\, I^\lambda_{0+}X(t), \lambda > 0$, where a is the fixed value. It leads to the non-integer-order integral equation [25–28], i.e., $X(t) - f(t)X_0 = -a^\lambda I^\lambda_{0+}X(t)$, involving an integrable function f(t). According to the discussion presented in [25], we obtain the subsequent non-integer singular integral equation by considering $f(\omega) = \Gamma_\nu(\omega)$ and $X = X_\nu$ in the above equation:

$$X_\nu(\omega) - X_0\Gamma_\nu(\omega) = -d^\lambda I^\lambda_{0+}X_\nu(\omega); \lambda > 0; 0 < \lambda < 1. \tag{43}$$

In the classical sense, Equation (43) is a singular integral equation containing infinitely many singularities at negative integers; $\omega = -n = \{0, -1, -2, \ldots..\}$ (consider (2)), when the fractional parameter $\nu = 1$. Due to this distributional representation, we can use the

Laplace transform, $X_\nu(s) = L[X_\nu(t) : s] = \int_0^\infty e^{-st}X(t)dt; \Re(s) > 0$, for Equation (43) to obtain the following:

$$L\{X_\nu(x+\iota y);s\} - X_0 L\{\Gamma_\nu(x+\iota y);s\} = L\Big\{-d^\lambda I_{0+}^\lambda X_\nu(x+\iota y);s\Big\}. \quad (44)$$

It leads to the subsequent result by considering Equation (41),

$$X_\nu(s) = 2\pi X_0 \sum_{l,m=0}^{\infty} \frac{(-1)^l(-\iota(x+l))^{\nu m}}{l!\Gamma(\nu m+1)} s^{\nu m} - \left(\frac{s}{d}\right)^{-\lambda} X_\nu(s), \quad (45)$$

where $L\Big\{I_{0+}^\lambda X_\nu(\omega);s\Big\} = s^{-\lambda}X_\nu(s)$, and we obtain

$$X_\nu(s)\left[1+\left(\frac{s}{d}\right)^{-\lambda}\right] = 2\pi X_0 \sum_{l,m=0}^{\infty} \frac{(-1)^l(-\iota(x+l))^{\nu m}}{l!\Gamma(\delta m+1)} s^{\lambda m}. \quad (46)$$

From the above discussion, we obtain the subsequent expression:

$$X_\nu(s) = 2\pi X_0 \sum_{l,m=0}^{\infty} \frac{(-1)^l(-\iota(x+l))^{\nu m}}{l!\Gamma(\nu m+1)} s^{\nu m} \sum_{n=0}^{\infty}\left[-\left(\frac{s}{d}\right)^{-\lambda}\right]^n. \quad (47)$$

Case 1: Moreover, we assume that $\lambda n - \nu m > 0; \lambda > 0$, and we use $L^{-1}\{s^{-\lambda};\omega\} = \frac{\omega^{\lambda-1}}{\Gamma(\omega)}$ to obtain the following solution:

$$X_\nu(t) = 2\pi X_0 \sum_{l,m=0}^{\infty} \frac{(-1)^l(-\iota(x+l))^{\nu m}}{l!\Gamma(\nu m+1)} t^{-\nu m-1} \times \sum_{n=0}^{\infty} \frac{(-d^\lambda t^\lambda)^n}{\Gamma(\lambda n-\nu m)} \quad (48)$$

$$X_\nu(t) = \frac{2\pi X_0}{t} \sum_{l,m=0}^{\infty} \frac{(-1)^l(-\frac{\iota(x+l)}{t})^{\nu m}}{l!\Gamma(\nu m+1)} E_{\lambda,-\nu m}\Big(-d^\lambda t^\lambda\Big).$$

Case 2: By considering [7], Equation (30), for $\lambda \in \mathbb{R}^+$, we obtain

$$\delta^{(\nu)}(t) = \frac{1}{\Gamma(-\lambda)t^{\lambda+1}} = L^{-1}\Big\{s^\lambda;t\Big\}, \quad (49)$$

and

$$L^{-1}\Big\{s^{\nu m-\lambda n};t\Big\} = \frac{1}{\Gamma(\lambda n)}\frac{d^{\nu m}}{dt^{\nu m}}(t^{\lambda n-1}) = \frac{1}{\Gamma(-\nu m+\lambda n)t^{\nu m-\lambda n+1}};\ \nu m-\lambda n \in \mathbb{R}^+. \quad (50)$$

