Consensus in Multi-Agent Systems and Bilateral Teleoperation with Communication Constraints

by

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B.Eng., Northwestern Polytechnical University, 2007

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ABSTRACT

With the advancement of communication technology, more and more control processes happen in networked environment. This makes it possible for us to deploy multiple systems in a spatially distributed way such that they could finish certain tasks collaboratively. While it brings about numerous advantages over conventional control, challenges arise in the mean time due to the imperfection of communication. This thesis is aimed to solve some problems in cooperative control involving multiple agents in the presence of communication constraints.

Overall, it is comprised of two main parts: Distributed consensus in multi-agent systems and bilateral teleoperation. Chapter 2 to Chapter 4 deal with the consensus problem in multi-agent systems. Our goal is to design appropriate control protocols such that the states of a group of agents will converge to a common value eventually. The robustness of multi-agent systems against various adverse factors in communication is our central concern. Chapter 5 copes with bilateral teleoperation with time delays. The task is to design control laws such that synchronization is reached between the master plant and slave plant. Meanwhile, transparency should be maintained within an acceptable level.

Chapter 2 investigates the consensus problem in a multi-agent system with directed communication topology. The time delays are modeled as a Markov chain,

thus more characteristics of delays are taken into account. A delay-dependent approach has been proposed to design the Laplacian matrix such that the system is robust against stochastic delays. The consensus problem is converted into stabilization of its equivalent error dynamics, and the mean square stability is employed to characterize its convergence property. One feature of Chapter 2 is redesign of the adjacency matrix, which makes it possible to adjust communication weights dynamically. In Chapter 3, average consensus in single-integrator agents with time-varying delays and random data losses is studied. The interaction topology is assumed to be undirected. The communication constraints lie in two aspects: 1) time-varying delays that are non-uniform and bounded; 2) data losses governed by Bernoulli processes with non-uniform probabilities. By considering the upper bounds of delays and probabilities of packet dropouts, sufficient conditions are developed to guarantee that the multi-agent system will achieve consensus. Chapter 4 is concerned with the consensus problem with double-integrator dynamics and non-uniform sampling. The communication topology is assumed to be fixed and directed. With the adoption of time-varying control gains and the theory on stochastic matrices, we prove that when the graph has a directed spanning tree and the control gains are properly selected, consensus will be reached.

Chapter 5 deals with bilateral teleoperation with probabilistic time delays. The delays are from a finite set and each element in the set has a probability of occurrence. After defining the tracking error between the master and slave, the input-to-state stability is used to characterize the system performance. By taking into account the probabilistic information in time delays and using the pole placement technique, the teleoperation system has achieved better position tracking and enhanced transparency.

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Acronyms

AUV	autonomous underwater vehicle
BIBO	bounded-input-bounded-output
ISS	input-to-state stability/stable
LMI	linear matrix inequality
LTI	linear time-invariant
MAS	multi-agent system
MJLS	Markov jump linear system
NCS	networked control system
UAV	unmanned aerial vehicle

Chapter 1

Introduction

In this chapter, an introduction to cooperative control and a review of its history and development will be presented. And then some major challenges and motivation for this research are stated.

1.1 An Overview on Cooperative Control

Traditionally, a control task is often confined within a local environment and one plant is only in charge of its own mission. When it comes to coordinating a number of systems, a simple way is to have a central computer or microcontroller to control each plant. While this centralized approach is easy in implementation, a big challenge arises with no surprise, especially when the scale of a system is becoming larger. The central control computer has to assume a huge load from both communication and computation, and must be highly reliable. From this perspective, a centralized system is fragile to failure of the central coordinator. When the scale of a system increases significantly, this strategy may not be implementable. These drawbacks call for the emergence of a new direction in control–distributed cooperative control. Compared with the conventional centralized control, this new strategy has many merits, which will be illustrated in the following sections.

Recently, with the development of communication technology, especially the mobile sensor and actuator networks, cooperative control that coordinates the motion of a group of dynamic systems has received a growing amount of attention. For example, in the workshop [62], cooperative control was discussed by many researchers, and the main concern was the decentralized implementation of cooperative control featuring collective behaviour and goal. In addition, leading international journals have held special issues on this topic; see *SIAM Journal on Control and Optimization* special issue on control and optimization in cooperative networks (Volume 48, Issue 1, 2009), *ASME Journal of Dynamic Systems, Measurement, and Control* special issue on analysis and control of multi-agent dynamic systems (Volume 129, Issue 5, 2007), *International Journal of Robust and Nonlinear Control* special issue on cooperative control of unmanned aerial vehicles (Volume 18, Issue 2, 2008), etc.

A fundamental topic in cooperative control is the so-called consensus problem [53] [107] [104] [134] [65]. Given a multi-agent system (MAS) consisting of multiple individual physical systems, it is required that the state of each agent converge to a common reference (a function or a value). Then we say that the MAS has reached consensus. Consensus has applications in many aspects: Rendezvous [146], flocking [95], formation control [63] [75], etc. At present, the consensus problem is being researched in the framework of distributed control. Coordination among different agents is achieved through communication networks and the network connection topology is defined by a graph. Each agent exchanges information with its neighbours. In fact, it is not necessary to require each agent to be able to directly communicate with all other agents. The mathematical theory on graph and stochastic matrices has played a crucial role in the analysis of convergence to consensus. A variety of control approaches can be applied to solve the consensus problem, such as the Lyapunov theory [96] [124], output regulation [136], among agents.

Bilateral teleoperation is a special type of application of cooperative control. Generally, for a bilateral teleoperation setup, there are a master group and a slave group. Often the master group consists of one manipulator, while the slave group may be composed of one or multiple robots. The goal is to design a control law such that (1) synchronization between these two groups is achieved, and (2) transparency is guaranteed. The master and slave do not have to be identical, but usually they possess the same model structure. The master sends its state information to the slave, and on the other hand, it receives feedback from the slave. In this way, the slave could follow the motion of the master. A comprehensive review on bilateral teleoperation can be found in [45]. A successful design of bilateral teleoperation control system is very beneficial for the work in hazardous workplace, such as in the mining tunnel [42], inside the nuclear power station [26] [64]. It is also applicable in tele-surgery [102] [111].

With a network as the communication medium, a variety of constraints pose great challenges to control engineers. These include time delays, data packet dropouts, quantization errors, etc. Not only do these factors degrade the system performance, but also they may even cause instability if the influence reaches a threshold. A rich literature can be found on communication constraints in control systems; see [2] [82] [89] [139] and the references therein. Therefore, how to appropriately model these constraints and find conditions to guarantee the prescribed performance requirements is the central issue of the thesis research. Figure 1.1 shows the framework of research in this thesis. It should be noted that we are only concerned with time delays and data losses in this research work. Consensus with other communication constraints (e.g., quantization) could be in the future work.

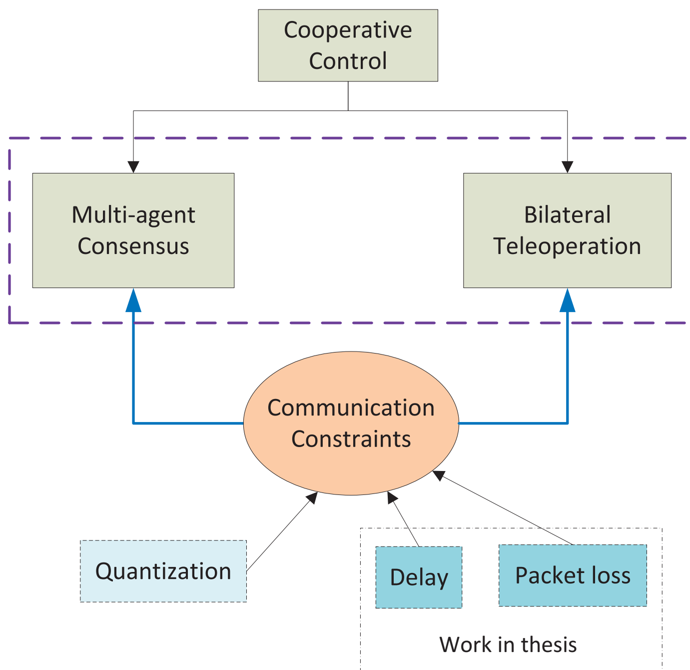

Figure 1.1: Framework of the PhD thesis research.

1.1.1 Details on Communication Constraints

In a network environment, many factors (communication constraints) will affect the system performance. In this subsection, let us look at some of them in more depth.

- **Time delays:** A delay is the time lag between when to send the information and when to use the received information. It is either because data packets cannot be sent and used at the same time, or because the controller needs time to process them. Time delays are ubiquitous in the natural world. How they influence the system performance depends on both the quantity of delays and the properties of system dynamics.

- **Quantization errors:** Real values cannot be directly transmitted with infinite bit rate through digital communication channels. Rather, they must be rounded off (a process of quantization) to be within certain range. This causes the difference between real data and transmitted data. If a control strategy is not designed properly, divergence of the system response may ensue.

- **Missing measurements:** Because of the unreliability of communication links, data transmission could fail at some time instants. In such a situation, the receiver node does not get any data. Therefore, how to deal with the incompleteness of data in the cooperative control design is a challenge.

- **Noise:** This is an unavoidable phenomenon in any practical applications. Similar to the effect of quantization, noise also causes inaccuracy in measurements, which degrades the system performance or even leads to instability. The most commonly seen type of noise is the additive noise, which is exerted as an exogenous input of a system.

- **Discontinuities in signal sampling:** This is mainly due to the implementation of controllers. In digital control, the concept "sampled-data" is widely used, meaning that the control signal is generated from the measurements that are periodically sampled. With a zero-order hold, there will be a jump in control input between two consecutive time intervals. Besides periodic sampling, there exists another type of sampling scheme: Non-uniform sampling. Or we can call it aperiodic sampling or irregular sampling as well.

In the next two sections, some recent research progress on cooperative control, including the consensus problem in MASs and bilateral teleoperation, is reviewed.

1.2 Consensus in Multi-Agent Systems

1.2.1 Background Knowledge

With the development of sensor networks, cooperative control in MASs has attracted more and more attention in recent years; see [53] [96] [107] [134] and the references therein. It has wide applications in both military and civilian fields [87], such as cooperative surveillance, rendezvous, and intelligent transportation systems. A significant amount of work has been devoted to this topic due to its potential in improving a nation's competence in this information rich world.

Cooperative control involves a number of agents connected by the communication network. An agent could be an unmanned aerial vehicle (UAV), autonomous underwater vehicle (AUV), or any type of dynamic systems. Each of them is autonomous, meaning that they have an onboard microcontroller to schedule their own tasks. In the mean time, they communicate with other agents (neighbours) around them, such that a mission will be completed collectively. Usually, this is assessed by the degree of consensus: The physical states of interest converge to a common decision value. Rendezvous is apparently a perfect example for consensus. Sometimes, for example, in formation control, it is desired that the relative distance between agents be kept constant. Under this circumstance, the difference between the states of agents converges to a constant value, and this kind of problem can still be dealt with under the consensus framework.

At present, most of the effort is put into the distributed consensus, which is in contrast with the centralized form. In the traditional centralized control, there is a high-level leader that coordinates the behaviour of different parts (agents) of the overall system. The leader directly sends commands to and collects information from all other agents. If we design a control law to meet the performance requirement of each individual, consensus can be achieved. However, the prerequisite is that both the leader and communication must be reliable, which could be a challenge in some situation. Distributed consensus aims at improving the robustness and reliability against node failures. For instance, even if a channel malfunctions or fails, the agents should continue to work collectively to achieve the preset goal. This manner of working distributes the work load to multiple agents more evenly, thus enhances the overall reliability. Meanwhile, a distributed algorithm is more scalable, which facilitates its use in large-scale networked systems.

Figure 1.2 shows the illustration of an MAS. There are five vehicles labeled from 1 to 5. Their information is transmitted over the communication channels, which may be subject to constraints. The arrows indicate the direction of information flow. For example, the directed arrow from agent 3 to agent 1 implies that agent 1 receives information from agent 3, but no information of agent 1 flows to agent 3. The up-down arrow between agent 1 and agent 2 represents the bidirectional information flow.

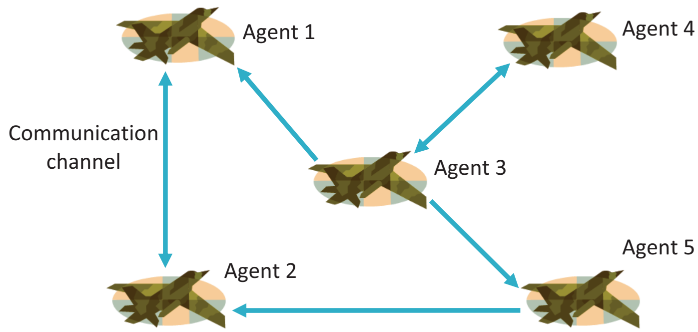

Figure 1.2: The schematic of a multi-agent system.

Mathematically, we can describe the information flow among the MAS in Figure 1.2 by a graph of five nodes, say $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$, where $\mathcal{V} = \{v_1, v_2, v_3, v_4, v_5\}$ is the node set, $\mathcal{E} = \{(v_1, v_2), (v_2, v_1), (v_3, v_1), (v_3, v_5), \ldots\} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set, and $A = [a_{ij}] \in \mathbb{R}^{5\times 5}$ is the adjacency matrix. Each node in $\mathcal{V}$ is associated with an agent, and each element (v_i, v_j) in $\mathcal{E}$ corresponds to a communication link from agent i to agent j. For instance, there is information flowing from agent 3 to agent 1, and thus $(v_3, v_1) \in \mathcal{E}$. The adjacency matrix A describes this relationship among agents numerically by assigning each edge a weight. That is, when $(v_i, v_j) \in \mathcal{E}$, we have $a_{ji} \neq 0$; otherwise $a_{ji} = 0$. For Figure 1.2, $a_{13} \neq 0$ because $(v_3, v_1) \in \mathcal{E}$, and $a_{23} = 0$ because $(v_3, v_2) \notin \mathcal{E}$. For more details on graph theory, refer to [37].

The dynamics of an MAS of N agents is described by differential equations. Suppose that each agent i has dynamics of the form

$$\begin{aligned} \dot{x}_i &= f(x_i, u_i), \\ y_i &= \phi(x_i), \quad i = 1, 2, \ldots, N, \end{aligned}$$

where $x_i \in \mathbb{R}^n$ is the state of agent i, $u_i \in \mathbb{R}^m$ is the control input, and $y_i \in \mathbb{R}^p$ is the output. Then the goal is to find the control law $u_i = g_i(y_1, y_2, \ldots, y_N)$ such that the following equation holds,

$$\lim_{t\to\infty} ||x_i(t) - x_j(t)|| = 0, \quad i, j \in \{1, 2, \ldots, N\}, i \neq j,$$

which characterizes the level of consensus. The information available to agent i is defined by its neighbour set.

1.2.2 Literature Review on Consensus and Consensus with Communication Constraints

In this subsection, the earlier work on consensus in MASs is reviewed. Then some interesting and challenging issues are summarized.

The research on consensus has received attention from academia as the agreement problem. In [7], the authors studied the agreement algorithm in the context of parallel computation, distributed optimization and signal processing. The study of consensus has not been surging until the appearance of [53], in which Jadbabaie *et al.* gave a theoretical explanation to the physical phenomenon of reaching a common heading angle among a group of particles [128]. The information exchange among agents was delineated by undirected graphs. The research work [53] features using a graph to characterize the interaction among agents and sufficient conditions on graph connectivity are developed such that consensus will be reached under these assumptions. Specifically, it is required that the union of graphs at all discrete time instants across any time interval of a given length should be jointly connected. Since there is no leader who is able to directly interact with all other agents, this strategy is essentially distributed (or decentralized).

Later, in [96] and [107], the relation between the graph structure and eigenvalues of the associated Laplacian matrix was further investigated. In particular, Olfati-Saber and Murray [96] dealt with the average consensus problem in directed networks, considering switching topology and constant time delay as well. The relationship between the connectivity of a directed graph that was strongly connected and the rank of its associated Laplacian matrix was uncovered. Different from the method in [53] to cope with switching topology, the work [96] uses the technique of common Lyapunov function with assumptions on graph connectivity. Both [53] and [96] made important contribution to the consensus problem in terms of graph connectivity. However, there are still some limitations: The graph in [53] is undirected, and the graph in [96] is directed and strongly connected. Ren and Beard [107] took a step further by studying the properties of general directed graphs. A necessary and sufficient condition was discovered between the connectivity of a directed graph and the eigenvalues of its corresponding graph Laplacian. The existence of a directed spanning tree is necessary and sufficient for reaching consensus in linear time-invariant (LTI) single-integrator agents. A spanning tree is in fact the minimum requirement on graph topology [107]. Similar results were developed in [78] [86]. These seminal work paved the way for subsequent research which was largely based on the conclusions about interaction topology.

More progress has been achieved to extend the previous results or consider more general settings. For example, earlier research on consensus mainly focused on the relatively simple single-integrator dynamics. After that, people are paying growing attention to more general dynamics, such as the double-integrator dynamics [104] [148], state-space models [109] [83], and Euler–Lagrange dynamics [105]. A variety of methods that have prevailed in control field have been used to solve the consensus problem, e.g., the Lyapunov theory, sliding mode control, model predictive control, output regulation, neural network. Nonlinearity is even taken into account and it may reside in system dynamics or the control law. By introducing the nonlinear function $\mathrm{sig}(\cdot)^{\alpha}$ in control protocol, the authors in [140] solved the finite-time consensus problem in first-order dynamics. Details on various extension work will be expanded in Subsection 1.2.3.

If we only deal with LTI systems without considering any restriction, the problem becomes deterministic and the analysis is relatively more straightforward. However, this does not comply with the real environment where there exist many factors that may destroy those ideal assumptions. There are many types of communication constraints in the consensus problem: time delays, data losses, quantization, switching topology, asynchrony in sampling and update, etc. Among them, let us look into three of them (time delays, data losses and switching topology) with more details.

In previous literature, the consensus problem with time delays has been widely investigated. Olfati-Saber and Murray [96] studied the average consensus in single-integrator agents in the presence of a single constant delay. The frequency domain approach was used to derive the stability condition dependent on the upper bound of delay. Afterwards, researchers studied the consensus problem with time-varying delays, e.g., [139] [124] dealing with time-varying delays and switching topology in first-order dynamics, [123] coping with the consensus problem in double-integrator dynamics with time-varying delays. When tackling time-varying delays, people shift from the classic frequency domain to time domain, and the Lyapunov technique is widely used. In the analysis, various information may be taken into consideration, such as the lower and upper bounds of delays. It has been shown in the research of networked control systems (NCSs) that when considering more information on delays, the obtained results are usually less conservative. For more work on consensus with time delays, refer to [74] [127] [138] and the references therein.

Besides delays, data loss (or packet dropout [29]) is another factor that affects system performance. It is due to the unreliability of communication channels, such as the temporary malfunction of transceivers. When the quantity of delay exceeds a threshold, we may also treat the long delay as data loss. Data loss is a major reason for switching topology, which has been widely investigated in literature [96] [107] [14]. Previously, switching topology is mainly treated as arbitrary switching without further considering more detailed stochastic features in switching patterns. However, in practice, communication channels often exhibit a corresponding probability of failure. Thus, it would be meaningful to incorporate that information into analysis. In [43] [133] [101], probabilities of the availability of communication links were taken into account and convergence results in the stochastic sense were established. The authors in [156] considered time delays and packet losses simultaneously. However, the delay was less than a sampling period. Recently, Zhang and Tian [157] considered the consensus problem in general identical linear agents with data losses. Sufficient conditions to guarantee consensus were found based on the analysis of maximum allowable loss rate.

Switching topology is mainly due to communication failures. As a result, agents are not able to receive data at some time instants. It is a common issue in the research on consensus in MASs. Jadbabaie *et al.* [53] investigated the coordination with switching topology, implying that the interaction between two neighboring agents was time dependent. It was proved that as long as the union of graphs was jointly connected, consensus could be achieved [53]. This condition was extended and generalized by Ren and Beard in [107], stating that a directed spanning tree for the joint graph is sufficient to reach consensus in single-integrator agents with directed interaction. In [96], consensus with switching topology was tackled using the technique of common Lyapunov function. Throughout the development of consensus theory, switching topology remains a hot topic and attracts much attention because of its practicality in describing imperfection in communication [40] [131].

In addition to the above mentioned communication constraints, there is also lots of work addressing the effects of other factors, such as quantization errors [4] [56] [58] [88] and noisy measurements [50]. Because they are not directly related to the current work in this thesis, details are omitted here.

1.2.3 Consensus Problem from Different Perspectives

With more and more researchers dedicated to the consensus problem, scenarios that are more general have been considered. In the following, consensus problem will be reviewed from different perspectives.

For example, depending on system properties, the consensus problem can be investigated in *linear* or *nonlinear* systems. Linear systems, due to their relative simplicity, have attracted much attention throughout, especially at the early stage [63] [96] [107]. In these systems, both the system dynamics and control protocols are in a linear form. But sometimes, the linearity may be destroyed due to various factors, e.g., the dynamics of an inverted pendulum [13]. Other sources of nonlinearity include the saturation in actuators or quantization in data transmission. Refer to [51] [79] [86] for more work on nonlinear consensus.

Looking back into the literature, we also notice a generalization of the system dynamics of agents. At the very beginning, *first-order* dynamics has been the central topic among researchers [5] [101] [141]. Because of its form ($\dot{x}_i = u_i$, where x_i is the state of agent i, and u_i is the control input), it is termed *single-integrator* dynamics as well. First-order dynamics finds applications in areas such as distributed linear averaging [141]. However, when it comes to more complex systems, e.g., the dynamic representation of a mobile robot, it is not enough to employ the single-integrator dynamics. As a result, people shifted more attention to the research on *second-order* dynamics, also known as *double-integrator* dynamics ($\dot{x}_i = v_i, \dot{v}_i = u_i$, where x_i is the position, v_i is the velocity, and u_i is the control input) [74] [80] [104]. As an application example, Ren and Atkins in [106] employed the second-order consensus theory to coordinate the motion of a group of nonholonomic mobile robots. In modern control systems, with the increasing complexity and the application of large-scale systems (e.g., in the aerospace industry), it is necessary to use a more sophisticated description for systems. With this background, the *state-space model* plays a crucial role. The state-space model encompasses both the first-order and second-order dynamics, and therefore is a more general representation of dynamic systems. Fax and Murray [31] investigated the vehicle formation control based on the algebraic graph theory and control theory. For more work on state-space consensus, refer to [83] [90] and the references therein.

From the viewpoint of the evolution of system states in time domain, we may deal with the consensus problem in three types of MASs: *continuous-time*, *discrete-time* and *sampled-data* systems. In continuous-time MASs, all variables involved are in continuous time [96] [107]. In nature, many dynamic processes occur in continuous time, e.g., the change in temperature. Likewise, when denoting control systems, such as mobile robots, it is natural to use a continuous-time model to represent its dynamics. In modern era, with the introduction of digital communication technology and digital signal processing, discrete-time systems become prevalent. In these systems, there is a working frequency and the system only works at discrete time instants. For instance, in the fast distributed linear averaging [141], the authors adopted a discrete-time scheme. For other related work, see [30] [34] [138], etc. In the third type–sampled-data systems [36] [156], the plant runs in continuous time, while the controller is in discrete time and works in a periodic fashion. This kind of dynamics is distinct from both the continuous-time and discrete-time dynamics. To cope with this situation, we can either transform the system dynamics into discrete-time dynamics and then study the asymptotic properties of the corresponding discrete-time system [139], or we can simply keep the sampled-data feature and look into the system's solution in time domain [149].

Depending on whether there exist any stochastic factors in an MAS, we have two categories: consensus in *deterministic* and *stochastic* systems. In deterministic MASs, there is no random factor involved in system dynamics [31] [96]. However, in stochastic systems, some stochastic processes exist in system parameters [49] [50] [71]. Take the switching of communication topology as an example. Huang *et al.* in [49] studied stochastic consensus with transmission noise and Markov data losses across communication channels. With state space decomposition, the consensus problem was converted into stabilization of the reduced-order error dynamics in the mean square sense and with probability one.

Regarding the communication topology, we have *fixed* and *switching* topology. If the communication relationship between agents is time-invariant [96] [125] [141], then the topology is fixed. For consensus with fixed topology and time-invariant system dynamics, we can derive some insightful results. For example, the authors in [96] and [107] related properties of the graph Laplacian to connectivity of the graph representing the interaction among agents. In case of switching topology [53] [86] [124] [130], it becomes more difficult since the information interaction among agents varies over time. The concept of joint connectivity has been widely used, which requires that the interaction among agents be frequent enough across each time interval of certain length. For switching topology, as aforementioned, we may also consider the stochastic nature in switching pattern. In this way, it is possible to incorporate more stochastic information into the convergence analysis.

Sometimes, we desire that there are one or more agents that act as the role of leaders, and the remaining agents should follow the behaviour of these leaders. This is called the *leader-follower* consensus [46] [90] [112]; otherwise, we call it *leaderless* consensus [107]. In the leader-follower scenario, if we analyze the communication graph of all agents including the leaders, we will find that the leaders are the root nodes of directed spanning trees of that graph. Therefore, the information of these leaders flows to the followers, while there is no influence from the followers that acts on the leaders. Here, it is necessary to point out the difference between the leader-follower control and centralized control. In centralized control, there is a leader that is able to directly communicate with all other agents. While in leader-follower control, the interaction from a leader to a follower can be conducted in an indirect way, e.g., through a chain of other agents. The leader-follower control is especially useful in formation control, in which the motion of leaders determines the trajectory of formation. In leaderless consensus, all the agents' initial states contribute to the final common decision value.

At the early stage of the research on consensus, people concentrate on MASs in which all agents have the same dynamics [107]: The same model structure and parameters. This type of systems is termed *homogeneous* MASs. Recently, researchers are paying more attention to another category: Consensus in *heterogeneous* systems [60] [61]. In a heterogeneous MAS, the agents have different dynamics (either with different model structure or parameters). For instance, the authors in [60] investigated the output consensus in a group of agents described by heterogeneous state-space models. With the aid of output regulation, observer-based consensus algorithms were designed.

To make the above statement clearer, Table 1.1 lists a brief summary on different categories of the consensus problem.

1.2.4 Theories and Approaches

Generally, the most commonly used theories involved in the research on consensus include the following three branches.

- **Control systems theory.** Like in any control systems, various control theories can be used in the consensus problem. For LTI systems, the frequency

Table 1.1: Classified representative research papers on consensus.

Types	**Description**	**Related work**
Linear:	Every part of the system is linear.	[63] [96] [107]
Nonlinear:	There are nonlinearities in system dynamics. They are inherent or caused by factors such as quantization.	[51] [79] [86]
First-order:	It is also known as single-integrator dynamics.	[5] [101] [141]
Second-order:	It is also known as double-integrator dynamics. More general dynamics is included.	[74] [80] [104]
State-space:	The system dynamics is described in state-space equations.	[83] [90]
Continuous-time:	All variables involved are in continuous time.	[79] [96]
Discrete-time:	All variables involved are in discrete time.	[30] [34] [138]
Sampled-data:	Some variables are periodically sampled, while others are still in continuous time.	[36] [156]
Deterministic:	There are no random factors in system dynamics.	[31] [96]
Stochastic:	Some stochastic processes exist in system parameters.	[49] [50] [71]
Fixed topology:	Communication relationship among agents is time-invariant.	[96] [125] [141]
Switching topology:	Information interaction among agents varies with time.	[86] [124] [130]
Leader-follower:	There is a leader that influences all other agents directly or indirectly.	[46] [90] [112]
Leaderless:	There does not exist a leader.	[107]
Homogeneous:	All agents have the same model structure and parameters.	[107]
Heterogeneous:	Agents have different dynamics.	[60] [61]

domain based approaches are suitable tools. In [96], a necessary and sufficient condition on upper bound of delay was derived by using the Nyquist criterion. Also see [126] for analysis of the discrete-time scenario in frequency domain. For time-varying systems, the Lyapunov method is a popular way for stability analysis after establishing the associated error dynamics [96]. In addition, the sliding mode control [103], model predictive control [33], neural network [18], and passivity based control [21] [3], etc., can be applied in consensus problem as well.

- **Matrix theory.** To establish the convergence results in consensus, Jadbabaie *et al.* [53] employed the theory on stochastic matrices. Its analysis was based on the asymptotic properties of the product of an infinite sequence of stochastic matrices [132]. This is particularly useful for purely discrete-time dynamics. For continuous-time systems with sampled measurements, we may first convert the dynamics into its equivalent discrete-time counterpart, then utilize the theory on stochastic matrices [139]. Besides the stochastic matrices, we can also use the spectral properties of matrices to establish the convergence to consensus. For example, in [96] [106], by analyzing the eigenvalues of the closed-loop system matrix, necessary and sufficient conditions were obtained.

- **Graph theory.** A graph is used to stand for the interaction or information flow among different agents. For example, consider a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ of N nodes, where $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$ is the node set, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set, and $A \in \mathbb{R}^{N\times N}$ is the corresponding adjacency matrix representing weights of the edges. Then each agent is denoted by a node of $\mathcal{G}$. Each edge $(v_i, v_j) \in \mathcal{E}$ implies that there is information transmission from agent i to agent j. The weight in communication for edge (v_i, v_j) is a_{ji} in A. By exploring the algebraic properties of a graph, we can infer the knowledge on how the graph connectivity affects the convergence to consensus. For more details on graph theory, refer to [37] [27].

It is interesting to note that different theories are often coupled when solving the consensus problem. For instance, the eigenvalues of the graph Laplacian matrix L are closely related to convergence, and L is determined by the graph topology.

In the literature on consensus, there are many approaches to solve the problem. But in general, there are mainly two methodologies: direct method and indirect method. Both of them are briefly reviewed in the following.

- **Direct method.** We study the solution to the differential (difference) equations governing an MAS in time domain and investigate its asymptotic properties. The application of stochastic matrices in discrete-time MASs is a good example of this type. For a continuous-time system, if it is time-invariant, we can analyze the eigenvalues of the closed-loop system matrix. If it is time-varying, an effective way is to first transform the continuous-time system into its discrete-time counterpart, and then apply the theory on stochastic matrices [107].

- **Indirect method.** In this category, we need to perform a system transformation to convert the consensus problem into another equivalent one. The most common approach is to establish the error dynamics which characterizes the difference between the states of different agents. For example, suppose we have N agents and x_i is the state of agent i. Apparently, $x_i(t) \to x_j(t)$ as $t \to \infty$, $\forall i \neq j, i, j = 1, 2, \ldots, N$, is equivalent to $\lim_{t\to\infty}[x_i(t) - x_j(t)] = 0$. The error vector is constructed at first, then the consensus problem is transformed into the stabilization of the resulting error dynamic system.

  In order to form the error dynamics, the first approach is to set one agent as reference and then take the difference between other agents' states and that of the reference agent. For example, if we treat agent 1 as a reference, then define $e_i(t) = x_i(t) - x_1(t), i = 2, \ldots, N$. The error vector is constructed as $e(t) = [e_2(t), \ldots, e_N(t)]^{\mathrm{T}}$. The consensus problem is now converted into the stabilization of the error dynamics with respect to $e(t)$. Refer to [156] [155] and the references therein. When dealing with average consensus, the average of agents' initial states can be set as a reference state [96]. The second approach to get error dynamics is to use the matrix related to the graph Laplacian as the state transformation matrix [36]. In this way, the reduced-order error dynamics is separated from system dynamics and the remaining task is to find conditions under which the error dynamics can be stabilized.

1.2.5 Applications of Consensus Theory

The research on consensus finds applications in many fields. The most obvious one is rendezvous, in which the states of all agents converge to a common value. This is desirable if all vehicles are expected to meet at one location. The second application is in formation control, where the relative position or heading between different agents must be maintained constant during the vehicles' maneuver [63]. This is realized by redefining agents' states according to the formation pattern.

In mobile sensor networks, to better coordinate different parts deployed at different locations, a common time-scale is necessary, and this requires clock synchronization. In a decentralized environment, it is often not possible to have a global node that directly communicates with all other nodes. Thus, how to achieve clock synchronization in a distributed way is the concern. With the help of consensus theory, we have more ways to implement synchronization in clocks and render the system more robust against node failures [110].

In addition to the control field, consensus theory can be employed for filtering as well. This type of filtering is in a distributed way and is called consensus filtering [97] [94] [73]. There are a group of filter sensor nodes and each of them gives an estimate of the target signal. The goal is to design control protocols such that they not only reach consensus on their own estimates, but also give a good estimate of the target signal.

Due to its inherently distributed feature, consensus approach has been used for decentralized parameter estimation [120] [57] [154]. This scheme of parameter estimation reduces the computational and communication load for each single node, therefore improves the system's efficiency. In the mean time, it enhances robustness of the overall system against node failure.

The area of smart grids also finds the research of consensus to boost its development. In [144], a distributed load restoration algorithm for microgrids was proposed, utilizing the average consensus. The more scalable decentralized scheme could potentially be deployed in large-scale applications. A distributed incremental cost consensus algorithm was proposed in [158] and the authors analyzed convergence of the algorithm. In particular, the relation between the rate of convergence and communication topology was explored.

From the above examples, we see that the wide applications of consensus theory come from its distributed nature with numerous merits. By distributing the communication and computation load to each node of a network, the system works more efficiently. Compared with the conventional centralized fashion, this approach is more robust against node failure and communication malfunction. Decentralized algorithms are more scalable, capable of dealing with networks of very large size.

1.3 Bilateral Teleoperation

1.3.1 A Brief Description

Bilateral teleoperation is a combination of robotics and control theory. It has been a hot topic for several decades. For a complete literature review, refer to [45]. In teleoperation, the human manipulates an operator (master) to simulate the motion desired to fulfil certain tasks. The position and velocity signals are then sent to the other operator (slave) so that the slave can track the motion of the master. It is not a simple task of trajectory tracking because the effect of environment force is also transmitted back to the master side so that we feel the presence of the remote environment. There are two major issues in bilateral teleoperation: Stability and transparency [66]. Stability requires that as a whole control system, the behavior of the master and slave should always satisfy the bounded-input-bounded-output (BIBO) stability. Transparency is a concept reflecting how well we feel the distant workplace. There is usually a trade-off between stability and transparency.

The first modern teleoperation system was reportedly built in 1945 in the Argonne National Laboratory [114]. In recent years, with the advancement of network technology, people are concentrating more and more on teleoperation through the Internet. As a result, many issues arise, e.g., communication delay, sampling and quantization over digital channels, distinct working rates of different sensors. Even worse, data may get lost. With the presence of these adverse factors, researchers have come up with various approaches to stabilize teleoperation systems and improve their robustness, without sacrificing too much transparency.

