End-of-the-world branes in AdS/BCFT

by

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Abstract

We investigate whether simplified models of anti-de Sitter space (AdS) terminated by an end-of-the-world (ETW) brane are a healthy dual to boundary conformal field theory (BCFT). Recent studies have shown that null trajectories starting from AdS boundary traveling to the ETW brane and back to AdS boundary can lead to singularities in the two-point function. Because such singularities does not exist on the BCFT side, they are detrimental to the healthiness of AdS/BCFT duality. We note that these singularities in BCFT two-point functions are unphysical if light takes infinite time to travel from the AdS boundary to the ETW brane in the gravitational dual. Hence, we propose the condition for which the light crossing time between the AdS boundary and the ETW brane takes infinite time as a potential criterion to determine whether a bulk gravitational theory is a healthy dual to a BCFT.

In order to justify our proposal, we tested this criterion in several configurations. We first show that simplified models of empty AdS space terminated by an ETW brane do not satisfy this criterion. Then, we uncovered that adding matter in the form of a massive scalar field pushes light crossing time in the healthy direction. Next, the criterion is tested in a known stable solution of non-SUSY Janus to confirm that our criterion is in agreement. Lastly, we embedded ETW branes in three examples of AdS/CFT to see if such configurations could exist. We found a solution but the ETW brane configurations were unstable, which is supported by our solution not satisfying the light crossing time criterion.

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Chapter 1

Introduction

Quantum gravity is a quantum theory whose low energy dynamics around weakly curved spacetime reduces to general relativity with fluctuations treated quantum mechanically. Such theories are needed to resolve outstanding problems in fundamental physics. For example, singularities in spacetimes are inevitable following general relativity, including black holes and potentially our universe. Additionally, it is a natural motivation to unify all interactions in one complete theory, especially with the success of the standard model.

One idea of implementing gravity into the standard model is to introduce the force carrier particle of graviton. String theory constructs a quantum theory of all interactions for which gravity emerges in some limit, with the graviton as an excitation of closed strings. Some of the features include supersymmetry (SUSY) as well as the presence of higher dimensions. The duality between Anti-de Sitter space (AdS) and conformal field theory (CFT) called AdS/CFT correspondence offers a tool for testing string theory predictions. Namely, the strongly coupled quantum field theory (QFT) can be described by weakly coupled gravity in AdS space and vice versa. Also, AdS/CFT offers an important realization of the holographic principle, providing a non-perturbative formulation of string theory. Because we can treate metric fluctuations in AdS space with AdS radius $L \ll$ string length ℓ_P as a background for gravitons weakly coupled at low energies, one needs quantum system that reproduces dynamics of low energy gravitons in large AdS.

In particular, we are interested in how spacetime can end. This question is relevant in big bang cosmology and the study of black holes as they serve as instances in which spacetime could end. Although there are top down models in string theory with terminating spacetime, such models are complicated. Therefore, one can consider an object called end-of-the-world (ETW) branes that terminate spacetime in the context of AdS/CFT. Takayanagi [1] proposed that such simplified configuration is dual to a CFT with a boundary, which is called boundary conformal field theory (BCFT). This duality has produced some promising results including the matching of entropy computations [1, 2]. Additionally, AdS/BCFT has been very useful for understanding the information loss problem in black hole evaporation. Namely, the second saddle point in Hawking radiation to produce the Page curve is justified through the use of double holography [3, 4].

However, it is still in question whether the bulk theory of AdS spacetime terminated by ETW brane is dual to a BCFT, as mulitple causality violation has been noted as an issue in AdS/BCFT [5, 6]. Therefore, it is important to investigate when such configuration admits a healthy duality.

1.1 Brief Overview of AdS/CFT

This section provides a brief overview of AdS/CFT following TASI Lectures on AdS/CFT [7]. The first subsection covers CFT while the second covers AdS space. By doing so, we will see the duality between the correlators in AdS and CFT, which leads to the AdS/CFT correspondence in the third section.

1.1.1 Conformal Field Theory

Conformal field theory is a theory that preserves conformal symmetry. Starting first with the conformal transformation, which is a coordinate transformation from $x \to \tilde{x}$ that preserves the metric up to a scale factor

$$g_{\mu\nu} \frac{d\tilde{x}^\mu}{dx^\alpha} \frac{d\tilde{x}^\nu}{dx^\beta} = \Omega^2(x) g_{\alpha\beta} \,. \tag{1.1}$$

The Euclidean conformal transformation forms the $SO(d+1,1)$ group with generators corresponding to translation, rotation, dilatation, and special conformal transformation. The commutation relations of these generators form the conformal algebra.

One can define local operators in such a field theory. Primary operators have the properties of being annihilated by special conformal transformation generators at the origin, eigenvectors of dilatation generator with eigenvalue Δ, and generate irreducible representation of rotation group $SO(d)$. Any derivatives of primary operators and their linear combinations are called descendant operators. When the insertion points of two operators approach each other, the operator product can be expressed as a linear combinations of local operators with coefficients C_{ijk} in the operator product expansion (OPE)

$$\mathcal{O}_i(x)\mathcal{O}_j(0) = \sum_k C_{ijk} |x|^{\Delta_k - \Delta_i - \Delta_j} \mathcal{O}_k(0) \,. \tag{1.2}$$

The correlation functions are important observables in CFT. Some explicit examples are

$$\begin{aligned}
&\langle \mathcal{O}(x) \rangle = 0\,, \\
&\langle \mathcal{O}_i(x_i)\mathcal{O}_j(x_j) \rangle = \frac{\delta_{ij}}{(x_i - x_j)^{2\Delta_i}}\,, \\
&\langle \mathcal{O}_i(x_i)\mathcal{O}_j(x_j)\mathcal{O}_k(x_k) \rangle = \frac{C_{ijk}}{|x_{ij}|^{\Delta_i+\Delta_j-\Delta_k}|x_{ik}|^{\Delta_i+\Delta_k-\Delta_j}|x_{jk}|^{\Delta_j+\Delta_k-\Delta_i}}\,, \\
&\langle \mathcal{O}(x_i)\mathcal{O}(x_j)\mathcal{O}(x_k)\mathcal{O}(x_l) \rangle = \frac{A(u,v)}{(x_{ik}^2 x_{jl}^2)^{\Delta}}\,,
\end{aligned} \tag{1.3}$$

where the four-point function is in terms of two conformally invariant cross ratios

$$u = \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}\,, \qquad v = \frac{x_{il}^2 x_{jk}^2}{x_{ik}^2 x_{jl}^2}\,. \tag{1.4}$$

Under the conformal transformation, correlation functions transform as

$$\langle \mathcal{O}_1(x_1)...\mathcal{O}_n(x_n) \rangle_{\Omega^2 g} = \frac{\langle \mathcal{O}_1(x_1)...\mathcal{O}_n(x_n) \rangle_g}{|\Omega(x_1)|^{\Delta_1}...|\Omega(x_n)|^{\Delta_n}}\,. \tag{1.5}$$

Using the OPE successively, one can derive any correlation function with the knowledge of CFT data: scaling dimensions Δ, $SO(d)$ irreducible representations of all primary operators, and the OPE coefficients C_{ijk}. These CFT data satisfy several constraints to give us more information. Firstly, the OPE must be associative, meaning that the different combinations of performing OPE must provide the same result. Imposing this constraint to the four-point function, we get the conformal bootstrap equation

$$\sum_k C_{12k} C_{k34} \mathcal{G}^{(12)(34)}_{\Delta_k, l_k}(x_1, ..., x_4) = \sum_q C_{13q} C_{q24} \mathcal{G}^{(13)(24)}_{\Delta_q, l_q}(x_1, ..., x_4)\,, \tag{1.6}$$

where $\mathcal{G}_{\Delta,l}$ are the conformal blocks encoding information of primary operator dimension Δ and spin l. Secondly, there must exist a stress-energy tensor as a primary operator of dimension $\Delta = d$ that is symmetric $T_{\mu\nu} = T_{\nu\mu}$, traceless, and conserved $\nabla_\mu T^{\mu\nu} = 0$. Lastly, the unitarity imposes lower bounds on scaling dimensions of local operators.

Now consider a $U(N)$ gauge theory with fields in the adjoint representation, giving the action

$$S = \frac{N}{\lambda} \int dx \mathrm{Tr}\left[(D\phi)^2 + c_3 \phi^3 + c_4 \phi^4 + ... \right]\,. \tag{1.7}$$

The constants $\lambda = g_{YM}^2 N$, g_{YM}, and c_i are the 't Hooft coupling, Yang-Mills coupling, and interaction vertex coupling, respectively. The vacuum diagram with V vertices, P propagators, and L lines scales as

$$\left(\frac{N}{\lambda}\right)^{V}\left(\frac{\lambda}{N}\right)^{P}N^{L}=\left(\frac{N}{\lambda}\right)^{\chi}\lambda^{L}\,, \tag{1.8}$$

where $\chi = V+L-P = 2-2g$. In large N limit, one can see that the dominant contribution comes from planar diagrams ($g=0$). This expansion of λ/N as string coupling is exactly realized in maximally supersymmetric Yang-Mills theory (SYM). Considering a single-trace local operators $\mathcal{O} = c_J \mathrm{Tr}(\phi^J)$ with normalization constant c_J, the connected correlators in large N expansion is given by

$$\langle \mathcal{O}_1 ... \mathcal{O}_n \rangle = \sum_{g=0}^{\infty} N^{2-n-2g} f_g(\lambda)\,. \tag{1.9}$$

The important result of large N factorization is that the two-point function is independent of N while higher point functions are suppressed by powers of N. Therefore, single-trace operators in large N CFT acts as a single particle state of weakly coupled theory.

1.1.2 Anti-de Sitter Space

This subsection gives a brief overview of AdS space, which is a negatively curved maximally symmetric space. Due to the maximal symmetry, the simplest term that can be added to the Einsten Hilbert action is the cosmological constant

$$S = \frac{1}{16\pi G}\int d^{d+1}x\sqrt{g}(R+2\Lambda)\,. \tag{1.10}$$

Then, we have the Einstein equation

$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} = \Lambda g_{\mu\nu}\,, \tag{1.11}$$

and the Ricci scalar

$$R = \frac{2(d+1)}{d-1}\Lambda\,, \tag{1.12}$$

which means that the curvature depends on the cosmological constant. The maximal symmetry also gives the form of Riemann tensor as

$$R_{\mu\nu\rho\sigma} = \frac{R}{d(d+1)}(g_{\nu\sigma}g_{\mu\rho} - g_{\nu\rho}g_{\mu\sigma})\,, \tag{1.13}$$

which gives

$$R_{\mu\nu} = \frac{R}{d+1} g_{\mu\nu} \,, \tag{1.14}$$

consistent with the Einstein's equation.

There are many forms of AdS metric with the AdS radius L given by the cosmological constant

$$\Lambda = -\frac{d(d-1)}{2L^2} \,. \tag{1.15}$$

Starting with the global AdS, we have

$$ds^2 = L^2 \left(-\cosh^2\rho \; dt^2 + d\rho^2 + \sinh^2\rho \; d\Omega_{d-1}^2\right) \,, \tag{1.16}$$

where the AdS boundary is located at $\rho = \infty$. By performing a coordinate transformation $\tanh\rho = \sin r$, we have another metric

$$ds^2 = \frac{L^2}{\cos^2 r}\left(-dt^2 + dr^2 + \sin^2 r \; d\Omega_{d-1}^2\right) \,, \tag{1.17}$$

which is conformal to a solid cylinder with radius r from 0 to the boundary $\frac{\pi}{2}$. Alternatively, we can use $\cos r = \sin\theta$ to get

$$ds^2 = \frac{L^2}{\sin^2\theta}\left(-dt^2 + d\theta^2 + \cos^2\theta \; d\Omega_{d-1}^2\right) \,. \tag{1.18}$$

Another useful coordinate is the Poincaré patch

$$ds^2 = L^2 \frac{dz^2 + g_{\mu\nu}dx^\mu dx^\nu}{z^2} \,. \tag{1.19}$$

It is important to note that the Poincaré patch covers the entire Euclidean AdS, but not Lorentzian AdS.

Due to the maximal symmetry, there are $d+1$ translations, d boosts, and $\frac{d(d-1)}{2}$ rotations for AdS_{d+1}. AdS spacetime has $SO(2,d)$ in Lorentzian signature and $SO(d+1,1)$ in Euclidean signature, which is same as the conformal isometry group.

Now consider a scalar field or particle in AdS. The action

$$S = \int d^{d+1}x \left(\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{3!}g\phi^3\right) \,, \tag{1.20}$$

has a cubic interaction term with coupling g, which gives the Klein Gordon equation

$$\nabla^2\phi = m^2\phi + \frac{g}{2}\phi^2 \,. \tag{1.21}$$

For $g = 0$, one can show that the quadratic Casimir of AdS isometry group acting on a scalar field satisfies

$$\frac{1}{2}J_{AB}J^{AB}\phi = L^2\nabla^2_{AdS}\phi \,, \tag{1.22}$$

where the linear combinations of J_{AB} can be used to define the conformal generators. In analogy with CFT primary operators, one can consider a scalar field annihilated by "special conformal transformation" generator and eigenstate of "dilatation" with eigenvalue Δ. Then, we can obtain the result

$$m^2L^2 = \Delta(\Delta - d) \,. \tag{1.23}$$

Without interaction, the energy of multi-particle state is just the sum of individual particle energy. However, turning on small interactions causes energy shifts of multi-particle state, similar to the space of local operators in large N CFT of single-trace operators. Adding weak interaction and sending all points to AdS boundary, the two-point function simplifies precisely to the CFT two-point function of primary operators

$$\langle \mathcal{O}(P_1)\mathcal{O}(P_2)\rangle = \frac{1}{(-2P_1\cdot P_2)^{\Delta}} + O(g^2) \,, \tag{1.24}$$

where P_i adapts the coordinates in embedding formalism of [8]. One can also check the associativity of OPE by using the fact that Hilbert space of bulk theory can be decomposed in irreducible representations of the conformal group. Then, the conformal block decomposition of four-point function can be performed as

$$\langle \mathcal{O}(P_1)...\mathcal{O}(P_4)\rangle = G_{0,0}(P_1,...,P_4) + \sum_{l=even}^{\infty}\sum_{n=0}^{\infty} c_{n,l}G_{2\Delta+2n+l,l}(P_1,...,P_4) \,, \tag{1.25}$$

Now what remains is to check the existence and the properties of stress-energy tensor. Consider AdS space scalar field

$$S = \frac{1}{\ell_P^{d-1}}\int d^{d+1}x\sqrt{G}\left(R - 2\Lambda + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right) \,, \tag{1.26}$$

satisfying the boundary conditions

$$ds^2 = L^2 \frac{dz^2 + dx^\mu dx^\nu [g_{\mu\nu} + O(z)]}{z^2}\,, \tag{1.27}$$

$$\phi = \frac{z^{d-\Delta}}{2\Delta - d}[\phi_0(x) + O(z)]\,. \tag{1.28}$$

The partition function is invariant under diffeomorphism of the boundary metric

$$Z[\Omega^2 g_{\mu\nu}, \Omega^{\Delta-d}\phi_0] = Z[g_{\mu\nu}, \phi_0]\,, \tag{1.29}$$

but the divergence of the partition function needs to be regulated by imposing a cutoff, which is not Weyl invariant. Luckly, the correlation functions are independent of the UV cutoff.

In semiclassical limit $\ell_P \ll L$, one can find the connected correlateors of the stress tensor as

$$\langle T_{\mu_1\nu_1} ... T_{\mu_n\nu_n} \rangle \sim \left(\frac{L}{\ell_P}\right)^{d-1}\,, \tag{1.30}$$

which matches the scaling found in large N factorization with $N^2 \sim \left(\frac{L}{\ell_P}\right)^{d-1}$. Hence, we can conclude that CFT is dual to semiclassical gravity in AdS with large degrees of freedom.

1.1.3 AdS/CFT Correspondence

The duality between CFT and gravity in AdS space is called AdS/CFT correspondence, a useful tool for understanding quantum gravity. As we have seen in previous sections, single-trace primary operator with scaling dimension Δ is dual to weakly coupled fields in AdS space with mass $m^2L^2 = \Delta(\Delta - d)$. Finding such a CFT with large N factorization, correct spectrum of single-trace operators, and stress tensors with required properties is not an easy task.

The first example of AdS/CFT introduced comes from type IIB string theory with N coincident D3-branes, interacting with closed strings propagating in 10d Minkowski spacetime. This interaction can be described in two different ways.

The first is closed strings interacting with D3-branes by breaking the string loop into open strings ending on D3-branes. In the low energy limit of string length $\ell_s \to 0$, this is described by $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) with $SU(N)$ gauge group. The second is D3-branes defined as solitons of closed string theory, forming curved background of the form

$$ds^2 = \frac{1}{\sqrt{1+\frac{L^4}{r^4}}}\eta_{\mu\nu}dx^\mu dx^\nu + \sqrt{1+\frac{L^4}{r^4}}\left(dr^2 + r^2 d\Omega_5^2\right) , \tag{1.31}$$

where $L^4 = 4\pi g_s N \ell_s^4$. In the low energy limit around $r = 0$, one can define a new coordinate $z = L^2/r$ to get the metric of $\mathrm{AdS}_5 \times S^5$ type IIB string theory

$$ds^2 = L^2 \frac{dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu}{z^2} + L^2 d\Omega_5^2 . \tag{1.32}$$

This leads to the AdS/CFT conjecture of Maldacena [9] that $SU(N)$ SYM with $g_{YM}^2 = 4\pi g_s$ is dual to $\mathrm{AdS}_5 \times S^5$ type IIB string theory with $\frac{L^4}{\ell_s^4} = g_{YM}^2 N = \lambda$. This duality can be confirmed by computing observables such as correlators. However, such computation are difficult and can only be performed in certain limits. One thing we can check is the Kaluza-Klein reduction on $\mathbb{S}^5$, which gives the Planck length to be $N^2 \sim L^3/\ell_P^3$. In large coupling $\lambda \gg 1$ limit, the only CFT operator with small scaling dimension is dual to massless string states of type IIB supergravity (SUGRA). Thus, SYM with $N \gg \lambda \gg 1$ provides a UV completion of $\mathrm{AdS}_5 \times S^5$ type IIB SUGRA.

1.2 AdS/BCFT

We now give a brief overview of AdS/BCFT, the main ingredient of this thesis. We first extend our understanding of CFT to a BCFT. Then, we define a holographic dual to BCFT by AdS/BCFT similar to how one can define holographic dual to CFT using AdS/CFT. Lastly, there is an overview of recent progress in AdS/BCFT relevant to this thesis.

1.2.1 Boundary Conformal Field Theory

Because boundary conditions have crucial impact on physical phenomena, it is important to consider a CFT with a boundary. Such systems form a bounday conformal field theory, and there are notable differences from ordinary CFTs. Consider a CFT on d-dimensions with $SO(2,d)$ symmetry cutoff by a $(d-1)$-dimensional time-like boundary. When the boundary maximally preserves the conformal symmetry, this gives rise to BCFT with $SO(2,d-1)$. Important results of BCFT are obtained by McAvity and Osborn [10], including the BCFT correlators which are an integral part of our analysis.

The spectrum of CFT operators (primary and descendant) and their algebra remains unchanged. Instead, there are additional, boundary operators localized on the boundary as representations of $SO(2,d-1)$ with boundary conformal dimension Δ_I as eigenvalue of unbroken dilatation operator. The CFT operators satisfy the usual OPE, but boundary operators obey the boundary operator expansion (BOE), where CFT operators can be expressed as linear combinations of boundary operators

$$\mathcal{O}_i(x) = \frac{A_{\mathcal{O}}}{(2x_\perp)^\Delta} + \sum_I \frac{B_{iI}}{(2x_\perp)^{\Delta_i+\Delta_I}} C[x_\perp, \partial_{\vec{x}}]\mathcal{O}_i(x_0, \vec{x})\,, \tag{1.33}$$

A BCFT one-point function behaves similar to a CFT two-point function

$$\langle \mathcal{O}(x) \rangle = \frac{A_{\mathcal{O}}}{(2x_\perp)^\Delta}\,, \tag{1.34}$$

and similarly, a BCFT two-point function behaves as a CFT four-point function

$$\langle \mathcal{O}_1(x)\mathcal{O}_2(y) \rangle = \frac{\mathcal{G}(\xi)}{|2x_\perp|^{\Delta_1}|2y_\perp|^{\Delta_2}}\,, \tag{1.35}$$

given in terms of the conformally invariant cross-ratio

$$\xi = \frac{(x-y)^2}{4x_\perp y_\perp}\,. \tag{1.36}$$

This can be explained using the Cardy's doubling trick, where a BCFT correlator is characterized by the insertion points and their mirrored images across the boundary, each with half the conformal weight.

The BCFT two-point function can be expanded in two equivalents limits: taking the operators to approach each other $\xi \to 0$ or taking the operators to the boundary $\xi \to \infty$. Both approaches should give the same result due to crossing symmetry. This gives $\mathcal{G}$ as a linear combinations of bulk or boundary conformal blocks, where the coefficient is determined by OPE and one-point function for bulk, BOE coefficient for boundary.

$$\begin{aligned} \text{Bulk: } \mathcal{G}(\xi) &= \sum_i C^i A_i g^B_{\Delta_i}(\xi) \\ \text{Boundary: } \mathcal{G}(\xi) &= \sum_I B_i^2 g^b_{\Delta_I}(\xi) \end{aligned} \tag{1.37}$$

The conformal blocks can be obtained by solving Casimir equations to be

$$\begin{aligned} \text{Bulk: } g^B_{\Delta_i}(\xi) &= \xi^{\Delta/2-\Delta_{ext}}\ {}_2F_1\left(\frac{\Delta}{2},\frac{\Delta}{2},\Delta-\frac{d}{2}+1,-\xi\right) \\ \text{Boundary: } g^b_{\Delta_I}(\xi) &= \xi^{-\Delta}\ {}_2F_1\left(\Delta,\Delta-\frac{d}{2}+1,2\Delta+2-d,-\xi^{-1}\right) \end{aligned} \tag{1.38}$$

Because Euclidean CFT correlator has singularities when two operators approach each other, BCFT correlator in Lorentzian signature have singularities when operators approach each other or operators approach the boundary (approach mirror). This amounts to the branch point singularities in $\mathcal{G}$ at $\xi \to 0, \infty$, which is also clearly visible from the conformal blocks. Additionally, there is another singularity associated with operators approaching lightcone of each other ($\xi \to 0$) or the mirror image ($\xi \to -1$ Regge Limit). The singular behavior at Regge limit is obtained in [11] by introducing the radial coordinate ρ with

$$\xi = \frac{(1-\rho)^2}{4\rho}\,, \tag{1.39}$$

in which the BCFT two-point function has singular behavior at Regge limit $\rho \to -1$ as

$$\mathcal{G}(\rho) \lesssim (1+\rho)^{-(\Delta_1+\Delta_2)}\,, \tag{1.40}$$

1.2.2 AdS/BCFT Double Holography

As both QFT with boundary and BCFT have been studies extensively, it is natural to investigate the corresponding AdS/CFT with a boundary. Let us first consider the gravitational dual to BCFT proposed by Tadashi Takayanagi in 2011 [1] known as AdS/BCFT.