The following solution is then obtained by utilizing Equation (49)–(50) in Equation (47):

$$X_\nu(t) = 2\pi X_0 \sum_{l,m=0}^{\infty} \frac{(-1)^l(-\iota(x+l))^{\nu m}}{l!\Gamma(\nu m+1)} t^{\nu m-1} \times \sum_{n=0}^{\infty} \frac{(-d^\lambda t^\lambda)^n}{\Gamma(\nu m-\lambda n)} \quad (51)$$

$$X_\nu(t) = \frac{2\pi X_0}{t} \sum_{l,m=0}^{\infty} \frac{(-1)^l(-\frac{\iota(x+l)}{t})^{\nu m}}{l!\Gamma(\nu m+1)} E_{-\lambda,\nu m}\Big(-d^\lambda t^\lambda\Big).$$

Case 3: Using the generalized form of [7], Equation (30), i.e., $L^{-1}\{s^{\nu m-\lambda n};t\} = \delta^{(\nu m-\lambda n)}(t)$, in (47), we obtain the following solution in terms of the non-integer derivatives of the delta function:

$$X_\nu(t) = 2\pi X_0 \sum_{l,m=0}^{\infty} \frac{(-1)^l(-\iota(x+l))^{\nu m}}{l!\Gamma(\nu m+1)} \sum_{n=0}^{\infty} (-d)^{\lambda n} \delta^{(\nu m-\lambda n)}(t). \quad (52)$$

This is a generalized (distributional) solution and using Equation (7), and we obtain

$$X_\nu(\xi) * \chi(\xi) = 2\pi X_0 \sum_{l,m=0}^{\infty} \frac{(-1)^l (-\iota(x+l))^{\nu m}}{l!\Gamma(\nu m+1)} \sum_{n=0}^{\infty} (-d)^{\lambda n} \chi^{(\nu m-\lambda n)}(t). \tag{53}$$

This generalized solution is valid only if the non-integer derivatives $\chi^{(\nu m-\lambda n)}(\xi)$ exist and are well defined. This was not achievable using the classical gamma function representations. If $\chi(\xi) = \xi^\mu; \mu > 0$, then $\chi^{(\nu m-\lambda n)}(\xi) = \frac{\Gamma(\nu m-\lambda n-\mu)}{\Gamma(-\mu)} \xi^{-\nu m+\lambda n+\mu}$ leads to

$$X_\nu(\xi) * \chi(\xi) = \frac{2\pi X_0 \xi^\mu}{\Gamma(-\mu)} \sum_{l,m=0}^{\infty} \frac{(-1)^l}{l!} \left(-\frac{d^\lambda}{t^\mu}\right)^m {}_1\Psi_1\left[\begin{matrix}(\lambda n-\mu,\nu)\\(1,\nu)\end{matrix}; (-\iota(x+l)t)^\nu\right], \tag{54}$$

and, for ν = 1, one can obtain the following:

$$\begin{aligned} X_\nu(\xi) * \chi(\xi) &= \tfrac{2\pi X_0 t^\mu}{\Gamma(-\mu)} \sum_{l,m=0}^{\infty} \tfrac{(-1)^l}{l!} \left(-\tfrac{d^\lambda}{t^\mu}\right)^m {}_1\Psi_1\left[\begin{matrix}(\lambda n-\mu,\nu)\\(1,\nu)\end{matrix}; (-\iota(x+l)t)^\nu\right] \\ &= \tfrac{2\pi X_0 t^\mu}{\Gamma(-\mu)} \sum_{m=0}^{\infty} \left(-\tfrac{d^\lambda}{t^\mu}\right)^m {}_1\Psi_1\left[\begin{matrix}(\lambda n-\mu,1)\\(1,1)\end{matrix}; -e^{-\iota(x+l)t}\right]. \end{aligned} \tag{55}$$