1.3.2 Review of Teleoperation with Delays

As in any NCSs, time delay is not new in teleoperation. It affects the performance of a teleoperation system. The system performance is degraded, or it even becomes unstable when time delay exceeds certain tolerance. Whenever there is information exchange through a network, time delay is unavoidable. As a result, a lot of attention has been paid to the stability and performance analysis in teleoperation systems in the presence of time delays. Anderson and Spong [2] solved the stability problem in a force-reflecting teleoperation system with constant delay, using the frequency domain approach. The introduction of scattering transformation increases robustness of the system. A similar method, wave variables, was studied in [91] to deal with constant delay. In a more recent work [67], the Parseval's identity and Lyapunov technique were employed to guarantee passivity of the teleoperation system subject to asymmetric delays with upper bounds. The teleoperators had the very general nonlinear Euler–Lagrange dynamics [67].

Besides constant delays, time-varying delays attract much attention as well. In

[22], the passivity of communication channels was preserved with the use of a time-varying gain in information transmission. Walker *et al.* [129] studied the teleoperation with time-varying and bounded delays. The lower and upper bounds of time delays were taken into account, and the mean exponential stability was used to characterize the system performance. Compared with the scenario of constant delays, the analysis in frequency domain is no longer directly applicable to the case with time-varying delays. The Lyapunov method has been widely utilized due to its versatility in the stability analysis of control systems.

Regarding the methodologies to cope with teleoperation, the passivity-based approach has been largely used; see [2] [20] [41] [67] [91] [98] and the references therein. Others include the H_∞ optimal control [68], input-to-output stability, small gain approach [100], stability in NCSs [129], etc. In our research, we will directly look into the stability of the error dynamic system which characterizes the difference between the states of the master and slave.

To compensate for the adverse effects and achieve better performance, much attention has been paid to preserving stability and enhancing transparency by proposing appropriate control strategies.

1.4 Motivation and Contribution

1.4.1 Consensus in Multi-Agent Systems

Due to the aforementioned communication constraints in MASs, how to design appropriate control schemes and find conditions to guarantee consensus is our main concern. In the first part of this thesis (Chapter 2 to Chapter 4), we deal with the consensus problem using proposed strategies. The motivation and objectives of each chapter are summarized below.

In previous work, the upper bound in delay provides important information to us and has been included in the convergence analysis [96] [14] [76]. Apart from that, delays sometimes exhibit stochastic characteristics as well. For example, the delay at current time instant may have some relation to that at previous instant [117]. Motivated by this observation, **Chapter 2** investigates the consensus problem in an MAS of single-integrator agents subject to random delays governed by a Markov chain. The communication topology is assumed to be directed and fixed. Under the sampled-data setting, we first convert the original system into its reduced-order error dynamics. Thus, the consensus problem is transformed into stabilization of the error dynamic system. Based on the theory in stochastic stability for time-delay systems, a sufficient condition is established in terms of a set of linear matrix inequalities (LMIs). Mean square stability of the error dynamics is shown to guarantee consensus of the MAS. By explicitly incorporating the transition probabilities of random delays into analysis, more information on delays is considered. A delay-dependent switching control scheme is developed by redesigning the adjacency matrix.

When dealing with data losses, earlier work has some limitation. For example, the authors in [43] [133] [101] considered probabilities of available communication channels without time delay. The work [156] took into account data loss and delay simultaneously, but the quantity of delay was less than a sampling period. Inspired by these works, in **Chapter 3**, average consensus with delays and data losses is investigated for MASs with undirected topology. The communication constraints are considered in two aspects: (1) time-varying delays are non-uniform and bounded; (2) data losses are governed by Bernoulli processes with non-uniform probabilities. We discretize the single-integrator dynamics and convert the consensus problem into stabilization of its corresponding error dynamics. By assuming symmetry in communication topology, conditions of ensuring the mean square stability of error dynamics are developed, by explicitly incorporating the probabilities of data losses. The developed scheme can be easily verified numerically.

Nowadays, more and more controllers are implemented in a digital manner. To get the measurements, a key issue is to sample the output of a plant from time to time. In previous sampled-data scheme to deal with consensus [36] [12], people adopted periodic sampling. This means, that all agents sample their outputs after a fixed period of time. Nevertheless, in real world, there are various factors that may result in aperiodic sampling [1]. For example, the information available to us may have different rates, or we can improve the overall system performance by adopting non-uniform sampling. In a recent work [137], the consensus problem with arbitrary sampling in double-integrator dynamics was investigated. **Chapter 4** is concerned with the consensus problem in MASs with double-integrator dynamics and non-uniform sampling. The communication topology is assumed to be fixed and directed. A control protocol with time-varying gains is proposed. The results on stochastic matrices play an important role in convergence analysis. We prove that when the directed graph has a spanning tree and the control gains are properly chosen, consensus can be achieved. Compared with [137], this chapter looks at the consensus problem with non-uniform sampling from a different point of view and with different settings.

1.4.2 Bilateral Teleoperation

There has been a lot of work concerned with time delays in bilateral teleoperation, as reviewed in Subsection 1.3.2. These works considered the lower and upper bounds of delays, but more information is still worth being included in analysis, such as the probabilistic distribution observed in experiments [129]. **Chapter 5** studies bilateral teleoperation over communication networks. Specifically, the network-induced random delays are from a finite set, and each delay in the set has a probability of occurrence. To fully utilize the stochastic information inherent with delays, a novel design scheme combining the probability information in delays and pole placement is proposed to achieve better tracking performance. The teleoperation problem is first formulated as stabilization of an error dynamic system where the error is the difference between the states of the master and slave. Then, by constructing a Lyapunov function, a sufficient condition to guarantee the input-to-state stability is established in terms of LMIs.

Chapter 2

Consensus in Multi-Agent Systems with Random Delays Governed by a Markov Chain

2.1 Introduction

Recently, multi-agent cooperative control has received a lot of attention. As a branch of mobile sensor and actuator networks, it has wide applications in the forms of formation control, flocking, swarming etc., where distributed coordination is the main concern [24] [121] [147]. For the consensus phenomenon among a group of particles using nearest neighbor rules (see [128]), Jadbabaie *et al.* provided a theoretical explanation [53]. The consensus problem with switching topology and constant time delay was discussed in [96], and a sufficient condition on graph topology was given to guarantee the convergence to a common value. Ren and Beard [107] extended the work in [53] to the scenario with a directed graph, proving that the existence of a directed spanning tree is a sufficient and necessary condition such that the Laplacian matrix has only one zero eigenvalue. This is also sufficient and necessary for consensus in the LTI first-order dynamics.

Time delay exists ubiquitously in practical systems. In traditional peer-to-peer control systems, the delay involved is usually very small compared with the system dynamics, and is thus often negligible. However, in a network environment, the effect of delay becomes significant. It makes the system be of nonminimum phase, and it degrades the performance of control systems. A system could even become unstable if the delay is too large. As a result, many researchers have been studying this subject to improve stability and robustness against time delay [39] [143]. When dealing with such problems, information on stochastic characteristics of delay would greatly facilitate the controller design and help to reduce the conservativeness. For instance, the authors in [35] considered the probabilistic distribution of delays in NCSs and presented an improved sufficient condition for stability.

In the area of distributed coordination of MASs, network-induced delay is also an important and practical issue to consider. The delay could be either constant or time-varying, uniform or diverse. The work in [96] allowed for consensus in continuous-time systems with a single constant delay. A necessary and sufficient condition on the upper bound of time delay was derived for achieving consensus, by analyzing the poles of the transfer function matrix. Other work includes [126], in which a constraint was imposed on the sum of absolute values of the elements in each row of the adjacency matrix, and the delays were assumed to be diverse and constant. Both conditions in [96] and [126] were *delay-dependent*. In [138], under the discrete-time framework, it was proved that consensus could be achieved as long as the time-varying delays had upper bounds and the union of communication topologies had a spanning tree. Such a condition is *delay-independent*.

However, it is worth pointing out that, in the aforementioned work on consensus, the statistical characteristics of delay have not been incorporated into the design, which motivates this work. It is conjectured that by considering the probabilistic distribution of the network-induced delays, less conservative results could be achieved.

As to the research on stabilization of a large class of stochastic systems, the Markov jump linear system (MJLS) has been well investigated; see, e.g., [25] [117] [118] and related references. This characterizes the model uncertainties and switching nature of the plant more explicitly. In [117] [118], the output feedback stabilization and H_2/H_∞ control of an NCS with time delays governed by a Markov process were studied. A dynamic control law depending on bilateral delays was designed with output feedback. Simulation showed the improvement in tracking performance compared with that using a classical approach considering only the sum of delays over two communication links.

In this work, we are concerned with consensus subject to delays from a Markov process. To the authors' best knowledge, no work has been done on this subject to date. It can be regarded as an extension of the result in [96]. The merits of using Markov chains to characterize delays are as follows. (1) The random delays in a network exhibit the feature that the occurrence of the current delay depends on the previous delay [93]. The Markov chain model can better characterize the random delay. (2) By considering the statistical characteristics of the delay in the design, conservativeness can be reduced, which results in improved system performance [117].

We assume that the delays over all the communication links are the same (uniform) but jumping. By transforming the original system into its reduced-order error counterpart, the consensus problem is converted into stabilization in the mean square sense. A sufficient condition is given through the feasibility of a set of LMIs. With this consideration, we expect that the maximum delay that the system can tolerate will be increased, depending on the probabilistic distribution of time delays. Moreover, from the theory of an MJLS [25], it is possible to combine several unstable modes to form a new stable system by switching among different modes. This leads to the idea of redesigning the adjacency matrices subsequently, which is another feature of this work.

The remainder of this work is organized as follows. In Section 2.2, some definitions and notation are given. Section 2.3 contains the details in problem formulation, the consensus problem being transformed into the stabilization of an error dynamic system. Section 2.4 presents the main results. The stochastic stability of the error dynamic system is guaranteed if the LMIs are feasible. In Section 2.5, simulation results are provided to verify the effectiveness of the proposed condition. Finally, we offer some conclusions in Section 2.6.

Notation: The superscript 'T' represents the matrix transpose. $\mathbb{E}$ denotes the mathematical expectation. We say that a matrix $P > 0$ if and only if P is symmetric and positive definite. '*' in a matrix stands for a term that is induced by symmetry. Matrices, if dimensions are not indicated explicitly, are assumed to be compatible with algebraic operations.

2.2 Preliminaries

For a graph of n nodes denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$, $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ is the node set; $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set. An edge $(v_j, v_i) \in \mathcal{E}$ represents the information flow from v_j to v_i. The *adjacency matrix* $A = [a_{ij}] \in \mathbb{R}^{n \times n}$, which is nonnegative ($a_{ij} \geq 0, \forall i, j = 1, 2, \ldots, n$) in many papers, models the communication topology among the agents. If there is a directed link from agent j to agent i, which means that i receives information from j, then $a_{ij} \neq 0$; otherwise, $a_{ij} = 0$. An *undirected* *graph* implies that the communication is bidirectional, i.e., a link from i to j means a link from j to i as well, or else the graph is *directed*. A *path* from i to j in a graph is a sequence of distinct nodes starting with i and ending with j such that consecutive nodes are adjacent [37]. The graph $\mathcal{G}_s$ is regarded as a *spanning subgraph* of $\mathcal{G}$ if $\mathcal{V}(\mathcal{G}_s) = \mathcal{V}(\mathcal{G})$ and $\mathcal{E}(\mathcal{G}_s) \subseteq \mathcal{E}(\mathcal{G})$. A *spanning tree* is a spanning subgraph without cycle. Obviously, in a graph with a spanning tree, there exists at least one node whose information flows to every other node. More details on graph theory can be found in [37].

The neighbor set of agent i is denoted by $\mathcal{N}_i$, from which i receives information. Thus, $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{E}\}$. Assume that there is no edge from an agent to itself. In most of the existing work, the adjacency matrix A associated with a graph has the property that $a_{ii} = 0$ and $a_{ij} \geq 0$ for $i \neq j$. The *graph Laplacian* $L \in \mathbb{R}^{n\times n}$ is defined as

$$l_{ij} = -a_{ij}, \forall i \neq j; l_{ii} = \sum_{j\in\mathcal{N}_i} a_{ij}, i, j = 1, 2, \ldots, n.$$

Obviously, A and L determine each other uniquely. In this work, we will comply with a similar definition except that the elements in A could be negative. Later, it will be observed that the existence of negative elements in the adjacency matrix provides more flexibility for design.

Next, some definitions on Markov process are presented.

Definition 2.1 ([38])**.** The Markov chain X with state space S is called *homogeneous* if

$$\mathbb{P}(X_{m+1} = j|X_m = i) = \mathbb{P}(X_1 = j|X_0 = i)$$

for all m, i, j. The *transition matrix* $\Pi = (\pi_{ij})$ is the $|S| \times |S|$ matrix of *transition probabilities*

$$\pi_{ij} = \mathbb{P}(X_{m+1} = j|X_m = i)$$

where $\mathbb{P}$ is the probability operator and $|S|$ is the cardinality of state space S.

The transition probability matrix Π satisfies

$$\sum_{j=1}^{|S|} \pi_{ij} = 1, i = 1, 2, \ldots, |S|.$$

The following assumption imposes an upper bound on the network-induced delays, which is reasonable and practical.

Assumption 2.1. The time delays $\{d_k\}$ are from a finite set of integers $\Gamma = \{\tau_1, \tau_2, \ldots, \tau_q\}$ and $0 \leq \tau_1 < \tau_2 < \cdots < \tau_q$.

Remark 2.1. This assumption is very general and does not impose very strong constraints on problem formulation. The fact that the set Γ is composed of integers rather than continuous real numbers results from the sampling and hold feature. Even if the actual delays can be real numbers, we only send and use data at discrete time instants. As a result, time delays will be integer multiple of the sampling period. In this work, not only the lower and upper bounds, but also the delay transition probabilities will be considered. The dimension of the transition probability matrix is q and $\{1, 2, \ldots, q\}$ constitute the state space of the Markov chain.

In practice, in order to determine the set of delays, we can first measure the delays across a time interval $[t_1\ t_2]$ that is large enough. From these measurements, we can extract the values of delays to form the set of delays. Then the next task is to compute the transition probabilities. To this end, we can start from one element (say τ_i) in the delay set Γ. Then check all the discrete time instants in $[t_1\ t_2]$ and decide the number of transition from current delay τ_i to all the delays in Γ. Suppose that the number of transition starting from τ_i to all elements in Γ across $[t_1\ t_2]$ is N_i, and the number of transition from τ_i to τ_j is N_{ij}. Then the transition probability from τ_i to τ_j is $\pi_{ij} = N_{ij}/N_i$. Similarly, all other transition probabilities can be calculated.

Sometimes, even if we make our best effort to determine the transition probabilities of the Markov chain, it is possible that we may not get the exact values of probabilities. This is due to either measurement errors or variation in system parameters. It causes uncertainty in the transition probability matrix. Under this circumstance, how to improve the robustness of the developed method is our concern. Typical work includes [153], in which the authors studied the analysis and synthesis of MJLSs with partially known transition probabilities.

2.3 Problem Formulation

Consider a group of n agents with first-order dynamics,

$$\dot{x}_i(t) = u_i(t), i = 1, 2, \ldots, n, \tag{2.1}$$

where $u_i(t)$ is the control input that can be generated by the following control law, as proposed in [96]:

$$u_i(t) = -\sum_{j \in \mathcal{N}_i} a_{ij}[x_i(t) - x_j(t)]. \tag{2.2}$$

In a sampled-data set-up, with a zero-order hold, the dynamics in (2.1) has the following equivalent form in discrete time:

$$x_i(k+1) = x_i(k) + hu_i(k),$$

where h is the sampling period.

The consensus problem is dealt with in the presence of uniform and random delays. Therefore, all the communication channels among the agents are subject to the same delay at one instant. *Consensus* is achieved if and only if there exists a common decision value $\alpha(x(0))$ which is the function of the initial state $x(0) = [x_1(0) \quad x_2(0) \quad \cdots \quad x_n(0)]^{\mathrm{T}}$, such that all the states in (2.1) converge to $\alpha(x(0))$. It is noteworthy that we do not need all the agents to have knowledge of their initial states such that $\alpha(x(0))$ is known to all agents. The final state of consensus even does not have to depend on the initial states of the agents. Sometimes, we may require the agent states to track an external reference signal. In this case, the consensus problem becomes the consensus tracking.

Assuming that the delay at time instant k is d_k, we have

$$x_i(k+1) = x_i(k) + hu_i(k),$$

where

$$u_i(k) = -\sum_{j \in \mathcal{N}_i} a_{ij}(d_k)[x_i(k-d_k) - x_j(k-d_k)].$$

Compared with (2.2), in the above equation a_{ij} is changed to $a_{ij}(d_k)$, which depends on the current delay d_k. In this way, the adapted system could be more robust against delay.

Define the error as $\delta_i(k) = x_i(k) - x_1(k), i = 2, 3, \ldots, n$, from which the error vector is obtained as follows

$$\delta(k) = [\delta_2(k) \quad \delta_3(k) \quad \cdots \quad \delta_n(k)]^{\mathrm{T}}.$$

After some algebraic manipulations, the error dynamic system is

$$\delta(k+1) = \delta(k) + h\hat{L}(d_k)\delta(k-d_k), \tag{2.3}$$

with

$$\begin{aligned}\hat{L}(d_k) &= -(L_{2:n,2:n}(d_k) - \mathbf{1}_{n-1}L_{1,2:n}(d_k)) \\ &= -\begin{bmatrix} l_{22}-l_{12} & l_{23}-l_{13} & \cdots & l_{2n}-l_{1n} \\ l_{32}-l_{12} & l_{33}-l_{13} & \cdots & l_{3n}-l_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ l_{n2}-l_{12} & l_{n3}-l_{13} & \cdots & l_{nn}-l_{1n} \end{bmatrix},\end{aligned} \tag{2.4}$$

where $\mathbf{1}_{n-1} = [1, 1, \ldots, 1]^{\mathrm{T}} \in \mathbb{R}^{(n-1)\times 1}$, $L_{2:n,2:n}(d_k) = [l_{ij}(d_k)] \in \mathbb{R}^{(n-1)\times(n-1)}$, $i, j = 2, 3, \ldots, n$, $L_{1,2:n}(d_k) = [l_{12}(d_k), l_{13}(d_k), \ldots, l_{1n}(d_k)]$. In (2.4), $l_{ij}, i = 1, 2, \ldots, n, j = 2, 3, \ldots, n$ are actually determined by the current delay and should be written as $l_{ij}(d_k)$. Here, d_k is omitted for the sake of succinctness.

Defining $\hat{L}_h(d_k) = h\hat{L}(d_k)$, Eqn. (2.3) can be rewritten in the following form

$$\delta(k+1) = \delta(k) + \hat{L}_h(d_k)\delta(k-d_k). \tag{2.5}$$

It is readily shown that reaching consensus in (2.1) with an initial condition is equivalent to the stability of the error system in (2.5). What we need to do is to determine whether the stochastic stability can be guaranteed under the given h and L. It is worth noting that for a fixed topology, one cannot alter the type of matrix L, but rather its elements can be adjusted to improve the system performance. The concept *type* delivers the information on the positions with nonzero entries in the adjacency matrix; it has been used in [132] to describe stochastic matrices. We adopt a switching control by choosing a different Laplacian matrix based on the current delay. In this way, a system that is not able to reach consensus with a single adjacency matrix could achieve agreement under the proposed delay-dependent strategy.

Before proceeding, we need the following definition of stability.

Definition 2.2 ([8])**.** The system in (2.5) is *mean square stable* if

$$\lim_{k\to\infty} \mathbb{E}\{\|\delta(k)\|^2\} = 0.$$

The above definition is widely used in the stability of an MJLS [32], and it can also be found in the research on consensus [72]. Based on Definition 2.2, the following lemma is readily obtained.

Lemma 2.1. *The consensus problem in (2.1) is solved if and only if the stochastic system in (2.5) is stabilized. Moreover, if (2.5) is stable in the mean square sense, we say that* mean square consensus *is achieved.*

Proof. The proof is straightforward, and thus we only provide a brief explanation. For the necessity part, when the MAS in (2.1) reaches consensus, $x_i \to x_j, \forall i \neq j, i, j = 1, 2, \ldots, n$. Therefore, $\delta_i = x_i - x_1 \to 0, i = 2, 3, \ldots, n$, and the error dynamics in (2.5) will be stable. For the sufficiency part, when the error dynamics in (2.5) is stable, we have $\delta \to 0$, which implies $x_i - x_1 = \delta_i \to 0, i = 2, 3, \ldots, n$. Thus $x_i \to x_1, i = 2, 3, \ldots, n$, and consensus is reached. □

Remark 2.2. In this work, we only study the single-integrator dynamics. For double-integrator dynamics, it is possible to apply the method in this chapter. That is, we first construct the error dynamics between one reference agent and other agents. Then the stability of the error dynamics will be investigated. However, the controller design process will become more involved as the double-integrator dynamics will introduce the Kronecker product.

To proceed, the following assumption is necessary.

Assumption 2.2. The directed graph has a spanning tree.

Remark 2.3. Assumption 2.2 is a necessary condition to guarantee consensus. It states the communication relation among the agents: At least one agent is able to affect all other agents through the links. If not, there always exist two subgroups of agents that cannot receive any information from each other, and thus consensus is not guaranteed. This assumption is a basic requirement for the fixed topology.

Then one may ask: What if the agents have the same initial states and they do not receive any external information from other agents? In this scenario, there is no communication among agents and the edge set of the graph is empty. It seems that the agents' states will always be the same and consensus is reached, so a spanning tree is not indispensable. However, if we consider different initial conditions, then the existence of a directed spanning tree becomes necessary. Otherwise, there is not enough information exchange. If the graph has a circle, which still contains a spanning tree, then every agent affects the states of all other agents directly or indirectly. In this situation, every agent can be the root of a spanning tree, and they will all contribute to the final state of consensus.

2.4 Main Results

Two important issues will be addressed in this section: (1) a sufficient condition to guarantee the stochastic consensus; (2) design of the adjacency matrices. We have the following theorem for consensus.

2.4.1 The Sufficient Condition for Consensus

Theorem 2.1. *For the system in (2.1) with random delays governed by a Markov chain, under Assumptions 2.1 and 2.2, mean square consensus is achieved if there exist matrices* $P > 0, Q_j > 0, Z_j > 0, M_j$, *and* $\hat{L}_h(\tau_j), j = 1, 2, \ldots, q$, *such that the following matrix inequality,*

$$\begin{bmatrix} Y_{11}(r) & Y_{12} \\ * & Y_{22} \end{bmatrix} < 0, \tag{2.6}$$

holds $\forall r = 1, 2, \ldots, q$, *where*

$$\begin{aligned}
Y_{11}(r) &= \begin{bmatrix} \Phi_0 & \Psi_1(r) & \cdots & \Psi_q(r) \\ * & \Phi_1(r) & \cdots & 0 \\ * & * & \ddots & \vdots \\ * & * & * & \Phi_q(r) \end{bmatrix} \\
&+ \begin{bmatrix} \sum_{j=1}^{q}(M_{0j} + M_{0j}^{\mathrm{T}}) & -M_{01} + \sum_{j=1}^{q} M_{1j}^{\mathrm{T}} & \cdots & -M_{0q} + \sum_{j=1}^{q} M_{qj}^{\mathrm{T}} \\ * & -M_{11} - M_{11}^{\mathrm{T}} & \cdots & -M_{1q} - M_{q1}^{\mathrm{T}} \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & -M_{qq} - M_{qq}^{\mathrm{T}} \end{bmatrix}, \\
Y_{12} &= \begin{bmatrix} \sqrt{\tau_1} M_1 & \sqrt{\tau_2} M_2 & \cdots & \sqrt{\tau_q} M_q \end{bmatrix} \\
&= \begin{bmatrix} \sqrt{\tau_1} M_{01} & \sqrt{\tau_2} M_{02} & \cdots & \sqrt{\tau_q} M_{0q} \\ \sqrt{\tau_1} M_{11} & \sqrt{\tau_2} M_{12} & \cdots & \sqrt{\tau_q} M_{1q} \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{\tau_1} M_{q1} & \sqrt{\tau_2} M_{q2} & \cdots & \sqrt{\tau_q} M_{qq} \end{bmatrix}, \\
Y_{22} &= -\mathrm{diag}\{Z_1 \quad Z_2 \quad \cdots \quad Z_q\},
\end{aligned}$$

$$\Phi_0 = \sum_{j=1}^{q} Q_j,$$

$$\Phi_i(r) = \pi_{ri}\hat{L}_h^{\mathrm{T}}(\tau_i)P\hat{L}_h(\tau_i) - Q_i + \pi_{ri}\hat{L}_h^{\mathrm{T}}(\tau_i)\left(\sum_{j=1}^{q}\tau_j Z_j\right)\hat{L}_h(\tau_i),$$

$$\Psi_i(r) = \pi_{ri}P\hat{L}_h(\tau_i), \quad i = 1, 2, \ldots, q.$$

Proof. Consider the following Lyapunov–Krasovskii functional candidate:

$$V(k) = V_1(k) + V_2(k) + V_3(k)$$

where

$$\begin{aligned} V_1(k) &= \delta^{\mathrm{T}}(k)P\delta(k), \\ V_2(k) &= \sum_{j=1}^{q}\sum_{i=k-\tau_j}^{k-1}\delta^{\mathrm{T}}(i)Q_j\delta(i), \\ V_3(k) &= \sum_{j=1}^{q}\sum_{i=-\tau_j}^{-1}\sum_{m=k+i}^{k-1}\eta^{\mathrm{T}}(m)Z_j\eta(m), \\ \eta(m) &= \delta(m+1) - \delta(m). \end{aligned}$$

For $V_3(k)$, considering (2.5), we have

$$V_3(k) = \sum_{j=1}^{q}\sum_{i=-\tau_j}^{-1}\sum_{m=k+i}^{k-1}\delta^{\mathrm{T}}(m-d_m)\hat{L}_h^{\mathrm{T}}(d_m)Z_j\hat{L}_h(d_m)\delta(m-d_m).$$

Define $d_{k-1} = \tau_r, d_k = \tau_s, r, s \in \{1, 2, \ldots, q\}$. Then the transition probability from d_{k-1} to d_k is

$$\mathbb{P}(d_k = \tau_s | d_{k-1} = \tau_r) = \pi_{rs}.$$

In what follows, when calculating the difference (conditional expectation) for each term of the Lyapunov function, we have the following vector,

$$\zeta(k) = \begin{bmatrix} \delta^{\mathrm{T}}(k) & \delta^{\mathrm{T}}(k-\tau_1) & \delta^{\mathrm{T}}(k-\tau_2) & \cdots & \delta^{\mathrm{T}}(k-\tau_q) \end{bmatrix}^{\mathrm{T}}$$

as the prior knowledge. Meanwhile, π_{rs} is taken into account.

$$\mathbb{E}\{\Delta V_1(k)\}$$

$$
\begin{aligned}
&= \mathbb{E}\{V_1(k+1) - V_1(k)\} \\
&= \mathbb{E}\left\{[\delta^{\mathrm{T}}(k) + \delta^{\mathrm{T}}(k-d_k)\hat{L}_h^{\mathrm{T}}(d_k)]P[\delta(k) + \hat{L}_h(d_k)\delta(k-d_k)]\right\} - \delta^{\mathrm{T}}(k)P\delta(k) \\
&= 2\sum_{s=1}^{q}\pi_{rs}\delta^{\mathrm{T}}(k)P\hat{L}_h(\tau_s)\delta(k-\tau_s) + \sum_{s=1}^{q}\pi_{rs}\delta^{\mathrm{T}}(k-\tau_s)\hat{L}_h^{\mathrm{T}}(\tau_s)P\hat{L}_h(\tau_s)\delta(k-\tau_s),
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\{\Delta V_2(k)\} \\
&\quad = \mathbb{E}\{V_2(k+1) - V_2(k)\} \\
&\quad = \sum_{j=1}^{q}[\delta^{\mathrm{T}}(k)Q_j\delta(k) - \delta^{\mathrm{T}}(k-\tau_j)Q_j\delta(k-\tau_j)],
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\{\Delta V_3(k)\} \\
&\quad = \mathbb{E}\{V_3(k+1) - V_3(k)\} \\
&\quad = \mathbb{E}\left\{\sum_{j=1}^{q}\sum_{i=-\tau_j}^{-1}\Big[\delta^{\mathrm{T}}(k-d_k)\hat{L}_h^{\mathrm{T}}(d_k)Z_j\hat{L}_h(d_k)\delta(k-d_k)\right. \\
&\qquad\quad \left. -\ \delta^{\mathrm{T}}(k+i-d_{k+i})\hat{L}_h^{\mathrm{T}}(d_{k+i})Z_j\hat{L}_h(d_{k+i})\delta(k+i-d_{k+i})\Big]\right\} \\
&\quad = \mathbb{E}\left\{\sum_{j=1}^{q}\sum_{i=-\tau_j}^{-1}\delta^{\mathrm{T}}(k-d_k)\hat{L}_h^{\mathrm{T}}(d_k)Z_j\hat{L}_h(d_k)\delta(k-d_k)\right. \\
&\qquad\quad \left. - \sum_{j=1}^{q}\sum_{l=k-\tau_j}^{k-1}\delta^{\mathrm{T}}(l-d_l)\hat{L}_h^{\mathrm{T}}(d_l)Z_j\hat{L}_h(d_l)\delta(l-d_l)\right\} \\
&\quad = \sum_{s=1}^{q}\sum_{j=1}^{q}\pi_{rs}\delta^{\mathrm{T}}(k-\tau_s)\hat{L}_h^{\mathrm{T}}(\tau_s)\tau_j Z_j\hat{L}_h(\tau_s)\delta(k-\tau_s) \\
&\qquad - \sum_{j=1}^{q}\sum_{l=k-\tau_j}^{k-1}\delta^{\mathrm{T}}(l-d_l)\hat{L}_h^{\mathrm{T}}(d_l)Z_j\hat{L}_h(d_l)\delta(l-d_l).
\end{aligned}
$$

For any matrices

$$
M_j = [M_{0j}^{\mathrm{T}} \quad M_{1j}^{\mathrm{T}} \quad M_{2j}^{\mathrm{T}} \quad \cdots \quad M_{qj}^{\mathrm{T}}]^{\mathrm{T}}, \quad j = 1, 2, \ldots, q
$$

with appropriate dimensions, we have the following identity:

$$
\zeta^{\mathrm{T}}(k)M_j\left[\delta(k) - \delta(k-\tau_j) - \sum_{l=k-\tau_j}^{k-1}\eta(l)\right] = 0.
$$

Then,

$$\begin{aligned}
&\mathbb{E}\{\Delta V(k)\} \\
&\quad = \mathbb{E}\{\Delta V_1(k)\} + \mathbb{E}\{\Delta V_2(k)\} + \mathbb{E}\{\Delta V_3(k)\} \\
&\quad \leq 2\sum_{s=1}^{q} \pi_{rs}\delta^{\mathrm{T}}(k)P\hat{L}_h(\tau_s)\delta(k-\tau_s) + \sum_{s=1}^{q} \pi_{rs}\delta^{\mathrm{T}}(k-\tau_s)\hat{L}_h^{\mathrm{T}}(\tau_s)P\hat{L}_h(\tau_s)\delta(k-\tau_s) \\
&\qquad + \sum_{j=1}^{q}[\delta^{\mathrm{T}}(k)Q_j\delta(k) - \delta^{\mathrm{T}}(k-\tau_j)Q_j\delta(k-\tau_j)] \\
&\qquad + \sum_{s=1}^{q} \pi_{rs}\delta^{\mathrm{T}}(k-\tau_s)\hat{L}_h^{\mathrm{T}}(\tau_s)\left(\sum_{j=1}^{q}\tau_j Z_j\right)\hat{L}_h(\tau_s)\delta(k-\tau_s) \\
&\qquad - \sum_{j=1}^{q}\sum_{l=k-\tau_j}^{k-1} \delta^{\mathrm{T}}(l-d_l)\hat{L}_h^{\mathrm{T}}(d_l)Z_j\hat{L}_h(d_l)\delta(l-d_l) \\
&\qquad + 2\sum_{j=1}^{q}\zeta^{\mathrm{T}}(k)M_j\left[\delta(k) - \delta(k-\tau_j) - \sum_{l=k-\tau_j}^{k-1}\eta(l)\right] \\
&\qquad + \sum_{j=1}^{q}\sum_{l=k-\tau_j}^{k-1}[\zeta^{\mathrm{T}}(k)M_j + \eta^{\mathrm{T}}(l)Z_j]Z_j^{-1}[M_j^{\mathrm{T}}\zeta(k) + Z_j\eta(l)].
\end{aligned}$$

In the above equation, we have added a nonnegative term (the last one) to form the inequality.

Let the right-hand side of the above equation be less than zero to ensure that $\mathbb{E}\{\Delta V(k)\} < 0$. We have

$$\mathbb{E}\{V(k+1) - V(k)\} \leq \zeta^{\mathrm{T}}(k)[Y_{11}(r) - Y_{12}Y_{22}^{-1}Y_{12}^{\mathrm{T}}]\zeta(k) < 0.$$

By the Schur complement [9], Eqn. (2.6) is derived for r in step k. The subsequent task is to show that the error dynamic system subject to random delays is stable in the mean square sense.

Define $W(r) = Y_{11}(r) - Y_{12}Y_{22}^{-1}Y_{12}^{\mathrm{T}} < 0$. Assume that

$$-\lambda_{\max}(r)I \leq \lambda(W(r)) \leq -\lambda_{\min}(r)I$$

where $\lambda(W(r))$ denotes the eigenvalue of $W(r)$. $\lambda_{\max}(r) > 0$ and $\lambda_{\min}(r) > 0$ are the largest and smallest eigenvalues of $-W(r)$. Then

$$\mathbb{E}\{V(k+1)-V(k)\} \leq \zeta^{\mathrm{T}}(k)W(r)\zeta(k) \leq -\lambda_{\min}(r)\|\zeta(k)\|^2 \leq -\beta\|\zeta(k)\|^2 \tag{2.7}$$

with $0 < \beta = \min\{\lambda_{\min}(r), r = 1, 2, \ldots, q\}$.