Consider a CFT on d-dimension manifold Σ with $SO(2,d)$ symmetry and a (d-1)-dimensional time-like boundary $\partial\Sigma$. When $\partial\Sigma$ maximally preserves the conformal symmetry, this gives rise to BCFT with $SO(2,d-1)$. Takayanagi proposed that such BCFT system (referred as BCFT picture) is dual to dynamical gravity living on (d+1) bulk M with boundaries composed of asymptotic boundary Σ and an ETW brane Q (referred as bulk picture). The correspondence of this duality is the AdS/BCFT.

This is also called double holography because there is another system (referred as intermediate picture) of d-dimensional CFT coupled to gravity on Q attached to a d-dimensional CFT on Σ with the common boundary $\partial\Sigma$. Performing ordinary AdS/CFT on gravitational system Q with boundary $\partial\Sigma$ gives back the BCFT picture, while AdS/CFT with $Q\cup\Sigma$ as a boundary of bulk M produces the bulk picture. Therefore, there are three equivalent dualities, giving the name double holography. The three pictures and the duality between each [3, 4] are summarized in Figure 1.1, following notations from Omiya and Wei [6].

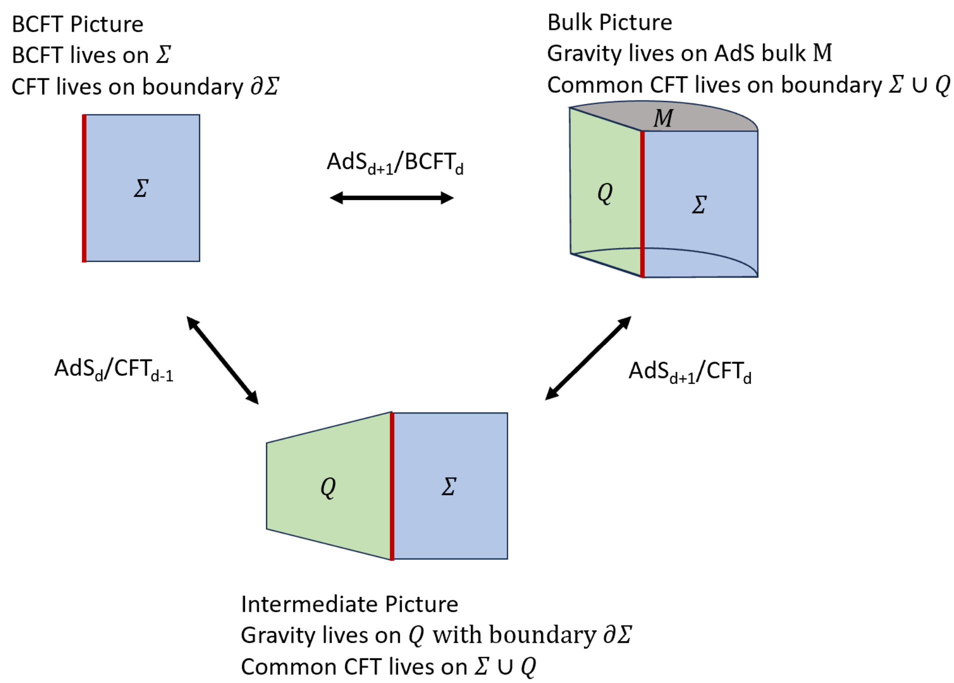

Figure 1.1: The three pictures in double holography with (d+1)-dimensional bulk M in gray, d-dimensional asymptotic boundary Σ in blue, d-dimensional ETW brane Q in green, and (d-1)-dimensional common boundary $\partial\Sigma = \partial Q$ in red.

The action in concern is the sum of Einsten-Hilbert with cosmological constant and two Gibbons-Hawking terms for the boundaries Σ and Q

$$S = \frac{1}{16\pi G}\int_M \sqrt{-G}(R - 2\Lambda) + \frac{1}{8\pi G}\int_\Sigma \sqrt{-\gamma}B + \frac{1}{8\pi G}\int_Q \sqrt{-h}(K - T)\,. \tag{1.41}$$

Here, G and R are bulk metric and Ricci scalar, γ and B are metric and trace of extrinsic curvature on Σ, h and K are metric and trace of extrinsic curvature on Q with tension T. Variation at the vicinity of Σ gives

$$\delta S = \frac{1}{16\pi G}\int_\Sigma \sqrt{-\gamma}(B_{ij} - B\gamma_{ij})\delta\gamma^{ij}\,, \tag{1.42}$$

which vanishes by imposing Dirichlet boundary condition $\delta\gamma^{ij} = 0$ on Σ. On the other hand, variation at the vicinity of Q gives

$$\delta S = \frac{1}{16\pi G}\int_Q \sqrt{-h}(K_{ab} - Kh_{ab} + Th_{ab})\delta h^{ab}\,, \tag{1.43}$$

which we impose Neumann boundary condition $K_{ab} - Kh_{ab} = -Th_{ab}$. Because the symmetry is broken only in direction perpendicular to the ETW brane, we can look for solutions of the form

$$ds^2 = d\rho^2 + e^{2A(\rho)}ds^2_{AdS_d}\,, \tag{1.44}$$

where $A(\rho) = \log\cosh(\rho)$ as $\rho \to \infty$ to satisfy the asymptotically empty AdS condition. In this coordinate, the ETW brane is located at $\rho = \bar{\rho}$.

Let the metric of the ETW brane be

$$h_{ab} = g_{\mu\nu}\left(X(\sigma)\right)\partial_a X^\mu \partial_b X^\nu\,. \tag{1.45}$$

Then, the extrinsic curvature K_{ab} is defined by

$$K_{ab} = n_\mu D_a \partial_b X^\mu\,, \tag{1.46}$$

where n_μ is the normal to the ETW brane, which is equivalent to

$$n_\mu \partial_a X^\mu = 0\,. \tag{1.47}$$

Using the product rule

$$n_\mu D_a \partial_b X^\mu = D_a\left(n_\mu \partial_b X^\mu\right) - \partial_b X^\mu D_a n_\mu\,, \tag{1.48}$$

and the fact that n_μ is normal to the brane, the extrinsic curvature simplifies to

$$K_{ab} = -\partial_a n_b + \Gamma^\nu_{ab} n_\nu\,. \tag{1.49}$$

Since the metric is diagonal, the non-zero component of Christoffel symbol is given by

$$\Gamma^\rho_{ab} = -\frac{1}{2}h^{\rho\rho}\frac{\partial h_{ab}}{\partial \rho}\,. \tag{1.50}$$

Using the fact that n_μ is normal to the brane which is $n = -\partial_\rho$, then we get

$$K_{ab} = -\frac{1}{2}\partial_\rho h_{ab} \,. \tag{1.51}$$

We now solve for the location of the ETW brane for this metric. First, the extrinsic curvature is given by

$$K_{ab} = -A' g_{ab} \,, \tag{1.52}$$

where g_{ab} is the non-$\rho\rho$ component of the bulk metric. By contracting, we arrive at the trace

$$K = g^{ab} K_{ab} = -A' d \,. \tag{1.53}$$

The boundary condition at the brane leads to

$$K_{ab} = (K - T) h_{ab} = -A' h_{ab} \,. \tag{1.54}$$

Then, the tension of the brane can be sovled as

$$T = (K + A') = (-d + 1) A' \,. \tag{1.55}$$

For empty AdS, we have $e^A = L \cosh(\frac{\rho}{L})$. This gives the extrinsic curvature

$$K_{ab} = \frac{-1}{L} \tanh(\frac{\rho}{L}) g_{ab} \,, \tag{1.56}$$

and the tension at the ETW brane

$$T = \frac{d-1}{L} \tanh(\frac{\rho}{L}) \,. \tag{1.57}$$

1.2.3 Relevant Research Progress

All of this so far has been theoretical proposal, and the double duality must be confirmed by checking that the observables match in both context. One of the supporting arguments for AdS/BCFT comes from the entropy calculations associated with the boundary degrees of freedom. Different calculations in (1+1)-dimensional AdS/BCFT with constant tension brane show that the boundary entropy is well-defined and independent of temperature or size of the bulk system, as expected from BCFT. In (2+1)-dimension, however, the boundary entropy is positive/negative for the ETW brane with negative/positive tension, which comes from the fact that the impurity causes repulsive/attractive interaction [2].

These results seem promising, but one still needs to be careful. It was noted that the effective theory of the brane in the intermediate picture breaks down at high energy, meaning that it is only valid for small fluctuations around a fixed background (large N limit) [12]. This means that the more studied bulk/boundary dictionary cannot be translated directly to brane/bulk dictionary.

There have also been inconsistencies in the causal structures between the dual theories. In ordinary AdS/CFT, scattering processes connected in the bulk can be described by the dual theory on the boundary as disconnected, which may seem like a breakdown of the duality. However, this can be resolved because the input regions are entangled through a generalized connected wedge theorem. An additional problem occurs when ETW brane is introduced because the HRT entanglement entropy calculation shows that no entanglement is favored. Mori and Yoshida [5] suggested that one can introduce fictitious boundary behind ETW brane to induce local excitation on the brane. Utilizing the induced light cone from this fictitious boundary ensures that the scattering process is disconnected but still entangled.

Additionally, Omiya and Wei [6] have studied the causal structures in double holography. They found that causality in bulk picture is compatible with BCFT picture by the

generalized Gao-Wald theorem, while the intermediate picture requires superluminal and nonlocal effect in the effective theory.

With these results in hand, it is natural to question the existence of a healthy holographic dual to BCFT. Because signals can propagate not only in the boundary where CFT is defined but also in the bulk, one must consider correlators connected through the bulk. Reeves et al. [13] investigated the behavior of BCFT correlators in large conformal dimension Δ for which the insertion points are separated by null rays traveling in the bulk and bouncing off the ETW brane. Imposing necessary boundary conditions to bulk operators, they found an unexpected singularity in the BCFT two-point function for $\xi = \sin^2 \phi_b$, where $\phi_b = \int_{\tilde{\rho}}^{\infty} e^{-A(\rho)} d\rho$ indicates the location of the brane. This new potential singularity in the two-point function

$$\mathcal{G}(\rho) \sim (\rho_b - \rho)^{-2\Delta + \frac{d-1}{2}}\,, \qquad \rho_b = e^{i2\phi_b}\,, \tag{1.58}$$

though they do not believe to be a true singularity due to the neglected fluctuation of the brane, is worth further investigation.

Hence, we have considered the bulk properties of null rays starting from AdS boundary traveling to the ETW brane to see their behavior. In particular, we have arrived at a criterion one could use to determine if simplified models of AdS/BCFT are healthy.

Chapter 2

Light Crossing Time

We argue that the parameter ϕ_b is a useful tool to determine whether the bulk theory permits a healthy dual to BCFT. One can see from the form of the two-point function that for $\phi_b \geq \pi$, the singularity is on a second sheet because ρ_b is in the complex plane. We would like to give a physical explanation as to why such transition occurs at $\phi_b = \pi$ by considering the light crossing time.

2.1 Light Crossing Time in Poincaré Patch

Consider the holographic dual of a BCFT, where the bulk spacetime has a line element

$$ds^2 = d\rho^2 + e^{2A(\rho)} ds^2_{AdS_d}\,, \tag{2.1}$$

only existing for $\rho \geq \bar{\rho}$ with an ETW brane located at $\rho = \bar{\rho}$. By considering the Poincaré patch

$$ds^2_{AdS_d} = \frac{-dt^2 + d\vec{x}^2 + dz^2}{z^2}\,, \tag{2.2}$$

and performing a coordinate transformation $d\phi = -e^{-A} d\rho$, the line element (2.1) becomes

$$ds^2 = \frac{e^{2A(\rho)}}{z^2}\left(-dt^2 + d\vec{x}^2 + dz^2 + z^2 d\phi^2\right)\,. \tag{2.3}$$

From the coordinate transformation, we have

$$\phi = \int_\rho^\infty e^{-A(\rho')} d\rho'\,. \tag{2.4}$$

In this coordinate system, $\rho = \infty$ corresponds to $\phi = 0$ and we define ϕ_b to be the location of ETW brane at $\rho = \bar{\rho}$. We note that $-dt^2 + d\vec{x}^2$ is just a Minkowski metric while $dz^2 + z^2 d\phi^2$

is a polar coordinate metric with z as radius and ϕ as angle. Because one can always boost with appropriate momentum to eliminate any change in $\vec{x}$, the coordinate system can be represented schematically with a cylinder sector. Because null trajectories in this geometry are that of a flat space, they are represented by straight line. Geometrically, the null ray starting from AdS boundary cannot travel to the ETW brane if

$$\phi_b = \int_{\bar{\rho}}^{\infty} e^{-A} d\rho \geq \pi \, . \tag{2.5}$$

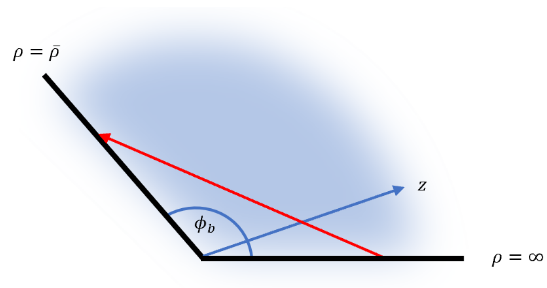

Figure 2.1: Null ray (red) traveling from the AdS boundary ($\rho = \infty$) to an ETW brane ($\rho = \bar{\rho}$) for $\phi_b < \pi$

We believe that this criterion that $\phi_b < \pi$ is a straightforward, yet strong condition to determine whether the bulk spacetime terminated by ETW brane is a healthy dual to BCFT. In the following chapters, this criterion is tested on various settings to support our argument.

2.2 Light Crossing Time in Global AdS

A similar computation can be performed with Poincaré patch replaced with global AdS to check consistency. The global AdS given in

$$ds^2_{AdS_d} = \frac{L^2}{\sin^2\theta}\left(-dt^2 + d\theta^2 + \cos^2\theta \; d\Omega^2_{d-2}\right) \, . \tag{2.6}$$

Then, the entire $d+1$-dimentional metric is given by

$$ds^2 = \frac{L^2 e^{2A}}{\sin^2\theta}\left(\sin^2\theta \; d\phi^2 - dt^2 + d\theta^2 + \cos^2\theta \; d\Omega^2_{d-2}\right) \, , \tag{2.7}$$

where

$$\phi_b = \int_{\bar{\rho}}^{\infty} e^{-A} d\rho \,, \tag{2.8}$$

as before. From this metric, one can see that null ray traveling on fixed position in $\mathbb{S}^{d-2}$ is a trajectory on a surface of sphere. At $\theta = \pi/2$, a null ray starting from $\phi = 0$ travels to $\phi = \phi_b$ with global time $t = \phi_b$. Because it takes global time $t = \pi$ for null ray to travel between AdS boundaries, we can see null ray cannot reach the ETW brane in finite Poincaré time for $\phi_b \geq \pi$.

2.3 Empty AdS Terminated by Tensionful Brane

Let us first check the light crossing time for empty AdS spacetime terminated by ETW brane as formulated by Takayanagi 2011[1]. Because empty AdS has $e^A = \cosh\rho$ in $L = 1$ units,

$$\int_{\bar{\rho}}^{\infty} e^{-A} d\rho = \int_{\bar{\rho}}^{\infty} \frac{1}{\cosh\rho} d\rho = \frac{\pi}{2} - 2\arctan\left(\tanh(\frac{\bar{\rho}}{2})\right) \,. \tag{2.9}$$

Since the $\bar{\rho} = -\infty$ gives exactly $\int_{\bar{\rho}}^{\infty} e^{-A} d\rho = \pi$, ETW branes in empty AdS are always connected to AdS boundary with null geodesics unless located at negative infinity. Our criterion then suggests that simplified models of tensionful ETW branes in empty AdS are not healthy dual of BCFT.

Chapter 3

Addition of a Real Scalar Field

Because we have seen that toy models of empty AdS terminated by ETW branes are unhealthy according to light crossing criterion, we investigate if adding matter helps. As a first example, we consider a linear perturbation in a massive scalar field.

3.1 Action and Equations of Motion

We now proceed to evaluate how adding matter content to the bulk affects the light crossing time. As a first example, we apply a real scalar field coupled to matter, which introduces extra terms in the bulk action

$$S = \int d^{d+1}x\sqrt{-g}\left(R - 2\Lambda - \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}m^2\Phi^2\right) . \tag{3.1}$$

By varying the bulk part of the action with respect to the metric, we recover the Einstein's field equation with a scalar field coupled to matter

$$\begin{aligned} E_{\mu\nu} &= T_{\mu\nu} \,, \\ E_{\mu\nu} &= R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} + \Lambda g_{\mu\nu} \,, \\ T_{\mu\nu} &= \partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}\left(\partial_\sigma\Phi\partial^\sigma\Phi + m^2\Phi^2\right) g_{\mu\nu} \,. \end{aligned} \tag{3.2}$$

Similarly, variation of the action with respect to the scalar field gives the Klein Gordon equation in curved spacetime

$$D^\mu D_\mu\Phi - m^2\Phi = 0 \,, \tag{3.3}$$

which can also be rearranged as

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(g^{\mu\nu}\sqrt{-g}\partial_\nu\Phi\right)-\Delta(\Delta-d)\Phi=0\,, \tag{3.4}$$

by defining $m^2=\Delta(\Delta-d)$.

This theory has an AdS_{d+1} vacuum, where the metric can be written as

$$ds^2=d\rho^2+\cosh^2\rho\left(\frac{-dt^2+d\vec{x}^2+dz^2}{z^2}\right)\,. \tag{3.5}$$

By AdS/CFT, this bulk theory is dual to a CFT with position dependant source J for operator $\mathcal{O}$. To define a CFT on Minkowski space at AdS boundary $\rho\to\pm\infty$, we define $f_\pm=2ze^{\mp\rho}$ to get

$$\lim_{\rho\to\pm\infty} f_\pm^2 P[ds^2]=-dt^2+d\vec{x}^2+dz^2\,. \tag{3.6}$$

As $z>0$, we have two half-Minkowski spaces at $\rho=\pm\infty$. By defining a coordinate $x_\perp=\pm z$ for $\rho=\pm\infty$, we can get a full Minkowski space with metric

$$ds^2=-dt^2+d\vec{x}^2+dx_\perp^2\,. \tag{3.7}$$

One can define a source on this combined Minkowski space as

$$J(x_\perp)=\begin{cases}\lim\limits_{\rho\to\infty} f_+^{d-\Delta}J_+\\ \lim\limits_{\rho\to-\infty} f_-^{d-\Delta}J_-\end{cases}=\frac{1}{|x_\perp|^{d-\Delta}}\begin{cases}J_+ & \text{for } x_\perp>0\\ J_- & \text{for } x_\perp<0\end{cases} \tag{3.8}$$

This is an interface CFT because the two sides of Minkowski space are sourced by different coupling $J_\pm$, joined together with an interface at $x_\perp=0$. For a position independent source, we end up with two sides of constant but different coupling, which we know as Janus configuration. Hence, we name this type of configuration with position dependent source as Janus as well.

3.2 Linearized Analysis for Janus Configuration

We first consider a Janus configuration with an interface at $\rho=0$. Each asymptotic boundary at $\rho=\pm\infty$ has position dependent source J for the CFT operator dual to the scalar field.

Linearized Analysis

The addition of a scalar field to the bulk is performed in linear perturbation. Consider a perturbation of scalar field defined by

$$\Phi(\rho) = 0 + \epsilon\Phi^{(1)} + O(\epsilon^2)\,. \tag{3.9}$$

The EOM must be satisfied for each order in perturbation. Under the unperturbed empty AdS background metric $g^{(0)}$ defined by $ds^2 = d\rho^2 + \cosh^2(\rho)\frac{-dt^2+dz^2+d\vec{x}^2}{z^2}$, the EOM simplifies to

$$d\tanh(\rho)\partial_\rho\Phi + \partial_\rho^2\Phi - \Delta(\Delta-d)\Phi = 0\,, \tag{3.10}$$

for which the solution is given in terms of associated Legendre polynomials

$$\Phi^{(1)} = \left(C_1 P_{\frac{d}{2}-1}^{\frac{d}{2}-\Delta}(\tanh\rho) + C_2 P_{\frac{d}{2}-1}^{\frac{d}{2}-\Delta}(-\tanh\rho)\right)(\cosh\rho)^{-d/2}\,. \tag{3.11}$$

Here, we have selected a basis of associated Legendre polynomials that respect the symmetry of our setup. Notably, the asymptotic behavior

$$\begin{aligned}
\Phi(\rho\to\infty) &\approx \alpha C_2 e^{(\Delta-d)\rho} + (\beta C_1 + \gamma C_2)e^{-\Delta\rho}\,,\\
\Phi(\rho\to-\infty) &\approx \alpha C_1 e^{(\Delta-d)|\rho|} + (\beta C_2 + \gamma C_1)e^{-\Delta|\rho|}\,,\\
\alpha &= -\frac{2^{d/2}\pi\csc\left(\frac{1}{2}\pi(d-2\Delta)\right)}{\Gamma\left(\frac{d}{2}-\Delta+1\right)\left(\Gamma(\Delta)\Gamma(-d+\Delta+1)\right)}\,,\\
\beta &= \frac{2^{d/2}}{\Gamma\left(-\frac{d}{2}+\Delta+1\right)}\,,\\
\gamma &= \frac{2^{d/2}\csc\left(\frac{1}{2}\pi(d-2\Delta)\right)\sin\left(\frac{d\pi}{2}\right)}{\Gamma\left(-\frac{d}{2}+\Delta+1\right)}\,.
\end{aligned} \tag{3.12}$$

tells us that irrelevant deformations $\Delta > d$ are non-normalizable at both $\rho = \pm\infty$, meaning that both sources must be turned off by setting $C_1 = C_2 = 0$.