Remark 2. *It is worth noting that a conventional solution approach is used. X(t) is generally expressed in the form of the Mittag–Leffler function, and the similar fact is obvious in case of the above solution. Consequently, the following sum is obtained for the coefficients in Equations (48), (51), and (52):*

$$C_\nu^x(t) = \sum_{l,m=0}^{\infty} \frac{(-1)^l \left(-\frac{\iota(x+l)}{t}\right)^{\nu m}}{l!\Gamma(\nu m+1)} = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} E_\nu \left(\frac{-\iota(x+l)}{t}\right)^\nu. \tag{56}$$

This sum is finite and well defined. Similarly,

$$\lim_{t\to\infty} C_\nu^x(t) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} E_\nu(0).$$

Furthermore, the stimulating special cases of these solutions can be computed for the non-fractional case when $\nu = 1$.

3.4. New q-Fractional Derivatives Involving Different Functions

A branch of mathematics called quantum calculus, commonly referred to as q-calculus, studies calculus without the concept of limits. Results of q-calculus can be linked to Euler's research. The concepts of the definite q-integrals and the q-derivatives were initially introduced by Jackson. Numerous branches of mathematics, including orthogonal polynomials, hypergeometric functions, number theory, and combinatorics, as well as physics subjects including mechanics, relativity theory, and quantum theory, have found use for quantum calculus [29].

We can obtain a number of q-fractional derivatives and integrals involving the Laplace and Fourier transforms of the new representation by using [29,30]

$$\left(\frac{d}{dz}\right)_q s^{\nu m} = [\nu m]_q s^{\nu m-1}, \tag{57}$$

i.e., by making the use of (41) and (42), we obtain

$$\begin{aligned}\left(\frac{d}{dz}\right)_q e^{-\iota xs} exp(-e^{-\iota s}) &= 2\pi \sum_{l,m=0}^{\infty} \frac{(-1)^l(-\iota(x+l))^m}{l!m!} \left(\frac{d}{dz}\right)_q (s)^m \\ &= 2\pi \sum_{l,m=0}^{\infty} \frac{(-1)^l(-\iota(x+l))^{\nu m}[\nu m]_q (s)^{m(n-1)}}{l!m!},\end{aligned} \tag{58}$$

and

$$\begin{aligned}\left(\frac{d}{dz}\right)_q L(\Gamma_\nu(x+\iota y); s) &= 2\pi \sum_{l,m=0}^{\infty} \frac{(-1)^l(-\iota(x+l))^{\nu m}}{l!\Gamma(\nu m+1)} \left(\frac{d}{dz}\right)_q (s)^{\nu m} \\ &= 2\pi \sum_{l,m=0}^{\infty} \frac{(-1)^l(-\iota(x+l))^{\nu m}[\nu m]_q (s)^{\nu m-1}}{l!\Gamma(\nu m+1)} = 2\pi \left(\frac{d}{dz}\right)_q \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} E_\nu(\iota(x+l)\omega)^\nu,\end{aligned} \tag{59}$$

and the corresponding fractional q-derivatives are

$$\begin{aligned}{}_0^{RL}D_q^\alpha (L(\Gamma_\nu(x+\iota y); s)) &= 2\pi \sum_{l,m=0}^{\infty} \frac{(-1)^l(-\iota(x+l))^{\nu m}}{l!\Gamma(\nu m+1)} {}_0^{RL}D_q^\alpha (s)^{\nu m} \\ &= \frac{2\pi}{s^\alpha} \sum_{l,m=0}^{\infty} \frac{(-1)^l \Gamma_q(\nu m+1)}{l!\Gamma(\nu m+1)\Gamma_q(\nu m-\alpha+1)} (-\iota(x+l)s)^{\nu m} \\ &= 2\pi {}_0^{RL}D_q^\alpha \left(\sum_{l=0}^{\infty} \frac{(-1)^l}{l!} E_\nu(\iota(x+l)\omega)^\nu\right),\end{aligned} \tag{60}$$