Summing (2.7) from $k = 0$, it can be obtained that

$$\mathbb{E}\{V(k+1)-V(0)\} \leq -\beta \sum_{m=0}^{k} \mathbb{E}\{\|\zeta(m)\|^2\},$$

$$\sum_{m=0}^{k} \mathbb{E}\{\|\zeta(m)\|^2\} \leq \frac{1}{\beta}\mathbb{E}\{V(0)-V(k+1)\}.$$

Let $k \to \infty$; then

$$\sum_{m=0}^{\infty} \mathbb{E}\{\|\zeta(m)\|^2\} \leq \frac{1}{\beta}\mathbb{E}\{V(0)-V(\infty)\} \leq \frac{1}{\beta}\mathbb{E}\{V(0)\}.$$

Meanwhile, note that $\|\zeta(m)\|^2 \geq \|\delta(m)\|^2, m = 0, 1, 2, \ldots$. Furthermore,

$$\sum_{m=0}^{\infty} \mathbb{E}\{\|\zeta(m)\|^2\} \geq \sum_{m=0}^{\infty} \mathbb{E}\{\|\delta(m)\|^2\}.$$

Therefore,

$$\sum_{m=0}^{\infty} \mathbb{E}\{\|\delta(m)\|^2\} \leq \frac{1}{\beta}\mathbb{E}\{V(0)\} < \infty,$$

from which we conclude that $\lim_{m\to\infty} \mathbb{E}\{\|\delta(m)\|^2\} = 0$, and thus the error dynamic system in (2.5) is mean square stable. From Lemma 2.1, consensus is guaranteed in the mean square sense. □

Remark 2.4. The Lyapunov function in the above proof is constructed in light of the work in [35]. Here, the novel idea for the consensus problem is to design the delay-dependent Laplacian matrices, or equivalently the adjacency matrices. At time instant k, the corresponding adjacency matrix $A(d_k)$ will be used based on the measured delay d_k. As long as there is a feasible solution for the matrix inequalities in Theorem 2.1, consensus will be reached with the designed switching control law. Note that the sufficient condition given in Theorem 2.1 is not in the form of LMIs. Thus, it is still necessary to derive an equivalent or stronger condition to guarantee consensus, such that it can be conveniently solved using existing tools, e.g., the MATLAB LMI Toolbox. We have the following theorem.

Theorem 2.2. *Consensus is achieved if there exist diagonal matrix* $\bar{P} > 0$ *and* $\bar{Q}_j > 0, \bar{Z}_j > 0, \bar{M}_j, \hat{\bar{L}}_h(\tau_j), j = 1, 2, \ldots, q,$ *such that the following LMI holds for* $r = 1, 2, \ldots, q$*:*

$$\begin{bmatrix} \Upsilon_0 + \Upsilon_1 + \Upsilon_1^{\mathrm{T}} & \Upsilon_2 & \Upsilon_4(r) & \Upsilon_6(r) \\ * & \Upsilon_3 & 0 & 0 \\ * & * & \Upsilon_5 & 0 \\ * & * & * & \Upsilon_7 \end{bmatrix} < 0 \tag{2.8}$$

where

$$\Upsilon_0 = \mathrm{diag}\left\{ -\bar{P} + \sum_{j=1}^{q} \bar{Q}_j, -\bar{Q}_1, -\bar{Q}_2, \ldots, -\bar{Q}_q \right\},$$

$$\Upsilon_1 = \begin{bmatrix} \sum_{j=1}^{q} \bar{M}_j & -\bar{M}_1 & -\bar{M}_2 & \cdots & -\bar{M}_q \end{bmatrix},$$

$$\Upsilon_2 = \begin{bmatrix} \sqrt{\tau_1}\bar{M}_1 & \sqrt{\tau_2}\bar{M}_2 & \cdots & \sqrt{\tau_q}\bar{M}_q \end{bmatrix},$$

$$\Upsilon_3 = -\mathrm{diag}\{\bar{Z}_1 \quad \bar{Z}_2 \quad \cdots \quad \bar{Z}_q\},$$

$$\Upsilon_4(r) = \begin{bmatrix} \sqrt{\pi_{r1}}\bar{P} & \sqrt{\pi_{r2}}\bar{P} & \cdots & \sqrt{\pi_{rq}}\bar{P} \\ \sqrt{\pi_{r1}}\hat{\bar{L}}_h^{\mathrm{T}}(\tau_1) & 0 & \cdots & 0 \\ 0 & \sqrt{\pi_{r2}}\hat{\bar{L}}_h^{\mathrm{T}}(\tau_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\pi_{rq}}\hat{\bar{L}}_h^{\mathrm{T}}(\tau_q) \end{bmatrix},$$

$$\Upsilon_5 = -\mathrm{diag}\{\bar{P} \quad \bar{P} \quad \cdots \quad \bar{P}\},$$

$$\Upsilon_6(r) = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \sqrt{\pi_{r1}}\hat{\bar{L}}_h^{\mathrm{T}}(\tau_1) & 0 & \cdots & 0 \\ 0 & \sqrt{\pi_{r2}}\hat{\bar{L}}_h^{\mathrm{T}}(\tau_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\pi_{rq}}\hat{\bar{L}}_h^{\mathrm{T}}(\tau_q) \end{bmatrix},$$

$$\Upsilon_7 = \mathrm{diag}\left\{ \sum_{j=1}^{q} \tau_j \bar{Z}_j, \sum_{j=1}^{q} \tau_j \bar{Z}_j, \ldots, \sum_{j=1}^{q} \tau_j \bar{Z}_j \right\} - 2(I_q \otimes \bar{P}).$$

The switching control law can be solved as $\hat{L}_h(\tau_j) = \hat{\bar{L}}_h(\tau_j)\bar{P}^{-1}$.

Proof. $Y_{11}(r)$ in (2.6) can be rewritten as

$$Y_{11}(r) = \Lambda_1 + \Omega_1 + \Omega_1^{\mathrm{T}} + \Omega_2(r)\Lambda_2\Omega_2^{\mathrm{T}}(r) + \Omega_3(r)\Lambda_3\Omega_3^{\mathrm{T}}(r)$$

where

$$\begin{aligned}
\Lambda_1 &= \operatorname{diag}\left\{-P + \sum_{j=1}^{q} Q_j, -Q_1, -Q_2, \ldots, -Q_q\right\}, \\
\Omega_1 &= \begin{bmatrix} \sum_{j=1}^{q} M_j & -M_1 & -M_2 & \cdots & -M_q \end{bmatrix}, \\
\Omega_2(r) &= \begin{bmatrix} \sqrt{\pi_{r1}} I_{n-1} & \sqrt{\pi_{r2}} I_{n-1} & \cdots & \sqrt{\pi_{rq}} I_{n-1} \\ \sqrt{\pi_{r1}} \hat{L}_h^{\mathrm{T}}(\tau_1) & 0 & \cdots & 0 \\ 0 & \sqrt{\pi_{r2}} \hat{L}_h^{\mathrm{T}}(\tau_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\pi_{rq}} \hat{L}_h^{\mathrm{T}}(\tau_q) \end{bmatrix}, \\
\Lambda_2 &= \operatorname{diag}\{P, P, \ldots, P\}, \\
\Omega_3(r) &= \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \sqrt{\pi_{r1}} \hat{L}_h^{\mathrm{T}}(\tau_1) & 0 & \cdots & 0 \\ 0 & \sqrt{\pi_{r2}} \hat{L}_h^{\mathrm{T}}(\tau_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\pi_{rq}} \hat{L}_h^{\mathrm{T}}(\tau_q) \end{bmatrix}, \\
\Lambda_3 &= \operatorname{diag}\left\{\sum_{j=1}^{q} \tau_j Z_j, \sum_{j=1}^{q} \tau_j Z_j, \ldots, \sum_{j=1}^{q} \tau_j Z_j\right\}.
\end{aligned}$$

Now, Eqn. (2.6) becomes

$$\begin{bmatrix} \Lambda_1 + \Omega_1 + \Omega_1^{\mathrm{T}} + \Omega_2(r)\Lambda_2\Omega_2^{\mathrm{T}}(r) + \Omega_3(r)\Lambda_3\Omega_3^{\mathrm{T}}(r) & Y_{12} \\ * & Y_{22} \end{bmatrix} < 0,$$

$$\begin{bmatrix} \Lambda_1 + \Omega_1 + \Omega_1^{\mathrm{T}} & Y_{12} \\ * & Y_{22} \end{bmatrix} + \begin{bmatrix} \Omega_2(r) & \Omega_3(r) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Lambda_2 & 0 \\ 0 & \Lambda_3 \end{bmatrix} \begin{bmatrix} \Omega_2^{\mathrm{T}}(r) & 0 \\ \Omega_3^{\mathrm{T}}(r) & 0 \end{bmatrix} < 0.$$

Using the Schur complement, we have

$$\begin{bmatrix} \Lambda_1 + \Omega_1 + \Omega_1^{\mathrm{T}} & Y_{12} & \Omega_2(r) & \Omega_3(r) \\ * & Y_{22} & 0 & 0 \\ * & * & -\Lambda_2^{-1} & 0 \\ * & * & * & -\Lambda_3^{-1} \end{bmatrix} < 0.$$

The term $-\Lambda_3^{-1}$ in the above equation makes it difficult to deal with. To proceed, perform congruence transformation by $\mathrm{diag}\{I_{1+q}\otimes P^{-1}, I_q\otimes P^{-1}, I_q\otimes I_{n-1}, I_q\otimes I_{n-1}\}$, where I_{q+1}, I_q, and I_{n-1} represent the identity matrices with dimensions $q+1, q$, and $n-1$, respectively.

Defining

$$\bar{P} = P^{-1}, \hat{\bar{L}}_h(\tau_j) = \hat{L}_h(\tau_j)\bar{P}, \bar{Q}_j = \bar{P}Q_j\bar{P}, \bar{M}_j = \bar{P}M_j\bar{P}, \bar{Z}_j = \bar{P}Z_j\bar{P},$$

the following inequality is obtained:

$$\begin{bmatrix} \Upsilon_0 + \Upsilon_1 + \Upsilon_1^{\mathrm{T}} & \Upsilon_2 & \Upsilon_4(r) & \Upsilon_6(r) \\ * & \Upsilon_3 & 0 & 0 \\ * & * & \Upsilon_5 & 0 \\ * & * & * & \Upsilon_{7t} \end{bmatrix} < 0$$

where $\Upsilon_i, i = 1, 2, \ldots, 6$, are as in the statement of Theorem 2.2, and

$$\Upsilon_{7t} = -\mathrm{diag}\left\{\sum_{j=1}^{q}\tau_j Z_j, \sum_{j=1}^{q}\tau_j Z_j, \ldots, \sum_{j=1}^{q}\tau_j Z_j\right\}^{-1}.$$

In the above equation, Υ_{7t} contains the inverse of matrix variables to be determined. It cannot be directly solved using existing numerical software. Thus, further manipulation is needed.

For any matrix $Z > 0$, we have

$$-PZ^{-1}P \leq Z - 2P. \tag{2.9}$$

Considering (2.9) gives rise to

$$\Upsilon_{7t} = (I_q \otimes \bar{P})(I_q \otimes P)\Upsilon_{7t}(I_q \otimes P)(I_q \otimes \bar{P})$$

$$\le (I_q \otimes \bar{P})(\Lambda_3 - 2I_q \otimes P)(I_q \otimes \bar{P})$$
$$= (I_q \otimes \bar{P})\Lambda_3(I_q \otimes \bar{P}) - 2(I_q \otimes \bar{P}) = \Upsilon_7.$$

Therefore, if the LMI in (2.8) holds, the error dynamic system is stable and consensus is reached. This completes the proof. □

Remark 2.5. Now we see that the feasibility of the LMI in (2.8) for $r = 1, 2, \dots, q$ implies consensus. The solution of the LMI results in $\hat{L}_h(\tau_j)$, from which the adjacency matrix can be further obtained. The details will be illustrated and exemplified in what follows. To let readers gain an intuitive understanding of the performance of this delay system, factors such as the magnitude of the delays and sampling period will be analyzed. Delays are adverse for consensus, as shown in [96] [126], and the bounds of delays impose constraints on the adjacency or Laplacian matrix. For a given interaction topology and adjacency matrix, when delays increase to a threshold, consensus cannot be guaranteed. On the other hand, a large sampling period also means significant retardation in the information exchange, which is unfavorable for consensus.

As can be observed in (2.8), when the number of agents increases, the computational complexity of the LMI also increases. This is mainly due to design of the Laplacian matrix. From this point of view, the method here is more suitable for small or medium scale systems. For systems of very large scale, we need to come up with new methods to distribute the computational load to individual systems. On the other hand, because the adjacency matrices are designed offline, the online computation in each agent has been greatly reduced.

2.4.2 Design of the Adjacency Matrices

After solving for the matrices $\hat{L}_h(\tau_j), j = 1, 2, \dots, q$, the remaining problem is how we can derive the corresponding Laplacian matrices $L(\tau_j)$. This can be done in the following steps: (1) get $\hat{L}(\tau_j) = \hat{L}_h(\tau_j)/h$; (2) solve (2.4) for the graph Laplacian $L(\tau_j)$.

Here, all the adjacency matrices corresponding to different delays must be of the same type to maintain the communication interaction. A question arises naturally: How can we ensure this after solving the LMIs? Apparently, in the transformation $\hat{\hat{L}}_h(\tau_j) = \hat{L}_h(\tau_j)\bar{P}$, for arbitrary $\bar{P} > 0$, the matrices $\hat{\hat{L}}_h(\tau_j)$ and $\hat{L}_h(\tau_j)$ may not be of the same type. To tackle this difficulty, we define $\bar{P}$ to be diagonal in the program.

This technique guarantees that $\hat{\hat{L}}_h(\tau_j)$ and $\hat{L}_h(\tau_j)$ will always be of the same type. Subsequently, the entries in the adjacency matrix $A(\tau_j)$ will be properly defined to reflect its type. It is worth noting that if we have information in advance about the structure of the adjacency matrix, it is possible to select a more general and thus better structure other than diagonal form for matrix $\bar{P}$. In this way, we can reduce the conservativeness when designing the controllers, thus improve the system performance.

For a known type of $A(\tau_j)$, there is an associated matrix $\hat{L}(\tau_j)$ in (2.4), which is a function of the elements $a_{km}(\tau_j), k, m = 1, 2, \ldots, n$. Then we can determine which elements are independent variables in $\hat{\hat{L}}_h(\tau_j)$ and each of them can be defined as a Type 1 scalar variable as in the MATLAB LMI Toolbox. Based on those variables, we define $\hat{\hat{L}}_h(\tau_j)$ as Type 3. After describing the LMIs in (2.8) term by term, the matrix $\hat{L}_h(\tau_j) = h\hat{L}(\tau_j)$ is obtained. Using back substitution, $A(\tau_j)$ can be solved. As a result, the type of the adjacency matrix $A(\tau_j)$ will not be affected.

Then we are in the position that there are $(n-1)^2$ equations with $n^2-n = n(n-1)$ unknowns. The equations are often underdetermined, and the difficulty comes from determining the values of $l_{12}, l_{13}, \ldots, l_{1n}$. In practice, with the adjacency matrix of a specific type, we may be able to determine those parameters uniquely. This case will be illustrated soon in the example. From the design process, we observe that not all the elements in the adjacency matrix need to be positive. On the contrary, the existence of some negative entries even provides certain benefits in some cases. This will be demonstrated through a comparison in Section 2.5.1.

2.5 Illustrative Examples

In this section, we provide illustrative examples to verify the design procedure and the effectiveness of the proposed method. Figure 2.1 shows the communication topology of five agents.

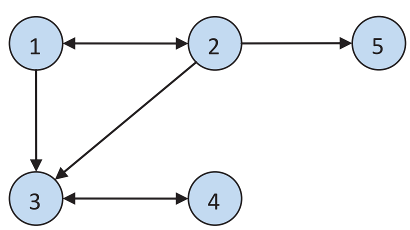

Figure 2.1: Communication topology with a directed spanning tree.

The sampling time is $h = 0.01$ s and the delay set is $\Gamma = \{65, 120, 180, 200\}$ steps $= \{0.65, 1.2, 1.8, 2.0\}$ s. Their transition probability matrix is

$$\Pi = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.3 & 0.4 & 0.3 & 0 \\ 0.3 & 0.2 & 0.3 & 0.2 \\ 0.2 & 0.4 & 0.3 & 0.1 \end{bmatrix}.$$

The initial state of the five agents is $x(0) = [0 \quad 1 \quad 2 \quad 3 \quad 4]^{\mathrm{T}}$. Two groups of simulations are conducted.

2.5.1 Group Coordination with Fixed Adjacency Matrix

In this example, the adjacency matrix below is used:

$$A_{e1} = \begin{bmatrix} 0 & 0.8 & 0 & 0 & 0 \\ 0.6 & 0 & 0 & 0 & 0 \\ 0.5 & 0.4 & 0 & 0.6 & 0 \\ 0 & 0 & 0.7 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}. \tag{2.10}$$

The evolution of states is shown in Figure 2.2, from which it can be seen that the states of the agents diverge, indicating that consensus is not achieved. The multi-agent system with a fixed communication topology cannot tolerate the random delays from the prescribed set. The delay-independent control strategy does not meet our goal. Figure 2.3 demonstrates the jumping delay within the first two seconds.

To make a comparison and show the merits when we allow the adjacency matrix to have negative elements, the following matrix is used in the simulation:

$$A_{e2} = \begin{bmatrix} 0 & 0.8 & 0 & 0 & 0 \\ 0.6 & 0 & 0 & 0 & 0 \\ 0.5 & -0.4 & 0 & 0.6 & 0 \\ 0 & 0 & 0.7 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

The sequence of delays is still the same and the simulation result is shown in Figure 2.4. As can be seen, with the negative element -0.4, consensus is reached. This

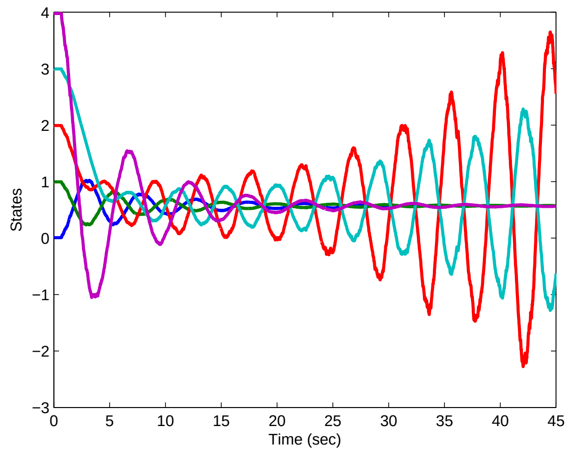

Figure 2.2: State evolution (x) of the agents with A_{e1}.

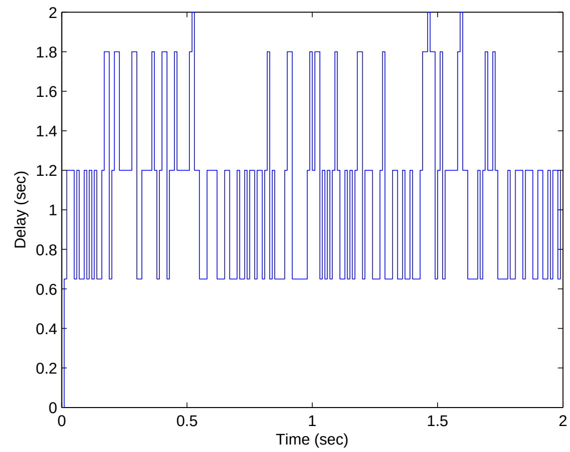

Figure 2.3: Time delay (d_k) over time.

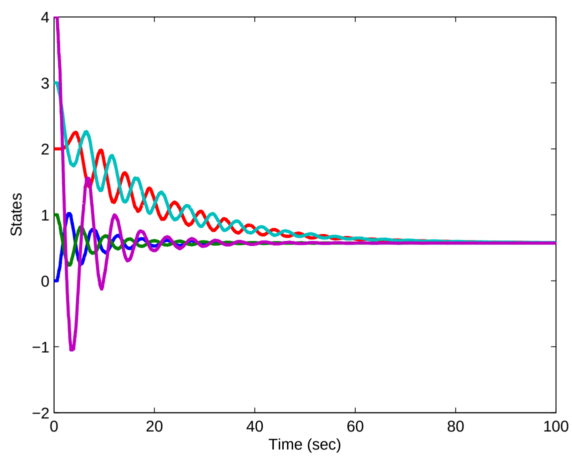

Figure 2.4: State evolution (x) of the agents with A_{e2}.

example shows the flexibility and effectiveness of allowing negative elements in the adjacency matrix. However, it does not provide a systematic way for controller design. In our method, we aim to propose a delay-dependent control design for consensus in a systematic way.

2.5.2 Group Coordination with Switching Adjacency Matrices

Here, we will show that, with the proposed switching control design, the original system in the previous example (in Section 2.5.1) which diverges could be stabilized by a delay-dependent controller. Based on the delay at each time instant, a different control law is utilized. In this way, consensus can be guaranteed. By solving the LMIs in Theorem 2.2, delay-dependent controllers corresponding to the delays are designed. This will be elaborated in detail subsequently.

An adjacency matrix A with the same type as the one in (2.10) has the following general form:

$$A = \begin{bmatrix} 0 & a_{12} & 0 & 0 & 0 \\ a_{21} & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & 0 & a_{34} & 0 \\ 0 & 0 & a_{43} & 0 & 0 \\ 0 & a_{52} & 0 & 0 & 0 \end{bmatrix}.$$

Its corresponding Laplacian matrix is

$$L = \begin{bmatrix} a_{12} & -a_{12} & 0 & 0 & 0 \\ -a_{21} & a_{21} & 0 & 0 & 0 \\ -a_{31} & -a_{32} & a_{31}+a_{32}+a_{34} & -a_{34} & 0 \\ 0 & 0 & -a_{43} & a_{43} & 0 \\ 0 & -a_{52} & 0 & 0 & a_{52} \end{bmatrix},$$

with $\hat{L}$ in (2.4) being

$$\hat{L} = -\begin{bmatrix} a_{21}+a_{12} & 0 & 0 & 0 \\ -a_{32}+a_{12} & a_{31}+a_{32}+a_{34} & -a_{34} & 0 \\ a_{12} & -a_{43} & a_{43} & 0 \\ -a_{52}+a_{12} & 0 & 0 & a_{52} \end{bmatrix}.$$

From the above equation, there are seven parameters to be determined: a_{12}, a_{21}, a_{31}, $a_{32}, a_{34}, a_{43}, a_{52}$. Solving the LMIs in (2.8), the following adjacency matrices are obtained:

$$A_1 = \begin{bmatrix} 0 & 0.01815 & 0 & 0 & 0 \\ 0.04491 & 0 & 0 & 0 & 0 \\ 0.04667 & 0.02101 & 0 & 0.01473 & 0 \\ 0 & 0 & 0.0318 & 0 & 0 \\ 0 & 0.03711 & 0 & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0.004697 & 0 & 0 & 0 \\ 0.0002828 & 0 & 0 & 0 & 0 \\ 0.0008104 & 0.004506 & 0 & 0.003306 & 0 \\ 0 & 0 & 0.005164 & 0 & 0 \\ 0 & 0.005586 & 0 & 0 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 0.005732 & 0 & 0 & 0 \\ -0.002495 & 0 & 0 & 0 & 0 \\ -0.002725 & 0.008094 & 0 & 0.006064 & 0 \\ 0 & 0 & 0.005945 & 0 & 0 \\ 0 & 0.00438 & 0 & 0 & 0 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 0 & 0.01233 & 0 & 0 & 0 \\ -0.006954 & 0 & 0 & 0 & 0 \\ -0.00771 & 0.01693 & 0 & 0.01469 & 0 \\ 0 & 0 & 0.01175 & 0 & 0 \\ 0 & 0.008514 & 0 & 0 & 0 \end{bmatrix}.$$

The four matrices A_1, A_2, A_3, and A_4 are associated with the delays 0.65 s, 1.2 s, 1.8 s, and 2.0 s, respectively. Note that there are some negative elements in the last two matrices. In fact, the condition that all the elements in matrix A should be nonnegative is not indispensable to guarantee consensus. As long as the error dynamic system is stable, consensus will be achieved asymptotically.

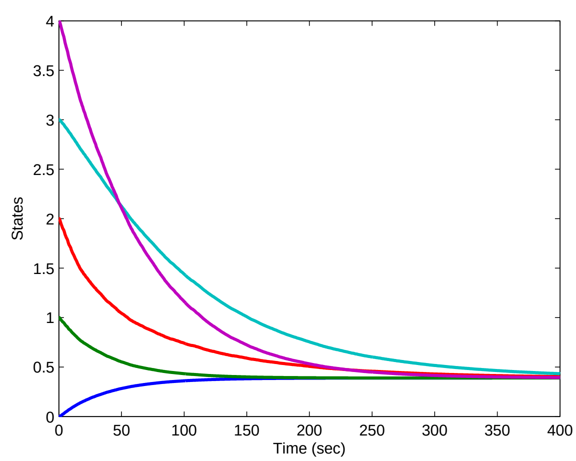

Figure 2.5: Consensus with switching adjacency matrices.

In Figure 2.5, we see that, with the same time delays, all the agents in the group reach consensus due to the use of delay-dependent adjacency matrices. It deserves further research to accelerate the convergence rate.

2.6 Conclusion

This work investigates the consensus problem for MASs with random delays governed by a Markov chain under the sampled-data scenario. By constructing a proper Lyapunov–Krasovskii functional, a sufficient condition is given in terms of the feasibility of some LMIs. Finally, the efficacy is verified by simulation results. The main contribution of the present work is the consideration of statistic features existing in delays. There could be some further improvements for this work. (1) In this work, we only consider uniform delay and this scenario can be extended to the case with nonuniform delays, and the delay across each communication link being subject to an independent Markov chain. (2) As noted in the literature [104], many researchers are studying higher-order dynamics, such as second-order dynamics or systems described by state-space equations. Moreover, consensus in nonlinear dynamic agents would be of great interest [47] [18].

Chapter 3

Average Consensus in Multi-Agent Systems with Non-uniform Time-Varying Delays and Random Packet Losses

3.1 Introduction

The MASs have been attracting increasing attention in recent years [53] [107] [104] [19]. Compared with traditional control systems, MASs adopt the distributed control framework to improve reliability and to allocate computational load more evenly. A central issue in MASs is the coordination of collective behavior, and it is often referred to as *consensus*, meaning that all the states of interest finally converge to a common decision value. Among different types of consensus, average consensus is especially interesting and there has been a lot of work on this topic [96] [124]. This requires that the states of all agents approach the average of their initial states. It can find applications in distributed estimation, resource and load allocation, and so on.

With the aid of communication networks (e.g., wireless communication), it is convenient to deploy an MAS and establish a cooperative interaction among them. Meanwhile, challenges also arise, mainly due to the intermittence of information transmission. For example, data need to be processed first and then sent out through a network, thus they will arrive at a delayed time. If the situation becomes worse, data packets may even get lost. To ameliorate the system performance, consensus with communication constraints has been investigated [96] [107] [156].

The authors in [96] [126] discussed consensus in single-integrator agents with constant delays using the Nyquist diagram in frequency domain. Consensus with time-varying delays was studied in [124] [122] [76] using the LMI based Lyapunov method. The work in [156] dealt with consensus with random delay and data loss governed by Bernoulli process. However, the delay was assumed to be no larger than one sampling period. In [29], average consensus was discussed and the convergence speed was analyzed with respect to the probability of data loss. Packet dropouts can also be modeled by a random graph process which assigns each edge a probability of existence. For instance, [43] [133] [101] studied stochastic consensus when the communication topology was subject to a random process, but they did not consider the effect of time delay that is often unavoidable.

In this work, we investigate average consensus in the presence of time delays and packet losses simultaneously. The delays are non-uniform, time-varying, and bounded. Every communication link has a different probability of failure. Each agent is equipped with a sampler and a zero-order hold, and the agents are synchronized in time. Then, by establishing the equivalent error dynamics, a Lyapunov function is constructed to analyze its stochastic stability, and thus to establish the conditions guaranteeing average consensus.

The contribution of this work mainly lies in the consideration of time delays and packet losses in a unified framework. The probabilities of data losses are explicitly taken into account. Some differences from previous work are: (1) In [156], data loss was treated as a switching topology with probability and the delay was less than one sampling period. We concentrate more on the link failure with a probability and the delay can be larger than one sampling period. (2) In [29] [43] [133] [101], the effect of delay was not considered. Here, we do not stress much on every single pattern of the communication topology due to data loss. The influence of data loss will be evaluated in the stochastic sense.

The remainder of this work is organized as follows. Section 3.2 contains some preliminaries on graph theory and the establishment of system dynamics. A sufficient condition to guarantee consensus is presented in Section 3.3. In Section 3.4, we provide a numerical example to verify the proposed approach. Finally, some conclusions and remarks are offered in Section 3.5.

Notation: $\mathbb{R}$ denotes the set of real numbers. For a matrix $P \in \mathbb{R}^{n\times n}$, $\mathrm{sym}(P) = P + P^{\mathrm{T}}$. $P > 0$ ($P < 0$) implies P is symmetric and positive definite (negative definite). '*' in a matrix stands for a term that is induced by symmetry. The set $\mathcal{I}_n = \{1, 2, \ldots, n\}$ includes integers from 1 to n. $\mathbf{1}_n = [1, 1, \ldots, 1]^{\mathrm{T}} \in \mathbb{R}^n$ is a vector with all elements equal to 1. We use $\mathbb{E}\{\cdot\}$ for the mathematical expectation and $\mathbb{P}\{\cdot\}$ is the probability operator. $[\![a_{ij}]\!]$ is the abbreviation for a matrix which has element a_{ij} at position (i, j). Here, $[\![a_{ij}]\!]$ is used instead of $[a_{ij}]$ as in Chapter 2, because $[\cdot]$ in this work might cause confusion between the notation of a matrix and that of a mathematical operator.

3.2 Problem Formulation

3.2.1 Preliminaries on Graph Theory

For a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ with n nodes, $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ is the node set and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set. An edge $(v_j, v_i) \in \mathcal{E}$ from v_j to v_i implies that node v_i receives information from v_j. If $(v_j, v_i) \in \mathcal{E}$ implies $(v_i, v_j) \in \mathcal{E}$, then the graph is *undirected*; otherwise, it is *directed*. A *path* from v_i to v_j in $\mathcal{G}$ is a sequence of edges from v_i to v_j. An undirected graph is *connected* if for any $v_i, v_j \in \mathcal{E}, v_i \neq v_j$, there exists a path from v_i to v_j. In an MAS, each agent i is represented by a node v_i in graph $\mathcal{G}$, and each edge (v_j, v_i) corresponds to a communication channel from agent j to agent i.

The neighbor set of agent i includes agents from which agent i receives information and it is denoted by $\mathcal{N}_i$. The adjacency matrix $A = [\![a_{ij}]\!] \in \mathbb{R}^{n \times n}, i, j \in \mathcal{I}_n$ is a matrix with nonnegative entries satisfying $a_{ij} > 0$ if $(v_j, v_i) \in \mathcal{E}$, and $a_{ij} = 0$ otherwise. Assume that there is no self loop from one node to itself, i.e., $a_{ii} = 0$. The Laplacian matrix $L = [\![l_{ij}]\!] \in \mathbb{R}^{n \times n}$ is defined as $l_{ij} = -a_{ij}, i \neq j; l_{ii} = \sum_{j \in \mathcal{N}_i} a_{ij}$. Apparently, L has at least one zero eigenvalue with the associated eigenvector $\mathbf{1}_n$, i.e., $L \cdot \mathbf{1}_n = 0$. For an undirected graph, we further have $L = L^{\mathrm{T}}, \mathbf{1}_n^{\mathrm{T}} L = 0$. More details on algebraic graph theory can be found in [37].

3.2.2 System Dynamics

Consider the single-integrator dynamics

$$\dot{x}_i = u_i, \quad i \in \mathcal{I}_n. \tag{3.1}$$

By adopting the scheme of periodic sampling, (3.1) can be discretized as

$$x_i(k+1) = x_i(k) + Tu_i(k),$$

where T is the sampling period. With time delays and data losses between agents considered, the following consensus protocol is employed,

$$u_i(k) = -\gamma_c \sum_{j\in\mathcal{N}_i} \gamma_{ij}(k)a_{ij}[x_i(k - d_{ij}(k)) - x_j(k - d_{ij}(k))], \tag{3.2}$$

where $\gamma_{ij}(k) = 1$ if there is no packet loss from agent j to agent i, and $\gamma_{ij}(k) = 0$ otherwise. $d_{ij}(k)$ is the delay from agent j to agent i at time instant k. γ_c is the control gain. Assume that packet loss occurs with the non-uniform probability $1 - p_{ij} \geq 0$, i.e.,

$$\mathbb{P}\{\gamma_{ij}(k) = 1\} = p_{ij}, \mathbb{P}\{\gamma_{ij}(k) = 0\} = 1 - p_{ij}, \forall i \neq j.$$

Apparently, $\mathbb{E}\{\gamma_{ij}(k)\} = p_{ij}$ is the probability of successfully receiving data.

Remark 3.1. It can be observed from (3.2) that this control law does not guarantee the use of most recent information since the time delay $d_{ij}(k)$ can vary arbitrarily. In addition, we may also consider another case in which the most recent information is used. Under that circumstance, we do not deal with data loss directly, but perceive it as a delay process. That is, if an agent does not receive any information from its neighbors, then it uses information received at the previous step. As a result, the delay increases by one step. The delay in (3.2) can change arbitrarily, while the delay with the most recent information can only rise by one at most.

Another scenario in which the information from an agent to itself is not delayed may also be considered. From the previous literature [85], this control strategy exhibits robustness against time delays. This will be studied in the future.

In (3.2), one may be wondering why we do not merge γ_c into a_{ij}. This is because the elements a_{ij} in the adjacency matrix are fixed so that we do not have to design the adjacency matrix A, which is often difficult. The introduction of γ_c provides flexibility in controller design. It is worth pointing out that if we let γ_c depend on i, that will lead to more freedom in choosing the controller gain. In Chapter 2, we let the adjacency matrix depend on current delay and design the delay-dependent adjacency matrices. In this chapter, A is fixed to avoid the difficulty to design A.

In order to determine the probabilities of data losses, an experiment can be performed. By sending out a number (N) of data packets, we calculate the number (N_s) of packets that arrive successfully on the receiver side. Then we calculate the ratio N_s/N, and the probability of data loss is obtained.

Assumption 3.1. The undirected topology is fully symmetric, i.e., for any pair of agents i and j, the link from i to j and that from j to i exist or vanish simultaneously, and they are subject to the same delay $d_{ij}(k) = d_{ji}(k)$ at each time instant.

Remark 3.2. The above assumption ensures that the symmetry of communication among the MAS is always maintained during the dynamic evolution. Thus, the average of agents' states is reserved in the system dynamics. The symmetric communication can happen in practice when the agents are equipped with transceivers of the same performance specifications. As a more general scenario, it is meaningful to investigate the asymmetric communication in the future.