This perturbative scalar field induces perturbation in the metric

$$A(\rho) = A^{(0)} + \epsilon^2 A^{(2)} + O(\epsilon^4) = \ln\cosh\rho + \epsilon^2 A^{(2)} + O(\epsilon^4)\,, \tag{3.13}$$

where $A(\rho)$ is the warp factor of perturbed metric given by

$$ds^2 = d\rho^2 + e^{2A(\rho)}\frac{-dt^2+dz^2+d\vec{x}^2}{z^2}\,. \tag{3.14}$$

As a result, the light crossing time also receives modification as follows

$$\int_{-\infty}^{\infty} e^{-A(\rho)} d\rho = \int_{-\infty}^{\infty} e^{-A^{(0)}(\rho)-\epsilon^2 A^{(2)}(\rho)} d\rho = \int_{-\infty}^{\infty} \frac{1}{\cosh\rho} d\rho - \epsilon^2 \int_{-\infty}^{\infty} \frac{A^{(2)}(\rho)}{\cosh\rho} d\rho \,. \tag{3.15}$$

The first term is the unperturbed ϕ_b while the second term is the change of the bulk. Because the unperturbed $\phi_b = \pi$ exactly, we would like to understand how matter perturbs the light crossing time.

The $\rho\rho$ component of Einstein's equation is given as

$$E_{\rho\rho} = d(d-1)\left(-1 + e^{-2A} + A'^2\right) \,. \tag{3.16}$$

Substituting the perturbed $A(\rho)$,

$$E_{\rho\rho} = \epsilon^2 d(d-1) \tanh^2\rho \left(\coth\rho A^{(2)}\right)' \,. \tag{3.17}$$

On the other hand, the stress energy tensor is given as

$$T_{\mu\nu} = \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} g_{\mu\nu} \left(\partial_\sigma \Phi \partial^\sigma \Phi + \Delta(\Delta - d)\Phi^2\right) \,. \tag{3.18}$$

Denoting the $\rho\rho$ component as $T(\rho)$,

$$T(\rho) = \frac{1}{2}\left[(\partial_\rho \Phi^{(1)})^2 - \Delta(\Delta-d){\Phi^{(1)}}^2\right] \,. \tag{3.19}$$

By setting $T_{\rho\rho} = E_{\rho\rho}$,

$$\left(\coth\rho A^{(2)}\right)' = \frac{1}{d(d-1)} \coth^2\rho T(\rho) \,, \tag{3.20}$$

which gives us the perturbed warp factor

$$A^{(2)} = \frac{\tanh\rho}{d(d-1)} \int_{\infty}^{\rho} \coth^2\rho' T(\rho') d\rho' + c \tanh\rho \,. \tag{3.21}$$

The integration constant can be eliminated by diffeomorphism and hence not considered.

Then, we define $S(\rho) = \coth^2\rho T(\rho)$ and the change in ϕ_b can be calculated as

$$\delta\phi_b = -\int_{-\infty}^{\infty} \frac{A^{(2)}(\rho)}{\cosh\rho} d\rho \propto \int_{-\infty}^{\infty} \frac{\sinh\rho}{\cosh^2\rho} \left(\int_{\rho}^{\infty} S(\rho') d\rho'\right) d\rho \propto -\int_{-\infty}^{\infty} \frac{S(\rho)}{\cosh\rho} d\rho \,, \tag{3.22}$$

where integration by parts was used in the final step.

3.2.1 Analytic Continuation

The obtained change in ϕ_b,

$$\delta\phi_b = \frac{-1}{d(d-1)} \int_{-\infty}^{\infty} \frac{S(\rho)}{\cosh\rho} d\rho \,, \tag{3.23}$$

is complicated due to $S(\rho)$ containing the stress energy tensor $T(\rho)$, which contains product of associated Legendre polynomials. As a result, analytical solution is only obtainable for $d = 2$, and any higher dimensions require numerical analysis.

Numerical analysis also cannot be performed simply. There is a singularity at $\rho = 0$ of $\coth\rho$ in $S(\rho)$, which blows up numerical integration of $S(\rho)$ since the integral is performed from negative infinity to positive infinity. Hence, we must perform analytical continuation to regularize the integral:

$$\begin{aligned}
\delta\phi_b &\propto -\int_{-\infty}^{\infty} \frac{\cosh\rho}{\sinh^2\rho} T(\rho) d\rho \\
&\propto -\int_{-\infty}^{\infty} \left(\frac{\cosh\rho}{\sinh^2\rho} T(\rho) - \frac{T(0)}{\rho^2} + \frac{T(0)}{\rho^2} \right) d\rho \\
&\propto \int_{-\infty}^{\infty} \left(\frac{T(0)}{\rho^2} - \frac{\cosh\rho}{\sinh^2\rho} T(\rho) \right) d\rho \,.
\end{aligned} \tag{3.24}$$

At $\rho = 0$, $S(\rho)$ behaves as $T(0)/\sinh^2(0) \sim T(0)/\rho^2$. By subtracting off this singular behavior, the integral becomes regularized. Additional term must also be introduced to keep the expression equivalent, but the added term is completely analytical. The obtained result seems completely known, but there are two arbitrary parameters C_1 and C_2 in the definition of scalar field Φ. Decomposing according to their dependence on the parameters,

$$\begin{aligned}
\delta\phi_b &= \delta\phi_{b,11} C_1^2 + 2\delta\phi_{b,12} C_1 C_2 + \delta\phi_{b,22} C_2^2 \\
&= \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} \delta\phi_{b,11} & \delta\phi_{b,12} \\ \delta\phi_{b,12} & \delta\phi_{b,22} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \\
&= \vec{C}^T M \vec{C} \,.
\end{aligned} \tag{3.25}$$

Although the sign of $\delta\phi_b$ depends on C_1 and C_2, there are some situations where the sign is determined independently of those parameters.

As mentioned before, $C_1 = C_2 = 0$ in the irrelevant regime, meaning that no perturbation is made. In the relevant regime, both C_1 and C_2 are non-zero in general. This complicates the sign of $\delta\phi_b$, but when both eigenvalues of M are positive, then $\delta\phi_b$ is also positive.

Similarly, when both eigenvalues of M are negative, then $\delta\phi_b$ is also negative. This can be proven as follows.

Let $\vec{C}_+, \vec{C}_-$ be the eigenvectors of M with eigenvalues λ_+, λ_-, respectively. Then, an arbitrary vector $\vec{C}$ can be written as a linear combination of the eigenvectors,

$$\vec{C} = K_+\vec{C}_+ + K_-\vec{C}_- \,. \tag{3.26}$$

Then,

$$\begin{aligned}\delta\phi_b &= (K_+\vec{C}_+ + K_-\vec{C}_-)^T M (K_+\vec{C}_+ + K_-\vec{C}_-) \\ &= (K_+\vec{C}_+ + K_-\vec{C}_-)^T (\lambda_+ K_+\vec{C}_+ + \lambda_- K_-\vec{C}_-) \\ &= \lambda_+ K_+^2 + \lambda_- K_-^2 \,.\end{aligned} \tag{3.27}$$

Since $K_i^2 \geq 0$, the sign of $\delta\phi_b$ is completely known if both eigenvalues have the same sign.

Numerical computation for the eigenvalues of M is performed over the relevant conformal dimensions $\frac{d}{2} < \Delta < d$. It turns out that both eigenvalues are always positive for $d = 2, 3, 4, 5, 6$ as shown in Figure 3.1, meaning that the change in light crossing time is positive for Janus type configuration. Because the unperturbed $\phi_b = \pi$, then the two AdS boundaries become even more causally disconnected.

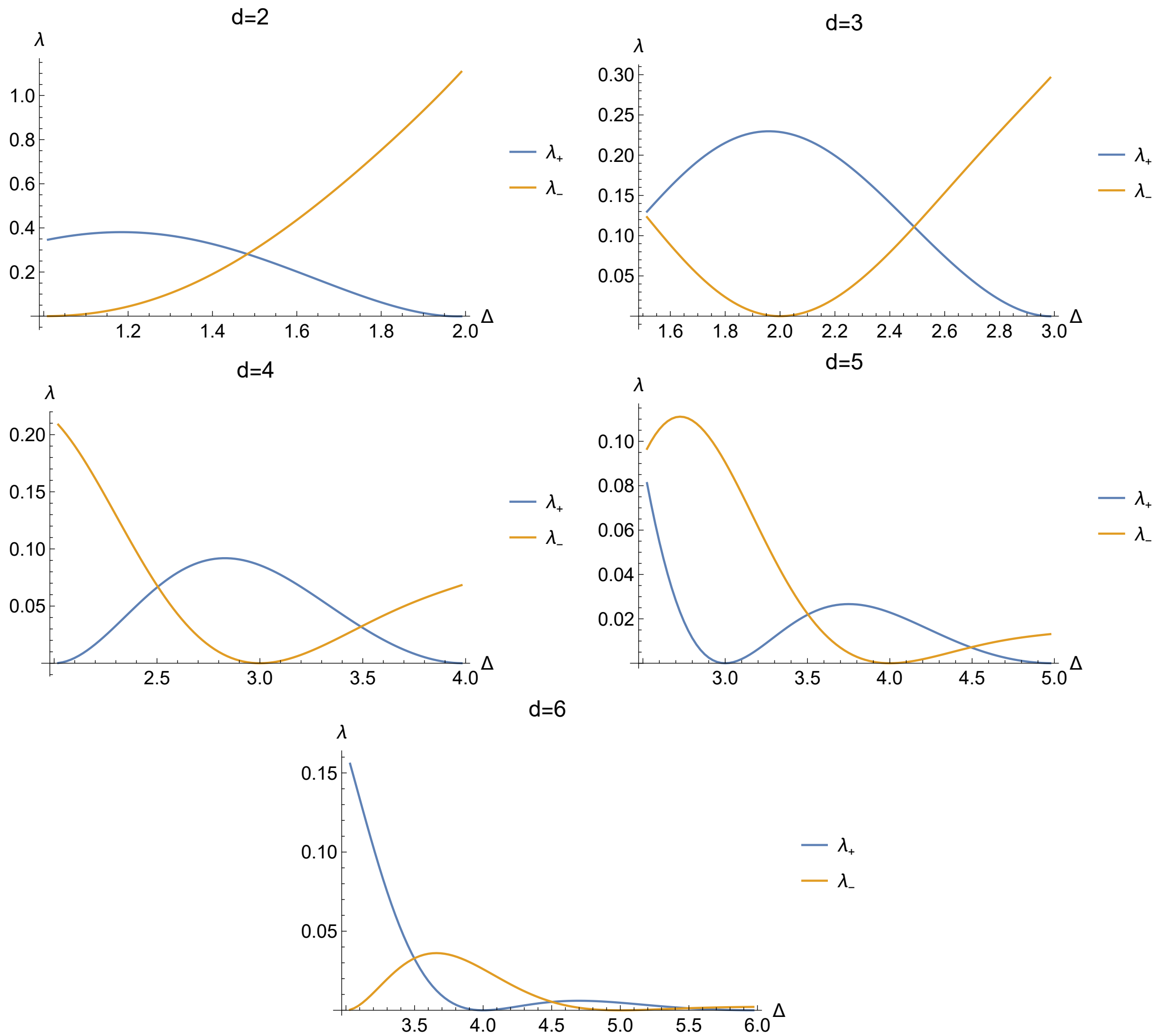

Figure 3.1: The eigenvalues of matrix M dictating the change ϕ_b for dimensions $d = 2, 3, 4, 5, 6$ in the relevant regime of $\frac{d}{2} < \Delta < d$.

3.3 Linearized Analysis for ETW Brane

Similar analysis can be performed for a toy model of empty AdS terminated by an ETW brane with a perturbative massive scalar field. We shall see that there are additional terms in the change in light crossing time due to the presence of the ETW brane, but the general outcome remains consistent.

3.3.1 Linearized Analysis

We consider perturbatation of scalar field in empty AdS again, but with an ETW brane this time. The ansatz for scalar field and the warp factor are equivalent to previous. However, we now need to consider the change in location of ETW brane $\bar{\rho}$ due to gravity backreacting. Hence, we have

$$\begin{aligned} \Phi(\rho) &= 0 + \epsilon\Phi^{(1)} + O(\epsilon^2)\,, \\ A(\rho) &= A^{(0)} + \epsilon^2 A^{(2)} + O(\epsilon^4)\,, \\ \bar{\rho} &= \bar{\rho}^{(0)} + \epsilon^2\bar{\rho}^{(2)} + O(\epsilon^4)\,, \end{aligned} \tag{3.28}$$

where the unperturbed location of ETW brane is given by

$$\bar{\rho}^{(0)} = \tanh^{-1}\left(\frac{T}{1-d}\right)\,. \tag{3.29}$$

As before, the scalar field is given as a linear combination of associated Legendre polynomials

$$\Phi^{(1)} = \left(C_1 P_{\frac{d}{2}-1}^{\frac{d}{2}-\Delta}(\tanh\rho) + C_2 Q_{\frac{d}{2}-1}^{\frac{d}{2}-\Delta}(\tanh\rho)\right)(\cosh\rho)^{-d/2}\,. \tag{3.30}$$

However, this configuration no longer has the $\mathbb{Z}^2$ symmetry of $\rho \to -\rho$. Thus, we choose a different basis of associated Legendre polynomials. The asymptotic behavior is given as

$$\begin{aligned} \Phi(\rho\to\infty) &\approx \alpha C_2 e^{(\Delta-d)\rho} + (\beta C_1 + \gamma C_2)e^{-\Delta\rho}\,, \\ \alpha &= \frac{2^{d/2}\mathrm{c}2\pi^2}{\left(\cos\left(\frac{3d\pi}{2} - 2\pi\Delta\right) - \cos\left(\frac{d\pi}{2}\right)\right)\Gamma\left(\frac{d}{2} - \Delta + 1\right)\Gamma(\Delta)\Gamma(-d+\Delta+1)}\,, \\ \beta &= \frac{2^{d/2}}{\Gamma\left(-\frac{d}{2} + \Delta + 1\right)}\,, \\ \gamma &= \frac{2^{\frac{d}{2}-1}\pi\cot\left(\frac{1}{2}\pi(d-2\Delta)\right)}{\Gamma\left(-\frac{d}{2} + \Delta + 1\right)}\,. \end{aligned} \tag{3.31}$$

We can see that C_2 must be turned off for irrelevant regime to ensure normalizability. We have also seen previously that C_1 acts as a source at $\rho = -\infty$. This time, however, we no longer have a source there due to the ETW brane terminating spacetime before that. As a result, there are no irrelevant deformations allowed in this analysis.

Combining all together, the light crossing time also receives modification as

$$\begin{aligned}\int_{\bar{\rho}}^{\infty} e^{-A(\rho)} d\rho &= \int_{\bar{\rho}^{(0)}+\epsilon^2\bar{\rho}^{(2)}}^{\infty} e^{-A^{(0)}(\rho)-\epsilon^2 A^{(2)}(\rho)} d\rho \\ &= \int_{\bar{\rho}^{(0)}}^{\infty} \frac{1}{\cosh\rho} d\rho - \epsilon^2 \int_{\bar{\rho}^{(0)}}^{\infty} \frac{A^{(2)}(\rho)}{\cosh\rho} d\rho - \epsilon^2 \frac{\bar{\rho}^{(2)}}{\cosh\bar{\rho}} \,.\end{aligned} \tag{3.32}$$

The first term is the unperturbed ϕ_b and the second term is change of bulk, exatly as Janus situation. However, there is another term which corresponds to the change in the location of ETW brane due to backreaction. We now compute how the total light crossing time changes due to perturbative scalar field.

The location of the ETW brane for a general warp factor $A(\rho)$ and tension T was derived as

$$T = (1-d)A'(\rho)\,. \tag{3.33}$$

Substituting the perturbed $A(\rho)$, we can recover the unperturbed location of ETW brane as well as the perturbed shift:

$$\begin{aligned} O(\epsilon^0): &\quad \bar{\rho}^{(0)} = \tanh^{-1}\left(\frac{T}{1-d}\right)\,, \\ O(\epsilon^2): &\quad \bar{\rho}^{(2)} = -A^{(2)\prime}(\bar{\rho}^{(0)})\cosh^2(\bar{\rho}^{(0)})\,. \end{aligned} \tag{3.34}$$

From previous section, we have

$$\left(\coth\rho A^{(2)}\right)' = \frac{1}{d(d-1)}\coth^2\rho T(\rho)\,. \tag{3.35}$$

Although we could solve for $A^{(2)}$ explicitly, it is not required for analysing the light crossing time, which we shall calculate now. By explicitly expanding the derivative,

$$\frac{-1}{\sinh^2\rho} A^{(2)} + \coth\rho A^{(2)\prime} = \frac{1}{d(d-1)}\coth^2\rho T(\rho)\,. \tag{3.36}$$

By rearranging terms,

$$\cosh\rho A^{(2)\prime} = \frac{A^{(2)}}{\sinh\rho} + \frac{1}{d(d-1)}\sinh\rho\coth^2\rho T(\rho)\,. \tag{3.37}$$

Then, we define $S(\rho) = \coth^2 \rho T(\rho)$ and the change in ϕ_b can be calculated as

$$\delta\phi_b = -\int_{\bar{\rho}^{(0)}}^{\infty} \frac{A^{(2)}(\rho)}{\cosh\rho} d\rho - \frac{\bar{\rho}^{(2)}}{\cosh\bar{\rho}^{(0)}} d\rho \propto \left\{ -\int_{\bar{\rho}^{(0)}}^{\infty} \frac{S(\rho)}{\cosh\rho} d\rho + \sinh\bar{\rho}^{(0)} S(\bar{\rho}^{(0)}) \right\} . \tag{3.38}$$

3.3.2 Analytic Continuation

Again, the obtained change in ϕ_b is complicated due to $S(\rho)$ containing the stress energy tensor $T(\rho)$, which contains product of associated Legendre polynomials. As a result, an analytical solution is only obtainable for $d = 2$, and any higher dimensions require numerical analysis. Additionally, analytic continuation is necessary to perform numerical computations.

$$\begin{aligned}
\delta\phi_b &\propto \left\{ -\int_{\bar{\rho}^{(0)}}^{\infty} \frac{S(\rho)}{\cosh\rho} d\rho + \sinh\bar{\rho}^{(0)} S(\bar{\rho}^{(0)}) \right\} \\
&\propto \left\{ -\int_{\bar{\rho}^{(0)}}^{\infty} \frac{\cosh\rho}{\sinh^2\rho} T(\rho) d\rho + \sinh\bar{\rho}^{(0)} \coth^2\bar{\rho}^{(0)} T(\bar{\rho}^{(0)}) \right\} \\
&\propto \left\{ -\int_{\bar{\rho}^{(0)}}^{\infty} \left(\frac{\cosh\rho}{\sinh^2\rho} T(\rho) - \frac{T(0)}{\sinh^2\rho} + \frac{T(0)}{\sinh^2\rho} \right) d\rho + \sinh\bar{\rho}^{(0)} \coth^2\bar{\rho}^{(0)} T(\bar{\rho}^{(0)}) \right\} \\
&\propto \left\{ \int_{\bar{\rho}^{(0)}}^{\infty} \frac{T(0) - \cosh\rho T(\rho)}{\sinh^2\rho} d\rho + T(0) - \coth\bar{\rho}^{(0)} \left(T(0) - \cosh\bar{\rho}^{(0)} T(\bar{\rho}^{(0)}) \right) \right\}
\end{aligned} \tag{3.39}$$

At $\rho = 0$, $S(\rho)$ behaves as $T(0)/\sinh^2(0)$. This time, we subtract off this singular behavior using $\sinh^2\rho$, as the boundary terms do not vanish anymore due to the dependence on $\bar{\rho}$. Decomposing the result according to their dependence on the parameters,

$$\begin{aligned}
\delta\phi_b &= \delta\phi_{b,11} C_1^2 + 2\delta\phi_{b,12} C_1 C_2 + \delta\phi_{b,22} C_2^2 \\
&= \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} \delta\phi_{b,11} & \delta\phi_{b,12} \\ \delta\phi_{b,12} & \delta\phi_{b,22} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \\
&= \vec{C}^T M \vec{C}
\end{aligned} \tag{3.40}$$

we can determine whether change in light crossing time is positive or negative.

We have argued that in irrelevant regime, all sources are turned off to ensure normalizability, meaning there are no perturbation of scalar field. In the relevant regime, both C_1 and C_2 are non-zero in general. Numerical computation for the eigenvalues of M is performed over relevant conformal dimension $\frac{d}{2} < \Delta < d$ and positive tension $0 < T < d - 1$ (which corresponds to the ETW brane located at $-\infty < \bar{\rho}^{(0)} < 0$). It turns out that one of the eigenvalues is always positive. Therefore, a contour plot of the other eigenvalue is produced in Figure 3.2 to showcase the region in which both eigenvalues are positive. This region corresponds to positive change in light crossing time, meaning that ETW branes become more causally disconnected. For AdS dimension $d = 2, 3$, there are regions in which one eigenvalue is negative. Higher dimension cases of $d = 4, 5, 6$ are not shown because both eigenvalues are positive in region, producing a monotone contour plot. The important result is that light crossing time always increases for $d = 4, 5, 6$.