as well as the corresponding fractional q-integrals are

$$\begin{aligned}{}_0^{RL}I_q^\alpha (L(\Gamma_\nu(x+\iota y); s)) &= 2\pi \sum_{l,m=0}^{\infty} \frac{(-1)^l(-\iota(x+l))^{\nu m}}{l!\Gamma(\nu m+1)} {}_0^{RL}I_q^\alpha (L(\Gamma_\nu(x+\iota y); s)) \\ &= 2\pi \sum_{l,m=0}^{\infty} \frac{(-1)^l(-\iota(x+l))^{\nu m}}{l!\Gamma(\nu m+1)} {}_0^{RL}I_q^\alpha (s)^{\nu m} \\ &= \frac{2\pi}{s^\alpha} \sum_{l,m=0}^{\infty} \frac{(-1)^l \Gamma_q(\nu m+1)}{l!\Gamma(\nu m+1)\Gamma_q(\nu m+\alpha+1)} (-\iota(x+l)s)^{\nu m} = 2\pi {}_0^{RL}I_q^\alpha \left(\sum_{l=0}^{\infty} \frac{(-1)^l}{l!} E_\nu(\iota(x+l)\omega)^\nu\right)\end{aligned} \tag{61}$$

Continuing in this manner, it is reasonable to apply the results for the solution of q-fractional integrodifferential equations. For example,

$$\left(\frac{d}{dz}\right)_q \mathcal{F}(\Gamma_\nu(x+\iota y); \omega) = 2\pi \sum_{l,m=0}^{\infty} \frac{(-1)^l((x+l))^{\nu m}}{l!\Gamma(\nu m+1)} \left(\frac{d}{dz}\right)_q (\omega)^{\nu m} = \frac{2\pi}{\omega} \sum_{l,m=0}^{\infty} \frac{(-1)^l(\omega(x+l))^{\nu m}[\nu m]_q}{l!\Gamma(\nu m+1)},$$

and the corresponding fractional q-derivatives are

$${}_0^{RL}D_q^\alpha F(\Gamma_\nu(x+\iota y); \omega) = 2\pi \sum_{l,m=0}^{\infty} \frac{(-1)^l((x+l))^{\nu m}}{l!\Gamma(\nu m+1)} {}_0^{RL}D_q^\alpha (\omega)^{\nu m} = \frac{2\pi}{\omega^\alpha} \sum_{l,m=0}^{\infty} \frac{(-1)^l(\omega(x+l))^{\nu m}\Gamma_q(\nu m+1)}{l!\Gamma(\nu m+1)\Gamma_q(\nu m-\alpha+1)},$$

as well as the corresponding fractional q-integrals are

$${}_0^{RL}I_q^\alpha \mathcal{F}(\Gamma_\nu(x+\iota y); \omega) = 2\pi \sum_{l,m=0}^{\infty} \frac{(-1)^l((x+l))^{\nu m}}{l!\Gamma(\nu m+1)} {}_0^{RL}I_q^\alpha (\omega)^{\nu m} = \frac{2\pi}{\omega^\alpha} \sum_{l,m=0}^{\infty} \frac{(-1)^l(\omega(x+l))^{\nu m}\Gamma_q(\nu m+1)}{l!\Gamma(\nu m+1)\Gamma_q(\nu m+\alpha+1)}.$$

4. Conclusions

We obtained a new representation of the gamma function over its complex domain in terms of the fractional derivatives of the delta function. The complex portion contributes to the coefficients of this series representation, and the fractional derivatives are used for the real part. Therefore, the new representation is also applicable to a class of functions whose fractional derivatives over the complex domain do not exist. Highly nontrivial examples, leading to the solution of the most general form of the fractional integrodifferential equation, are also discussed. Using the Fourier transform of the gamma function, novel identities containing the gamma function are obtained using multiple Erdélyi–Kober (E-K) fractional derivatives, and their fractional integral transforms are computed. Besides the conventional solution of the singular fractional integral equation, the generalized solution is also obtained using the non-integer derivatives of the delta function. By doing so, we obtain a large class of integrals and derivatives in a uniform way, i.e., q-fractional derivatives using the new representation are also computed. It demonstrates how the employed approach may enhance the upcoming applications of the delta function in big data, machine learning, and artificial intelligence [31]. We conclude that this work is more fruitful over the others presented in [32] and the references provided therein, because those contain only the delta function, but here, we use the fractional derivatives of the Dirac delta function. This study sheds more light on the potential of the new representations developed in the references [32–34] and their related works, to achieve a broader applicability compared to other results, for example, refer [35–37].