Define $L(k) = [\![l_{ij}(k)]\!] \in \mathbb{R}^{n\times n}, l_{ij}(k) = -\gamma_{ij}(k)a_{ij}, i \neq j, l_{ii}(k) = \sum_{j=1}^{n} \gamma_{ij}(k)a_{ij}$, and

$$
\begin{aligned}
x(k) &= [x_1(k), x_2(k), \ldots, x_n(k)]^{\mathrm{T}}, \\
u(k) &= [u_1(k), u_2(k), \ldots, u_n(k)]^{\mathrm{T}}.
\end{aligned}
$$

Then the system dynamics can be written as

$$
x(k+1) = x(k) - \gamma_c T \sum_{m=1}^{r} L^{(m)}(k)x(k - d_m(k)), \tag{3.3}
$$

where $d_m \in \{d_{ij} : i, j \in \mathcal{I}_n\}$ and $L(k) = \sum_{m=1}^{r} L^{(m)}(k)$. Here, r denotes the number of different delay sequences, i.e., some delays may have the same quantity throughout. From Assumption 3.1, it is easy to get that $\mathbf{1}_n^{\mathrm{T}} L^{(m)}(k) = 0$ and $\mathbf{1}_n^{\mathrm{T}} L(k) = 0$.

Remark 3.3. Above we consider d_m instead of d_{ij}. This is more general, since it is possible that some delay sequences may be the same throughout the dynamic process, e.g., $d_{12}(k) = d_{13}(k)$ for all $k \geq 0$. In order to get $L^{(m)}$, we can first figure out the number of delay sequences d_m, i.e., r in this work. Then, for the communication channels with one specific delay sequence, say $d_m(k)$, collect the elements at the corresponding positions of the adjacency matrix $A = [a_{ij}]_{n\times n}$. This process splits the matrix A into the sum of r matrices. As a result, $L(k)$ has also been written into the sum of $L^{(1)}(k), \ldots, L^{(r)}(k)$.

Assumption 3.2. The time-varying delays are bounded: $0 \leq d_m(k) \leq h_m$.

Remark 3.4. In practice, in order to determine the bounds of time delays, a very straightforward way is to measure time delays over a large time interval, and then pick the maximum one among these quantities. That maximum quantity is exactly the upper bound of time delays. Now the question is: How do we measure time delays accurately? The key technique for this purpose is to use the time stamp. Before everything is operated, the first step is to synchronize the clocks mounted in different sensors and actuators. After that, for each data packet that is going to be sent, we append the time information to it. In this way, after receiving the data, the precise time delay can be calculated.

Note that in this chapter, we only consider the lower and upper bounds in delays. Unlike Chapter 2, where the transition probabilities between delays are taken into account, this chapter does not incorporate stochastic information in delays.

Let $\mathcal{G}^{(m)}$ be the graph associated with the set $\{L^{(m)}(k), k = 0, 1, 2, \ldots\}$. Then its edge set $\mathcal{E}(\mathcal{G}^{(m)})$ includes all possible edges with delay $d_m(k)$. At each time instant k, the graph of $L^{(m)}(k)$ will take some edges from $\mathcal{E}(\mathcal{G}^{(m)})$ with certain probability. By taking the mathematical expectation of $L^{(m)}(k)$, we have

$$\hat{L}^{(m,0)} = \mathbb{E}\{L^{(m)}(k)\} = \left[\!\left[\hat{l}_{ij}^{(m,0)}\right]\!\right] :$$
$$\hat{l}_{ij}^{(m,0)} = p_{ij} l_{ij}^{(m,0)}, i \neq j; \hat{l}_{ii}^{(m,0)} = -\sum_{j=1, j\neq i}^{n} \hat{l}_{ij}^{(m,0)}.$$

$L^{(m,0)} = \left[\!\left[l_{ij}^{(m,0)}\right]\!\right]$ can be interpreted as the Laplacian matrix of full weights associated with $\mathcal{G}^{(m)}$. It is defined as: $l_{ij}^{(m,0)} = -a_{ij}$ if $i \neq j$ and the edge $(v_j, v_i) \in \mathcal{E}(\mathcal{G}^{(m)})$; $l_{ii}^{(m,0)} = \sum_{j \in \mathcal{N}_i^{(m)}} a_{ij}$, where $\mathcal{N}_i^{(m)}$ denotes the neighbors of agent i in $\mathcal{G}^{(m)}$. The adjacency matrix corresponding to $L^{(m,0)}$ is denoted as $A^{(m,0)}$. Since each edge is subject to a unique random process, there is at most one nonzero entry at each off-diagonal position of the matrices from the set $\{L^{(m,0)}, m \in \mathcal{I}_r\}$.

Remark 3.5. When taking the mathematical expectation of $L^{(m)}(k)$, one may think that $\mathbb{E}\{L^{(m)}(k)\}$ depends on the probabilistic distribution of the time delay $d_m(k)$. This is in fact not the case. The time index k denotes the status of data loss. It corresponds to the random variables $\gamma_{ij}(k)$. Once the delays are grouped into r random variables $d_1(k), \ldots, d_r(k)$, the matrices $L^{(1)}(k), \ldots, L^{(r)}(k)$ are independent of the delays. Therefore, $L^{(m)}(k)$ is a matrix for data loss only. For any time instant

k, when calculating $\mathbb{E}\{L^{(m)}(k)\}$, since the data loss across each communication link is subject to a Bernoulli process, we will always have the same expectation $L^{(m,0)}$ for a specific m.

Before proceeding to the main results, the following assumption is necessary for consensus.

Assumption 3.3. The expected communication topology $\mathcal{G}^{(0)}$ associated with $L^{(0)} = \sum_{m=1}^{r} L^{(m,0)}$ is connected.

This assumption guarantees the connectivity on communication graph: There should be at least one agent that is able to affect all other agents directly or indirectly. If the undirected graph $\mathcal{G}^{(0)}$ is not connected, then there exist at least two nonempty, disjoint groups of agents that have no communication with each other at any time. Thus, consensus cannot be reached.

Remark 3.6. The loss and recovery of communication links can also be characterized by switching topology; following this line, some studies have been conducted in the literature [53] [107] [76]. In these works, the authors consider the very general switching pattern with some graph connectivity constraints. To better characterize the variation of topology, we explicitly incorporate the probability information into the theoretical analysis.

In what follows, the main results will be presented, and a sufficient condition to guarantee the average consensus is given in terms of LMIs.

3.3 Main Results

From (3.3), we can see that the average of agents' states $\alpha = \frac{1}{n}\sum_{i=1}^{n} x_i(k) = \frac{1}{n}\mathbf{1}_n^{\mathrm{T}} x(k)$ is an invariant quantity. Then perform decomposition of the following form:

$$x(k) = \alpha \mathbf{1}_n + \delta(k),$$

where $\delta(k) = [\delta_1(k), \delta_2(k), \ldots, \delta_n(k)]^{\mathrm{T}}$ satisfying $\mathbf{1}_n^{\mathrm{T}}\delta(k) = 0$. Following some manipulation, the error dynamics is obtained:

$$\delta(k+1) = \delta(k) - \gamma_c T \sum_{m=1}^{r} L^{(m)}(k)\delta(k - d_m(k)). \tag{3.4}$$

Obviously, the stability of (3.4) is equivalent to the consensus in (3.1) or (3.3). In the following, we will study the stochastic stability of the system in (3.4). The definition of mean square stability is introduced first [35].

Definition 3.1. The error dynamic system in (3.4) is mean square stable if

$$\lim_{k\to\infty} \mathbb{E}\{\|\delta(k)\|^2\} = 0.$$

The following lemma plays an important role in the stability analysis of (3.4).

Lemma 3.1. *Suppose that $L^{(i)}(k)$ and $L^{(j)}(k), i, j \in \mathcal{I}_r$ are Laplacian matrices corresponding to two undirected graphs. Then, for a matrix $P = P^{\mathrm{T}}$, $\mathbb{E}\{L^{(i)}(k)PL^{(j)}(k)\}$ can be evaluated as follows:*
(1) *$i \neq j$. We have*

$$\mathbb{E}\{L^{(i)}(k)PL^{(j)}(k)\} = \hat{L}^{(i,0)}P\hat{L}^{(j,0)};$$

(2) *$i = j$. We have*

$$\mathbb{E}\{L^{(i)}(k)PL^{(i)}(k)\} = \hat{L}^{(i,0)}P\hat{L}^{(i,0)} + \Xi(i,P), \tag{3.5}$$

where $\Xi(i,P) = \Xi_1(i,P) + \Xi_2(i,P)$, being a function of i and P, is defined as

$$\begin{aligned}
\Xi_1(i,P) = {} & \sum_{m=1}^{n}\sum_{q=1}^{n}(p_{mq} - p_{mq}^2)\left[E_{(m,q)}^{\mathrm{T}}PE_{(m,q)}\right. \\
& + E_{(m,m)}^{\mathrm{T}}PE_{(m,m)} \\
& \left.-\mathrm{sym}\left(E_{(m,q)}^{\mathrm{T}}PE_{(m,m)}\right)\right]\left(A_{m,q}^{(i,0)}\right)^2, \\
\Xi_2(i,P) = {} & \sum_{m=1}^{n}\sum_{t=m+1}^{n}(p_{mt} - p_{mt}^2)\mathrm{sym}\left(2E_{(m,m)}^{\mathrm{T}}PE_{(t,t)}\right. \\
& \left.-E_{(m,m)}^{\mathrm{T}}PE_{(t,m)} - E_{(m,t)}^{\mathrm{T}}PE_{(t,t)}\right)\left(A_{m,t}^{(i,0)}\right)^2.
\end{aligned}$$

$E_{(m,q)} \in \mathbb{R}^{n\times n}$ has only one nonzero entry equal to 1 at the position (q,m) and entries at other positions are zero. The product $E_{(m_1,q_1)}^{\mathrm{T}}PE_{(m_2,q_2)}$ is a matrix with the sole nonzero entry at position (m_1,m_2) and that entry equals P_{q_1,q_2}. $A_{m,q}^{(i,0)}$ represents the element at (m,q) in $A^{(i,0)}$.

Proof. The proof is included in Subsection 3.3.1. □

The following theorem provides a sufficient condition for consensus.

Theorem 3.1. *Given the control gain γ_c, consensus is achieved in (3.3) if there exist matrices $P > 0, R_i > 0, M_i, i = 1, 2, \ldots, r$, such that the following LMI holds:*

$$\begin{bmatrix} W^{\mathrm{T}}(\Upsilon_1 + \Upsilon_2 + \Upsilon_3 + \Upsilon_4 + \Upsilon_4^{\mathrm{T}})W & W^{\mathrm{T}}\Phi \\ * & \Lambda \end{bmatrix} < 0, \tag{3.6}$$

where

$$W = I_{r+1} \otimes F, \quad F = \begin{bmatrix} I_{n-1} \\ -\mathbf{1}_{n-1}^{\mathrm{T}} \end{bmatrix},$$

$$\Upsilon_1 = -\gamma_c T \begin{bmatrix} 0 & P\hat{L}^{(1,0)} & \cdots & P\hat{L}^{(r,0)} \\ * & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & 0 & \cdots & 0 \end{bmatrix},$$

$$\Upsilon_2 = \gamma_c^2 T^2 \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \hat{L}^{(1,0)}\Omega\hat{L}^{(1,0)} & \cdots & \hat{L}^{(1,0)}\Omega\hat{L}^{(r,0)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \hat{L}^{(r,0)}\Omega\hat{L}^{(1,0)} & \cdots & \hat{L}^{(r,0)}\Omega\hat{L}^{(r,0)} \end{bmatrix},$$

$$\Omega = P + \sum_{m=1}^{r} h_m R_m,$$

$$\Upsilon_3 = \gamma_c^2 T^2 \mathrm{diag}\left\{0, \Xi(1, P) + \Xi\left(1, \sum_{m=1}^{r} h_m R_m\right),\right.$$
$$\left.\ldots, \Xi(r, P) + \Xi\left(r, \sum_{m=1}^{r} h_m R_m\right)\right\},$$

$$\Upsilon_4 = \begin{bmatrix} \sum_{m=1}^{r} M_m & -M_1 & \cdots & -M_r \end{bmatrix},$$

$$\Phi = \begin{bmatrix} h_1 M_1 & \cdots & h_r M_r \end{bmatrix},$$

$$\Lambda = \mathrm{diag}\{-h_1 R_1, \ \ldots, \ -h_r R_r\}.$$

In the matrix W, '$\otimes$' denotes the Kronecker product.

Proof. Construct the Lyapunov function candidate $V(k) = V_1(k) + V_2(k)$,

$$V_1(k) = \delta^{\mathrm{T}}(k) P \delta(k),$$

$$V_2(k) = \sum_{m=1}^{r} \sum_{j=-h_m}^{-1} \sum_{i=k+j}^{k-1} v^{\mathrm{T}}(i) R_m v(i),$$
$$v(i) = \delta(i+1) - \delta(i),$$

where $P > 0$ and $R_m > 0$ are to be determined. Then from (3.4) we have

$$\begin{aligned}
&\mathbb{E}\{\Delta V_1(k)\} = \mathbb{E}\{V_1(k+1) - V_1(k)\} \\
&= \mathbb{E}\{\delta^{\mathrm{T}}(k+1) P \delta(k+1) - \delta^{\mathrm{T}}(k) P \delta(k)\} \\
&= \mathbb{E}\left\{ -2\gamma_c T \delta^{\mathrm{T}}(k) P \sum_{m=1}^{r} L^{(m)}(k) \delta(k - d_m(k)) \right. \\
&\qquad + \gamma_c^2 T^2 \sum_{i=1}^{r} \sum_{j=1}^{r} \delta^{\mathrm{T}}(k - d_i(k)) L^{(i)}(k) P \\
&\qquad\qquad \left. \times\, L^{(j)}(k) \delta(k - d_j(k)) \right\}.
\end{aligned}$$

Now the difficulty lies in calculating $\mathbb{E}\{L^{(i)}(k) P L^{(j)}(k)\}$ due to the existence of correlation in the elements of $L^{(i)}(k)$ and those of $L^{(j)}(k)$. Using Lemma 3.1, $\mathbb{E}\{\Delta V_1(k)\}$ becomes

$$\begin{aligned}
&\mathbb{E}\{\Delta V_1(k)\} \\
&= -2\gamma_c T \cdot \delta^{\mathrm{T}}(k) P \sum_{m=1}^{r} \hat{L}^{(m,0)} \delta(k - d_m(k)) \\
&+ \gamma_c^2 T^2 \sum_{i=1}^{r} \sum_{j=1}^{r} \delta^{\mathrm{T}}(k - d_i(k)) \hat{L}^{(i,0)} P \hat{L}^{(j,0)} \delta(k - d_j(k)) \\
&+ \gamma_c^2 T^2 \sum_{i=1}^{r} \delta^{\mathrm{T}}(k - d_i(k)) \Xi(i, P) \delta(k - d_i(k)).
\end{aligned}$$

For $V_2(k)$,

$$\begin{aligned}
&\mathbb{E}\{\Delta V_2(k)\} = \mathbb{E}\{V_2(k+1) - V_2(k)\} \\
&= \mathbb{E}\left\{ v^{\mathrm{T}}(k) \left(\sum_{m=1}^{r} h_m R_m \right) v(k) \right. \\
&\qquad \left. - \sum_{m=1}^{r} \sum_{j=k-h_m}^{k-1} v^{\mathrm{T}}(j) R_m v(j) \right\}
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\left\{\gamma_c^2 T^2 \sum_{i=1}^{r}\sum_{j=1}^{r} \delta^{\mathrm{T}}(k-d_i(k)) L^{(i)}(k) \left(\sum_{m=1}^{r} h_m R_m\right)\right. \\
&\qquad \left. \times L^{(j)}(k)\delta(k-d_j(k)) - \sum_{m=1}^{r}\sum_{j=k-h_m}^{k-1} v^{\mathrm{T}}(j) R_m v(j)\right\} \\
&= \gamma_c^2 T^2 \sum_{i=1}^{r}\sum_{j=1}^{r} \delta^{\mathrm{T}}(k-d_i(k)) \hat{L}^{(i,0)} \left(\sum_{m=1}^{r} h_m R_m\right) \\
&\qquad \times \hat{L}^{(j,0)} \delta(k-d_j(k)) \\
&\quad + \gamma_c^2 T^2 \sum_{i=1}^{r} \delta^{\mathrm{T}}(k-d_i(k)) \Xi\left(i, \sum_{m=1}^{r} h_m R_m\right) \\
&\qquad \times \delta(k-d_i(k)) - \sum_{m=1}^{r}\sum_{j=k-h_m}^{k-1} v^{\mathrm{T}}(j) R_m v(j).
\end{aligned}
$$

Define

$$
\xi(k) = [\delta^{\mathrm{T}}(k), \delta^{\mathrm{T}}(k-d_1(k)), \ldots, \delta^{\mathrm{T}}(k-d_r(k))]^{\mathrm{T}}.
$$

For any matrices $M_m, m \in \mathcal{I}_r$ with appropriate dimensions, consider the nonnegative term

$$
\begin{aligned}
0 &\le \sum_{m=1}^{r}\sum_{j=k-d_m(k)}^{k-1} [\xi^{\mathrm{T}}(k) M_m + v^{\mathrm{T}}(j) R_m] \\
&\qquad \times R_m^{-1} [M_m^{\mathrm{T}} \xi(k) + R_m v(j)] \\
&= \sum_{m=1}^{r} \Big[d_m(k) \xi^{\mathrm{T}}(k) M_m R_m^{-1} M_m^{\mathrm{T}} \xi(k) \\
&\quad + 2\xi^{\mathrm{T}}(k) M_m \sum_{j=k-d_m(k)}^{k-1} v(j) + \sum_{j=k-d_m(k)}^{k-1} v^{\mathrm{T}}(j) R_m v(j) \Big] \\
&\le \xi^{\mathrm{T}}(k) \left(\sum_{m=1}^{r} h_m M_m R_m^{-1} M_m^{\mathrm{T}}\right) \xi(k) \\
&\quad + 2\xi^{\mathrm{T}}(k) \sum_{m=1}^{r} M_m [\delta(k) - \delta(k-d_m(k))] \\
&\quad + \sum_{m=1}^{r}\sum_{j=k-h_m}^{k-1} v^{\mathrm{T}}(j) R_m v(j) = Y(k).
\end{aligned} \tag{3.7}
$$

Eq. (3.7) has exploited the free-weighting technique [142], [151], which facilitates the derivation of an upper bound of $\mathbb{E}\{\Delta V(k)\}$ in the quadratic form of $\xi(k)$. Summing up $\mathbb{E}\{\Delta V_1(k)\}$, $\mathbb{E}\{\Delta V_2(k)\}$ and (3.7), we have

$$\begin{aligned}
&\mathbb{E}\{\Delta V(k)\} = \mathbb{E}\{\Delta V_1(k)\} + \mathbb{E}\{\Delta V_2(k)\} \\
&\leq \mathbb{E}\{\Delta V_1(k)\} + \mathbb{E}\{\Delta V_2(k)\} + Y(k) \\
&= \xi^{\mathrm{T}}(k)(\Upsilon_1 + \Upsilon_2 + \Upsilon_3 + \Upsilon_4 + \Upsilon_4^{\mathrm{T}} - \Phi\Lambda^{-1}\Phi^{\mathrm{T}})\xi(k).
\end{aligned} \tag{3.8}$$

If Eq. (3.8) is negative, i.e., $\mathbb{E}\{\Delta V(k)\} < 0$, then we can prove that (3.4) is mean square stable by following the similar line to that in [35]. Therefore, the states of all the agents will converge to their average state, that is, average consensus is reached.

However, from the previous literature, e.g., Theorem 9 in [96] and Theorem 1 in [124], we know that (3.8) being negative does not necessarily require

$$\Upsilon_1 + \Upsilon_2 + \Upsilon_3 + \Upsilon_4 + \Upsilon_4^{\mathrm{T}} - \Phi\Lambda^{-1}\Phi^{\mathrm{T}} < 0.$$

This is because in $\delta(k)$, there are only $n-1$ independent variables due to the fact that $\sum_{i=1}^{n} \delta_i(k) = 0$. Thus, we define

$$\begin{aligned}
\bar{\delta}(k) &= [\delta_1(k), \delta_2(k), \ldots, \delta_{n-1}(k)]^{\mathrm{T}}, \\
\bar{\xi}(k) &= [\bar{\delta}^{\mathrm{T}}(k), \bar{\delta}^{\mathrm{T}}(k - d_1(k)), \ldots, \bar{\delta}^{\mathrm{T}}(k - d_r(k))]^{\mathrm{T}},
\end{aligned}$$

and have $\delta(k) = F\bar{\delta}(k), \xi(k) = W\bar{\xi}(k)$, where F and W are defined in Theorem 3.1. It follows that

$$\begin{aligned}
\mathbb{E}\{\Delta V(k)\} \leq \bar{\xi}^{\mathrm{T}}(k)W^{\mathrm{T}}(&\Upsilon_1 + \Upsilon_2 + \Upsilon_3 + \Upsilon_4 + \Upsilon_4^{\mathrm{T}} \\
&- \Phi\Lambda^{-1}\Phi^{\mathrm{T}})W\bar{\xi}(k).
\end{aligned}$$

A sufficient condition for $\mathbb{E}\{\Delta V(k)\} < 0$ is

$$W^{\mathrm{T}}(\Upsilon_1 + \Upsilon_2 + \Upsilon_3 + \Upsilon_4 + \Upsilon_4^{\mathrm{T}} - \Phi\Lambda^{-1}\Phi^{\mathrm{T}})W < 0.$$

Using the Schur complement, Theorem 3.1 is readily obtained. □

Remark 3.7. Theorem 3.1 provides a sufficient condition for average consensus that can be solved efficiently with the MATLAB LMI toolbox. The selection of the control gain γ_c is achieved by certain search methods, e.g., the bisection method. Since it is only sufficient, conservativeness exists to some extent and may be reduced by choosing a better Lyapunov function. It is also observed that γ_c and T in (3.6) always appear in the form of paired multiplication with the same order ($\gamma_c T$ and $(\gamma_c T)^2$). This reveals that if there is a pair of γ_c and T such that the LMI in Theorem 3.1 is feasible, then for any other sampling period T, there always exists a corresponding γ_c that renders (3.6) feasible.

3.3.1 Proof of Lemma 3.1

Proof. We prove this lemma in two parts.

(Part 1) $i \neq j$. $L^{(i)}(k)$ and $L^{(j)}(k)$ are subject to independent stochastic processes, so

$$\begin{aligned}\mathbb{E}\{L^{(i)}(k)PL^{(j)}(k)\} &= \mathbb{E}\{L^{(i)}(k)\}P\mathbb{E}\{L^{(j)}(k)\} \\ &= \hat{L}^{(i,0)}P\hat{L}^{(j,0)}.\end{aligned}$$

(Part 2) $i = j$. This becomes more complicated. Write $L^{(i)}(k)$ as

$$L^{(i)}(k) = \left[\begin{array}{cccc} \left(l_{1,:}^{(i)}\right)^{\mathrm{T}} & \left(l_{2,:}^{(i)}\right)^{\mathrm{T}} & \cdots & \left(l_{n,:}^{(i)}\right)^{\mathrm{T}} \end{array} \right]^{\mathrm{T}},$$

where $\left(l_{m,:}^{(i)}\right), m \in \mathcal{I}_n$ is the mth row of $L^{(i)}(k)$. Then

$$L^{(i)}(k)PL^{(i)}(k) = \left[\!\left[\left(l_{m,:}^{(i)}\right) P \left(l_{t,:}^{(i)}\right)^{\mathrm{T}} \right]\!\right] \in \mathbb{R}^{n\times n}.$$

The time step k is dropped for simplicity. We consider probabilistic dependence in two types of entries: diagonal and off-diagonal entries.

<u>Case 1)</u> In the diagonal entry

$$H_m = \left(l_{m,:}^{(i)}\right) P \left(l_{m,:}^{(i)}\right)^{\mathrm{T}} = \sum_{q=1}^{n}\sum_{s=1}^{n} \left(l_{m,q}^{(i)}\right) P_{qs} \left(l_{m,s}^{(i)}\right),$$

probabilistic dependence in five subcases is taken into account:

(i) $q = s \neq m$:

$$\mathbb{E}\left\{\left(l_{m,q}^{(i)}\right) P_{qs} \left(l_{m,s}^{(i)}\right)\right\} = p_{mq}P_{qq}\left(l_{m,q}^{(i,0)}\right)^2 = P_{qq}\left[\left(\hat{l}_{m,q}^{(i,0)}\right)^2 + \left(p_{mq} - p_{mq}^2\right)\left(l_{m,q}^{(i,0)}\right)^2\right].$$

(ii) $q=s=m$. From $\left(l_{m,m}^{(i)}\right)=-\sum_{s\in\mathcal{N}_m^{(i)}}\left(l_{m,s}^{(i)}\right)$, it is obtained that

$$
\begin{aligned}
&\mathbb{E}\left\{\left(l_{m,q}^{(i)}\right)P_{qs}\left(l_{m,s}^{(i)}\right)\right\}=\mathbb{E}\left\{\left(l_{m,m}^{(i)}\right)P_{mm}\left(l_{m,m}^{(i)}\right)\right\}\\
&=P_{mm}\cdot\mathbb{E}\left\{\sum_{s\in\mathcal{N}_m^{(i)}}\left(l_{m,s}^{(i)}\right)^2+\sum_{s_1\in\mathcal{N}_m^{(i)}}\sum_{s_2\in\mathcal{N}_m^{(i)},s_2\neq s_1}\left(l_{m,s_1}^{(i)}\right)\left(l_{m,s_2}^{(i)}\right)\right\}\\
&=P_{mm}\cdot\left[\left(\hat{l}_{m,m}^{(i,0)}\right)^2+\sum_{s\in\mathcal{N}_m^{(i)}}\left(p_{ms}-p_{ms}^2\right)\left(l_{m,s}^{(i,0)}\right)^2\right].
\end{aligned}
$$

Following a similar way, we have the remaining three results.

(iii) $q\neq s,q\neq m,s\neq m$: $\mathbb{E}\left\{\left(l_{m,q}^{(i)}\right)P_{qs}\left(l_{m,s}^{(i)}\right)\right\}=P_{qs}\cdot\left(\hat{l}_{m,q}^{(i,0)}\right)\left(\hat{l}_{m,s}^{(i,0)}\right)$.

(iv) $q\neq s,q=m,s\neq m$:

$$
\mathbb{E}\left\{\left(l_{m,q}^{(i)}\right)P_{qs}\left(l_{m,s}^{(i)}\right)\right\}=P_{ms}\left[\left(\hat{l}_{m,m}^{(i,0)}\right)\left(\hat{l}_{m,s}^{(i,0)}\right)-\left(p_{ms}-p_{ms}^2\right)\left(l_{m,s}^{(i,0)}\right)^2\right].
$$

(v) $q\neq s,q\neq m,s=m$:

$$
\mathbb{E}\left\{\left(l_{m,q}^{(i)}\right)P_{qs}\left(l_{m,s}^{(i)}\right)\right\}=P_{qm}\left[\left(\hat{l}_{m,q}^{(i,0)}\right)\left(\hat{l}_{m,m}^{(i,0)}\right)-\left(p_{mq}-p_{mq}^2\right)\left(l_{m,q}^{(i,0)}\right)^2\right].
$$

Combining (i)–(v) above yields

$$
\begin{aligned}
\mathbb{E}\{H_m\}=&\left(\hat{l}_{m,:}^{(i,0)}\right)P\left(\hat{l}_{m,:}^{(i,0)}\right)^{\mathrm{T}}\\
&+\sum_{q=1,q\neq m}^{n}(p_{mq}-p_{mq}^2)P_{qq}\left(l_{m,q}^{(i,0)}\right)^2\\
&+\sum_{s\in\mathcal{N}_m^{(i)}}(p_{ms}-p_{ms}^2)P_{mm}\left(l_{m,s}^{(i,0)}\right)^2\\
&-\sum_{s=1,s\neq m}^{n}(p_{ms}-p_{ms}^2)P_{ms}\left(l_{m,s}^{(i,0)}\right)^2\\
&-\sum_{q=1,q\neq m}^{n}(p_{mq}-p_{mq}^2)P_{qm}\left(l_{m,q}^{(i,0)}\right)^2.
\end{aligned}
$$

We have

$$
\mathbb{E}\{\mathrm{diag}\{H_1,H_2,\ldots,H_n\}\}
$$

$$= \text{diag}\left\{ \left(\hat{l}_{m,:}^{(i,0)}\right) P \left(\hat{l}_{m,:}^{(i,0)}\right)^{\text{T}}, m \in \mathcal{I}_n \right\}$$
$$+ D_1 + D_2 - D_3 - D_4, \tag{3.9}$$

where the term diag$\{\cdot\}$ should be interpreted as such that we compute every element for m from 1 to n, then use these numbers to form a diagonal matrix. D_1, D_2, D_3 and D_4 will be shown in (3.10), (3.12), (3.13) and (3.14), respectively.

$$D_1 = \text{diag}\left\{ \sum_{q=1,q\neq m}^{n} (p_{mq} - p_{mq}^2) P_{qq} \left(l_{m,q}^{(i,0)}\right)^2, m \in \mathcal{I}_n \right\}$$
$$= \sum_{m=1}^{n} \sum_{q=1,q\neq m}^{n} (p_{mq} - p_{mq}^2) E_{(m,q)}^{\text{T}} P E_{(m,q)} \left(A_{m,q}^{(i,0)}\right)^2, \tag{3.10}$$

and $E_{(m,q)}$ is defined in Lemma 3.1. $A^{(i,0)}$ is the adjacency matrix associated with the Laplacian matrix $L^{(i,0)}$. Since $A_{m,m}^{(i,0)} = 0$, we also have

$$D_1 = \sum_{m=1}^{n} \sum_{q=1}^{n} (p_{mq} - p_{mq}^2) E_{(m,q)}^{\text{T}} P E_{(m,q)} \left(A_{m,q}^{(i,0)}\right)^2. \tag{3.11}$$

Similarly,

$$D_2 = \text{diag}\left\{ \sum_{q\in\mathcal{N}_m^{(i)}} (p_{mq} - p_{mq}^2) P_{mm} \left(l_{m,q}^{(i,0)}\right)^2 \right\}$$
$$= \sum_{m=1}^{n} \sum_{q=1}^{n} (p_{mq} - p_{mq}^2) E_{(m,m)}^{\text{T}} P E_{(m,m)} \left(A_{m,q}^{(i,0)}\right)^2, \tag{3.12}$$
$$D_3 = \text{diag}\left\{ \sum_{s=1,s\neq m}^{n} (p_{ms} - p_{ms}^2) P_{ms} \left(l_{m,s}^{(i,0)}\right)^2 \right\}$$
$$= \sum_{m=1}^{n} \sum_{s=1}^{n} (p_{ms} - p_{ms}^2) E_{(m,m)}^{\text{T}} P E_{(m,s)} \left(A_{m,s}^{(i,0)}\right)^2, \tag{3.13}$$
$$D_4 = \text{diag}\left\{ \sum_{q=1,q\neq m}^{n} (p_{mq} - p_{mq}^2) P_{qm} \left(l_{m,q}^{(i,0)}\right)^2 \right\}$$
$$= \sum_{m=1}^{n} \sum_{q=1}^{n} (p_{mq} - p_{mq}^2) E_{(m,q)}^{\text{T}} P E_{(m,m)} \left(A_{m,q}^{(i,0)}\right)^2. \tag{3.14}$$

Note that the transpose of (3.13) is (3.14). Substituting (3.11)–(3.14) into (3.9), we have

$$\begin{aligned}&\mathbb{E}\{\mathrm{diag}\{H_1, H_2, \ldots, H_n\}\}\\&= \mathrm{diag}\left\{\left(\hat{l}^{(i,0)}_{m,:}\right) P \left(\hat{l}^{(i,0)}_{m,:}\right)^{\mathrm{T}}, m \in \mathcal{I}_n\right\} + \Xi_1(i,P).\end{aligned} \tag{3.15}$$

Case 2) The off-diagonal entry

$$\left(l^{(i)}_{m,:}\right) P \left(l^{(i)}_{t,:}\right)^{\mathrm{T}} = \sum_{q=1}^{n}\sum_{s=1}^{n} \left(l^{(i)}_{m,q}\right) P_{qs} \left(l^{(i)}_{t,s}\right), m \neq t.$$

Since $L^{(i)}(k)$ is symmetric for an undirected graph, $\left(l^{(i)}_{m,t}\right)$ and $\left(l^{(i)}_{t,m}\right)$ are subject to the same random process from Assumption 3.1. Therefore, dependence in four terms are to be considered: $\left(l^{(i)}_{m,m}\right) P_{mt} \left(l^{(i)}_{t,t}\right)$, $\left(l^{(i)}_{m,m}\right) P_{mm} \left(l^{(i)}_{t,m}\right)$, $\left(l^{(i)}_{m,t}\right) P_{tm} \left(l^{(i)}_{t,m}\right)$, $\left(l^{(i)}_{m,t}\right) P_{tt} \left(l^{(i)}_{t,t}\right)$. After some calculations similar to those in Case 1), we get the four terms of expectation:

$$\begin{aligned}&\mathbb{E}\left\{\left(l^{(i)}_{m,m}\right) P_{mt} \left(l^{(i)}_{t,t}\right)\right\}\\&= \left(\hat{l}^{(i,0)}_{m,m}\right) P_{mt} \left(\hat{l}^{(i,0)}_{t,t}\right) + (p_{mt} - p_{mt}^2)\left(l^{(i,0)}_{m,t}\right)^2 P_{mt},\\&\mathbb{E}\left\{\left(l^{(i)}_{m,m}\right) P_{mm} \left(l^{(i)}_{t,m}\right)\right\}\\&= \left(\hat{l}^{(i,0)}_{m,m}\right) P_{mm} \left(\hat{l}^{(i,0)}_{t,m}\right) - (p_{mt} - p_{mt}^2)\left(l^{(i,0)}_{m,t}\right)^2 P_{mm},\\&\mathbb{E}\left\{\left(l^{(i)}_{m,t}\right) P_{tm} \left(l^{(i)}_{t,m}\right)\right\}\\&= \left(\hat{l}^{(i,0)}_{m,t}\right)^2 P_{mt} + (p_{mt} - p_{mt}^2)\left(l^{(i,0)}_{m,t}\right)^2 P_{mt},\\&\mathbb{E}\left\{\left(l^{(i)}_{m,t}\right) P_{tt} \left(l^{(i)}_{t,t}\right)\right\}\\&= -(p_{mt} - p_{mt}^2)\left(l^{(i,0)}_{m,t}\right)^2 P_{tt} + \left(\hat{l}^{(i,0)}_{m,t}\right) P_{tt} \left(\hat{l}^{(i,0)}_{t,t}\right).\end{aligned}$$

If we denote $\mathcal{M} = \{(m,t),(m,m),(t,m),(t,t)\}$, then

$$\begin{aligned}\left(l^{(i)}_{m,:}\right) P \left(l^{(i)}_{t,:}\right)^{\mathrm{T}} &= \sum_{q=1}^{n}\sum_{s=1,(q,s)\notin\mathcal{M}}^{n} \left(l^{(i)}_{m,q}\right) P_{qs} \left(l^{(i)}_{t,s}\right)\\&\quad + \sum_{(q,s)\in\mathcal{M}} \left(l^{(i)}_{m,q}\right) P_{qs} \left(l^{(i)}_{t,s}\right).\end{aligned}$$

Considering the expectation of the four terms, it is obtained that

$$\begin{aligned}&\mathbb{E}\left\{\left(l_{m,:}^{(i)}\right) P \left(l_{t,:}^{(i)}\right)^{\mathrm{T}}\right\}\\&=\left(\hat{l}_{m,:}^{(i,0)}\right) P \left(\hat{l}_{t,:}^{(i,0)}\right)^{\mathrm{T}}\\&\quad+\left(p_{mt}-p_{mt}^{2}\right)\left(2P_{mt}-P_{mm}-P_{tt}\right)\left(l_{m,t}^{(i,0)}\right)^{2}.\end{aligned}$$

In the above equation, we have used the symmetry in P and $L^{(i)}(k)$. For $m \neq t$, we have $\left(l_{m,t}^{(i,0)}\right)^2 = \left(A_{m,t}^{(i,0)}\right)^2$. Then the expectation of the matrix comprised of off-diagonal entries is

$$\begin{aligned}&\mathbb{E}\left\{\left[\!\left[\left(l_{m,:}^{(i)}\right) P \left(l_{t,:}^{(i)}\right)^{\mathrm{T}}\right]\!\right]_{m\neq t}\right\}\\&=\left[\!\left[\left(\hat{l}_{m,:}^{(i,0)}\right) P \left(\hat{l}_{t,:}^{(i,0)}\right)^{\mathrm{T}}\right]\!\right]_{m\neq t}\\&\quad+\left[\!\left[\left(p_{mt}-p_{mt}^{2}\right)\left(2P_{mt}-P_{mm}-P_{tt}\right)\left(A_{m,t}^{(i,0)}\right)^{2}\right]\!\right]_{m\neq t}\\&=\left[\!\left[\left(\hat{l}_{m,:}^{(i,0)}\right) P \left(\hat{l}_{t,:}^{(i,0)}\right)^{\mathrm{T}}\right]\!\right]_{m\neq t}\\&\quad+\sum_{m=1}^{n}\sum_{t=1}^{n}\left(p_{mt}-p_{mt}^{2}\right)\left[2E_{(m,m)}^{\mathrm{T}}PE_{(t,t)}\right.\\&\qquad\left.-E_{(m,m)}^{\mathrm{T}}PE_{(t,m)}-E_{(m,t)}^{\mathrm{T}}PE_{(t,t)}\right]\left(A_{m,t}^{(i,0)}\right)^{2}\\&=\left[\!\left[\left(\hat{l}_{m,:}^{(i,0)}\right) P \left(\hat{l}_{t,:}^{(i,0)}\right)^{\mathrm{T}}\right]\!\right]_{m\neq t}+\Xi_{2}(i,P).\end{aligned}\tag{3.16}$$

Combining (3.15) in Case 1) and (3.16) in Case 2), (3.5) is obtained readily. This completes the proof. □

3.4 Simulation Results

A numerical example is provided to verify the design method. Consider an MAS of four agents with the expected topology $\mathcal{G}^{(0)}$ and $\mathcal{E}(\mathcal{G}^{(0)}) = \{(v_1, v_2), (v_1, v_3), (v_3, v_4)\}$ (all possible edges). Suppose that delays across the three channels are independent of each other, so $r = 3$. Then, we have the following adjacency matrices with full weights:

$$A^{(1,0)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A^{(2,0)} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A^{(3,0)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The weights are set to unity for simplicity and other weights are applicable. The sampling period is $T = 0.05$ s. Delays have the upper bounds $h_1 = h_2 = h_3 = 0.6$ s. Figure 3.1 shows the sequence of the delay d_{12} within the first five seconds. The initial condition is $x(0) = [1,\ 2,\ 3,\ 4]^{\mathrm{T}}$. The probabilities of successfully receiving data are $p_{12} = 0.4, p_{13} = 0.9, p_{34} = 0.6$, respectively. Set $\gamma_c = 0.6$ and solve the LMI in Theorem 3.1. The result shows that it is feasible, therefore average consensus can be reached. Figure 3.2 shows state evolution of the four agents.