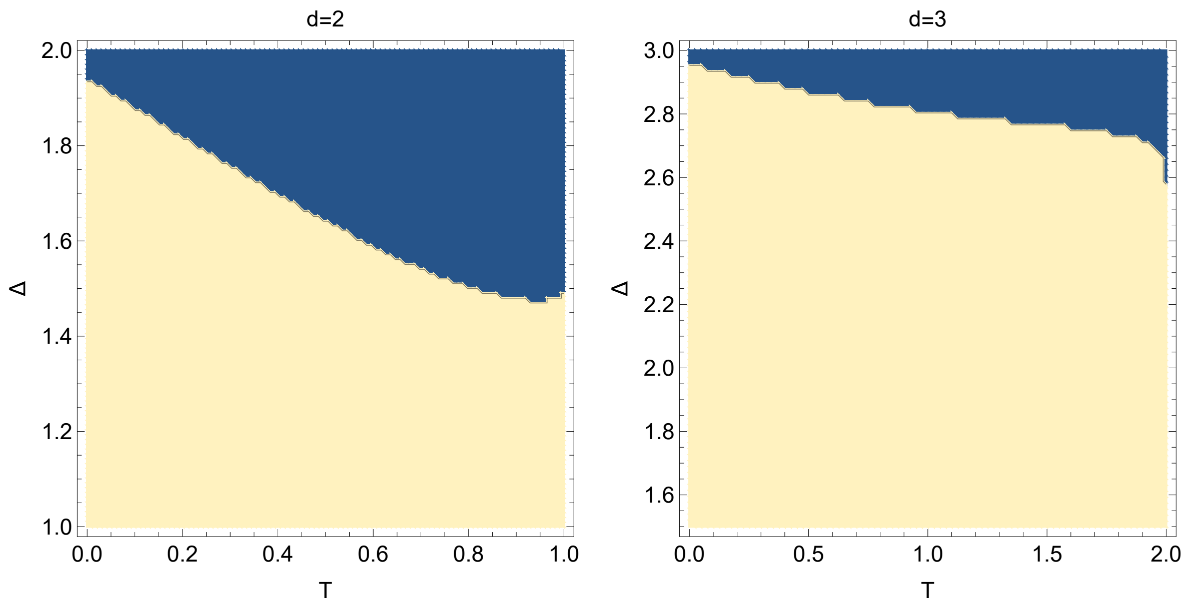

Figure 3.2: The blue region denotes the part of the (T, Δ) plane where perturbations of the scalar can increase the light crossing time.

Chapter 4

Non-SUSY Janus

Now we consider how our criterion for light crossing time performs in string theory settings. Let us consider Janus solutions in type IIB SUGRA, as it provides simple deformations of maximally supersymmetric $\mathrm{AdS}_5 \times S^5$ solution with a domain wall. The Janus solution numerically obtained by Bak, Gutperle and, Hirano [14], later analytically solved by D'Hoker, Estes, and Gutperle [15] breaks all supersymmetries (hence non-SUSY) with vanishing 3-form fields. This configuration has dilaton fields approaching different values on the two sides of the domain wall. The holographic dual to Janus solutions are interface CFT with two half-space of different coupling constant joined together with an interface.

The non-SUSY Janus solution to type IIB SUGRA is given by the Einstein-frame metric

$$ds^2 = L^2 \left(\gamma^{-1} h(\xi)^2 d\xi^2 + h(\xi) ds^2_{AdS_4}\right) + L^2 ds^2_{\mathbb{S}^5} \,. \tag{4.1}$$

Here, the AdS radius is $L^4 = 4\pi N \alpha'^2$ where N is the number of D3-branes, γ is a real parameter satisfying $3/4 \leq \gamma \leq 1$, and $h(\xi)$ is the warp factor. The warp factor

$$h(\xi) = \gamma \left(1 + \frac{4\gamma - 3}{\mathcal{P}(\xi) + 1 - 2\gamma}\right) \,, \tag{4.2}$$

is given in terms of Weierstrass elliptic function $\mathcal{P}(\xi)$ with elliptic invariants

$$g_2 = 16\gamma(1-\gamma), \qquad g_3 = 4(\gamma - 1) \,. \tag{4.3}$$

The metric has AdS component that can be matched with the AdS slicing we have been working with $\rho = \rho(\xi)$

$$ds^2 = \gamma^{-1} h(\xi)^2 d\xi^2 + h(\xi) ds^2_{AdS_d} = d\rho^2 + e^{2A(\rho)} ds^2_{AdS_d} \,. \tag{4.4}$$

By comparison,

$$h(\xi) = e^{2A(\rho)}, \qquad \gamma^{-1} h(\xi)^2 = \left(\frac{d\rho}{d\xi}\right)^2 . \tag{4.5}$$

Then, ϕ_b can be calculated in the same manner

$$\begin{aligned} \phi_b &= \int_{\infty}^{\infty} e^{-A} d\rho \\ &= \int_{\infty}^{\infty} h(\xi)^{-\frac{1}{2}} \frac{d\rho}{d\xi} d\xi \\ &= \gamma^{-\frac{1}{2}} \int_{\infty}^{\infty} h(\xi)^{\frac{1}{2}} d\xi \\ &= \int_{-\xi_0}^{\xi_0} \left(1 + \frac{4\gamma - 3}{\mathcal{P}(\xi) + 1 - 2\gamma}\right)^{\frac{1}{2}} d\xi \end{aligned} \tag{4.6}$$

The integration bounds have changed since the metric should be well behaved (smooth, real, and finite). This occurs for ξ within the two poles at $\pm\xi_0$ where $\mathcal{P}(\xi_0) = 2\gamma - 1$. The numerical computation demonstrates that $\phi_b - \pi \geq 0$ for the allowed region of parameter γ, as shown in Figure 4.1. Hence, the two AdS boundaries connected by interface CFT in non-SUSY Janus cannot be connected with a null geodesic.

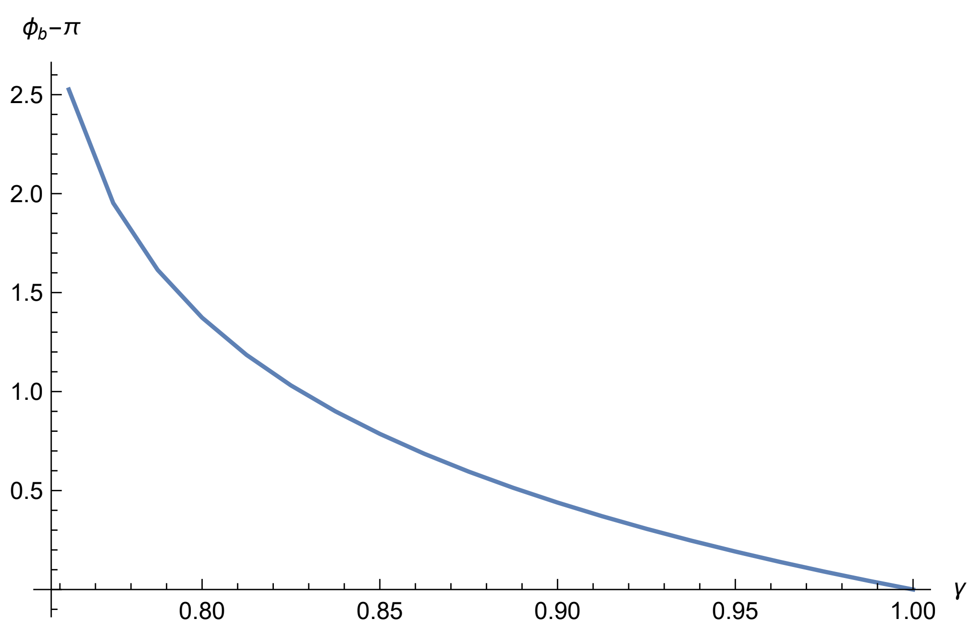

Figure 4.1: Numerical computation of $\phi_b - \pi$ results in non-negative value for $3/4 \leq \gamma \leq 1$

Chapter 5

$\mathrm{AdS}_5 \times \mathbb{S}^5$ type IIB SUGRA Ending on $SO(6)$ Invariant Smeared ETW Brane

Now we attempt to embed the concept of ETW branes in examples of AdS/CFT, starting with $\mathrm{AdS}_5 \times \mathbb{S}^5$ vacuum of IIB supergravity (SUGRA).

5.1 $\mathrm{AdS}_5 \times \mathbb{S}^5$ Vacuum of IIB Supergravity

We first summarize the important results of type IIB SUGRA, adapting the convensions following Polchinski [16]. The bosonic action of type IIB SUGRA is

$$\begin{aligned}
S_{\mathrm{IIB}} &= S_{\mathrm{NS}} + S_{\mathrm{RR}} + S_{\mathrm{CS}}\,, \\
S_{\mathrm{NS}} &= \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G_S}\,e^{-2\Phi}\left(R + 4(\partial\Phi)^2 - \frac{1}{2}|H_3|^2\right)\,, \\
S_{\mathrm{RR}} &= -\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-G_S}\left(|F_1|^2 + |\widetilde{F}_3|^2 + \frac{1}{2}|\widetilde{F}_5|^2\right)\,, \\
S_{\mathrm{CS}} &= -\frac{1}{4\kappa_{10}^2}\int C_4 \wedge H_3 \wedge F_3\,,
\end{aligned} \tag{5.1}$$

where $2\kappa_{10}^2 = (2\pi)^7\ell_s^8$, $(G_S)_{MN}$ is the string frame metric, Φ is the dilaton, B_2 is the Neveu-Schwarz B-field, and (C_0, C_2, C_4) are the Ramond-Ramond (RR) potentials. The field strengths are defined by

$$\begin{aligned}
F_1 &= dC_0\,, & H_3 &= dB_2\,, \\
F_3 &= dC_2\,, & \widetilde{F}_3 &= F_3 - C_0 \wedge H_3\,, \\
F_5 &= dC_4\,, & \widetilde{F}_5 &= F_5 + \frac{1}{2}(B_2 \wedge F_3 - C_2 \wedge H_3)\,,
\end{aligned} \tag{5.2}$$

with the on-shell five-form flux being self-dual, $\widetilde{F}_5 = \star\widetilde{F}_5$.

We can go to Einstein frame

$$G_{MN} = e^{-\frac{1}{2}\Phi}(G_S)_{MN}\,, \tag{5.3}$$

and combine the 0-form RR potential with the dilaton into a complex axiodilaton $\tau = C_0 + ie^{-\Phi}$. Then

$$S_{\rm IIB} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G}\left(R(G) - \frac{|\partial\tau|^2}{2(\mathrm{Im}(\tau))^2} - \frac{\mathcal{M}_{ij}}{2}F_3^i\cdot F_3^j - \frac{1}{4}|\widetilde{F}_5|^2\right) + S_{\rm CS}\,, \tag{5.4}$$

where $F_3^i = (H_3, F_3)$ and

$$\mathcal{M}_{ij} = \frac{1}{\mathrm{Im}(\tau)}\begin{pmatrix} |\tau|^2 & -\mathrm{Re}(\tau) \\ -\mathrm{Re}(\tau) & 1\end{pmatrix}\,. \tag{5.5}$$

Type IIB SUGRA has an $\mathrm{AdS}_5\times\mathbb{S}^5$ solution supported by $N = \frac{L^4}{4\pi\ell_s^4}$ units of five-form flux through a $\mathbb{S}^5$,

$$ds^2 = L^2(ds^2_{\mathrm{AdS}_5} + d\Omega_5^2)\,, \qquad F_5 = 4L^4\left(\mathrm{vol}_{\mathrm{AdS}_5} + \mathrm{vol}_{\mathbb{S}^5}\right)\,, \tag{5.6}$$

with constant τ and all other fields vanishing. Here $ds^2_{\mathrm{AdS}_5}$ is the line element on a unit radius AdS_5, $\mathrm{vol}_{\mathrm{AdS}_5}$ its volume form, $d\Omega_5^2$ is the line element on a round unit radius $\mathbb{S}^5$ and $\mathrm{vol}_{\mathbb{S}^5}$ its volume form.

5.2 Reduction on $\mathbb{S}^m$

This section reviews the Freund-Rubin compactification [17] of D-dimensional gravity onto m-dimensional sphere obtained by Bremer et al. [18]. The examples of string theory and supergravity solutions we consider are in the form $\mathrm{AdS}_{d+1}\times\mathbb{S}^m$, hence these compactifications are useful to simplify the calculations.

Consider a theory of gravity in D-dimensional spacetime with metric G existing for $\rho\geq\bar{\rho}$. The D-dimension action consists of Einstein-Hilbert and Gibbons-Hawking terms

$$S_D = \int d^Dx\sqrt{-G}\,\mathcal{R} + 2\int d^{D-1}x\sqrt{-H}\,K(H)\,, \tag{5.7}$$

with H the induced metric on the boundary and $K(H)$ the trace of its extrinsic curvature.

The D-dimensional Einstein's equations are

$$\mathcal{R}_{MN} - \frac{\mathcal{R}}{2} G_{MN} = 0 \,. \tag{5.8}$$

We now compactify this theory of gravity on a sphere. Consider a D-dimensional metric

$$ds_D^2 = e^{-\frac{2m}{d-1}\phi} ds^2_{\mathrm{AdS}_{d+1}} + e^{2\phi} d\Omega_m^2 \,, \tag{5.9}$$

which is separated into unit radius $\mathbb{S}^m$ with metric $\tilde{g}$ and $d+1$-dimensional AdS space with metric g. We label the indices of $\mathbb{S}^m$ by $\{\alpha, \beta, ...\}$ while the AdS part by $\{\mu, \nu, ...\}$. The warpfactor in front of the $\mathbb{S}^m$ is parametrized by $\phi(x^\mu)$, while the factor in front of AdS part is necessary to reduce D-dimentional Einstein-Hilbert and Gibbons-Hawking action into $d+1$-dimensional theory of gravity coupled to scalar ϕ with the action

$$\begin{aligned} S = \mathrm{vol}(\mathbb{S}^m) \Bigg(\int d^{d+1}x \sqrt{-g} \Big(R + m(m-1) e^{-\frac{2(d+m-1)}{d-1}\phi} \\ - \frac{m(d+m-1)}{d-1} (\partial\phi)^2 \Big) + 2 \int d^d x \sqrt{-h}\, K(h) \Bigg) \,. \end{aligned} \tag{5.10}$$

Then, the D-dimensional Einstein's equations simplify to

$$\begin{aligned} R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = \frac{m(d+m-1)}{d-1} \left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}(\partial\phi)^2 g_{\mu\nu} \right) + \frac{m(m-1)}{2} e^{-\frac{2(d+m-1)}{d-1}\phi} g_{\mu\nu} \,, \\ e^{\frac{2(d+m-1)}{d-1}\phi} \left((m-1) e^{-\frac{2(d+m-1)}{d-1}\phi} - \Box\phi \right) \tilde{g}_{\alpha\beta} = \frac{R}{2} e^{2\phi} \tilde{g}_{\alpha\beta} \,, \end{aligned} \tag{5.11}$$

while the curvatures are related by

$$K(H) = e^{\frac{m\phi}{d-1}} \left(K(h) - \frac{m}{d-1} \partial_\mu n^\mu \phi \right) \,. \tag{5.12}$$

5.3 General Einstein's Equation for AdS Slicing

With the compactification on a sphere, the D-dimensional Einstein's equation reduces to d+1-dimensional form. As a result, it is useful to oultline the Einstein's equation for AdS_{d+1} with AdS slicing. For the metric and scalar field

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = d\rho^2 + e^{2A(\rho)} ds^2_{\mathrm{AdS}_d} \,, \qquad \phi = \phi(\rho) \,, \tag{5.13}$$

the result of Einstein's equations for general AdS space gives us

$$\begin{aligned} R_{\mu\nu} &= \left((1-d) - (dA'^2 + A'')e^{2A}\right) h_{\mu\nu}\,, \\ R_{\rho\rho} &= -d(A'' + A'^2)\,, \\ R &= -d(d-1)e^{-2A} - d(d+1)A'^2 - 2dA''\,. \end{aligned} \tag{5.14}$$

Then, we have

$$\begin{aligned} R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} &= \left(\frac{(d-1)(d-2)}{2}e^{-2A} + \frac{d(d-1)}{2}A'^2 + (d-1)A''\right) h_{\mu\nu}\,, \\ R_{\rho\rho} - \frac{R}{2}g_{\rho\rho} &= \left(-d + \frac{d(d+1)}{2}\right) A'^2 + \frac{d(d-1)}{2}e^{-2A}\,, \\ \Box\phi &= \phi'' + dA'\phi'\,. \end{aligned} \tag{5.15}$$

Similarly, the extrinsic curvatures are given by

$$K(h)_{\mu\nu} = -A'h_{\mu\nu}\,, \qquad K(h) = -dA'\,. \tag{5.16}$$

5.4 $SO(6)$ Invariant Reduction on $\mathbb{S}^5$

Our first example of a reduction is type IIB SUGRA with N units of D3-brane flux, where we compactify on the $\mathbb{S}^5$. In terms of the previous section, this corresponds to $d = 4$ and $m = 5$ so

$$ds_{10}^2 = L^2\left(ds_{AdS_5}^2 + d\Omega_5^2\right) = e^{-\frac{10}{3}\phi}g_{\mu\nu}dx^\mu dx^\nu + e^{2\phi}d\Omega_5^2\,, \tag{5.17}$$

which gives us

$$e^\phi = \widetilde{L}\,, \qquad g_{\mu\nu} = \tilde{L}^{\frac{16}{3}}\gamma_{\mu\nu}\,, \tag{5.18}$$

with $\gamma_{\mu\nu}$ the metric on a unit-radius AdS$_5$. The five form flux is fixed by Gauss' Law with N units of flux through the $\mathbb{S}^5$ together with the self-duality condition as

$$F_{\mu\nu\rho\sigma\tau} = 4\widetilde{L}^4 e^{-\frac{40}{3}\phi}\varepsilon_{\mu\nu\rho\sigma\tau}\,, \qquad F_{\alpha\beta\gamma\delta\epsilon} = 4\widetilde{L}^4\varepsilon_{\alpha\beta\gamma\delta\epsilon}\,, \tag{5.19}$$

where $\varepsilon_{\mu\nu\rho\sigma\tau}$ is the Levi-Civita tensor associated with $g_{\mu\nu}$ and $\varepsilon_{\alpha\beta\gamma\delta\epsilon}$ is the Levi-Civita tensor associated with a unit $\mathbb{S}^5$. Raising the indices,

$$F^{\mu\nu\rho\sigma\tau} = 4\widetilde{L}^4 e^{\frac{10}{3}\phi}\varepsilon^{\mu\nu\rho\sigma\tau}\,, \qquad F^{\alpha\beta\gamma\delta\epsilon} = 4\widetilde{L}^4 e^{-10\phi}\varepsilon^{\alpha\beta\gamma\delta\epsilon}\,. \tag{5.20}$$

To construct the stress tensor we note

$$\begin{aligned} \frac{1}{4!}F_{\mu MNPQ}F_{\nu}{}^{MNPQ} &= \frac{1}{4!}F_{\mu\rho_1..\rho_4}F_{\nu}{}^{\rho_1..\rho_4} = -16\widetilde{L}^8 e^{-\frac{40}{3}\phi}g_{\mu\nu}\,, \\ \frac{1}{4!}F_{\alpha MNPQ}F_{\beta}{}^{MNPQ} &= \frac{1}{4!}F_{\alpha\gamma_1..\gamma_4}F_{\beta}{}^{\gamma_1..\gamma_4} = 16\widetilde{L}^8 e^{-8\phi}\tilde{g}_{\alpha\beta}\,. \end{aligned} \tag{5.21}$$

This implies that

$$\begin{aligned} |F_5|^2 &= \frac{1}{5!}F_{MNPQR}F^{MNPQR} \\ &= \frac{1}{5}\left(e^{\frac{10}{3}\phi}g^{\mu\nu}\frac{1}{4!}F_{\mu MNPQ}F_{\nu}{}^{MNPQ} + e^{-2\phi}\tilde{g}^{\alpha\beta}\frac{1}{4!}F_{\alpha MNPQ}F_{\beta}{}^{MNPQ}\right) \\ &= \frac{16\widetilde{L}^8}{5}e^{-10\phi}\left(-g^{\mu\nu}g_{\mu\nu} + \tilde{g}^{\alpha\beta}\tilde{g}_{\alpha\beta}\right) = 0\,. \end{aligned} \tag{5.22}$$

The 10d stress tensor is then

$$\begin{aligned} T_{\mu\nu} &= \frac{1}{4\times 4!}F_{\mu MNPQ}F_{\nu}{}^{MNPQ} - \frac{1}{8}|F_5|^2 e^{-\frac{10}{3}\phi}g_{\mu\nu} = -4\widetilde{L}^8 e^{-\frac{40}{3}\phi}g_{\mu\nu}\,, \\ T_{\alpha\beta} &= \frac{1}{4\times 4!}F_{\alpha MNPQ}F_{\beta}{}^{MNPQ} - \frac{1}{8}|F_5|^2 e^{2\phi}\tilde{g}_{\alpha\beta} = 4\widetilde{L}^8 e^{-8\phi}\tilde{g}_{\alpha\beta}\,, \end{aligned} \tag{5.23}$$

so that the 10d Einstein's equations are

$$\begin{aligned} \mathcal{R}_{\mu\nu} - \frac{\mathcal{R}}{2}G_{\mu\nu} &= -4\widetilde{L}^8 e^{-\frac{40}{3}\phi}g_{\mu\nu}\,, \\ \mathcal{R}_{\alpha\beta} - \frac{\mathcal{R}}{2}G_{\alpha\beta} &= 4\widetilde{L}^8 e^{-8\phi}\tilde{g}_{\alpha\beta}\,. \end{aligned} \tag{5.24}$$

Note that the trace of the 10d Einstein's equations is simply

$$-4\mathcal{R} = 0\,. \tag{5.25}$$

From the 10d action

$$S = \mathrm{vol}(\mathbb{S}^5)\int d^5x\sqrt{-g}\left(R + 20e^{-\frac{16}{3}\phi} - \frac{40}{3}(\partial\phi)^2\right) + (\mathrm{bdy})\,, \tag{5.26}$$

the AdS component of EOM gives

$$-4\widetilde{L}^8 e^{-\frac{40}{3}\phi}g_{\mu\nu} = R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} - \frac{40}{3}\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}(\partial\phi)^2 g_{\mu\nu}\right) - 10e^{-\frac{16}{3}\phi}g_{\mu\nu}\,, \tag{5.27}$$

which simplifies to

$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} = \frac{40}{3}\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}(\partial\phi)^2 g_{\mu\nu}\right) - \frac{1}{2}4e^{-\frac{16}{3}\phi}\left(2\widetilde{L}^8 e^{-8\phi} - 5\right) g_{\mu\nu}\,. \tag{5.28}$$

The angular part gives

$$4\widetilde{L}^8 e^{-8\phi}\tilde{g}_{\alpha\beta} = e^{\frac{16}{3}\phi}\left(4e^{-\frac{16}{3}\phi} - \Box\phi\right)\tilde{g}_{\alpha\beta}\,, \tag{5.29}$$

which simplifies to

$$\Box\phi = 4e^{-\frac{16}{3}\phi}\left(1 - \widetilde{L}^8 e^{-8\phi}\right)\,. \tag{5.30}$$

These equations follow from the effective 5d action

$$\begin{aligned} S_5 &= \frac{\mathrm{vol}(\mathbb{S}^5)}{2\kappa_{10}^2}\int d^5x\sqrt{-g}\left(R - \frac{40}{3}(\partial\phi)^2 - V(\phi)\right) + (\mathrm{bdy})\,, \\ V(\phi) &= 4e^{-\frac{16}{3}\phi}\left(2\widetilde{L}^8 e^{-8\phi} - 5\right)\,. \end{aligned} \tag{5.31}$$

The effective potential for ϕ has a global minimum at $e^\phi = \widetilde{L}$. Freezing ϕ to that value we have Einstein gravity with a cosmological constant,

$$S_5 = \frac{\mathrm{vol}(\mathbb{S}^5)}{2\kappa_{10}^2}\int d^5x\sqrt{-g}\left(R + 12\widetilde{L}^{-\frac{16}{3}}\right)\,, \tag{5.32}$$

which has an AdS_5 vacuum with radius of curvature $L_{\mathrm{eff}}^2 = \widetilde{L}^{\frac{16}{3}}$ as expected.