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References

1. Gel'fand, I.M.; Shilov, G.E. *Generalized Functions: Properties and Operations*; Academic Press: New York, NY, USA, 1969; Volumes I–V.
2. Lebedev, N.N. *Special Functions and Their Applications*; Prentice Hall: Englewood Cliffs, NJ, USA, 1965.
3. Chaudhry, M.A.; Qadir, A. Fourier transform and distributional representation of gamma function leading to some new identities. *Int. J. Math. Math. Sci.* **2004**, *37*, 2091–2096. [CrossRef]
4. Tassaddiq, A.; Srivastava, R.; Kasmani, R.M.; Almutairi, D.K. Distributional Representation of a Special Fox–Wright Function with an Application. *Mathematics* **2023**, *11*, 3372. [CrossRef]
5. Zamanian, A.H. *Distribution Theory and Transform Analysis*; Dover Publications: New York, NY, USA, 1987.
6. Osler, T.J. Taylor's series generalized for fractional derivatives and applications. *SIAM J. Math. Anal.* **1971**, *2*, 37–47. [CrossRef]
7. Makris, N. The Fractional Derivative of the Dirac Delta Function and Additional Results on the Inverse Laplace Transform of Irrational Functions. *Fractal Fract.* **2021**, *5*, 18. [CrossRef]
8. Tassaddiq, A.; Cattani, C. Fractional distributional representation of gamma function and the generalized kinetic equation. *Alex. Eng. J.* **2023**, *82*, 577–586. [CrossRef]
9. Li, C.K.; Li, C.P. Remarks on fractional derivatives of distributions. *Tbil. Math. J.* **2017**, *10*, 1–18. [CrossRef]
10. Li, M. Integral Representation of Fractional Derivative of Delta Function. *Fractal Fract.* **2020**, *4*, 47. [CrossRef]
11. Feng, Z.; Ye, L.; Zhang, Y. On the fractional derivative of Dirac delta function and its application. *Adv. Math. Phys.* **2020**, *7*, 1842945. [CrossRef]
12. Kiryakova, V. Unified Approach to Fractional Calculus Images of Special Functions—A Survey. *Mathematics* **2020**, *8*, 2260. [CrossRef]
13. Tassaddiq, A.; Srivastava, R. New Results Involving Riemann Zeta Function Using Its Distributional Representation. *Fractal Fract.* **2022**, *6*, 254. [CrossRef]
14. Tassaddiq, A.; Srivastava, R. New Results Involving the Generalized Krätzel Function with Application to the Fractional Kinetic Equations. *Mathematics* **2023**, *11*, 1060. [CrossRef]
15. Srivastava, H.M. Some Parametric and Argument Variations of the Operators of Fractional Calculus and Related Special Functions and Integral Transformations. *J. Nonlinear Convex Anal.* **2021**, *22*, 1501–1520.