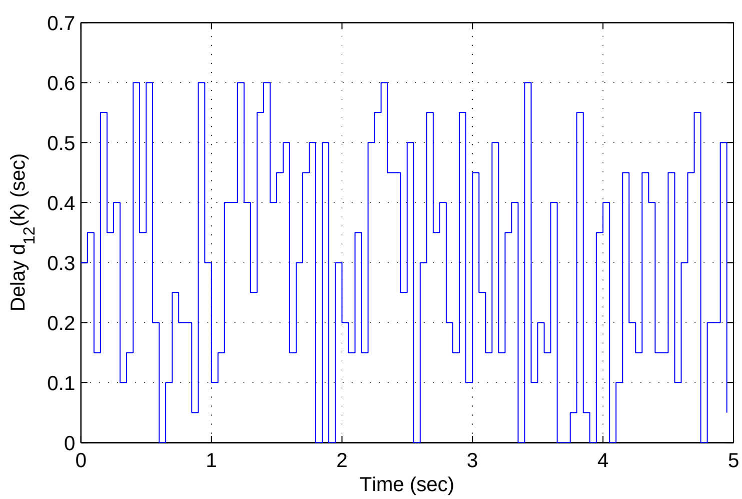

Figure 3.1: The time delay $d_{12}(k)$ in seconds.

As we perform more tests with different simulation conditions, a decrease in convergence rate is observed when the probability of data loss rises. With lower availability of data, the system evolves more slowly. The graphical results are not shown here for brevity.

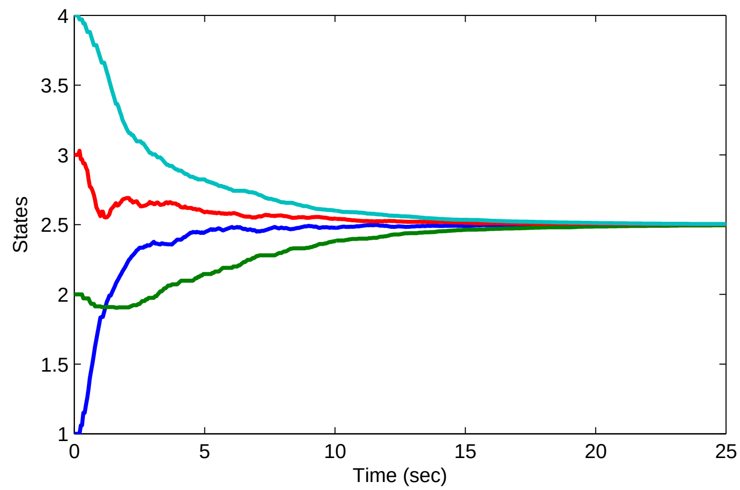

Figure 3.2: The agents' states versus time.

3.5 Conclusion

In this work, we investigate the average consensus in first-order dynamics with non-uniform time-varying delays and packet dropouts. Both the bounds of delays and probabilities of data losses are taken into consideration. An LMI-based Lyapunov method is used to derive a sufficient condition for consensus. In the future, it will be interesting to extend the agent dynamics to more general form and study its performance.

Chapter 4

Consensus in Multi-Agent Systems with Non-uniform Sampling

4.1 Introduction

The last decade has witnessed a surge of research interest in MASs [17] [53] [86] [107] [134]. An important concern in MASs is the consensus problem, which means that we design proper control laws to drive the states of interest to a common decision value. Traditionally, due to the limits in communication, this may only be implemented in a centralized way, i.e., we have a central system to regulate the dynamic performance of all plants. Nowadays, as the technology advances, researchers are concentrating on the distributed implementation based on local information exchange. Under this new scheme, no agent is able to interact with all other agents. Each of them communicates with its neighbours to coordinate the collective behavior. Consensus theory has many applications, such as rendezvous [107], formation control [63] [77], flocking [95] [152].

Researchers started the research on consensus with relatively simple dynamics: the single-integrator (first-order) dynamics [53] [96] [107] [138]. In particular, Ren and Beard in [107] find the necessary and sufficient conditions for first-order LTI dynamics. Their conclusions only require the existence of a spanning tree, which is the weakest condition on graph connectivity. Later on, researchers have extended the results to double-integrators, which are more practical since they involve both the position and velocity in the consensus problem. For related work, refer to [19] [104] [106] and the references therein. In recent years, the research on higher-order dynamics such as the state-space models has been attracting growing attention [17]

[83] [109]. In addition, consensus with nonlinearities (e.g., Euler–Lagrange dynamics [105]) has also been studied in literature.

Depending on how MASs operate in time domain, the consensus problem can be coped with in continuous time [55] [96] [107], discrete time [53] [138], or with sampled data [36] [149]. In practice, most of the plants work in continuous time and have continuous states. However, many controllers that are used today are digital and adopt the sampling/hold scheme. They receive data in an intermittent way and apply the control signal through a zero-order hold. Therefore, it is necessary to explore the conditions for consensus when sampled data are used. For instance, the authors in [12] [36] investigate the sampled-data consensus with double-integrator dynamics and fixed/switching topology. Sufficient conditions are derived with some restrictions on the sampling period, graph connectivity, and control gains. In these works, the sampled-data system has been transformed into its equivalent discrete-time counterpart. Then, we can use some existing results (e.g., [53]) to solve the consensus problem.

In the aforementioned work on sample-data consensus, the sampling is assumed to be periodic. However, due to various reasons, such as delay, limited communication bandwidth, it is not always possible to guarantee periodic communication [1]. In this situation, the introduction of irregular sampling makes the system time-varying and poses challenges for us. The research results on LTI systems cannot be directly applied to this scenario. How to adapt the current control scheme to the case with non-uniform sampling remains a question. Motivated by this, we are going to study the consensus problem with non-uniform sampling.

In this work, the agent dynamics is assumed to be double-integrators, which have been researched widely in literature [74] [104] [149]. Compared with the single-integrator dynamics, it is more practical and represents the physical characteristics of a larger class of systems. The communication topology is directed and fixed. The sampling periods are assumed to be from a finite set. At each sampling instant, the sampling period is selected randomly from that set. A control protocol with time-dependent gains is developed, meaning that the control gains depend on the sampling period explicitly. We prove that consensus can be reached if the directed graph has a spanning tree and the control gains are properly designed.

The main contribution of this work is twofold: (1) The non-uniform sampling scheme is considered, which characterizes communication more practically. The sampled-data MAS is converted into its equivalent discrete-time system between every two consecutive sampling instants. Then consensus is studied in discrete time with the aid of stochastic matrices. (2) The use of time-varying control gains provides flexibility for reaching consensus. Rather than employing fixed control gains, the strategy here lets us have more choices at each sampling instant to regulate the system performance.

The remainder of this work is organized as follows. In Section 4.2, some preliminaries are reviewed and the problem is formulated. Section 4.3 contains the main theoretical results on the conditions for consensus. A numerical example is given in Section 4.4 to verify the effectiveness of the proposed controller. Finally, some conclusions and remarks are offered in Section 4.5.

Notation: The superscript 'T' denotes the matrix transpose. The space of real numbers is represented by $\mathbb{R}$. The N-dimensional vector $\mathbf{1}_N = [1\ 1\ \cdots\ 1]^{\mathrm{T}}$ has all of its entries equal to one.

4.2 Statement of the Problem

4.2.1 Preliminaries

In this subsection, some basic concepts in algebraic graph theory and nonnegative matrices will be reviewed.

A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ of N nodes consists of the node set $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$, edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and the adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{N \times N}$. In an MAS, each node in $\mathcal{V}$ corresponds to an agent. An edge $e_{ij} = (v_j, v_i) \in \mathcal{E}$ implies that agent i is able to get information from agent j. For the adjacency matrix, we have $a_{ij} > 0$ if $(v_j, v_i) \in \mathcal{E}$, and $a_{ij} = 0$ otherwise. Assume that there is no edge from an agent to itself, i.e., $a_{ii} = 0$. If $(v_j, v_i) \in \mathcal{E}$ also implies $(v_i, v_j) \in \mathcal{E}$, then the graph $\mathcal{G}$ is *undirected*; otherwise, $\mathcal{G}$ is a *directed* graph. A path from node v_i to v_j is a sequence of edges $(v_i, v_{m_1}), (v_{m_1}, v_{m_2}), \ldots, (v_{m_p}, v_j)$ such that each of these edges is in $\mathcal{E}$. A graph $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ is a *subgraph* of $\mathcal{G}$ if $\mathcal{V}' \subseteq \mathcal{V}$ and $\mathcal{E}' \subseteq \mathcal{E}$. If $\mathcal{V}' = \mathcal{V}$, then $\mathcal{G}'$ is termed a *spanning subgraph* of $\mathcal{G}$. A *directed spanning tree* of a graph is a spanning subgraph such that there is a root in the tree that has a directed path to any other node in the tree. For more details on graph theory, refer to [37] and other related references.

The Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{N\times N}$ is defined as

$$l_{ij} = -a_{ij}, i \neq j; \quad l_{ii} = -\sum_{j=1, j\neq i}^{N} l_{ij}.$$

Obviously, it follows from the definition of L that $L\mathbf{1}_N = 0$.

Now let us recall some knowledge on nonnegative matrices. A matrix $M \in \mathbb{R}^{n\times n}$ is *nonnegative* if all of its elements are nonnegative, and it is denoted as $M \geq 0$. A *stochastic* matrix is a nonnegative matrix with row sum equal to one for each row. For two matrices A and B, we say $A \geq B$ if and only if $A - B \geq 0$. A stochastic matrix M is indecomposable and aperiodic (SIA) if $\lim_{k\to\infty} M^k = \mathbf{1}_n c^{\mathrm{T}}$, where c is a column vector [132].

4.2.2 Problem Formulation

Consider an MAS of N agents with the following double-integrator dynamics,

$$\dot{x}_i = v_i, \dot{v}_i = u_i, \quad i = 1, 2, \ldots, N, \tag{4.1}$$

where $x_i \in \mathbb{R}^n$ and $v_i \in \mathbb{R}^n$ are the position and velocity of agent i, respectively. $u_i \in \mathbb{R}^n$ is the control input. For simplicity, we only consider the case with $n = 1$ in this work. The scenario of $n > 1$ can be dealt with by introducing the Kronecker product. The graph associated with the MAS is directed.

With sampling period h, we can discretize (4.1) as

$$\begin{aligned} x_i(k+1) &= x_i(k) + hv_i(k) + \frac{h^2}{2}u_i(k), \\ v_i(k+1) &= v_i(k) + hu_i(k). \end{aligned}$$

Define

$$\begin{aligned} x(k) &= [x_1(k), x_2(k), \ldots, x_N(k)]^{\mathrm{T}}, \\ v(k) &= [v_1(k), v_2(k), \ldots, v_N(k)]^{\mathrm{T}}, \\ u(k) &= [u_1(k), u_2(k), \ldots, u_N(k)]^{\mathrm{T}}. \end{aligned}$$

Then

$$\begin{bmatrix} x(k+1) \\ v(k+1) \end{bmatrix} = \begin{bmatrix} I_N & hI_N \\ 0 & I_N \end{bmatrix} \begin{bmatrix} x(k) \\ v(k) \end{bmatrix} + \begin{bmatrix} \frac{h^2}{2} I_N \\ hI_N \end{bmatrix} u(k).$$

In this work, the update scheme is assumed to be synchronous, i.e., all agents update their states at the same time. The asynchronous consensus has also been investigated in the past [139]. Suppose that the sequence $\{t_k, k = 0, 1, \ldots\}$ consists of the time instants at which the agents update their states. Figure 4.1 demonstrates the non-uniform sampling.

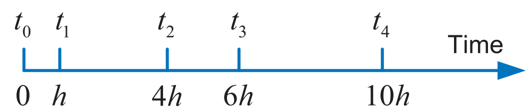

Figure 4.1: Schematic of non-uniform sampling.

To characterize the phenomenon of non-uniform sampling, we make the following assumption:

Assumption 4.1. The length of the time interval between any two consecutive update instants is from a finite set. That is, there exists a positive integer τ such that $t_{k+1} - t_k = m_k h, m_k \in \Gamma = \{n_1, n_2, \ldots, n_\tau\}, \forall k \geq 0$, where Γ is a set of integers.

Remark 4.1. If we synchronize the clocks of all agents using some external sources such as GPS, then it is possible to implement the synchronous update scheme. The sampling period h acts as the base sampling period as in multirate control. The assumption that the number of non-uniform sampling periods is finite is reasonable, otherwise, there will be some time after which the MAS stops receiving new data.

The following control algorithm is used,

$$u_i(t) = -\alpha_k v_i(t_k) - \beta_k \sum_{j=1}^{N} a_{ij}[x_i(t_k) - x_j(t_k)], \qquad t_k \leq t < t_{k+1}, \tag{4.2}$$

where α_k and β_k are time-varying gains depending on $t_{k+1} - t_k$. Here, we assume that $t_{k+1} - t_k$ is known to us at time t_k. This may cause an issue of implementation in practice due to the constraint of causality. In the future, causality will be addressed from the perspective of practicality. From Assumption 4.1, we know that the number of values of α_k and β_k is also finite. Compared with the control law in [12], we have different gains for the damping term and relative measurement, respectively. In what follows, it will be seen that these time-varying gains provide much flexibility in driving the MAS to consensus.

From (4.2), we have

$$u(t) = -\alpha_k v(t_k) - \beta_k L x(t_k), \quad t_k \le t < t_{k+1}.$$

If considering the discretization between t_k and t_{k+1}, it is obtained that

$$\begin{bmatrix} x(t_{k+1}) \\ v(t_{k+1}) \end{bmatrix} = \Pi_k \begin{bmatrix} x(t_k) \\ v(t_k) \end{bmatrix}, \tag{4.3}$$

where

$$\Pi_k = \begin{bmatrix} I_N - \frac{1}{2}\beta_k(m_k h)^2 L & (m_k h - \frac{1}{2}\alpha_k(m_k h)^2) I_N \\ -\beta_k(m_k h) L & (1 - \alpha_k(m_k h)) I_N \end{bmatrix}. \tag{4.4}$$

We are going to study the consensus property of (4.3) in the next section.

4.3 Main Results

In this section, conditions to guarantee the convergence of consensus will be explored.

From (4.4), we have

$$\begin{aligned} \begin{bmatrix} x(t_{k+1}) \\ v(t_{k+1}) \end{bmatrix} &= \Pi_k \Pi_{k-1} \cdots \Pi_1 \Pi_0 \begin{bmatrix} x(t_0) \\ v(t_0) \end{bmatrix} \\ &= \begin{bmatrix} B_k & C_k \\ D_k & E_k \end{bmatrix} \begin{bmatrix} x(t_0) \\ v(t_0) \end{bmatrix}. \end{aligned}$$

Similar to Lemma 3.1 in [12], we have the following lemma.

Lemma 4.1. *Assume that* $\alpha_k m_k h \neq 2$ *for any nonnegative integer* k*. Then* $x_i(t_k) \to x_j(t_k), v_i(t_k) \to 0$ *if* $\lim_{k\to\infty} B_k$ *exists and all rows of* $\lim_{k\to\infty} B_k$ *are the same for any initial matrices* B_0 *and* D_0*.*

Proof. The proof can be completed by following the similar line as in [12]. $\square$

The next step is to derive an iterative equation for the evolution of B_k. Denote

$$\Pi_k = \begin{bmatrix} \Pi_{k,11} & \Pi_{k,12} \\ \Pi_{k,21} & \Pi_{k,22} \end{bmatrix},$$

where $\Pi_{k,11}, \Pi_{k,12}, \Pi_{k,21}$ and $\Pi_{k,22}$ are equal to the four blocks of Π_k in (4.4), respectively. Apparently, $\Pi_{k,12}$ and $\Pi_{k,22}$ are scalar matrices. Then

$$\begin{aligned} B_k &= \Pi_{k,11} B_{k-1} + \Pi_{k,12} D_{k-1}, \\ B_{k-1} &= \Pi_{k-1,11} B_{k-2} + \Pi_{k-1,12} D_{k-2}, \\ D_{k-1} &= \Pi_{k-1,21} B_{k-2} + \Pi_{k-1,22} D_{k-2}. \end{aligned}$$

From the above three equations, we have

$$B_k = \Phi_{k1} B_{k-1} + \Phi_{k2} B_{k-2}, \tag{4.5}$$

where

$$\begin{aligned} \Phi_{k1} &= (\Pi_{k-1,12}^{-1} \Pi_{k,12}) \Pi_{k-1,22} + \Pi_{k,11}, \\ \Phi_{k2} &= \Pi_{k,12} \Pi_{k-1,21} - (\Pi_{k-1,12}^{-1} \Pi_{k,12}) \Pi_{k-1,22} \Pi_{k-1,11}. \end{aligned}$$

Eq. (4.5) can be written in another form as

$$\begin{bmatrix} B_k \\ B_{k-1} \end{bmatrix} = H_k \begin{bmatrix} B_{k-1} \\ B_{k-2} \end{bmatrix} = \begin{bmatrix} \Phi_{k1} & \Phi_{k2} \\ I & 0 \end{bmatrix} \begin{bmatrix} B_{k-1} \\ B_{k-2} \end{bmatrix}. \tag{4.6}$$

Lemma 4.2. *There exist α_k and β_k such that Φ_{k1} and Φ_{k2} are nonnegative matrices with positive diagonal elements, and the coefficient matrix H_k in (4.6) is a stochastic matrix.*

Proof. Let $\alpha_k m_k h = \gamma \neq 2, \forall k$. Then $\Pi_{k,12} = \left(1 - \frac{1}{2}\gamma\right) m_k h I_N, \Pi_{k,22} = (1-\gamma) I_N$, $\Pi_{k-1,12}^{-1} \Pi_{k,12} = \frac{m_k}{m_{k-1}} I_N$.

$$\begin{aligned} \Phi_{k1} &= \frac{m_k}{m_{k-1}} \Pi_{k-1,22} + \Pi_{k,11} \\ &= \frac{m_k}{m_{k-1}} (1-\gamma) I_N + I_N - \frac{1}{2} \beta_k (m_k h)^2 L \\ &= \left(1 + (1-\gamma) \frac{m_k}{m_{k-1}}\right) I_N - \frac{1}{2} \beta_k (m_k h)^2 L, \end{aligned} \tag{4.7}$$

$$\begin{aligned}
\Phi_{k2} &= \Pi_{k,12}\Pi_{k-1,21} - \frac{m_k}{m_{k-1}}\Pi_{k-1,22}\Pi_{k-1,11} \\
&= \left(1 - \frac{1}{2}\gamma\right) m_k h I_N(-\beta_{k-1}(m_{k-1}h)L) \\
&\quad - \frac{m_k}{m_{k-1}}(1-\gamma)I_N\left(I_N - \frac{1}{2}\beta_{k-1}(m_{k-1}h)^2 L\right) \\
&= \frac{m_k}{m_{k-1}}\left[-\left(1 - \frac{1}{2}\gamma\right)\beta_{k-1}(m_{k-1}h)^2 L \right. \\
&\qquad \left. -(1-\gamma)\left(I_N - \frac{1}{2}\beta_{k-1}(m_{k-1}h)^2 L\right)\right] \\
&= \frac{m_k}{m_{k-1}}\left[-(1-\gamma)I_N - \frac{1}{2}\beta_{k-1}(m_{k-1}h)^2 L\right].
\end{aligned} \tag{4.8}$$

When $\beta_k > 0$, to guarantee that Φ_{k1} and Φ_{k2} are nonnegative matrices with positive diagonal elements, we first require that

$$1 + (1-\gamma)\frac{m_k}{m_{k-1}} > 0, \quad -(1-\gamma) > 0,$$

which results in

$$1 < \gamma < 1 + \frac{m_{k-1}}{m_k},$$

i.e.,

$$1 < \alpha_k m_k h = \gamma < 1 + \frac{m_{k-1}}{m_k}. \tag{4.9}$$

In addition, if we choose β_k such that

$$\frac{1}{2}\beta_k(m_k h)^2 l_{ii} < \min\left\{-(1-\gamma), 1 + (1-\gamma)\frac{m_k}{m_{k-1}}\right\}, \quad \forall i = 1, 2, \ldots, N, \tag{4.10}$$

then Φ_{k1} and Φ_{k2} are nonnegative matrices with positive diagonal elements. Thus H_k is a nonnegative matrix. From (4.9) and (4.10), it can be seen that we can always choose α_k and β_k to meet these two conditions.

Considering $L\mathbf{1}_N = 0$, we have $\Pi_{k,11}\mathbf{1}_N = \mathbf{1}_N, \Pi_{k,21}\mathbf{1}_N = 0$. Then,

$$\begin{aligned}
\Phi_{k1}\mathbf{1}_N &= [(\Pi_{k-1,12}^{-1}\Pi_{k,12})\Pi_{k-1,22} + \Pi_{k,11}]\mathbf{1}_N \\
&= (\Pi_{k-1,12}^{-1}\Pi_{k,12})\Pi_{k-1,22}\mathbf{1}_N + \mathbf{1}_N, \\
\Phi_{k2}\mathbf{1}_N &= [\Pi_{k,12}\Pi_{k-1,21}
\end{aligned}$$

$$- (\Pi_{k-1,12}^{-1}\Pi_{k,12})\Pi_{k-1,22}\Pi_{k-1,11}]\mathbf{1}_N$$
$$= -(\Pi_{k-1,12}^{-1}\Pi_{k,12})\Pi_{k-1,22}\mathbf{1}_N.$$

Therefore, $(\Phi_{k1} + \Phi_{k2})\mathbf{1}_N = \mathbf{1}_N$, and H_k is a matrix with row sum equal to one. It follows from the above analysis that with (4.9) and (4.10) satisfied, H_k is a stochastic matrix. □

The subsequent lemmas are important for the convergence analysis of consensus.

Lemma 4.3 (Lemma 2 in [53]). *Let $m \geq 2$ be a positive integer, and $M_1, M_2, \ldots, M_m \in \mathbb{R}^{n\times n}$ are nonnegative matrices with positive diagonal elements. Then there exists a positive number ε such that*

$$M_1 M_2 \cdots M_m \geq \varepsilon(M_1 + M_2 + \cdots + M_m).$$

Lemma 4.4 (Corollary 3.4 in [12]). *Suppose that the nonnegative matrix $M \in \mathbb{R}^{n\times n}$ has the same row sum for each row and the directed graph associated with M has a spanning tree. Then the directed graph associated with $\begin{bmatrix} M & M \\ M & M \end{bmatrix}$ also has a spanning tree.*

Lemma 4.5 (Corollary 3.5 & Lemma 3.7 in [107]). *Let $M \in \mathbb{R}^{n\times n}$ be a stochastic matrix with positive diagonal elements. If the directed graph associated with M has a spanning tree, then M is SIA.*

Lemma 4.6 ([132]). *Let $\{M_1, M_2, \ldots, M_k\}$ be a finite set of stochastic matrices of the same dimension such that any sequence of matrix product $M_{i_m} \cdots M_{i_2} M_{i_1}$ of positive length is SIA. Then for any product $M_{i_m} M_{i_{m-1}} \cdots$ of infinite length, there exists a corresponding vector y such that*

$$\lim_{m\to\infty} M_{i_m} M_{i_{m-1}} \cdots M_{i_1} = \mathbf{1} y^{\mathrm{T}}.$$

The following theorem summarizes the main results of this work.

Theorem 4.1. *Assume that α_k and β_k satisfy the conditions in Lemma 4.2. The MAS in (4.1) with non-uniform sampling reaches consensus if the directed graph $\mathcal{G}$ has a spanning tree.*

Proof. From (4.6), we have

$$H_kH_{k-1} = \begin{bmatrix} \Phi_{k1}\Phi_{(k-1)1} + \Phi_{k2} & \Phi_{k1}\Phi_{(k-1)2} \\ \Phi_{(k-1)1} & \Phi_{(k-1)2} \end{bmatrix}$$
$$\geq \begin{bmatrix} \Phi_{k2} & \Phi_{k1}\Phi_{(k-1)2} \\ \Phi_{(k-1)1} & \Phi_{(k-1)2} \end{bmatrix}.$$

From Lemma 4.2 and Lemma 4.3, there exists $\varepsilon_1 > 0$ such that

$$\Phi_{k1}\Phi_{(k-1)2} \geq \varepsilon_1(\Phi_{k1} + \Phi_{(k-1)2}).$$

Hence

$$H_kH_{k-1} \geq \begin{bmatrix} \Phi_{k2} & \varepsilon_1(\Phi_{k1} + \Phi_{(k-1)2}) \\ \Phi_{(k-1)1} & \Phi_{(k-1)2} \end{bmatrix}$$
$$\geq \begin{bmatrix} \Phi_{k2} & \varepsilon_1\Phi_{k1} \\ \Phi_{(k-1)1} & \Phi_{(k-1)2} \end{bmatrix}.$$

It can be seen from (4.7) and (4.8) that both Φ_{k1} and Φ_{k2} have structure of the form $r_{1k}I_N - r_{2k}L$, where r_{1k} and r_{2k} are positive scalars. As a result, the two nonnegative matrices Φ_{k1} and Φ_{k2} have positive and zero elements at the same positions. Since L is fixed, $\Phi_{k1}, \Phi_{k2}, \Phi_{(k-1)1}$ and $\Phi_{(k-1)2}$ all have the same structure (with positive and zero elements at the same positions). Then there exists $\varepsilon > 0$ such that

$$H_kH_{k-1} \geq \varepsilon \begin{bmatrix} \Phi_{k1} & \Phi_{k1} \\ \Phi_{k1} & \Phi_{k1} \end{bmatrix}.$$

When the graph $\mathcal{G}$ has a directed spanning tree, so does the graph associated with Φ_{k1}. From Lemma 4.4, we know that the graph associated with H_kH_{k-1} has a spanning tree as well. It is obtained from Lemma 4.5 that H_kH_{k-1} is SIA. Then, if we consider the sequence $H_kH_{k-1}\cdots H_2H_1 = (H_kH_{k-1})\cdots(H_2H_1)$ and Lemma 4.6, we know that there exists a vector c such that

$$\lim_{k\to\infty} H_kH_{k-1}\cdots H_2H_1 = \mathbf{1}c^{\mathrm{T}}.$$

Thus, all rows of B_k in (4.6) will be the same as $k \to \infty$. In view of Lemma 4.1, consensus can be reached. □

Remark 4.2. From the above process of proof, it can be seen that the values of α_k and β_k play an important role in the convergence of consensus. They can be calculated online, dependent on the measured length of sampling interval. In practice, if the computational capacity does not allow, we can choose the smallest values of α_k and β_k, and use these fixed values throughout. The property of finiteness in Assumption 4.1 guarantees that their smallest values exist.

With the two adjustable gains α_k and β_k, we do not have to make a compromise on the sampling period $m_k h$. This is a distinct feature from that in [12], where the sampling period is also a design parameter. In practice, because of some physical restrictions, the sampling period may not be arbitrarily selected.

4.4 Simulation

In this example, we will verify the effectiveness of the developed method through an example. Figure 4.2 shows an MAS of four agents with fixed topology. For each edge

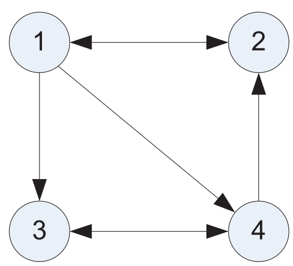

Figure 4.2: A multi-agent system with four agents.

$e_{ij} \in \mathcal{E}$, let $a_{ij} = 1$ (e.g., $a_{31} = 1$); otherwise, $a_{ij} = 0$. Hence,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, L = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}.$$

The base sampling period is $h = 0.1$ sec, and the set for non-uniform sampling is $\Gamma = \{1, 2, 3, 4, 5\}$. At each sampling instant, the sampling period is randomly selected from Γ with equal probability. The values of α_k and β_k are chosen according to (4.9) and (4.10). In particular,

$$\alpha_k = \frac{1}{m_k h}\frac{1}{2}\left(1 + 1 + \frac{m_{k-1}}{m_k}\right) = \frac{1}{m_k h}\left(1 + \frac{m_{k-1}}{2m_k}\right).$$

The initial position and velocity for the four agents are $x(t_0) = [1\ 2\ 3\ 4]^{\mathrm{T}}$ and $v(t_0) = [0\ 0\ 0\ 0]^{\mathrm{T}}$, respectively. Figure 4.3 shows the evolution of the four agents' states. We can see that after some time, consensus has been reached under the proposed control scheme. Figure 4.4 demonstrates the sequence of sampling instants

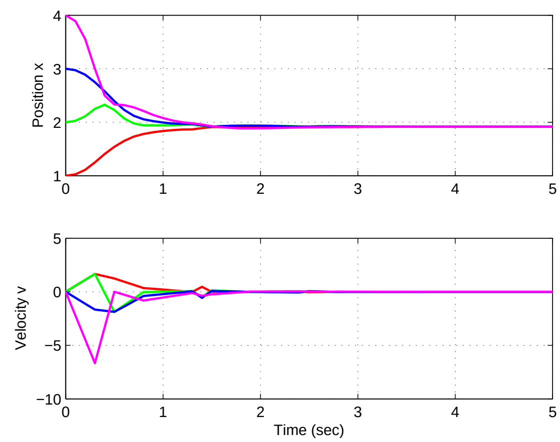

Figure 4.3: The evolution of agents' states.

in the MAS. The time instants associated with a value equal to one correspond to the sampling instants. The interval between two neighboring sampling instants represents the length of sampling period.

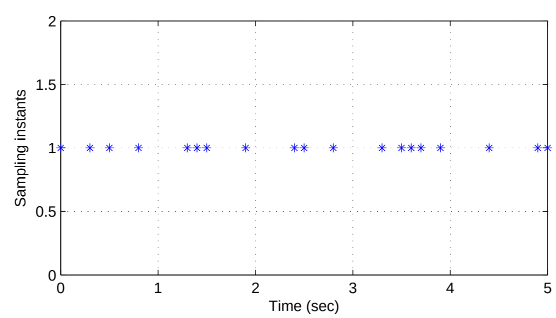

Figure 4.4: Sequence of sampling instants. Each asterisk indicates an instant at which the agents sample their states.

4.5 Conclusion

In this work, we studied the consensus problem with non-uniform sampling and fixed directed communication topology. By discretizing the sampled-data MAS, sufficient conditions were developed with some mild assumptions. Both the graph connectivity and control gains have proven to be crucial to reaching consensus. In the future, it would be interesting to look into the consensus problem with non-uniform sampling under switching topology.