Consider a small fluctuation of ϕ around this saddle point value,

$$\phi = \ln\widetilde{L} + \delta\phi\,. \tag{5.33}$$

The quadratic effective action for $\delta\phi$ is

$$S_{\mathrm{quad}} = -\frac{80\mathrm{vol}(\mathbb{S}^5)}{6\kappa_{10}^2}\int d^5x\sqrt{-g}\left(\frac{1}{2}(\partial\delta\phi)^2 + \frac{16}{L_{\mathrm{eff}}^2}\delta\phi^2\right)\,, \tag{5.34}$$

so that $\delta\phi$ is a massive field with $m^2 L_{\mathrm{eff}}^2 = 32$. The dual operator has dimension $\Delta = 8$ by the usual expression $m^2 L_{\mathrm{eff}}^2 = \Delta(\Delta - d)$.

5.5 $\mathbf{AdS}_5 \times \mathbb{S}^5$ Ending on Smeared 3-Branes

We would like to find $SO(6)$-invariant spacetimes with a boundary. If type IIB SUGRA has an action principle, then we would simply deduce Neumann-like boundary conditions on the SUGRA fields by mandating a consistent variational principle. Since such an action does not exist, we work at the level of equations of motion. We first consider the dual of a conformal interface connecting a region with N_+ units of five-form flux to the one with N_- units where we take $N_+ > N_-$. The total isometry is then $SO(2,3) \times SO(6)$. To get a boundary we will then take $N_- = 0$ and cut off the geometry on that side of the interface in such a way as to respect the Neumann boundary condition on the metric, the one that follows from pure 10d Einstein gravity. To start we allow the five-form flux to vary in a continuous way corresponding to an $SO(2,3) \times SO(6)$-invariant density of 3-brane charge. We wish to consider 10-dimensional spacetimes of the form

$$ds_{10}^2 = e^{-\frac{10}{3}\phi(\rho)} \left(e^{2A(\rho)} \left(\frac{-dt^2 + dx^2 + dy^2 + dz^2}{z^2} \right) + d\rho^2 \right) + e^{2\phi(\rho)} d\Omega_5^2 \,, \qquad (5.35)$$

and the five-form flux

$$F_{txyz\rho} = \frac{4e^{-\frac{40}{3}\phi(\rho)+4A(\rho)}}{z^4} L^4(\rho) \,, \qquad F_{\alpha\beta\gamma\delta\epsilon} = 4L^4(\rho) \,, \qquad (5.36)$$

where $\frac{L^4}{4\pi\ell_s^4}$ interpolates between N_+ at large positive ρ and N_- at large negative ρ.

From the divergence of the five-form flux we infer a density J^{txyz} of 3-brane charge,

$$\frac{1}{(2\pi\ell_s)^4} D_M F^{Mtxyz} = -\frac{1}{4(\pi\ell_s)^4} \frac{\sqrt{g}}{\sqrt{-G}} \frac{\partial L^4(\rho)}{\partial \rho} = J^{txyz}(\rho) \,, \qquad (5.37)$$

where the indices are raised with the 10d Einstein metric. Integrating the 3-brane density over all of space and using $\mathrm{vol}(\mathbb{S}^5) = \pi^3$ we find that the brane distribution carries $N_- - N_+ < 0$ units of D3-brane charge as it ought.

Let us suppose that this distribution is made up out of a density of $\overline{\mathrm{D3}}$-branes. The 10d stress tensor then receives a contribution from the 3-branes and the 5-form flux. It reads

$$2\kappa_{10}^2 T_{MN} dx^M dx^N = -\frac{\partial L^4}{\partial \rho} e^{-\frac{20\phi}{3}+2A} ds^2_{\mathrm{AdS}_4} - 4L^8 e^{-\frac{20\phi}{3}} ds_5^2 + 4L^8 e^{-2\phi} d\Omega_5^2 \,. \qquad (5.38)$$

The conservation of this stress tensor restricts the form of $L^4(\rho)$ such that only constant L^4 on either side of the interface is allowed.

We now endeavor to find suitable boundary conditions at $\rho = \bar{\rho}$. To do so we could consider a more general problem than that of an ETW brane. Namely we extend the spacetime past $\rho = \bar{\rho}$ and impose that there is a sharp interface connecting the region with $\rho > \bar{\rho}$, supported by N_+ units of five-form flux, to the new $\rho < \bar{\rho}$ region, supported by N_- units of flux. The flux background now has

$$F_{txyz\rho} = \frac{4e^{-\frac{40}{3}\phi(\rho)+4A(\rho)}}{z^4}\left(L_+^4\Theta(\rho-\bar{\rho}) + L_-^4\Theta(\bar{\rho}-\rho)\right), \tag{5.39}$$

with $\frac{L_\pm^4}{\ell_P^4} = 4\pi N_\pm$ and ℓ_P is the 10-dimensional Planck length. The flux is not conserved at $\rho = \bar{\rho}$, indicating the presence of a uniform density of $N_- - N_+$ units of D3-brane charge there, smeared homogeneously over the $\mathbb{S}^7$. This gives the smeared $\overline{\text{D3}}$ action

$$S_{\text{ETW}} = \frac{1}{2\kappa_{10}^2}\left(2\int d^9x\sqrt{-H}\,K(H) - 4\left(\int d^4\sigma\sqrt{-\text{P}[G]}\wedge\Sigma - \int C_4\wedge\Sigma\right)\right), \tag{5.40}$$

where we have included the Gibbons-Hawking term as well.

The last two terms act as an effective potential

$$V_{\text{eff}} \propto \sqrt{-\text{P}[G]} - \int C_4\,, \tag{5.41}$$

for a single $\overline{\text{D3}}$ located at some point in the distribution. Using $F_5 = dC_4$ and the pull-back from above,

$$V_{\text{eff}} \propto e^{-\frac{20\phi}{3}+4A} - 4\int_{\bar{\rho}}^{\rho} e^{-\frac{40\phi(\rho')}{3}+4A(\rho')}d\rho'\,. \tag{5.42}$$

This potential has an unstable maximum at $\rho = \bar{\rho} = 0$ as seen in Figure 5.1, indicating that smeared distribution of the $\overline{\text{D3}}$ brane is unstable. We will now show that this can also be realized through light crossing time.

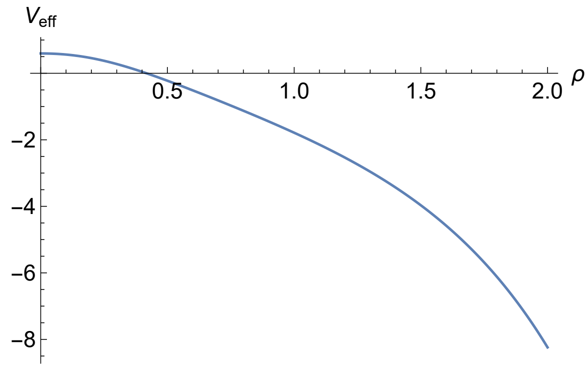

Figure 5.1: The effective potential of a single $\overline{\mathrm{D3}}$ brane as a function of ρ. The potential is globally maximized at $\rho = \bar{\rho} = 0$, indicating an instability to separating from smeared distribution of branes.

The boundary conditions for the metric are modified by the smeared ETW brane, which acts as an object that carries tension in the AdS_5 part of the geometry but, being a superposition of branes located at particular angles, not along the $\mathbb{S}^5$. This gives the boundary conditions

$$\begin{aligned} K(H)_{\mu\nu} - K(H)H_{\mu\nu} &= -2e^{-5\phi}H_{\mu\nu}\,, \\ K(H)_{\alpha\beta} - K(H)H_{\alpha\beta} &= 0\,. \end{aligned} \tag{5.43}$$

Using $N^M\partial_M = -e^{\frac{5\phi}{3}}\partial_\rho$ and $h_{\mu\nu} = e^{\frac{10\phi}{3}}H_{\mu\nu}$, we have

$$\begin{aligned} K(H)_{\mu\nu} &= \frac{1}{2}N^M\partial_M H_{\mu\nu} = -\frac{e^{-\frac{5\phi}{3}}}{2}\left(\partial_\rho h_{\mu\nu} - \frac{10}{3}\partial_\rho\phi\, h_{\mu\nu}\right)\,, \\ K(H)_{\alpha\beta} &= \frac{1}{2}N^M\partial_M H_{\alpha\beta} = -e^{\frac{11\phi}{3}}\partial_\rho\phi\,\tilde{g}_{\alpha\beta}\,, \end{aligned} \tag{5.44}$$

combined with the result from reduction

$$K(H) = e^{\frac{5\phi}{3}}K(h) + \frac{5}{3}e^{\frac{5\phi}{3}}\partial_\rho\phi\,. \tag{5.45}$$

Then the 5d boundary conditions are

$$\begin{aligned} K(H)_{\mu\nu} - K(H)H_{\mu\nu} &= e^{-\frac{5\phi}{3}}\left(K(h)_{\mu\nu} - K(h)h_{\mu\nu}\right) = -2e^{-\frac{25\phi}{3}}h_{\mu\nu}\,, \\ K(H)_{\alpha\beta} - K(H)H_{\alpha\beta} &= -e^{\frac{11\phi}{3}}\left(\frac{8}{3}\partial_\rho\phi + K(h)\right)\tilde{g}_{\alpha\beta} = 0\,, \end{aligned} \tag{5.46}$$

which simplify to

$$\begin{aligned} K(h)_{\mu\nu} - K(h)h_{\mu\nu} &= -2e^{-\frac{20\phi}{3}}h_{\mu\nu}\,, \\ K(h) &= -\frac{8}{3}\partial_\rho\phi\,. \end{aligned} \tag{5.47}$$

These conditions arise from the effective reduced action

$$\tilde{S}_{\text{ETW}} = \frac{\text{vol}(\mathbb{S}^5)}{2\kappa_{10}^2}\left(2\int d^4x\sqrt{-h}\left(K(h) - 2e^{-\frac{20\phi}{3}}\right)\right)\,. \tag{5.48}$$

Using the results of the general Einstein's equation, the boundary conditions become

$$\begin{aligned} K(h)_{\mu\nu} - K(h)h_{\mu\nu} = 3A'h_{\mu\nu} &= -2e^{-\frac{20\phi}{3}}h_{\mu\nu}\,, \\ K(h) = -4A' &= -\frac{8}{3}\phi'\,, \end{aligned} \tag{5.49}$$

which simplify to

$$\phi'(\bar{\rho}) = -e^{-\frac{20\phi}{3}}\,, \qquad A' = -\frac{2}{3}e^{-\frac{20\phi}{3}}\,. \tag{5.50}$$

We are dealing with the 5d equations of motion

$$\begin{aligned} R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} &= \frac{40}{3}\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}(\partial\phi)^2 g_{\mu\nu}\right) - \frac{1}{2}4e^{-\frac{16\phi}{3}}\left(2e^{-8\phi} - 5\right)g_{\mu\nu}\,, \\ \Box\phi &= 4e^{-\frac{16\phi}{3}}\left(1 - e^{-8\phi}\right)\,. \end{aligned} \tag{5.51}$$

The $\rho\rho$ component equation of motion is

$$A'^2 + e^{-2A} = \frac{10}{9}\phi'^2 - \frac{1}{3}e^{-\frac{16\phi}{3}}\left(2e^{-8\phi} - 5\right)\,, \tag{5.52}$$

which is not suitable for numerical analysis as first order equations terminate when $A' = 0$. Thus, we substitute the boundary conditions to get another boundary condition at the ETW brane

$$e^{-2A(\bar{\rho})} = \frac{5}{3}e^{-\frac{16\phi(\bar{\rho})}{3}}\,. \tag{5.53}$$

The scalar equation of motion becomes

$$\phi'' + 4A'\phi' = 4e^{-\frac{16\phi}{3}}\left(1 - e^{-8\phi}\right)\,, \tag{5.54}$$

while the non-$\rho\rho$ component of the Einstein's equations contracted with the null vector

$u^\mu \partial_\mu = z e^{-A} \partial_t + \partial_\rho$ reads

$$A'' - e^{-2A} = -\frac{40}{9}\phi'^2 \,. \tag{5.55}$$

In summary, we are solving the equations

$$\begin{aligned} A'' - e^{-2A} &= -\frac{40}{9}\phi'^2 \,, \\ \phi'' + 4A'\phi' &= 4e^{-\frac{16\phi}{3}}\left(1 - e^{-8\phi}\right) \,, \end{aligned} \tag{5.56}$$

with the boundary conditions

$$\phi'(\bar{\rho}) = -e^{-\frac{20\phi(\bar{\rho})}{3}} \,, \qquad A'(\bar{\rho}) = -\frac{2}{3}e^{-\frac{20\phi(\bar{\rho})}{3}} \,, \qquad e^{-2A(\bar{\rho})} = \frac{5}{3}e^{-\frac{16\phi(\bar{\rho})}{3}} \,, \tag{5.57}$$

by tuning the parameter $\phi(\bar{\rho})$.

Performing the numerical computation with $\bar{\rho} = 0$, we have been able to obtain a solution with the desired property of approximating the UV boundary condition $\phi \to 0$, as shown in Figure 5.2.

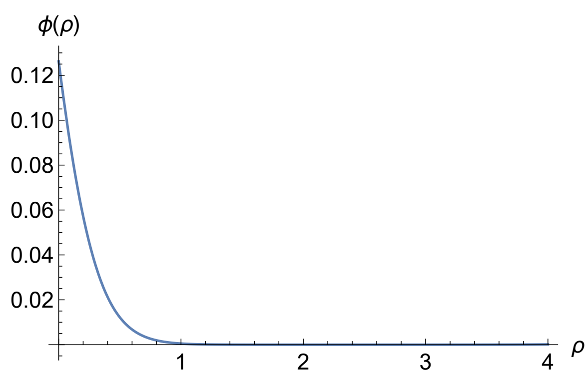

Figure 5.2: The profile for $\phi(\rho)$ in the numerical solution obeying the $\overline{\text{D3}}$ boundary condition at $\rho = 0$.

The solution for ϕ deviates after $\rho \approx 5$ due to the exponentially growing mode starting to dominate. Comparing the solution for $A(\rho)$ with pure AdS_5 cut off by a tensionful brane, the empty AdS solution is given by

$$ds^2 = L_{\text{eff}}^2 \cosh^2\left(\frac{\rho}{L_{\text{eff}}}\right) ds^2_{\text{AdS}_4} + d\rho^2 \,. \tag{5.58}$$

Because we have $L_{\text{eff}}^2 = L^{\frac{16}{3}}$, then

$$e^A = \cosh(\rho) \tag{5.59}$$

in $L = 1$ unit. Placing a tensionful ETW brane at $\rho = \bar{\rho}$ is a shift in the solution

$$e^{A} = \cosh\left(\rho - \rho_0\right) . \tag{5.60}$$

The amount of shift ρ_0 is given by the maximum of numerical solution e^A since it has maximum at $A' = 0$ and $\rho = \rho_0$. This maximum occurs at $\rho_0 = 0.453$, which corresponds to the tension $T \approx 1.27$ using the equation

$$T = \frac{d-1}{L_{\text{eff}}} \tanh\left(\frac{\rho_0}{L_{\text{eff}}}\right) . \tag{5.61}$$

Plotting this approximate solution against the numerical solution in Figure 5.3, we find that the approximation is close to the numerical except for near the ETW brane.

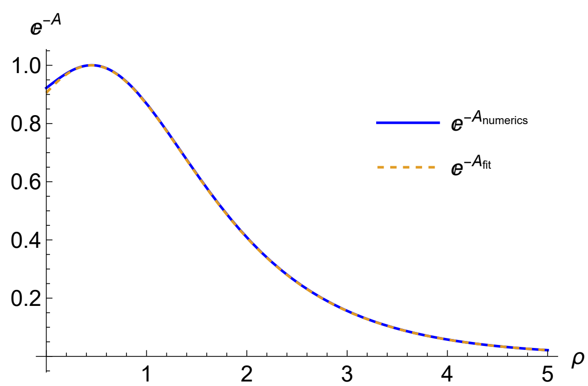

Figure 5.3: A fit of e^{-A} to $\frac{1}{\cosh(\rho-\rho_0)}$. The value of ρ_0 is fixed by the tension $T \approx 1.27$ (which has $\phi_b = 2.01$) to be $\rho_0 \approx 0.453$

Therefore, attaching tensionful brane solution after $\rho = 4$ and numerically evaluating the light-crossing integral, we have

$$\phi_b = \int_{\bar{\rho}=0}^{\infty} d\rho\, e^{-A(\rho)} \approx 2.01 < \pi\, . \tag{5.62}$$

As discussed, $\phi_b < \pi$ indicates that the AdS boundary and the smeared ETW brane are in causal contact by a null geodesic.

5.6 $\mathbf{AdS}_5 \times \mathbb{S}^5$ Interface

Now consider two $SO(6)$ invariant compactification of type IIB SUGRA on a $\mathbb{S}^5$ joined together with an interface at $\rho = \bar{\rho}$. One side is supported by N_+ units of five-form flux while the other is supported by N_- units. The domain wall is a smeared distribution of $\overline{D3}$ branes sourcing this difference in the flux.

The five-form fluxes are

$$\begin{aligned} F_{\mu\nu\rho\sigma\tau} &= 4\left(\widetilde{L}_+^4\Theta(\rho-\bar{\rho}) + \widetilde{L}_-^4\Theta(\bar{\rho}-\rho)\right)e^{-\frac{40}{3}\phi}\varepsilon_{\mu\nu\rho\sigma\tau}\,, \\ F_{\alpha\beta\gamma\delta\epsilon} &= 4\left(\widetilde{L}_+^4\Theta(\rho-\bar{\rho}) + \widetilde{L}_-^4\Theta(\bar{\rho}-\rho)\right)\varepsilon_{\alpha\beta\gamma\delta\epsilon}\,. \end{aligned} \tag{5.63}$$

Then, we can calculate the following as,

$$\begin{aligned} \frac{1}{4!}F_{\mu MNPQ}F_\nu{}^{MNPQ} &= -16\left(\widetilde{L}_+^8\Theta(\rho-\bar{\rho}) + \widetilde{L}_-^8\Theta(\bar{\rho}-\rho)\right)e^{-\frac{40}{3}\phi}g_{\mu\nu}\,, \\ \frac{1}{4!}F_{\alpha MNPQ}F_\beta{}^{MNPQ} &= 16\left(\widetilde{L}_+^8\Theta(\rho-\bar{\rho}) + \widetilde{L}_-^8\Theta(\bar{\rho}-\rho)\right)e^{-8\phi}\tilde{g}_{\alpha\beta}\,. \end{aligned} \tag{5.64}$$

This implies

$$|F_5|^2 = \frac{1}{5!}F_{MNPQR}F^{MNPQR} = 0\,. \tag{5.65}$$

The 10d stress tensor is then

$$\begin{aligned} T_{\mu\nu} &= \frac{1}{4\times 4!}F_{\mu MNPQ}F_\nu{}^{MNPQ} - \frac{1}{8}|F_5|^2e^{-\frac{10}{3}\phi}g_{\mu\nu} = -4\left(\widetilde{L}_+^8\Theta(\rho-\bar{\rho}) + \widetilde{L}_-^8\Theta(\bar{\rho}-\rho)\right)e^{-\frac{40}{3}\phi}g_{\mu\nu}\,, \\ T_{\alpha\beta} &= \frac{1}{4\times 4!}F_{\alpha MNPQ}F_\beta{}^{MNPQ} - \frac{1}{8}|F_5|^2e^{2\phi}\tilde{g}_{\alpha\beta} = 4\left(\widetilde{L}_+^8\Theta(\rho-\bar{\rho}) + \widetilde{L}_-^8\Theta(\bar{\rho}-\rho)\right)e^{-8\phi}\tilde{g}_{\alpha\beta}\,, \end{aligned} \tag{5.66}$$

so that the 10d Einstein's equations are

$$\begin{aligned} \mathcal{R}_{\mu\nu} - \frac{\mathcal{R}}{2}G_{\mu\nu} &= -4\left(\widetilde{L}_+^8\Theta(\rho-\bar{\rho}) + \widetilde{L}_-^8\Theta(\bar{\rho}-\rho)\right)e^{-\frac{40}{3}\phi}g_{\mu\nu}\,, \\ \mathcal{R}_{\alpha\beta} - \frac{\mathcal{R}}{2}G_{\alpha\beta} &= 4\left(\widetilde{L}_+^8\Theta(\rho-\bar{\rho}) + \widetilde{L}_-^8\Theta(\bar{\rho}-\rho)\right)e^{-8\phi}\tilde{g}_{\alpha\beta}\,. \end{aligned} \tag{5.67}$$

Note that the trace of the 10d Einstein's equations is simply

$$-4\mathcal{R} = 0\,. \tag{5.68}$$

The effective 5d equations are then

$$
\begin{aligned}
R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} &= \frac{40}{3}\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}(\partial\phi)^2 g_{\mu\nu}\right) - \frac{1}{2}V(\phi) g_{\mu\nu}\,, \\
V(\phi) &= 4e^{-\frac{16}{3}\phi}\left(2\left(\widetilde{L}_+^8\Theta(\rho-\bar{\rho}) + \widetilde{L}_-^8\Theta(\bar{\rho}-\rho)\right)e^{-8\phi} - 5\right)\,, \\
\mathcal{R}_{\alpha\beta} &= e^{\frac{16}{3}\phi}\left(4e^{-\frac{16}{3}\phi} - \Box\phi\right)\tilde{g}_{\alpha\beta} = 4\left(\widetilde{L}_+^8\Theta(\rho-\bar{\rho}) + \widetilde{L}_-^8\Theta(\bar{\rho}-\rho)\right)e^{-8\phi}\tilde{g}_{\alpha\beta}\,,
\end{aligned}
\tag{5.69}
$$

and the last line simplifies to

$$
\Box\phi = 4e^{-\frac{16}{3}\phi}\left(1 - \left(\widetilde{L}_+^8\Theta(\rho-\bar{\rho}) + \widetilde{L}_-^8\Theta(\bar{\rho}-\rho)\right)e^{-8\phi}\right)\,. \tag{5.70}
$$