16. Srivastava, H.M.; Tomovski, Ž. Fractional calculus with an integral operator containing a generalized Mittag–Leffler function in the kernel. *Appl. Math. Comput.* **2009**, *211*, 198–210. [CrossRef]
17. Kilbas, A.A.; Saigo, M. *H-Transforms: Theory and Applications*; Chapman & Hall/CRC: Boca Raton, FL, USA; London, UK; New York, NY, USA, 2004.
18. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*, 1st ed.; Elsevier/North-Holland: Amsterdam, The Netherlands, 2006; ISBN 9780444518323.
19. Srivastava, H.M. An Introductory Overview of Fractional-Calculus Operators Based Upon the Fox-Wright and Related Higher Transcendental Functions. *J. Adv. Eng. Comput.* **2021**, *5*, 135–166. [CrossRef]
20. Mittag-Leffler, M.G. Sur la nouvelle fonction E(x). *C. R. Acad. Sci. Paris* **1903**, *137*, 554–558.
21. Garrappa, R.; Kaslik, E.; Popolizio, M. Evaluation of Fractional Integrals and Derivatives of Elementary Functions: Overview and Tutorial. *Mathematics* **2019**, *7*, 407. [CrossRef]
22. Srivastava, H.M. A Survey of Some Recent Developments on Higher Transcendental Functions of Analytic Number Theory and Applied Mathematics. *Symmetry* **2021**, *13*, 2294. [CrossRef]
23. Srivastava, H.M.; Fernandez, A.; Baleanu, D. Some New Fractional-Calculus Connections between Mittag–Leffler Functions. *Mathematics* **2019**, *7*, 485. [CrossRef]
24. Tomovski, Ž.; Hilfer, R.; Srivastava, H.M. Fractional and operational calculus with generalized fractional derivative operators and Mittag-Leffler type functions. *Integral Transform. Spec. Funct.* **2010**, *21*, 797–814. [CrossRef]
25. Haubold, H.J.; Mathai, A.M. The fractional kinetic equation and thermonuclear functions. *Astrophys. Space Sci.* **2000**, *273*, 53–63. [CrossRef]
26. Saxena, R.K.; Mathai, A.M.; Haubold, H.J. Unified fractional kinetic equations and a fractional diffusion equation. *Astrophys. Space Sci.* **2004**, *290*, 299–310. [CrossRef]
27. Hilfer, R.; Anton, L. Fractional master equations and fractal time random walks. *Phys. Rev. E* **1995**, *51*, R848–R851. [CrossRef] [PubMed]
28. Kumar, D.; Choi, J.; Srivastava, H.M. Solution of a general family of kinetic equations associated with the Mittag-Leffler function. *Nonlinear Funct. Anal. Appl.* **2018**, *23*, 455–471.
29. Wongsantisuk, P.; Ntouyas, S.K.; Passary, D.; Tariboon, J. Hilfer Fractional Quantum Derivative and Boundary Value Problems. *Mathematics* **2022**, *10*, 878. [CrossRef]
30. Srivastava, H.M. Fractional-Order Derivatives and Integrals: Introductory Overview and Recent Developments. *Kyungpook Math. J.* **2020**, *60*, 73–116.
31. Niu, H.; Chen, Y.; West, B.J. Why Do Big Data and Machine Learning Entail the Fractional Dynamics? *Entropy* **2021**, *23*, 297. [CrossRef]
32. Tassaddiq, A.; Qadir, A. Fourier transform and distributional representation of the generalized gamma function with some applications. *Appl. Math. Comput.* **2011**, *218*, 1084–1088. [CrossRef]
33. Tassaddiq, A.; Alruban, A. On Modifications of the Gamma Function by Using Mittag-Leffler Function. *J. Math.* **2021**, *2021*, 9991762. [CrossRef]
34. Al-Lail, M.H. A Fourier Transform Representation of the Generalized Hypergeometric Functions with Applications to the Confluent and Gauss Hypergeometric Functions. Ph.D. Thesis, KFUPM, Dhahran, Saudi Arabia, 2015.
35. Chahma, Y.; Chen, H. Infinitely many high energy solutions for fourth-order elliptic equations with p-Laplacian in bounded domain. *J. Math. Comput. SCI-JM* **2023**, *32*, 109–121. [CrossRef]
36. Abdullah, T.Q.S.; Xiao, H.; Huang, G.; Al-Sadi, W. Stability and existence results for a system of fractional differential equations via Atangana-Baleanu derivative with $\phi_{\{p\}}$-Laplacian operator. *J. Math. Comput. SCI-JM* **2022**, *27*, 184–195. [CrossRef]
37. Nikan, O.; Molavi-Arabshai, S.M.; Jafari, H. Numerical simulation of the nonlinear fractional regularized long-wave model arising in ion acoustic plasma waves. *Discret. Contin. Dyn. Syst. S* **2021**, *14*, 3685–3701. [CrossRef]