Chapter 5

Stochastic Stabilization for Bilateral Teleoperation over Networks with Probabilistic Delays

5.1 Introduction

Bilateral teleoperation, which involves a master and a slave, has been attracting increasing attention from both the academia and industry over the past few decades [45] [2] [150]. On the master side, the human operates a manipulator to obtain the desired trajectory to accomplish the task. Meanwhile, the slave robot located in the working environment follows the motion of the master. The teleoperation has wide applications in practice, e.g., remote operation in mining vehicles and systems [42], remotely controlled robots in nuclear station [52], minimal invasive surgery [102], etc. Two important measures characterizing the teleoperation performance are stability and transparency [66]. That is, the closed-loop system should be stable when interacting with human and environment, and we expect to have a feeling of tele-presence. There is usually a trade-off between these two indices.

The communication network acts as the bridge to connect the master and slave. As a result, constraints such as delay, data package dropouts, and quantization error, are inevitable. Time delay degrades the system performance, or even leads to instability. Depending on if we utilize the information on delay for controller design, two schemes emerge: *delay-independent* and *delay-dependent* control. The latter employs the delay information and involves more complicated design process. In practice, time delays are bounded and occur with *non-uniform* probabilities at different values. This feature is valuable for controller design, e.g., a less conservative result was obtained in [35] after considering the probability distribution of delay. Also see [135] for probabilistic delays in state estimation.

Time delay is not new for bilateral teleoperation. Anderson and Spong [2] for the first time tackled the stability issue in a force-reflecting teleoperator with a constant delay. Passivity was achieved by introducing the scattering transformation. Another similar approach, called wave variables, appeared in [91] to passify the communication channel; see also [92] [145]. These works all dealt with constant and uniform delay. The Lyapunov–Krasovskii technique was applied in [67] to guarantee passivity for Euler–Lagrange dynamics when delays were constant and non-uniform. The work in [82] [22] considered the time-varying delay by applying a varying gain on the communication link. Stochastic delays are coped with in [129] under the scheme of Markov jump linear systems. Apart from the passivity-based methods, an alternative approach is to study the stability and stabilization problem based on the research on NCSs [48]. For more work dealing with time delay in teleoperation, one can refer to [81] [108].

The main contribution of this work comes from the integration of delay distribution and pole placement into the controller design for bilateral teleoperation. The authors in [129] studied the teleoperation over network with stochastic delays. However, the stochastic characteristics of delays were not used, with only lower and upper bounds considered. A natural question arises: How to incorporate more information on delays and adopt new techniques such that the performance can be improved? Motivated by this, the main objectives of this work are threefold:

- To practically characterize the *non-uniform* probability distribution of network-induced delays. The delays are assumed to be from a finite set, and each delay within the set has its own probability of occurrence.

- To derive the error dynamic system with error being defined as the difference between the states of the master and slave.

- To employ a novel technique (pole placement) and establish the sufficient condition of guaranteeing the input-to-state stability of the error dynamic system.

The rest of this work is organized as follows. In Section 5.2, the problem of teleoperation is formulated and converted into the stabilization of an error dynamic system. Section 5.3 details the controller design dependent on delay probabilities. A sufficient condition expressed as LMIs is provided to guarantee bounded error under bounded external input. The simulation in Section 5.4 demonstrates effectiveness of the proposed approach and enhanced performance. Finally, we offer some concluding remarks in Section 5.5.

Notation: $\mathbb{E}\{\cdot\}$ denotes the mathematical expectation, and $\mathbb{P}\{\cdot\}$ is the probability operator. '$*$' in a matrix stands for a term induced by symmetry. A matrix $Q > 0$ ($Q < 0$) means Q is symmetric and positive definite (negative definite). '$\otimes$' denotes the Kronecker product. '$\|\cdot\|$' represents the Euclidean norm for a vector or the corresponding induced norm for a matrix. Matrices, even if their dimensions are not specified, are assumed to be compatible with algebraic operations.

5.2 Problem Formulation

In this section, we establish the mathematical model of the networked bilateral teleoperation system, illustrated in Figure 5.1. The master robot is operated by human, while the slave interacts with the environment on a remote side. The task is to let the slave follow the trajectory of the master with desired accuracy. Usually, the two robots have dynamics of the same structure, with identical or discrepant parameters. For example, they can be two DC motors or robotic manipulators. To achieve better control performance, both the master and slave send their force and velocity information to each other. As shown in Figure 5.1, the information transmission is subjected to delay d_k. Here it is assumed that the two channels (from master to slave and from slave to master) have the same delay at any time instant.

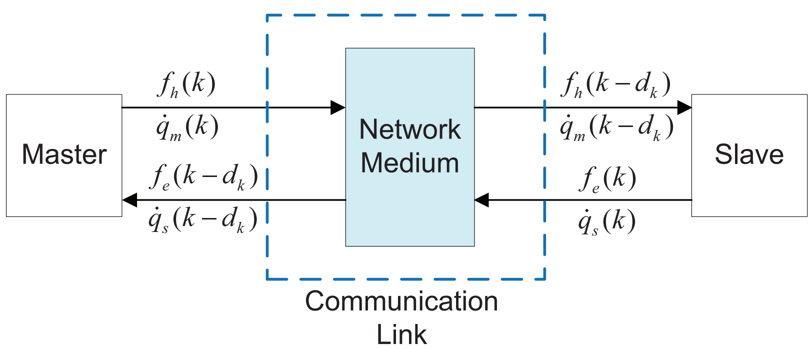

Figure 5.1: Schematic of bilateral teleoperation with delayed communication.

The dynamics of the master and slave is described by

$$M\ddot{q}_m + b\dot{q}_m = u_m + f_h \tag{5.1a}$$

$$M\ddot{q}_s + b\dot{q}_s = u_s - f_e \tag{5.1b}$$

where M is the moment of inertia, b is the damping coefficient, q_m and q_s are angular displacements of the master and slave, respectively. f_h is the torque from human, while f_e is the torque resulting from the contact with the environment. u_m and u_s are the respective control signals for master and slave. Equation (5.1) represents a large class of practical systems. For instance, in a mass-spring-damper system, M is the mass, q_m and q_s are translational displacements. This type of dynamics has also been studied in [22] [84] with heterogeneous master and slave dynamics, and in [129] with identical dynamics.

Define the states for master and slave,

$$x_m = \begin{bmatrix} q_m \\ \dot{q}_m \end{bmatrix}, x_s = \begin{bmatrix} q_s \\ \dot{q}_s \end{bmatrix}.$$

Then we can write (5.1a) and (5.1b) in the state-space form,

$$\dot{x}_m = Ax_m + B(u_m + f_h)$$
$$\dot{x}_s = Ax_s + B(u_s - f_e)$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{b}{M} \end{bmatrix}, B = \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}.$$

To facilitate the digital controller design, it is necessary to derive the discrete-time model of this system. With sampling period h and zero-order hold for the control u_m, the master dynamics in continuous time is converted into its equivalent counterpart in discrete time,

$$\begin{aligned} x_m(k+1) &= A_d x_m(k) + \int_{kh}^{(k+1)h} e^{[(k+1)h-\tau]A} B[u_m(\tau) + f_h(\tau)]d\tau \\ &= A_d x_m(k) + \int_{kh}^{(k+1)h} e^{[(k+1)h-\tau]A} B[u_m(kh) + f_h(\tau)]d\tau \\ &= A_d x_m(k) + B_d u_m(k) + \int_{kh}^{(k+1)h} e^{[(k+1)h-\tau]A} B f_h(\tau) d\tau \end{aligned} \tag{5.2}$$

where

$$A_d = e^{Ah}, B_d = \int_{kh}^{(k+1)h} e^{[(k+1)h-\tau]A} B d\tau = \int_0^h e^{A\tau} B d\tau.$$

Similarly, for the slave, we have

$$x_s(k+1) = A_d x_s(k) + B_d u_s(k) - \int_{kh}^{(k+1)h} e^{[(k+1)h-\tau]A} B f_e(\tau) d\tau. \tag{5.3}$$

Note that in (5.2) and (5.3), f_h and f_e are still in continuous time, which is different from the previous literature [129] [119]. Since the human and environmental torques do not work in a discrete-time way and they are exerted on the master and slave continuously, it is more reasonable to treat f_h and f_e as continuous-time signals. These terms involving f_h and f_e are perceived as exogenous disturbances, and we will prove their norm boundedness for further stability analysis.

With the communication scheme in Figure 5.1, the following PD (proportional-derivative) control laws are utilized for the master and slave,

$$u_m(k) = K[x_m(k) - x_s(k-d_k)] - f_e(k-d_k), \tag{5.4a}$$

$$u_s(k) = K[x_s(k) - x_m(k-d_k)] + f_h(k-d_k). \tag{5.4b}$$

The above two equations have the same form as those in [129] when there is no data loss. This PD based control is also widely used in the previous literature [67] [100]. Since they assume the same dynamics, it is natural to use the same K for the master and slave. It should also be noted that with force/torque sensors, it is possible to get the delayed values of f_h and f_e periodically.

Remark 5.1. This controller inherits a typical four-channel structure which has been widely studied in the literature [66] [84]. Both the displacements and forces are transmitted to controllers. In [66] [84], the delay is constant and passivity is analyzed in frequency domain. However, the time delay here is randomly varying and this makes it difficult to analyze the performance in frequency domain. We adopt the same control scheme but will use different analysis methodology to address the issue of delay.

Assumption 5.1. Delays are from the finite set of integers: $\Gamma = \{\tau_1, \tau_2, \dots, \tau_q\}$, $0 \le \tau_1 < \tau_2 < \cdots < \tau_q$. At each time instant k, the delay $d_k = \tau_i, i = 1, 2, \dots, q$, occurs with the probability

$$\mathbb{P}\{d_k = \tau_i\} = \pi_i \ge 0$$

where $\sum_{i=1}^{q} \pi_i = 1$.

Remark 5.2. For discrete-time or sampled-data systems, it is reasonable to assume that the delay is integer multiple of the sampling period h. Data package may arrive at arbitrary time instants. However, by introducing periodic sampling and zero-order hold, we can ensure that both the measurement and control action will only occur at instants that are integer multiple of h. It is reported in [129] that delays do not occur with uniform probability. Rather, they are more likely to happen within a relatively smaller interval. Controller design following such a formulation apparently involves more information compared with the case of only considering the lower and upper bounds.

In practice, to determine the probabilities of occurrence of different delays, we can measure the delays across a time interval that is long enough. Then we count the number of occurrences for each element in the delay set. For example, suppose that the total number of time instants considered is N and the number for the occurrence of time delay τ_i is N_i. The ratio N_i/N is the probability for delay τ_i. Due to the inevitable imperfection of experiment and the effect of various random factors, it is possible that there might exist some uncertainties in the probabilities. In that case, we need to deal with robustness of the designed controllers against uncertainties.

Define the error vector

$$e(k) = x_m(k) - x_s(k).$$

Then the closed-loop error dynamics is obtained from (5.2) and (5.3) as

$$\begin{aligned} e(k+1) &= A_d e(k) + B_d[u_m(k) - u_s(k)] \\ &\quad + \int_{kh}^{(k+1)h} e^{[(k+1)h-\tau]A} B[f_h(\tau) + f_e(\tau)] d\tau \\ &= A_d e(k) + B_d K[e(k) + e(k-d_k)] + B_d[-f_h(k-d_k) - f_e(k-d_k)] \\ &\quad + \int_{kh}^{(k+1)h} e^{[(k+1)h-\tau]A} B[f_h(\tau) + f_e(\tau)] d\tau \\ &= (A_d + B_d K) e(k) + B_d K e(k-d_k) + w(k) \end{aligned} \tag{5.5}$$

where

$$w(k) = \int_{kh}^{(k+1)h} e^{[(k+1)h-\tau]A} B[f_h(\tau) - f_h(k-d_k) + f_e(\tau) - f_e(k-d_k)] d\tau.$$

For forces from the human operator and environment, we adopt the similar boundedness assumption to that in [129].

Assumption 5.2. The rate of change of f_h and f_e is bounded: $|\dot{f}_h| \leq f_h^{\max}, |\dot{f}_e| \leq f_e^{\max}$.

The boundedness of $\dot{f}_h$ and $\dot{f}_e$ can be achieved by using a low-pass filter as stated in [129]. The constraint of bounded rate is quite general and does not fully capture the characteristics of the human and environment. Intuitively, if we consider the dynamics with more details, better results should be obtained. Thus, it is of interest to take into consideration more dynamic features of f_h and f_e in the future. This assumption is vital for the boundedness of $w(k)$, which plays an important role in the stability analysis. The following lemma proves the boundedness of $w(k)$.

Lemma 5.1. *The term $w(k) \in \mathbb{R}^2$ in (5.5) is norm bounded, i.e., $\|w(k)\| < \infty$.*

Proof. From the properties of norm in matrix theory,

$$\begin{aligned}
&\|w(k)\| \\
&= \left\| \int_{kh}^{(k+1)h} e^{[(k+1)h-\tau]A} B[f_h(\tau) - f_h(k-d_k) + f_e(\tau) - f_e(k-d_k)]d\tau \right\| \\
&\leq \int_{kh}^{(k+1)h} \left\|e^{[(k+1)h-\tau]A}\right\| \cdot \|B\| \cdot |[f_h(\tau) - f_h(k-d_k) + f_e(\tau) - f_e(k-d_k)]|d\tau \\
&\leq \int_{kh}^{(k+1)h} e^{[(k+1)h-\tau]\|A\|} \cdot \|B\| \cdot |[f_h(\tau) - f_h(k-d_k) + f_e(\tau) - f_e(k-d_k)]|d\tau.
\end{aligned}$$

Because $\tau \in [kh,\ (k+1)h]$, we have $0 \leq (k+1)h - \tau \leq h$, which yields

$$e^{[(k+1)h-\tau]\|A\|} \leq e^{h\|A\|}. \tag{5.6}$$

Using the integral equality

$$f_h(\tau) - f_h(k-d_k) = \int_{(k-d_k)h}^{\tau} \dot{f}_h(s)ds$$

and Assumption 5.2, it is obtained that

$$\begin{aligned}
|f_h(\tau) - f_h(k-d_k)| &\leq \int_{k-d_k}^{\tau} |\dot{f}_h(s)|ds \leq \int_{(k-d_k)h}^{\tau} f_h^{\max} ds \\
&= (\tau - kh + d_k h) f_h^{\max} \leq (1+\tau_q)h f_h^{\max}.
\end{aligned} \tag{5.7}$$

Similarly, for the environmental force,

$$|f_e(\tau) - f_e(k - d_k)| \leq (1 + \tau_q)hf_e^{\max}. \tag{5.8}$$

Hence, considering (5.7) and (5.8) gives

$$\begin{aligned} &|[f_h(\tau) - f_h(k - d_k) + f_e(\tau) - f_e(k - d_k)]| \\ &\leq |f_h(\tau) - f_h(k - d_k)| + |f_e(\tau) - f_e(k - d_k)| \\ &\leq (1 + \tau_q)hf_h^{\max} + (1 + \tau_q)hf_e^{\max} \\ &= (1 + \tau_q)h(f_h^{\max} + f_e^{\max}). \end{aligned} \tag{5.9}$$

Combining (5.6) and (5.9), we get

$$\begin{aligned} \|w(k)\| &\leq \int_{kh}^{(k+1)h} e^{h\|A\|} \cdot \|B\| \cdot (1 + \tau_q)h(f_h^{\max} + f_e^{\max})d\tau \\ &= (1 + \tau_q)h^2(f_h^{\max} + f_e^{\max})e^{h\|A\|}\|B\|. \end{aligned}$$

Since all the variables involved in the above equation are constant, we conclude that $w(k)$ is norm bounded. □

Remark 5.3. For the discretization in much of the previous literature [129] [119], researchers treat f_h and f_e as constant during the time interval $[kh,\ (k+1)h]$. In contrast, f_h and f_e are still continuous-time signals here. Obviously, this is more practical since the forces from human operator and environment will act on the master and slave continuously in time domain. To facilitate the stability analysis, the property of norm boundedness of the exogenous signal is proved first.

Introduce the indicator function

$$\mathbf{1}_{\{d_k=\tau_j\}} = \begin{cases} 1, & d_k = \tau_j \\ 0, & d_k \neq \tau_j \end{cases}$$

for which we have $\mathbb{E}\{\mathbf{1}_{\{d_k=\tau_j\}}\} = \pi_j, j = 1, 2, \ldots, q$. Then, (5.5) could be written as

$$e(k+1) = (A_d + B_dK)e(k) + \sum_{j=1}^{q} \mathbf{1}_{\{d_k=\tau_j\}} B_dKe(k - \tau_j) + w(k). \tag{5.10}$$

Instead of coping with the position tracking directly, we study the stability of the error dynamics in (5.10). Since this is a teleoperation problem, the exogenous term $w(k)$ has to be taken into consideration. Therefore, the input-to-state stability will be investigated. Specifically, we must achieve that

- The unforced system of (5.10) with $w(k) = 0$ is asymptotically stable;
- The state (tracking error $e(k)$) is bounded for any bounded $w(k)$.

Remark 5.4. In this work, we are only concerned with the boundedness of the tracking error. How the external torques affect the tracking error quantitatively is not discussed. Sometimes, it may be necessary to characterize the relation between the external torques and tracking error in more detail. In this situation, we can consider using other methods such as H_∞ optimal control to characterize the relationship. As will be mentioned below, the exponential stability of the unforced system with $w(k) = 0$ guarantees the system's performance.

The following definition on input-to-state stability is adapted from the theory on nonlinear systems [59]. It is also used in [129].

Definition 5.1. The system in (5.10) with an exogenous term $w(k)$ is *input-to-state stable* (ISS) if there exists a Lyapunov function $V(k)$ such that the following equation holds:

$$\mathbb{E}\{\Delta V(k)\} = \mathbb{E}\{V(k+1) - V(k)\} \leq -\xi^{\mathrm{T}}(k)P_1\xi(k) + w^{\mathrm{T}}(k)P_2 w(k), \tag{5.11}$$

where $\xi(k)$ is a vector stacked by the error state $e(k)$ and its delayed states $e(k - \tau_j), j = 1, 2, \ldots, q$, P_1 and P_2 are symmetric and positive definite matrices.

The reason why we can adopt this definition is that when (5.11) holds, if we let $w(k) = 0$, then $\mathbb{E}\{\Delta V(k)\} \leq -\xi^{\mathrm{T}}(k)P_1\xi(k) < 0, \forall \xi(k) \neq 0$. Therefore, the unforced system of (5.10) will be exponentially stable. From Lemma 4.6 in [59], we know that (5.10) is ISS. If the system is ISS, then for any bounded input, we will get bounded state ($e(k)$ in (5.10)).

The goal of this work is to find a control gain K such that (5.10) satisfies the ISS property in Definition 5.1. If K exists, we can always guarantee that the tracking error $e(k)$ is bounded for any bounded $w(k)$.

Remark 5.5. It is worth noting that in a teleoperation system, the dynamics of the human and environment and the dynamics of the master and slave are closely coupled.

A more comprehensive analysis of stability should consider the specific model structure of the human and environment. Here, because our emphasis is on the stochastic stability when the system is subject to communication constraints with probabilistic features, a more general assumption on $w(k)$ is adopted. When the forces from human and environment satisfy the condition in Assumption 5.2, the term $w(k)$ will be norm bounded. Therefore, it is a very general way to treat $w(k)$ as an exogenous signal in this work. In the future, we can try to further improve the system performance by including more details of the human and environment dynamics.

5.3 Main Results

In this section, we focus on design of the controller K to guarantee the ISS stability.

5.3.1 Initial Design

The following theorem provides a sufficient condition for ISS of the teleoperation system.

Theorem 5.1. *The system in (5.10) is ISS if there exist matrices $P > 0, Q_j > 0, Z_j > 0, M_j, j = 1, 2, \ldots, q,$ and a scalar $\lambda > 0$, such that the following matrix inequality holds,*

$$\begin{bmatrix} \Lambda_1 + \Phi_1 + \Phi_1^{\mathrm{T}} & \Phi_2 & \Phi_3 & \Phi_4 & \Phi_5 & \Phi_6 \\ * & -\Lambda_2^{-1} & & & & \\ * & & -\Lambda_3^{-1} & & & \\ * & & & -\Lambda_4 & & \\ * & & & & -\lambda P & \\ * & & & & & -\lambda \sum_{m=1}^{q} \tau_m Z_m \end{bmatrix} < 0 \qquad (5.12)$$

where

$$\Lambda_1 = \mathrm{diag}\{-P + \sum_{j=1}^{q} Q_j, -Q_1, -Q_2, \ldots, -Q_q\},$$

$$\Lambda_2 = I_q \otimes P, \quad \Lambda_3 = I_q \otimes \left(\sum_{m=1}^{q} \tau_m Z_m \right), \quad \Lambda_4 = \mathrm{diag}\{Z_1,\ Z_2,\ \ldots,\ Z_q\},$$

$$\Phi_1 = \left[\sum_{j=1}^{q} M_j,\ -M_1,\ -M_2,\ \ldots,\ -M_q \right],$$

$$\Phi_2 = \begin{bmatrix} \sqrt{\pi_1}A_{cl}^{\mathrm{T}} & \sqrt{\pi_2}A_{cl}^{\mathrm{T}} & \cdots & \sqrt{\pi_q}A_{cl}^{\mathrm{T}} \\ \sqrt{\pi_1}K^{\mathrm{T}}B_d^{\mathrm{T}} & 0 & \cdots & 0 \\ 0 & \sqrt{\pi_2}K^{\mathrm{T}}B_d^{\mathrm{T}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\pi_q}K^{\mathrm{T}}B_d^{\mathrm{T}} \end{bmatrix},$$

$$\Phi_3 = \begin{bmatrix} \sqrt{\pi_1}(A_{cl}-I)^{\mathrm{T}} & \sqrt{\pi_2}(A_{cl}-I)^{\mathrm{T}} & \cdots & \sqrt{\pi_q}(A_{cl}-I)^{\mathrm{T}} \\ \sqrt{\pi_1}K^{\mathrm{T}}B_d^{\mathrm{T}} & 0 & \cdots & 0 \\ 0 & \sqrt{\pi_2}K^{\mathrm{T}}B_d^{\mathrm{T}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\pi_q}K^{\mathrm{T}}B_d^{\mathrm{T}} \end{bmatrix},$$

$$\Phi_4 = \left[\sqrt{\tau_1}M_1,\ \sqrt{\tau_2}M_2,\ \ldots,\ \sqrt{\tau_q}M_q \right],$$

$$\Phi_5 = \begin{bmatrix} A_{cl}^{\mathrm{T}}P \\ \pi_1 K^{\mathrm{T}}B_d^{\mathrm{T}}P \\ \vdots \\ \pi_q K^{\mathrm{T}}B_d^{\mathrm{T}}P \end{bmatrix}, \Phi_6 = \begin{bmatrix} (A_{cl}-I)^{\mathrm{T}}\left(\sum_{m=1}^{q}\tau_m Z_m\right) \\ \pi_1 K^{\mathrm{T}}B_d^{\mathrm{T}}\left(\sum_{m=1}^{q}\tau_m Z_m\right) \\ \vdots \\ \pi_q K^{\mathrm{T}}B_d^{\mathrm{T}}\left(\sum_{m=1}^{q}\tau_m Z_m\right) \end{bmatrix}, A_{cl} = A_d + B_d K.$$

Proof. For the error dynamic system in (5.10), consider the following Lyapunov function candidate [35],

$$\begin{aligned} V(k) &= V_1(k) + V_2(k) + V_3(k), \\ V_1(k) &= e^{\mathrm{T}}(k)Pe(k), \\ V_2(k) &= \sum_{j=1}^{q} \sum_{i=k-\tau_j}^{k-1} e^{\mathrm{T}}(i)Q_j e(i), \\ V_3(k) &= \sum_{j=1}^{q} \sum_{i=-\tau_j}^{-1} \sum_{m=k+i}^{k-1} \eta^{\mathrm{T}}(m)Z_j\eta(m), \end{aligned}$$

where P, Q_j, Z_j are positive definite matrices to be determined, and

$$\begin{aligned} \eta(m) &= e(m+1) - e(m) \\ &= (A_d + B_d K - I)e(m) + \sum_{j=1}^{q} \mathbf{1}_{\{d_k=\tau_j\}} B_d K e(m-\tau_j) + w(m). \end{aligned}$$

Then, take the mathematical expectation of the difference of each term in $V(k)$.

$$\begin{aligned}
&\mathbb{E}\{\Delta V_1(k)\} = \mathbb{E}\{V_1(k+1) - V_1(k)\} \\
&= \mathbb{E}\left\{\left[e^{\mathrm{T}}(k)(A_d + B_dK)^{\mathrm{T}} + \sum_{j=1}^{q} \mathbf{1}_{\{d_k=\tau_j\}} e^{\mathrm{T}}(k-\tau_j)K^{\mathrm{T}}B_d^{\mathrm{T}} + w^{\mathrm{T}}(k)\right] P \right. \\
&\qquad \left. \times \left[(A_d + B_dK)e(k) + \sum_{j=1}^{q} \mathbf{1}_{\{d_k=\tau_j\}} B_dKe(k-\tau_j) + w(k)\right] - e^{\mathrm{T}}(k)Pe(k)\right\} \\
&= e^{\mathrm{T}}(k)[(A_d + B_dK)^{\mathrm{T}}P(A_d + B_dK) - P]e(k) \\
&\quad + 2e^{\mathrm{T}}(k)\sum_{j=1}^{q} \pi_j (A_d + B_dK)^{\mathrm{T}}PB_dKe(k-\tau_j) \\
&\quad + \sum_{j=1}^{q} \pi_j e^{\mathrm{T}}(k-\tau_j)K^{\mathrm{T}}B_d^{\mathrm{T}}PB_dKe(k-\tau_j) \\
&\quad + 2e^{\mathrm{T}}(k)(A_d + B_dK)^{\mathrm{T}}Pw(k) \\
&\quad + 2\sum_{j=1}^{q} \pi_j e^{\mathrm{T}}(k-\tau_j)K^{\mathrm{T}}B_d^{\mathrm{T}}Pw(k) + w^{\mathrm{T}}(k)Pw(k), \qquad (5.13)
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}\{\Delta V_2(k)\} = \mathbb{E}\{V_2(k+1) - V_2(k)\} \\
&= \sum_{j=1}^{q} [e^{\mathrm{T}}(k)Q_je(k) - e^{\mathrm{T}}(k-\tau_j)Q_je(k-\tau_j)], \qquad (5.14)
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}\{\Delta V_3(k)\} = \mathbb{E}\{V_3(k+1) - V_3(k)\} \\
&= \mathbb{E}\left\{\eta^{\mathrm{T}}(k)\left(\sum_{m=1}^{q} \tau_m Z_m\right)\eta(k) - \sum_{j=1}^{q}\sum_{m=k-\tau_j}^{k-1} \eta^{\mathrm{T}}(m)Z_j\eta(m)\right\} \\
&= e^{\mathrm{T}}(k)(A_d + B_dK - I)^{\mathrm{T}}\left(\sum_{m=1}^{q} \tau_m Z_m\right)(A_d + B_dK - I)e(k) \\
&\quad + 2e^{\mathrm{T}}(k)(A_d + B_dK - I)^{\mathrm{T}}\left(\sum_{m=1}^{q} \tau_m Z_m\right)\sum_{j=1}^{q} \pi_j B_dKe(k-\tau_j) \\
&\quad + \sum_{j=1}^{q} \pi_j e^{\mathrm{T}}(k-\tau_j)K^{\mathrm{T}}B_d^{\mathrm{T}}\left(\sum_{m=1}^{q} \tau_m Z_m\right)B_dKe(k-\tau_j) \\
&\quad + 2e^{\mathrm{T}}(k)(A_d + B_dK - I)^{\mathrm{T}}\left(\sum_{m=1}^{q} \tau_m Z_m\right)w(k)
\end{aligned}$$

$$+ 2\sum_{j=1}^{q}\pi_j e^{\mathrm{T}}(k-\tau_j)K^{\mathrm{T}}B_d^{\mathrm{T}}\left(\sum_{m=1}^{q}\tau_m Z_m\right)w(k) + w^{\mathrm{T}}(k)\left(\sum_{m=1}^{q}\tau_m Z_m\right)w(k)$$
$$-\sum_{j=1}^{q}\sum_{m=k-\tau_j}^{k-1}\eta^{\mathrm{T}}(m)Z_j\eta(m). \tag{5.15}$$

For any matrices

$$M_j = \left[\begin{array}{ccccc} M_{0j}^{\mathrm{T}} & M_{1j}^{\mathrm{T}} & M_{2j}^{\mathrm{T}} & \cdots & M_{qj}^{\mathrm{T}} \end{array}\right]^{\mathrm{T}}, j = 1, 2, \ldots, q,$$

with appropriate dimensions, we have

$$\xi^{\mathrm{T}}(k)M_j\left[e(k) - e(k-\tau_j) - \sum_{m=k-\tau_j}^{k-1}\eta(m)\right] = 0,$$

where

$$\xi(k) = \left[e^{\mathrm{T}}(k), e^{\mathrm{T}}(k-\tau_1), e^{\mathrm{T}}(k-\tau_2), \ldots, e^{\mathrm{T}}(k-\tau_q)\right]^{\mathrm{T}}.$$

Then, $\mathbb{E}\{\Delta V(k)\}$ can be obtained as

$$\mathbb{E}\{\Delta V(k)\} = \mathbb{E}\{\Delta V_1(k)\} + \mathbb{E}\{\Delta V_2(k)\} + \mathbb{E}\{\Delta V_3(k)\}$$
$$\leq \mathbb{E}\{\Delta V_1(k)\} + \mathbb{E}\{\Delta V_2(k)\} + \mathbb{E}\{\Delta V_3(k)\}$$
$$+ 2\sum_{j=1}^{q}\xi^{\mathrm{T}}(k)M_j\left[e(k) - e(k-\tau_j) - \sum_{m=k-\tau_j}^{k-1}\eta(m)\right]$$
$$+ \mathbb{E}\left\{\sum_{j=1}^{q}\sum_{m=k-\tau_j}^{k-1}[\xi^{\mathrm{T}}(k)M_j + \eta^{\mathrm{T}}(m)Z_j]Z_j^{-1}[M_j^{\mathrm{T}}\xi(k) + Z_j\eta(m)]\right\}. \tag{5.16}$$

The last term in (5.16) is nonnegative. Using the inequality $2a^{\mathrm{T}}c \leq a^{\mathrm{T}}Qa + c^{\mathrm{T}}Q^{-1}c$, where a, c are vectors and $Q > 0$, we get

$$2e^{\mathrm{T}}(k)(A_d + B_dK)^{\mathrm{T}}Pw(k) + 2\sum_{j=1}^{q}\pi_j e^{\mathrm{T}}(k-\tau_j)K^{\mathrm{T}}B_d^{\mathrm{T}}Pw(k)$$
$$= 2\xi^{\mathrm{T}}(k)\Phi_5 w(k) \leq \xi^{\mathrm{T}}(k)\Phi_5(\lambda P)^{-1}\Phi_5^{\mathrm{T}}\xi(k) + w^{\mathrm{T}}(k)(\lambda P)w(k), \tag{5.17}$$
$$2e^{\mathrm{T}}(k)(A_d + B_dK - I)^{\mathrm{T}}\left(\sum_{m=1}^{q}\tau_m Z_m\right)w(k)$$

$$
\begin{aligned}
&+2\sum_{j=1}^{q}\pi_j e^{\mathrm{T}}(k-\tau_j)K^{\mathrm{T}}B_d^{\mathrm{T}}\left(\sum_{m=1}^{q}\tau_m Z_m\right)w(k)\\
&=2\xi^{\mathrm{T}}(k)\Phi_6 w(k)\\
&\leq \xi^{\mathrm{T}}(k)\Phi_6\left(\lambda\sum_{m=1}^{q}\tau_m Z_m\right)^{-1}\Phi_6^{\mathrm{T}}\xi(k)+w^{\mathrm{T}}(k)\left(\lambda\sum_{m=1}^{q}\tau_m Z_m\right)w(k),
\end{aligned}
\tag{5.18}
$$

where Φ_5 and Φ_6 are defined in this theorem. $\lambda > 0$ is a weighting factor that can be tuned and plays a role to adjust the quadratic terms of $\xi(k)$ in (5.17) and (5.18). Considering (5.13), (5.14), (5.15), (5.17) and (5.18), it follows that

$$
\begin{aligned}
\mathbb{E}\{\Delta V(k)\} \leq \xi^{\mathrm{T}}(k)[&\Lambda_1+\Phi_1+\Phi_1^{\mathrm{T}}+\Phi_2\Lambda_2\Phi_2^{\mathrm{T}}+\Phi_3\Lambda_3\Phi_3^{\mathrm{T}}+\Phi_4\Lambda_4^{-1}\Phi_4^{\mathrm{T}}\\
&+\Phi_5(\lambda P)^{-1}\Phi_5^{\mathrm{T}}+\Phi_6\left(\lambda\sum_{m=1}^{q}\tau_m Z_m\right)^{-1}\Phi_6^{\mathrm{T}}\Bigg]\xi(k)\\
&+(1+\lambda)w^{\mathrm{T}}(k)\left(P+\sum_{m=1}^{q}\tau_m Z_m\right)w(k).
\end{aligned}
\tag{5.19}
$$

Now (5.19) possesses the same form as that in Definition 5.1. A sufficient condition to guarantee ISS is

$$
\begin{aligned}
&\Lambda_1+\Phi_1+\Phi_1^{\mathrm{T}}+\Phi_2\Lambda_2\Phi_2^{\mathrm{T}}+\Phi_3\Lambda_3\Phi_3^{\mathrm{T}}+\Phi_4\Lambda_4^{-1}\Phi_4^{\mathrm{T}}\\
&+\Phi_5(\lambda P)^{-1}\Phi_5^{\mathrm{T}}+\Phi_6\left(\lambda\sum_{m=1}^{q}\tau_m Z_m\right)^{-1}\Phi_6^{\mathrm{T}}<0.
\end{aligned}
\tag{5.20}
$$

By applying the Schur complement [9] on (5.20), Eq. (5.12) is obtained. From Definition 5.1 and the boundedness of $w(k)$ in Lemma 5.1, we know that (5.10) is ISS. Thus, the tracking error $e(k)$ will always be bounded for any bounded $w(k)$. The proof of Theorem 5.1 is completed. □

Remark 5.6. If the error dynamic system is stabilized, it implies that the slave can track the motion of the master with bounded error for any bounded exogenous input. Thus, the stability and performance of bilateral teleoperation is achieved. It is intuitive that when the magnitude of $w(k)$ becomes larger, the tracking error $e(k)$ will also increase. We do not investigate the quantitative relationship between them in this work, and this could be looked into in the future. The effectiveness of ensuring the performance of teleoperation will be validated in the simulation part. The condition in (5.12) is not an LMI, and this necessitates further manipulation to derive a sufficient condition that is convenient to solve with existing tools. Theorem 5.2 below gives a sufficient condition for ISS.