These equations follow from the effective 5d action

$$
S_5 = \frac{\mathrm{vol}(\mathbb{S}^5)}{2\kappa_{10}^2}\int d^5x\sqrt{-g}\left(R - \frac{40}{3}(\partial\phi)^2 - V(\phi)\right) + (\mathrm{bdy})\,. \tag{5.71}
$$

The boundary condition for interface configuration was actually derived in the previous section. The effective action was given as

$$
S_{\mathrm{ETW}} = \frac{1}{2\kappa_{10}^2}\left(2\int d^9x\sqrt{-H}\,K(H) - 4\left(\int d^4\sigma\sqrt{-\mathrm{P}[G]}\wedge\Sigma - \int C_4\wedge\Sigma\right)\right)\,, \tag{5.72}
$$

and the 10d Israel junction condition at the interface is

$$
\begin{aligned}
\Delta\left(K(H)_{\mu\nu} - K(H)H_{\mu\nu}\right) &= -2\left(\widetilde{L}_+^4 - \widetilde{L}_-^4\right)e^{-5\phi}H_{\mu\nu}\,, \\
\Delta\left(K(H)_{\alpha\beta} - K(H)H_{\alpha\beta}\right) &= 0\,.
\end{aligned}
\tag{5.73}
$$

These conditions reduce to

$$
\begin{aligned}
\Delta\left(K(h)_{\mu\nu} - K(h)h_{\mu\nu}\right) &= -2\left(\widetilde{L}_+^4 - \widetilde{L}_-^4\right)e^{-\frac{20\phi}{3}}h_{\mu\nu}\,, \\
\Delta\left(\frac{8}{3}\phi' + K(h)\right) &= 0\,,
\end{aligned}
\tag{5.74}
$$

where h is the induced metric on the interface as determined from the 5d metric g. The effective action at the interface is

$$
\tilde{S} = -\frac{4\mathrm{vol}(\mathbb{S}^5)}{2\kappa_{10}^2}\left(\int d^4x\sqrt{-h}\,\left(\widetilde{L}_+^4 - \widetilde{L}_-^4\right)e^{-\frac{20\phi}{3}} - \int C_4\right)\,. \tag{5.75}
$$

The boundary conditions at the interface $\rho = \bar{\rho}$ read

$$\begin{aligned} A'(\bar{\rho}_+) - A'(\bar{\rho}_-) &= -\frac{2}{3}\left(\widetilde{L}_+^4 - \widetilde{L}_-^4\right) e^{-\frac{20\phi(\bar{\rho})}{3}} \,, \\ \phi'(\bar{\rho}_+) - \phi'(\bar{\rho}_-) &= -\left(\widetilde{L}_+^4 - \widetilde{L}_-^4\right) e^{-\frac{20\phi(\bar{\rho})}{3}} \,. \end{aligned} \tag{5.76}$$

Similar to previous sections, we use the $\rho\rho$ component of Einstein's equation

$$\begin{aligned} A'(\bar{\rho}_+)^2 + e^{-2A(\bar{\rho})} &= \frac{10}{9}\phi'(\bar{\rho}_+)^2 + \frac{e^{-\frac{16\phi(\bar{\rho})}{3}}}{3}\left(5 - 2\widetilde{L}_+^8 e^{-8\phi(\bar{\rho})}\right) \,, \\ A'(\bar{\rho}_-)^2 + e^{-2A(\bar{\rho})} &= \frac{10}{9}\phi'(\bar{\rho}_-)^2 + \frac{e^{-\frac{16\phi(\bar{\rho})}{3}}}{3}\left(5 - 2\widetilde{L}_-^8 e^{-8\phi(\bar{\rho})}\right) \,, \end{aligned} \tag{5.77}$$

to give further boundary conditions. Solving these four boundary condition equations, there are six unknowns $\phi(\bar{\rho}), \phi'(\bar{\rho}_+), \phi'(\bar{\rho}_-), A(\bar{\rho}), A'(\bar{\rho}_+), A'(\bar{\rho}_-)$. Thus, there will be two parameters to be tuned as initial variables of numerical computation. We chose $\phi(\bar{\rho})$ and $\phi'(\bar{\rho}_-)$ to be the parameters and the boundary conditions for these parameters are

$$\begin{aligned} \phi'(\bar{\rho}_+) &= \phi'(\bar{\rho}_-) - \left(\widetilde{L}_+^4 - \widetilde{L}_-^4\right) e^{-\frac{20\phi(\bar{\rho})}{3}} \,, \\ A'(\bar{\rho}_-) &= \frac{5}{3}\phi'(\bar{\rho}_-) + \widetilde{L}_-^4 e^{-\frac{20\phi(\bar{\rho})}{3}} \,, \\ A'(\bar{\rho}_+) &= \frac{5}{3}\phi'(\bar{\rho}_-) + \left(\frac{5}{3}\widetilde{L}_-^4 - \frac{2}{3}\widetilde{L}_+^4\right) e^{-\frac{20\phi(\bar{\rho})}{3}} \,, \\ e^{-2A(\bar{\rho})} &= -\frac{5}{3}\left(\phi'(\bar{\rho}_-) + \widetilde{L}_-^4 e^{-\frac{20\phi(\bar{\rho})}{3}}\right)^2 + \frac{5}{3} e^{-\frac{16\phi(\bar{\rho})}{3}} \,. \end{aligned} \tag{5.78}$$

We are now ready to solve the the equations of motion for the warpfactors $A(\rho)$ and $\phi(\rho)$, subject to this boundary condition. The scalar equation of motion and the 5d Einstein's equation contracted with the null vector $u^\mu \partial_\mu = z e^{-A}\partial_t + \partial_\rho$ read

$$\begin{aligned} \phi'' + 4A'\phi' &= 4e^{-\frac{16\phi}{3}}\left(1 - \left(\widetilde{L}_+^8 \Theta(\rho - \bar{\rho}) + \widetilde{L}_-^8 \Theta(\bar{\rho} - \rho)\right) e^{-8\phi}\right) \,, \\ A'' - e^{-2A} &= -\frac{40}{9}\phi'^2 \,. \end{aligned} \tag{5.79}$$

Performing the numerical computation for a range of values with initial conditions $\phi(\bar{\rho})$ and $\phi'(\bar{\rho}_-)$ for each side of the interface, we found that the parameter space in which both sides have $\phi \to 0$ away from the interface is very restricted. In the case where both sides of the interface has same magnitude but opposite sign of the five-form flux or $\widetilde{L}_-^4 = -\widetilde{L}_+^4$, the solution is exactly the same as the previous section except mirrored on both sides of the interface.

Chapter 6

11-Dimensional Supergravity Ending on Smeared ETW Brane

A procedure similar to type IIB SUGRA can be followed to embed smeared ETW branes in 11-dimensional supergravity.

6.1 Vacuum of 11-Dimensional Supergravity

We begin with a brief overview of 11d supergravity following Polchinski [16]. There are two bosonic fields, the metric $G_{\mu\nu}$ and a Ramond-Ramond potential C_3. The bosonic action is given by

$$S_{11} = \frac{1}{2\kappa_{11}^2}\int d^{11}x\sqrt{-G}\left(R - \frac{1}{2}|F_4|^2\right) - \frac{1}{12\kappa_{11}^2}\int C_3 \wedge F_4 \wedge F_4\,, \tag{6.1}$$

where the last term is the Chern-Simons term of 11d supergravity. By the variation of the action, one can obtain the field equations

$$\begin{aligned} R_{MN} - \frac{R}{2}G_{MN} &= \frac{1}{2}F_{MPQR}F_N^{PQR} - \frac{1}{96}G_{MN}F_{PQRS}F^{PQRS}\,, \\ D_M F^{MNPQ} &= \frac{1}{2\times 4!^2}\epsilon^{NPQS_1\ldots S_4 T_1\ldots T_4}F_{S_1\ldots S_4}F_{T_1\ldots T_4}\,, \end{aligned} \tag{6.2}$$

where $\epsilon^{NPQS_1\ldots S_{4,7}T_1\ldots T_{4,7}}$ is the 11d Levi-Civita tensor. These equations have $\mathrm{AdS}_4\times\mathbb{S}^7$ solution

$$ds^2 = \frac{L^2}{4}ds^2_{\mathrm{AdS}_4} + L^2 d\Omega_7^2\,, \qquad F_{\mu\nu\rho\sigma} = \frac{3L^3}{8}\varepsilon_{\mu\nu\rho\sigma}\,, \tag{6.3}$$

with $\varepsilon_{\mu\nu\rho\sigma}$ being the Levi-Civita tensor on a unit-radius AdS_4 as well as $\mathrm{AdS}_7\times\mathbb{S}^4$ solution

$$ds^2 = 4L^2 ds^2_{\mathrm{AdS}_7} + L^2 d\Omega_4^2\,, \qquad F_{\alpha\beta\gamma\delta} = 3L^3\varepsilon_{\alpha\beta\gamma\delta}\,, \tag{6.4}$$

with $\varepsilon_{\alpha\beta\gamma\delta}$ being the Levi-Civita tensor on a unit radius $\mathbb{S}^4$, supported by a stack of M2 or M5 branes, respectively. The flux background satisfies the RR 3-form equation of motion while the Einstein's equations are satisfied together with the metric.

6.2 AdS$_4 \times \mathbb{S}^7$ 11d Supergravity Ending on $SO(8)$ Invariant Smeared ETW Brane

Here let us take 11d SUGRA with N units of M2-brane flux, with the AdS$_4 \times \mathbb{S}^7$ vacuum above, and compactify on the $\mathbb{S}^7$. Comparing the solutions to AdS$_4 \times \mathbb{S}^7$ 11d SUGRA field equations with reduced metric for $d = 3$ and $m = 7$

$$ds_{11}^2 = \frac{L^2}{4} ds_{AdS_4}^2 + L^2 d\Omega_7^2 = e^{-7\phi} g_{\mu\nu} dx^\mu dx^\nu + e^{2\phi} d\Omega_7^2 \,, \tag{6.5}$$

we have

$$F^{\mu\nu\rho\sigma} = 6L^6 e^{7\phi} \varepsilon^{\mu\nu\rho\sigma} \,, \qquad F_{\mu\nu\rho\sigma} = 6L^6 e^{-21\phi} \varepsilon_{\mu\nu\rho\sigma} \,, \tag{6.6}$$

with $\varepsilon^{\mu\nu\rho\sigma}$ the Levi-Civita tensor associated with $g_{\mu\nu}$ essentially by Gauss' Law, the requirement that $F_7 = \star F_4$ has N units of flux through the $\mathbb{S}^7$. Then we can calculate the effective stress tensor can be calculated as

$$\begin{aligned} T_{\mu\nu} &= \frac{1}{12} F_{\mu MNP} F_\nu{}^{MNP} - \frac{1}{96} F_{MNPQ} F^{MNPQ} G_{\mu\nu} = -9L^{12} e^{-21\phi} g_{\mu\nu} \,, \\ T_{\alpha\beta} &= \frac{1}{12} F_{\alpha MNP} F_\beta{}^{MNP} - \frac{1}{96} F_{MNPQ} F^{MNPQ} G_{\alpha\beta} = 9L^{12} e^{-12\phi} \tilde{g}_{\alpha\beta} \,, \end{aligned} \tag{6.7}$$

giving us the 11-dimensional Einstein's equations

$$\begin{aligned} \mathcal{R}_{\mu\nu} - \frac{\mathcal{R}}{2} G_{\mu\nu} &= -9L^{12} e^{-21\phi} g_{\mu\nu} \,, \\ \mathcal{R}_{\alpha\beta} - \frac{\mathcal{R}}{2} G_{\alpha\beta} &= 9L^{12} e^{-12\phi} \tilde{g}_{\alpha\beta} \,, \end{aligned} \tag{6.8}$$

and the curvature

$$\mathcal{R} = -6L^{12} e^{-14\phi} \,. \tag{6.9}$$

Using the result of reduction on sphere, we arrive at the action

$$S = \mathrm{vol}(\mathbb{S}^7) \int d^4x \sqrt{-g} \left(R + 42 e^{-9\phi} - \frac{63}{2} (\partial\phi)^2 \right) + (\mathrm{bdy}) \,, \tag{6.10}$$

and the equations of motion

$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} = \frac{63}{2}\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}(\partial\phi)^2 g_{\mu\nu}\right) - \frac{1}{2}6e^{-9\phi}\left(3L^{12}e^{-12\phi} - 7\right)g_{\mu\nu}\,,$$
$$\Box\phi = 6e^{-9\phi}\left(1 - L^{12}e^{-12\phi}\right)\,. \tag{6.11}$$

The effective potential $V(\phi)$ of the action has a unique minimum at $e^\phi = L$. At this minima, the action is Einstein-Hilbert with cosmological constant

$$S = \frac{\mathrm{vol}(\mathbb{S}^7)}{2\kappa_{11}^2}\int d^4x\sqrt{-g}\left(R + \frac{24}{L^9}\right) + (\mathrm{bdy})\,, \tag{6.12}$$

with radius $L_{\mathrm{eff}}^2 = \frac{L^9}{4}$.

Adding a small fluctuations around this minimum $\phi = \ln L + \delta\phi$ give the quadratic action

$$S_{\mathrm{quad}} = -\frac{63\mathrm{vol}(\mathbb{S}^7)}{2\kappa_{11}^2}\int d^4x\sqrt{-g}\left(\frac{1}{2}(\partial\delta\phi)^2 + \frac{9}{L_{\mathrm{eff}}^2}\delta\phi^2\right)\,, \tag{6.13}$$

corresponding to an effective mass of $m^2L_{\mathrm{eff}}^2 = 18$ and a dual operator of dimension $\Delta = 6$.

Similar to type IIB SUGRA situation, we would like to put an $SO(8)$-invariant ETW brane into this geometry. The procedure is simpler because there is an action principle for 11d SUGRA. As in our type IIB SUGRA analysis we study backgrounds that preserve the isometry of the internal space. Because we also have in mind the dual of a conformal boundary condition, we then impose that the bulk geometry has a $SO(2,2)\times SO(8)$ isometry, so that we can parameterize the 4d line element as

$$ds_{11}^2 = e^{-7\phi(\rho)}\left(e^{2A(\rho)}ds_{\mathrm{AdS}_3}^2 + d\rho^2\right) + e^{2\phi(\rho)}d\Omega_7^2\,, \qquad ds_{\mathrm{AdS}_3}^2 = \frac{-dt^2 + dx^2 + dz^2}{z^2} \tag{6.14}$$

where the warpfactor A is fixed to be a function of ρ alone, and we take the spacetime to exist only for $\rho \geq \bar{\rho}$. With this parameterization we have

$$F_{txz\rho} = \frac{6e^{-21\phi+3A}}{z^3}\,, \qquad F^{txz\rho} = -6z^3e^{7\phi-3A}\,. \tag{6.15}$$

We proceed to find boundary conditions at $\rho = \bar{\rho}$ such that, with suitable boundary terms, the SUGRA effective action has a consistent variational principle. In backgrounds where

$F_4 \wedge F_4 = 0$ the variation of the RR field has the boundary term

$$\delta S_{11} = -\frac{1}{2\kappa_{11}^2 \times 3!} \int d^{10}x \sqrt{-H}\, \delta C_{MNP} N_Q F^{QMNP} \,, \tag{6.16}$$

where indices are raised with the 11-dimensional metric and N_Q is a normalized outward-pointing unit vector. We would like to impose a Neumann-like condition on C_3, but this cannot be achieved without a RR current density on the boundary, since at $\rho = \bar{\rho}$ we have $\sqrt{-H} N_Q F^{Qtxz} = -6\sqrt{\tilde{g}}$ where $\tilde{g}$ is the metric on a unit $\mathbb{S}^7$. Allowing a RR charge density J^{txz} at $\rho = \bar{\rho}$ introduces an extra boundary term in the variation of the action, so that

$$\delta S_{11} = \int d^{10}x \sqrt{-g}\, \delta C_{txz} \left(\frac{1}{2\kappa_{11}^2} F^{\rho txz} - J^{txz} \right) . \tag{6.17}$$

We achieve a Neumann-like condition by setting the term in parentheses to vanish, which implies a uniform density of N units of M2-brane charge smeared homogeneously over the $\mathbb{S}^7$. We then assume that this distribution of charge is built from N M2 branes smeared over the boundary, which leads to a contribution to the stresse-energy there. Including the Gibbons-Hawking term as usual, we also impose a Neumann-like condition on the metric, the analogue of the Israel junction condition, $K(H)_{MN} - K(H)H_{MN} = -T_{MN}$,which guarantees that the boundary term in the variation of the action with respect to metric fluctuations vanishes. If we let $\Sigma = \mathrm{vol}_{\mathbb{S}^7} = d^7\theta\, \sqrt{\tilde{g}}$ denote the smearing form, then the total boundary term, the sum of the Gibbons-Hawking term and the action of this distribution of M2-branes, reads

$$S_{\rm ETW} = \frac{1}{2\kappa_{11}^2} \left(2 \int d^{10}x \sqrt{-H}\, K(H) - 6 \left(\int d^3\sigma \sqrt{-\mathrm{P}[G]} \wedge \Sigma + \int C_3 \wedge \Sigma \right) \right) . \tag{6.18}$$

The last two terms act as an effective potential

$$V_{\rm eff} \propto \sqrt{-\mathrm{P}[G]} - \int C_3 \propto e^{-\frac{21\phi}{2}+3A} - 6 \int_{\bar{\rho}}^{\rho} e^{-21\phi(\rho')+3A(\rho')} d\rho' \,, \tag{6.19}$$

for a single M2 brane located at some point in the distribution. This potential has an unstable maximum at $\rho = \bar{\rho} = 0$ as seen in Figure 6.1, indicating that smeared distribution of M2 brane is unstable. We will now show that this can also be realized through light crossing time.

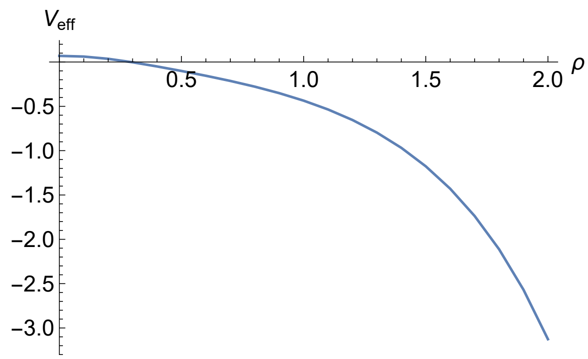

Figure 6.1: The effective potential of a single M2 brane as a function of ρ. The potential is globally maximized at $\rho = \bar{\rho} = 0$, indicating an instability to separating from smeared distribution of branes.

The boundary conditions in 11-dimensions are

$$\begin{aligned} K(H)_{\mu\nu} - K(H)H_{\mu\nu} &= -3e^{-7\phi}H_{\mu\nu}\,, \\ K(H)_{\alpha\beta} - K(H)H_{\alpha\beta} &= 0\,, \end{aligned} \tag{6.20}$$

which reduce to the 4-dimensional compactified theory with

$$\begin{aligned} K(h)_{\mu\nu} - K(h)h_{\mu\nu} &= -3e^{-\frac{21\phi}{2}}h_{\mu\nu}\,, \\ K(h) &= -\frac{9}{2}\partial_\rho\phi\,. \end{aligned} \tag{6.21}$$

The first line is the boundary condition we would anticipate from the dimensional reduction of the 11d action

$$\tilde{S}_{\text{ETW}} = \frac{\text{vol}(\mathbb{S}^7)}{2\kappa_{11}^2}\left(2\int d^3x\sqrt{-h}\left(K(h) - 3e^{-\frac{21\phi}{2}}\right)\right)\,. \tag{6.22}$$

The angular part vanishing is due to the ETW brane carrying energy momentum only in the AdS components. Using the result of the general Eintesin's equation, we can get the boundary conditions

$$\phi'(\bar{\rho}) = -e^{-\frac{21\phi}{2}}\,, \qquad A'(\bar{\rho}) = -\frac{3}{2}e^{-\frac{21\phi}{2}}\,. \tag{6.23}$$

From the reduction, we have

$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} = \frac{63}{2}\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}(\partial\phi)^2 g_{\mu\nu}\right) - \frac{1}{2}6e^{-9\phi}\left(3e^{-12\phi} - 7\right)g_{\mu\nu}\,, \qquad \square\phi = 6e^{-9\phi}\left(1 - e^{-12\phi}\right)\,. \tag{6.24}$$

The $\rho\rho$ component of Einstein's equation together with the two boundary conditions give

$$e^{-2A(\bar\rho)} = 7e^{-9\phi(\bar\rho)}\,. \tag{6.25}$$

which gives us additional constraint on $A(\bar\rho)$. Therefore, the only remaining boundary condition is $\phi(\bar\rho)$, which is the parameter we tune to reach a solution with an asymptotically AdS_4 region at large ρ, which requires $\phi \to 0$. The equations of motion to be solved are the scalar equation of motion and non-$\rho\rho$ component of Einstein's equation contracted by null vector $u^\mu\partial_\mu = ze^{-A}\partial_t + \partial_\rho$.

In summary, we are solving the equations

$$A'' - e^{-2A} = -\frac{63}{4}\phi'^2\,, \qquad \phi'' + 3A'\phi' = 6e^{-9\phi}\left(1 - e^{-12\phi}\right)\,, \tag{6.26}$$

with the boundary conditions

$$\phi'(\bar\rho) = -e^{-\frac{21\phi(\bar\rho)}{2}}\,, \qquad A'(\bar\rho) = -\frac{3}{2}e^{-\frac{21\phi(\bar\rho)}{2}}\,, \qquad e^{-2A(\bar\rho)} = 7e^{-9\phi(\bar\rho)}\,, \tag{6.27}$$

by tuning the parameter $\phi(\bar\rho)$.