Theorem 5.2. *The error dynamic system in (5.10) for bilateral teleoperation is ISS if there exist* $\bar{P} > 0, \bar{Q}_i > 0, \bar{Z}_i > 0, \bar{M}_i, i = 1, 2, \ldots, q,$ *and a scalar* $\lambda > 0$*, such that the following LMI holds,*

$$\begin{bmatrix} \bar{\Lambda}_1 + \Upsilon_1 + \Upsilon_1^{\mathrm{T}} & \Upsilon_2 & \Upsilon_3 & \Upsilon_4 & \Upsilon_5 & \Upsilon_6 \\ * & \bar{\Lambda}_2 & & & & \\ * & & \bar{\Lambda}_3 & & & \\ * & & & \bar{\Lambda}_4 & & \\ * & & & & \bar{\Lambda}_5 & \\ * & & & & & \bar{\Lambda}_6 \end{bmatrix} < 0 \tag{5.21}$$

where

$$\bar{\Lambda}_1 = \mathrm{diag}\{-\bar{P} + \sum_{j=1}^{q} \bar{Q}_j, -\bar{Q}_1, -\bar{Q}_2, \ldots, -\bar{Q}_q\},$$

$$\bar{\Lambda}_2 = -I_q \otimes \bar{P}, \quad \bar{\Lambda}_3 = I_q \otimes \left(\sum_{m=1}^{q} \tau_m \bar{Z}_m - 2\bar{P} \right),$$

$$\bar{\Lambda}_4 = \mathrm{diag}\{-\bar{Z}_1, -\bar{Z}_2, \ldots, -\bar{Z}_q\}, \quad \bar{\Lambda}_5 = -\lambda \bar{P}, \quad \bar{\Lambda}_6 = \lambda \left(\sum_{m=1}^{q} \tau_m \bar{Z}_m - 2\bar{P} \right),$$

$$\Upsilon_1 = \left[\sum_{j=1}^{q} \bar{M}_j, \ -\bar{M}_1, \ -\bar{M}_2, \ \ldots, \ -\bar{M}_q \right],$$

$$\Upsilon_2 = \begin{bmatrix} \sqrt{\pi_1}\bar{A}_{cl}^{\mathrm{T}} & \sqrt{\pi_2}\bar{A}_{cl}^{\mathrm{T}} & \cdots & \sqrt{\pi_q}\bar{A}_{cl}^{\mathrm{T}} \\ \sqrt{\pi_1}\bar{K}^{\mathrm{T}}B_d^{\mathrm{T}} & 0 & \cdots & 0 \\ 0 & \sqrt{\pi_2}\bar{K}^{\mathrm{T}}B_d^{\mathrm{T}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\pi_q}\bar{K}^{\mathrm{T}}B_d^{\mathrm{T}} \end{bmatrix},$$

$$\Upsilon_3 = \begin{bmatrix} \sqrt{\pi_1}(\bar{A}_{cl} - \bar{P})^{\mathrm{T}} & \sqrt{\pi_2}(\bar{A}_{cl} - \bar{P})^{\mathrm{T}} & \cdots & \sqrt{\pi_q}(\bar{A}_{cl} - \bar{P})^{\mathrm{T}} \\ \sqrt{\pi_1}\bar{K}^{\mathrm{T}}B_d^{\mathrm{T}} & 0 & \cdots & 0 \\ 0 & \sqrt{\pi_2}\bar{K}^{\mathrm{T}}B_d^{\mathrm{T}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\pi_q}\bar{K}^{\mathrm{T}}B_d^{\mathrm{T}} \end{bmatrix},$$

$$\Upsilon_4 = \left[\sqrt{\tau_1}\bar{M}_1,\ \sqrt{\tau_2}\bar{M}_2,\ \ldots,\ \sqrt{\tau_q}\bar{M}_q\right],$$

$$\Upsilon_5 = \begin{bmatrix} \bar{A}_{cl}^{\mathrm{T}} \\ \pi_1\bar{K}^{\mathrm{T}}B_d^{\mathrm{T}} \\ \vdots \\ \pi_q\bar{K}^{\mathrm{T}}B_d^{\mathrm{T}} \end{bmatrix}, \quad \Upsilon_6 = \begin{bmatrix} (\bar{A}_{cl}-\bar{P})^{\mathrm{T}} \\ \pi_1\bar{K}^{\mathrm{T}}B_d^{\mathrm{T}} \\ \vdots \\ \pi_q\bar{K}^{\mathrm{T}}B_d^{\mathrm{T}} \end{bmatrix}, \quad \bar{A}_{cl} = A_d\bar{P} + B_d\bar{K}.$$

Furthermore, the stabilizing controller is solved as $K = \bar{K}\bar{P}^{-1}$.

Proof. Perform congruence transformation on (5.12) by

$$W = \mathrm{diag}\left\{ I_{1+q}\otimes P^{-1}, I_q\otimes I_2, I_q\otimes I_2, I_q\otimes P^{-1}, P^{-1}, \left(\sum_{m=1}^{q}\tau_m Z_m\right)^{-1} \right\}$$

where I_{1+q}, I_q and I_2 denote the identity matrices with dimensions $q+1$, q and 2, respectively. It is obtained that

$$\begin{bmatrix} \bar{\Lambda}_1+\Upsilon_1+\Upsilon_1^{\mathrm{T}} & \Upsilon_2 & \Upsilon_3 & \Upsilon_4 & \Upsilon_5 & \Upsilon_6 \\ * & \bar{\Lambda}_2 & & & & \\ * & & \bar{\Lambda}_{3t} & & & \\ * & & & \bar{\Lambda}_4 & & \\ * & & & & \bar{\Lambda}_5 & \\ * & & & & & \bar{\Lambda}_{6t} \end{bmatrix} < 0 \tag{5.22}$$

where $\bar{P} = P^{-1}$ and

$$\begin{aligned} \bar{Q}_j &= P^{-1}Q_jP^{-1} = \bar{P}Q_j\bar{P}, \\ \bar{M}_j &= (I_{1+q}\otimes P^{-1})M_jP^{-1} = (I_{1+q}\otimes\bar{P})M_j\bar{P}, \\ \bar{Z}_j &= P^{-1}Z_jP^{-1} = \bar{P}Z_j\bar{P}, \\ \bar{K} &= KP^{-1} = K\bar{P}. \end{aligned}$$

The matrices $\Upsilon_i, i = 1, 2, \ldots, 6$, are defined in this theorem. In (5.22),

$$\bar{\Lambda}_{3t} = -I_q\otimes\left(\sum_{m=1}^{q}\tau_m Z_m\right)^{-1}, \quad \bar{\Lambda}_{6t} = -\lambda\left(\sum_{m=1}^{q}\tau_m Z_m\right)^{-1}. \tag{5.23}$$

The above equations contain the inverse of the sum of matrix variables, thus are not solvable using the existing numerical tools (e.g., MATLAB LMI Toolbox). This necessitates a further step to make the problem more tractable.

In light of the following relationship

$$(Z-P)Z^{-1}(Z-P) \geq 0 \;\Leftrightarrow\; -PZ^{-1}P \leq Z-2P, \tag{5.24}$$

we have

$$\begin{aligned} -\left(\sum_{m=1}^{q}\tau_m Z_m\right)^{-1} &= -P^{-1}P\left(\sum_{m=1}^{q}\tau_m Z_m\right)^{-1} PP^{-1} \\ &\leq P^{-1}\left(\sum_{m=1}^{q}\tau_m Z_m - 2P\right)P^{-1} = \sum_{m=1}^{q}\tau_m \bar{Z}_m - 2\bar{P}. \end{aligned}$$

Therefore, $\bar{\Lambda}_{3t} \leq \bar{\Lambda}_3, \bar{\Lambda}_{6t} \leq \bar{\Lambda}_6$, and we conclude that (5.21) is a sufficient condition to (5.22). Note that (5.22) is equivalent to (5.12) in Theorem 5.1, thus Theorem 5.2 is sufficient to Theorem 5.1. Now, the stability condition has been expressed in terms of LMIs, which can be easily solved in MATLAB. □

Remark 5.7. Thus far, the bilateral teleoperation has been guaranteed by the controller in (5.4) whose parameters are obtained by solving a set of LMIs. Notice that when solving the matrix inequality in (5.21), in order for it to be an LMI, λ should be given first. Apparently, from Theorem 5.1 to Theorem 5.2, the result becomes more conservative due to the manipulation of matrix inequalities. It is worth noting that by constructing a better Lyapunov function or using the augmentation technique, less conservative results may be obtained [44] [23]. The input-to-state stability of the error dynamics ensures the performance of teleoperation.

5.3.2 Improvement with Pole Placement

Pole placement is very effective in the control of LTI systems. If a system is controllable, then we can find a state-feedback controller such that the poles of the closed-loop system can be put at arbitrary locations in the complex plane. For the stochastic system investigated in this work, the approach cannot be directly applied. However, we can benefit from this classical idea through some adaptation.

Figure 5.2 illustrates the basic idea: to design an appropriate K such that the poles of A_d+B_dK are within the circular region centered at $(\sigma, 0)$ with radius r. This circular region must be located within the unit circle. In the simulation part, we will

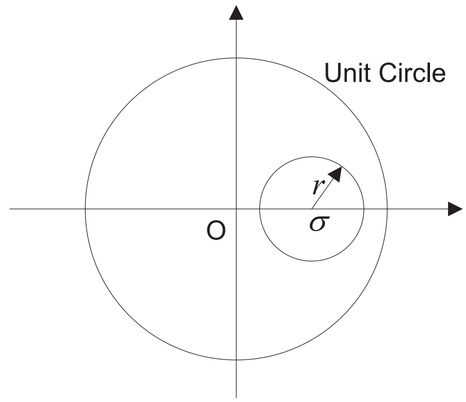

Figure 5.2: Illustration of pole placement.

demonstrate its effectiveness through an example. The following theorem is obtained readily.

Theorem 5.3. *The error dynamic system in (5.10) for bilateral teleoperation is ISS if there exist $\bar{P} > 0, \bar{Q}_i > 0, \bar{Z}_i > 0, \bar{M}_i, i = 1, 2, \ldots, q,$ and scalars $\lambda > 0, 0 < \sigma < 1, r > 0$, such that the following LMI holds,*

$$\begin{bmatrix} \bar{\Lambda}_1 + \Upsilon_1 + \Upsilon_1^{\mathrm{T}} & \Upsilon_2 & \Upsilon_3 & \Upsilon_4 & \Upsilon_5 & \Upsilon_6 \\ * & \bar{\Lambda}_2 & & & & \\ * & & \bar{\Lambda}_3 & & & \\ * & & & \bar{\Lambda}_4 & & \\ * & & & & \bar{\Lambda}_5 & \\ * & & & & & \bar{\Lambda}_6 \end{bmatrix} < 0 \tag{5.25}$$

where

$$\bar{\Lambda}_1 = \mathrm{diag}\{-\bar{P} + \sum_{j=1}^{q} \bar{Q}_j, -\bar{Q}_1, -\bar{Q}_2, \ldots, -\bar{Q}_q\},$$

$$\bar{\Lambda}_2 = -I_q \otimes \bar{P}, \quad \bar{\Lambda}_3 = I_q \otimes \left(\sum_{m=1}^{q} \tau_m \bar{Z}_m - 2\bar{P} \right),$$

$$\bar{\Lambda}_4 = \mathrm{diag}\{-\bar{Z}_1, -\bar{Z}_2, \ldots, -\bar{Z}_q\}, \quad \bar{\Lambda}_5 = -\lambda \bar{P}, \quad \bar{\Lambda}_6 = \lambda \left(\sum_{m=1}^{q} \tau_m \bar{Z}_m - 2\bar{P} \right),$$

$$\Upsilon_1 = \left[\sum_{j=1}^{q} \bar{M}_j, \ -\bar{M}_1, \ -\bar{M}_2, \ \ldots, \ -\bar{M}_q \right],$$

$$\Upsilon_2 = \begin{bmatrix} \sqrt{\pi_1}\Omega^{\mathrm{T}} & \sqrt{\pi_2}\Omega^{\mathrm{T}} & \cdots & \sqrt{\pi_q}\Omega^{\mathrm{T}} \\ \sqrt{\pi_1}\bar{K}^{\mathrm{T}}B_d^{\mathrm{T}} & 0 & \cdots & 0 \\ 0 & \sqrt{\pi_2}\bar{K}^{\mathrm{T}}B_d^{\mathrm{T}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\pi_q}\bar{K}^{\mathrm{T}}B_d^{\mathrm{T}} \end{bmatrix},$$

$$\Upsilon_3 = \begin{bmatrix} \sqrt{\pi_1}(\Omega-\bar{P})^{\mathrm{T}} & \sqrt{\pi_2}(\Omega-\bar{P})^{\mathrm{T}} & \cdots & \sqrt{\pi_q}(\Omega-\bar{P})^{\mathrm{T}} \\ \sqrt{\pi_1}\bar{K}^{\mathrm{T}}B_d^{\mathrm{T}} & 0 & \cdots & 0 \\ 0 & \sqrt{\pi_2}\bar{K}^{\mathrm{T}}B_d^{\mathrm{T}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\pi_q}\bar{K}^{\mathrm{T}}B_d^{\mathrm{T}} \end{bmatrix},$$

$$\Upsilon_4 = \left[\sqrt{\tau_1}\bar{M}_1,\ \sqrt{\tau_2}\bar{M}_2,\ \ldots,\ \sqrt{\tau_q}\bar{M}_q\right],$$

$$\Upsilon_5 = \begin{bmatrix} \Omega^{\mathrm{T}} \\ \pi_1\bar{K}^{\mathrm{T}}B_d^{\mathrm{T}} \\ \vdots \\ \pi_q\bar{K}^{\mathrm{T}}B_d^{\mathrm{T}} \end{bmatrix}, \quad \Upsilon_6 = \begin{bmatrix} (\Omega-\bar{P})^{\mathrm{T}} \\ \pi_1\bar{K}^{\mathrm{T}}B_d^{\mathrm{T}} \\ \vdots \\ \pi_q\bar{K}^{\mathrm{T}}B_d^{\mathrm{T}} \end{bmatrix}, \quad \Omega = (A_d\bar{P} + B_d\bar{K} - \sigma\bar{P})/r.$$

Furthermore, the stabilizing controller is solved as $K = \bar{K}\bar{P}^{-1}$.

Proof. The proof can be completed by replacing $A_d + B_dK$ with $(A_d + B_dK - \sigma I)/r$ in Theorem 5.1, then following the similar line in Theorem 5.2. □

Remark 5.8. In Theorem 5.3, if we set $\sigma = 0, r = 1$, then it reduces to Theorem 5.2. Thus, Theorem 5.3 is in a more general form than Theorem 5.2 and provides much more flexibility in controller design. By shifting the poles of $A_d + B_dK$ closer to the origin to some degree, the system response becomes faster and we can get better tracking performance between the master and slave.

Since this theorem is still a sufficient condition, conservativeness is inevitable. From Theorem 5.1 to Theorem 5.2, in order to derive a condition that is easy to solve using existing numerical tools, the condition becomes more conservative through (5.24). However, we have also made some effort to reduce the conservativeness. For instance, the introduction of λ in (5.17) (5.18) of Theorem 1 and the pole placement in Theorem 5.3 reduce the conservativeness to some extent.

Now the question is: How should we select these parameters λ, σ and r? For λ, it can be observed from (5.20) that choosing a larger value of λ will lead to less conservative results as that reduces the weight for the matrix terms that are positive definite. For σ and r, we only consider the case where the center of the smaller circular region for poles is located on the real axis, as shown in Figure 5.2. For discrete-time systems, the first requirement is that the circular region centered at $(\sigma, 0)$ with radius r should be within the unit disk. To get faster convergence speed, the circular region needs to be close to the origin. But if the region is too close to the origin, the LMIs may not be feasible and the control input will be large. Thus, we must make a trade-off between the convergence speed and the feasibility of LMIs. In MATLAB, this can be done by trial and error.

5.4 Simulation Results

For the purpose of comparison, the system parameters are chosen to be the same as those in [129], i.e., $M = 8.4796 \times 10^{-3}$ kg·m^2, $b = 114.6 \times 10^{-6}$ N·m·s/rad. The sampling period is $h = 0.001$ s. After discretization, the two matrices are

$$A_d = \begin{bmatrix} 1 & 9.999932 \times 10^{-4} \\ 0 & 0.999986 \end{bmatrix}, B_d = \begin{bmatrix} 5.896478 \times 10^{-5} \\ 0.117929 \end{bmatrix}.$$

The set for delays is $\Gamma = \{44, 50, 55, 60, 64\} \times h = \{0.044, 0.05, 0.055, 0.06, 0.064\}$ s, with the corresponding probability set $\Pi = \{0.05, 0.3, 0.4, 0.2, 0.05\}$. The lower and upper bounds here are 0.044 s and 0.064 s, respectively, the same as those in [129]. To be clearer, they are shown in Table 5.1.

Table 5.1: The set of delays and their associated probabilities.

Delay (h)	44	50	55	60	64
Delay (s)	0.044	0.05	0.055	0.06	0.064
Probability	0.05	0.3	0.4	0.2	0.05

The initial position of the master and slave is $q_m = q_s = 0$ rad, with zero initial velocities. The weighting factor is $\lambda = 10^3$. By setting $\sigma = 0.88, r = 0.04$, and solving the LMIs in Theorem 5.3, the controller $K = [-83.5104, -1.6467]$ is obtained. For the environment, it has stiffness $k_e = 40$ N·m/rad and damping coefficient $b_e = 5$ N·m·s/rad.

To assess the system performance, the relative error is defined:

$$E_r = \left| \frac{\max(q_m) - \max(q_s)}{\max(q_m)} \right| \times 100\%. \tag{5.26}$$

Figure 5.3 and Figure 5.4 show the trajectories and control torques for the master and slave, respectively. The relative error is $E_r = 3.43\%$, and the magnitude of torques is at a reasonable level. For a teleoperation system, in addition to the stability, we must also consider transparency when evaluating its performance. To this end, a comparison of f_h and f_e is shown in Figure 5.5. The ideal case for transparency would be $q_m = q_s$ and $f_h = f_e$. As can be observed from Figure 5.3 and Figure 5.5, we still get acceptable transparency in spite of the time-varying delays. Figure 5.6 shows the evolution of delay during the first 200 steps.

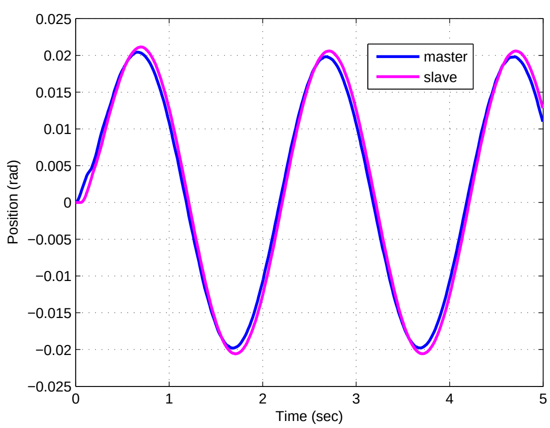

Figure 5.3: Trajectories of the master and slave with controller K.

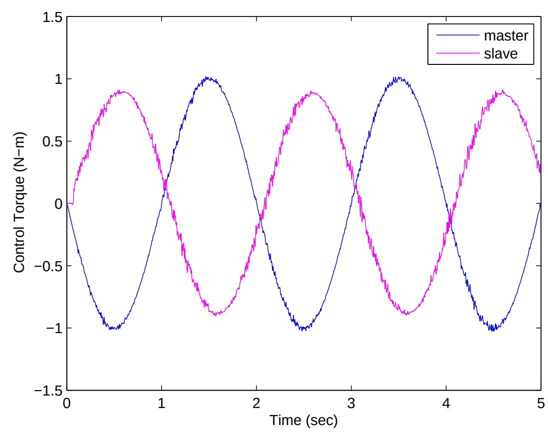

Figure 5.4: Control torques of the master and slave with controller K.

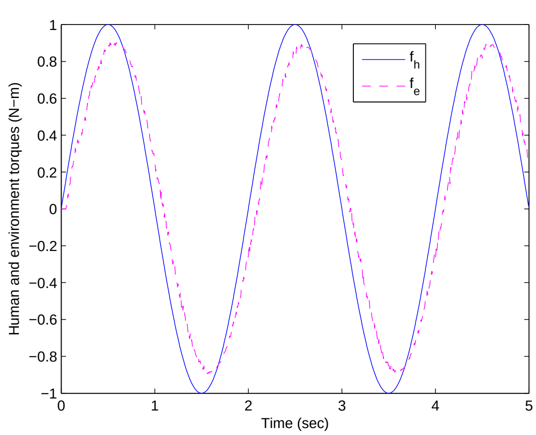

Figure 5.5: The human and environment torques f_h and f_e with controller K.

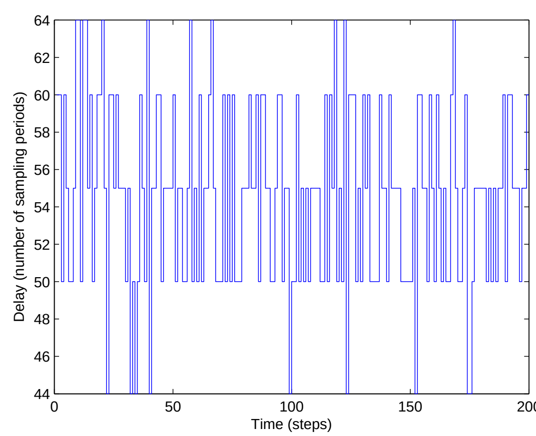

Figure 5.6: History of the time delay.

Next, we make a comparison with two controllers: $K_1 = [-1.4812, -9.1312]$ from [129] for the case without delay, and $K_2 = [-0.7206, -12.2508]$ that is obtained by selecting the same Lyapunov function and following the same derivation in [129]. These two controllers still use the same delay sequence. Figure 5.7 to Figure 5.12 demonstrate the simulation results using controllers K_1 and K_2. As can be seen from those figures, with controller K, the slave tracks the master better. By comparing Figure 5.5, Figure 5.11 and Figure 5.12, it is seen that the controller K exhibits satisfactory performance of force tracking. This, together with the comparison in position tracking, indicates that a higher level of transparency has been achieved.

In control problems, the energy consumption is also our concern. To address this issue, a comparison is listed in Table 5.2, where $\|\cdot\|_2$ and $\|\cdot\|_\infty$ denote the vector 2-norm and ∞-norm, respectively.

Table 5.2: Comparison of different controllers.

Controller	Relative error	$\|u_m\|_2$	$\|u_s\|_2$	$\|u_m\|_\infty$	$\|u_s\|_\infty$
K	3.43%	50.0746	44.5112	1.0393	0.9181
K_1	17.19%	50.1327	56.4138	1.0467	1.1954
K_2	13.50%	50.1437	58.7578	1.0689	1.2826

From Table 5.2, it is observed that the controller K achieves better tracking while requiring less control energy.

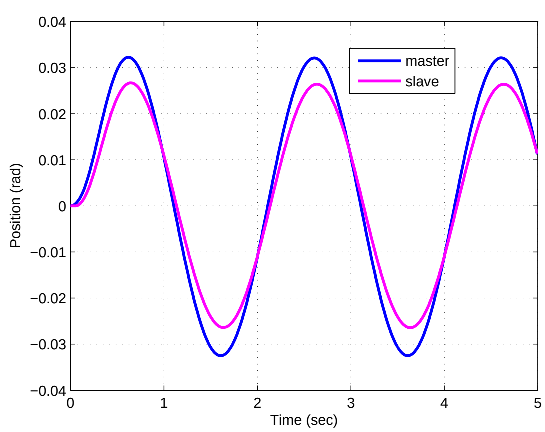

Figure 5.7: Trajectories of the master and slave with controller K_1.

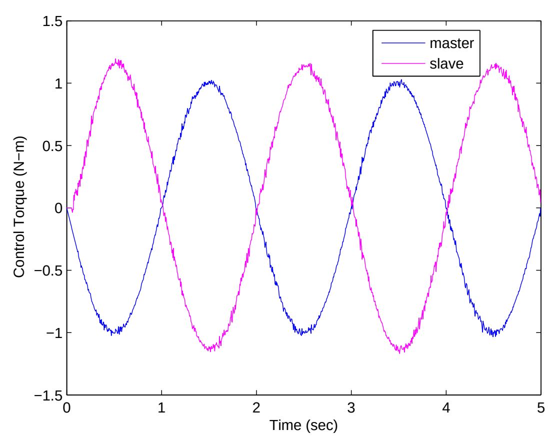

Figure 5.8: Control torques of the master and slave with controller K_1.

Remark 5.9. From the above simulation results and comparison, it is seen that with the consideration of stochastic features in time delay and the technique of pole placement, better tracking performance has been achieved. This validates the effectiveness

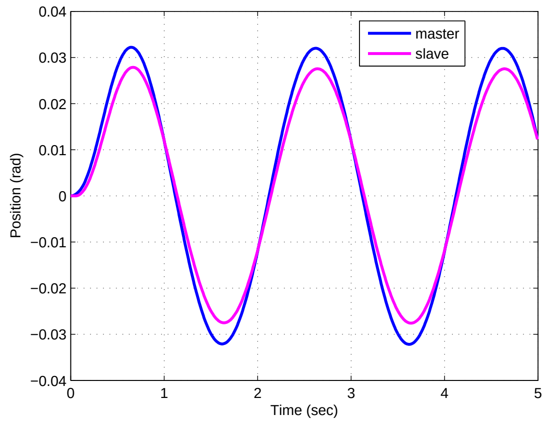

Figure 5.9: Trajectories of the master and slave with controller K_2.

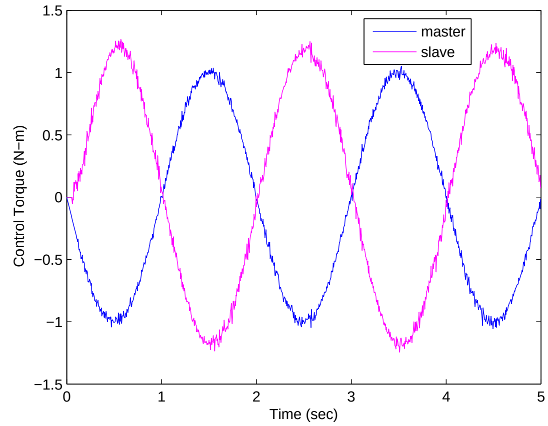

Figure 5.10: Control torques of the master and slave with controller K_2.

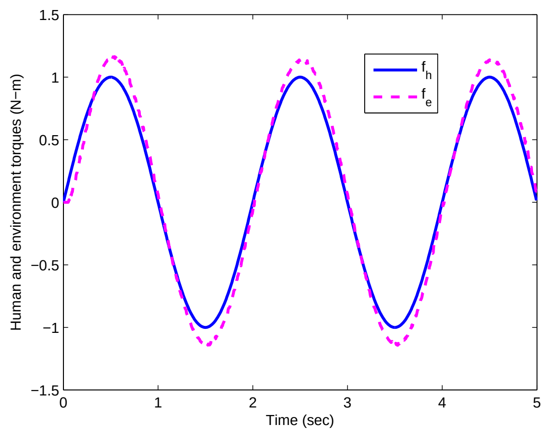

Figure 5.11: The human and environment torques f_h and f_e with controller K_1.

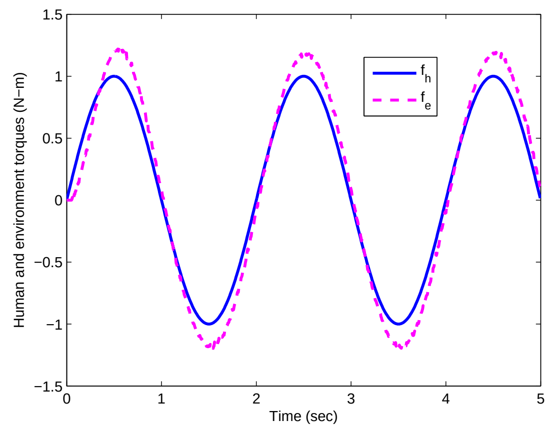

Figure 5.12: The human and environment torques f_h and f_e with controller K_2.

of the developed method. The designed controller is less conservative and leads to smaller tracking error.

For teleoperation, in addition to stability, transparency is also crucial. In much of the previous literature, this can be characterized by how close the transmitted impedance matches the environmental impedance [66]. The closer those two impedances match, the better transparency that we get. Another index is to see how well the slave tracks the master. In this work, we adopt the criterion of relative tracking error. With the proposed controller, the relative error has been decreased while with acceptable force tracking, thus better transparency has been achieved.

5.5 Conclusion

We have investigated the bilateral teleoperation in a network environment subject to stochastic time delays. By defining the tracking error between the master and slave, the teleoperation problem was formulated as the stabilization of the corresponding error dynamics. A PD controller with pole placement that incorporated statistic characteristics in delay was designed for both the master and slave. The input-to-state stability was used to assess the system performance, and the simulation results showed a decrease in tracking error.

It is worthwhile to further consider the following interesting topics of the teleoperation with random delays.

- In real applications, the model is always subject to uncertainties. The proposed methods in this work can be further extended to teleoperation systems subject to model uncertainties [118] [70].
- In practice, the delay from master to slave and that from slave to master are independent of each other. Therefore, it is more practical to investigate the robust teleoperation subject to asymmetrical delays [117].
- To further tackle the model uncertainties and the parameter variations in real-time, it would be useful to develop the online identification and adaptive control scheme over network for the teleoperation system [116] [115].

Chapter 6

Conclusions

6.1 Summary of the Thesis

This thesis investigates the consensus problem in MASs and bilateral teleoperation with various communication constraints. By designing control protocols appropriately, expected system performance has been guaranteed, resulting in an improvement in system's robustness.

For an MAS of single-integrator dynamics with time delays governed by a homogeneous Markov chain, Chapter 2 incorporates the statistical information into control design. In a directed graph, although its communication topology is fixed, we have introduced the scheme of delay-dependent adjacency matrices. With state transformation, the consensus problem is converted into the stabilization of its equivalent error dynamics. Then sufficient conditions in terms of LMIs and graph connectivity (existence of a spanning tree) are derived. Simulation results prove effectiveness of the proposed strategy. One important feature of the work in Chapter 2 lies in that the time-dependent adjacency matrices can be numerically designed by solving the established LMIs. Thus, it provides flexility in choosing controller parameters to drive the system states into consensus.

In Chapter 3, average consensus with time-varying delays and random data losses is studied. The agent dynamics is first-order and subject to delays with upper bounds. The communication topology is assumed to be fully symmetric, meaning that it is undirected and the communication link between any two agents exists and vanishes concurrently. This leads to the fact that the average of the initial states of agents is an invariant quantity, based on which the error dynamics is formed. Then sufficient conditions for consensus have been derived. It is observed in simulation that the probability of data loss affects the convergence speed to consensus. The higher the availability of data is, the faster the convergence speed becomes.

To address the phenomenon of non-uniform sampling, we study the consensus problem with double-integrator dynamics in Chapter 4. The communication graph is assumed to be directed and fixed. With the introduction of time-varying gains, conditions of ensuring consensus have been developed. The existence of a directed spanning tree is sufficient for connectivity of the interaction topology. Convergence analysis is conducted based on the theory on stochastic matrices.

It should be noted that in Chapter 2 and Chapter 3, single-integrator dynamics is considered, while in Chapter 4, double-integrator dynamics is studied. Compared with single-integrator dynamics, double-integrator dynamics exhibits a more general form and represents a larger class of systems. However, single-integrator dynamics also has applications in practice, such as the averaging algorithm. By studying single-integrator dynamics first, we can get some insight to the problem and then think about extending the current results. The study of single-integrator dynamics contributed a lot to the development of consensus theory in the early stage. Later, people moved to more complex dynamics to generalize the established theory in new scenarios.

In Chapter 5, bilateral teleoperation with stochastic time-varying delays is investigated. One of the features of this work is to make full use of the available delay information: Incorporation of probabilities of occurrence in delays into the analysis. The analysis is conducted by constructing an error vector between states of the master and salve. Then we derive conditions to guarantee the ISS stability of an error dynamic system with exogenous input. By virtue of the Lyapunov theory and pole placement technique, the system has achieved improved position tracking and enhanced transparency.

6.2 Future Work

In previous chapters, there are some assumptions in problem formulation. They may limit the spectrum of applications of the developed algorithms in practice. Therefore, it is necessary to consider further extension of the current work to adapt them to more general scenarios.

6.2.1 Convergence Rate Characterization in Average Consensus with Directed Graph

The work in Chapter 3 provides sufficient conditions for consensus, but does not uncover the quantitative relationship between the rate of convergence and availability of data. According to more simulation in Chapter 3, when the probability of data loss is lower, we can achieve faster convergence rate. This observation complies with our intuition. It motivates us for further in-depth research to see how the probability of data loss affects the system performance. Xiao and Boyd [141] used the spectral radius and 2-norm of the update matrix $W - \mathbf{1}\mathbf{1}^{\mathrm{T}}/n$ to quantify convergence speed of the deviation error from the average value. In [43], for the agreement problem subject to communication failure with uniform probability, the authors characterized the convergence rate in terms of the number of agents and the probability of data loss. For distributed average consensus with connected and undirected graph in [99], the convergence rate was characterized as the decay rate of worst-case variance of deviation from average. Each link failed independently with different probability from others.

To reach average consensus, the most common assumption in literature is that the communication topology is undirected. In this case, the average of states of the agents will be an invariant quantity, and we can research the properties of deviation of the system states from this invariant average value. Another assumption on topology is that the graph is strongly connected and balanced [96]. Under this condition, it can be shown that the average of agents' states is still invariant. In fact, a strongly connected graph should be enough for sufficient information exchange, because with that type of topology, any agent is able to influence any other agent directly or indirectly. A question arises naturally: Can we further extend the existing results on average consensus? The answer is affirmative and there has already been some work on this extension. It is noted in [10] [11] that in order to reach deterministic or quantized average consensus, the graph does not have to be undirected. The authors in [10] [11] introduced an additional quantity called "surplus" to track the record of state updates of all agents, and this information is transmitted to other neighboring agents. The authors in [28] proposed an adaptive algorithm to dynamically adjust the communication weights to reach average consensus. In that way, the weight matrix would finally be doubly stochastic.

Motivated by the above analysis, we have two potential directions to work towards as future work: 1) to characterize and quantify the convergence speed of consensus when the communication network is subject to random failure; 2) to relax the requirement on connectivity of the graph by extending from undirected graph to strongly connected graph.

6.2.2 Consensus with Multiple Sampling Rates

Due to the limited capacity of digital communication channels, all signals have to sampled in order for them to be transmitted. Then we face a practical problem: How fast should we sample the measurements so as to achieve the expected system performance? Obviously, sampling too fast is not always necessary and could be a waste of computational resource, while sampling too slowly is inadequate to satisfy the prescribed performance. When deploying a large-scale and distributed system, it is often not possible to let all sensors and actuators work at the same sampling period. Even in a local environment such as the hard disk drive servo system [54], the sampling frequency is limited in order to maximize the data capacity, and this results in the multi-rate phenomenon.

The multi-rate problem has been encountered in many fields, such as system control [16], filtering [113], and system identification [69]. Since a multi-rate system is a more general form compared with the single-rate one, if we utilize this property well, we can make better use of the available resource, while maintaining the performance. With the potential benefits, however, challenges also come along. An obvious issue is the increasing system complexity. Consequently, the design process for a multi-rate system is more involved. In previous literature, the most commonly used approach is the so-called lifting method [16] [15]. Its basic idea is to augment the system within a larger sampling period that is determined by different sampling rates. Then the augmented system becomes single-rate. During this process, one important concern is causality. If causality is not properly addressed, we may finally end up with a controller that is not physically implementable due to the use of future signals.

In a spatially distributed MAS equipped with different types of sensors and actuators, it is practical to consider multiple sampling rates. If we explicitly take into account the multirate feature, it is hopeful that the system performance will be improved. It is a choice to treat multirate control similarly as in the case of asynchronous consensus, in which agents update their states at different time instants. But that manner does not catch the essence of multiple rates. How to adapt that idea to MASs remains a concern and will be a direction for future work.

6.2.3 Bilateral Teleoperation with Asymmetric Delays

For the bilateral teleoperation considered in Chapter 5, it is assumed that the communication link from master to slave and that from slave to master are subject to the same delay at each time instant. In practice, there might be some difference between the two delays in two directions. To generalize this scheme, we can consider asymmetric delays across bidirectional communication channels. There exists some work dealing with time-varying and asymmetric delays. The authors in [22] [82] [20] studied the bilateral teleoperation with time-varying delays. By incorporating a time-varying gain across each communication channel and applying the scattering transformation, the teleoperation system is passified. In [48], teleoperation in Euler–Lagrange dynamics with time-varying and asymmetric delays was investigated. Stability with respect to the equilibria was established by constructing an appropriate Lyapunov–Krasovskii functional. Numerical conditions in terms of LMIs were provided to guarantee stability of the teleoperation system and calculate the maximum allowable delay. In particular, the work in [6] discussed the passivity preservation in teleoperation with discrete-time dynamics. A proper control strategy was proposed to handle the increasing delay and packet loss.

From the above review, we know that there are various ways to cope with teleoperation with asymmetric and time-varying delays, using either the passivity-based approach or others. The remaining question is whether we can utilize more delay information, such as the probabilistic distribution, to improve the system performance. This could be a research direction in the next step.

It is also meaningful to look into the teleoperation with more general dynamics, such as heterogeneous dynamics for the master and slave. In Chapter 5, the master and slave are assumed to have the same dynamics. In real world, due to various factors, they may possess different dynamics. For example, the components from which they are manufactured could have different properties, or we cannot identify the same model because of the existing model uncertainties. In addition, the nonlinear Euler–Lagrange dynamics that especially represents robotic manipulators deserves further investigation in the presence of probabilistic delays. Then potentially the previously developed scheme can be adapted to a larger family of dynamic systems.

Bibliography

[1] P. Albertos and J. Salt. Non-uniform sampled-data control of MIMO systems. *Annual Reviews in Control*, 35(1):65–76, 2011.

[2] R. J. Anderson and M. W. Spong. Bilateral control of teleoperators with time delay. *IEEE Transactions on Automatic Control*, 34(5):494–501, May 1989.

[3] M. Arcak. Passivity as a design tool for group coordination. *IEEE Transactions on Automatic Control*, 52(8):1380–1390, August 2007.

[4] T. C. Aysal, M. J. Coates, and M. G. Rabbat. Distributed average consensus with dithered quantization. *IEEE Transactions on Signal Processing*, 56(10):4905–4918, October 2008.

[5] D. Bauso, L. Giarré, and R. Pesenti. Non-linear protocols for optimal distributed consensus in networks of dynamic agents. *Systems & Control Letters*, 55(11):918–928, 2006.

[6] P. Berestesky, N. Chopra, and M. W. Spong. Discrete time passivity in bilateral teleoperation over the Internet. In *Proceedings of the 2004 IEEE International Conference on Robotics and Automation*, volume 5, pages 4557–4564, New Orleans, LA, April 2004.

[7] D. P. Bertsekas and J. N. Tsitsiklis. *Parallel and Distributed Computation: Numerical Methods*. Englewood Cliffs, NJ: Prentice-Hall, 1989.

[8] E.-K. Boukas. *Stochastic Switching Systems: Analysis and Design*. Control Engineering. Birkhäuser Boston, 2006.

[9] S. Boyd, L. Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*, volume 15 of *SIAM studies in applied mathematics*. Philadelphia: SIAM, 1994.

[10] K. Cai and H. Ishii. Quantized consensus and averaging on gossip digraphs. *IEEE Transactions on Automatic Control*, 56(9):2087–2100, September 2011.

[11] K. Cai and H. Ishii. Average consensus on general strongly connected digraphs. *Automatica*, 48(11):2750–2761, November 2012.

[12] Y. Cao and W. Ren. Sampled-data discrete-time coordination algorithms for double-integrator dynamics under dynamic directed interaction. *International Journal of Control*, 83(3):506–515, March 2010.

[13] C.-T. Chen. *Linear System Theory and Design*. Oxford University Press, New York, 3rd edition, 1999.

[14] G. Chen and F. L. Lewis. Robust consensus of multiple inertial agents with coupling delays and variable topologies. *International Journal of Robust and Nonlinear Control*, 21(6):666–685, 2011.

[15] T. Chen and B. Francis. *Optimal Sampled-Data Control Systems*. New York: Springer-Verlag, 1995.

[16] T. Chen and L. Qiu. H_∞ design of general multirate sampled-data control systems. *Automatica*, 30(7):1139–1152, July 1994.

[17] L. Cheng, Z.-G. Hou, Y. Lin, M. Tan, and W. Zhang. Solving a modified consensus problem of linear multi-agent systems. *Automatica*, 47(10):2218–2223, 2011.

[18] L. Cheng, Z.-G. Hou, M. Tan, Y. Lin, and W. Zhang. Neural-network-based adaptive leader-following control for multiagent systems with uncertainties. *IEEE Transactions on Neural Networks*, 21(8):1351–1358, August 2010.

[19] L. Cheng, Z.-G. Hou, M. Tan, and X. Wang. Necessary and sufficient conditions for consensus of double-integrator multi-agent systems with measurement noises. *IEEE Transactions on Automatic Control*, 56(8):1958–1963, August 2011.

[20] N. Chopra, P. Berestesky, and M. W. Spong. Bilateral teleoperation over unreliable communication networks. *IEEE Transactions on Control Systems Technology*, 16(2):304–313, March 2008.

[21] N. Chopra and M. W. Spong. Passivity-based control of multi-agent systems. In Sadao Kawamura and Mikhail Svinin, editors, *Advances in Robot Control*, pages 107–134. Springer Berlin Heidelberg, 2006.

[22] N. Chopra, M. W. Spong, S. Hirche, and M. Buss. Bilateral teleoperation over the internet: the time varying delay problem. In *Proceedings of the American Control Conference*, volume 1, pages 155–160, Denver, Colorado, June 2003.

[23] M. B. G. Cloosterman, L. Hetel, N. van de Wouw, W. P. M. H. Heemels, J. Daafouz, and H. Nijmeijer. Controller synthesis for networked control systems. *Automatica*, 46(10):1584–1594, 2010.

[24] J. Cortés. Global and robust formation-shape stabilization of relative sensing networks. *Automatica*, 45(12):2754–2762, December 2009.

[25] O. L. V. Costa, M. D. Fragoso, and R. P. Marques. *Discrete-Time Markov Jump Linear Systems*. Probability and Its Applications. London: Springer-Verlag, 2005.

[26] P. Desbats, F. Geffard, G. Piolain, and A. Coudray. Force-feedback teleoperation of an industrial robot in a nuclear spent fuel reprocessing plant. *Industrial Robot: An International Journal*, 33(3):178–186, 2006.

[27] R. Diestel. *Graph Theory*, volume 173 of *Graduate Texts in Mathematics*. New York: Springer-Verlag, 3rd edition, 2005.

[28] A. D. Domínguez-García and C. N. Hadjicostis. Distributed strategies for average consensus in directed graphs. In *2011 50th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC)*, pages 2124–2129, Orlando, FL, USA, December 2011.

[29] F. Fagnani and S. Zampieri. Average consensus with packet drop communication. *SIAM Journal on Control and Optimization*, 48(1):102–133, 2009.

[30] L. Fang and P. J. Antsaklis. Asynchronous consensus protocols using nonlinear paracontractions theory. *IEEE Transactions on Automatic Control*, 53(10):2351–2355, November 2008.

[31] J. A. Fax and R. M. Murray. Information flow and cooperative control of vehicle formations. *IEEE Transactions on Automatic Control*, 49(9):1465–1476, September 2004.

[32] J.-E Feng, J. Lam, and Z. Shu. Stabilization of Markovian systems via probability rate synthesis and output feedback. *IEEE Transactions on Automatic Control*, 55(3):773–777, March 2010.

[33] G. Ferrari-Trecate, L. Galbusera, M. P. E. Marciandi, and R. Scattolini. Model predictive control schemes for consensus in multi-agent systems with single- and double-integrator dynamics. *IEEE Transactions on Automatic Control*, 54(11):2560–2572, November 2009.

[34] P. Frasca, R. Carli, F. Fagnani, and S. Zampieri. Average consensus on networks with quantized communication. *International Journal of Robust and Nonlinear Control*, 19(16):1787–1816, November 2009.

[35] H. Gao, X. Meng, and T. Chen. Stabilization of networked control systems with a new delay characterization. *IEEE Transactions on Automatic Control*, 53(9):2142–2148, October 2008.

[36] Y. Gao, L. Wang, G. Xie, and B. Wu. Consensus of multi-agent systems based on sampled-data control. *International Journal of Control*, 82(12):2193–2205, December 2009.

[37] C. Godsil and G. Royle. *Algebraic Graph Theory*, volume 207 of *Graduate Texts in Mathematics*. New York: Springer-Verlag, 2001.

[38] G. Grimmett and D. Stirzaker. *Stability and Random Process*. Oxford University Press Inc. New York, 3rd edition, 2001.

[39] K. Gu and S.-I. Niculescu. Survey on recent results in the stability and control of time-delay systems. *Journal of Dynamic Systems, Measurement, and Control*, 125(2):158–165, June 2003.

[40] W. Guo, J. Lü, S. Chen, and X. Yu. Second-order tracking control for leader–follower multi-agent flocking in directed graphs with switching topology. *Systems & Control Letters*, 60(12):1051–1058, December 2011.

[41] A. Haddadi and K. Hashtrudi-Zaad. Bounded-impedance absolute stability of bilateral teleoperation control systems. *IEEE Transactions on Haptics*, 3(1):15–27, January-March 2010.

[42] D. W. Hainsworth. Teleoperation user interfaces for mining robotics. *Autonomous Robots*, 11(1):19–28, 2001.

[43] Y. Hatano and M. Mesbahi. Agreement over random networks. *IEEE Transactions on Automatic Control*, 50(11):1867–1872, November 2005.

[44] L. Hetel, J. Daafouz, and C. Iung. Equivalence between the Lyapunov-Krasovskii functionals approach for discrete delay systems and that of the stability conditions for switched systems. *Nonlinear Analysis: Hybrid Systems*, 2(3):697–705, 2008.

[45] P. F. Hokayem and M. W. Spong. Bilateral teleoperation: An historical survey. *Automatica*, 42(12):2035–2057, 2006.

[46] Y. Hong, G. Chen, and L. Bushnell. Distributed observers design for leader-following control of multi-agent networks. *Automatica*, 44(3):846–850, 2008.

[47] Z.-G. Hou, L. Cheng, and M. Tan. Decentralized robust adaptive control for the multiagent system consensus problem using neural networks. *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, 39(3):636–647, June 2009.

[48] C.-C. Hua and P. X. Liu. Convergence analysis of teleoperation systems with unsymmetric time-varying delays. *IEEE Transactions on Circuits and Systems II: Express Briefs*, 56(3):240–244, March 2009.

[49] M. Huang, S. Dey, G. N. Nair, and J. H. Manton. Stochastic consensus over noisy networks with Markovian and arbitrary switches. *Automatica*, 46(10):1571–1583, 2010.

[50] M. Huang and J. H. Manton. Coordination and consensus of networked agents with noisy measurements: Stochastic algorithms and asymptotic behavior. *SIAM Journal on Control and Optimization*, 48(1):134–161, 2009.

[51] Q. Hui, W. M. Haddad, and S. P. Bhat. On robust control algorithms for nonlinear network consensus protocols. *International Journal of Robust and Nonlinear Control*, 20(3):269–284, 2010.

[52] A. Iborra, J. A. Pastor, B. Alvarez, C. Fernandez, and J. M. F. Merono. Robots in radioactive environments. *IEEE Robotics & Automation Magazine*, 10(4):12–22, December 2003.

[53] A. Jadbabaie, J. Lin, and A. S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions on Automatic Control*, 48(6):988–1001, June 2003.

[54] Q. Jia. A new method of multirate state feedback control with application to an hdd servo system. *Mechatronics*, 18(1):13–20, 2008.

[55] F. Jiang and L. Wang. Finite-time weighted average consensus with respect to a monotonic function and its application. *Systems & Control Letters*, 60(9):718–725, 2011.

[56] S. Kar and J. M. F. Moura. Distributed consensus algorithms in sensor networks: Quantized data and random link failures. *IEEE Transactions on Signal Processing*, 58(3):1383–1400, March 2010.

[57] S. Kar, J. M. F. Moura, and K. Ramanan. Distributed parameter estimation in sensor networks: Nonlinear observation models and imperfect communication. *IEEE Transactions on Information Theory*, 58(6):3575–3605, June 2012.

[58] A. Kashyap, T. Başar, and R. Srikant. Quantized consensus. *Automatica*, 43(7):1192–1203, 2007.

[59] H. K. Khalil. *Nonlinear Systems*. Prentice Hall, Upper Saddle River, NJ, 3rd edition, 2002.

[60] H. Kim, H. Shim, and J. H. Seo. Output consensus of heterogeneous uncertain linear multi-agent systems. *IEEE Transactions on Automatic Control*, 56(1):200–206, January 2011.

[61] M. Kumar, D. P. Garg, and V. Kumar. Segregation of heterogeneous units in a swarm of robotic agents. *IEEE Transactions on Automatic Control*, 55(3):743–748, March 2010.

[62] V. Kumar, N. Leonard, and A. S. Morse, editors. *Cooperative Control: A Post-Workshop Volume 2003 Block Island Workshop on Cooperative Control*, volume 309 of *Lecture Notes in Control and Information Sciences*. Springer-Verlag Berlin Heidelberg, 2005.

[63] G. Lafferriere, A. Williams, J. Caughman, and J. J. P. Veerman. Decentralized control of vehicle formations. *Systems & Control Letters*, 54(9):899–910, September 2005.

[64] J.-P. Launay. Teleoperation and nuclear services advantages of computerized operator-assistance tools. *Nuclear Engineering and Design*, 180(1):47–52, 1998.

[65] J. Lavaei and R. M. Murray. Quantized consensus by means of gossip algorithm. *IEEE Transactions on Automatic Control*, 57(1):19–32, January 2012.

[66] D. A. Lawrence. Stability and transparency in bilateral teleoperation. *IEEE Transactions on Robotics and Automation*, 9(5):624–637, October 1993.

[67] D. Lee and M. W. Spong. Passive bilateral teleoperation with constant time delay. *IEEE Transactions on Robotics*, 22(2):269–281, April 2006.

[68] G. M. H. Leung, B. A. Francis, and J. Apkarian. Bilateral controller for teleoperators with time delay via μ-synthesis. *IEEE Transactions on Robotics and Automation*, 11(1):105–116, February 1995.

[69] D. Li, S. L. Shah, and T. Chen. Identification of fast-rate models from multirate data. *International Journal of Control*, 74(7):680–689, 2001.

[70] H. Li and Y. Shi. Robust H_∞ filtering for nonlinear stochastic systems with uncertainties and Markov delays. *Automatica*, 48(1):159–166, 2012.

[71] T. Li and J.-F. Zhang. Mean square average-consensus under measurement noises and fixed topologies: Necessary and sufficient conditions. *Automatica*, 45(8):1929–1936, August 2009.

[72] T. Li and J.-F. Zhang. Consensus conditions of multi-agent systems with time-varying topologies and stochastic communication noises. *IEEE Transactions on Automatic Control*, 55(9):2043–2057, September 2010.

[73] W. Li and Y. Jia. Consensus-based distributed multiple model UKF for jump Markov nonlinear systems. *IEEE Transactions on Automatic Control*, 57(1):227–233, January 2012.

[74] P. Lin and Y. Jia. Consensus of a class of second-order multi-agent systems with time-delay and jointly-connected topologies. *IEEE Transactions on Automatic Control*, 55(3):778–784, March 2010.

[75] P. Lin and Y. Jia. Distributed rotating formation control of multi-agent systems. *Systems & Control Letters*, 59(10):587–595, October 2010.

[76] P. Lin and Y. Jia. Multi-agent consensus with diverse time-delays and jointly-connected topologies. *Automatica*, 47(4):848–856, 2011.

[77] P. Lin, K. Qin, Z. Li, and W. Ren. Collective rotating motions of second-order multi-agent systems in three-dimensional space. *Systems & Control Letters*, 60(6):365–372, 2011.

[78] Z. Lin, B. Francis, and M. Maggiore. Necessary and sufficient conditions for formation control of unicycles. *IEEE Transactions on Automatic Control*, 50(1):121–127, January 2005.

[79] Z. Lin, B. Francis, and M. Maggiore. State agreement for continuous-time coupled nonlinear systems. *SIAM Journal on Control and Optimization*, 46(1):288–307, 2007.

[80] Y. Liu and K. M. Passino. Cohesive behaviors of multiagent systems with information flow constraints. *IEEE Transactions on Automatic Control*, 51(11):1734–1748, November 2006.

[81] W.-T. Lo, Y. Liu, I. H. Elhajj, N. Xi, Y. Wang, and T. Fukuda. Cooperative teleoperation of a multirobot system with force reflection via internet. *IEEE/ASME Transactions on Mechatronics*, 9(4):661–670, December 2004.

[82] R. Lozano, N. Chopra, and M. W. Spong. Passivation of force reflecting bilateral teleoperators with time varying delay. In *Proceedings of the 8th Mechatronics Forum*, pages 954–962, Netherlands, 2002.

[83] C.-Q. Ma and J.-F. Zhang. Necessary and sufficient conditions for consensusability of linear multi-agent systems. *IEEE Transactions on Automatic Control*, 55(5):1263–1268, May 2010.

[84] F. Mobasser and K. Hashtrudi-Zaad. Transparent rate mode bilateral teleoperation control. *The International Journal of Robotics Research*, 27(1):57–72, 2008.

[85] L. Moreau. Stability of continuous-time distributed consensus algorithms. In *43rd IEEE Conference on Decision and Control*, volume 4, pages 3998–4003, Atlantis, Paradise Island, Bahamas, December 2004.

[86] L. Moreau. Stability of multiagent systems with time-dependent communication links. *IEEE Transactions on Automatic Control*, 50(2):169–182, February 2005.

[87] R. M. Murray. Recent research in cooperative control of multivehicle systems. *Journal of Dynamic Systems, Measurement, and Control*, 129(5):571–583, September 2007.

[88] A. Nedić, A. Olshevsky, A. Ozdaglar, and J. N. Tsitsiklis. On distributed averaging algorithms and quantization effects. *IEEE Transactions on Automatic Control*, 54(11):2506–2517, November 2009.

[89] A. Nedić, A. Ozdaglar, and P. A. Parrilo. Consensus of a class of second-order multi-agent systems with time-delay and jointly-connected topologies. *IEEE Transactions on Automatic Control*, 55(4):922–938, April 2010.

[90] W. Ni and D. Cheng. Leader-following consensus of multi-agent systems under fixed and switching topologies. *Systems & Control Letters*, 59(3-4):209–217, 2010.

[91] G. Niemeyer and J.-J. E. Slotine. Stable adaptive teleoperation. *IEEE Journal of Oceanic Engineering*, 16(1):152–162, January 1991.

[92] G. Niemeyer and J.-J. E. Slotine. Telemanipulation with time delays. *The International Journal of Robotics Research*, 23(9):873–890, 2004.

[93] J. Nilsson. *Real-Time Control Systems with Delays*. PhD thesis, Lund Institute of Technology, Lund, Sweden, 1998.

[94] R. Olfati-Saber. Distributed Kalman filter with embedded consensus filters. In *44th IEEE Conference on Decision and Control, and European Control Conference 2005*, pages 8179–8184, December 2005.

[95] R. Olfati-Saber. Flocking for multi-agent dynamic systems: algorithms and theory. *IEEE Transactions on Automatic Control*, 51(3):401–420, March 2006.

[96] R. Olfati-Saber and R. M. Murray. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, 49(9):1520–1533, September 2004.

[97] R. Olfati-Saber and J. S. Shamma. Consensus filters for sensor networks and distributed sensor fusion. In *44th IEEE Conference on Decision and Control, and European Control Conference 2005*, pages 6698–6703, December 2005.

[98] C. D. Onal and M. Sitti. Teleoperated 3-D force feedback from the nanoscale with an atomic force microscope. *IEEE Transactions on Nanotechnology*, 9(1):46–54, January 2010.

[99] S. Patterson, B. Bamieh, and A. El Abbadi. Convergence rates of distributed average consensus with stochastic link failures. *IEEE Transactions on Automatic Control*, 55(4):880–892, April 2010.

[100] I. G. Polushin, P. X. Liu, and C.-H. Lung. A force-reflection algorithm for improved transparency in bilateral teleoperation with communication delay. *IEEE/ASME Transactions on Mechatronics*, 12(3):361–374, June 2007.

[101] M. Porfiri and D. J. Stilwell. Consensus seeking over random weighted directed graphs. *IEEE Transactions on Automatic Control*, 52(9):1767–1773, September 2007.

[102] C. Preusche, T. Ortmaier, and G. Hirzinger. Teleoperation concepts in minimal invasive surgery. *Control Engineering Practice*, 10(11):1245–1250, 2002.

[103] S. Rao and D. Ghose. Sliding mode control-based algorithms for consensus in connected swarms. *International Journal of Control*, 84(9):1477–1490, September 2011.

[104] W. Ren. On consensus algorithms for double-integrator dynamics. *IEEE Transactions on Automatic Control*, 53(6):1503–1509, July 2008.

[105] W. Ren. Distributed leaderless consensus algorithms for networked Euler-Lagrange systems. *International Journal of Control*, 82(11):2137–2149, November 2009.

[106] W. Ren and E. Atkins. Distributed multi-vehicle coordinated control via local information exchange. *International Journal of Robust and Nonlinear Control*, 17(10-11):1002–1033, 2007.

[107] W. Ren and R. W. Beard. Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Transactions on Automatic Control*, 50(5):655–661, May 2005.

[108] J.-H. Ryu, J. Artigas, and C. Preusche. A passive bilateral control scheme for a teleoperator with time-varying communication delay. *Mechatronics*, 20(7):812–823, 2010.

[109] L. Scardovi and R. Sepulchre. Synchronization in networks of identical linear systems. *Automatica*, 45(11):2557–2562, 2009.

[110] L. Schenato and G. Gamba. A distributed consensus protocol for clock synchronization in wireless sensor network. In *46th IEEE Conference on Decision and Control*, pages 2289–2294, New Orleans, LA, USA, December 2007.

[111] S. Seung, B. Kang, H. Je, J. Park, K. Kim, and S. Park. Tele-operation master-slave system for minimal invasive brain surgery. In *Proceedings of the 2009 IEEE International Conference on Robotics and Biomimetics*, pages 177–182, Guilin, China, December 2009.

[112] J. Shao, G. Xie, and L. Wang. Leader-following formation control of multiple mobile vehicles. *IET Control Theory & Applications*, 1(2):545–552, March 2007.

[113] J. Sheng, T. Chen, and S. L. Shah. Optimal filtering for multirate systems. *IEEE Transactions on Circuits and Systems–II: Express Briefs*, 52(4):228–232, April 2005.

[114] T. B. Sheridan. Telerobotics. *Automatica*, 25(4):487–507, 1989.

[115] Y. Shi and H. Fang. Kalman filter-based identification for systems with randomly missing measurements in a network environment. *International Journal of Control*, 83(3):538–551, 2010.

[116] Y. Shi, H. Fang, and M. Yan. Kalman filter-based adaptive control for networked systems with unknown parameters and randomly missing outputs. *International Journal of Robust and Nonlinear Control*, 19(18):1976–1992, 2009.

[117] Y. Shi and B. Yu. Output feedback stabilization of networked control systems with random delays modeled by Markov chains. *IEEE Transactions on Automatic Control*, 54(7):1668–1674, July 2009.

[118] Y. Shi and B. Yu. Robust mixed H_2/H_∞ control of networked control systems with random time delays in both forward and backward communication links. *Automatica*, 47(4):754–760, 2011.

[119] S. Sirouspour and A. Shahdi. Discrete-time linear quadratic Gaussian control for teleoperation under communication time delay. *The International Journal of Robotics Research*, 25(2):187–202, 2006.

[120] S. S. Stanković and, M. S. Stanković, and D. M. Stipanović. Decentralized parameter estimation by consensus based stochastic approximation. *IEEE Transactions on Automatic Control*, 56(3):531–543, March 2011.

[121] H. Su, X. Wang, and Z. Lin. Flocking of multi-agents with a virtual leader. *IEEE Transactions on Automatic Control*, 54(2):293–307, February 2009.

[122] Y. G. Sun and L. Wang. Consensus of multi-agent systems in directed networks with nonuniform time-varying delays. *IEEE Transactions on Automatic Control*, 54(7):1607–1613, July 2009.

[123] Y. G. Sun and L. Wang. Consensus problems in networks of agents with double-integrator dynamics and time-varying delays. *International Journal of Control*, 82(10):1937–1945, 2009.

[124] Y. G. Sun, L. Wang, and G. Xie. Average consensus in networks of dynamic agents with switching topologies and multiple time-varying delays. *Systems & Control Letters*, 57(2):175–183, 2008.

[125] H. G. Tanner, A. Jadbabaie, and G. J. Pappas. Flocking in fixed and switching networks. *IEEE Transactions on Automatic Control*, 52(5):863–868, May 2007.

[126] Y.-P. Tian and C.-L. Liu. Consensus of multi-agent systems with diverse input and communication delays. *IEEE Transactions on Automatic Control*, 53(9):2122–2128, October 2008.

[127] Y.-P. Tian and C.-L. Liu. Robust consensus of multi-agent systems with diverse input delays and asymmetric interconnection perturbations. *Automatica*, 45(5):1347–1353, May 2009.

[128] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, and O. Schochet. Novel type of phase transitions in a system of self-driven particles. *Physical Review Letters*, 75(6):1226–1229, August 1995.

[129] K. C. Walker, Y.-J. Pan, and J. Gu. Bilateral teleoperation over networks based on stochastic switching approach. *IEEE/ASME Transactions on Mechatronics*, 14(5):539–554, October 2009.

[130] J.-L. Wang and H.-N. Wu. Leader-following formation control of multi-agent systems under fixed and switching topologies. *International Journal of Control*, 85(6):695–705, June 2012.

[131] X. Wang, W. Ni, and X. Wang. Leader-following formation of switching multirobot systems via internal model. *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, 42(3):817–826, June 2012.

[132] J. Wolfowitz. Products of indecomposable, aperiodic, stochastic matrices. *Proceedings of the American Mathematical Society*, 14(5):733–737, October 1963.

[133] C. W. Wu. Synchronization and convergence of linear dynamics in random directed networks. *IEEE Transactions on Automatic Control*, 51(7):1207–1210, July 2006.

[134] J. Wu and Y. Shi. Consensus in multi-agent systems with random delays governed by a Markov chain. *Systems & Control Letters*, 60(10):863–870, 2011.

[135] J. Wu, H. Zhang, and Y. Shi. H_2 state estimation for network-based systems subject to probabilistic delays. *Signal Processing*, 92(11):2700–2705, 2012.

[136] J. Xiang, W. Wei, and Y. Li. Synchronized output regulation of linear networked systems. *IEEE Transactions on Automatic Control*, 54(6):1336–1341, June 2009.

[137] F. Xiao and T. Chen. Sampled-data consensus for multiple double integrators with arbitrary sampling. *IEEE Transactions on Automatic Control*, 57(12):3230–3235, December 2012.

[138] F. Xiao and L. Wang. State consensus for multi-agent systems with switching topologies and time-varying delays. *International Journal of Control*, 79(10):1277–1284, October 2006.

[139] F. Xiao and L. Wang. Asynchronous consensus in continuous-time multi-agent systems with switching topology and time-varying delays. *IEEE Transactions on Automatic Control*, 53(8):1804–1816, September 2008.

[140] F. Xiao, L. Wang, J. Chen, and Y. Gao. Finite-time formation control for multi-agent systems. *Automatica*, 45(11):2605–2611, November 2009.

[141] L. Xiao and S. Boyd. Fast linear iterations for distributed averaging. *Systems & Control Letters*, 53(1):65–78, 2004.

[142] J. Xiong and J. Lam. Stabilization of networked control systems with a logic ZOH. *IEEE Transactions on Automatic Control*, 54(2):358–363, February 2009.

[143] S. Xu and J. Lam. A survey of linear matrix inequality techniques in stability analysis of delay systems. *International Journal of Systems Science*, 39(12):1095–1113, December 2008.

[144] Y. Xu and W. Liu. Novel multiagent based load restoration algorithm for microgrids. *IEEE Transactions on Smart Grid*, 2(1):152–161, March 2011.

[145] Y. Ye and P. X. Liu. Improving trajectory tracking in wave-variable-based teleoperation. *IEEE/ASME Transactions on Mechatronics*, 15(2):321–326, April 2010.

[146] J. Yu, S. M. LaValle, and D. Liberzon. Rendezvous without coordinates. *IEEE Transactions on Automatic Control*, 57(2):421–434, February 2012.

[147] W. Yu, G. Chen, and M. Cao. Distributed leader-follower flocking control for multi-agent dynamical systems with time-varying velocities. *Systems & Control Letters*, 59(9):543–552, September 2010.

[148] W. Yu, G. Chen, and M. Cao. Some necessary and sufficient conditions for second-order consensus in multi-agent dynamical systems. *Automatica*, 46(6):1089–1095, 2010.

[149] W. Yu, W. X. Zheng, G. Chen, W. Ren, and J. Cao. Second-order consensus in multi-agent dynamical systems with sampled position data. *Automatica*, 47(7):1496–1503, 2011.

[150] C. J. Zandsteeg, D. J. H. Bruijnen, and M. J. G. van de Molengraft. Haptic tele-operation system control design for the ultrasound task: A loop-shaping approach. *Mechatronics*, 20(7):767–777, 2010.

[151] H. Zhang, Y. Shi, and A. S. Mehr. Robust H_∞ PID control for multivariable networked control systems with disturbance/noise attenuation. *International Journal of Robust and Nonlinear Control*, 22(2):183–204, January 2012.

[152] H.-T. Zhang, C. Zhai, and Z. Chen. A general alignment repulsion algorithm for flocking of multi-agent systems. *IEEE Transactions on Automatic Control*, 56(2):430–435, February 2011.

[153] L. Zhang, E.-K. Boukas, and J. Lam. Analysis and synthesis of Markov jump linear systems with time-varying delays and partially known transition probabilities. *IEEE Transactions on Automatic Control*, 53(10):2458–2464, November 2008.

[154] Q. Zhang and J.-F. Zhang. Distributed parameter estimation over unreliable networks with Markovian switching topologies. *IEEE Transactions on Automatic Control*, 57(10):2545–2560, October 2012.

[155] Y. Zhang and Y.-P. Tian. Consentability and protocol design of multi-agent systems with stochastic switching topology. *Automatica*, 45(5):1195–1201, May 2009.

[156] Y. Zhang and Y.-P. Tian. Consensus of data-sampled multi-agent systems with random communication delay and packet loss. *IEEE Transactions on Automatic Control*, 55(4):939–943, April 2010.

[157] Y. Zhang and Y.-P. Tian. Maximum allowable loss probability for consensus of multi-agent systems over random weighted lossy networks. *IEEE Transactions on Automatic Control*, 57(8):2127–2132, August 2012.

[158] Z. Zhang and M.-Y. Chow. Convergence analysis of the incremental cost consensus algorithm under different communication network topologies in a smart grid. *IEEE Transactions on Power Systems*, 27(4):1761–1768, November 2012.

Appendix A

Publications

The following is a list of publications during the PhD studies.

Journal papers (published)

1. **J. Wu**, Y. Shi, J. Huang, and D. Constantinescu. Stochastic stabilization for bilateral teleoperation over networks with probabilistic delays. *Mechatronics*, 22(8):1050–1059, December 2012.

2. **J. Wu**, H. Zhang, and Y. Shi. H_2 state estimation for network-based systems subject to probabilistic delays. *Signal Processing*, 92(11):2700–2705, November 2012.

3. **J. Wu** and Y. Shi. Consensus in multi-agent systems with random delays governed by a Markov chain. *Systems & Control Letters*, 60(10):863–870, October 2011.

4. J. Huang, Y. Shi, and **J. Wu**. Transparent virtual coupler design for networked haptic systems with a mixed virtual wall. *IEEE/ASME Transactions on Mechatronics*, 17(3):480–487, June 2012.

5. H. Fang, **J. Wu**, and Y. Shi. Genetic adaptive state estimation with missing input/output data. *Proceedings of the Institution of Mechanical Engineers, Part I: Journal of Systems and Control Engineering*, 224(5):611–617, 2010.

Conference papers (published and accepted)

1. **J. Wu** and Y. Shi. Consensus in multi-agent systems with non-uniform sampling. Accepted by *2013 American Control Conference*, January 2013.

2. **J. Wu** and Y. Shi. Average consensus in multi-agent systems with time-varying delays and packet losses. In *Proceedings of the 2012 American Control Conference*, pages 1579–1584, Montréal, Canada, June 27-29, 2012.

3. **J. Wu**, B. Mu, and Y. Shi. Stable bilateral teleoperation over networks with random delays. In *Proceedings of the 2012 CSME International Congress*, Winnipeg, Manitoba, Canada, June 4-6, 2012.

4. W. Li, M. Liu, **J. Wu**, and Y. Shi. Cooperative navigation for autonomous underwater vehicles. In *Proceedings of the 23rd Canadian Congress of Applied Mechanics*, Vancouver, BC, Canada, June 5-9, 2011.

5. H. Fang, Y. Shi, and **J. Wu**. Parameter estimation with missing input/output data. In *Proceedings of the 2009 American Control Conference*, pages 5061–5066, St. Louis, MO, USA, June 10-12, 2009.