Performing the numerical computation with $\bar\rho = 0$, we have been able to obtain a solution shown in Figure 6.2 with the desired property of approximating the AdS boundary condition of $\phi \to 0$ as $\rho \to \infty$.

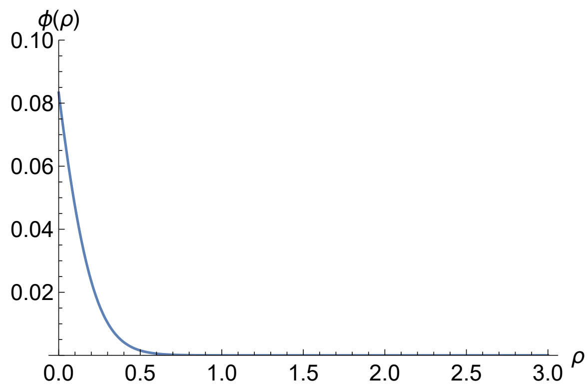

Figure 6.2: The profiles for $\phi(\rho)$ in the numerical solution obeying the $\overline{\text{M2}}$ boundary condition at $\rho = 0$.

Comparing the solution for $A(\rho)$ with pure AdS_4 cut off by a tensionful brane, we have $L_{\text{eff}}^2 = \frac{L^9}{4}$ giving us

$$e^A = \frac{1}{2}\cosh\left(2(\rho - \rho_0)\right) . \tag{6.28}$$

where $\rho_0 = 0.252$, which corresponds to the tension $T \approx 1.86$ using the equation

$$T = \frac{d-1}{L_{\text{eff}}}\tanh\left(\frac{\rho_0}{L_{\text{eff}}}\right) . \tag{6.29}$$

Plotting this approximate solution against the numerical solution in Figure 6.3, we find that the approximation is close to the numerical except for near the ETW brane.

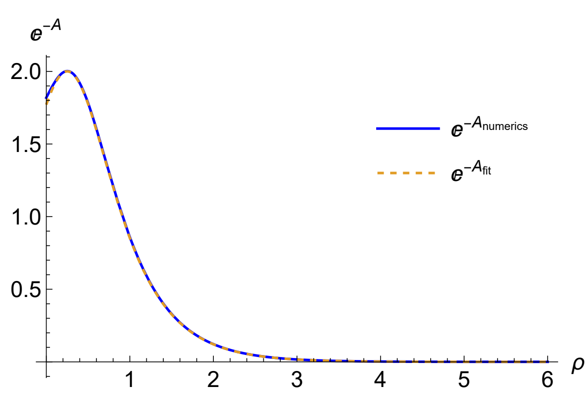

Figure 6.3: A fit of e^{-A} to $\frac{2}{\cosh(2(\rho-\rho_0))}$. The value of ρ_0 is fixed by the tension $T \approx 1.86$ (which has $\phi_b = 2.07$) to be $\rho_0 \approx 0.252$

Therefore, attaching tensionful brane solution after $\rho = 3$ and numerically evaluating the light-crossing integral, we have

$$\phi_b = \int_{\bar{\rho}=0}^{\infty} d\rho\, e^{-A(\rho)} \approx 2.07 < \pi\,. \tag{6.30}$$

As discussed, $\phi_b < \pi$ indicates that the AdS boundary and the smeared ETW brane are connected by null geodesics.

6.3 $\mathbf{AdS}_4 \times \mathbb{S}^7$ Interface

We may also consider interfaces connecting two asymptotically $\mathrm{AdS}_4 \times \mathbb{S}^7$ regions in a $SO(2,2) \times SO(8)$ invariant way. As in IIB SUGRA, such interfaces must be sharp so as to be consistent with the Einstein's equations. Let the region with $\rho > \bar{\rho}$ be supported by N_+ units of four-form flux, and the region with $\rho < \bar{\rho}$ be supported with N_- units. The flux background is

$$F_{\mu\nu\rho\sigma} = 6\left(\widetilde{L}_+^6 \Theta(\rho - \bar{\rho}) + \widetilde{L}_-^6 \Theta(\bar{\rho} - \rho)\right) e^{-21\phi} \varepsilon_{\mu\nu\rho\sigma}\,, \qquad F_{\alpha\beta\gamma\delta} = 0\,, \tag{6.31}$$

giving the stress tensors

$$\begin{aligned} T_{\mu\nu} &= -9\left(\widetilde{L}_+^{12} \Theta(\rho - \bar{\rho}) + \widetilde{L}_-^{12} \Theta(\bar{\rho} - \rho)\right) e^{-21\phi} g_{\mu\nu}\,, \\ T_{\alpha\beta} &= 9\left(\widetilde{L}_+^{12} \Theta(\rho - \bar{\rho}) + \widetilde{L}_-^{12} \Theta(\bar{\rho} - \rho)\right) e^{-12\phi} \tilde{g}_{\alpha\beta}\,, \end{aligned} \tag{6.32}$$

and thus, the 10d Einstein's equations

$$\begin{aligned} \mathcal{R}_{\mu\nu} - \frac{\mathcal{R}}{2} G_{\mu\nu} &= -9\left(\widetilde{L}_+^{12} \Theta(\rho - \bar{\rho}) + \widetilde{L}_-^{12} \Theta(\bar{\rho} - \rho)\right) e^{-21\phi} g_{\mu\nu}\,, \\ \mathcal{R}_{\alpha\beta} - \frac{\mathcal{R}}{2} G_{\alpha\beta} &= 9\left(\widetilde{L}_+^{12} \Theta(\rho - \bar{\rho}) + \widetilde{L}_-^{12} \Theta(\bar{\rho} - \rho)\right) e^{-12\phi} \tilde{g}_{\alpha\beta}\,. \end{aligned} \tag{6.33}$$

The effective 4d equations are then

$$\begin{aligned} R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} &= \frac{63}{2}\left(\partial_\mu\phi \partial_\nu\phi - \frac{1}{2}(\partial\phi)^2 g_{\mu\nu}\right) - \frac{1}{2} V(\phi) g_{\mu\nu}\,, \\ V(\phi) &= 6e^{-9\phi}\left(3\left(\widetilde{L}_+^{12} \Theta(\rho - \bar{\rho}) + \widetilde{L}_-^{12} \Theta(\bar{\rho} - \rho)\right) e^{-12\phi} - 7\right)\,, \\ \Box\phi &= 6e^{-9\phi}\left(1 - \left(\widetilde{L}_+^{12} \Theta(\rho - \bar{\rho}) + \widetilde{L}_-^{12} \Theta(\bar{\rho} - \rho)\right) e^{-12\phi}\right)\,. \end{aligned} \tag{6.34}$$

These equations follow from the effective 4d action

$$S_4 = \frac{\mathrm{vol}(\mathbb{S}^7)}{2\kappa_{11}^2} \int d^5x \sqrt{-g} \left(R - \frac{63}{2} (\partial\phi)^2 - V(\phi) \right) + (\mathrm{bdy})\,. \tag{6.35}$$

This flux background is not conserved, implying the existence of $N_+ - N_-$ units of M2-brane charge at $\rho = \bar{\rho}$, smeared homogeneously over the $\mathbb{S}^7$. Assuming that this charge density is made up by $N_+ - N_-$ M2 branes, and their associated stress energy, one ends up with 11-dimensional SUGRA together with this brane source at $\rho = \bar{\rho}$. For $N_+ > N_-$ we have

$$\begin{aligned} S &= S_{11} + S_{\mathrm{interface}}\,, \\ S_{\mathrm{interface}} &= -\frac{N_+ - N_-}{\mathrm{vol}(\mathbb{S}^7)} T_3 \left(\int d^3\sigma \sqrt{\mathrm{P}[G]} \wedge \Sigma + \int C_3 \wedge \Sigma \right)\,, \end{aligned} \tag{6.36}$$

where $T_3 = \frac{1}{(2\pi)^2 \ell_P^3}$ is the tension of a single M2 brane. There is an Israel junction condition at $\rho = \bar{\rho}$ that comes from integrating the Einstein's equations in a pillbox around $\rho = \bar{\rho}$ giving

$$\begin{aligned} &\Delta \left(K(H)_{\mu\nu} - K(H) H_{\mu\nu} \right) = -3 \left(\widetilde{L}_+^6 - \widetilde{L}_-^6 \right) e^{-7\phi} H_{\mu\nu}\,, \\ &\Delta \left(K(H)_{\alpha\beta} - K(H) H_{\alpha\beta} \right) = 0\,. \end{aligned} \tag{6.37}$$

These conditions reduce to

$$\begin{aligned} \Delta \left(K(h)_{\mu\nu} - K(h) h_{\mu\nu} \right) &= -3 \left(\widetilde{L}_+^6 - \widetilde{L}_-^6 \right) e^{-\frac{21\phi}{2}} h_{\mu\nu}\,, \\ \Delta \left(\frac{9}{2}\phi' + K(h) \right) &= 0\,, \end{aligned} \tag{6.38}$$

where h is the induced metric on the boundary as determined from the 4d metric g. The effective action at the interface is

$$\tilde{S} = -\frac{\mathrm{vol}(\mathbb{S}^7)}{2\kappa_{11}^2} \left(2 \int d^3x \sqrt{-h} \left(K(h) - 3 \left(\widetilde{L}_+^6 - \widetilde{L}_-^6 \right) e^{-\frac{21\phi}{2}} \right) \right)\,. \tag{6.39}$$

The boundary conditions at the interface $\rho = \bar{\rho}$ read

$$\begin{aligned} A'(\bar{\rho}_+) - A'(\bar{\rho}_-) &= -\frac{3}{2} \left(\widetilde{L}_+^6 - \widetilde{L}_-^6 \right) e^{-\frac{21\phi(\bar{\rho})}{2}}\,, \\ \phi'(\bar{\rho}_+) - \phi'(\bar{\rho}_-) &= -\left(\widetilde{L}_+^6 - \widetilde{L}_-^6 \right) e^{-\frac{21\phi(\bar{\rho})}{2}}\,. \end{aligned} \tag{6.40}$$

Similar to previous sections, we use the $\rho\rho$ component of Einstein's equation

$$\begin{aligned} A'(\bar{\rho}_+)^2 + e^{-2A(\bar{\rho})} &= \frac{21}{4}\phi'(\bar{\rho}_+)^2 + e^{-9\phi(\bar{\rho})}\left(7 - 3\widetilde{L}_+^{12} e^{-12\phi(\bar{\rho})}\right) , \\ A'(\bar{\rho}_-)^2 + e^{-2A(\bar{\rho})} &= \frac{21}{4}\phi'(\bar{\rho}_-)^2 + e^{-9\phi(\bar{\rho})}\left(7 - 3\widetilde{L}_-^{12} e^{-12\phi(\bar{\rho})}\right) , \end{aligned} \tag{6.41}$$

to give further boundary conditions. We chose $\phi(\bar{\rho})$ and $\phi'(\bar{\rho}_-)$ to be the parameters and the boundary conditions for these parameters are

$$\begin{aligned} \phi'(\bar{\rho}_+) &= \phi'(\bar{\rho}_-) - \left(\widetilde{L}_+^6 - \widetilde{L}_-^6\right) e^{-\frac{21\phi(\bar{\rho})}{2}} , \\ A'(\bar{\rho}_-) &= \frac{7}{2}\phi'(\bar{\rho}_-) + 2\widetilde{L}_-^6 e^{-\frac{21\phi(\bar{\rho})}{2}} , \\ A'(\bar{\rho}_+) &= \frac{7}{2}\phi'(\bar{\rho}_-) + \left(\frac{7}{2}\widetilde{L}_-^6 - \frac{3}{2}\widetilde{L}_+^6\right) e^{-\frac{21\phi(\bar{\rho})}{2}} , \\ e^{-2A(\bar{\rho})} &= -7\left(\phi'(\bar{\rho}_-) + \widetilde{L}_-^6 e^{-\frac{21\phi(\bar{\rho})}{2}}\right)^2 + 7e^{-9\phi(\bar{\rho})} . \end{aligned} \tag{6.42}$$

We are now ready to solve the the equations of motion for the warpfactors $A(\rho)$ and $\phi(\rho)$, subject to this boundary condition. The scalar equation of motion and the 4d Einstein's equation contracted with the null vector $u^\mu \partial_\mu = ze^{-A}\partial_t + \partial_\rho$ read

$$\begin{aligned} \phi'' + 3A'\phi' &= 6e^{-9\phi}\left(1 - \left(\widetilde{L}_+^{12}\Theta(\rho - \bar{\rho}) + \widetilde{L}_-^{12}\Theta(\bar{\rho} - \rho)\right) e^{-12\phi}\right) , \\ A'' - e^{-2A} &= -\frac{63}{2}\phi'^2 . \end{aligned} \tag{6.43}$$

Performing the numerical computation for a range of values with initial conditions $\phi(\bar{\rho})$ and $\phi'(\bar{\rho}_-)$ for each side of the interface, we found that the parameter space in which both sides have $\phi \to 0$ away from the interface is very restricted. In the case where both sides of the interface has same magnitude but opposite sign of the five-form flux or $\widetilde{L}_-^6 = -\widetilde{L}_+^6$, the solution is exactly the same as the previous section except mirrored on both sides of the interface.

6.4 AdS$_7 \times \mathbb{S}^4$ 11d Supergravity Ending on $SO(5)$ Invariant Smeared ETW Brane

Here, we perform a similar compactification of AdS$_7 \times \mathbb{S}^4$ vacuum on the $\mathbb{S}^4$ in the presence of N units of M5-brane charge. Comparing the solutions to AdS$_7 \times \mathbb{S}^4$ 11d SUGRA field equations with reduced metric for $m = 4$ and $d = 6$, we have the metric

$$ds_{11}^2 = 4L^2 ds_{AdS_7}^2 + L^2 d\Omega_4^2 = e^{-\frac{8}{5}\phi} g_{\mu\nu} dx^\mu dx^\nu + e^{2\phi} d\Omega_4^2 \,, \tag{6.44}$$

and the four-form flux

$$F_{\alpha\beta\gamma\delta} = 3L^3 \varepsilon_{\alpha\beta\gamma\delta} \,, \tag{6.45}$$

with $\varepsilon_{\alpha\beta\gamma\delta}$ the Levi-Civita tensor on a unit-radius $\mathbb{S}^4$. Then one arrives at the stress tensor

$$\begin{aligned} T_{\mu\nu} &= \frac{1}{12} F_{\mu MNP} F_\nu^{MNP} - \frac{1}{96} F_{MNPQ} F^{MNPQ} G_{\mu\nu} = -\frac{9L^6}{4} e^{-\frac{48\phi}{5}} g_{\mu\nu} \\ T_{\alpha\beta} &= \frac{1}{12} F_{\alpha MNP} F_\beta{}^{MNP} - \frac{1}{96} F_{MNPQ} F^{MNPQ} G_{\alpha\beta} = \frac{9L^6}{4} e^{-6\phi} \tilde{g}_{\alpha\beta} \,, \end{aligned} \tag{6.46}$$

and 11-dimensional Einstein's equations

$$\begin{aligned} \mathcal{R}_{\mu\nu} - \frac{\mathcal{R}}{2} G_{\mu\nu} &= -\frac{9L^6}{4} e^{-\frac{48\phi}{5}} g_{\mu\nu} \\ \mathcal{R}_{\alpha\beta} - \frac{\mathcal{R}}{2} G_{\alpha\beta} &= \frac{9L^6}{4} e^{-6\phi} \tilde{g}_{\alpha\beta} \,. \end{aligned} \tag{6.47}$$

The equations simplify after reduction to

$$\begin{aligned} R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} &= \frac{36}{5} \left(\partial_\mu\phi \partial_\nu\phi - \frac{1}{2} (\partial\phi)^2 g_{\mu\nu} \right) - \frac{1}{2} 3 e^{-\frac{18}{5}\phi} \left(\frac{3}{2} L^6 e^{-6\phi} - 4 \right) g_{\mu\nu} \,, \\ \Box\phi &= 3 e^{-\frac{18}{5}\phi} \left(1 - L^6 e^{-6\phi} \right) \,, \end{aligned} \tag{6.48}$$

which follow from variation of a 7d effective action

$$S_7 = \frac{\mathrm{vol}(\mathbb{S}^4)}{2\kappa_{11}^2} \int d^7x \sqrt{-g} \left(R - \frac{36}{5} (\partial\phi)^2 - V(\phi) \right) + (\mathrm{bdy}) \,, \qquad V = 3 e^{-\frac{18\phi}{5}} \left(\frac{3L^6}{2} e^{-6\phi} - 4 \right) . \tag{6.49}$$

The potential has a unique minimum at $e^\phi = L$. At this minima, the action behaves as Einstein-Hilbert with cosmological constant

$$S = \frac{\mathrm{vol}(\mathbb{S}^4)}{2\kappa_{11}^2} \int d^7x \sqrt{-g} \left(R + \frac{15}{2L^{\frac{18}{5}}} \right) + (\mathrm{bdy}) \,, \tag{6.50}$$

with effective radius $L_{\rm eff}^2 = 4L^{\frac{18}{5}}$. Adding a small fluctuations of ϕ around its minimum $\phi = \ln L + \delta\phi$, there is a shift in effective potential

$$S_{\rm quad} = -\frac{72{\rm vol}(\mathbb{S}^4)}{10\kappa_{11}^2}\int d^7x\sqrt{-g}\left(\frac{1}{2}(\partial\delta\phi)^2 + \frac{36}{L_{\rm eff}^2}\delta\phi^2\right), \tag{6.51}$$

corresponding to an effective mass of $m^2L_{\rm eff}^2 = 72$ and, by $\Delta(\Delta-d) = m^2L_{\rm eff}^2$, a dual operator of dimension $\Delta = 12$.

For a solution invariant under an $SO(2,5)\times SO(5)$ isometry, the metric takes the form

$$ds^2 = e^{-\frac{8\phi(\rho)}{5}}\left(e^{2A(\rho)}ds^2_{\rm AdS_6} + d\rho^2\right) + e^{2\phi(\rho)}d\Omega_4^2\,, \qquad ds^2_{\rm AdS_6} = \frac{-dt^2 + d\vec{x}^2 + dz^2}{z^2}\,. \tag{6.52}$$

Then, the flux is given by

$$F_{t1234z\rho} = \frac{3e^{-\frac{48\phi(\rho)}{5}+6A(\rho)}}{z^6}\,, \qquad F^{t12345z\rho} = -\frac{3\sqrt{\tilde{g}}}{\sqrt{-G}}\,. \tag{6.53}$$

We then proceed to find boundary conditions at $\rho = \bar{\rho}$ such that, with suitable boundary terms, the SUGRA effective action has a consistent variational principle. The calculation goes in the same way as in the discussion in the previous sections. Imposing a Neumann-like boundary condition on the RR potential at $\rho = \bar{\rho}$ implies the existence of a uniform density of $-N$ units of M5-brane charge smeared homogeneously over the $\mathbb{S}^4$. We posit that this distribution of M5 brane charge is made up by a uniform distribution of $\overline{\rm M5}$ branes, which leads to a particular boundary contribution to the stress-energy. Including it we then impose a Neumann condition on the metric, which includes the stress tensor of the brane distribution. If we let $\Sigma = {\rm vol}_{\mathbb{S}^4} = d^4\theta\,\sqrt{\tilde{g}}$ denote the smearing form, then the total boundary term, the sum of the Gibbons-Hawking term and the action of this distribution of $\overline{\rm M5}$-branes, reads

$$S_{\rm ETW} = \frac{1}{2\kappa_{11}^2}\left(2\int d^{10}x\sqrt{-H}\,K(H) - 3\left(\int d^6\sigma\sqrt{-{\rm P}[G]}\wedge\Sigma - \int C_6\wedge\Sigma\right)\right), \tag{6.54}$$

with an effective potential

$$V_{\rm eff} \propto \sqrt{-{\rm P}[G]} - \int C_6 \propto e^{-\frac{24\phi}{5}+6A} - 6\int_{\bar{\rho}}^{\rho} e^{-\frac{48\phi(\rho')}{5}+6A(\rho')}d\rho'\,, \tag{6.55}$$

for a single $\overline{\rm M5}$ located at some point in the distribution. This potential has an unstable maximum at $\rho = \bar{\rho} = 0$ as shown in Figure 6.4, indicating that smeared distribution of $\overline{\rm M5}$

brane is unstable. We will now show that this can also be realized through light crossing time.

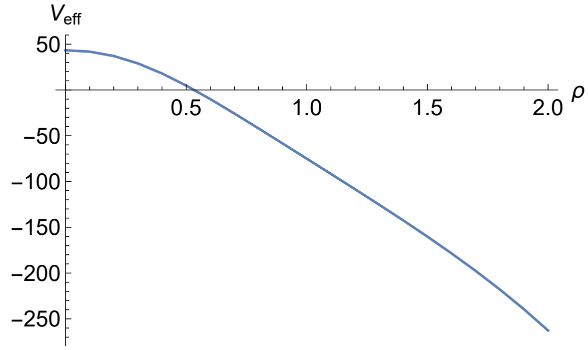

Figure 6.4: The effective potential of a single $\overline{\rm M5}$ brane as a function of ρ. The potential is globally maximized at $\rho = \bar{\rho} = 0$, indicating an instability to separating from smeared distribution of branes.

The effective Israel-junction conditions in 11-dimensions are

$$\begin{aligned} K(H)_{\mu\nu} - K(H)H_{\mu\nu} &= -\frac{3}{2}e^{-4\phi}H_{\mu\nu}\,, \\ K(H)_{\alpha\beta} - K(H)H_{\alpha\beta} &= 0\,, \end{aligned} \tag{6.56}$$

which reduce to

$$\begin{aligned} K(h)_{\mu\nu} - K(h)h_{\mu\nu} &= -\frac{3}{2}e^{-\frac{24\phi}{5}}h_{\mu\nu}\,, \\ K(h) &= -\frac{9}{5}\partial_\rho\phi\,. \end{aligned} \tag{6.57}$$

The first line is the boundary condition from the reduced action

$$\tilde{S}_{\rm ETW} = \frac{{\rm vol}(\mathbb{S}^4)}{2\kappa_{11}^2}\left(2\int d^6x\sqrt{-h}\left(K(h) - \frac{3}{2}e^{-\frac{24\phi}{5}}\right)\right)\,, \tag{6.58}$$

with the angular part carrying no energy momentum. The boundary conditions obtained are

$$\phi'(\bar{\rho}) = -e^{-\frac{24\phi}{5}}\,, \qquad A'(\bar{\rho}) = -\frac{3}{10}e^{-\frac{24\phi}{5}}\,. \tag{6.59}$$

The 7d equations of motion in $L=1$ units are

$$R_{\mu\nu}-\frac{R}{2}g_{\mu\nu}=\frac{36}{5}\left(\partial_\mu\phi\partial_\nu\phi-\frac{1}{2}(\partial\phi)^2g_{\mu\nu}\right)-\frac{1}{2}3e^{-\frac{18\phi}{5}}\left(\frac{3}{2}e^{-6\phi}-4\right)g_{\mu\nu}\,,$$
$$\Box\phi=3e^{-\frac{18\phi}{5}}\left(1-e^{-6\phi}\right)\,. \tag{6.60}$$

The $\rho\rho$ component upon substituting the boundary conditions simplifies to

$$e^{-2A(\bar\rho)}=\frac{2}{5}e^{-\frac{18\phi(\bar\rho)}{3})}\,, \tag{6.61}$$

while the non-$\rho\rho$ component contracted by null vector and the scalar equation give the equations of motion to be solved.

In summary, we are solving the equations

$$A''-e^{-2A}=-\frac{36}{25}\phi'^2\,,$$
$$\phi''+6A'\phi'=3e^{-\frac{18\phi}{5}}\left(1-e^{-6\phi}\right)\,, \tag{6.62}$$

with the boundary conditions

$$\phi'(\bar\rho)=-e^{-\frac{24\phi(\bar\rho)}{5}}\,,\qquad A'(\bar\rho)=-\frac{3}{10}e^{-\frac{24\phi(\bar\rho)}{5}}\,,\qquad e^{-2A(\bar\rho)}=\frac{2}{5}e^{-\frac{18\phi(\bar\rho)}{3}}\,, \tag{6.63}$$

by tuning the parameter $\phi(\bar\rho)$.

Performing the numerical computation with $\bar\rho=0$, we have been able to obtain a solution shown in Figure 6.5 with the desired property of approximating the AdS boundary condition of $\phi\to 0$ as $\rho\to\infty$.

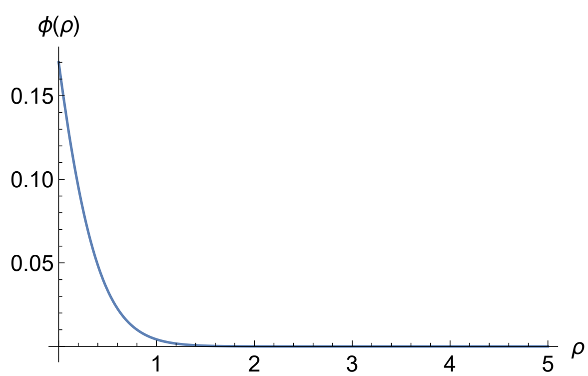

Figure 6.5: The profiles for $\phi(\rho)$ in the numerical solution obeying the $\overline{\text{M5}}$ boundary condition at $\rho=0$.

The solution for ϕ deviates after $\rho \approx 7$ due to the exponentially growing mode starting to dominate. Comparing the solution for $A(\rho)$ with pure AdS_7 cut off by a tensionful brane, we have

$$e^A = 2\cosh\left(\frac{\rho - \rho_0}{2}\right), \tag{6.64}$$

with $L_{\text{eff}}^2 = 4L^{\frac{18}{5}}$. The maximum occurs at $\rho_0 = 0.822$, which corresponds to the tension $T \approx 0.974$ using the equation

$$T = \frac{d-1}{L_{\text{eff}}} \tanh\left(\frac{\rho_0}{L_{\text{eff}}}\right). \tag{6.65}$$

Plotting this approximate solution against the numerical solution in Figure 6.6, we find that the approximation is close to the numerical except for near the ETW brane.

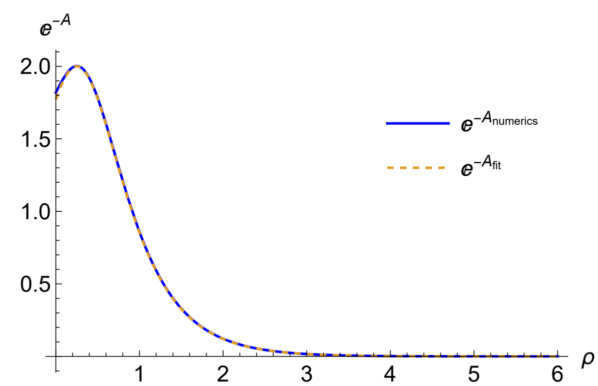

Figure 6.6: A fit of e^{-A} to $2\cosh\left(\frac{\rho-\rho_0}{2}\right)$. The value of $\bar{\rho}$ is fixed by the tension $T \approx 0.974$ (which has $\phi_b = 2.26$) to be $\rho_0 \approx 0.822$

Therefore, attaching tensionful brane solution after $\rho = 6$ and numerically evaluating the light-crossing integral, we have

$$\phi_b = \int_{\bar{\rho}=0}^{\infty} d\rho\, e^{-A(\rho)} \approx 2.26 < \pi\,. \tag{6.66}$$

Based on our criterion, $\phi_b < \pi$ indicates that the AdS boundary and the smeared ETW brane can be connected by a null geodesics.

6.5 $\mathbf{AdS_7 \times \mathbb{S}^4}$ Interface

We can also arrive at these boundary conditions and the 5-brane part of the ETW brane action by first considering a sharp interface connecting two asymptotically $\mathrm{AdS}_7 \times \mathbb{S}^4$ regions in a $SO(2,5) \times SO(5)$ invariant way, one supported by N_+ units of flux and the other by N_- units. The dual seven-form flux is given by

$$F_{\mu\nu\rho\sigma\tau\lambda\kappa} = 3\left(\widetilde{L}_+^3\Theta(\rho-\bar{\rho}) + \widetilde{L}_-^3\Theta(\bar{\rho}-\rho)\right) e^{-\frac{48}{5}\phi}\varepsilon_{\mu\nu\rho\sigma\tau\lambda\kappa} \,. \tag{6.67}$$

Then, we can calculate the stress tensor as

$$\begin{aligned} T_{\mu\nu} &= -\frac{9}{4}\left(\widetilde{L}_+^6\Theta(\rho-\bar{\rho}) + \widetilde{L}_-^6\Theta(\bar{\rho}-\rho)\right) e^{-\frac{48}{5}\phi} g_{\mu\nu} \,, \\ T_{\alpha\beta} &= \frac{9}{4}\left(\widetilde{L}_+^6\Theta(\rho-\bar{\rho}) + \widetilde{L}_-^6\Theta(\bar{\rho}-\rho)\right) e^{-6\phi}\tilde{g}_{\alpha\beta} \,, \end{aligned} \tag{6.68}$$

giving the 11d Einstein's equations

$$\begin{aligned} \mathcal{R}_{\mu\nu} - \frac{\mathcal{R}}{2}G_{\mu\nu} &= -\frac{9}{4}\left(\widetilde{L}_+^6\Theta(\rho-\bar{\rho}) + \widetilde{L}_-^6\Theta(\bar{\rho}-\rho)\right) e^{-\frac{48}{5}\phi} g_{\mu\nu} \,, \\ \mathcal{R}_{\alpha\beta} - \frac{\mathcal{R}}{2}G_{\alpha\beta} &= \frac{9}{4}\left(\widetilde{L}_+^6\Theta(\rho-\bar{\rho}) + \widetilde{L}_-^6\Theta(\bar{\rho}-\rho)\right) e^{-6\phi}\tilde{g}_{\alpha\beta} \,. \end{aligned} \tag{6.69}$$

The effective 7d equations are then

$$\begin{aligned} R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} &= \frac{36}{5}\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}(\partial\phi)^2 g_{\mu\nu}\right) - \frac{1}{2}V(\phi)g_{\mu\nu} \,, \\ V(\phi) &= 3e^{-\frac{18}{5}\phi}\left(\frac{3}{2}\left(\widetilde{L}_+^6\Theta(\rho-\bar{\rho}) + \widetilde{L}_-^6\Theta(\bar{\rho}-\rho)\right) e^{-6\phi} - 4\right) \,, \\ \Box\phi &= 3e^{-\frac{18}{5}\phi}\left(1 - \left(\widetilde{L}_+^6\Theta(\rho-\bar{\rho}) + \widetilde{L}_-^6\Theta(\bar{\rho}-\rho)\right) e^{-6\phi}\right) \,. \end{aligned} \tag{6.70}$$

These equations follow from the effective 7d action

$$S_7 = \frac{\mathrm{vol}(\mathbb{S}^4)}{2\kappa_{11}^2}\int d^7x\sqrt{-g}\left(R - \frac{36}{5}(\partial\phi)^2 - V(\phi)\right) + (\mathrm{bdy}) \,. \tag{6.71}$$

The flux background is not conserved, implying the existence of $N_+ - N_-$ units of $\overline{\mathrm{M5}}$-brane charge at $\rho = \bar{\rho}$, smeared homogeneously over the $\mathbb{S}^4$. Assuming that this charge density is made up by $N_+ - N_-$ $\overline{\mathrm{M5}}$ branes, and their associated stress energy, one ends up with 11-dimensional SUGRA together with this brane source at $\rho = \bar{\rho}$. There is an Israel junction condition at $\rho = \bar{\rho}$ that comes from integrating the Einstein's equations in a pillbox around $\rho = \bar{\rho}$ giving

$$\begin{aligned} \Delta\left(K(H)_{\mu\nu} - K(H)H_{\mu\nu}\right) &= -\frac{3}{2}\left(\widetilde{L}_{+}^{3} - \widetilde{L}_{-}^{3}\right) e^{-4\phi} H_{\mu\nu}\,, \\ \Delta\left(K(H)_{\alpha\beta} - K(H)H_{\alpha\beta}\right) &= 0\,. \end{aligned} \tag{6.72}$$

These conditions reduce to

$$\begin{aligned} \Delta\left(K(h)_{\mu\nu} - K(h)h_{\mu\nu}\right) &= -\frac{3}{2}\left(\widetilde{L}_{+}^{3} - \widetilde{L}_{-}^{3}\right) e^{-\frac{24\phi}{5}} h_{\mu\nu}\,, \\ \Delta\left(\frac{9}{5}\phi' + K(h)\right) &= 0\,, \end{aligned} \tag{6.73}$$

where h is the induced metric on the boundary as determined from the 7d metric g. The effective action at the interface is

$$\tilde{S} = -\frac{\mathrm{vol}(\mathbb{S}^4)}{2\kappa_{11}^2} 2\left(\int d^6x\sqrt{-h}\left(K(h) - \frac{3}{2}\left(\widetilde{L}_{+}^{3} - \widetilde{L}_{-}^{3}\right) e^{-\frac{24\phi}{5}}\right)\right)\,. \tag{6.74}$$

The boundary conditions at $\rho = \bar{\rho}$ read

$$\begin{aligned} A'(\bar{\rho}_+) - A'(\bar{\rho}_-) &= -\frac{3}{10}\left(\widetilde{L}_{+}^{3} - \widetilde{L}_{-}^{3}\right) e^{-\frac{24\phi(\bar{\rho})}{5}}\,, \\ \phi'(\bar{\rho}_+) - \phi'(\bar{\rho}_-) &= -\left(\widetilde{L}_{+}^{3} - \widetilde{L}_{-}^{3}\right) e^{-\frac{24\phi(\bar{\rho})}{5}}\,. \end{aligned} \tag{6.75}$$

Similar to previous sections, we use the $\rho\rho$ component of Einstein's equation

$$\begin{aligned} A'(\bar{\rho}_+)^2 + e^{-2A(\bar{\rho})} &= \frac{6}{25}\phi'(\bar{\rho}_+)^2 + \frac{e^{-\frac{18\phi(\bar{\rho})}{5}}}{10}\left(4 - \frac{3}{2}e^{-6\phi(\bar{\rho})}\right)\,, \\ A'(\bar{\rho}_-)^2 + e^{-2A(\bar{\rho})} &= \frac{6}{25}\phi'(\bar{\rho}_-)^2 + \frac{e^{-\frac{18\phi(\bar{\rho})}{5}}}{10}\left(4 - \frac{3}{2}e^{-6\phi(\bar{\rho})}\right)\,, \end{aligned} \tag{6.76}$$

to give further boundary conditions. We chose $\phi(\bar{\rho})$ and $\phi'(\bar{\rho}_-)$ to be the parameters and the boundary conditions for these parameters are

$$\begin{aligned} \phi'(\bar{\rho}_+) &= \phi'(\bar{\rho}_-) - \left(\widetilde{L}_+^3 - \widetilde{L}_-^3\right) e^{-\frac{24\phi(\bar{\rho})}{5}} \,, \\ A'(\bar{\rho}_-) &= \frac{4}{5}\phi'(\bar{\rho}_-) + \frac{1}{2}\widetilde{L}_-^3 e^{-\frac{24\phi(\bar{\rho})}{5}} \,, \\ A'(\bar{\rho}_+) &= \frac{4}{5}\phi'(\bar{\rho}_-) + \left(\frac{4}{5}\widetilde{L}_-^3 - \frac{3}{10}\widetilde{L}_+^3\right) e^{-\frac{24\phi(\bar{\rho})}{5}} \,, \\ e^{-2A(\bar{\rho})} &= -\frac{2}{5}\left(\phi'(\bar{\rho}_-) + \widetilde{L}_-^3 e^{-\frac{24\phi(\bar{\rho})}{5}}\right)^2 + \frac{2}{5} e^{-\frac{18\phi(\bar{\rho})}{5}} \,. \end{aligned} \tag{6.77}$$

We are now ready to solve the the equations of motion for the warpfactors $A(\rho)$ and $\phi(\rho)$, subject to this boundary condition. The scalar equation of motion and the 7d Einstein's equation contracted with the null vector $u^\mu \partial_\mu = ze^{-A}\partial_t + \partial_\rho$ read

$$\begin{aligned} \phi'' + 6A'\phi' &= 3e^{-\frac{18\phi}{5}}\left(1 - \left(\widetilde{L}_+^6 \Theta(\rho - \bar{\rho}) + \widetilde{L}_-^6 \Theta(\bar{\rho} - \rho)\right) e^{-6\phi}\right) \,, \\ A'' - e^{-2A} &= -\frac{36}{25}\phi'^2 \,. \end{aligned} \tag{6.78}$$

Performing the numerical computation for a range of values with initial conditions $\phi(\bar{\rho})$ and $\phi'(\bar{\rho}_-)$ for each side of the interface, we found that the parameter space in which both sides have $\phi \to 0$ away from the interface is very restricted. In the case where both sides of the interface has same magnitude but opposite sign of the five-form flux or $\widetilde{L}_-^3 = -\widetilde{L}_+^3$, the solution is exactly the same as the previous section except mirrored on both sides of the interface.

Chapter 7

Conclusion

This thesis motivates a potential criterion to determine the healthiness of gravitational dual theory to a BCFT. We have tested such criterion in various examples including empty AdS, linear perturbative massive scalar field in AdS, interface structure of non-SUSY Janus configuration, and smeared distribution of ETW branes in $AdS_5 \times \mathbb{S}^5$ vacuum of type IIB SUGRA, as well as $AdS_4 \times \mathbb{S}^7$ and $AdS_7 \times \mathbb{S}^4$ vacua of 11-dimensional SUGRA.

We have demonstrated that when the parameter $\phi_b \geq \pi$, then it takes more than π global AdS time and hence infinite Poincaré time for null geodesics to travel from the AdS boundary to the ETW brane or the other asymptotic boundary. We argue that emergend singularities in BCFT two-point functions for such configurations are unphysical due to infinite light crossing time, hence the dual theory is healthy. However, configurations with $\phi_b < \pi$ permit null geodesics to travel from AdS boundary to ETW brane in finite Poincaré time, leading to an unhealthy dual to BCFT. The examples we have considered so far seem to follow this general trend.

The first example we considered is the toy model of empty AdS terminated by tensionful ETW brane. By considering the light crossing time, we determined that $\phi_b < \pi$ unless the ETW brane is located at the opposite AdS boundary. This means that the toy model proposed by Takayanagi [1] does not permit a healthy gravitational dual to a BCFT, and other configurations need to be considered.

To understand how addition of matter impacts the light crossing time, we considered a perturbation of a massive scalar field in empty AdS. For a Janus like configuration, the addition of scalar field caused change in light crossing time due to the change in bulk geometry. By numerical computation, we demosntrated that light crossing time always increases for relevant deformation. Because $\phi_b = \pi$ for empty AdS, a Janus configuration with perturbative matter has $\phi_b > \pi$, meaning that the dual bulk theory to BCFT is healthy according to our criterion. For a situation where empty AdS is terminated by an ETW brane, there is an additional change in light crossing time due to the change in location of ETW brane. Similar computation was performed to show that the light crossing time increases as a result of additional scalar field in most of the relevant regime. Because $\phi_b < \pi$ for empty AdS, this suggests that a healthy bulk theory of BCFT should have some matter content to increase the light crossing time. In both cases, irrelevant deformations lead to non-normalizable scalar fields, which are set to zero to satisfy the boundary conditions.

The next example we considered was an interface configuration of non-SUSY Janus type solution. Because this is a stable solution in string theory, we expect the bulk theory to be a healthy dual to interface CFT. Indeed, the computation results in $\phi_b > \pi$ as expected. This result provides a strong supportive argument to the proposed criterion, as we have demonstrated that a known stable configuration in string theory is in agreement.

Then, we attempted to embed the simple model of ETW branes in holography into the lamppost examples of AdS/CFT. We impose rotational invariance in $\mathbb{S}^5$ for the $\mathrm{AdS}_5 \times \mathbb{S}^5$ vacuum of type IIB SUGRA, with a smeared distribution of ETW brane dual to a conformal boundary of the BCFT. We present a solution of asymptotically $\mathrm{AdS}_5 \times \mathbb{S}^5$ type IIB SUGRA ending on a smeared distribution of D3-branes, respecting the appropriate boundary conditions. The obtained geometry is nearly $\mathrm{AdS}_5 \times \mathbb{S}^5$ for most of the spacetime up to a region very close to the smeared ETW brane. The light crossing time of this solution reveals that $\phi_b < \pi$, meaning that our solutions are unstable. This is expected as we found an instability in the effective potential of D3-branes making up the ETW brane.

We also adapted our methods to search for $SO(6)$-invariant domain wall of smeared D3-branes connecting two asymptotically $\mathrm{AdS}_5 \times \mathbb{S}^5$ regions, supported by N_+ and N_- units of five-form flux respectively. Interestingly, we do not find solutions that respect the boundary conditions far away from the interface for both sides. The only exception is where $N_- = -N_+$, in which the solution is two copies of the geometry with a smeared ETW brane describe before, merged together at the interface.

Similar computations are also performed in $\mathrm{AdS}_4 \times \mathbb{S}^7$ and $\mathrm{AdS}_7 \times \mathbb{S}^4$ vacua of 11-dimensional SUGRA. All computations, including the results of the interface are exactly the same as type IIB SUGRA. The smeared ETW configurations do not permit healthy dual to a BCFT according to our criterion, which is supported by the instability in the potential of M-branes stacked at ETW brane.

Our proposed light crossing criterion seems to be a powerful tool for potentially predicting the healthiness of AdS/BCFT duality. Although we have tested our criterion in several notable examples, more examples need to be considered for a better understanding. Alternatively, one can perhaps show this light crossing criterion arising from BCFT computations, which would strengthen our argument.

Bibliography

[1] T. Takayanagi, "Holographic dual of a boundary conformal field theory," *Physical Review Letters*, vol. 107, Aug. 2011.

[2] A. G. Cavalcanti, D. Melnikov, and M. R. O. Silva, "Studies of boundary entropy in AdS/BCFT," 2019.

[3] G. Penington, "Entanglement wedge reconstruction and the information paradox," 2020.

[4] A. Almheiri, N. Engelhardt, D. Marolf, and H. Maxfield, "The entropy of bulk quantum fields and the entanglement wedge of an evaporating black hole," *Journal of High Energy Physics*, vol. 2019, Dec. 2019.

[5] T. Mori and B. Yoshida, "Exploring causality in braneworld/cutoff holography via holographic scattering," *Journal of High Energy Physics*, vol. 2023, Oct. 2023.

[6] H. Omiya and Z. Wei, "Causal structures and nonlocality in double holography," *Journal of High Energy Physics*, vol. 2022, July 2022.

[7] J. Penedones, "TASI lectures on AdS/CFT," in *New Frontiers in Fields and Strings*, WORLD SCIENTIFIC, Nov. 2016.

[8] M. S. Costa, J. Penedones, D. Poland, and S. Rychkov, "Spinning conformal correlators," *Journal of High Energy Physics*, vol. 2011, Nov. 2011.

[9] J. Maldacena *International Journal of Theoretical Physics*, vol. 38, no. 4, p. 1113–1133, 1999.

[10] D. McAvity and H. Osborn, "Conformal field theories near a boundary in general dimensions," *Nuclear Physics B*, vol. 455, p. 522–576, Sept. 1995.

[11] D. Mazáč, L. Rastelli, and X. Zhou, "An analytic approach to BCFT_d," *Journal of High Energy Physics*, vol. 2019, Dec. 2019.

[12] D. Neuenfeld, "The dictionary for double holography and graviton masses in d dimensions," 2021.

[13] W. Reeves, M. Rozali, P. Simidzija, J. Sully, C. Waddell, and D. Wakeham, "Looking for (and not finding) a bulk brane," *Journal of High Energy Physics*, vol. 2021, Dec. 2021.

[14] D. Bak, M. Gutperle, and S. Hirano, "A dilatonic deformation of AdS_5 and its field theory dual," *Journal of High Energy Physics*, vol. 2003, p. 072–072, May 2003.

[15] E. D'Hoker, J. Estes, and M. Gutperle, "Ten-dimensional supersymmetric janus solutions," *Nuclear Physics B*, vol. 757, p. 79–116, Nov. 2006.

[16] J. Polchinski, *String Theory*. Cambridge Monographs on Mathematical Physics, Cambridge University Press, 1998.

[17] P. G. Freund and M. A. Rubin, "Dynamics of dimensional reduction," *Physics Letters B*, vol. 97, no. 2, pp. 233–235, 1980.

[18] M. Bremer, M. Duff, H. Lü, C. Pope, and K. Stelle, "Instanton cosmology and domain walls from M-theory and string theory," *Nuclear Physics B*, vol. 543, p. 321–364, Mar. 1999.